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Frequency and Severity Modelling Using Multifractal Processes: An Application to Tornado Occurrence in the USA and CAT Bonds

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Abstract This paper proposes a statistical model for insurance claims arising from climatic events, such as tornadoes in the USA, that exhibit a large variability both in frequency and intensity. To represent this variability and seasonality, the claims process modelled by a Poisson process of intensity equal to the product of a periodic function, and a multifractal process is proposed. The size of claims is modelled in a similar way, with gamma random variables. This method is shown to enable simulation of the peak times of damage. A two-dimensional multifractal model is also investigated. The work concludes with an analysis of the impact of the model on the yield of weather bonds linked to damage caused by tornadoes.

Keywords Multifractal process · Claims process · Poisson · Gamma · Dependence · CAT bonds

1 Introduction

As reported by Barrieu and Scaillet [1], weather is not only an environmental issue but also a key economic factor. W. Daley, the former US commerce secretary, stated in 1998 that at least $1 trillion of the world economy is weather-sensitive. There are two main solutions to financially hedge against the economic losses caused by the weather. The first is to take out an insurance policy, but this is not always an appropriate solution in the case of such climatic events such as storm or drought, or for those events that exhibit a huge variability in the frequency of occurrence, such as tornadoes. The second way to hedge against weather risk is to purchase financial contracts depending on weather conditions. This type of contract is for most of the time a tailor made transaction, traded on the over-the-counter market. Some basic weather derivatives (mainly designed for the US) are however also traded on the Chicago Mercantile Exchange. The Weather Risk Management Association conducts a survey every year of the weather derivatives market. The value of trade in the year to March 2011 totalled $11.8 billion, which is nearly 20 % more than the previous year, though far below the peak reached before the financial crisis dampened down business activity. In 2005–2006, the value of contracts had reached $45 billion.

The first weather contract was concluded in 1997 between Enron and Kock Industries and was based upon temperature indices. In parallel to the development of futures and options, of which the price is mainly related to the progression of indices, weather and catastrophe (CAT) bonds have also appeared on the market. These bonds deliver payments that are directly related to the occurrence of climatic events. Such weather or CAT bonds are interesting tools of investment for investors looking for diversifaction, given that they only have a very small correlation with traditional financial markets. The work of Schmock [2] gives a detailed analysis of the WINCAT bond, a CAT bond linked to damages caused by hail and storm to motor vehicles insured with Winterthur in Switzerland. A survey of products and their applications is given in Barrieu and Dischel [3].

Physical models for the analysis and forecasting of damage related to recurrent meteorological events have a limited tractability for financial applications such as the pricing of climatic products, given their complexity. For this reason, the existing literature on the pricing of weather derivatives mainly develop statistical models. In his dissertation, Lopez Cabrera [4] provides a detailed survey. In Vaugirard [5] or in Lee and Yub [6], claims caused by weather catastrophes are modelled by a jump diffusion process. In Alaton et al. [7] or Campbell

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and Diebold [8], the index of temperature is modelled by a Brownian motion applying a seasonal drift. Other climatic indices are modelled by an Ornstein-Uhlenbeck process such as used by Dornier and Queruel [9] and Benth and Benth [10,11]. In Hainaut [12], a similar approach is used to model the arrival process of seasonal claims.

The first objective of this paper is to propose a new statistical model for the claims arrival and cost processes that reflects seasonality and the large variability of meteorologic events. The second objective is to illustrate how this model can be subsequently used to price weather derivatives such as CAT bonds. The novelty of this approach is that it considers that the parameters defining the claims numbers and costs are stochastic multifractal processes. In other words, a multifractal process is a combination of hidden similar basic processes, here two states Markov chains. These chains exhibit the same pattern but at different scales of time. More precisely, Markov processes have increasing frequencies of oscillation. This type of multifractal processes is also called multiplicative cascade. For calibration purposes, these processes are converted into Markov Switching Poisson models with multiple hidden states. In addition, they are, in this sense, related to the two states Bayesian model of Chiera et al. [13]. Published literature about this type of statistical model is rather sparse, even if multifractals have been used since the early 1960s within geophysics. One may refer to the survey of Lovejoy and Schertzer [14] for an overview. Recent applications of fractals to meteorology may be found in Sachs et al. [15] and Tchiguirinskaia et al. [16]. The model proposed here is based on Markov-Switching Multifractals processes that have been presented by Calvet and Fisher [17]. These are similar to on-off processes used to model data transmission (e.g., Resnick and Samorodnitsky [18]). Some applications of on–off processes in weather prediction have been studied by Mu and Zheng [19]. This process is well adapted to reproduce memory effects that are often exhibited by empirical observations of weather indices (e.g., Brody et al. [20] with respect to his attempt to model these effects using a fractional Brownian). To illustrate the utility of this model, it is shown that it is particularly efficient to simulate damage caused by tornadoes in the US. Next, it is shown that the fitted model may be used to design a CAT bond linked to these climatic events. The word "fractal' emerged on the scientific scene with the work of Mandelbrot [21] in the 1960s and 1970s. Subsequently, multifractal processes became a popular means of modelling financial times series. Mandelbrot expands on this in numerous publications (e.g., [22,23]), for applications of these processes to finance. Otherwise, in current scientific study, apart from the work of Major and Lantsman [24] (that proposes methods to fit and simulate multifractal models in the context of two-dimensional fields), there are very few applications. The current study endeavors to fill this gap.

The first part of this study ("Section 2") focuses solely on the claims arrivals: (1) the time series is modelled by a Poisson process with random intensity which is the product of a periodic function and of a multifractal process; (2) the model is reformulated as a Markov Switching Poisson model the Hamilton's filter used for calibration is also presented. (3) The results of this procedure and the tests of goodness of fit are then presented. In the second part of this work ("Section 3"), a similar approach is used to model the claims size. A Gamma process is developed for which the average is the product of a deterministic trend and a multifractal random factor. Related numerical results are then given. A two-dimensional framework developed by Calvet and Fisher [17] is applied to eliminate the assumption of independence between costs and frequencies. Numerical illustrations are included. The last part of this paper deals with the pricing of CAT bonds. In particular, the impact of adopting the multifractal model on bonds spreads is shown.

2 The Number of Claims

2.1 The Model

The frequency of many natural phenomena like tornadoes or hurricanes exhibit seasonality combined with a large variability. Figure 1 presents the monthly numbers of tornadoes that hit the USA between 1990 and 2008 (data retrieved on Sheldus[1]). It shows that most of the tornadoes are observed during the second quarter compared with the remainder of the year. Modeling the number of claims by a Poisson process with a constant intensity, as usually done for claim arrivals process, is consequently insufficient to capture this trend.

The calculation of the smoothed average number of claims per month on data from 1990 to 2008 confirms a peak in May as shown in Table 1. Replacing the constant intensity of the Poisson process by a piecewise constant function partly improves the modeling of the claims arrival process. However, the volatility of this process is subsequently shown to still be significantly lower than the overdispersion exhibited by real data.

One way to model overdispersion is to add a stochastic component in the intensity of a jump process. This category of counting processes is detailed by Jung and Tremayne [25] who identified several classes of models, such static regression or autoregressive models. Previously, Hainaut [12] developed a similar approach in continuous time fitting it to the claims process of US tornadoes. However, the results were not satisfactory. The overdispersion brought about by this method is still lower than that observed. In addition, the high volatility of the Brownian motion leads to a negative intensity with a significant probability. On the other hand, multifractal processes have been successfully applied in econometrics and in data transmissions to model stochastic volatility of time series.

[1] http://webra.cas.sc.edu/hvriapps/sheldus_setup/sheldus_login.aspx

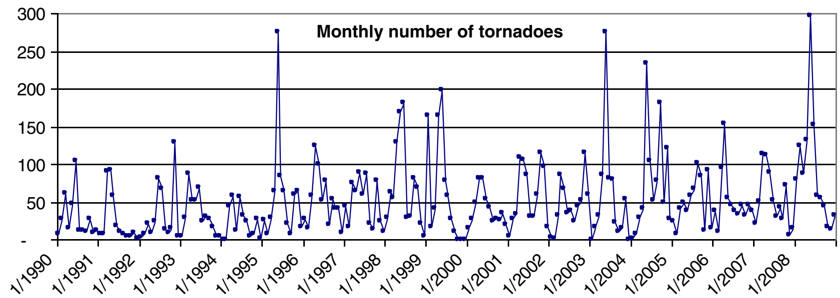


Fig. 1 Number of tornadoes per month from January 1990 to December 2008

Furthermore, in finance, Calvet and Fisher ([17], chapter 3) have shown that, in many cases, multifractal volatility models outperform the Garch model. For these reasons, we decided to explore a new type of counting: the Poisson-multifractal process.

In this new approach, the monthly number of tornadoes is modelled by a Poisson random variable where the intensity is driven by a multifractal process having some memory effect. As illustrated in numerical results, this approach is efficient in generating the peaks of tornadoes occurrences.

The number of claims observed over a period t is denoted by N_t in this paper. This process is defined on a filtration F_t, in a probability space Ω coupled with a probability measure, denoted by P. The term F_t is a sigma algebra of Ω that carries on information about the history of N_t. The intensity of N_t is a stochastic process, denoted by λ_t, and defined on a filtration H_t, containing solely the history of λ_t. We note Δt, the length of the period, during which the intensity is constant. Conditionally on $H_t \vee F_0$ (the initial filtration augmented by the filtration of λ_t), the process N_t is a Poisson process for which the probability of observing n jumps is given by the formula:

$$P(N_t = n|\mathrm{H}_t \vee \mathrm{F}_0) = \frac{\left(\sum_{i=0}^{t} \lambda_i \Delta t\right)^n}{n!} e^{-\sum_{i=0}^{t} \lambda_i \Delta t}. \tag{1}$$

Table 1 $\lambda(t)$ average number of claims per month

Month	λ_i
January	20
February	25
March	53
April	102
May	118
June	103
July	60
August	42
September	33
October	25
November	21
December	21

Bremaud [26] and Lecki and Rutkowski [27] describe in detail this kind of process, also called doubly stochastic. The intensity of the Poisson process used here is modelled as the product of a piecewise constant function $\lambda(t)$, with a multifractal process F_t^N that will be defined later. The function $\lambda(t)$ will be constructed to replicate the seasonality observed in the claims arrival process while the process F_t^N introduces volatility in the claims frequency:

$$\lambda_t = \lambda(t) F_t^N. \tag{2}$$

The function $\lambda(t)$ is piecewise constant and periodic. In the remainder of this work, a monthly basis is used. $\lambda(t)$ is in this case equal to:

$$\lambda(t) = \lambda_i \quad i \equiv t \bmod 12 \tag{3}$$

The term λ_i is set to the smoothed average of claims observed during the i^{th} month of the year, between 1990 and 2008 (see Table 1).

The process F_t^N, adding volatility in the intensity λ_t, is modelled by a multifractal process, as in the framework proposed by Calvet and Fisher [17]. In Appendix A, a visual analysis of auto-covariances in levels is demonstrated and justifies the fractal nature of F_t^N.This process is defined on the filtration H_t. It is assumed that there exists m climatic factors affecting the frequency of tornado occurences. These climatic factors are unobservable and are modelled by a vector of Markov processes, M_t^N, of m components:

$$M_t^N = \left(M_{1,t}^N, M_{2,t}^N \ldots M_{m,t}^N\right) \in \mathrm{R}_+^m.$$

The process F_t^N is the product of such climatic factors:

$$\lambda_t = \lambda_i F_t^N = \lambda_i \prod_{k=1}^{m} M_{k,t}^N \quad i \equiv t \bmod 12.$$

M_t^N is built in a recursive manner. It is assumed that the vector M_t^N has been built until period t. For each $k=\{1,\ldots,m\}$, the next period multiplier $M_{k,t}^N$ is drawn from a fixed distribution M with probability γ_k and is otherwise equal to its previous value $M_{k,t}^N = M_{k,t-1}^N$. The distribution of M is positive and is such that $E(M_{k,t}^N)=1$. This last constraint ensures that, on average, the intensity λ_t is equal to $\lambda(t)$. Calvet and Fisher [17] recommend the following distribution for M:

$$M = \begin{pmatrix} m_0 & p_0 = \frac{1}{2} \\ 2-m_0 & 1-p_0 = \frac{1}{2} \end{pmatrix} \tag{4}$$

which is fully determined by the parameter $m_0 \in [0,1]$. A Markov process $M_{i,t}^N$, that equals 2-m_0, increases the intensity. Conversely, if $M_{i,t}^N = m_0$, the intensity of the claims arrival process is reduced. The probabilities $\gamma_{k=1\ldots m}$ depend on the two parameters $\gamma_1 \in (0,1)$ and $b \in (1,\infty)$ as follows:

$$\gamma_k \equiv 1-(1-\gamma_1)^{b^{k-1}} \quad k = 1, \ldots, m \tag{5}$$

This rule of construction guarantees that $\gamma_1 \leq \ldots \leq \gamma_m < 1$. If we note that $1_{k,t}{}^N$ the indicator function will equal 1 if there is a new sampling from the distribution M at time t, then for the k^{th} components we have

$$P\left(\mathbf{1}_{k,t}^N = 1\right) = \gamma_k \quad k = 1\ldots m$$

This means that the last climatic factor $M_{m,t}^N$ changes value more frequently than the first climatic factor $M_{m,t}^N$ as illustrated by Fig. 2. This shows simulated changes in the three climatic factors, involved in the evolution of the number of damages (the calibration of these factors is detailed in the next section). Clearly, the third component oscillates more frequently than the first one.

The main advantage of this model is its ability to capture low-frequency regime shifts and long volatility cycles of the damages arrival process. Furthermore, it allows a parsimonious representation (only three parameters) of a high-dimensional state space. If one considers that there are $m=8$ hidden climatic factors, the intensity at time t can have $2^8=256$ values.

2.2 Calibration

As mentioned in the previous section, the function F_t^N can take $d=2^m$ values. The Markov state vector $M_t{}^N$ then takes values $m^1, \ldots m^d \in \mathrm{R}_+{}^m$. The transition matrix $A = (a_{i,j})_{1\leq i,j\leq d}$ is fully determined by $\gamma_{k=1\ldots m}$. It has the following components:

$$\begin{aligned} a_{i,j} &= P\left(M_{t+1}^N = m^j \middle| M_t^N = m^i\right) \\ &= \prod_{k=1}^{m} \left(\gamma_k \frac{1}{2} + (1-\gamma_k)\mathrm{I}_{\{m^j(k)=m^i(k)\}}\right). \end{aligned} \tag{6}$$

For a given combination m^j, the number of tornadoes occurring over the time interval $[t, t+\Delta t]$ is distributed as a Poisson random variable with intensity:

$$\lambda_t^j = \lambda_i F_t^N = \lambda_i \prod_{k=1}^{m} m^j(k) \quad i \equiv t \bmod 12 \quad j = 1\ldots d., \tag{7}$$

where $m^j(k)$ is the k^{th} elements of the vector m^j. In this case, the probability of observing n_t claims, given $\lambda_t{}^j$, during this period is:

$$p(t, j, n_t) = P\left(N_{t+h} - N_t = n_t | \lambda_t^j\right) = \frac{\left(\lambda_t^j \Delta t\right)^n}{n!} e^{-\lambda_t^j \Delta t}. \tag{8}$$

At time t, the vector of these probabilities for each combination of climatic factors is denoted by $p(t,n_t) = (p(t,j,n_t))_{j=1\ldots d}$. The climatic factors $(M_{k,t}^N)_{k=1\ldots m}$ are not directly observable, but the filtering technique developed by Hamilton [28] and derived from Kalman's filter [29] allows one to retrieve the probabilities of being in a state given all the previous observations. Summarizing this filter: $n_{i=0,\ldots,t}$ is noted as the number of tornadoes observed in previous periods and the probabilities of presence in a certain state j is defined as:

$$\Pi_t^j = P\left(M_t^N = m^j | n_1, \ldots, n_t\right)$$

Hamilton proved that the vector $\Pi_t = (\Pi_t{}^j)_{j=1\ldots m}$ can be calculated as a function of the probabilities of presence during the previous period:

$$\Pi_t^j = \frac{p(t, n_t) * (\Pi_{t-1} A)}{\langle p(t, n_t) * (\Pi_{t-1} A), \mathbf{1} \rangle}, \tag{9}$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathrm{R}^d$ and $x * y$ is the Hadamard product $(x_1 y_1, \ldots, x_d y_d)$. To start the recursion, we assume that the Markov processes have reached their stable distribution. The

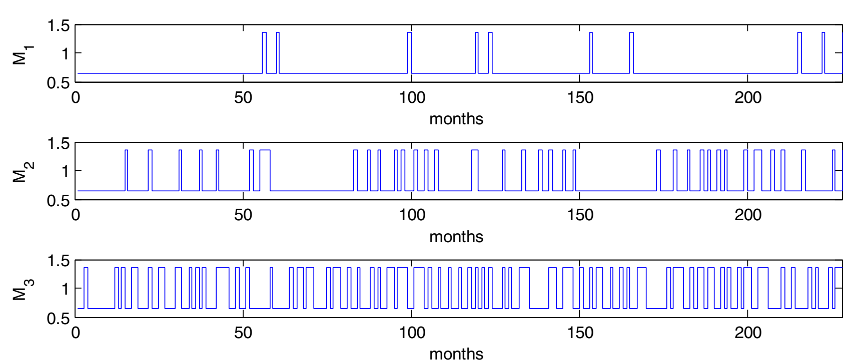


Fig. 2 Frequencies of fractal components

vector of probabilities Π_0 is then set to the ergodic distribution, which is the eigenvector of the matrix A, coupled to the eigenvalue equal to 1. Observing the claims process on T months, the log of Likelihood is:

$$lnL(n_1 \ldots n_T | m_0, \gamma_1, b) = \sum_{t=0}^{T} ln\langle p(t, n_t), (\Pi_{t-1} A)\rangle. \tag{10}$$

The most likely parameters are obtained by numerical maximization of Eq. 10.

At the current time, there do not appear to exist any exact statistics to test the goodness of fit of a regime switching model but an approached chi square test can be used. The term $\bar{\lambda}_t = \sum_{j=1}^{m} \lambda_t^j \Pi_t^j$ is defined as the intensity weighted by filtered probabilities. It may be seen as the average intensity and volatility of the Poisson process at time t. Empirical tests reveal that the statistic Z (computed on k periods of time) defined as follows

$$Z = \sum_{t=1}^{k} \frac{(Nt - \bar{\lambda}_t)^2}{\bar{\lambda}_t}$$

is approximately $\chi_k^2 - 3$ distributed, where 3 is the number of parameters. $\bar{\lambda}_t$ depends on the estimated probabilities of presence Π_t^j that are initially set to the ergodic distribution of the Markov chain. To ensure that this assumption does not influence the statistics, it must then be calculated using the last observations of the sample. In numerical applications, p values relating to this statistic and computed for the last 100 observations are reported.

2.3 Empirical Illustration

The calibration of the claims process has been performed on monthly data over a period from 1990 to 2008 comprising 228 observations. A basic Poisson process was fitted first with a constant intensity and using log Likelihood maximization. On average, 51 tornadoes per month were observed, for a log Likelihood of −5,246. Next, the log Likelihood of a Poisson process having a time-dependent intensity given in Table 1 was computed: The log Likelihood was improved. Nonetheless, a comparison of simulated N_t with real number of claims indicates that the volatility of this model is significantly lower than the real situation.

Table 2 presents the calibrated parameters of multifractal models, counting five to nine hidden climatic factors. Calculations have been performed in SAS. The highest Likelihood is obtained with nine components (512 states). The highest p value of the χ^2 test is obtained with eight Markov chains. This test disqualifies models with only four and five components. Note that a certain stability of parameter values between models is observed.

Table 2 Parameters of N_t

M	Parameter	Estimate	SE	Loglik. and p values
9	γ_1	0.1247	0.0875	Loglik., −1,110.849
	b	1.7750	0.5521	χ^2 p value, 0.7121
	m_0	0.7405	0.0043	
8	γ_1	0.2218	0.1170	Loglik., −1,133.168
	b	1.5524	0.3532	χ^2 p value, 0.8725
	m_0	0.7477	0.0059	
7	γ_1	0.2620	0.1558	Loglik., −1,159.252
	b	1.5696	0.4701	χ^2 p value, 0.6320
	m_0	0.7448	0.0055	
6	γ_1	0.2300	0.1372	Loglik., −1,150.353
	b	1.7850	0.6402	χ^2 p value, 0.2075
	m_0	0.7056	0.0049	
5	γ_1	0.2460	0.1323	Loglik., −1,194.588
	b	1.7283	0.5362	χ^2 p value, 3.4682e-07
	m_0	0.6412	0.0055	
4	γ_1	0.1295	0.0617	Loglik., −1,311.508
	b	4.1602	2.8746	χ^2 p value, 0.0000
	m_0	0.5784	0.0082	

In Fig. 3, the term $\bar{\lambda}_t = \sum_{j=1}^{m} \lambda_t^j \Pi_t^j$, for $m = 9$, is compared with the observed number of tornadoes from January 2001 to December 2008. Both curves closely correspond which confirms the goodness of fit of our model.

In Table 3, the first two moments of the monthly number of tornadoes (from 1990 to 2008) is compared with the moments of a sample of 1,000 simulations. These figures tend to confirm that the model duplicates the seasonality and volatility of the number of claims reasonably well.

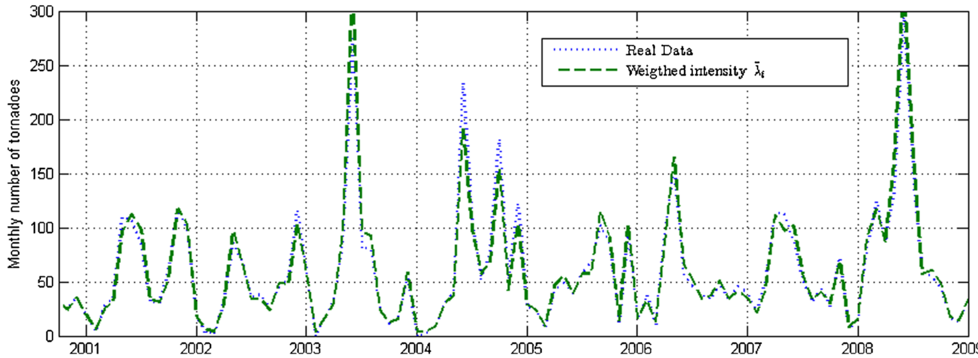
3 The Size of Claims

3.1 The Model

Figure 4 presents the mean monthly cost of damage incurred per tornado. The size of claims varies considerably between months. An analysis of the smoothed average inflation adjusted cost per tornado per month (Table 4, second column), computed over a period from 1990 to 2008, shows that damage costs appear higher in April and May. From June to October, the average costs exhibit small oscillations.

In this study, C_t, denotes the cost of damage caused per tornado, during the period $[t, t + \Delta t]$. The choice of the probability distribution for claim costs should ideally take into account a certain degree of seasonality. Given the observations

Fig. 3 Comparison of $\overline{\lambda_t}$ and N_t



reported, damages are more severe in April and May. Furthermore, costs exhibit huge variability. To represent these trends, the cost process is modelled by a gamma random variable, whose mean parameter is the product of a time dependent function and a multifractal process. Note that different laws such as exponential or Pareto were tested to model the costs of tornadoes, but none of them were satisfactory.

C_t as N_t is defined on the filtration F_t. The mean cost of C_t at time t is denoted by τ_t and is a stochastic process defined on a filtration E_t. The term τ_t will be defined later but for now, E_t is noted as a sigma algebra of Ω that contains only the information about the history of τ_t. Conditionally to $E_t \vee F_0$, the density of costs per tornado C_t occurring in the period $[t,t+\Delta t]$, is gamma distributed:

$$f(|c|\tau_t) = \frac{1}{\Gamma(\nu)}\left(\frac{\nu c}{\tau_t}\right)^{\nu}\exp\left(-\frac{c\nu}{\tau_t}\right)\frac{1}{c}I_{(0,\infty)}$$

where $\nu \in R^+$. The variance here is equal to $\frac{(\tau_t)^2}{\nu}$. The mean cost of damage is defined as the product of a piecewise constant periodic function $\tau(t)$ with a multifractal process F_t^C:

$$\tau_t = \tau(t)F_t^C. \tag{11}$$

In Appendix A, a visual analysis of autocovariances in levels of C_t justifies the choice of a multifractal process. Progression is by steps Δt of 1 month, then $\tau(t)$ is in this case is equal to:

$$\tau(t) = \tau_i \quad i \equiv t \bmod 12 \tag{12}$$

where the chosen τ_i comes from the third column of Table 4. The values of $\tau_{i=1\ldots12}$ differs slightly from the smoothed average cost, presented in the second column of the same table. In particular, the oscillations observed from July to October have been removed.

The scale of cost is assumed to be independent from the claims arrival process. In this scenario, the cost of claims is also influenced by m^C unobservable factors, which are independent from those driving the claims arrival process. These factors are modelled by a Markov state vector, M_t^C

$$M_t^C = \left(M_{1,t}^C, M_{2,t}^C \ldots, M_{m,t}^C\right) \in \mathrm{R}_+^n.$$

which is built in a similar way to the state vector M_t affecting the number of tornadoes, and it is fully parametrized by three parameters (m_0^C, b^C, γ_1^C). The multipliers $M_{k,t}^C$ are drawn

Table 3 Average number of claims and standard deviations per month

Month	Historical mean	Historical SD	Simulated mean	Simulated SD
January	26	39	20	17
February	25	29	25	20
March	57	27	54	43
April	86	44	101	81
May	120	92	121	95
June	77	39	100	80
July	48	23	59	50
August	33	23	42	36
September	40	43	32	26
October	38	30	24	20
November	48	43	22	18
December	15	15	20	15

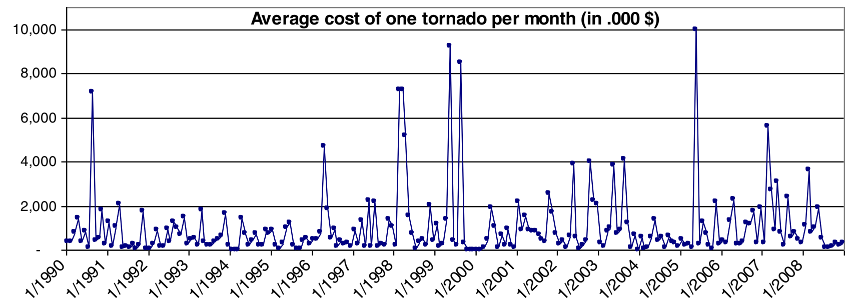


Fig. 4 Observed mean monthly cost per tornado

from a fixed distribution M^C, with probability γ_k^C, as defined by Eqs. 4 and 5. Otherwise, the multiplier is equal to its previous value $M_{k,t}^C$=$M_{k,t-1}^C$. The distribution of M^C is also such that $E(M_{k,t}^C)$=1. The process $F_t{}^C$ is the product of these factors:

$$F_t^C = \prod_{k=1}^{m^C} M_{k,t}^C \quad i \equiv t mod 12.$$

This process is adapted to the filtration E_t. The calibration is done in the same way as for the claims arrival process using the Hamilton filter [28]. $M_t{}^C$ can take $d = 2^{m^C}$ values $m^{C,j=1..d}$. The term τ_t^j, the realisation of τ_t for each $m^{C,j=1..d}$, is noted. The matrix of transition probabilities between these states is denoted by A^C. The vector of probabilities of presence is denoted by $\Pi_t{}^C$ and is computed by Eq. 9 above. If $c_1,\ldots,c_T$ are the costs observed over the past T periods, the log Likelihood is given by:

$$lnL\left(c_1 \ldots c_T \middle| m_0^C, \gamma_1^C, b^C, \nu\right) = \sum_{t=0}^{T} ln\left\langle f^C(c_t), \left(\Pi_{t-1}^C A^C\right)\right\rangle$$

where $f^C(c_t)$ is the density vector $(f^C(c_t|\tau_j))_{j=1\ldots d}$. The parameters $m_0{}^C$,$\gamma_1{}^C$,b^C and ν are obtained numerically by the maximization of this log Likelihood.

Table 4 $\tau(t)$ average claim cost per month in dollars

Month	Observed average costs ($)	Model costs τi ($)
January	673,123	673,123
February	673,123	673,123
March	1,059,905	1,059,905
April	1,351,749	1,351,749
May	1,241,208	1,241,208
June	840,978	840,978
July	676,593	676,593
August	645,338	645,338
September	703,493	645,338
October	620,101	645,338
November	702,757	645,338
December	669,092	645,338

Following calibration, the goodness of fit was checked by a χ^2 test. The average expected cost and volatility are defined respectively as $\overline{\tau_t} = \sum_{j=1}^{m^C} \tau_t^j \Pi_t^j$ and $\overline{\sigma}_t^2 = \sum_{j=1}^{m^C} \left(\tau_t^j\right)^2 \frac{1}{\nu} \Pi_t^j$. The following statistic Z^C was computed over k periods

$$Z^C = \sum_{t=1}^{k} \frac{\left(C_t - \overline{\tau_t}\right)^2}{\overline{\sigma}_t^2}$$

was compared with a χ_{k-4}^2 distribution (4 being the number of parameters). With respect to the number of tornadoes, this statistic and p value are computed for the last 100 observations.

3.2 Empirical Illustration

Considering now the claim arrival process, the cost process is fitted on monthly data from 1990 to 2008 that is 576 observations. A basic gamma distribution was first fitted to claim costs. In this model, a claim caused by a single tornado costs on average \$1.0314 million and has volatility equal to \$1.108 million: The log Likelihood is −3264. Table 5 presents the log of Likelihoods and parameters of multifractal models with five to nine components. Increasing the number of volatility components does not produce a significant improvement. Furthermore, all p values of the χ^2 test are very close to 1.

In Fig. 5, the average cost $\overline{\tau_t}$ versus the mean monthly cost of damage per tornado is given from January 2001 to December 2008 (for m=9). This graph adds weight that the proposed model is able to predict observed peaks in damage claims.

4 Multivariate Analysis

4.1 Two-Dimensional Multifractal Process

Instead of modelling frequency and severity independently (as reported earlier), it was interesting to analyze a modelling strategy for simultaneous fitting. A bivariate model was tested in which the multifractal processes influencing claims costs, and numbers are dependent.

Table 5 Parameters for the claims cost

M	Parameter	Estimate	SE	Loglik. and p values
9	γ_1	0.1283	0.2132	Loglik., −3,223.746
	b	4.0087	5.5231	χ^2 p value, 1.0000
	m_0	0.7181	0.0326	
	v	2.4828	0.8762	
8	γ_1	0.1491	0.2017	Loglik., −3,223.805
	b	4.9044	19.0991	χ^2 p value, 1.0000
	m_0	0.7028	0.0326	
	v	2.4802	0.8332	
7	γ_1	0.1646	0.1617	Loglik., −3,223.846
	b	7.3779	13.9051	χ^2 p value, 1.0000
	m_0	0.6816	0.0368	
	v	2.5341	0.8875	
6	γ_1	0.1822	0.1545	Loglik., −3,223.898
	b	8.5550	13.4898	χ^2 p value, 1.0000
	m_0	0.6602	0.0345	
	v	2.4658	0.7193	
5	γ_1	0.2094	0.1705	Loglik., −3,224.011
	b	9.4178	17.9029	χ^2 p value, 1.0000
	m_0	0.6260	0.0439	
	v	2.5280	0.9224	
4	γ_1	0.2335	0.1721	Loglik., −3,224.115
	b	9.9730	20.6153	χ^2 p value, 1.0000
	m_0	0.5852	0.0383	
	v	2.4140	0.5996	

A Poisson-Gamma model is now considered whose means arise from dependent multifractal processes. As previously explained, the number of claims N_t, observed in the period $[t,t+\Delta t]$ is assumed to be Poisson, and its intensity is modelled as the product of a piecewise constant function $\lambda(t)$, and a multifractal process ${F_t}^N$:

$$\begin{aligned}\lambda_t &= \lambda(t)F_t^N \\ &= \lambda_i \prod_{k=1}^{m} M_{k,t}^N \quad i\equiv t mod 12.\end{aligned} \tag{13}$$

where λ_i are those data presented in Table 1. The size of claims, C_t, caused by a single tornado observed in the period $[t,t+\Delta t]$ is modelled as in "Section 3" by a Gamma random variable. Its shape parameter is denoted by v, and its mean is the product of a piecewise constant function and of a multifractal process ${F_t}^C$, having as much components, m, as ${F_t}^N$:

$$\begin{aligned}\tau_t &= \tau(t)F_t^C \\ &= \tau_i \prod_{k=1}^{m} M_{k,t}^C \quad i\equiv t mod 12\end{aligned}$$

where τ_i are those data presented in Table 4. $M_{k,t}^N$ and $M_{k,t}^C$ are characterized by the same triplet (m_0,γ_1,b).

The assumption of independence between frequencies and costs is dropped here. A two-dimensional model allowing dependence between claims and costs with multifractal mean is used. The same multifractal structure as in "Sections 2 and 3" is applied. In the bidimensional multifractal process, the frequency and severity models share some common parameters and unconditional correlation between the arrival of $M_{k,t}^C$ and $M_{k,t}^N$.

As shown previously, the probability of having a new sampling from the distribution M, for the component ${M_{k,t}}^C$, is denoted by γ_k:

$$P\left(\mathbf{1}_{k,t}^N = 1\right) = \gamma_k \quad k = 1\ldots m$$

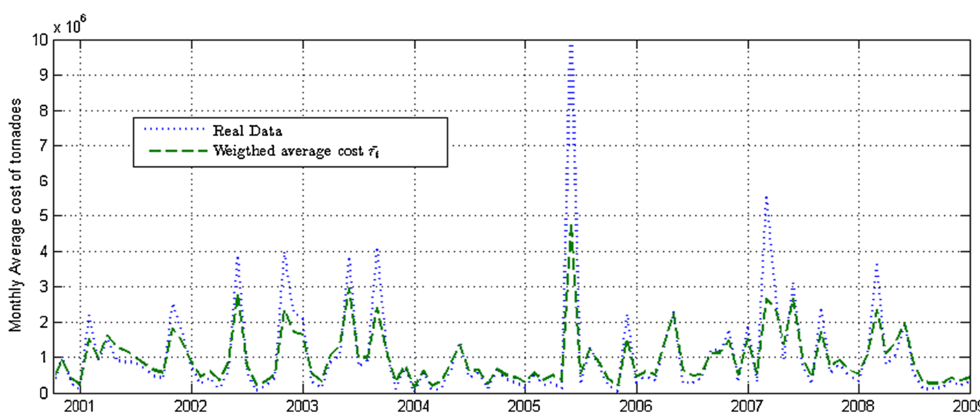
Based on the work of Calvet and Fisher [17], it is assumed that there exists θ∈[0,1] such that the condition

$$\left(\begin{aligned} P\left(\mathbf{1}_{k,t}^C = 1|\mathbf{1}_{k,t}^N = 1\right) &= (1-\theta)\gamma_k + \theta \\ P\left(\mathbf{1}_{k,t}^C = 0|\mathbf{1}_{k,t}^N = 1\right) &= (1-\theta)(1-\gamma_k)\end{aligned}\right. \quad k = 1\ldots m$$

is satisfied. If $\theta=0$ or if $\theta=1$, ${M_{k,t}}^N$ and ${M_{k,t}}^C$ are respectively independent or dependent. Furthermore, it is assumed that the arrivals vector is symmetrically distributed:

$$\left(\mathbf{1}_{k,t}^C, \mathbf{1}_{k,t}^N\right) \overset{d}{=} \left(\mathbf{1}_{k,t}^N, \mathbf{1}_{k,t}^C\right)$$

Fig. 5 Comparison of $\overline{\lambda_t}$ with C_t



Its distribution is then defined as follows:

$$\begin{pmatrix} p_{11}^{k} = P\left(\mathbf{1}_{k,t}^{C} = 1, \mathbf{1}_{k,t}^{N} = 1\right) & = \gamma_k((1-\theta)\gamma_k + \theta) \\ p_{01}^{k} = P\left(\mathbf{1}_{k,t}^{C} = 0, \mathbf{1}_{k,t}^{N} = 1\right) & = \gamma_k(1-\theta)(1-\gamma_k) \\ p_{10}^{k} = P\left(\mathbf{1}_{k,t}^{C} = 1, \mathbf{1}_{k,t}^{N} = 0\right) & = \gamma_k(1-\theta)(1-\gamma_k) \\ p_{00}^{k} = P\left(\mathbf{1}_{k,t}^{C} = 0, \mathbf{1}_{k,t}^{N} = 0\right) & = (1-\gamma_k)(1-\gamma_k(1-\theta)) \end{pmatrix} \tag{14}$$

The third equality is a direct consequence of the symmetry of $(\mathbf{1}_{k,t}^{C},\mathbf{1}_{k,t}^{N})$. The expression for p_{00}^{k} is obtained from the relation $p_{10}^{k}+p_{00}^{k}=1-\gamma_{k\cdot}$. The marginal distributions of $\mathbf{1}_{k,t}^{C}$ is identical to the marginal distribution of $1_{k,t}^{N}$:

$$P\left(\mathbf{1}_{k,t}^{C} = 1\right) = \gamma_k \quad k = 1\ldots m$$

Furthermore, from Eq. 14, the conditional probabilities are inferred when $\mathbf{1}_{k,t}^{N}=0$:

$$\begin{pmatrix} P\left(\mathbf{1}_{k,t}^{C} = 1|\mathbf{1}_{k,t}^{N} = 0\right) = (1-\theta)\gamma_k \\ P\left(\mathbf{1}_{k,t}^{C} = 0|\mathbf{1}_{k,t}^{N} = 0\right) = 1-(1-\theta)\gamma_k \end{pmatrix} \quad k = 1\ldots m$$

Denoting $M_t=(M_{1,t}^{N}\ldots M_{m,t}^{N},M_{1,t}^{C}\ldots M_{m,t}^{C})$ which is the m vector of volatility components. M_t can take $d=4^m$ possible values, $m^1,\ldots,m^d\in\mathrm{R}_+^{2m}$. The probabilities of switching from one state to another are given by the transition matrix $A=(a_{i,j})_{1\le i,j\le d}$ where

$$\begin{aligned} a_{i,j} &= P\left(M_{t+1} = m^j | M_{t+1} = m^i\right) \\ &= \prod_{k=1}^{m}\left[\left(((1-\theta)\gamma_k + \theta)\frac{1}{2} + ((1-\theta)(1-\gamma_k))\mathrm{I}_{\{m^j(m+k)=m^i(m+k)\}}\right)\gamma_k\frac{1}{2} + (1-\theta)\left(\gamma_k\frac{1}{2} + (1-(1-\theta)\gamma_k)\mathrm{I}_{\{m^j(m+k)=m^i(m+k)\}}\right)(1-\gamma_k)\mathrm{I}_{\{m^j(k)=m^i(k)\}}\right] \end{aligned}$$

The observed number and costs of tornadoes on the past period are given as $o_{i=0,\ldots,t}=(n_i,c_i)_{i=0,\ldots,t}$. The probabilities of the presence in a certain state j are denoted as in the previous section by $\Pi_t^j=P(M_t=m^j|o_1,\ldots,o_t)$. The vector $\Pi_t=(\Pi_t^j)_{j=1\ldots m}$. If the arrivals and costs processes are observed over T months, the log Likelihood is:

$$\mathrm{lnL}(o_1\ldots o_T|m_0,\gamma_1,b,\theta,\nu) = \sum_{t=0}^{T} \ln\langle p(t,o_t),(\Pi_{t-1}A)\rangle \tag{15}$$

where $p(t,o_t)$ is the vector of probability density functions of the claim arrivals and costs processes. The parameters are obtained numerically by maximization of this log Likelihood. The goodness of fit is tested by comparison of statistics Z and Z^C such as defined in the earlier "Sections 2.2 and 3.1" applying a χ^2_{k-4} distribution.

4.2 Empirical Illustration

Table 6 presents the parameters fitting the bidimension process to the frequency and size of claims caused by tornadoes. With an equivalent number of fractal components and fewer parameters, the 2D model has a log Likelihood slightly lower than the sum of log Likelihoods of standalone arrivals and claims models. Note that the dependence parameter θ is close to zero for two fractal components and increases with m. In the opinion of the authors, this reveals a higher degree of dependence between high frequency fractal components than between low frequency components. The χ^2 test for the number of tornadoes leads to the rejection of models with less than five components.

Figure 6 shows for $m=6$ the average intensity (frequence of tornados per month), real data versus modelled; this graph is compared with a second showing the average insurance claim, real data versus modelled. For each graph, the time series (real and modelled) follow each other closely, thus supporting the validity of the model.

It would be interesting to analyze a bimensional multifractal model with $Nbk\geq 7$, but this would need a transition matrix of more than 4^7 rows. Calvet and Fisher propose the use of a numerical procedure for the inference, via a particle filter. This area of research warrants further study.

5 Pricing of CAT Bonds

5.1 Spread Calculations

A reinsurer can securitize a portfolio of reinsurance contracts so as to tranfer the risk to other potential investors who are looking for diversification. The reinsurance agreements are transferred to a special purpose vehicle, and in exchange for collateral, investors receive a periodic floating payment, linked to the amount of claims covered by agreements. The success of issuing such weather derivatives depends on the pricing and the specification of the model chosen to replicate the costs caused by the natural catastrophes covered by reinsurance.

Two objectives are now set out: firstly to underline the impact of working with multifractal models on the pricing of Catastrophe bonds linked to claims caused by hurricanes

Table 6 Calibration of a 2D multifractal process

M	Parameter	Estimate	SE	Loglik. and p values
6	γ_1	0.1934	0.0699	Loglik., −4,364.056
	b	2.2967	0.4677	χ^2 p value N_t, 0.1919
	m_0	0.7038	0.0048	χ^2 p value C_t, 1.0000
	θ	0.4215	0.7360	
	v	1.9573	0.2700	
5	γ_1	0.1870	0.0650	Loglik., −4,409.264
	b	2.3501	0.4241	χ^2 p value N_t, 0.0000
	m_0	0.6394	0.0052	χ^2 p value C_t, 1.0000
	θ	0.5512	0.3485	
	v	2.1318	0.3201	
4	γ_1	0.1654	0.0596	Loglik., −4,529.002
	b	3.5859	0.8235	χ^2 p value N_t, 0.0000
	m_0	0.5796	0.0084	χ^2 p value C_t, 1.0000
	θ	0.0692	0.5654	
	v	2.2928	0.3774	
3	γ_1	0.2827	0.0791	Loglik., −4,671.987
	b	3.6959	0.9883	χ^2 p value N_t, 0.0000
	m_0	0.5890	0.0066	χ^2 p value C_t, 1.0000
	θ	0.0000	0.5317	
	v	1.9070	0.2399	
2	γ_1	0.4416	0.0892	Loglik., −4,925.479
	b	3.8246	2.1702	χ^2 p value N_t, 0.0000
	m_0	0.5150	0.0080	χ^2 p value C_t, 1.0000
	θ	0.0000	0.4331	
	v	1.5721	0.1802	

hitting the USA. The results from this study are compared with those obtained from a basic Poisson Gamma model. The second objective is to exhibit the influence of seasonality on pricing.

Table 7 Spreads in percent, independent fractal models

	Quarterly	Semi-annual	Annual
1 year	1.03 %	2.06 %	4.11 %
2 years	2.72 %	5.45 %	10.89 %
3 years	5.42 %	10.89 %	21.77 %
4 years	8.54 %	17.20 %	34.40 %
5 years	11.35 %	22.89 %	45.84 %

The assumption is made that the risk faced by an insurance bondholder is inherent to his exposure to accumulated insured property losses. This process of accumulated losses, which is denoted by X_t, depends both upon the frequency of claims N_t and on the magnitude of claims. The process of aggregated losses is defined by the following expression:

$$X_{t_n} = \sum_{t=t_1}^{t_n} N_t C_t$$

The insurance bond priced in this section pays a periodic coupon equal to a constant percentage of the nominal sum reduced by the amount of aggregated losses, exceeding a certain trigger level. At maturity, what is left of the nominal is repaid. To compensate for this possible loss of nominal, the coupon rate always exceeds the risk free rate. If fewer claims occur, the bondholder is rewarded at a higher rate than the one obtained by investing in risk-free assets with the same maturities. Conversely, in the case of catastrophic losses, the nominal value of the bond can fall to zero, and the payment of coupons can be interrupted. To understand how the spread of

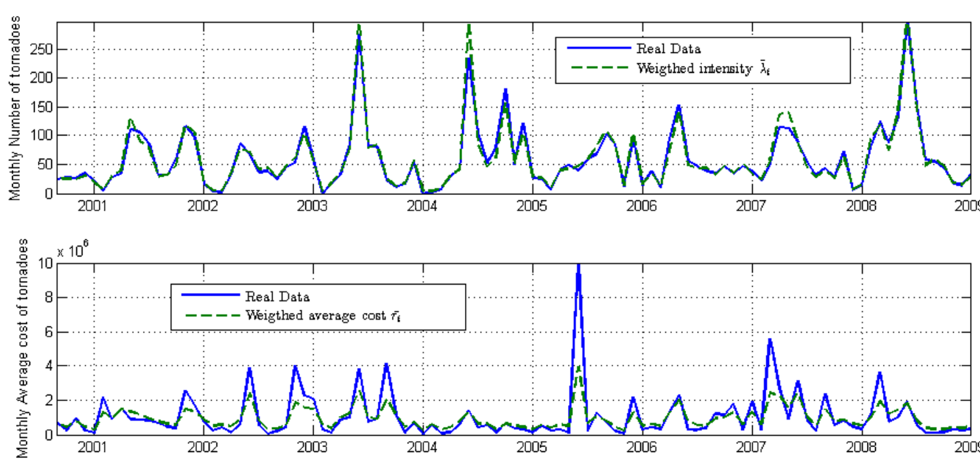


Fig. 6 Comparisons of $\overline{\lambda_t}$, N_t and of $\overline{\lambda_t}$, C_t

Table 8 Spreads in percent, bivariate fractal models

	Quarterly	Semi-annual	Annual
1 year	0.96 %	1.92 %	3.84 %
2 years	5.94 %	11.94 %	23.89 %
3 years	11.25 %	22.69 %	45.47 %
4 years	14.93 %	30.19 %	60.55 %
5 years	17.57 %	35.55 %	71.36 %

this bond is priced, some additional mathematical notations need to be introduced.

The term $BN$is the initial nominal of the bond. The level above which the excess of aggregated losses is deduced from the nominal is denoted by K_1 and called the "attachment point." If the total insured losses reach the amount of K_2=K_1+BN, before maturity, the bond stops delivering coupons, and the nominal is depleted. The bond, issued at time t_0, pays n coupons, at regular intervals of time, Δt, ranging from t_1 to t_n. The coupon rate is the sum of the constant risk-free rate of maturity t_n and of a spread that are respectively denoted by r and sp. The coupons paid at times $t_{i=1\ldots n}$ are denoted by $cp(t_i)$ and defined as follows:

$$cp(t_i) = (r+sp)\Delta t \underbrace{\left[(K_2 - K_1)I_{X_{t_i}\in[0,K_1]} + (K_2 - X_{t_i})I_{X_{t_i}\in(K_1,K_2]}\right]}_{BN_{t_i}}. \quad (16)$$

The term between the brackets in Eq. 16 is the (stochastic) nominal of bond at time t_i and is written BN_{t_i} in the following developments. Note that BN_{t0} is worth BN. Based upon the principle of absence of arbitrage, the spread of the insurance bond is chosen such that the expectations of future discounted spreads and of future discounted cutbacks of nominal are equal. The expectations of future discounted spreads and the reductions in the nominal are respectively termed the "spreads leg" and the "claims leg" (this terminology is inspired by that of credit derivatives). They are defined by the following expressions:

$$\text{Spreads leg}(t_0) = sp\Delta t \sum_{t_i=t_1}^{t_n} e^{-r(t_i-t_0)} \mathrm{E}(BN_{t_i}|\mathrm{F}_{t_0}), \quad (17)$$

$$\text{Claims leg}(t_0) = \sum_{t_i=t_1}^{t_n} e^{-r(t_i-t_0)} \mathrm{E}(BN_{t_{i-1}} - BN_{t_i}|\mathrm{F}_{t_0}). \quad (18)$$

As the two legs, Eqs. 17 and 18, must be equal to avoid any arbitrage, the following spread is inferred, added to the risk-free rate, at the issuance of the insurance bond:

$$sp = \frac{\sum_{t_i=t_1}^{t_n} e^{-r(t_i-t_0)} \mathrm{E}(BN_{t_{i-1}} - BN_{t_i}|\mathrm{F}_{t_0})}{\Delta t \sum_{t_i=t_1}^{t_n} e^{-r(t_i-t_0)} \mathrm{E}(BN_{t_i}|\mathrm{F}_{t_0})} \quad (19)$$

The expected future nominals are not calculable by a closed form equation, and we have to rely on numerical methods to appraise them. Among the numerical tools available, a Monte Carlo method has been chosen.

5.2 Numerical Applications

At this point of the study, the effort now moves to price insurance bonds linked to the aggregated costs caused by

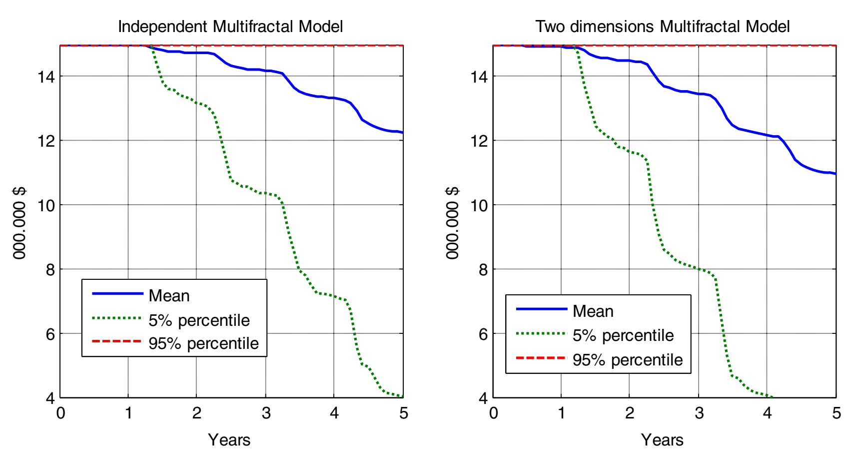


Fig. 7 Evolution of nominal

tornadoes in the USA. The exposure of the insurer issuing the insurance bonds is assumed to be 1/1,000 of the total claims cost. Following the method of Vaugirard [5], computation was by Monte Carlo simulations, the spreads of insurance bonds of maturities ranging from 1 to 5 years and paying quarterly, biannual, and annual coupons. The risk-free rate is constant and set to 3 %. The nominal, NB, is 15 million $ and is reduced if the aggregated losses breach the trigger of 5 million $.

Two approaches are compared. In the first one, the claims and arrivals are modelled by independent multifractal processes, with nine fractal components. Parameters used to simulate claims scenarios are those presented earlier in Tables 2 and 5. In the second approach, the claims and arrivals processes are modelled by a two-dimensional multifractal process, with six components. Parameters used in this simulation are those of Table 6.

Tables 7 and 8 presents the spreads obtained with 10, 000 simulations. The spreads obtained with the 2D model are clearly higher than those obtained under the assumption of independence between claims and costs processes. An analysis of simulations points to positive dependence between the number of losses and the extent of damage caused by tornadoes. It is also observed that for long-term bonds, the spreads are very high. This is directly related to the fact that the attachement point is breached in most scenarios after just 1 year. To confirm this hypothesis, the average development of the nominals for the two considered models has been plotted in Fig. 7. On average, the nominal is reduced by 3 or 5 million after 5 years, depending on the model chosen. This graph reveals the influence of seasonality on the progression of the nominal. From July to March, the nominal decreases more slowly than during the spring. Note that the volatility of the nominal is high. The 5 % percentile of the nominal distribution after 5 years is null. The insurance bonds have also been priced using a simple Poisson Gamma process. However, this method widely underestimates spreads that are close to zero.

6 Conclusions

This paper proposes a new statistical model based on fractals able to represent the seasonality of meteorologic events and the wide variability exhibited by the frequency and size of insurance claims. The innovation of this approach is that it considers a traditional Poisson-gamma model for claims aggregated costs, in which parameters are Markov switching multifractals. To justify the utility of this model, it has been fitted to the claims process caused by tornadoes that have hit the USA in the recent decades. A significant improvement of log Likelihoods has been observed with this model, when compared with traditional Poisson-Gamma models.

In the first part of this study, the claims arrivals and costs were assumed to be independent. In the second part of this work, a multivariate analysis of the claim process was performed. The two-dimensional framework developed by Calvet and Fisher [17] was attempted in order to rule out the assumption of independence between costs and frequencies. This last model followed the occurence of tornadoes with fewer parameters and much better than with previous methods. Apparently, dependence appears when the number of fractals increases.

In the last part of this work, the impact of adopting a multifractal model on the pricing of generic catastrophe bonds, linked to tornadoes in the USA was investigated. The results reveal that the two-dimensional multifractal model leads to the highest spreads. This was due to the positive dependence between the number and size of claims. Using a multifractal model is believed to lead to a better pricing of a wide category of insurance bonds, but this point needs further investigation.

Appendix A.

Proving the fractal nature of the number and cost of claims can be done by a visual analysis of autocovariance in levels. For any integrated process Z (number or cost of claims), the autocovariance in levels is defined as:

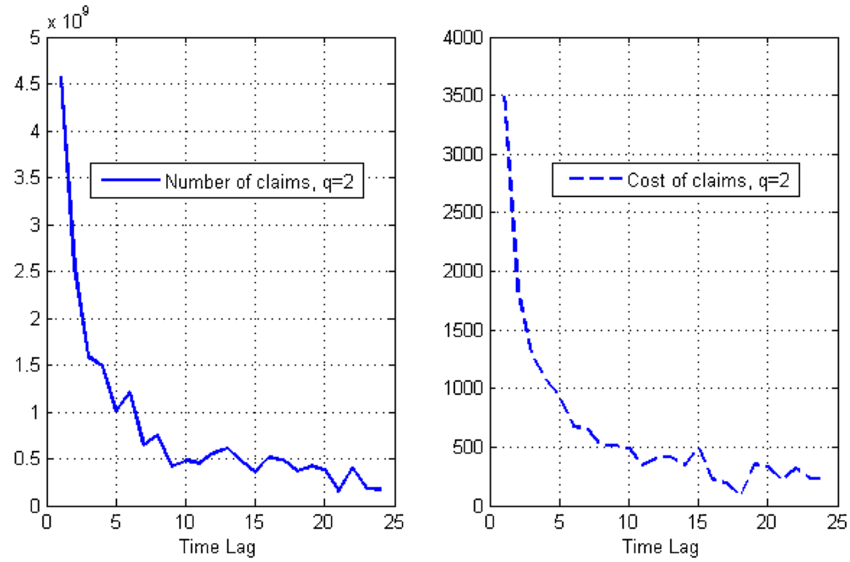
$$\delta_Z(t,q) = Cov(|Z(a,\Delta t)|^q, |Z(a+t,\Delta t)|^q)$$

and which quantifies the dependence of the size of increments

$$Z(a+t,\Delta t) = Z(a+t) - Z(t)|q$$

such that statistical moments exist. Calvet and Fisher [30] show that multifractal processes have hyperbolically declining autocovariances in levels when $t/\Delta t \to \infty$. In Fig. 8, these autocovariances ($q=2$) are plotted for $N_t - \lambda(t)$ and for $C_t - \tau(t)$. Autocovariances are well hyperbolically decreasing. The same trends are observed if $q=3$.

Fig. 8 Autocovariances in levels for different time lags



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