

UNIVERSITÉ CATHOLIQUE DE LOUVAIN

Equimeasurable functions and their application to vortex dynamics in lakes

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Chapter 1

General introduction

This thesis is the reflection of my researches in mathematical sciences. As a PhD. student at the Université catholique de Louvain, I worked on different topics in the vast field of analysis and calculus of variations. The main part of my researches is about the use of equimeasurability notions in the study of solutions of the lake equations.

The lake equation is a physical model describing the motion of vortices in a two dimensional basin of varying topography. In the absence of viscosity, equimeasurability appears as a preliminary conservation law on the vortex: the vortex is transported along the flow, and transforms itself during the motion by “rearrangement” of itself. In comparison with the 2D Euler’s equations, the varying topography induces a different constraint: rearrangements should be understood with respect to a different *measure*.

The above mentioned work is presented in the second part of this thesis. We present there three chapters, focused on the study of singular single vortex or vortex pairs, in a steady or dynamical context. Each of the chapters reflects a paper submitted or published.

I have chosen to dedicate the first part of this text to two other problems I have been working, namely: Markov process of symmetrizations; and a characterization of mass convergence via range convergence for vector valued measures. Those problems give the opportunity to softly introduce measure theoretic concepts. Each of those chapters reflects a paper that has been published or submitted.

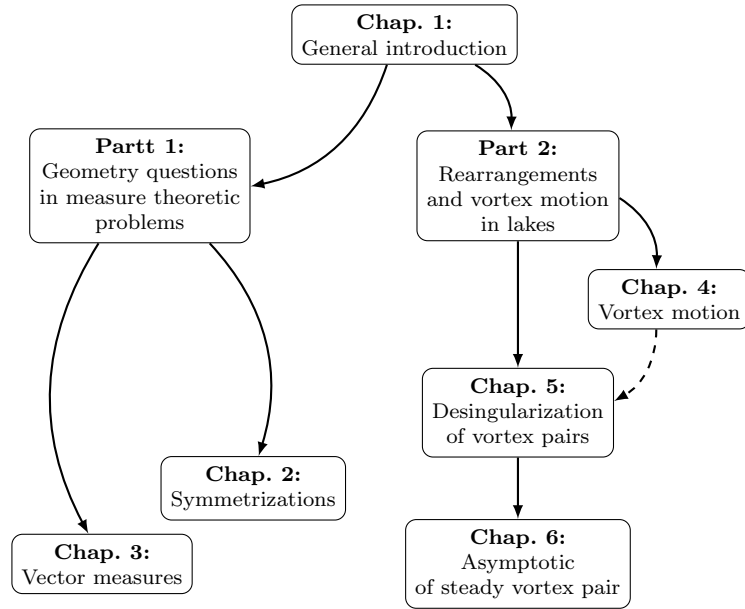


Figure 1.1: Chapter dependencies

The reader should be aware that the chapter ordering does not follow the real timeline of the thesis project. We have sketched below the natural timeline.

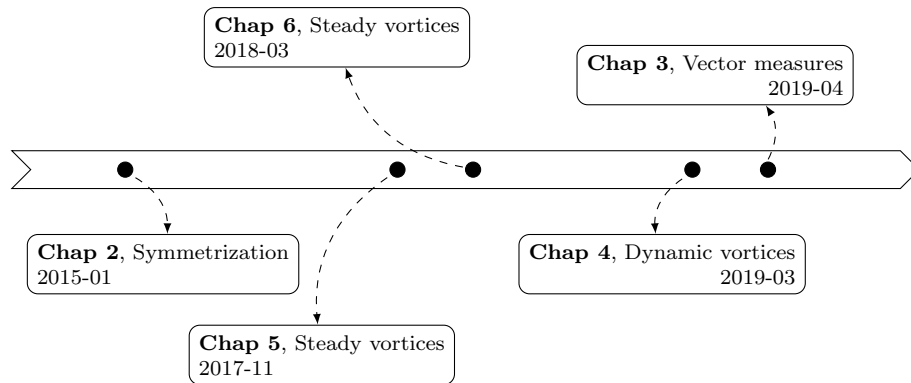


Figure 1.2: Timeline and approximate production dates

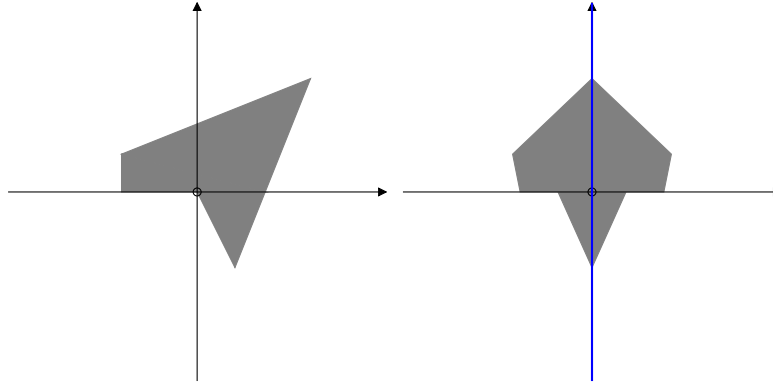


Figure 1.3: A Steiner symmetrization with respect to the vertical direction in the plane

1.1 Advances around symmetrizations

Steiner symmetrizations are geometric transformations of shapes and functions that were first introduced by Steiner [49] in order to give a proof of the famous isoperimetric problem. The idea of Steiner symmetrization is to follow the next algorithm:

1. Choose a direction u ;
2. Slice the initial shape X in the perpendicular direction $u^\perp$;
3. Compact the slice as a single segment of same length, centered on the line directed by u .

The resulting shape X^u is symmetric with respect to u and, because of the Fubini's theorem for the integral, has same area. In fact, Steiner proved that the perimeter of the resulting shape is smaller than the initial one. The process is shown in figure 1.3

Figure 1.4 illustrates what happens when we keep symmetrizing the shape X along different directions u_1, u_2, u_3, u_4 . As the reader may see, the resulting shape $X^{u_1 u_2 u_3 u_4}$ is more spherical than the initial one and an easy guess would be that, the more we symmetrize, more spherical the shape becomes. This phenomenon is the starting point of our work on successive symmetrizations taken at random.

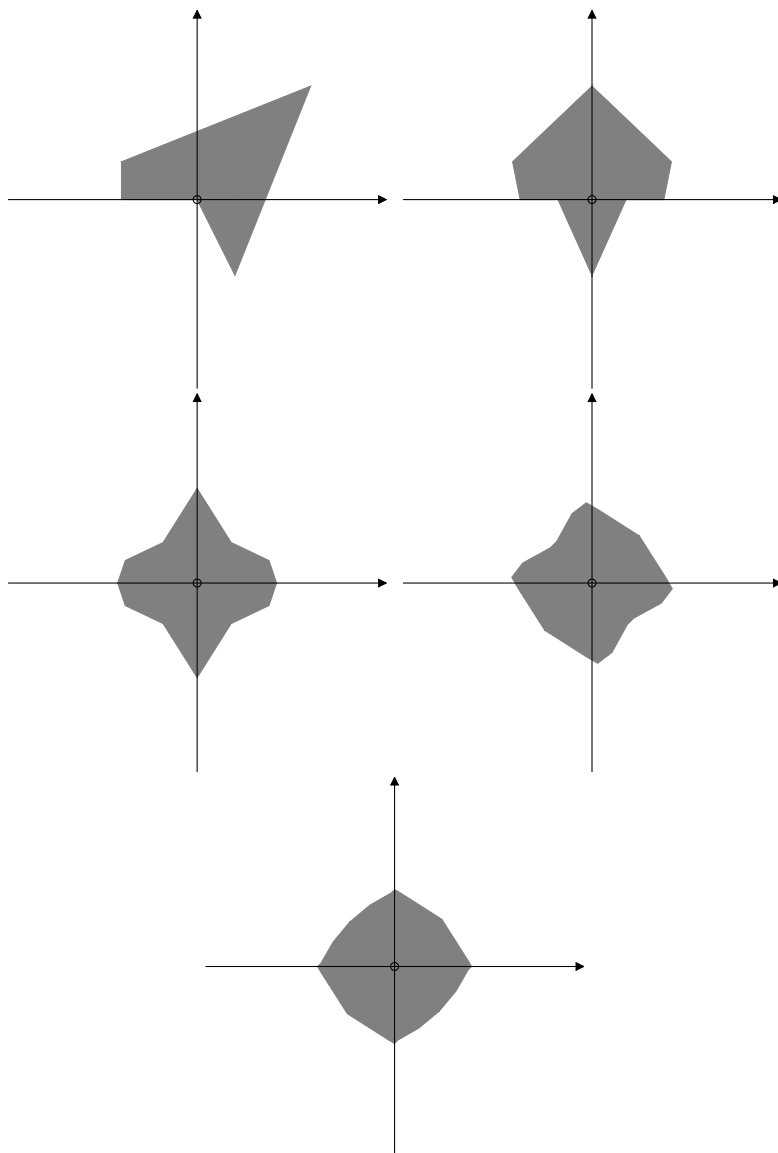


Figure 1.4: Successive symmetrizations (from left to right, then next line) of an initial shape (top left) in the plane. Successive directions are vertical, horizontal, of angle $\pi/4$, then $3\pi/4$. The result (bottom middle) tends to be spherical.

1.1.1 Previously known results

Symmetrizations are operations that transform a shape to another one of same measure. In this respect, symmetrizations are a special kind of a more general family of transformations: *rearrangements*. The initial shape X and its symmetrized X^u have that in common that they share the same measure: they are *equimeasurable*. Symmetrizations are also expected to bring symmetries, which make them useful in solving variational problems.

A natural question that have been asked is to give conditions for successive symmetrizations to converge to a ball. The transformation that maps a set X to the ball X^* of same radius (centered on the origin) is usually referred to as the spherical nonincreasing rearrangement. Steiner symmetrizations are rich enough to approximate the spherical non increasing rearrangement (see for example [9; 34, proof of theorem 3.7]), and in fact, the process is robust enough to allow universal choices of symmetrizations [53], and even *random choices of successive directions*. The field of random successive symmetrizations really began with the following result of Mani-Levitska [36]:

Known result (Mani-Levitska [36]). Giving a random process $(S_n)_{n \in \mathbf{N}}$ uniformly distributed in the sphere, and composed of independent random variables, any sequence of the form $(K^{S_1 S_2 \dots S_n})_{n \in \mathbf{N}}$ with K compact and convex, converges almost-surely to K^* in the sense of the Hausdorff distance.

Mani-Levitska's result was the starting point of various improvements [5, 11, 21, 53–55], concerning the assumptions on K (convexity is not necessary), mode of convergence (the Hausdorff distance and K compact may be replaced *without any down- or upgrading the result* by A measurable and the L^1 -distance), uniform distribution may be replaced by any distribution leaving sufficiently many choices (in the plane, a support of two points may already be sufficient), depending or not on $n \in \mathbf{N}$. Hence a typical statement of the form:

Known result (Burchard & Fortier[11]). Assume that $(\mu_n)_{n \in \mathbf{N}}$ is a sequence of probability measures on the sphere, that leaves “sufficiently

many choices of directions” and that does not degenerate too quickly. Giving a random process $(S_n)_{n \in \mathbf{N}}$ made of independent random variables, and $S_n \sim \mu_n$, any sequence of the form $(A^{S_1 S_2 \dots S_n})_{n \in \mathbf{N}}$ with A measurable, converges almost-surely to A^* in the sense of L^1 -convergence.

1.1.2 New contributions

Our main concern in the study of random sequences of symmetrizations was the improvement of the “independence assumption”. Indeed, independence of the process is an important assumption in the previously known results in the literature. We have shown that a *Markov process of directions* is a sufficient assumption to obtain an almost-sure convergence result.

A Markov process on the sphere $\mathcal{S}$ is a random process driven by a transition function (see [52])

$$P : \mathcal{S} \times \mathcal{B}(\mathcal{S}) \rightarrow [0, 1] : (s, A) \mapsto P_s(A),$$

representing the probability $P_s(A)$ that the process enters a set A knowing it is already in state s . Similarly, the iterated transition function $P_s^k(A_1 \times \dots \times A_k)$ represents the probability for the process to enter successively in states $A_1, \dots, A_k$, knowing it starts from state s . In contrast with processes made up of independent and identically distributed variables, successive Steiner symmetrizations that form a Markov process are mutually correlated. We have obtained the following new result:

New result. Let $(S_n)_{n \in \mathbf{N}}$ be a time-homogeneous Markov process of Steiner symmetrizations with initial distribution μ . If there exists a limit direction u^* such that

1. (*Recurrence*) the process comes back near u^* almost-surely infinitely often,
2. (*Discrimination*) starting from point u^* , we can reach any relevant symmetrization almost-surely in a finite number of steps,
3. (*Continuity*) for every $n \in \mathbf{N}$ and for every open sets $\mathcal{O}_1, \dots, \mathcal{O}_n$,

the map

$$s \in \mathcal{S} \mapsto P_s^n(\mathcal{O}_1 \times \dots \times \mathcal{O}_n)$$

is lower semi-continuous at u^* ,

then for every Lebesgue measurable set X of finite measure, the sequence $(X^{S_1 \dots S_n})_{n \in \mathbf{N}}$ converges to X^* , almost-surely. In particular, the sequence is almost surely universal.

The recurrence and discrimination conditions taken together offer the knowledge of a point u^* that acts as an attraction state, and from which sufficiently many directions are available in finite time. The continuity condition makes the link between the two.

In some sense, our result explains that successive symmetrizations are expected to be robust enough so that even if the process acts with finite memory against it, convergence to a ball should always occur with probability 1.

A corollary of the above result concerned sequences of *deterministic choices*. We were able to prove the following new result:

New result. Assume that $\phi : \mathcal{S} \rightarrow \mathcal{S}$ is a continuous function from the circle to itself, with the property that the orbit

$$\{\phi^n(u^*) = (\phi \circ \dots \circ \phi)(u^*) : n \in \mathbf{N}\}$$

is dense in the circle, for some starting point u^* . Then any sequence of successive Steiner symmetrizations $(X^{\phi^n(u^*)})_{n \in \mathbf{N}}$ converges to X^* .

This result extends the particular case of the Kronecher sequence [4, Section 5]

$$\phi(e^{i\theta}) = e^{i(\theta+\alpha)}, \quad \alpha \text{ is an irrational multiple of } \pi$$

to more general continuous functions.

We finally point out that the results exposed in this introduction are not confined in the study of Steiner symmetrizations. In fact, we were also able to approximate the spherical nonincreasing rearrangement via cap symmetrizations, polarizations; or Steiner and cap symmetrizations by polarizations. All this results are presented in details in part 1, chapter 2.

1.2 Advances around vector measures

Mass convergence of vector measures often appears in calculus of variation [22, 26] as an in-between notion between strong and weak convergence of measures.

To illustrate the concept, let us consider an elementary yet fundamental example. Assume we are given in a sequence $(x_n)_{n \in \mathbf{N}}$ in $\mathbf{R}$, converging to some point $x \in \mathbf{R}$: $\lim_{n \rightarrow \infty} x_n = x$. We are interested in the convergence of the sequence of Dirac masses

$$\lim_{n \rightarrow \infty} \delta_{x_n} = \delta_x.$$

Weak convergence means that, for every continuous function with compact support $\varphi \in C_c(\mathbf{R}, [0, 1])$, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbf{R}} \varphi(y) d\delta_{x_n}(y) = \int_{\mathbf{R}} \varphi(y) d\delta_x(y).$$

In the case of Dirac masses, this reduces to

$$\lim_{n \rightarrow \infty} \varphi(x_n) = \varphi(x),$$

which holds by continuity of φ . In comparison, strong convergence means that

$$\lim_{n \rightarrow \infty} \sup_{\substack{\varphi \in C_c(\mathbf{R}) \\ |\varphi| \leq 1}} \int_{\mathbf{R}} \varphi(y) d(\delta_{x_n} - \delta_x)(y) = 0;$$

or equivalently, talking with measure-theoretic notations:

$$\lim_{n \rightarrow \infty} |\delta_{x_n} - \delta_x| = 0.$$

Strict convergence (or mass convergence) is an in-between notion, as it reads:

$$\lim_{n \rightarrow \infty} \sup_{\substack{\varphi \in C_c(\mathbf{R}) \\ |\varphi| \leq 1}} \int_{\mathbf{R}} \varphi(y) d\delta_{x_n}(y) = \sup_{\substack{\varphi \in C_c(\mathbf{R}) \\ |\varphi| \leq 1}} \int_{\mathbf{R}} \varphi(y) d\delta_x(y);$$

or equivalently:

$$\lim_{n \rightarrow \infty} |\delta_{x_n}| = |\delta_x|.$$

It is readily checked that the mass of a finite combination of N Dirac masses with coefficients 1 or -1 , always equals N . This is due to the fact that we have sufficiently many continuous functions to find one with exact value 1 or -1 at each point mass, with sign according to the coefficient sign. In particular, we cannot hope for strong convergence, since

$$|\delta_{x_n} - \delta_x| = 2;$$

although we certainly have strict convergence. A contrario, we loose mass convergence if we consider the sequence of Dirac combinations

$$\mu_n = \delta_{x_n} - \delta_{-x_n},$$

with $\lim_{n \rightarrow \infty} x_n = 0$. In this case, the mass of μ_n equals 2, but the mass of the weak limit (which is the null measure) is 0. For real values measures, things stays quite easy, as mass convergence roughly appears when the limit points are colliding.

The situation is more dramatic in higher dimensions, where the choice of the norm used in the mass definition alters the conclusion of mass convergence, even in the case of a colliding pair. Changing the norm in $\mathbf{R}^N$ does not alter the topology of the space (hence the set of test continuous functions), but it changes its geometry. Two famous examples of norms in the plane are given by the euclidean norm

$$|x|_2 = \sqrt{(x_1)^2 + (x_2)^2},$$

and the Manhattan norm

$$|x|_1 = |x_1| + |x_2|.$$

To see that the choice of geometry affects mass convergence, consider for all $n \in \mathbf{N}$ the vector valued measure

$$\mu_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \delta_{1/n} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_{-1/n}.$$

The weak limit $(\mu_n)_{n \in \mathbf{N}}$ is easily computed as

$$\mu = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \delta_0.$$

With respect to the euclidean norm, the mass of μ_n is computed as

$$\|\mu_n\|_2 = |(1, 0)|_2 + |(0, 1)|_2 = 2,$$

while the mass of μ is computed as $\|\mu\|_2 = \sqrt{2}$. Considering the euclidean norm, we should say that $(\mu_n)_{n \in \mathbf{N}}$ does not converge, in the sense of mass convergence, to μ . Note that the mass of μ is strictly smaller to the mass of μ_n , which sounds legit in term of mass collision. Let us compare this result with the choice of the Manhattan norm. We compute

$$\|\mu_n\|_1 = |(1, 0)|_1 + |(0, 1)|_1 = 2,$$

while $\|\mu\|_1 = 2$. Hence, with respect to the Manhattan norm, the sequence $(\mu_n)_{n \in \mathbf{N}}$ converges strictly to μ .

A similar behavior occurs in the field of functions of bounded variations, if one considers the sequence

$$u_n : [0, 1] \times [0, 1] \rightarrow \mathbf{R} : u_n(x) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} x + \frac{1}{n} \chi(nx_2) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $\chi(t) = \frac{1}{2} - |t - \frac{1}{2}|$ on $[0, 1]$, and repeats periodically. This is a standard laminar construction as one can find in [19, Section 2], for example. The sequence $(u_n)_{n \in \mathbf{N}}$ is weakly* converging to

$$u : [0, 1] \times [0, 1] \rightarrow \mathbf{R} : u(x) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} x,$$

but the derivatives keep oscillating between

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

on equimeasurable subspaces of the cube, so that $\|Du_n\|_2 = 2 = \|Du\|_2$.

Nevertheless, if we equip $\mathbf{R}^{2 \times 2}$ with the Frobenius norm we have

$$\|Du\|_{\text{Fr}} = \sqrt{\text{Tr}(Du^* Du)} = \sqrt{5},$$

while $\|Du_n\|_{\text{Fr}} = 1 + \sqrt{2}$ for all $n \in \mathbf{N}$.

1.2.1 Previously known results

A first important result is due to Reshetnyak [42] and Delladio [23] (see also [47]), who have proved the following statement:

Known result (Reshetnyak [42]). Assume that $(\mu_n)_{n \in \mathbf{N}}$ is a sequence of vector valued measures weakly converging to some vector measure μ . If we have

$$\lim_{n \rightarrow \infty} \|\mu_n\| = \|\mu\|$$

with respect to the mass induced by a strictly convex norm, then mass convergence occurs for any other choice of norm.

The above result naturally rises the question to know if it is possible to encode mass convergence via a purely topological notion. To do so, we propose to consider the notion of *range of a vector measure* [25, 35, 43], defined as the closed convex hull of the algebraic range

$$\{\mu(A) : A \in \Sigma\}.$$

When the measures are valued in $\mathbf{R}$, we have [20, Appendix C]

$$\text{range}(\mu) = [-\mu^-(X), \mu^+(X)]$$

where μ^+, μ^- form the Hahn-Jordan decomposition of μ , while

$$|\mu|(X) = \mu^+(X) + \mu^-(X),$$

so that for any widely converging sequence of vector measures, the convergence of the range is equivalent with mass convergence.

Our choice is motivated by the following characterization of the mass, due to Rodriguez-Piazza:

Known result (Rodriguez-Piazza [43]). Let μ, ν be two vector measures with bounded ranges in $\mathbf{R}^n$, such that

$$\text{range}(\mu) = \text{range}(\nu).$$

Then we have $\|\mu\| = \|\nu\|$, for any choice of norm on $\mathbf{R}^n$.

The above mentioned result shows that somehow, the range of a vector measure characterizes its mass. In this perspective, the choice of the range as a relevant topological object to characterize mass convergence is natural, and it also leads to the following question: can we hope for some monotony property of the mass with respect to inclusion of the ranges?

1.2.2 New contributions

A first result we prove is an extension of Reshetnyak theorem:

New result. Let $(\mu_n)_{n \in \mathbf{N}}$ be a sequence of vector valued measures, weakly converging to some μ , and with bounded ranges. If the sequence of ranges $(\text{range}(\mu_n))_{n \in \mathbf{N}}$ converges to $\text{range}(\mu)$ in the sense of the Hausdorff distance between compact and convex sets, then

$$\lim_{n \rightarrow \infty} \|\mu_n\| = \|\mu\|$$

for any choice of norm on $\mathbf{R}^N$. Furthermore, the converse implication holds if, and only if, the chosen norm is strictly convex.

The above result shows that strict convexity is required for range convergence, and encoded mass convergence (for strictly convex norms) with a purely topological object that does not depend on the geometry of the space. It also explains why the Manhattan norm lead to mass convergence although the masses were colliding: the Manhattan norm is not strictly convex, and therefore cannot catch all the information concerning the ranges (and mass collision). A contrario, the Euclidean norm is strictly convex, and catches the information of mass collision.

When the chosen norm on the target space $\mathbf{R}^N$ is euclidean, we are able to prove a monotony result for the mass:

New result. Let μ, ν be two vector measures of bounded ranges, such that

$$\text{range}(\mu) \subseteq \text{range}(\nu).$$

If the norm $\|\cdot\|$ on $\mathbf{R}^N$ derives from a scalar product, then we have

$$\|\mu\| \leq \|\nu\|,$$

with equality if and only if the ranges coincide.

The requirement that the norm derives from a scalar product comes with the technique we have used to solve the above problem. In fact, we have proved a representation formula of the form

$$\|\mu\| = \int_{\mathcal{S}} |(\mu|u)| d\sigma(u),$$

for some measure σ on the sphere $\mathcal{S}$ of $\mathbf{R}^N$, and where $|(\mu|u)|$ denotes the mass of the projected measure with the endowed scalar product. Such representation formula holds for euclidean norms, and more generally, its validity is linked with the zonoid property of the unit ball in $\mathbf{R}^N$ [6, 18]. We mention that in dimension 2, $\mathbf{R}^N = \mathbf{R}^2$, such formula *always holds* and in this case, we have brought a interesting geometric interpretation of the mass:

New result. Assume that μ is a vector measure valued in $\mathbf{R}^2$. For any choice of norm, we have

$$\frac{1}{2} \|\mu\| = \text{Perimeter}(\text{range}(\mu)),$$

where the perimeter is computed with respect to the distance induced by the chosen norm $\|\cdot\|$.

Convergence of the perimeters is sufficient to prove convergence of the convex sets in the sense of Hausdorff; and the monotony result may be derived from an isoperimetric inequality (seen also [48]).

A detailed presentation of the above results is exposed in part 1, chapter 3.

1.3 Advances around singular vortex in lakes

In this introduction, we present the lake problem in its potential vorticity formulation, as it is one of the formulations that is common to our three chapters, chapters 4, 5, 6 in part 2.

A lake $D \subseteq \mathbf{R}^2$ is a bounded planar domain, endowed with a depth function $b : D \rightarrow [0, \infty)$ [16, 27, 40]. In general, one may think of b as being a smooth function at least continuously differentiable.

The velocity $\mathbf{u}$ in the lake satisfies the incompressibility equation

$$\nabla \cdot (b\mathbf{u}) = 0,$$

and the time-evolution equation

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla h,$$

where h is the water elevation from position at rest. In general, one drops the dependence in h by taking the curl of the above time-dependent equation, which leads, after some algebra, to

$$\partial_t \frac{\zeta}{\Gamma} + \mathbf{u} \cdot \nabla \frac{\zeta}{\Gamma} = 0,$$

where $\zeta = b^{-1} \text{curl}(\mathbf{u})$ is the *potential vorticity* associated with $\mathbf{u}$,

$$\Gamma = \int_D \zeta b(x) dx = \int_D \text{curl}(\mathbf{u}) = \oint_{\partial D} \mathbf{u} \circ \tau$$

is the total circulation of the velocity. It is a conserved quantity of the motion (because ζ is transported along the flow, and the incompressibility condition on $\mathbf{u}$ makes the weighted measure $b(x)dx$ the *invariant measure* of the system).

Giving a potential vorticity ζ , it is always possible to reconstruct the velocity $\mathbf{u}$ by mean of some operator K , in such a way that

$$\mathbf{u} = \frac{1}{b} \nabla^\perp K \zeta = \frac{1}{b} (\partial_2 K \zeta, -\partial_1 K \zeta).$$

The scalar field $K\zeta$ is referred to as the *stream function* of the problem.

We have been interested in answering a simple yet challenging question

concerning the lake equation: is it possible to mathematically justify the existence of singular vortices, and to foresee their evolution in the lake? The link with measure theory is done through our wish to solve the above problem by relying on the transport equation for ζ , which may be interpreted by stating that $\zeta(t)$ at time t is linked with the initial value $\zeta(0)$ by the fact that there are $b(x)dx$ -rearrangements of each other:

$$\int_{\{\zeta(t)>\lambda\}} b(x)dx = \int_{\{\zeta(0)>\lambda\}} b(x)dx.$$

This conservation law allows us to get profit from the rich structure of rearrangements, but the expert reader should keep in mind that we cannot expect any of the usual symmetrization techniques to be useful here, because the topography b need not satisfy symmetry properties.

1.3.1 Previously known results

It is known that weak solutions of lake equations exist globally [29, 31, 33, 38, 39]; these solutions are unique [7, 31, 38] and as smooth as the data permits it [29, 32, 39].

When $b \equiv 1$, the lake equation reduces to the well-known two-dimensional Euler's equation. It is known since the works of Helmholtz, Kirchhoff and Routh, that the two-dimension Euler's equation have singular vortex-point solutions whose vorticity is a linear combination of Dirac deltas. The position of those vortices are described through a dynamical system whose Hamiltonian is the Kirchhoff–Routh stream function [28, §5; 30, Zwanzigste Vörlung, §2–3; 45, §23]. Marchioro and Pulvirenti have proved that the singular vortex-point solutions are in fact the *limits of solutions* of the planar Euler equations whose initial data's vorticity concentrates into Dirac masses [37]. Such constructions of singular point-vortices are usually referred to as *desingularization methods*.

Much less is known concerning point-vortices in the lake equation. Richardson has formally derived the following concentration law for a system of concentrated vortices (see also [10, 17, 41]):

Known result (Richardson [44, (5.1)]). If $x_\epsilon : [0, \infty) \rightarrow D$ describes the position of an isolated vortex obeying the lake equation with total circulation Γ_ϵ , concentrating in a region of width $\epsilon > 0$, then the function x_ϵ should satisfy the following ordinary differential equation:

$$\dot{x}_\epsilon(t) \simeq \frac{\Gamma_\epsilon^2}{4\pi} \ln \frac{1}{\epsilon} (\nabla^\perp \ln b)(x_\epsilon(t)).$$

Richardson's law tells us that point-vortices should follow isobath curves. This behavior is different from the typical behavior of point-vortices in the Euler's equations, where the motion of a point-vortex is driven by the whole geometry of the domain D .

Unfortunately, Richardson's work did not provide a proof that an initially concentrated vortex remains indeed concentrated. In fact, the techniques used by Richardson is more about matching asymptotics. We would like to give a more robust justification of the law of evolution.

It should be noted that those questions have received an answer in the particular case of *stationary solutions*, where the point-vortex is proved to locate on the point of maximal depth. Existence of such solutions were studied by Van Schaftingen and de Valeriola [24] and were already known in the flat Euler case [3, 14, 46, 50, 51]:

Known result (de Valeriola, Van Schaftingen [24]). There exists a family of solutions $(\zeta_\epsilon)_{\epsilon>0}$ of the lake equation, obtained by an energy maximization principle, weakly converging to a point-vortex located on a point of maximal depth.

The techniques used to prove the above was to solve an energy maximization on the stream function $\psi = K\zeta$, with an a priori constrained of the form

$$\zeta = F(\psi),$$

for some prescribed increasing function F . According to Arnold's interpretation [1, 2] of the transport equation, maximizer should follow such a relation, and one may wonder if it is possible to construct steady solutions by solving an energy maximization problem on a class of rearrangements of the vorticity ζ directly, without imposing the shape of

F and working with a maximization problem in the space of ψ . Such approach was successfully implemented for the Euler's equations and for Euler's rings [12, 13, 15], where it brought results that were robust enough to handle the cases of non smooth lakes. It is natural to hope that such approach may be used successfully as well for the lake equation.

1.3.2 New contributions

In this introduction, we will focus on the particular solutions prepared from an initial state as follows. We fix a nonnegative function $f \in C_c^\infty(\mathbf{R}^2)$ such that $\int_{\mathbf{R}^2} f = 1$, a sequence of positive numbers $(\varepsilon_n)_{n \in \mathbf{N}}$ converging to 0, $x_0 \in \mathbf{R}^2$ and a sequence $(\Gamma_n)_{n \in \mathbf{N}}$ such that for every $n \in \mathbf{N}$ $B(x_0, \varepsilon_n) \subset D$, we set for each $x \in D$

$$b(x)\zeta_n(0, x) = \Gamma_n f\left(\frac{x - x_0}{\varepsilon_n}\right). \quad (1.3.1)$$

Thus $\zeta_n(0)$ appears to be a rescaled version of the same model positive function f . It can be computed that

$$\int_D \zeta_n(0, x) b(x) dx = \Gamma_n,$$

and the following weak convergence holds:

$$\frac{1}{\Gamma_n} \zeta_n(0) \rightharpoonup \delta_{x(0)}.$$

New result. Assume that the depth function b is constant on each component of ∂D and non vanishing, and assume that we are given a family of classical solutions of the lake equations

$$\{\zeta_n : n \in \mathbf{N}\},$$

prepared from the initial conditions $\zeta_n(0, \cdot)$ of definition (1.3.1). For any $s \in \mathbf{R}$, we have weakly

$$\lim_{n \rightarrow \infty} \frac{1}{\Gamma_n} \zeta_n\left(\frac{\Gamma_n s}{E_n}\right) = \delta_{x(s)},$$

where E_n is the (conserved) energy of the system, and $x : [0, \infty) \rightarrow D$ is

the unique solution of the Cauchy problem

$$\begin{cases} x'(s) = -(\nabla^\perp b^{-1})(x(s)), \\ x(0) = x_0. \end{cases}$$

The energy E_n of the system is classically defined as

$$E_n = \frac{1}{2} \int_D |\mathbf{u}_n|^2 b(x) dx = \frac{1}{2} \int_D \zeta_n K \zeta_n b(x) dx.$$

This result actually shows *two improvements* of Richardson's work. First, it proves that the vortex remains shrunk as time evolves, which was a *serious gap* in Richardson's analysis. Secondly, it shows that there exists a typical time-scale for the observation of the singular vortex motion. It may be computed that the energy is estimated as

$$E_n(0) \sim \frac{\Gamma_n^2}{4\pi} \ln \frac{1}{\varepsilon_n},$$

so that the time-scale factor is estimated at first order by

$$\frac{\Gamma_n}{E_n} \sim \frac{1}{\Gamma_n \log \frac{1}{\varepsilon_n}}.$$

The blow-up of this factor depends on the blow-up of the total circulation Γ_n compared to the logarithmic divergence.

For steady vortices, we are able to perform the analysis further, by proving the following result:

New result. Let f^+, f^- be two positive functions of integral 1, and let $\zeta_\varepsilon^+, \zeta_\varepsilon^-$ be constructed from f^+, f^- as in definition (1.3.1), with

$$\Gamma^+ = \tau\Gamma, \quad \Gamma^- = (1 - \tau)\Gamma; \quad \Gamma > 0,$$

and distinct points $x_0^+ \neq x_0^-$. Let $\mathcal{R}_\varepsilon$ be the family of all $b(x)dx$ -rearrangements of $\zeta_\varepsilon^+ - \zeta_\varepsilon^-$. There exists a maximizer $\zeta_{\star;\varepsilon}^+ - \zeta_{\star;\varepsilon}^-$ of the energy over $\mathcal{R}_\varepsilon$; and every maximizer of $\mathcal{R}_\varepsilon$ is a stationary solution of the lake equation that shrinks to a vortex pair:

$$\lim_{\varepsilon \rightarrow 0} \zeta_{\star;\varepsilon}^+ = \delta_{x^+}, \quad \lim_{\varepsilon \rightarrow 0} \zeta_{\star;\varepsilon}^- = \delta_{x^-};$$

with x^+, x^- are both maximizers of b .

This result is interesting because it shows that the maximization problem on classes of rearrangements is rich enough to construct solutions. It also makes echo with Arnold's interpretation of the transport equations. As expected, the above result will actually hold for a wide class of non smooth lakes, allowing vanishing conditions on b , and robust enough to take into account a Coriolis term.

Note that, a contrario with the Euler equation, the limit vortex pair may collide if the lake has only one point of maximal depth, in which case both parts of the singular-vortex are attracted to the same point. The energy maximization principle, however, ensures that the rate of convergence is sufficiently small.

When the lake has more than one point of maximal depth, the pair never collide, and we can give further estimates:

New result. In the assumption above, if b admits at least two maximizers in D , then we always have $x^+ \neq x^-$ and the pair (x^+, x^-) maximizes a function H that reduces to the Kirchoff-Routh function arising from the Euler's equations.

Moreover, the asymptotic shape of the vortices is a disk, and if the maximization class $\mathcal{R}$ is enlarged to a ball of L^∞ , then the optimal choice for the vortices are vortex patches.

This result improves the asymptotic sketch of an optimal pair in the energy maximization problem. The function H that drives the location of the maximal pair is not easily computed with an explicit formula, but this statement fills a gap between the lake model and the Euler's two-dimensional equations: at second order, the lake equation is sensitive to the geometry of ∂D .

As mentionned above, the energy maximization principle is robust enough to take into account a Coriolis force acting on the system. We have used such artifact to construct the following solution of the lake model, formed by a singular rotating vortex pair in a rotation-invariant lake:

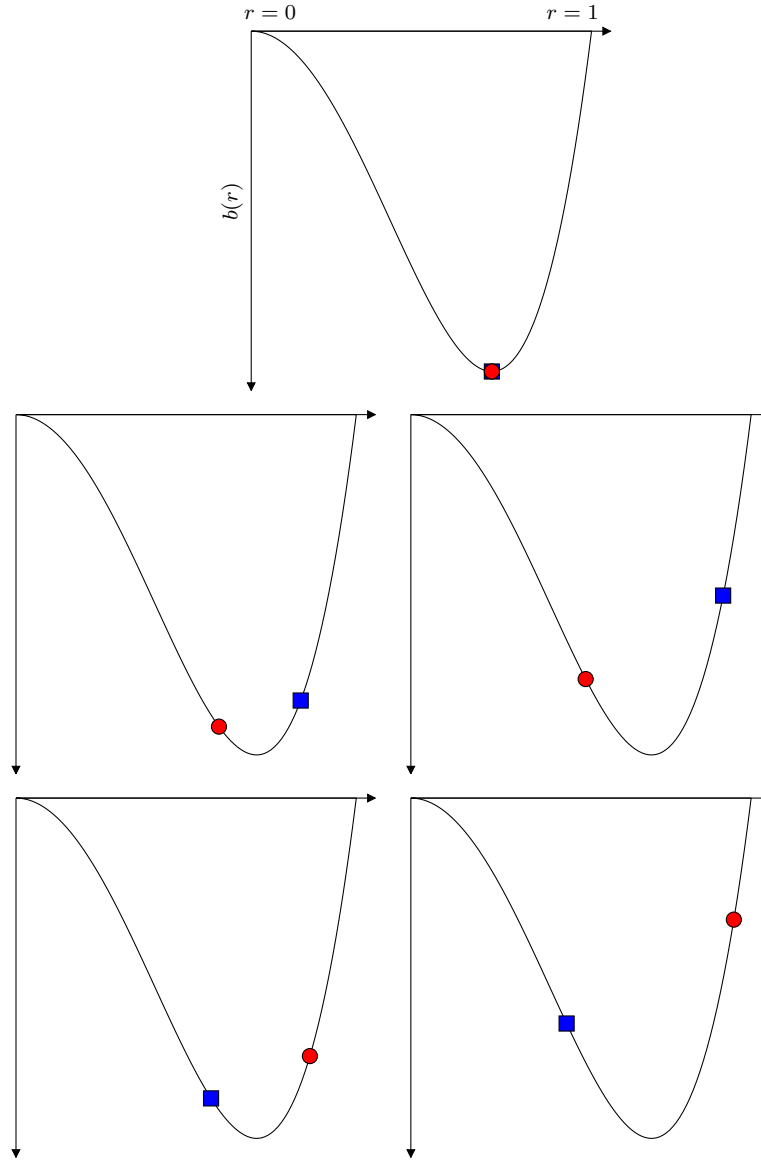


Figure 1.5: Illustration of the deviation of an optimal vortex pair in a rotating disk. The radii from 0 to 1 are represented. At rest, $\nu = 0$, the pair are on the same level curve. If ν is increased (or decreased), the vortex parts tends to separated from this curve, according to the disk direction of rotation.

New result. If $\delta_{x^+(t)} - \delta_{x^-(t)}$ is a rotating vortex pair in a disk D with radial depth function $b(x) = b(|x|^2)$, obtained by a desingularization process of maximizers of the energy in their reference frame, rotating at angular speed $8\pi\nu$ for $\nu \in (-1, 1)$, then $|x^+(t)|^2$ maximizes the function

$$r \in [0, 1] \mapsto \varphi^+(r) = b(r) - \frac{\nu}{2} \int_0^r b(s) ds,$$

while $|x^-(t)|^2$ maximizes

$$r \in [0, 1] \mapsto \varphi^-(r) = b(r) + \frac{\nu}{2} \int_0^r b(s) ds.$$

On figure (1.5), we have sketched a situation for a radially symmetric lake and a radial depth function b parametrized through the distance r to the center of the disk.

The position $x^+(t)$ of the positive vortex core is marked down by a circle, while the position $x^-(t)$ of the negative one is marked down by a square.

When the rotation speed of the disk equals 0, the two cores tend to be localized near a point of maximal depth. Note that in our example, the two pairs will not collide; but in cases where the lake admits a unique point of maximal depth, collision occurs asymptotically. In chapter 6 we prove that the collision rate is small with respect to the concentration rate.

As the pair rotates with speed $8\pi\nu$, the positive part of the pair gets delocalized, and so gets the negative part. The direction of the deviation depends on the sign of ν , that is: depends on the orientation of the rotation. If $\nu > 0$, then the positive part is attracted to the center of the lake, while the negative part gets pushed away from the center.

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Part I

Geometry questions in measure theoretic problems

Chapter 2

Approximation of successive symmetrizations by Markov processes

*The following chapter reflects a paper that was published in the
Indiana University Mathematics Journal.
Collaborator: Jean Van Schaftingen.*

2.1 Introduction and main results

2.1.1 Approximation by Steiner symmetrizations

Steiner symmetrizations are measure-preserving transformations of sets that bring symmetry with respect to one direction $u \in \mathbf{P}_{\mathbf{R}}^{d-1}$ in the Euclidean space [1]. The resulting set X^u is symmetric with respect to the direction u . It was observed in the study of the classical isoperimetric inequality that any Borel measurable set $X \subseteq \mathbf{R}^d$ which is left invariant under all Steiner symmetrizations must be an Euclidean ball centered on the origin [17].

A natural question is whether the spherical nonincreasing rearrangement, which associates to each Borel measurable set X the unique Euclidean ball X^* centered on 0 and with the same measure as X , can be approximated by Steiner symmetrizations, that is whether there ex-

ists a sequence $(u_n)_{n \in \mathbf{N}}$ such that the sequence of successive Steiner symmetrizations $(X^{u_1 \dots u_n})_{n \in \mathbf{N}}$ converges somehow to the spherical non-increasing rearrangement X^* . Such results have been obtained in order to prove various properties of symmetrizations [5; 11, proof of theorem 3.7]. The approximation procedure seems quite robust, and this brings the question whether *random sequences* of symmetrizations approximate symmetrizations.

Independent random symmetrizations of sets and functions were studied in various settings [7, 8, 12, 20, 23], and rates of convergence were recently discovered [7, Corollary 5.4, Proposition 6.2; 8, Theorem 3]. A typical result for the convergence of independent Steiner symmetrizations is:

Theorem 2.1.1. *Let $(S_n)_{n \in \mathbf{N}}$ be a sequence of independent and μ -identically distributed sequence of Steiner symmetrizations. We have*

$$\mu(\{u \in \mathbf{P}_{\mathbf{R}}^{d-1} : m(X \Delta X^u) > 0\}) > 0, \quad (2.1.1)$$

for every Lebesgue measurable set $X \subseteq \mathbf{R}^d$ of finite measure with $m(X \Delta X^) > 0$, if and only if, for every Lebesgue measurable set $X \subseteq \mathbf{R}^d$ of finite measure, the sequence of successive Steiner symmetrizations $(X^{S_1 \dots S_n})_{n \in \mathbf{N}}$ converges almost-surely in measure to X^* .*

In this chapter we investigate the approximation by *time-homogeneous Markov processes*. A stochastic process $(S_n)_{n \in \mathbf{N}}$ valued in a topological space $\mathcal{S}$ is a time-homogeneous Markov process if there exists a transition function

$$P : \mathcal{S} \times \mathcal{B}(\mathcal{S}) \rightarrow [0, 1] : (s, A) \mapsto P_s(A),$$

satisfying some measurability conditions (see [13] or section 2.2.4 in this text), and such that if, for every $n \in \mathbf{N}$, and for every Borel measurable set $A \in \mathcal{B}(\mathcal{S})$, we have almost-surely

$$\mathbb{P}\{S_{n+1} \in A | S_1, \dots, S_n\} = P_{S_n}(A).$$

The iterated kernels P^k are then defined to satisfy ([13] or section 2.2.4

in this text)

$$P_{S_n}^k(A_1 \times \dots \times A_k) = \mathbb{P}\{S_{n+1} \in A_1, \dots, S_{n+k} \in A_k | S_1, \dots, S_n\},$$

almost-surely for all $n, k \in \mathbf{N}$ and all Borel measurable sets $A_1, \dots, A_k \in \mathcal{B}(\mathcal{S})$. In contrast with processes made up of independent and identically distributed variables, successive Steiner symmetrizations that form a Markov process are mutually correlated, and there are known examples [3, 7, 24] of deterministic sequences for which convergence does not occur. We obtain the following result for Steiner symmetrizations:

Theorem 2.1.2. *Let $(S_n)_{n \in \mathbf{N}}$ be a time-homogeneous Markov process of Steiner symmetrizations with initial distribution μ . If there exists $\mathfrak{s}_\star \in \mathbf{P}_{\mathbf{R}}^{d-1}$ such that*

(i) (Recurrence) *for every nonempty open set $\mathcal{O} \subseteq \mathbf{P}_{\mathbf{R}}^{d-1}$, we have*

$$\mathbb{P}((S_n)_{n \in \mathbf{N}} \text{ enters } \mathcal{O} \text{ infinitely many often}) = 1,$$

(ii) (Continuity) *for every $n \in \mathbf{N}$ and for every open set $\mathcal{O} \subseteq (\mathbf{P}_{\mathbf{R}}^{d-1})^n$, the map*

$$s \in \mathbf{P}_{\mathbf{R}}^{d-1} \mapsto P_s^n(\mathcal{O})$$

is lower semi-continuous at $\mathfrak{s}_\star$,

(iii) (Discrimination) *for every Lebesgue measurable set $X \subseteq \mathbf{R}^d$ of finite measure, with $m(X \Delta X^\star) > 0$, the process of Steiner symmetrizations starting at $\mathfrak{s}_\star$ reaches in finite time the set $\{u \in \mathbf{P}_{\mathbf{R}}^{d-1} : m(X \Delta X^u) > 0\}$, that is:*

$$\sum_{n \in \mathbf{N}} P_{\mathfrak{s}_\star}^n((\mathbf{P}_{\mathbf{R}}^{d-1})^{n-1} \times \{u \in \mathbf{P}_{\mathbf{R}}^{d-1} : m(X \Delta X^u) > 0\}) > 0;$$

then for every Lebesgue measurable set $X \subseteq \mathbf{R}^d$ of finite measure, the sequence $(X^{S_1 \dots S_n})_{n \in \mathbf{N}}$ converges in measure to $X^\star$, almost-surely.

The discrimination condition (iii) in theorem 2.1.2 is similar to condition (2.1.1) in theorem 2.1.1, and they are equivalent for independent sequences of random symmetrizations. A necessary condition for the

conclusion of theorem 2.1.2 to hold is that for each Lebesgue measurable set X of finite measure and with $m(X \Delta X^\star) > 0$, we should have

$$\sum_{n \in \mathbf{N}} \mathbb{P}(\{m(X \Delta X^{S_n}) > 0\}) > 0. \quad (2.1.2)$$

By Fubini's theorem, for each X , condition (2.1.2) implies that condition (iii) holds for some $s \in \mathbf{P}_{\mathbf{R}}^{d-1}$, but s may depend on X and n . Therefore, condition (2.1.2) is close, but not equivalent, to condition (iii) in theorem 2.1.2. The initial distribution μ of the process is only involved in the recurrence condition (i): the recurrent point $\mathfrak{s}_\star$ is assumed to be deterministic, so that its existence does not simply follow from the compactness of the projective plane $\mathbf{P}_{\mathbf{R}}^{d-1}$. The continuity condition in theorem 2.1.2 is stronger than the usual weak-Feller continuity at $\mathfrak{s}_\star$, but still weaker than the usual strong Feller-continuity everywhere (see also [13] for definitions, and proposition 2.2.14 and discussion below for details). The recurrence condition (i) and continuity condition (ii) ensure together that the asymptotic behaviour of the process is closed to an independent process with distribution $P_{\mathfrak{s}_\star}$.

In the proof of theorems 2.1.1 and 2.1.2, we do not study directly the distance between sets, in order to prove the convergence. We rather measure the convergence with an asymmetry, which is a functional of the form

$$\mathcal{A}(X) = \int_X \frac{|x|^2}{1 + |x|^2} dx. \quad (2.1.3)$$

The asymmetry function strictly decreases along Steiner symmetrizations of X , and reaches a minimum at $X^\star$. The idea of using such a function to measure the asymmetry of sets is a standard technique in the field of symmetrizations (see for example [2, 7, 22, 23]).

2.1.2 Other symmetrizations

The spherical nonincreasing rearrangement has been approximated by other symmetrizations, such as cap symmetrizations [15] and polarizations [1, 6, 20, 21, 24]. Other symmetrizations such as the cap symmetrization [16] and discrete symmetrizations [14] have also been approximated

in the deterministic case in order to prove isoperimetric theorems.

Theorem 2.1.2 *fails* for the approximation of the spherical nonincreasing rearrangement by successive cap symmetrizations or polarizations. Polarizations and cap symmetrizations can be thought of as elements in $S^{d-1} \times [0, +\infty)$ (see also section 2.3.1 for accurate definitions). In fact, any cap symmetrization is characterized by a affine half line passing through the origin $0 \in \mathbf{R}^d$, which yields $S^{d-1} \times [0, +\infty)$ as parametrization space. Although cap symmetrizations from $S^{d-1} \times (0, +\infty)$ – those cap symmetrizations whose half line has edge different of 0 – strictly decrease the asymmetry (2.1.3) of non spherical sets, cap symmetrizations from $S^{d-1} \times \{0\}$ may act as isometries. Such symmetrization could conspire to a non convergent behaviour. On the other hand, they could also be necessary in the realization of the process (see section 2.3.4 for more details).

In order to prevent bad behaviour without just throwing away cap symmetrizations from $S^{d-1} \times \{0\}$, we strengthen the discrimination condition (iii) from theorem 2.1.2. This leads us to the following theorem:

Theorem 2.1.3. *Let $(S_n)_{n \in \mathbf{N}}$ be a time-homogeneous Markov process of cap symmetrizations (resp. polarizations) with initial distribution μ . If there exists $\mathfrak{s}_\star \in S^{d-1} \times [0, +\infty)$ such that*

- (i) *(Recurrence) for every nonempty open set $\mathcal{O} \subseteq S^{d-1} \times [0, +\infty)$, we have*

$$\mathbb{P}((S_n)_{n \in \mathbf{N}} \text{ enters } \mathcal{O} \text{ infinitely many often}) = 1,$$

- (ii) *(Continuity) for every $n \in \mathbf{N}$ and for every open set $\mathcal{O} \subseteq (S^{d-1} \times [0, +\infty))^n$, the map*

$$s \in \mathbf{P}_{\mathbf{R}}^{d-1} \mapsto P_{\mathfrak{s}_\star}^n(\mathcal{O})$$

is lower semi-continuous at $\mathfrak{s}_\star$,

- (iii) *(Discrimination) for every Lebesgue measurable set $X \subseteq \mathbf{R}^d$ of finite measure, we have*

$$\sum_{n \in \mathbf{N}} P_{\mathfrak{s}_*}^n((S^{d-1} \times (0, +\infty))^{n-1} \times \{u \in S^{d-1} \times [0, +\infty) : m(X \Delta X^u) > 0\}) > 0,$$

then for every Lebesgue measurable set $X \subseteq \mathbf{R}^d$ of finite measure, the sequence $(X^{S_1 \dots S_n})_{n \in \mathbf{N}}$ converges in measure to X^* , almost-surely.

While conditions (i) and (ii) in theorem 2.1.2 and theorem 2.1.3 are similar, the discrimination condition (iii) takes into account the bad symmetrizations from $S^{d-1} \times \{0\}$. Condition (iii) means that the process starting at $\mathfrak{s}_*$ reaches in finite time, any set of the form $\{u \in \mathbf{P}_{\mathbf{R}}^{d-1} : m(X \Delta X^u) > 0\}$ without passing through $S^{d-1} \times \{0\}$. In the case of independent and identically distributed cap symmetrizations and polarizations, condition (2.1.1) tells us that the boundary set $S^{d-1} \times \{0\}$ is not allowed to support the measure μ ; thus the discrimination condition (iii) in theorem 2.1.3 reduces to (2.1.1).

2.1.3 Organization of the chapter

In order to emphasize the main properties of symmetrizations that we use, we work in section 2.2 with an abstract notion of symmetrizations that covers Steiner and cap symmetrizations, and polarizations. We draw the reader attention to the fact that the abstract framework we work with, is mainly aimed to strip the proofs of non pertinent particularities. Without any assumptions on the process $(S_n)_{n \in \mathbf{N}}$, not even the Markov property, we prove an abstract criterion to test the convergence, proposition 2.2.7 (section 2.2.2). The strategy here is a summability trick used by Burchard and Fortier [7]. This abstract criterion is then particularized in two directions. We first deduce an abstract version of 2.1.1. A second particularization is a general result about Markov processes, from which we derive theorems 2.1.2 and 2.1.3. We finally give several more explicit examples and results. A particular attention is drawn to a *new example* for cap symmetrizations, where the boundary $S^{d-1} \times \{0\}$ is needed for the universal convergence and, thus, can not be simply removed from the parameter space for the cap symmetrizations and polarizations.

2.2 Abstract convergence result

2.2.1 Abstract symmetrizations

We fix a metric space $(\mathcal{X}, d)$ and a nonexpansive projector $\star$ in $(\mathcal{X}, d)$, that is a map $[X \in \mathcal{X} \mapsto X^\star \in \mathcal{X}]$ such that, for all $X, Y \in \mathcal{X}$, we have $d(X^\star, Y^\star) \leq d(X, Y)$ and $X^{\star\star} = X^\star$. We introduce the following abstract setting for symmetrizations.

Definition 2.2.1. A *symmetrization space* is a nonempty set $\mathcal{S}$ of maps $[X \in \mathcal{X} \mapsto X^s \in \mathcal{X}]$ endowed with a metrizable topology with countable basis, such that

- (a) (*Continuity*) the map $[(X, s) \in \mathcal{X} \times \mathcal{S} \mapsto X^s]$ is continuous,
- (b) (*Idempotence*) for every $X \in \mathcal{X}$, $X^{ss} = X^s$,
- (c) (*Nonexpansiveness*) for all $X, Y \in \mathcal{X}$, $d(X^s, Y^s) \leq d(X, Y)$.

The elements of $\mathcal{X}$ are called *objects*, and elements of $\mathcal{S}$ are called *symmetrizations*.

In view of the nonexpansiveness (c), the continuity (a) can be deduced from the apparently weaker assumption that, for every $X \in \mathcal{X}$, the map $[s \in \mathcal{S} \mapsto X^s]$ is continuous.

Definition 2.2.2. We say that a symmetrization space $\mathcal{S}$ is $\star$ -*compatible* if

- (a) for every $s \in \mathcal{S}$, for every $X \in \mathcal{X}$, we have $X^{s\star} = X^\star = X^{\star s}$,
- (b) for every $X \in \mathcal{X}$, if $X = X^s$ for every $s \in \mathcal{S}$, then $X = X^\star$.

Definition 2.2.3. Let $\mathcal{S}$ be a symmetrization space. A function $\mathcal{A} : \mathcal{X} \rightarrow \mathbf{R}$ is an *asymmetry* on $\mathcal{S}$ if $\mathcal{A}$ is continuous and if for every $s \in \mathcal{S}$, for every $X \in \mathcal{X}$, we have $\mathcal{A}(X^s) \leq \mathcal{A}(X)$. An asymmetry function $\mathcal{A}$ is said to be a *strict asymmetry* on $\mathcal{S}$ when for every $X \in \mathcal{X}$, for every $s \in \mathcal{S}$, the equality $\mathcal{A}(X^s) = \mathcal{A}(X)$ implies $X^s = X$.

Definition 2.2.4. Let $\mathcal{S}$ be a symmetrization space and $\mathcal{A}$ be an asymmetry function on $\mathcal{S}$. We say that $\mathcal{A}$ is $\star$ -*compatible* if for every $X \in \mathcal{X}$ satisfying $\mathcal{A}(X) \leq \mathcal{A}(X^\star)$, we have $X = X^\star$.

If a symmetrization space $\mathcal{S}$ on $(\mathcal{X}, d)$ is $\star$ -compatible, then the function

$$\mathcal{A} : \mathcal{X} \rightarrow \mathbf{R} : \mathcal{A}(X) = d(X, X^\star)$$

always defines a $\star$ -compatible asymmetry function on $\mathcal{S}$. However, this asymmetry function might not be the best choice in convergence theory. Another direct consequence of the definitions is that every strict asymmetry function on a $\star$ -compatible symmetrization space $\mathcal{S}$, is itself $\star$ -compatible.

The next proposition characterizes the convergence in $\mathcal{X}$ of iterated symmetrizations in terms of the asymmetry.

Proposition 2.2.5. *Let $\mathcal{S}$ be a symmetrization space, $\mathcal{A}$ be an asymmetry function on $\mathcal{S}$, $(s_n)_{n \in \mathbf{N}}$ be a sequence in $\mathcal{S}$ and $X \in \mathcal{X}$. If $\mathcal{S}$ and $\mathcal{A}$ are $\star$ -compatible, then*

$$\liminf_{n \rightarrow +\infty} d(X^\star, X^{s_1 \dots s_n}) = 0$$

if and only if, the set $\{X^{s_1 \dots s_n} : n \in \mathbf{N}\}$ has compact closure in $\mathcal{X}$ and

$$\liminf_{n \rightarrow +\infty} \mathcal{A}(X^{s_1 \dots s_n}) = \mathcal{A}(X^\star).$$

Proof. The “only if” part is a consequence of the continuity of $\mathcal{A}$, and the fact that the closure of a convergence sequence is always compact. For the converse, assume that the set $\{X^{s_1 \dots s_n} : n \in \mathbf{N}\}$ has compact closure, and that

$$\liminf_{n \rightarrow +\infty} \mathcal{A}(X^{s_1 \dots s_n}) = \mathcal{A}(X^\star).$$

Since the asymmetry function decreases along symmetrizations, the sequence $(\mathcal{A}(X^{s_1 \dots s_n}))_{n \in \mathbf{N}}$ converges to $\mathcal{A}(X^\star)$. By compactness assumption, there exists a subsequence $(X^{s_1 \dots s_{n_k}})_{k \in \mathbf{N}}$ that converges to some $Y \in \mathcal{X}$. By continuity of $\mathcal{A}$, we have

$$\mathcal{A}(X^\star) = \lim_{k \rightarrow +\infty} \mathcal{A}(X^{s_1 \dots s_{n_k}}) = \mathcal{A}(Y).$$

The $\star$ -compatibility of $\mathcal{S}$ then implies

$$d(X^\star, Y^\star) = \lim_{k \rightarrow +\infty} d(X^\star, X^{s_1 \dots s_{n_k} \star}) = d(X^\star, X^\star) = 0,$$

so that $X^\star = Y^\star$ and $\mathcal{A}(Y^\star) = \mathcal{A}(Y)$. Since $\mathcal{A}$ is $\star$ -compatible, we deduce that $Y = X^\star$. Since this is true for each accumulation point of the sequence $(X^{s_1 \dots s_n})_{n \in \mathbf{N}}$, by compactness, this sequence converges to $X^\star$. $\square$

2.2.2 Abstract result for random symmetrizations

From now on, we fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $(S_n)_{n \in \mathbf{N}}$ a sequence of measurable maps from $(\Omega, \mathcal{A})$ to $\mathcal{S}$, which is assumed to be a symmetrization space, endowed with its Borel σ -algebra $\mathcal{B}(\mathcal{S})$. For every $n \in \mathbf{N}$, we write $\mathcal{F}_n$ the sub- σ -algebra of $\mathcal{A}$ induced by $\{S_1, \dots, S_n\}$, and $\mathcal{F}$ the smallest sub- σ -algebra of $\mathcal{A}$ that contains $\bigcup_{n \in \mathbf{N}} \mathcal{F}_n$. If N is a stopping time adapted to $(S_n)_{n \in \mathbf{N}}$, its induced filtration is denoted by $\mathcal{F}_N$. Throughout the text, we write $\mathbb{P}(\cdot)$ (resp. $\mathbb{E}$) probabilities (resp. expectations), and $\mathbb{E}\{\cdot|\cdot\}$ conditional expectations.

The next technical lemma allows to reduce the randomness by taking the infimum; it follows from the classical properties of conditional expectation.

Lemma 2.2.6. *Let $\mathcal{S}$ be a symmetrization space and $\mathcal{B} \subseteq \mathcal{F}$ be a σ -algebra. Let $X \in \mathcal{X}$, and set*

$$X^\mathcal{S} = \{X^{s_1 \dots s_n} : n \in \mathbf{N}, s_1, \dots, s_n \in \mathcal{S}\}.$$

If $f : X^\mathcal{S} \times \mathcal{S} \rightarrow \mathbf{R}$ is continuous and bounded, and if $\mathfrak{G} : (\Omega, \mathcal{B}) \rightarrow X^\mathcal{S}$ and $S : (\Omega, \mathcal{F}) \rightarrow \mathcal{S}$ are measurable, then we have for every $U \in \mathcal{X}$, almost-surely on $\mathfrak{G}^{-1}(U)$,

$$\mathbb{E}\{f(\mathfrak{G}, S)|\mathcal{B}\} \geq \inf_{Y \in U} \mathbb{E}\{f(Y, S)|\mathcal{B}\}.$$

Proof. Without loss of generality, we can assume that f is positive. The topological space $X^\mathcal{S} \times \mathcal{S}$ is second-countable. Therefore it is not difficult to prove that, since f is bounded and continuous on $X^\mathcal{S} \times \mathcal{S}$, for all probability measure μ on $\mathcal{B}(X^\mathcal{S} \times \mathcal{S})$, there exists a decreasing sequence $(f_n)_{n \in \mathbf{N}}$ of simple functions, converging μ -almost-everywhere to f , and whose level sets are finite unions of disjoint Borel rectangles.

We apply this approximation scheme with the conjoint distribution μ

of the random vector $(\mathfrak{G}, S)$. Let $(f_n)_{n \in \mathbf{N}}$ be the corresponding approximation sequence. According to standard properties of the conditional expectation [4, Theorem 34.3], we have for every $n \in \mathbf{N}$, almost-surely on $\mathfrak{G}^{-1}(U)$,

$$\mathbb{E}\{f_n(\mathfrak{G}, S)|\mathcal{B}\} \geq \inf_{Y \in U} \mathbb{E}\{f_n(Y, S)|\mathcal{B}\}.$$

It now follows from the monotone convergence theorem for the conditional expectation that, almost-surely on $\mathfrak{G}^{-1}(U)$,

$$\begin{aligned} \mathbb{E}\{f(\mathfrak{G}, S)|\mathcal{B}\} &= \inf_{n \in \mathbf{N}} \mathbb{E}\{f_n(\mathfrak{G}, S)|\mathcal{B}\} \geq \inf_{n \in \mathbf{N}} \inf_{Y \in U} \mathbb{E}\{f_n(Y, S)|\mathcal{B}\} \\ &\geq \inf_{n \in \mathbf{N}} \inf_{Y \in U} \mathbb{E}\{f(Y, S)|\mathcal{B}\} = \inf_{Y \in U} \mathbb{E}\{f(Y, S)|\mathcal{B}\}. \quad \square \end{aligned}$$

We are now ready to prove the main result about general stochastic processes of symmetrizations, which need not be Markov processes.

Proposition 2.2.7 (Convergence by divergence). *Let $\mathcal{S}$ be a symmetrization space and $\mathcal{A}$ be an asymmetry function on $\mathcal{S}$, such that $\mathcal{S}$ and $\mathcal{A}$ are both $\star$ -compatible. Let be $X \in \mathcal{X}$. If the set*

$$X^{\mathcal{S}} = \{X^{s_1 \dots s_n} : n \in \mathbf{N}, s_1, \dots, s_n \in \mathcal{S}\}$$

has compact closure in $(\mathcal{X}, d)$, and if there exists an increasing and almost-surely finite sequence of stopping times $(N_n)_{n \in \mathbf{N}}$ adapted to the process $(S_n)_{n \in \mathbf{N}}$, such that we have almost-surely for all $\epsilon > 0$

$$\begin{aligned} \sum_{n \in \mathbf{N}} \inf \left\{ \mathbb{E}\{\mathcal{A}(Y) - \mathcal{A}(Y^{s_{N_n+1} \dots s_{N_{(n+1)}}}) | \mathcal{F}_{N_n}\} : \right. \\ \left. Y \in \overline{X^{\mathcal{S}}}, \mathcal{A}(Y) \geq \mathcal{A}(X^{\star}) + \epsilon \right\} = +\infty, \quad (2.2.1) \end{aligned}$$

then the sequence $(X^{s_1 \dots s_n})_{n \in \mathbf{N}}$ converges almost-surely to $X^{\star}$.

The proof is based on a summability trick found by Burchard and Fortier [7].

Proof. If $X = X^{\star}$, there is nothing to prove. Otherwise, by $\star$ -compatibility of $\mathcal{A}$, there exists $\epsilon > 0$ such that $\mathcal{A}(X) \geq \mathcal{A}(X^{\star}) + \epsilon$. For $m \in \mathbf{N}$,

we write almost-surely

$$\begin{aligned}
\mathcal{A}(X) - \mathcal{A}(X^*) &= \mathcal{A}(X) - \mathcal{A}(X^{S_1 \dots S_{N_1}}) + \mathcal{A}(X^{S_1 \dots S_{N(m+1)}}) - \mathcal{A}(X^*) \\
&\quad + \sum_{n=1}^m \mathcal{A}(X^{S_1 \dots S_{N_n}}) - \mathcal{A}(X^{S_1 \dots S_{N(n+1)}}) \\
&\geq \sum_{n=1}^m \mathcal{A}(X^{S_1 \dots S_{N_n}}) - \mathcal{A}(X^{S_1 \dots S_{N(n+1)}}) \\
&\geq \sum_{n=1}^m \chi_{\Theta_n} \cdot (\mathcal{A}(X^{S_1 \dots S_{N_n}}) - \mathcal{A}(X^{S_1 \dots S_{N(n+1)}})).
\end{aligned}$$

where, for every $n \in \mathbf{N}$, we have defined the set

$$\Theta_n^\varepsilon = \{\mathcal{A}(X^{S_1 \dots S_{N_n}}) \geq \mathcal{A}(X^*) + \varepsilon\} \in \mathcal{F}_{N_n}.$$

(Here the symbol χ denotes indicator functions of sets.) Taking the expectation on both sides, we compute

$$\begin{aligned}
\mathcal{A}(X) - \mathcal{A}(X^*) &\geq \sum_{n=1}^m \mathbb{E}(\chi_{\Theta_n^\varepsilon} \cdot (\mathcal{A}(X^{S_1 \dots S_{N_n}}) - \mathcal{A}(X^{S_1 \dots S_{N(n+1)}}))) \\
&= \sum_{n=1}^m \mathbb{E}(\chi_{\Theta_n^\varepsilon} \mathbb{E}\{\mathcal{A}(X^{S_1 \dots S_{N_n}}) - \mathcal{A}(X^{S_1 \dots S_{N(n+1)}}) | \mathcal{F}_{N_n}\}).
\end{aligned}$$

According to lemma 2.2.6, and writing $\overline{X_\varepsilon^\mathcal{J}} = \{Y \in \overline{X^\mathcal{J}} : \mathcal{A}(Y) \geq \mathcal{A}(X^*) + \varepsilon\}$, we get

$$\begin{aligned}
\mathcal{A}(X) - \mathcal{A}(X^*) &\geq \sum_{n=1}^m \mathbb{E} \left(\chi_{\Theta_n^\varepsilon} \inf_{Y \in \overline{X_\varepsilon^\mathcal{J}}} \mathbb{E}\{\mathcal{A}(Y) \right. \\
&\quad \left. - \mathcal{A}(Y^{S_{N_n+1} \dots S_{N(n+1)}}) | \mathcal{F}_{N_n}\} \right) \\
&= \mathbb{E} \left(\sum_{n=1}^m \chi_{\Theta_n^\varepsilon} \inf_{Y \in \overline{X_\varepsilon^\mathcal{J}}} \mathbb{E}\{\mathcal{A}(Y) \right. \\
&\quad \left. - \mathcal{A}(Y^{S_{N_n+1} \dots S_{N(n+1)}}) | \mathcal{F}_{N_n}\} \right).
\end{aligned}$$

Letting $m \in \mathbf{N}$ tend to $+\infty$, the monotone convergence theorem ensures

$$\mathcal{A}(X) - \mathcal{A}(X^*) \geq \mathbb{E} \left(\sum_{n \in \mathbb{N}} \chi_{\Theta_n^\varepsilon} \inf_{Y \in X_\varepsilon^\mathcal{S}} \mathbb{E} \{ \mathcal{A}(Y) - \mathcal{A}(Y^{S_{N_n+1} \dots S_{N_{(n+1)}}}) | \mathcal{F}_{N_n} \} \right).$$

Therefore, we have almost-surely

$$\sum_{n \in \mathbb{N}} \chi_{\Theta_n^\varepsilon} \inf_{Y \in X_\varepsilon^\mathcal{S}} \mathbb{E} \{ \mathcal{A}(Y) - \mathcal{A}(Y^{S_{N_n+1} \dots S_{N_{(n+1)}}}) | \mathcal{F}_{N_n} \} < +\infty.$$

According to identity (2.2.1) and the monotonicity of $\mathcal{A}$ along symmetrizations, the sequence $(\chi_{\Theta_n^\varepsilon})_{n \in \mathbb{N}}$ reaches 0 almost-surely after finitely many steps, so that we have almost-surely

$$\lim_{n \rightarrow \infty} \mathcal{A}(X^{S_1 \dots S_n}) \leq \mathcal{A}(X^*) + \varepsilon.$$

Now, considering only rational $\varepsilon > 0$, we deduce that the sequence $(\mathcal{A}(X^{S_1 \dots S_n}))_{n \in \mathbb{N}}$ converges almost-surely to $\mathcal{A}(X^*)$. By proposition 2.2.5, the sequence $(X^{S_1 \dots S_n})_{n \in \mathbb{N}}$ converges almost-surely to X^* . $\square$

2.2.3 Convergence of independent processes

In the context of processes made up from independent and identically distributed symmetrizations, we obtain an abstract result from which theorem 2.1.1 directly follows.

Theorem 2.2.8. *Let $\mathcal{S}$ be a $\star$ -compatible symmetrization space. Let $(S_n)_{n \in \mathbb{N}}$ be a sequence of independent and μ -identically distributed variables. If, for every $X \in \mathcal{X}$, the set*

$$X^\mathcal{S} = \{X^{s_1 \dots s_n} : n \in \mathbb{N}, s_1, \dots, s_n \in \mathcal{S}\}$$

has compact closure in $(\mathcal{X}, d)$, then the following conditions are equivalent:

- (i) *for every $X \in \mathcal{X}$, the sequence $(X^{S_1 \dots S_n})_{n \in \mathbb{N}}$ converges almost-surely to X^* ,*
- (ii) *for every $X \in \mathcal{X}$ with $X \neq X^*$, we have $\mu(\{s \in \mathcal{S} : X \neq X^s\}) > 0$.*

Proof. Let us first assume that, for every $X \in \mathcal{X}$ with $X \neq X^\star$, we have

$$\mu(\{s \in \mathcal{S} : X \neq X^s\}) > 0.$$

We apply proposition 2.2.7 with the $\star$ -compatible asymmetry

$$\mathcal{A} : \mathcal{X} \rightarrow \mathbf{R} : \mathcal{A}(X) = d(X, X^\star).$$

We consider the increasing sequence of deterministic stopping times $(N_n)_{n \in \mathbf{N}} = (n)_{n \in \mathbf{N}}$. Let be $X \in \mathcal{X}$ with $d(X, X^\star) \geq \varepsilon > 0$ (otherwise there is nothing to prove) and set

$$\overline{X_\varepsilon^\mathcal{S}} = \{Y \in \overline{X^\mathcal{S}} : d(Y, X^\star) \geq \varepsilon\}.$$

Proposition 2.2.7 shows that it is sufficient to prove

$$\inf_{Y \in \overline{X_\varepsilon^\mathcal{S}}} \int_{\mathcal{S}} d(Y, Y^\star) - d(Y^s, Y^\star) \mu(ds) > 0.$$

By continuity of $\mathcal{A}$ and compactness of $\overline{X^\mathcal{S}}$, it suffices to prove that for all $Y \in \overline{X_\varepsilon^\mathcal{S}}$, we have

$$\int_{\mathcal{S}} d(Y, Y^\star) - d(Y^s, Y^\star) \mu(ds) > 0.$$

Let us fix $Y \in \overline{X_\varepsilon^\mathcal{S}}$. By assumption, there exists some $\delta > 0$ such that

$$\mu(\{s \in \mathcal{S} : d(Y, Y^\star) \geq \delta\}) > 0.$$

Therefore, we have

$$\int_{\mathcal{S}} d(Y, Y^\star) - d(Y^s, Y^\star) \mu(ds) \geq \delta \mu(\{s \in \mathcal{S} : d(Y, Y^\star) \geq \delta\}) > 0.$$

This concludes the first part of the alternative.

For the converse, assume that for every $X \in \mathcal{X}$, the sequence $(X^{S_1 \dots S_n})_{n \in \mathbf{N}}$ converges almost-surely to $X^\star$. Assume by contradiction

that there exists $X \in \mathcal{X}$, $X \neq X^*$, such that

$$\mu(\{s \in \mathcal{S} : X \neq X^s\}) = 0$$

or, equivalently,

$$\mathbb{P}(\{\omega \in \Omega : X = X^{S_1(\omega)}\}) = 1.$$

Then the sequence $(X^{S_1 \dots S_n})_{n \in \mathbf{N}}$ is almost-surely constant and equals X , hence $X = X^*$ by assumption on $(S_n)_{n \in \mathbf{N}}$. $\square$

2.2.4 Markov processes

In order to deal with Markov processes, we recall some classical terminology about transition functions. In a metrizable topological space $\mathcal{S}$ with countable basis, a transition function on $\mathcal{S}$ is a function

$$P : \mathcal{S} \times \mathcal{B}(\mathcal{S}) \rightarrow [0, 1] : (s, A) \mapsto P_s(A)$$

such that

- (a) for every $s \in \mathcal{S}$, the function

$$P_s : \mathcal{B}(\mathcal{S}) \rightarrow [0, 1] : A \mapsto P_s(A)$$

is a probability measure,

- (b) for every Borel measurable and bounded function $f : \mathcal{S} \rightarrow \mathbf{R}$, the function

$$Pf : \mathcal{S} \rightarrow [0, 1] : s \mapsto P_s f = \int_{\mathcal{S}} f(y) P_s(dy)$$

is (bounded and) Borel measurable.

For every $n \in \mathbf{N}$ and every rectangle $A_1 \times \dots \times A_n \in \mathcal{B}(\mathcal{S})^n$, the iterated kernel P^n is

$$P^n(A_1 \times \dots \times A_n) : \mathcal{S} \rightarrow \mathbf{R} :$$

$$s \mapsto \int_{\mathcal{S}} \dots \int_{\mathcal{S}} \left(\prod_{i=1}^n \chi_{A_i}(s_i) \right) P_{s_{n-1}}(ds_n) \dots P_{s_1}(ds_2) P_s(ds_1) \quad (2.2.2)$$

We recall [13, Chapter 3] that the stochastic process $(S_n)_{n \in \mathbf{N}}$ is a time-homogeneous Markov process on $\mathcal{S}$ if there exists a transition function P on $\mathcal{S}$ such that for all $n, k \in \mathbf{N}$ with $k > 1$, for every Borel set $A_1 \times \dots \times A_k \subseteq \mathcal{S}^k$, we have (almost-surely)

$$\mathbb{E}\{\chi_{A_1 \times \dots \times A_k}(S_{n+1}, \dots, S_{n+k}) | \mathcal{F}_n\} = P_{S_n}^n(A_1 \times \dots \times A_k). \quad (2.2.3)$$

Since we deal with discrete time processes, this equality extends to stopping times. Furthermore, identities (2.2.2) and (2.2.3) directly extend to bounded and continuous functions $f : \mathcal{S}^n \rightarrow \mathbf{R}$, following the same approximation scheme of lemma 2.2.6.

Proposition 2.2.9. *Let $\mathcal{S}$ be a $\star$ -compatible symmetrization space such that for every $X \in \mathcal{X}$, the set*

$$X^{\mathcal{S}} = \{X^{s_1 \dots s_n} : n \in \mathbf{N}, s_1, \dots, s_n \in \mathcal{S}\}$$

has compact closure in $(\mathcal{X}, d)$. Let $\mathcal{A}$ be a $\star$ -compatible asymmetry function on $\mathcal{S}$. Assume that there exists $\mathcal{I}^{(\mathcal{S})} \in \mathcal{B}(\mathcal{S})$ such that $\mathcal{A}$ is strict on $\mathcal{S} \setminus \mathcal{I}^{(\mathcal{S})}$. If there exists $\mathfrak{s}_\star \in \mathcal{S}$ such that

(i) *(Recurrence) for every nonempty open set $\mathcal{O} \subseteq \mathcal{S}$, we have*

$$\mathbb{P}((S_n)_{n \in \mathbf{N}} \text{ enters } \mathcal{O} \text{ infinitely many often}) = 1,$$

(ii) *(Continuity) for every $n \in \mathbf{N}$, for every bounded and continuous function $f : \mathcal{S}^n \rightarrow \mathbf{R}$, the function $P^n f$ is continuous at $\mathfrak{s}_\star$,*

(iii) *(Discrimination) for every $X \in \mathcal{X}$ with $X \neq X^\star$, we have*

$$\sum_{n \in \mathbf{N}} P_{\mathfrak{s}_\star}^n((\mathcal{S} \setminus \mathcal{I}^{(\mathcal{S})})^{n-1} \times \{s \in \mathcal{S} : X \neq X^s\}) > 0,$$

then for every $X \in \mathcal{X}$, the sequence $(X^{s_1 \dots s_n})_{n \in \mathbf{N}}$ converges almost-surely to $X^\star$.

Proof. We apply proposition 2.2.7. Let us consider a sequence $(\mathcal{O}_n)_{n \in \mathbf{N}}$

of nonempty open sets in $\mathcal{S}$ decreasing to the limit point $\mathfrak{s}_\star$, that is

$$\bigcap_{n \in \mathbf{N}} \mathcal{O}_n = \{\mathfrak{s}_\star\}.$$

We define the stopping time $N_1 = \min\{k \in \mathbf{N} : S_k \in \mathcal{O}_1\}$, and for every $n \in \mathbf{N}$,

$$N_{n+1} = \min\{k \in \mathbf{N} : k \geq N_n + n, S_k \in \mathcal{O}_{n+1}\}.$$

By recurrence assumption (i), the sequence $(N_n)_{n \in \mathbf{N}}$ is a sequence of stopping times which is almost-surely finite and satisfies almost-surely $S_{N_n} \in \mathcal{O}_n$ and $N_{n+1} - N_n \geq n$. Fix $\omega \in \Omega$ such that the previous relations hold, and write $(s_n)_{n \in \mathbf{N}} = (S_{N_n}(\omega))_{n \in \mathbf{N}}$ and $\ell_n = N_{n+1}(\omega) - N_n(\omega)$ for every $n \in \mathbf{N}$. The sequence $(s_n)_{n \in \mathbf{N}}$ converges to $\mathfrak{s}_\star$. Fix $X \in \mathcal{X}$ with $\mathcal{A}(X) \geq \mathcal{A}(X^\star) + \varepsilon$ (otherwise there is nothing to prove). By proposition 2.2.7, we only need to show

$$\sum_{n \in \mathbf{N}} \inf \left\{ \int_{\mathcal{S}^{\ell_n}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1 \dots u_{\ell_n}}) P_{s_n}^{\ell_n}(du_1, \dots, du_{\ell_n}) : \right. \\ \left. Y \in \overline{X^\mathcal{S}}, \mathcal{A}(Y) \geq \mathcal{A}(X^\star) + \varepsilon \right\} = +\infty, \quad (2.2.4)$$

By compactness of $\overline{X_\varepsilon^\mathcal{S}}$, for every $n \in \mathbf{N}$, there exists $Y_n \in \overline{X_\varepsilon^\mathcal{S}}$ such that

$$\inf_{Y \in \overline{X_\varepsilon^\mathcal{S}}} \int_{\mathcal{S}^{\ell_n}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1 \dots u_{\ell_n}}) P_{s_n}^{\ell_n}(du_1, \dots, du_{\ell_n}) \\ = \int_{\mathcal{S}^{\ell_n}} \mathcal{A}(Y_n) - \mathcal{A}(Y_n^{u_1 \dots u_{\ell_n}}) P_{s_n}^{\ell_n}(du_1, \dots, du_{\ell_n})$$

Since $\overline{X_\varepsilon^\mathcal{S}}$ is compact, there exists a subsequence $(Y_{n_k})_{k \in \mathbf{N}}$ that converges to some $Y \in \overline{X_\varepsilon^\mathcal{S}}$. Without loss of generality, we can assume

$$\int_{\mathcal{S}^{\ell_{n_k}}} \mathcal{A}(Y_{n_k}) - \mathcal{A}(Y_{n_k}^{u_1 \dots u_{\ell_{n_k}}}) P_{s_{n_k}}^{\ell_{n_k}}(du_1, \dots, du_{\ell_{n_k}}) \\ \geq \int_{\mathcal{S}^{\ell_{n_k}}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1 \dots u_{\ell_{n_k}}}) P_{s_{n_k}}^{\ell_{n_k}}(du_1, \dots, du_{\ell_{n_k}}) - \frac{1}{2k}.$$

It is now sufficient to check that

$$\sum_{k \in \mathbf{N}} \int_{\mathcal{J}^{\ell_{n_k}}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1 \dots u_{\ell_{n_k}}}) P_{s_{n_k}}^{\ell_{n_k}}(du_1, \dots, du_{\ell_{n_k}}) = +\infty.$$

By the continuity assumption (ii), for all $k \in \mathbf{N}$, there exists a smaller integer $j_k \in \mathbf{N}$ such that for all $j \geq j_k$,

$$\left| \int_{\mathcal{J}^{\ell_{n_k}}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1 \dots u_{\ell_{n_k}}}) P_{s_j}^{\ell_{n_k}}(du_1, \dots, du_{\ell_{n_k}}) - \int_{\mathcal{J}^{\ell_{n_k}}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1 \dots u_{\ell_{n_k}}}) P_{s_*}^{\ell_{n_k}}(du_1, \dots, du_{\ell_{n_k}}) \right| \leq \frac{1}{2^k}.$$

Define $m_1 = \min\{n_k : n_k \geq j_1\}$, and by recurrence

$$m_{k+1} = \min\{n_k : n_k \geq \max\{m_k + 1, j_{k+1}, n_{(k+1)}\}\}.$$

By construction, $(m_k)_{k \in \mathbf{N}}$ is a subsequence of $(n_k)_{n \in \mathbf{N}}$ such that $m_k \geq j_k$ for all $k \in \mathbf{N}$. Since the asymmetry decreases along symmetrizations, we have for every $k \in \mathbf{N}$

$$\begin{aligned} & \int_{\mathcal{J}^{\ell_{m_k}}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1 \dots u_{\ell_{m_k}}}) P_{s_{m_k}}^{\ell_{m_k}}(du_1, \dots, du_{\ell_{m_k}}) \\ & \geq \int_{\mathcal{J}^{\ell_{n_k}}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1 \dots u_{\ell_{n_k}}}) P_{s_{m_k}}^{\ell_{n_k}}(du_1, \dots, du_{\ell_{n_k}}) \\ & \geq \int_{\mathcal{J}^{\ell_{n_k}}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1 \dots u_{\ell_{n_k}}}) P_{s_*}^{\ell_{n_k}}(du_1, \dots, du_{\ell_{n_k}}) - \frac{1}{2^k} \\ & \geq \int_{\mathcal{J}^k} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1 \dots u_k}) P_{s_*}^k(du_1, \dots, du_k) - \frac{1}{2^k}. \end{aligned}$$

In the previous line, we have used the fact that $\ell_{n_k} \geq n_k \geq k$ by construction. If we prove the strict inequality

$$\sup_{n \in \mathbf{N}} \int_{\mathcal{J}^n} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1 \dots u_n}) P_{s_*}^n(du_1, \dots, du_n) > 0, \quad (2.2.5)$$

then by comparison of series, condition (2.2.4) would then hold. Let us thus prove (2.2.5), where $Y \in \mathcal{X}$ satisfies $Y \neq Y^\star$. By the discrimination assumption (iii), there exists $n \in \mathbf{N}$ such that

$$P_{\mathfrak{s}_\star}^n((\mathcal{S} \setminus \mathcal{I}^{(\mathcal{S})})^{n-1} \times \{s \in \mathcal{S} \setminus \mathcal{I}^{(\mathcal{S})} : Y \neq Y^s\}) > 0.$$

Since $\mathcal{A}$ is strict on $\mathcal{S} \setminus \mathcal{I}^{(\mathcal{S})}$, there exists $\delta > 0$ such that $P_{\mathfrak{s}_\star}^n(H) > 0$, with

$$H = (\mathcal{S} \setminus \mathcal{I}^{(\mathcal{S})})^{n-1} \times \{s \in \mathcal{S} : \mathcal{A}(Y) \geq \mathcal{A}(Y^s) + \delta\}.$$

Let us assume by contradiction that we have

$$\int_{\mathcal{S}^n} \mathcal{A}(Y) - \mathcal{A}(Y^{s_1 \dots s_n}) P_{\mathfrak{s}_\star}^n(ds_1, \dots, ds_n) = 0.$$

There exists a set $E \in \mathcal{B}(\mathcal{S}^n)$ of $P_{\mathfrak{s}_\star}^n$ -measure 1 such that, for every $(s_1, \dots, s_n) \in E$, we have $\mathcal{A}(Y) = \mathcal{A}(Y^{s_1 \dots s_n})$. For $(s_1, \dots, s_n) \in E$, with $s_1 \notin \mathcal{I}^{(\mathcal{S})}$, we have $\mathcal{A}(Y) = \mathcal{A}(Y^{s_1})$ and, since $\mathcal{A}$ is a strict asymmetry function on $\mathcal{S} \setminus \mathcal{I}^{(\mathcal{S})}$, we have $Y = Y^{s_1}$, so that we also have $\mathcal{A}(Y) = \mathcal{A}(Y^{s_2})$. By recurrence, we have $Y = Y^{s_i}$, $i = 1, \dots, n-1$, for every $(s_1, \dots, s_n) \in E$ with $s_i \notin \mathcal{I}^{(\mathcal{S})}$ for each $i \in \{1, \dots, n-1\}$. We now have

$$\begin{aligned} 0 &= \int_{\mathcal{S}^n} \mathcal{A}(Y) - \mathcal{A}(Y^{s_1 \dots s_n}) P_{\mathfrak{s}_\star}^n(ds_1, \dots, ds_n) \\ &\geq \int_{H \cap E} \mathcal{A}(Y) - \mathcal{A}(Y^{s_n}) P_{\mathfrak{s}_\star}^n(ds_1, \dots, ds_n) \geq \delta P_{\mathfrak{s}_\star}^n(H) > 0, \end{aligned}$$

which is the desired contradiction. This proves

$$\sup_{n \in \mathbf{N}} \int_{\mathcal{S}^n} \mathcal{A}(Y) - \mathcal{A}(Y^{s_1 \dots s_n}) P_{\mathfrak{s}_\star}^n(ds_1, \dots, ds_n) > 0,$$

and (2.2.5) holds. $\square$

In practice, the proof of the existence of a recurrent point $\mathfrak{s}_\star$ requires some additional work. We recall a notion of stability that is useful for locally compact symmetrization spaces.

Definition 2.2.10 ([13, Section 9.2]). Let $\mathcal{S}$ be a locally compact

metrizable topological space with countable basis, and $(S_n)_{n \in \mathbf{N}}$ a Markov process on $\mathcal{S}$. The process $(S_n)_{n \in \mathbf{N}}$ is said to be *nonevanescant* if

$$\mathbb{P}(\exists K \subseteq \mathcal{S} : K \text{ compact, } (S_n)_{n \in \mathbf{N}} \text{ enters } K \text{ infinitely many often}) = 1.$$

The topological assumptions on $\mathcal{S}$ ensure that the above definition makes sense.

Proposition 2.2.11 ([13, Theorem 9.1.3]). *Let $\mathcal{S}$ be a metrizable topological space with countable basis, and $(S_n)_{n \in \mathbf{N}}$ be a Markov process on $\mathcal{S}$. For every $A \in \mathcal{B}(\mathcal{S})$, the sequence $(\inf_{k \in \mathbf{N}} P_{S_n}^k((\mathcal{S} \setminus A)^k))_{n \in \mathbf{N}}$ converges almost-surely to the indicator function of the set*

$$\bigcup_{n \in \mathbf{N}} \bigcap_{k \geq n} \{S_{k+1} \notin A\}.$$

Proof. We fix a Borel measurable set $A \in \mathcal{B}(\mathcal{S})$, and we consider the function

$$H : \mathcal{S} \mapsto [0, 1] : H(s) = 1 - \inf_{k \in \mathbf{N}} P_s^k((\mathcal{S} \setminus A)^k).$$

We write, for every $n \in \mathbf{N}$,

$$\Omega_n = \bigcup_{i \geq n} \{S_{i+1} \in A\} \in \mathcal{F},$$

and

$$\Omega_\infty = \bigcap_{m \in \mathbf{N}} \bigcup_{i \geq m} \{S_{i+1} \in A\}.$$

For every $n \in \mathbf{N}$, the Markov property ensures that almost-surely $H(S_n) = \mathbb{E}\{\chi_{\Omega_\infty} | \mathcal{F}_n\}$. Moreover, we also have for each $m \leq n$

$$\mathbb{E}\{\chi_{\Omega_\infty} | \mathcal{F}_n\} \leq H(S_n) \leq \mathbb{E}\{\chi_{\Omega_m} | \mathcal{F}_n\}.$$

For a fixed $m \in \mathbf{N}$, the martingale convergence theorem ensures that the left side converges almost-surely to $\mathbb{E}\{\chi_{\Omega_\infty} | \mathcal{F}\} = \chi_{\Omega_\infty}$, and that the right side converges almost-surely to $\mathbb{E}\{\chi_{\Omega_m} | \mathcal{F}\} = \chi_{\Omega_m}$ as $n \rightarrow +\infty$.

Therefore, we have

$$\chi_{\Omega_\infty} \leq \liminf_{n \rightarrow +\infty} H(S_n) \leq \limsup_{n \rightarrow +\infty} H(S_n) \leq \chi_{\Omega_m},$$

almost-surely for every $m \in \mathbf{N}$. Letting $m \in \mathbf{N}$ tends to $+\infty$, we have almost-surely

$$\lim_{n \rightarrow +\infty} \inf_{k \in \mathbf{N}} P_{S_n}^k((\mathcal{S} \setminus A)^k) = 1 - \lim_{n \rightarrow +\infty} H(S_n) = 1 - \chi_{\Omega_\infty}. \quad \square$$

Corollary 2.2.12. *Let $\mathcal{S}$ be a $\star$ -compatible and locally compact symmetrization space such that for every $X \in \mathcal{X}$, the set*

$$X^{\mathcal{S}} = \{X^{s_1 \dots s_n} : n \in \mathbf{N}, s_1, \dots, s_n \in \mathcal{S}\}$$

has compact closure in $(\mathcal{X}, d)$. Let $\mathcal{A}$ be a $\star$ -compatible asymmetry function on $\mathcal{S}$. Assume that there exists $\mathcal{I}^{(\mathcal{S})} \subset \mathcal{S}$ such that $\mathcal{A}$ is strict on $\mathcal{S} \setminus \mathcal{I}^{(\mathcal{S})}$. Let $P : \mathcal{S} \times \mathcal{B}(\mathcal{S}) \rightarrow [0, 1]$ be a transition function, and $(S_n)_{n \in \mathbf{N}}$ be a time-homogeneous Markov process with transition function P . If

- (i) (Stability) *the process $(S_n)_{n \in \mathbf{N}}$ is nonevanescant and $\mathcal{I}^{(\mathcal{S})}$ is closed,*
- (ii) (Continuity) *for all $n \in \mathbf{N}$, for every bounded and continuous function $f : \mathcal{S}^n \rightarrow \mathbf{R}$, the function $P^n f$ is continuous,*
- (iii) (Discrimination) *for every $s \in \mathcal{S}$, for every nonempty open set $\mathcal{O} \subseteq \mathcal{S}$,*

$$\sum_{n \in \mathbf{N}} P_s^n((\mathcal{S} \setminus \mathcal{I}^{(\mathcal{S})})^{n-1} \times \mathcal{O}) > 0,$$

then for every $X \in \mathcal{X}$, the sequence $(X^{S_1 \dots S_n})_{n \in \mathbf{N}}$ converges almost-surely to $X^\star$.

Proof. Let us fix $\mathfrak{s}_\star \in \mathcal{S}$. According to proposition 2.2.9, it is sufficient to check that for every nonempty open set $\mathcal{O} \subseteq \mathcal{S}$ containing $\mathfrak{s}_\star$, we have

$$\mathbb{P}(\{(S_n)_{n \in \mathbf{N}} \text{ enters } \mathcal{O} \text{ infinitely many often}\}) = 1.$$

We thus fix a nonempty open set $\mathcal{O} \subseteq \mathcal{S}$ that contains $\mathfrak{s}_\star$, and we define

the function

$$H : \mathcal{S} \mapsto [0, 1] : H(s) = 1 - \inf_{k \in \mathbf{N}} P_s^k((\mathcal{S} \setminus \mathcal{O})^k).$$

By proposition 2.2.11, we have almost-surely $\lim_{n \rightarrow +\infty} H(S_n) = \chi_{\Omega_\infty}$, where

$$\Omega_\infty = \bigcap_{m \in \mathbf{N}} \bigcup_{i \geq m} \{S_{i+1} \in \mathcal{O}\} = \{(S_n)_{n \in \mathbf{N}} \text{ enters } \mathcal{O} \text{ infinitely many often}\}.$$

Assume by contradiction that there exists $s \in \mathcal{S}$ with $H(s) = 0$, that is: for every $k \in \mathbf{N}$, we have $P_s^k((\mathcal{S} \setminus \mathcal{O})^k) = 1$. Since there exists $n \in \mathbf{N}$ such that $P_s^n((\mathcal{S} \setminus \mathcal{I}^{(\mathcal{S})})^{n-1} \times \mathcal{O}) > 0$, it follows that

$$0 < P_s^n((\mathcal{S} \setminus \mathcal{I}^{(\mathcal{S})})^{n-1} \times \mathcal{O}) \cap (\mathcal{S} \setminus \mathcal{O})^n = P_s^n(\emptyset) = 0,$$

which is a contradiction. This proves that the function H is strictly positive. Using Urysohn's lemma, the function H is also lower semi-continuous as supremum of semi-continuous functions. Therefore, for all compact set $K \subseteq \mathcal{S}$, the function H attains a strictly positive minimal value on K , which has to equal 1. Hence, we have proven the essential inclusion

$$\{(S_n)_{n \in \mathbf{N}} \text{ enters } K \text{ infinitely many often}\} \subseteq \Omega_\infty.$$

Since there exists a countable basis of nonempty open sets with compact closure for $\mathcal{S}$, the nonevanescence of $(S_n)_{n \in \mathbf{N}}$ ensures

$$\mathbb{P}(\{(S_n)_{n \in \mathbf{N}} \text{ enters } \mathcal{O} \text{ infinitely many often}\}) = 1. \quad \square$$

In view of examples, we prove that a strong Feller transition function P always satisfy our continuity assumption for $\{P^n : n \in \mathbf{N}\}$.

Definition 2.2.13. Let $\mathcal{S}$ be a topological space. A transition function P on $\mathcal{S}$ is *strong Feller continuous* at $\mathfrak{s}_\star$ if, for every bounded and Borel measurable function $f : \mathcal{S} \rightarrow \mathbf{R}$, the map $[s \in \mathcal{S} \mapsto P_s f]$ is continuous at $\mathfrak{s}_\star$.

Proposition 2.2.14 ([13, Proposition 6.1.1]). *Let $\mathcal{S}$ be a metrizable*

topological space with countable basis, $\mathfrak{s}_\star \in \mathcal{S}$ and P a transition function on $\mathcal{S}$. If P is strong Feller continuous at $\mathfrak{s}_\star$, then for every $n \in \mathbf{N}$, for every bounded and continuous function $f : \mathcal{S}^n \rightarrow \mathbf{N}$, the function $P^n f$ is continuous at $\mathfrak{s}_\star$.

Proof. If $f : \mathcal{S}^n \rightarrow \mathbf{R}$ is bounded and continuous, the function

$$g : \mathcal{S} \rightarrow \mathbf{R} : g(x) = P_x^n f(\cdot, x)$$

is bounded and Borel measurable. The measurability follows from the monotone class theorem. Since there holds

$$P_s^{n+1} f = \int_{\mathcal{S}} g(x) P_s(dx),$$

it suffices to prove that for every bounded and Borel measurable function $g : \mathcal{S} \rightarrow \mathbf{R}$, Pg is continuous at $\mathfrak{s}_\star$. Let $M > 0$ be a bound for g , and define $h_1 = g + M$. Then h_1 is Borel measurable, positive and bounded. There exists an increasing sequence $(\phi_n)_{n \in \mathbf{N}}$ of simple functions that converges to h_1 , and such that the level sets of ϕ_n are disjoint Borel measurable sets. By linearity, the functions $P\phi_n$ are continuous at $\mathfrak{s}_\star$, for every $n \in \mathbf{N}$. Since the sequence $(P\phi_n)_{n \in \mathbf{N}}$ increases to Ph_1 by the monotone convergence theorem, the function Ph_1 is lower semi-continuous, and so is Pg . The same conclusions hold for the function $h_2 = -g + M$, so that Ph_2 is lower semi-continuous, and thus Pg is also upper semi-continuous. $\square$

The converse of proposition 2.2.14 is false in general. Consider for example a continuous map $\phi : \mathcal{S} \rightarrow \mathcal{S}$ which is not trivial, and the transition function

$$P : \mathcal{S} \times \mathcal{B}(\mathcal{S}) \rightarrow [0, 1] : P_s(A) = \delta_{\phi(s)}(A).$$

The iterated kernels are given by the formula

$$P_s^n f = f(\phi(s), \dots, \phi^n(s)).$$

The functions $P^n f$ are continuous whenever f is itself continuous and

bounded, but the strong Feller continuity may fail in general.

2.3 Examples

In this section, we give examples for various symmetrizations as Steiner and cap symmetrizations and polarizations. We first recall standard definitions, and then we give examples of application of our abstract method.

2.3.1 Various symmetrizations

We write $\mathcal{H}^k$ the k -dimensional Hausdorff measure in $\mathbf{R}^d$. We work within the metric space $(\mathcal{M}_{\sharp}(\mathbf{R}^d), d_1)$ of equivalence classes of Borel measurable subsets of $\mathbf{R}^d$ with finite $\mathcal{H}^d$ measure, endowed with the $\mathcal{H}^d$ -metric. The space $\mathbf{R}^d$ is equipped with its usual metric.

Spherical nonincreasing rearrangement. The simplest symmetrization transform sets into balls.

Definition 2.3.1. Let $A \in \mathcal{B}(\mathbf{R}^d)$ be a Borel measurable set. The spherical nonincreasing rearrangement of A is the open ball $A^\star$ centered on the origin 0, that satisfies $\mathcal{H}^d(A^\star) = \mathcal{H}^d(A)$.

The induced map on the quotient space $\mathcal{M}_{\sharp}(\mathbf{R}^d)$ is denoted by $\star$. The fact that $\star$ is an involution is direct. For the nonexpansiveness, one can simply observe that since $\star$ is measure-preserving and monotone, we directly have for all $A, B \in \mathcal{B}(\mathbf{R}^d)$

$$\begin{aligned} \mathcal{H}^d(A^\star \Delta B^\star) &= \mathcal{H}^d(A^\star \setminus B^\star) + \mathcal{H}^d(B^\star \setminus A^\star) \\ &\leq \mathcal{H}^d(A^\star \setminus (B \cap A)^\star) + \mathcal{H}^d(B^\star \setminus (A \cap B)^\star) \\ &= \mathcal{H}^d(A^\star) + \mathcal{H}^d(B^\star) - 2\mathcal{H}^d((B \cap A)^\star) \\ &= \mathcal{H}^d(A) + \mathcal{H}^d(B) - 2\mathcal{H}^d(B \cap A) = \mathcal{H}^d(A \Delta B). \end{aligned}$$

Steiner symmetrizations. Let be $u \in \mathbf{P}_{\mathbf{R}}^{d-1}$ and $\langle u \rangle$ be its linear span. We write $u^\perp$ the orthogonal complement subspace in $\mathbf{R}^d$. For every

$x \in u^\perp$, we write the section

$$A|_x = A \cap (x + \langle u \rangle).$$

Definition 2.3.2. Given $u \in \mathbf{P}_{\mathbf{R}}^{d-1}$ and a Borel measurable set $A \in \mathcal{B}(\mathbf{R}^d)$, the Steiner symmetrization of A with respect to u is the unique $A^u \in \mathcal{B}(\mathbf{R}^d)$ such that

- for every $x \in u^\perp$, $A^u|_x$ is an open ball of $(x + \langle u \rangle)$,
- for every $x \in u^\perp$, $A^u|_x$ is centered on 0,
- for every $x \in u^\perp$, $\mathcal{H}^1(A^u|_x) = \mathcal{H}^1(A|_x)$.

The induced maps on the quotient $\mathcal{M}_\#(\mathbf{R}^d)$ are the usual Steiner symmetrizations. The set $\mathcal{S}$ of Steiner symmetrizations being in one-to-one correspondence with $\mathbf{P}_{\mathbf{R}}^{d-1}$, we equip it with the induced topology of $\mathbf{P}_{\mathbf{R}}^{d-1}$, so that $\mathcal{S}$ is a metrizable and compact topological space with countable basis.

Polarizations. Fix $\hat{e} \in S^{d-1}$ and $r \geq 0$. The corresponding affine half-subspace is defined by

$$H^{\hat{e},r} = \{x \in \mathbf{R}^d : x \cdot \hat{e} \leq r\}.$$

There is a unique nontrivial reflection $\sigma^{\hat{e},r}$ of $\mathbf{R}^d$ that leaves the boundary of $H^{\hat{e},r}$ invariant.

Definition 2.3.3. Given $\hat{e} \in S^{d-1}$, $r \geq 0$ and a Borel measurable set $A \in \mathcal{B}(\mathbf{R}^d)$, the polarization of A is defined as the unique $A^{\hat{e},r} \in \mathcal{B}(\mathbf{R}^d)$ satisfying the following axioms:

- if $x \in H^{\hat{e},r}$, then $x \in A^{\hat{e},r}$ if, and only if, $x \in H^{\hat{e},r} \cup \sigma^{\hat{e},r}(H^{\hat{e},r})$,
- if $x \notin H^{\hat{e},r}$, then $x \in A^{\hat{e},r}$ if, and only if, $x \in H^{\hat{e},r} \cap \sigma^{\hat{e},r}(H^{\hat{e},r})$.

The induced maps on the quotient $\mathcal{M}_\#(\mathbf{R}^d)$ are the usual polarizations. The set $\mathcal{H}$ of polarizations is in one-to-one correspondence with $S^{d-1} \times [0, +\infty)$. We equip it with the induced topology of $S^{d-1} \times [0, +\infty)$,

so that $\mathcal{H}$ is a metrizable and locally compact topological space with countable basis. Through this identification, we define the closed subset

$$\mathcal{I}^{(\mathcal{H})} = S^{d-1} \times \{0\}.$$

It corresponds to affine half-subspaces that contains 0 in their usual boundary.

Cap symmetrizations. Fix $\hat{e} \in S^{d-1}$ and $r \geq 0$. For $t \geq 0$, define the spherical section of a set $A \subseteq \mathbf{R}^d$ as

$$A|_t = A \cap \partial B(r\hat{e}, t).$$

Definition 2.3.4. Given $r \geq 0$ and a Borel measurable set $A \in \mathcal{B}(\mathbf{R}^d)$, the cap symmetrization of A with respect to $(\hat{e}, r)$ is the unique $A^{\hat{e}, r} \in \mathcal{B}(\mathbf{R}^d)$ that satisfies the following axiom:

- for every $t \geq 0$, $A^{\hat{e}, r}|_t$ is an open ball of $\partial B(r\hat{e}, t)$,
- for every $x \in u^\perp$, $A^{\hat{e}, r}|_t$ is centered on $(r - t)\hat{e}$
- for every $t \geq 0$, $\mathcal{H}^{d-1}(A^{\hat{e}, r}|_t) = \mathcal{H}^{d-1}(A|_t)$.

The induced maps on the quotient $\mathcal{M}_\#(\mathbf{R}^d)$ form the usual set of cap symmetrizations. The set $\mathcal{L}$ of cap symmetrizations being in one-to-one correspondence with $S^{d-1} \times [0, +\infty)$, we equip it with the induced topology of $S^{d-1} \times [0, +\infty)$, so that $\mathcal{L}$ is a metrizable and locally compact topological space with countable basis. We define the closed subset

$$\mathcal{I}^{(\mathcal{L})} = S^{d-1} \times \{0\}.$$

It corresponds to cap symmetrizations with respect to half-lines whose initial points are the origin 0.

Common properties of the examples of symmetrizations. Steiner symmetrizations, cap symmetrizations and polarizations enjoy important common properties, which we recall in the next proposition. The following result being classical in the field of symmetrizations, we omit the proof.

Proposition 2.3.5. *The sets $\mathcal{S}, \mathcal{H}$ and $\mathcal{L}$ acting on $(\mathcal{M}_\#(\mathbf{R}^d), d_1)$ are $\star$ -compatible symmetrization spaces and for every $X \in \mathcal{M}_\#(\mathbf{R}^d)$, the sets*

$$\begin{aligned} X^\mathcal{S} &= \{X^{s_1 \dots s_n} : n \in \mathbf{N}, s_1, \dots, s_n \in \mathcal{S}\}, \\ X^\mathcal{H} &= \{X^{s_1 \dots s_n} : n \in \mathbf{N}, s_1, \dots, s_n \in \mathcal{H}\}, \\ X^\mathcal{L} &= \{X^{s_1 \dots s_n} : n \in \mathbf{N}, s_1, \dots, s_n \in \mathcal{L}\} \end{aligned}$$

have compact closure in $(\mathcal{M}_\#(\mathbf{R}^d), d_1)$. The function

$$\mathcal{A} : \mathcal{M}_\#(\mathbf{R}^d) \rightarrow \mathbf{R}^+ : \mathcal{A}(X) = \int_X \frac{|x|^2}{1 + |x|^2} dx$$

is a $\star$ -compatible asymmetry function on $\mathcal{S}$, on $\mathcal{H}$ and on $\mathcal{L}$. Moreover, $\mathcal{A}$ is a strict asymmetry function on $\mathcal{S}$, on $\mathcal{H} \setminus \mathcal{I}^{(\mathcal{H})}$ and on $\mathcal{L} \setminus \mathcal{I}^{(\mathcal{L})}$.

The previous proposition is straightforward for polarizations [6; 7, Polarization identity]. The compactness property follows from the Kolmogorov-Riesz compactness theorem. Once one has the result for polarizations, one can extend it to Steiner and cap symmetrizations by an approximation argument [22].

2.3.2 Markov Steiner symmetrizations

In view of the discussion above, we can prove directly theorem 2.1.2.

Proof of theorem 2.1.2. The function

$$\mathcal{A} : \mathcal{M}_\#(\mathbf{R}^d) \rightarrow \mathbf{R}^+ : \mathcal{A}(X) = \int_X \frac{|x|^2}{1 + |x|^2} dx$$

is a strict asymmetry function on $\mathcal{S}$, according to proposition 2.3.5. The fact that the continuity condition (ii) in theorem 2.1.2 is equivalent to the continuity condition (ii) of proposition 2.2.9 is a consequence of Urysohn's lemma. We can apply proposition 2.2.9 to get the desired result. $\square$

Example 2.3.6 (Random walk in $\mathbf{P}_\mathbf{R}^{d-1}$). We fix $r > 0$ and we write σ for the Haar measure on $\mathbf{P}_\mathbf{R}^{d-1}$. The map $[e \in \mathbf{P}_\mathbf{R}^{d-1} \mapsto \sigma(B(e, r))]$ is

constant. We fix $e \in \mathbf{P}_{\mathbf{R}}^{d-1}$ and we define the transition function

$$P : \mathbf{P}_{\mathbf{R}}^{d-1} \times \mathcal{B}(\mathbf{P}_{\mathbf{R}}^{d-1}) \rightarrow [0, 1] : P_s(A) = \int_{B(s,r)} \chi_A(x) \frac{\sigma(dx)}{\sigma(B(e,r))}.$$

The transition function P is strong Feller continuous everywhere on $\mathbf{P}_{\mathbf{R}}^{d-1}$. According to proposition 2.2.14, the family $\{P^n : n \in \mathbf{N}\}$ enjoys the usual continuity assumption at every point. We also have, for every nonempty open set $\mathcal{O} \subseteq \mathbf{P}_{\mathbf{R}}^{d-1}$ and for every $s \in \mathbf{P}_{\mathbf{R}}^{d-1}$,

$$\sum_{n \in \mathbf{N}} P_s^n((\mathbf{P}_{\mathbf{R}}^{d-1})^{n-1} \times \mathcal{O}) > 0.$$

This last inequality can be proven by noting that, for every $n \in \mathbf{N}$, for every $s \in \mathbf{P}_{\mathbf{R}}^{d-1}$, the probability measure

$$H_s^n : \mathcal{B}(\mathbf{P}_{\mathbf{R}}^{d-1}) \rightarrow [0, 1] : H_s^n(A) = P_s^n((\mathbf{P}_{\mathbf{R}}^{d-1})^{n-1} \times A),$$

has $\overline{B(s, nr)}$ as support. Since $\mathbf{P}_{\mathbf{R}}^{d-1}$ is compact, any Markov process on $\mathbf{P}_{\mathbf{R}}^{d-1}$ is nonevanescant. Corollary 2.2.12 ensures that any time-homogeneous Markov process $(S_n)_{n \in \mathbf{N}}$ with transition function P satisfies that for every $X \in \mathcal{M}_{\sharp}(\mathbf{R}^d)$, the sequence $(X^{S_1 \dots S_n})_{n \in \mathbf{N}}$ converges almost-surely to X^* .

Proposition 2.3.7 (Deterministic Steiner symmetrizations). *Let $\phi : \mathbf{P}_{\mathbf{R}}^{d-1} \rightarrow \mathbf{P}_{\mathbf{R}}^{d-1}$ be a continuous map and $s \in \mathbf{P}_{\mathbf{R}}^{d-1}$ such that*

$$\overline{\{\phi^n(s) : n \in \mathbf{N}\}} = \mathbf{P}_{\mathbf{R}}^{d-1}.$$

For every $X \in \mathcal{M}_{\sharp}(\mathbf{R}^d)$, the sequence $(X^{\phi(s) \dots \phi^n(s)})_{n \in \mathbf{N}}$ converges in measure to X^ .*

Proof. Let us define for all $n \in \mathbf{N}$, $S_n = \phi^n(s)$. The sequence $(S_n)_{n \in \mathbf{N}}$ is a time-homogeneous Markov process on $\mathbf{P}_{\mathbf{R}}^{d-1}$. The iterated kernels are given for every $n \in \mathbf{N}$ by the formula

$$P_s^n f = f(\phi(s), \dots, \phi^n(s)).$$

The recurrence condition (i) and the discrimination condition (iii) of

proposition 2.2.9 both follow from the assumption that the orbit of s under ϕ is dense in $\mathbf{P}_{\mathbf{R}}^{d-1}$. Since ϕ is continuous, the continuity condition (ii) is satisfied whenever f is continuous and bounded. We can thus apply proposition 2.2.9 with limit point $\mathfrak{s}_\star = s$. The proof is done. $\square$

The previous proposition can be thought of as a generalization of [2, Theorem 5.1], where the authors studied Kronecker sequence of deterministic Steiner symmetrizations of the form $((\cos(n\alpha), \sin(n\alpha)))_{n \in \mathbf{N}}$, with α/π irrational. Their analysis is based on the convergence in shape. Our result can be applied for other deterministic sequence, and has straightforward generalizations for cap symmetrizations and polarizations.

Counterexample 2.3.8. The continuity condition (ii) in theorem 2.1.2 is necessary. Let us consider a sequence $(\alpha_n)_{n \in \mathbf{N}}$ in $\mathbf{P}_{\mathbf{R}}^{d-1}$ such that $\alpha_i \neq \alpha_j$ for $i \neq j$, and such that the set $\{\alpha_n : n \in \mathbf{N}\}$ is dense in $\mathbf{P}_{\mathbf{R}}^{d-1}$. We define the transition function on $\mathbf{P}_{\mathbf{R}}^{d-1}$ through

$$P_s = \begin{cases} \delta_{\alpha_{n+1}} & \text{if } s = \alpha_n \text{ for some } n \in \mathbf{N}, \\ \delta_{\alpha_1} & \text{otherwise} \end{cases}.$$

If $(S_n)_{n \in \mathbf{N}}$ is a time-homogeneous Markov process associated with the transition function P , then almost-surely $S_n = \alpha_n$ for every $n \in \mathbf{N}$. Hence, it is a straightforward computation to check that the hypothesis of theorem 2.1.2 are satisfied with $\mathfrak{s}_\star = \alpha_1$, except that we miss the continuity property. Actually, in general, there exists $X \in \mathcal{M}_\#(\mathbf{R}^d)$ such that $(X^{S_1 \dots S_n})_{n \in \mathbf{N}}$ fails to convergence to $X^\star$ almost-surely [3, 24].

In this example, note that one could endow $\mathbf{P}_{\mathbf{R}}^{d-1}$ with the discrete topology. The continuity assumption is then trivial, but the recurrence condition forces the process to have a finite cycle. This is the situation of iterated Steiner symmetrizations using a finite number of directions [10].

2.3.3 Markov cap symmetrizations and polarizations on $(0, +\infty)$

Proof of theorem 2.1.3. Since we identify $\mathcal{L}$ and $\mathcal{H}$ with $S^{d-1} \times [0, +\infty)$, let us recall that $\mathcal{I}^{(\mathcal{L})} = S^{d-1} \times \{0\} = \mathcal{I}^{(\mathcal{H})}$. We consider the usual

asymmetry function

$$\mathcal{A} : \mathcal{M}_{\sharp}(\mathbf{R}^d) \rightarrow \mathbf{R}^+ : \mathcal{A}(X) = \int_X \frac{|x|^2}{1 + |x|^2} dx.$$

It is a $\star$ -compatible asymmetry function on $\mathcal{L}$ and $\mathcal{H}$ which is strict on the subsets $\mathcal{L} \setminus \mathcal{I}^{(\mathcal{L})}$ and $\mathcal{H} \setminus \mathcal{I}^{(\mathcal{H})}$ (proposition 2.3.5). In these settings, the result follows from proposition 2.2.9. $\square$

For polarizations, the following elementary result shows that the nonevanescence assumption in corollary 2.2.12 is actually required for the sequence to approximate the spherical nonincreasing rearrangement. This property fails in general for cap symmetrizations.

Proposition 2.3.9. *Let $(s_n)_{n \in \mathbf{N}}$ be a sequence of polarizations written for every $n \in \mathbf{N}$ as $s_n = (\hat{e}_n, r_n)$ with $e_n \in S^{d-1}$ and $r_n \in [0, +\infty)$. If $\liminf_{n \rightarrow +\infty} r_n > 0$, then there exists $X \in \mathcal{M}_{\sharp}(\mathbf{R}^d)$ of finite measure such that the sequence $(X^{s_1 \dots s_n})_{n \in \mathbf{N}}$ does not converge.*

Proof. The condition $\liminf_{n \rightarrow +\infty} r_n > 0$ shows that there exists $N \in \mathbf{N}$ such that, for every $n \geq N$, we have $r_n \geq \delta$, for some $\delta > 0$. We define

$$r = \min\{\delta, \min\{r_i : i \in \{1, \dots, N-1\}, r_i > 0\}\},$$

and $X \in \mathcal{M}_{\sharp}(\mathbf{R}^d)$ the (equivalent class of the) annulus

$$X = B(0, r) \setminus B(0, r/2).$$

Then clearly $X \neq X^*$. For $n \in \mathbf{N}$, two cases may occur. If $r_n \geq r$, then X is contained in the half-space $(\hat{e}_n, r_n)$, and thus $X = X^{(\hat{e}_n, r_n)}$. If $r_n < r$, then we must have by construction $r_n = 0$. Since X is radially symmetric, we have $X = X^{(\hat{e}_n, 0)}$, so that X remains fixed by all polarizations of the set $\{s_n : n \in \mathbf{N}\}$. $\square$

Example 2.3.10 (Brownian motion on $(0, +\infty)$). Let $(Z_n)_{n \in \mathbf{N}}$ be a sequence of independent and identically distributed variables on $(0, +\infty)$

with density function ρ , and define for every $n \in \mathbf{N}$

$$\begin{cases} W_1 &= Z_1, \\ W_{n+1} &= Z_{n+1} \cdot W_n. \end{cases}$$

Let $(U_n)_{n \in \mathbf{N}}$ be a sequence of independent and identically distributed variables on S^{d-1} with distribution μ . We assume that

$$\text{supp}(\mu) \times \text{supp}(\rho) = S^{d-1} \times (0, +\infty)$$

and

$$\int_{(0, +\infty)} s \rho(s) \, ds = 1.$$

The transition function of the time-homogeneous Markov process $(W_n)_{n \in \mathbf{N}}$ is given by

$$P : \mathcal{B}((0, +\infty)) \times (0, +\infty) \rightarrow [0, 1] : P_x(A) = \int_{(0, +\infty)} \chi_A(sx) \rho(s) \, ds.$$

For every continuous and bounded function $f : (0, +\infty) \rightarrow \mathbf{R}$, we have

$$P_x f = \int_{(0, +\infty)} f(xs) \rho(s) \, ds,$$

which is continuous with respect to the parameter $x \in (0, +\infty)$. By assumption on the support of ρ , every nonempty open set of $(0, +\infty)$ is reachable with positive probability from every point $x \in (0, +\infty)$, in one step. We now study the nonevanescence of $(W_n)_{n \in \mathbf{N}}$. It remains to prove that the process $((U_n, W_n))_{n \in \mathbf{N}}$ is nonevanescence. Since S^{d-1} is compact, it suffices to show that $(W_n)_{n \in \mathbf{N}}$ is nonevanescence in $(0, +\infty)$. We proceed by following the classical drift criterion [13, chapters 8, 9]. By assumption, we have

$$\int_{(0, +\infty)} s P_x(ds) = x \int_{(0, +\infty)} s \rho(s) \, ds = x.$$

We write

$$E = \{\exists K \subseteq \mathcal{S} : K \text{ compact,} \\ (W_n)_{n \in \mathbf{N}} \text{ enters } K \text{ infinitely many often}\} \in \mathcal{F}.$$

Assume by contradiction that $\mathbb{P}(E) < 1$, and let $K \subseteq (0, +\infty)$ be a compact set. Then there exists $k \in \mathbf{N}$ such that $0 < \mathbb{P}(\{\forall i \geq k : W_i \notin K\} \setminus E)$. Denoting by μ the initial distribution of the process, the distribution of W_k is given by the measure

$$\mu P^k : \mathcal{B}(\mathcal{S}) \rightarrow [0, 1] : \mu P^k(A) = \int_{\mathcal{S}} P_s^k(\mathcal{S}^{k-1} \times A) \mu(ds).$$

Hence, the new process $(G_n)_{n \in \mathbf{N}}$ defined for every $n \in \mathbf{N}$ by $G_n = W_{n+k-1}$ is a time-homogeneous Markov process with transition function P , and initial distribution μP^k . The random variable

$$T = \min\{n \in \mathbf{N} : G_n \in K\},$$

is adapted to the filtration $(\mathcal{F}_{n+k-1})_{n \in \mathbf{N}}$ and it satisfies $\{T = +\infty\} = \{\forall i \geq k : W_i \notin K\}$. We use the martingale convergence theorem [4, Theorem 35.5] to show that the set $(\{T = +\infty\} \setminus E) \in \mathcal{F}$ has null $\mathbb{P}$ -measure [13, Proposition 9.4.1]. To see this, observe that the stochastic process $(M_n)_{n \in \mathbf{N}}$ defined for every $n \in \mathbf{N}$ by

$$M_n = G_n \chi_{\{T \geq n\}},$$

is a positive martingale with respect to the filtration $(\mathcal{F}_{n+k-1})_{n \in \mathbf{N}}$, by construction of c . By the martingale convergence theorem, it converges $\mathbb{P}$ -almost-surely to some $\mathcal{F}$ -measurable random variable M_∞ , which is $\mathbb{P}$ -almost-surely finite. Therefore, we have $\mathbb{P}$ -almost-surely

$$\chi_{\{T=+\infty\}} M_\infty = \chi_{\{T=+\infty\}} \lim_{n \rightarrow +\infty} W_{n+k-1},$$

so that $\{T = +\infty\} \setminus E$ has null $\mathbb{P}$ -measure, which contradicts the construction of $k \in \mathbf{N}$. Therefore, the process $(W_n)_{n \in \mathbf{N}}$ is nonevanescant, and so is $((U_n, W_n))_{n \in \mathbf{N}}$. We deduce from corollary 2.2.12 that for every $X \in \mathcal{M}_\#(\mathbf{R}^d)$, the sequence of successive cap symmetrizations (resp. polarizations) $(X^{(U_1, W_1) \dots (U_n, W_n)})_{n \in \mathbf{N}}$ converges in measure to $X^\star$.

2.3.4 Example for cap symmetrizations on $[0, +\infty)$

The previous examples concern random walks on S^{d-1} or $S^{d-1} \times (0, +\infty)$, viewed as symmetrization spaces equipped with a strict asymmetry function. In this section, we present a *new example* of cap symmetrization space $\mathcal{L}^\# \subset \mathcal{L}$. This new symmetrization space satisfies that the subset

$$\overline{\mathcal{L}^\# \setminus \mathcal{I}^{(\mathcal{L})}} \in \mathcal{L}^\#$$

is *not* a symmetrization space. In other words, our example occurs in a subset $\mathcal{L}^\#$ where the nonstrict symmetrization of $\mathcal{I}^{(\mathcal{L})}$ are needed for the convergence. To our knowledge, such symmetrization spaces are unknown from the literature. We will see that it is possible to construct a continuous and nonevanescient time-homogeneous Markov process in $\mathcal{L}^\#$ such that the universal convergence fails. This will illustrate the necessity of the discrimination condition (iii) in proposition 2.2.9.

As a preparation for the results of this paragraph, we first recall a known model [13] of random walk in $[0, +\infty)$. Although the assumptions in the following lemma can be weakened, we keep them as simple as possible to make the analysis easy.

Lemma 2.3.11 (Brownian motion on $[0, +\infty)$). *Let $(Z_n)_{n \in \mathbf{N}}$ be a sequence of independent and identically distributed variables on $\mathbf{R}$ with density function ρ , and define for every $n \in \mathbf{N}$*

$$\begin{cases} W_1 &= Z_1, \\ W_{n+1} &= \max\{W_n + Z_{n+1}, 0\}. \end{cases}$$

If $\text{supp}(\rho) = \mathbf{R}$ and if there exists $\delta > 0$ such that

$$\int_{(0, +\infty)} s \rho(s) \, ds < 0 < \inf_{[-\delta, \delta]} \rho,$$

then $(W_n)_{n \in \mathbf{N}}$ is nonevanescient time-homogeneous Markov process with strong Feller continuous transition function P such that for every $s \in$

$[0, +\infty)$, for every nonempty open set $\mathcal{O} \subseteq [0, +\infty)$, we have

$$\sum_{n \in \mathbf{N}} P_s^n((0, +\infty)^{n-1} \times \mathcal{O}) > 0.$$

Proof. The process $(W_n)_{n \in \mathbf{N}}$ is a Markov process, whose transition function is given by

$$P : \mathcal{B}([0, +\infty)) \times [0, +\infty) \rightarrow [0, 1] :$$

$$P_s(A) = \Gamma(A \setminus \{0\} - s) + \Gamma((-\infty, -s])\delta_0(A).$$

Here Γ stands for the Lebesgue measure on $\mathbf{R}$ weighted by ρ . The continuity of translations [25, Lemma 4.3.8] and the dominated convergence theorem that P is strong Feller continuous. The nonevanesence of $(W_n)_{n \in \mathbf{N}}$ is known [13, Proposition 9.4.5, Theorem 9.4.1] and the proof is similar to the proof given for the brownian motion on $(0, +\infty)$. We omit the details.

Let us prove that for every $s \in [0, +\infty)$, for every nonempty open set $\mathcal{O} \subseteq [0, +\infty)$, we have

$$\sum_{n \in \mathbf{N}} P_s^n((0, +\infty)^{n-1} \times \mathcal{O}) > 0.$$

First observe that it is sufficient to prove the claim for every nonempty open set of $(0, +\infty)$. By assumption on ρ , there exists $C > 0$ such that for every Borel measurable set $A \subseteq [0, +\infty)$,

$$\int_A \rho(x) dx \geq C \int_A \chi_{[-\delta, \delta]} dx.$$

Therefore, by comparison of series, we can assume without loss of generality that the distribution Γ is the uniform distribution on the interval $[-\delta, \delta]$. A straightforward computation then shows that for every $s \in [0, +\infty)$ and for every $n \in \mathbf{N}$, the measure

$$H_s^n : \mathcal{B}((0, +\infty)) \rightarrow [0, 1] : H_s^n(A) = P_s^n((0, +\infty)^{n-1} \times A)$$

has support $[\max\{0, s - n\delta\}, s + n\delta]$, which concludes the proof. $\square$

Definition 2.3.12. Fix $\hat{e} \in S^{d-1}$. The truncated cap symmetrization space $\mathcal{L}^\# \subseteq \mathcal{L}$ is defined from $\{-1, 1\} \times [0, +\infty)$ through the action

$$\begin{cases} (1, r) \mapsto \text{the cap symmetrization } (\hat{e}, r) \\ (-1, r) \mapsto \text{the cap symmetrization } (-\hat{e}, 0). \end{cases}$$

Observe that $\mathcal{L}^\#$ posses two connected components, one of them being $\{-1\} \times [0, +\infty)$ whose elements act the same way on $\mathcal{M}_\#(\mathbf{R}^d)$.

Proposition 2.3.13. *The symmetrization space $\mathcal{L}^\#$ acting on the space $(\mathcal{M}_\#(\mathbf{R}^d), d)$ as in definition 2.3.12, is $\star$ -compatible.*

Proof. The algebraic relation $s \circ \star = \star = \star \circ s$ is valid for every $s \in \mathcal{L}$, so it is true in $\mathcal{L}^\#$. Let $A \in \mathcal{B}(\mathbf{R}^d)$ be a Borel measurable set such that $\mathcal{H}^d(A^s \Delta A) = 0$ for every $s \in \mathcal{L}^\#$. Assume by contradiction that $\mathcal{H}^d(A^\star \Delta A) > 0$. Consider the Borel measurable set

$$B = A^{(-\hat{e}, 0)(\hat{e}, 0)}.$$

Since $A = B$ in $\mathcal{M}_\#(\mathbf{R}^d)$, we clearly have $B^\star = A^\star$, $\mathcal{H}^d(B^\star \Delta B) > 0$ and $\mathcal{H}^d(B^s \Delta B) = 0$ for every $s \in \mathcal{L}^\#$. Moreover, the set $B \cap \partial B(0, t)$ equals $\emptyset$ or $\partial B(0, t)$, for every $t > 0$, by construction of cap symmetrizations and by assumption on A . Since $\mathcal{H}^d(B) = \mathcal{H}^d(B^\star)$, the sets $B \setminus B^\star$ and $B^\star \setminus B$ share the same positive Hausdorff measure. There exists $x_1 = (\hat{e}, r_1) \in B^\star \setminus B$ and $x_2 = (\hat{e}, r_2) \in B \setminus B^\star$, with $r_1, r_2 > 0$, and such that for every $r > 0$, we have

$$\mathcal{H}^d(B(x_1, r) \cap (B^\star \setminus B)) > 0, \quad \mathcal{H}^d(B(x_2, r) \cap (B \setminus B^\star)) > 0.$$

But then we get $\mathcal{H}^d(B^{(\hat{e}, \frac{r_1+r_2}{2})} \Delta B) > 0$, which contradicts the construction of B . Therefore, we should have $\mathcal{H}^d(B \Delta B^\star) = 0$. $\square$

Example 2.3.14 (Random walk on $\mathcal{L}^\#$). Consider a random walk $(W_n)_{n \in \mathbf{N}}$ on $[0, +\infty)$ constructed as in lemma 2.3.11. We write

$$K : [0, +\infty) \times \mathcal{B}([0, +\infty)) \rightarrow [0, 1]$$

the transition function of $(W_n)_{n \in \mathbf{N}}$. Consider a Markov process $(U_n)_{n \in \mathbf{N}}$ on $\{-1, 1\}$, independent of $(W_n)_{n \in \mathbf{N}}$, and whose transition function D

has the form

$$H_i(A) = \beta_i \delta_{\{i\}}(A) + (1 - \beta_i) \delta_{\{-i\}}(A),$$

where $\beta_1, \beta_{-1} \in (0, 1)$. The product process $((U_n, W_n))_{n \in \mathbf{N}}$ is a Markov process on $\{-1, 1\} \times [0, +\infty)$, which satisfies the following conditions:

- the transition function P of the product process satisfies

$$P_{(i,r)}(A) = H_i(\{1\})K_r(\pi^+(A)) + H_i(\{-1\})K_r(\pi^-(A)),$$

where $\pi^-(A), \pi^+(A) \in \mathcal{B}([0, +\infty))$ are the unique Borel measurable sets that satisfies

$$A \cap \{-1\} \times [0, +\infty) = \{-1\} \times \pi^-(A),$$

and

$$A \cap \{1\} \times [0, +\infty) = \{1\} \times \pi^+(A).$$

- P is strong Feller continuous and for every nonempty open set $\mathcal{O} \subseteq \{-1, 1\} \times [0, +\infty)$, we have for every $(i, r) \in \{-1, 1\} \times [0, +\infty)$

$$\sum_{n \in \mathbf{N}} P_{(i,r)}^n((\{-1, 1\} \times (0, +\infty))^{n-1} \times \mathcal{O}) > 0.$$

- the process $((U_n, W_n))_{n \in \mathbf{N}}$ is nonevanescant, since so is $(W_n)_{n \in \mathbf{N}}$.

According to corollary 2.2.12, for every $X \in \mathcal{M}_\#(\mathbf{R}^d)$, the sequence of successive symmetrizations $(X^{(U_1, W_1) \dots (U_n, W_n)})_{n \in \mathbf{N}}$ converges almost-surely in measure to $X^\star$.

Counterexample 2.3.15 (Convergence failure for Markov process on $\mathcal{L}^\#$). We illustrate the necessity of the discrimination condition (iii) in proposition 2.2.9. We define the transition function

$$P : \mathcal{L}^\# \times \mathcal{B}(\mathcal{L}^\#) \rightarrow [0, 1]$$

through the formula

$$\begin{aligned} P_{(\hat{e}, r)}(A) &= \frac{1}{2}(\delta_{\hat{e}} \otimes \mu + \delta_{-\hat{e}} \otimes \delta_0)(A), \\ P_{(-\hat{e}, r)}(A) &= \frac{1}{2}(\delta_{-\hat{e}} \otimes \mu + \delta_{\hat{e}} \otimes \delta_0)(A). \end{aligned}$$

where $\mu : \mathcal{B}([0, +\infty)) \rightarrow [0, 1]$ is a strictly positive distribution. For example, one could choose μ given by a half normal distribution

$$\mu(A) = \sqrt{\frac{2}{\pi}} \int_A e^{-\frac{x^2}{2}} dx.$$

The continuity and discrimination properties are easily checked. The definition of P forces the process to go on the boundary before to jump on the other half of $\mathcal{L}^\sharp$. Considering half-disks

$$\begin{aligned} D^+ &= \{(z_1, z_2) \in \mathbf{R}^2 : z_1^2 + z_2^2 < 1, z_2 > 0\}, \\ D^- &= \{(z_1, z_2) \in \mathbf{R}^2 : z_1^2 + z_2^2 < 1, z_2 < 0\}, \end{aligned}$$

it is a straightforward computation to see that the sequences of successive cap symmetrizations $((D^+)^{S_1 \dots S_n})_{n \in \mathbf{N}}$ and $((D^-)^{S_1 \dots S_n})_{n \in \mathbf{N}}$ alternate between D^+ and D^- , and the almost-sure convergence does not occur.

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Chapter 3

Range and mass convergence for vector measures

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3.1 Introduction

A vector measure μ maps some Σ -measurable subsets of a space X to vectors in a linear space V [1, Chapter 1; 8; 13]. Vector measures appear naturally in calculus of variations and in geometric measure theory, as derivatives of functions of bounded variation (BV) [6] and as currents of finite mass [10]. In these contexts, it is natural to work with sequences of finite Radon measures and to study their convergence.

The *wide convergence* of a sequence $(\mu_n)_{n \in \mathbb{N}}$ of finite vector measures to some finite vector measure μ is defined by the condition that for every compactly supported continuous test function $\varphi \in C_c(X, \mathbb{R})$, one has

$$\lim_{n \rightarrow \infty} \int_X \varphi \, d\mu_n = \int_X \varphi \, d\mu.$$

The *strict convergence*, which requires the total variation to converge

also, is an important concept that brings one back into a setting similar to tight sequences of probability measures.

The *total variation* of a vector measure μ is defined in terms of a seminorm $|\cdot|_V$ on V by

$$|\mu|_V(X) \triangleq \sup \left\{ \sum_{E \in \mathcal{P}} |\mu(E)|_V : \mathcal{P} \text{ is a finite } \Sigma\text{-measurable partition of } X \right\}.$$

For real-valued signed Radon measures, $V = \mathbb{R}$ and the mass depends only, up to a constant positive multiple, on the vector structure of the space V . This is not any more the case when V is a finite-dimensional vector space of dimension greater than 1 ($\dim V > 1$). The norms on the space V are all equivalent, but induce total variations whose values are not connected mutually beyond the bounds given by equivalence of seminorms.

This lack of connection between various total variations is the starting question of the present work. Reshetnyak [16] and Delladio [7] have proved that strict convergence with respect to a strictly convex norm is a sufficient assumption to obtain strict convergence with respect to any another norm on V (see also [21] for the mass with respect to a Euclidean norm).

In order to characterize strict convergence through a topological object, independent of a choice of norm, we propose to describe strict convergence with the *range* of a vector measure μ , which is defined as ¹

$$\text{rg } \mu(X) \triangleq \overline{\text{conv } \{\mu(A) : A \in \Sigma\}}.$$

In particular, when $V = \mathbb{R}$, we have

$$\text{rg } \mu(X) = [\mu^-(X), \mu^+(X)], \quad (3.1.1)$$

¹Classically, the range of a vector measure is defined [8] as the closure of the set $\{\mu(A) : A \subset X \text{ is } \Sigma\text{-measurable}\}$. This definition coincides with ours as soon as μ has no atoms [8, Corollary 9.5; 14]. Our notion has the advantage of being always a convex set and is natural in view of the link between range and the total variation [18, Theorem 3].

where $\mu = \mu^+ + \mu^-$, with $\mu^+ \geq 0$, $\mu^- \leq 0$, and μ^+ and μ^- are mutually singular, so that for any widely converging sequence of vector measures, the convergence of the range is equivalent with the convergence of the mass [5, Appendix C].

In general, the range of a vector measure characterizes uniquely the total variation of a measure [18]. This result is stable under convergence as the range convergence implies in general the mass convergence:

Theorem 3.1.1. *Let X be a locally compact separable metric space endowed with its Borel σ -algebra $\mathcal{B}(X)$, let V be a finite-dimensional space and let $|\cdot|_V$ be a seminorm on V . If $(\mu_n)_{n \in \mathbb{N}}$ is a sequence of Borel vector measures from X to V that converges widely to some vector Borel measure μ , and if the sequence of ranges $(\text{rg } \mu_n(X))_{n \in \mathbb{N}}$ converges to $\text{rg } \mu(X)$ in Hausdorff distance, then*

$$\lim_{n \rightarrow \infty} |\mu_n|_V(X) = |\mu|_V(X).$$

The assumption on the convergence is described in terms of Hausdorff distance; this choice is in practice inconsequential: since the space V is finite-dimensional, the Hausdorff convergence of compact convex sets is equivalent to the Kuratowski convergence and to the convergence of the support functions [20, Corollaries 3A and P4.A] (see also [15]).

The converse statement of theorem 3.1.1 does not hold in general. In fact, the strict convexity of the norm appears as a necessary condition to have range convergence:

Proposition 3.1.2. *If the locally compact separable metric space X is not discrete and if the seminorm $|\cdot|_V$ is not strictly convex, then there exists a sequence $(\mu_n)_{n \in \mathbb{N}}$ of Borel vector measures from X to V converging widely to some vector measure μ such that $(|\mu_n|_V(X))_{n \in \mathbb{N}}$ converges to $|\mu|_V(X)$ but $(\text{rg } \mu_n(X))_{n \in \mathbb{N}}$ is constant and different from $\text{rg } \mu(X)$.*

We prove that strict convexity is in fact a necessary and sufficient condition for the equivalence between range convergence and mass convergence:

Theorem 3.1.3. *Let X be a locally compact separable metric space and $(V, |\cdot|_V)$ be a finite-dimensional Euclidean space. If the norm $|\cdot|_V$ is strictly convex and if $(\mu_n)_{n \in \mathbb{N}}$ is a sequence of Borel vector measures from X to V that converges widely to some vector measure μ , then*

$$\lim_{n \rightarrow \infty} \operatorname{rg} \mu_n(X) = \operatorname{rg} \mu(X)$$

if and only if

$$\lim_{n \rightarrow \infty} |\mu_n|_V(X) = |\mu|_V(X).$$

The above result completely characterizes the strict convergence in terms of range convergence. In comparison to mass convergence, the definition of the rank depends on V only as a *topological vector space*, and does not require any specific choice of norm.

The proof of theorem 3.1.3 follows a more general approach due to Reshetnyak [16], relying on layerwise decomposition technique.

Theorem 3.1.3 shows that mass convergence implies range convergence in a strictly convex setting. It is also known [18] that equality of the range implies equality of the variation. This leads to the question whether the mass of a measure depends monotonically on its range, that is, whether the inclusion of sets $\operatorname{rg} \nu(X) \subseteq \operatorname{rg} \mu(X)$ does imply the inequality $|\nu|_V(X) \leq |\mu|_V(X)$.

In the case where the norm $|\cdot|_V$ is Euclidean, we prove such a relation:

Theorem 3.1.4. *Assume that V is a finite-dimensional vector space, equipped with a semi-norm $|\cdot|_V$. If either $\dim V \leq 2$ or $|\cdot|_V$ is Euclidean, then if Σ is a σ -algebra on X and the vector measures $\nu, \mu : \Sigma \rightarrow V$ satisfy*

$$\operatorname{rg} \nu(X) \subseteq \operatorname{rg} \mu(X),$$

then

$$|\nu|_V(X) \leq |\mu|_V(X).$$

If moreover the norm $|\cdot|_V$ is strictly convex, equality in the above conclusion occurs if and only if the ranges are equal.

In higher dimensions, we do not know if such relation still holds.

Open problem 1. If $\dim V \geq 3$ and if V is endowed with an arbitrary (respectively strictly convex) norm, is the mass (respectively strictly) monotone with respect to range inclusion?

Our approach to prove theorem 3.1.4 when V is a Euclidean space directly derives from the a representation formula

$$|v|_V = \int_{\mathcal{S}} |\langle \eta, v \rangle| d\sigma(\eta) \quad (3.1.2)$$

for some suitable multiple of the uniform measure σ on the unit sphere $\mathcal{S}$ of V^* . We will prove that (3.1.2) implies the representation formula

Theorem 3.1.5. *Assume that V is a finite-dimensional vector space, equipped with a Euclidean norm $|\cdot|_V$, then for every σ -algebra Σ on X , every vector measure $\mu : \Sigma \rightarrow V$ and every $A \in \Sigma$,*

$$|\mu|_V(A) = \int_{\mathcal{S}} |\langle \eta, v \rangle|(A) d\sigma(\eta).$$

The representation formula (3.1.2) means that the norm is an L^1 -norm in the sense of Choquet [4, Paragraph 41], and that the unit ball of the dual norm is a zonoid [2, Theorem 2.1; 4]. In the two-dimensional case, any centrally symmetric convex set is a zonoid [2, Theorem 5.4; 4] and, in fact, any norm has a similar representation for some positive even measure σ this implies a counterpart of theorem 3.1.5 for any two-dimensional norm. The latter characterization of zonoids fails in higher dimensions: for example the ℓ^∞ norm whose whose dual ℓ^1 ball is hyperoctahedron, which is not a zonoid.

In dimension 2, one has similar representation for any choice of norm, and strict convexity translates by density of the support for the representing measure σ .

When the vector space V is two-dimensional, we improve the geometric interplay between range convergence and mass convergence, by drawing the following link:

Theorem 3.1.6. *Assume that V is a two-dimensional vector space, equipped with a seminorm $|\cdot|_V$. If Σ is a σ -algebra on X and if $\mu : \Sigma \rightarrow V$*

is a vector measure, then

$$|\mu|_V(X) = \frac{1}{2} \operatorname{Per}_V(\operatorname{rg} \mu(X)).$$

Here $\operatorname{Per}_V(\operatorname{rg} \mu(X))$ is the perimeter of the closed convex set $\operatorname{rg} \mu(X)$, computed with respect to the seminorm $|\cdot|_V$. When the target space V is equipped with a strictly convex norm, the perimeter becomes strictly monotone and theorem 3.1.3 may be derived from theorem 3.1.6. This kind of isoperimetric inequalities for convex sets were also investigated in [12]. Theorem 3.1.6 is the extension of the one-dimensional formula $|\mu|(X) = \operatorname{diam} \operatorname{rg} \mu(X)$ that follows from (3.1.1).

For the two-dimensional case, theorem 3.1.3 may be derived also from the integral representation (3.1.2).

3.2 Total variation and range

3.2.1 Vector measures

We assume that X is a space endowed with a σ -algebra Σ and that the space V is a finite-dimensional vector space on the real numbers. The space V^* is the algebraic dual of V and $\langle \eta, v \rangle = \eta(v)$ denotes the duality product defined by the action of the linear form $\eta \in V^*$ on the vector $v \in V$.

A *vector measure* $\mu : \Sigma \rightarrow V$ is a countably additive function; when $V = \mathbb{R}$ it is a (finite) *signed measure*. A vector measure μ induces definitions of integrals

$$\int_A f \, d\mu \in V \quad \text{and} \quad \int_A \langle g, d\mu \rangle \in \mathbb{R}.$$

for all Σ -measurable and bounded functions $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow V^*$ and sets $A \in \Sigma$. These integrals satisfy for every $E \in \Sigma$ and $\eta \in V^*$,

$$\int_A \chi_E \, d\mu = \mu(A \cap E) \quad \text{and} \quad \int_A \langle \eta \chi_E, d\mu \rangle = \langle \eta, \mu(A \cap E) \rangle,$$

and the usual additivity, linearity and dominated convergence properties.

3.2.2 Total variation

The total variation of a vector measure is defined as follows (see [1, Definition 1.4 (b)]).

Definition 3.2.1. Let Σ be a σ -algebra on X . The total variation of a vector measure $\mu : \Sigma \rightarrow X$ with respect to a seminorm $|\cdot|_V$ is the function $|\mu|_V : \Sigma \rightarrow [0, +\infty]$ defined for each set $A \in \Sigma$ by

$$|\mu|_V(A) \triangleq \sup \left\{ \sum_{n=1}^m |\mu(E_n)|_V : E_1, \dots, E_m \in \Sigma, E_1, \dots, E_m \subseteq A \text{ and the sets } E_1, \dots, E_m \text{ are pairwise disjoint} \right\}.$$

The total variation enjoys properties that make of it a finite measure [1, Theorem 1.6].

Proposition 3.2.2 (Finiteness). *Let Σ be a σ -algebra on X and let $\mu : \Sigma \rightarrow V$ be a vector measure. For every $A \in \Sigma$, $|\mu|_V(A) < +\infty$.*

Proposition 3.2.3 (Monotonicity). *Let Σ be a σ -algebra on X , let $\mu : \Sigma \rightarrow V$ be a vector measure and let $A, B \in \Sigma$. If $A \subseteq B$, then*

$$|\mu|_V(A) \subseteq |\mu|_V(B).$$

Proposition 3.2.4 (Countable additivity). *Let Σ be a σ -algebra on X and let $\mu : \Sigma \rightarrow V$ be a vector measure. For every sequence $(A_n)_{n \in \mathbb{N}}$ of disjoint sets in Σ ,*

$$|\mu|_V(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} |\mu|_V(A_n).$$

We also have some algebraic properties:

Proposition 3.2.5 (Triangle inequality). *Let Σ be a σ -algebra on X and let $\mu, \nu : \Sigma \rightarrow V$ be vector measures. For every $A \in \Sigma$,*

$$|\mu + \nu|_V(A) \leq |\mu|_V(A) + |\nu|_V(A).$$

Proposition 3.2.6 (Homogeneity). *Let Σ be a σ -algebra on X and let $\mu : \Sigma \rightarrow V$ be a vector measure. For every $\lambda \in \mathbb{R}$ and every $A \in \Sigma$,*

$$|\lambda\mu|_V(A) = |\lambda| |\mu|_V(A).$$

It follows from the previous properties that if the function $f : X \rightarrow \mathbb{R}$ is bounded and Σ -measurable, then

$$\left| \int_A f \, d\mu \right|_V \leq \int_A |f| \, d|\mu|_V.$$

If X is a locally compact separable metric space, we endow it with its Borel σ -algebra $\mathcal{B}(X)$. This topological setting on X is common when thinking of $C_0(X)^*$ as the space of finite measures on X , every finite measure being Radon measures because X is second-countable and metric. In particular, whatever the choice of the norm $|\cdot|_V$ on V , the variation measure $|\mu|_V$ is a Radon measure.

Proposition 3.2.7. *Let X be a locally compact separable metric space. If $\mu : \mathcal{B}(X) \rightarrow V$ is a vector measure and if $A \subset X$ is open, then*

$$|\mu|_V(A) = \sup \left\{ \int_A \langle \psi, d\mu \rangle : \psi \in C_c(A, V') \text{ and } |\psi|_{V'} \leq 1 \right\}.$$

Here

$$V' \triangleq \{\eta \in V^* : |\eta|_{V'} < +\infty\},$$

where V^* is the algebraic dual space of the vector space V and

$$|\eta|_{V'} \triangleq \sup \{ \langle \eta, v \rangle : v \in V \text{ and } |v|_V \leq 1 \}. \quad (3.2.1)$$

Equivalently, $\eta \in V'$ if and only if for every $v \in V$, $|v|_V = 0$ implies $\langle \eta, v \rangle = 0$. By the Hahn–Banach theorem for seminorms (see for example [3, Theorem 1.1]), one has

$$|v|_V = \sup \{ \langle \eta, v \rangle : \eta \in V' \text{ and } |\eta|_{V'} \leq 1 \}. \quad (3.2.2)$$

Since V and V' are finite-dimensional, the suprema in (3.2.1) and (3.2.2) are achieved in view of the Weierstrass extreme value theorem.

The proof of proposition 3.2.7 is a transposition of the proof when V

is Euclidean and X is separable [1, Proposition 1.47].

Proof of proposition 3.2.7. Let $E_1, \dots, E_m \in \mathcal{B}(X)$ be disjoint subsets of A and, in view of (3.2.2), let $\eta_1, \dots, \eta_m \in V'$ be linear forms such that for every $i \in \{1, \dots, m\}$ $|\eta_i|_{V'} = 1$ and $\langle \eta_i, \mu(E_i) \rangle = |\mu(E_i)|_V$. Since μ is a vector measure, $|\mu|_V$ is a finite measure (propositions 3.2.2 and 3.2.4) and thus by Lusin's theorem there exists a function $\psi \in C(A, V')$ such that $|\mu|_V(\{x \in A : \psi(x) \neq \sum_{n=1}^m \eta_n \chi_{E_n}(x)\}) < \varepsilon$. Without loss of generality we can assume that $|\psi|_{V'} \leq 1$. Since the set A is σ -compact and in view of Urysohn's lemma for locally compact spaces [19, lemma 2.12], there exists a sequence of functions $(\theta_j)_{j \in \mathbb{N}}$ in $C_c(A, [0, 1])$ that converges everywhere to 1 in the set A . Therefore, by multiplying ψ by θ_j for $j \in \mathbb{N}$ large enough, we can assume that $\psi \in C_c(A, V')$ and $|\psi|_{V'} \leq 1$ everywhere in A . We then have

$$\left| \sum_{n=1}^m |\mu(E_n)|_V - \int_A \langle \psi, d\mu \rangle \right| \leq \varepsilon |\mu|_V(A).$$

This implies thus that

$$\sum_{n=1}^m |\mu(E_n)|_V \in \overline{\left\{ \int_A \langle \psi, d\mu \rangle : \psi \in C_c(A, V') \text{ and } |\psi|_{V'} \leq 1 \right\}}.$$

Conversely, if $\psi \in C_c(A, V')$ and $|\psi|_{V'} \leq 1$ on A , then there exist an integer $m \in \mathbb{N}$, forms $\eta_1, \dots, \eta_m \in V'$ and pairwise disjoint sets $E_1, \dots, E_m \in \mathcal{B}(A)$ such that for each $n \in \{1, \dots, m\}$, $|\eta_n|_{V'} \leq 1$ and $E_n \subseteq A$ and such that if $g = \sum_{n=1}^m \eta_n \chi_{E_n}$, one has $|g - \psi|_{V'} \leq \varepsilon$ everywhere on A . We then have

$$\left| \int_A \langle g, d\mu \rangle - \int_A \langle \psi, d\mu \rangle \right| \leq \varepsilon |\mu|_V(A)$$

and by definition 3.2.1

$$\int_A |g| d\mu = \sum_{n=1}^m \langle \eta_n, \mu(E_n) \rangle \leq \sum_{n=1}^m |\mu(E_n)|_V \leq |\mu|_V(A). \quad \square$$

From now on, X will be assumed to be a locally compact separable metric space endowed with its Borel σ -algebra $\Sigma = \mathcal{B}(X)$.

3.2.3 Range

The range of a measure has a definition that has some analogies with the total variation.

Definition 3.2.8. The *range* of the vector measure $\mu : \Sigma \rightarrow V$ on a set $A \in \Sigma$ is the set

$$\text{rg } \mu(A) \triangleq \overline{\text{conv} \{ \mu(E) : E \in \Sigma \text{ and } E \subset A \}} \subseteq V.$$

Here, $\overline{\text{conv } H}$ denotes the closed convex hull of a set $H \subset V$. The range defined by definition 3.2.8 only depends on the structure of V as a topological vector space.

If $\mu = \sum_{i=1}^k v_i \delta_{x_i}$, with $x_1, \dots, x_k \in X$ distinct points, then definition 3.2.8 gives for every $A \in \Sigma$

$$\text{rg } \mu(A) = \left\{ \sum_{i=1}^d t_i v_i : 0 \leq t_i \leq \chi_A(x_i) \right\}.$$

If $X \subset \mathbb{R}^N$ is a Borel-measurable set and if $\mu \triangleq w \mathcal{L}^N$, where $w : X \rightarrow V$ is a Borel-measurable function, then

$$\text{rg } \mu(A) = \left\{ \int_X f w \, d\mathcal{L}^N : f : X \rightarrow \mathbb{R} \text{ is Borel measurable} \right. \\ \left. \text{and } 0 \leq f \leq \chi_A \text{ in } X \right\}.$$

It follows immediately from the definition 3.2.8 that the set $\text{rg } \mu(A)$ is a closed convex subset of the space V and that $0 \in \text{rg } \mu(A)$. Moreover, $\text{rg } \mu(A) = \{0\}$ if and only if $\mu = 0$.

Since for every $E \in \Sigma$ such that $E \subset A$, $\mu(A \setminus E) = \mu(A) - \mu(E)$, we have $\text{rg } \mu(A) = \mu(A) - \text{rg } \mu(A)$, that is, the set $\text{rg } \mu(A)$ is symmetric with respect to the vector $\mu(A)/2 \in V$.

In a finite-dimensional space V , the range is always bounded (see proposition 3.2.2).

Proposition 3.2.9. *Let Σ be a σ -algebra on X . If $\mu : \Sigma \rightarrow V$ is a vector measure, then for every $A \in \Sigma$, the set $\text{rg } \mu(A)$ is bounded.*

This property fails in general if the domain is not a σ -algebra [8, Example 8, Proposition 11].

Proof of proposition 3.2.9. Since V is finite-dimensional, there exist vectors $v_1, \dots, v_d \in V$ and linear forms $\eta_1, \dots, \eta_d \in V'$ such that $d = \dim V$ and for every $i, j \in \{1, \dots, d\}$, $\langle \eta_i, v_j \rangle = \delta_{ij}$, where δ_{ij} denotes Kronecker's delta. In particular, we have for every $v \in V$, $v = \sum_{i=1}^d v_i \langle \eta_i, v \rangle$, and thus, for every set $E \in \Sigma$, $\mu(E) = \sum_{i=1}^d v_i \mu_i(E)$, where $\mu_i(E) = \langle \eta_i, \mu(E) \rangle$. For every $i \in \{1, \dots, d\}$, the set function μ_i is a finite signed measure and thus by the Hahn–Jordan decomposition theorem it can be decomposed as $\mu_i = \mu_i^+ + \mu_i^-$, where $\mu_i^+ \geq 0$, $\mu_i^- \leq 0$ are finite Borel measures [5, Appendix C]. Hence

$$\operatorname{rg} \mu(A) \subseteq \left\{ \sum_{i=1}^n \tau_i v_i : \text{for every } i \in \{1, \dots, d\}, \mu_i^-(A) \leq \tau_i \leq \mu_i^+(A) \right\},$$

and thus the set $\operatorname{rg} \mu(A)$ is bounded. $\square$

The range is monotone with respect to the set, as was the total variation (proposition 3.2.3).

Proposition 3.2.10 (Monotonicity). *Let Σ be a σ -algebra on X , let $\mu : \Sigma \rightarrow V$ be a vector measure and let $A, B \in \Sigma$. If $A \subseteq B$, then*

$$\operatorname{rg} \mu(A) \subseteq \operatorname{rg} \mu(B).$$

Proof. This follows directly from definition 3.2.8. $\square$

Finally the range has a countable additivity property as the total variation (proposition 3.2.4).

Proposition 3.2.11 (Countable additivity). *Let Σ be a σ -algebra on X , let $\mu : \Sigma \rightarrow V$ be a vector measure and let $(A_n)_{n \in \mathbb{N}}$ be a sequence of disjoint sets in Σ . Then*

$$\operatorname{rg} \mu \left(\bigcup_{n \in \mathbb{N}} A_n \right) = \overline{\sum_{n \in \mathbb{N}} \operatorname{rg} \mu(A_n)}.$$

Here $\sum_{n \in \mathbb{N}} \operatorname{rg} \mu(A_n)$ denotes the set of *finite sums* of elements in the sets $\operatorname{rg} \mu(A_n)$.

Since the range is compact (proposition 3.2.9), proposition 3.2.11 implies that if $A_1, \dots, A_k \in \Sigma$ are disjoint, we have

$$\operatorname{rg} \mu \left(\bigcup_{n=1}^k A_n \right) = \sum_{n=1}^k \operatorname{rg} \mu (A_n).$$

If we drop in proposition 3.2.11 the assumption that the sets $(A_n)_{n \in \mathbb{N}}$ are disjoint we obtain

$$\operatorname{rg} \mu \left(\bigcup_{n \in \mathbb{N}} A_n \right) \subseteq \overline{\sum_{n \in \mathbb{N}} \operatorname{rg} \mu (A_n)}.$$

Proof of proposition 3.2.11. We first assume that $v \in \sum_{n \in \mathbb{N}} \operatorname{rg} \mu (A_n)$. Then there exists $k \in \mathbb{N}$ and $v_0, \dots, v_k$ such that for each $n \in \{0, \dots, k\}$, $v_n \in \operatorname{rg} \mu (A_n)$ and $v = \sum_{i=0}^k v_i$. By definition of the range, for every $n \in \{0, \dots, k\}$, $v_n = \lim_{\ell \rightarrow \infty} v_n^\ell$, where

$$v_n^\ell \in \operatorname{conv} \{ \mu(E) : E \in \Sigma \text{ and } E \subset A_n \} \subseteq V.$$

Hence we have by additivity of the measure,

$$v^\ell \triangleq \sum_{n=0}^k v_n^\ell \in \operatorname{conv} \left\{ \mu(E) : E \in \Sigma \text{ and } E \subset \bigcup_{i=0}^n A_i \right\}$$

and, by closedness of the range, $v \in \operatorname{rg} \mu \left(\bigcup_{n \in \mathbb{N}} A_n \right)$. We have thus proved that $\sum_{n \in \mathbb{N}} \operatorname{rg} \mu (A_n) \subseteq \operatorname{rg} \mu \left(\bigcup_{n \in \mathbb{N}} A_n \right)$. Since the range is a closed set by definition, we deduce that $\overline{\sum_{n \in \mathbb{N}} \operatorname{rg} \mu (A_n)} \subseteq \operatorname{rg} \mu \left(\bigcup_{n \in \mathbb{N}} A_n \right)$.

Conversely, we observe that if $E \subset \bigcup_{n \in \mathbb{N}} A_n$, then by countable additivity of the measure μ , we have $\mu(E) = \sum_{n \in \mathbb{N}} \mu(E \cap A_n) \in \sum_{n \in \mathbb{N}} \operatorname{rg} \mu (A_n)$. It then follows that $\operatorname{rg} \mu \left(\bigcup_{n \in \mathbb{N}} A_n \right) \subseteq \overline{\sum_{n \in \mathbb{N}} \operatorname{rg} \mu (A_n)}$. $\square$

The range operation is also compatible with the linear structure of vector measures. First it is sublinear with respect to the sum of measures, as was the total variation proposition 3.2.5.

Proposition 3.2.12 (Triangle inequality). *Let Σ be a σ -algebra on X and let $\mu : \Sigma \rightarrow V$ be a vector measure. For every $A \in \Sigma$,*

$$\operatorname{rg}(\mu + \nu)(A) \subseteq \operatorname{rg} \mu(A) + \operatorname{rg} \nu(A).$$

The range is not linear. Indeed, in general one has $\text{rg}(\mu - \nu)(A) = \{0\} \neq \text{rg} \mu(A) + \text{rg}(-\nu)(A)$.

Proof of proposition 3.2.12. One has for every $E \in \Sigma$ such that $E \subset A$, $\mu(E) \in \text{rg} \mu(A) + \text{rg} \nu(A)$. Since the range is convex and bounded (proposition 3.2.9), the set $\text{rg} \mu(A) + \text{rg} \nu(A)$ is closed and convex and the conclusion follows. $\square$

The range operation is also stable under linear mappings in the target.

Proposition 3.2.13 (Composition with linear maps). *Let Σ be a σ -algebra on X , let V, W be finite-dimensional spaces and let $\mu : \Sigma \rightarrow V$ be a vector measure. For every $T \in \text{Lin}(V, W)$,*

$$\text{rg}(T \circ \mu)(X) = T(\text{rg} \mu(X)).$$

In particular, if $\lambda \in \mathbb{R}$ and $A \in \Sigma$, we have

$$\text{rg}(\lambda \mu)(A) = \lambda \text{rg} \mu(A), \quad (3.2.3)$$

which is the counterpart of proposition 3.2.6.

Proof of proposition 3.2.13. If $E \in \Sigma$ and if $E \subseteq A$, then we have $(T \circ \mu)(E) = T(\mu(E))$. It follows then that

$$\begin{aligned} \overline{\text{conv}(\{T \circ \mu(E) : E \in \Sigma \text{ and } E \subseteq A\})} \\ = \overline{T(\text{conv}(\{\mu(E) : E \in \Sigma \text{ and } E \subseteq A\}))} \end{aligned}$$

We observe now that if the set $H \subset V$ is convex and bounded, then by continuity and compactness in the finite-dimensional space V , $T(\overline{H}) = \overline{T(H)}$. $\square$

The topological structure on the space X can be used to characterize the range by duality on test functions.

Proposition 3.2.14. *If X is a locally compact separable metric space and if $\mu : \mathcal{B}(X) \rightarrow V$ is a vector Borel measure and if the set $A \subset X$ is open, then*

$$\text{rg} \mu(A) = \overline{\left\{ \int_A \varphi d\mu : \varphi \in C_c(A, [0, 1]) \right\}}.$$

Proposition 3.2.14 is the counterpart of proposition 3.2.7 for the range.

Proof of proposition 3.2.14. Let $E \in \mathcal{B}(X)$. Since μ is a vector measure, $|\mu|_V(X)$ is a finite measure (propositions 3.2.2 and 3.2.4) and thus by Lusin's theorem there exists a function $\varphi \in C(X)$ such that $|\mu|_V(\{x \in X : \varphi(x) \neq \chi_E(x)\}) \leq \varepsilon$. Since the set A is open and σ -compact, we can assume without loss of generality that $\varphi \in C_c(A, [0, 1])$. We then have

$$\left| \mu(E) - \int_X \varphi d\mu \right| \leq \varepsilon |\mu|_V(A).$$

This implies thus that

$$\mu(E) \in \overline{\{\langle \mu, \varphi \rangle : \varphi \in C_c(X, [0, 1])\}}.$$

Conversely, if $\varphi \in C_c(A)$, there exists a Borel-measurable function $f : A \rightarrow [0, 1]$ that takes only finitely many values and such that $|f - \varphi| \leq \varepsilon$ on A . We then have

$$\left| \int_A f d\mu - \int_A \varphi d\mu \right| \leq \varepsilon |\mu|_V$$

and

$$\int_A f d\mu \in \text{conv} \{ \mu(E) : E \in \mathcal{B}(X) \text{ and } E \subset X \},$$

so that by definition 3.2.8

$$\int_A \varphi d\mu \in \text{rg } \mu(A). \quad \square$$

3.2.4 Links between the range and the total variation

In the one-dimensional case, there is a straightforward relationship between the range and the mass.

Proposition 3.2.15. *Assume that $\dim(V) = 1$. If $\mu : \Sigma \rightarrow V$ is a vector measure and $A \in \Sigma$, then*

$$|\mu|_V(A) = \text{diam}(\text{rg } \mu(A)).$$

Proof. Without loss of generality, we can assume that $V = \mathbb{R}$. Let $\mu = \mu^+ + \mu^-$ be the Hahn–Jordan decomposition (see e.g. [5, Appendix C]) into measures $\mu^+ \geq 0$ and $\mu^- \leq 0$. One has then

$$|\mu|(A) = \mu^+(A) - \mu^-(A)$$

and

$$\text{rg } \mu(A) = [\mu^-(A), \mu^+(A)],$$

and the conclusion follows. $\square$

In higher dimensions, the mass and the range are still related by a quantitative bound.

Proposition 3.2.16 (Comparison between total variation and range). *Let $(V, |\cdot|_V)$ be a seminormed space. There exists a constant $C > 0$ such that for every Σ -algebra on the space X and every vector measure $\mu : \Sigma \rightarrow V$ and every $A \in \Sigma$, we have*

$$\text{rg } \mu(A) \subset \overline{B_V(0, |\mu|_V(A))}$$

and

$$|\mu|_V(A) \leq C \sup \{|v|_V : v \in \text{rg } \mu(A)\}.$$

Diestel and Uhl have a similar statement for the *semivariation* of a vector measure in a Banach space [8, Proposition I.1.11].

Proof of proposition 3.2.16. For the first part of the proposition, by definition 3.2.1, for every set $E \in \Sigma$ such that $E \subseteq A$ we have

$$|\mu(E)|_V \leq |\mu(E)|_V + |\mu(A \setminus E)|_V \leq |\mu|_V(A),$$

and the assertion follows from definition 3.2.8.

For the second part, let $\eta_1, \dots, \eta_d$ be a basis of V' . By (3.2.2), there exists a constant $C_1 > 0$ such that for every $v \in V$,

$$|v|_V \leq C_1 \sum_{i=1}^d |\langle \eta_i, v \rangle|.$$

It follows then that if the sets $E_1, \dots, E_\ell \in \Sigma$ are pairwise disjoint and if $E_1, \dots, E_\ell \subset A$, we have

$$\sum_{n=1}^{\ell} |\mu(E_i)|_V \leq C_1 \sum_{i=1}^d \sum_{n=1}^{\ell} |\langle \eta_i, \mu(E_i) \rangle| \leq C_1 \sum_{i=1}^d |\langle \eta_i, \mu(E_i) \rangle|_V(A),$$

and using proposition 3.2.15 we infer

$$\sum_{n=1}^{\ell} |\mu(E_i)|_V \leq C_2 \sup \{ |v|_V : v \in \text{rg } \mu(A) \}. \quad \square$$

Proposition 3.2.16 implies the continuity of the range with respect to the seminorm induced by the total variation.

Proposition 3.2.17. *If $\mu, \nu : \Sigma \rightarrow V$ and $A \in \Sigma$, then*

$$d_{\mathfrak{H}}(\text{rg } \nu(A), \text{rg } \mu(A)) \leq |\nu - \mu|_V(A).$$

Here $d_{\mathfrak{H}}(\text{rg } \nu(A), \text{rg } \mu(A))$ denotes the Hausdorff distance between non-empty compact subsets of V : if $K_1, K_2 \subset V$ are non-empty compact sets,

$$d_{\mathfrak{H}}(K_1, K_2) \triangleq \inf \{ \varepsilon > 0 : K_1 \subseteq K_2 + \overline{B(0, \varepsilon)} \text{ and } K_2 \subseteq K_1 + \overline{B(0, \varepsilon)} \}.$$

Proposition 3.2.17 implies in particular that if the sequence $(\mu_n)_{n \in \mathbb{N}}$ converges in total variation to μ on A , then the associated sequence of ranges $(\text{rg } \mu_n(A))_{n \in \mathbb{N}}$ converges to $\text{rg } \mu(A)$ in Hausdorff distance.

Proof of proposition 3.2.17. By the triangle inequality for the range (proposition 3.2.12), we have

$$\text{rg } \nu(X) \subseteq \text{rg}(\nu - \mu)(X) + \text{rg } \mu(X).$$

By proposition 3.2.16, we have

$$\text{rg}(\nu - \mu)(X) \subseteq \overline{B_V(0, |\nu - \mu|_V)}.$$

We conclude then that

$$\text{rg } \nu(X) \subseteq \overline{B_V(0, |\nu - \mu|_V)} + \text{rg } \mu(X).$$

By symmetry, we also have

$$\operatorname{rg} \mu(X) \subseteq \overline{B_V(0, |\nu - \mu|_V)} + \operatorname{rg} \nu(X),$$

and the conclusion follows from the definition of Hausdorff distance. $\square$

A second property is the fact that the range determines univocally the image: if two measures share the same range on a set A , then their total variation on that set coincides [18].

Proposition 3.2.18 (Rodríguez-Piazza). *Let $\mu, \nu : \Sigma \rightarrow V$ be vector measures. If $\operatorname{rg} \mu(X) = \operatorname{rg} \nu(X)$, then $|\mu|_V(X) = |\nu|_V(X)$.*

The next lemma highlights the key property used by Rodríguez-Piazza to prove proposition 3.2.18. It is an adaptation, in terms of total masses, of a representation principle for the norm by finite combination of positive values of linear functionals [18, Lemma 2].

Lemma 3.2.19. *Let $|\cdot|_V$ be a seminorm on V . For each $\varepsilon > 0$, there exists $\eta_1, \dots, \eta_m \in V'$ and $\alpha_1, \dots, \alpha_m \in \{-1, 1\}$ such that, for every Σ -algebra on the space X and every vector measure $\mu : \Sigma \rightarrow V$, we have*

$$\left| |\mu|_V(A) - \sum_{i=1}^m \alpha_i |\langle \eta_i, \mu \rangle|(A) \right| \leq \varepsilon |\mu|_V(A).$$

The proof follows the strategy of [18, Theorem 3].

Proof of lemma 3.2.19. Fix $\varepsilon > 0$. There exist forms $\eta_1, \dots, \eta_m \in V'$ and $\alpha_1, \dots, \alpha_m \in \{-1, 1\}$ such that for each $v \in V$, we have

$$\left| |v|_V - \sum_{i=1}^m \alpha_i |\langle \eta_i, v \rangle| \right| \leq \varepsilon |v|_V$$

(see [2, Theorem 2.8; 4, p. 53; 17, Lemma 1].) Let $\delta > 0$. By definition of total mass there exists a finite partition $\mathcal{P}_* \subset \Sigma$ of A such that

$$\sum_{E \in \mathcal{P}_*} |\mu(E)|_V \geq |\mu|_V(A) - \delta.$$

and for every $i \in \{1, \dots, m\}$, there exists a finite partition $\mathcal{P}_i \subset \Sigma$ of A such that

$$\sum_{E \in \mathcal{P}_i} |\langle \eta_i, \mu(E) \rangle| \geq |\langle \eta_i, \mu \rangle|(A) - \delta.$$

If we now take $\mathcal{P}$ to be a common refinement the partitions of the partitions $\mathcal{P}_*, \mathcal{P}_1, \dots, \mathcal{P}_m$, we have

$$\sum_{E \in \mathcal{P}} |\mu(E)|_V \geq |\mu|_V(A) - \delta$$

and

$$\sum_{E \in \mathcal{P}} |\langle \eta_i, \mu(E) \rangle| \geq |\langle \eta_i, \mu \rangle|(A) - \delta.$$

It follows then that

$$\begin{aligned} \left| |\mu|_V(A) - \sum_{i=1}^m \alpha_i |\langle \eta_i, \mu \rangle|(A) \right| &\leq \varepsilon \sum_{E \in \mathcal{P}} |\mu(E)|_V + \left| \sum_{E \in \mathcal{P}} |\mu(E)|_V - |\mu|_V(A) \right| \\ &\quad + \sum_{i=1}^m |\alpha_i| \left| \sum_{E \in \mathcal{P}} |\langle \eta_i, \mu(E) \rangle| - |\langle \eta_i, \mu \rangle|(A) \right| \\ &\leq \varepsilon |\mu|_V(A) + \left(1 + \sum_{i=1}^m |\alpha_i| \right) \delta. \end{aligned}$$

Since $\delta > 0$ is arbitrary, the conclusion follows. $\square$

Rodríguez-Piazza's proposition 3.2.18 can be deduced immediately from lemma 3.2.19.

Proof of proposition 3.2.18. Let $\varepsilon > 0$ and let then $\eta_1, \dots, \eta_m \in V'$ and $\alpha_1, \dots, \alpha_m \in \{-1, 1\}$ be given by lemma 3.2.19. Since $\text{rg } \mu(A) = \text{rg } \nu(A)$, we have for every $i \in \{1, \dots, m\}$, in view of proposition 3.2.15,

$$|\langle \eta_i, \mu \rangle|(A) = \text{diam}(\text{rg} \langle \eta_i, \mu \rangle(A)) = \text{diam}(\text{rg} \langle \eta_i, \nu \rangle(A)) = |\langle \eta_i, \nu \rangle|(A),$$

and therefore by lemma 3.2.19,

$$||\nu|_V(A) - |\mu|_V(A)| \leq \varepsilon (|\nu|_V(A) + |\mu|_V(A));$$

since $\varepsilon > 0$ is arbitrary, the conclusion follows. $\square$

3.2.5 Wide convergence

The notion of wide convergence of measures is classically defined by testing the measure against compactly supported functions.

Definition 3.2.20. Let X be a locally compact separable metric space. A sequence of vector measures $(\mu_n)_{n \in \mathbb{N}}$ from X to V converges widely to some vector measure μ whenever for every $\varphi \in C_c(X)$,

$$\lim_{n \rightarrow \infty} \int_X \varphi \, d\mu_n = \int_X \varphi \, d\mu.$$

The wide convergence is compatible with the linear structure of V . If $T \in \text{Lin}(V, W)$ and if the sequence $(\mu_n)_{n \in \mathbb{N}}$ converges widely to μ , then $(T \circ \mu_n)_{n \in \mathbb{N}}$ converges widely to $T \circ \mu$.

A classical result is the lower semicontinuity of the mass under weak convergence.

Proposition 3.2.21. *Let X be a locally compact separable metric space and let $(V, |\cdot|_V)$ be a seminormed finite-dimensional space. If $(\mu_n)_{n \in \mathbb{N}}$ is a sequence of vector measures from X to V widely converging to some vector measure $\mu : \mathcal{B}(X) \rightarrow V$, then*

$$|\mu|_V(X) \leq \liminf_{n \rightarrow \infty} |\mu_n|_V(X).$$

Proof. Let $\psi \in C_c(X, V')$ and assume that $|\psi|_{V'} \leq 1$ everywhere in X . By proposition 3.2.7 for every $n \in \mathbb{N}$,

$$\int_X \langle \psi, d\mu_n \rangle \leq |\mu_n|_V,$$

and thus, in view of definition 3.2.20

$$\int_X \langle \psi, d\mu \rangle = \lim_{n \rightarrow \infty} \int_X \langle \psi, d\mu_n \rangle \leq \liminf_{n \rightarrow \infty} |\mu_n|_V.$$

We conclude by proposition 3.2.7. □

In order to have an analogue for the range, we define the Kuratowski lower limit of a sequence of set.

Definition 3.2.22. The Kuratowski lower limit of a sequence $(C_n)_{n \in \mathbb{N}}$ of subsets of V is defined as

$$\liminf_{n \rightarrow \infty} C_n \triangleq \left\{ \lim_{n \rightarrow \infty} x_n : \begin{array}{l} \text{the sequence } (x_n)_{n \in \mathbb{N}} \text{ converges in } V \\ \text{and for every } n \in \mathbb{N}, x_n \in C_n \end{array} \right\}.$$

By a diagonal argument, the Kuratowski lower limit $\liminf_{n \rightarrow \infty} C_n$ is always a closed set. Moreover, if C_n is convex for every $n \in \mathbb{N}$, then the Kuratowski lower limit $\liminf_{n \rightarrow \infty} C_n$ is convex. Similarly, if for every $n \in \mathbb{N}$, $0 \in C_n$, then $0 \in \liminf_{n \rightarrow \infty} C_n$.

Proposition 3.2.23 (Weak lower-semi continuity of the range). *If $(\mu_n)_{n \in \mathbb{N}}$ is a sequence of vector measures from X to V that converges widely to some vector measure $\mu : \mathcal{B}(X) \rightarrow V$, then*

$$\text{rg } \mu(X) \subseteq \liminf_{n \rightarrow \infty} \text{rg } \mu_n(X).$$

Moreover, for every $\varepsilon > 0$, if $n \in \mathbb{N}$ is large enough

$$\text{rg } \mu(X) \subseteq \text{rg } \mu_n(X) + \overline{B(0, \varepsilon)}.$$

In general, the inclusion is strict. Indeed if the space X is not discrete and $V \neq \{0\}$, there exists a point $x \in X$ and a sequence $(x_n)_{n \in \mathbb{N}}$ in $X \setminus \{x\}$ converging to x . If $v \in V \setminus \{0\}$, the sequence $(v\delta_x - v\delta_{x_n})_{n \in \mathbb{N}}$ converges widely to 0, but the range of any of its terms is $[-v, v]$ while the range of the limit is 0, so that the inclusion is strict.

Proof of proposition 3.2.23. For every $\varphi \in C_c(X, [0, 1])$, we observe that for every $n \in \mathbb{N}$, in view of proposition 3.2.14

$$\lim_{n \rightarrow \infty} \int_X \varphi d\mu_n \in \text{rg } \mu_n(X)$$

Hence in view of definitions 3.2.20 and 3.2.22, we have

$$\int_X \varphi d\mu = \lim_{n \rightarrow \infty} \int_X \varphi d\mu_n \in \liminf_{n \rightarrow \infty} \text{rg } \mu_n(X).$$

Since the Kuratowski lower limit $\liminf_{n \rightarrow \infty} \text{rg } \mu_n(X)$ is closed and convex, the conclusion follows from proposition 3.2.14.

For the additional assertion, let $\varepsilon > 0$. Since the set $\text{rg } \mu(X)$ is compact, there exists a finite set $F \subseteq \text{rg } \mu(X)$ such that

$$\text{rg } \mu(X) \subseteq F + \overline{B(0, \varepsilon/2)}. \quad (3.2.4)$$

Since the set F is finite, by the first part of the proof, when $n \in \mathbb{N}$ is large enough

$$F \subset \text{rg } \mu_n(X) + \overline{B(0, \varepsilon/2)} \quad (3.2.5)$$

and thus by (3.2.4) and (3.2.5) we have

$$\text{rg } \mu(X) \subseteq \text{rg } \mu_n(X) + \overline{B(0, \varepsilon)}. \quad (3.2.6)$$

□

3.3 Convergence of the total variation and of the range

We consider in this section the relationship, for widely converging sequences between convergence of the range and convergence of the total variation.

3.3.1 Range convergence implies mass convergence

If a sequence of vector measures converges widely and if its ranges converge, then its total variations converge.

Proof of theorem 3.1.1. We fix $\varepsilon \in (0, 1)$ and we let $\eta_1, \dots, \eta_m \in V'$ and $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ be given by lemma 3.2.19, so that

$$\begin{aligned} & (1 - \varepsilon)|\mu_n|_V(X) \\ & \leq (1 + \varepsilon)|\mu|_V(X) + \sum_{i=1}^m |\alpha_i| |\langle \eta_i, \mu \rangle|_V(X) - \alpha_i |\langle \eta_i, \mu_n \rangle|_V(X). \end{aligned}$$

We observe that for every $n \in \mathbb{N}$ and every $i \in \{1, \dots, m\}$,

$$\begin{aligned} & |\text{diam}(\langle \eta_i, \text{rg } \mu_n(X) \rangle) - \text{diam}(\langle \eta_i, \text{rg } \mu(X) \rangle)| \\ & \leq 2 |\eta_i|_{V'} d_{\mathfrak{H}}(\text{rg } \mu_n(X), \text{rg } \mu(X)). \end{aligned}$$

By proposition 3.2.21 and by proposition 3.2.15, we have, as $n \rightarrow \infty$,

$$\begin{aligned} |\langle \eta_i, \mu_n \rangle|_V(X) &= \text{diam}(\langle \eta_i, \text{rg } \mu_n(X) \rangle) \\ &\rightarrow \text{diam}(\langle \eta_i, \text{rg } \mu(X) \rangle) = |\langle \eta_i, \mu \rangle|_V(X). \end{aligned}$$

If $\varepsilon < 1$, it follows that

$$\limsup_{n \rightarrow \infty} |\mu_n|_V(X) \leq \frac{1 + \varepsilon}{1 - \varepsilon} |\mu|_V(X).$$

The conclusion follows by letting $\varepsilon \rightarrow 0$. $\square$

3.3.2 One-dimensional case

When $\dim V = 1$, the convergence of the range is equivalent to the convergence of the total variation.

Proposition 3.3.1. *Let X be a locally compact separable metric space and let V be a one-dimensional space. Let $(\mu_n)_{n \in \mathbb{N}}$ be sequence from X to V converging widely to some vector measure μ . If*

$$\limsup_{n \rightarrow \infty} |\mu_n|(X) \leq |\mu|(X),$$

then the sequence $(\text{rg } \mu_n(X))_{n \in \mathbb{N}}$ converges to $\text{rg } \mu(X)$ in Hausdorff distance.

Proof. Let $\varepsilon > 0$. By proposition 3.2.23, if $n \in \mathbb{N}$ is large enough, we have

$$\text{rg } \mu(X) \subseteq \text{rg } \mu_n(X) + \overline{B(0, \varepsilon)}. \quad (3.3.1)$$

It follows then that

$$\begin{aligned} \text{rg } \mu_n(X) &\subseteq \text{rg } \mu(X) \\ &\quad + \overline{B(0, (\varepsilon + \text{diam}(\text{rg } \mu_n(X)) - \text{diam}(\text{rg } \mu(X)))_+)} . \end{aligned} \quad (3.3.2)$$

The conclusion follows from (3.3.1) and (3.3.2) since $\varepsilon > 0$ is arbitrary. $\square$

A corollary of the one dimensional case described above in proposition 3.3.1 is the following equivalence criterion for range convergence, in

terms of masses of projection measures:

Corollary 3.3.2. *Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of vector measures. The sequence $(\text{rg } \mu_n(X))_{n \in \mathbb{N}}$ converges to $\text{rg } \mu(X)$ in Hausdorff distance if, and only if, for every $\eta \in V'$,*

$$\limsup_{n \rightarrow \infty} |\langle \eta, \mu_n \rangle|(X) \leq |\langle \eta, \mu \rangle|(X).$$

Proof. Let $\varepsilon > 0$. For every $\eta \in V'$, the sequence $(\langle \eta, \mu_n \rangle)_{n \in \mathbb{N}}$ of signed measures converges widely to the signed measure $\langle \eta, \mu \rangle$. By proposition 3.3.1, we have then

$$\lim_{n \rightarrow \infty} d_{\mathfrak{H}}(\text{rg } \langle \eta_i, \mu_n \rangle(X), \text{rg } \langle \eta_i, \mu \rangle(X)) = 0. \quad (3.3.3)$$

Since the set $\text{rg } \mu(X) \subset V$ is compact and convex, there exist $\eta_1, \dots, \eta_m \in V'$ and $\delta > 0$ such that

$$\text{rg } \mu(X) \subseteq \{v \in V : \text{for each } i \in \{1, \dots, m\}, \langle \eta_i, v \rangle \leq 1\} \quad (3.3.4)$$

and

$$\begin{aligned} \{v \in V : \text{for each } i \in \{1, \dots, m\} \langle \eta_i, v \rangle \leq 1 + \delta\} \\ \subseteq \text{rg } \mu(X) + \overline{B(0, \delta)}. \end{aligned} \quad (3.3.5)$$

In particular, if $n \in \mathbb{N}$ is large enough, we have by proposition 3.2.13 and by (3.3.3) and (3.3.4),

$$\sup \langle \eta_i, \text{rg } \mu_n(X) \rangle = \sup \text{rg } \langle \eta_i, \mu_n \rangle(X) \leq 1 + \delta,$$

and thus by (3.3.5)

$$\text{rg } \mu_n(X) \subseteq \text{rg } \mu(X) + \overline{B(0, \varepsilon)}. \quad (3.3.6)$$

We conclude the first implication by combining the inclusion (3.3.6) with proposition 3.2.23. The converse claim of the proposition follows from the fact that range convergence always implies mass convergence, and

$$|\langle \eta, \text{rg } \mu(X) \rangle| = |\mu|_{\tilde{V}},$$

where $\tilde{V}$ is the vector space V endowed with the degenerate norm

$$|x|_{\tilde{V}} = |(x, \eta)|. \quad \square$$

3.3.3 Higher-dimensional case

In this section, we prove an equivalence result between convergence of the range and convergence of the total mass, when the mass is taken with respect to any strictly convex norm.

A seminorm $|\cdot|_V$ is *strictly convex* whenever if the vectors $v, w \in V$ are linearly independent, then

$$|v + w|_V < |v|_V + |w|_V.$$

As claimed in proposition 3.1.2 stated in the introduction, strict convexity of the norm is necessary to enforce the convergence of ranges. The proof of proposition 3.1.2 we propose is based on an straightforward counter-example involving Dirac masses:

Proof of proposition 3.1.2. Since by our assumption the seminorm $|\cdot|_V$ is not strictly convex, there exist two linearly independent vectors $v, w \in V \setminus \{0\}$ such that $|v + w|_V = |v|_V + |w|_V$. Since the space X is not discrete, there exists a point $x \in X$ and a sequence of points $(x_n)_{n \in \mathbb{N}}$ in $X \setminus \{x\}$ that converges to x . We define for each $n \in \mathbb{N}$, the vector measure

$$\mu_n \triangleq v \delta_x + w \delta_{x_n}.$$

The sequence $(\mu_n)_{n \in \mathbb{N}}$ then widely converges to the measure $\mu = (v + w) \delta_x$ and we have for each $n \in \mathbb{N}$

$$|\mu|_V(X) = |v + w|_V = |v|_V + |w|_V = |\mu_n|_V(X).$$

On the other hand the range of $\text{rg } \mu_n(X) = \text{conv } \{0, v, w, v + w\}$ while $\text{rg } \mu(X) = \text{conv } \{0, v + w\}$ corresponds to the diagonal of the parallelogram $\text{rg}((X) \mu_n)$. Since the vectors v and w are linearly independent, the sequence of ranges does not converge. $\square$

In particular, by proposition 3.1.2 and proposition 3.3.1 there exists

a linear map $T : V \rightarrow \mathbb{R}$ such that the sequence $(|T \circ \mu_n|(\Omega))_{n \in \mathbb{N}}$ does not converge to $|T \circ \mu|(\Omega)$. That is, in contrast with wide convergence and with convergence of ranges, the convergence of total variations is not invariant under linear changes of variables.

Theorem 3.3.3. *Assume that X is a locally compact separable metric space, and let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of vector measures weakly converging to some vector measure μ . If $|\cdot|_V$ is a strictly convex norm on V such that*

$$\lim_{n \rightarrow \infty} |\mu_n|_V(X) = |\mu|_V(X),$$

then the sequence of ranges $(\text{rg } \mu_n(X))_{n \in \mathbb{N}}$ converges to $\text{rg } \mu(X)$ in Hausdorff distance.

The proof follows lines of thought from the work of Reshetnyak [16, Theorem 3], Since the approach requires some tools from measure theory, we repeat the proof with explanations on how the tools are used, and we chain the argument with corollary 3.3.2 to obtain Hausdorff convergence of the ranges.

Proof of theorem 3.3.3. For all $n \in \mathbb{N}$, let χ_{μ_n} be the *indicator measure* [16] of $|\mu|_V$, defined as the unique measure

$$\chi_\mu : \mathcal{B}(X \times \mathcal{S}) \rightarrow [0, +\infty)$$

acting on any continuous function $\varphi \in C(X \times \mathcal{S}, \mathbb{R})$ as

$$\int_{X \times \mathcal{S}} \varphi(x, e) d\chi_\mu(x, e) = \int_X \varphi\left(x, \frac{d\mu_n}{d|\mu|_V}(x)\right) d|\mu|_V(x).$$

Since we are working with vector measures, and because the ranges and the masses remain uniformly bounded for $n \in \mathbb{N}$, we may pass to subsequences and assume without loss of generality that the sequence $(\chi_{\mu_n})_{n \in \mathbb{N}}$ weakly converges to some measure ρ . As in [16, Theorem 3], we are going to prove that the limit ρ actually equals χ_μ , the indicator function of $|\mu|_V$. To that purpose, we use [16, Theorem 1] to obtain a layerwise decomposition (ω, λ) of the indicator measure ρ [16], in the sense that $\omega : X \rightarrow [0, +\infty)$ is a positive measure and $\lambda : X \rightarrow \mathcal{M}(\mathcal{B}(\mathcal{S}))$

is a function such that for every $\varphi \in C(X \times \mathcal{S})$,

$$\int_{X \times \mathcal{S}} \varphi(x, e) \, d\rho(x, e) = \int_X \int_{\mathcal{S}} \varphi(x, e) \, d\lambda(x, e) \, d\omega(x).$$

Moreover, for this layerwise decomposition, the function

$$x \in X \mapsto u(x) = \int_{\mathcal{S}} e \, d\lambda(x, e)$$

is $\mathcal{B}(X)$ -measurable and for every $A \in \mathcal{B}(X)$,

$$\int_A u(x) \, d\omega(x) = \int_A \int_{\mathcal{S}} e \, d\lambda(x, e) \, d\omega(x) = \mu(A). \quad (3.3.7)$$

The existence of such decomposition with relation (3.3.7) only uses the weak convergence assumption $\mu_n \rightharpoonup \mu$. By convexity of the norm, we have

$$|u(x)|_V \leq \int_{\mathcal{S}} d\lambda(x, e) \leq 1. \quad (3.3.8)$$

Those inequality follow from [16, Theorem 1], where it is proved that $\lambda(x, \mathcal{S}) = 1$ for ω -almost-all $x \in X$. Integrating left sides of the above inequality (3.3.8) with respect to the measure ω also yields

$$\int_X |u(x)|_V \, d\omega(x) \leq \int_X \int_{\mathcal{S}} d\lambda(x, e) \, d\omega(x) = \int_{X \times \mathcal{S}} d\rho(x, e),$$

and by construction of ρ :

$$\int_{X \times \mathcal{S}} d\rho(x, e) = \lim_{n \rightarrow \infty} \int_{X \times \mathcal{S}} d\chi_{\mu_n}(x, e) = \lim_{n \rightarrow \infty} |\mu_n|_V(X) = |\mu|_V(X).$$

Thus

$$\int_X |u(x)|_V \, d\omega(x) \leq |\mu|_V.$$

For $A \in \mathcal{B}(X)$ an arbitrary Borel set, we observe that

$$\int_A \frac{d\mu}{d|\mu|_V} \, d|\mu|_V = \mu(A) = \int_A u \, d\omega.$$

Considering the Radon-Nikodym derivatives of $|\mu|_V$ and ω with respect

to the measure $\sigma \triangleq (|\mu|_V + \omega)$, we obtain

$$\int_A \frac{d\mu}{d|\mu|_V} \frac{d|\mu|_V}{d(\omega + |\mu|_V)} d\sigma = \int_A u \frac{d\omega}{d(\omega + |\mu|_V)} d\sigma,$$

and since the set $A \in \mathcal{B}(X)$ is arbitrary, we obtain

$$u \frac{d\omega}{d(\omega + |\mu|_V)} = \frac{d\mu}{d|\mu|_V} \frac{d|\mu|_V}{d(\omega + |\mu|_V)} \quad (3.3.9)$$

$(|\mu|_V + \omega)$ -almost-everywhere on X . Using the fact that ω and $|\mu|_V$ are positive measures, we take norms on both sides of (3.3.9) and we integrate both sides with respect to σ to obtain

$$\int_X |u(x)|_V d\omega(x) \int_X \left| \frac{d\mu}{d|\mu|_V}(x) \right|_V d|\mu|_V(x) = |\mu|_V(X)$$

and equality holds in the inequality (3.3.8): we have $|u(x)|_V = 1$ for ω -almost-all $x \in X$. In particular, $\omega = |\mu|_V$ and $u = \frac{d\mu}{d|\mu|_V}$ on X .

The strict convexity of the norm $|\cdot|_V$ may be used to observe that in fact, $\lambda(x, \cdot)$ is a Dirac mass at $u(x)$, for $|\mu|_V$ -almost-all $x \in X$. Indeed, this follows from the fact that

$$\left| \int_S e d\lambda(x, e) \right|_V = \int_S |e|_V d\lambda(x, e),$$

which is possible if and only if $\lambda(x, \cdot)$ is a multiple of a Dirac mass. The concentration point should thus be $u(x)$, and the multiple is 1.

Now we fix an arbitrary direction $\eta \in \mathcal{S}$. Considering again Radon-Nikodym derivatives, it is straightforward to check the identity

$$|\langle \mu, \eta \rangle|(X) = \int_X \left| \left\langle \frac{d\mu}{d|\mu|_V}(x), \eta \right\rangle \right| d|\mu|_V(x),$$

or equivalently

$$|\langle \mu, \eta \rangle|(X) = \int_X |\langle u(x), \eta \rangle| d\omega(x).$$

Using the fact that $\lambda(x, \cdot)$ is a Dirac mass at $u(x)$, the inner absolute value commutes the integral to give

$$\begin{aligned} \int_X \left| \langle u(x), \eta \rangle \right| d\omega(x) &= \int_X \int_{\mathcal{S}} |\langle e, \eta \rangle| d\lambda(x, e) d\omega(x) \\ &= \int_{X \times \mathcal{S}} |\langle e, \eta \rangle| d\rho(x, e). \end{aligned}$$

Now we always have

$$|\langle \mu_n, \eta \rangle|(X) = \int_X \left| \left\langle \frac{d\mu}{d|\mu|_V}(x), \eta \right\rangle \right| d|\mu|_V(x) = \int_{X \times \mathcal{S}} |\langle e, \eta \rangle| d\chi_{\mu_n}(x, e),$$

and therefore, by definition of ρ , we obtain

$$\lim_{n \rightarrow \infty} |\langle \mu_n, \eta \rangle|(X) = |\langle \mu, \eta \rangle|(X).$$

Since this holds for all $\eta \in \mathcal{S}$, we obtain Hausdorff convergence of the range, by corollary 3.3.2. $\square$

Theorem 3.1.1 and theorem 3.3.3 imply immediately theorem 3.1.3.

3.4 Zonal representation of masses

In this section, we develop a geometric approach to manage the relation between range convergence and mass convergence. Although the geometric approach will give much less results when $\dim(V) \geq 3$, it provides an interesting geometric point of view of the relation between mass convergence and strict convergence when $\dim(V) = 2$.

3.4.1 Zonal representation formula

We will show that seminorms having a zonal representation have an associated mass that has an integral representation.

Definition 3.4.1. A locally finite non-negative Borel measure

$$\sigma : \mathcal{B}(V') \rightarrow [0, +\infty]$$

is a *zonal representation* of a seminorm $|\cdot|_V$ whenever

$$\int_{V'} |\eta|_{V'} d\sigma(\eta) < +\infty,$$

and for every $v \in V$,

$$|v|_V = \int_{V'} |\langle \eta, v \rangle| d\sigma(\eta).$$

As an example, we can consider the standard Euclidean norm in $\mathbb{R}^2$ and the uniform measure ν on the unit circle $\mathcal{S} \subset \mathbb{R}^2$, giving

$$\int_{\mathcal{S}} |\langle v, e \rangle| d\rho(e) = \frac{|v|_2}{2\pi} \int_0^{2\pi} |\cos(\theta)| d\theta = |v|_2 \frac{2}{\pi},$$

and the zonal representation holds with $\sigma = \frac{\pi}{2}\rho$.

More generally, every Euclidean norm admits a zonal representation.

Theorem 3.4.2 (Representation of Euclidean norms). *If $|\cdot|_V$ is a Euclidean semi-norm on V , then it admits a zonal representation.*

Proof. Up to a linear isometry and an orthogonal projection, we can assume that $V = \mathbb{R}^d$ and $|\cdot|_V = |\cdot|_2$ is the standard Euclidean norm. Then we have

$$|v|_2 = \alpha \int_{\mathcal{S}} |\langle v, e \rangle| d\sigma(e),$$

where σ is the uniform surface measure on the Euclidean sphere $\mathcal{S}$, and $\alpha = \alpha(d) > 0$ is a normalization constant. The above identity directly follows from the fact that the uniform surface measure is the unique (up to multiplication constants) Borel measure on $\mathcal{S}$ that is stable under isometries of $\mathbb{R}^d$. $\square$

A norm $|\cdot|_V$ has a zonal representation if and only if the associated unit ball in V^* is a *zonoid*, which can be defined as the ranges of vector measure [2, Theorem 2.1; 4].

If V is a two-dimensional space, every norm has a zonal representation in the sense of definition 3.4.1 ([2, Theorem 5.4; 4, 41.2]). This follows from the fact that, in the plane $\mathbb{R}^2$, every symmetric compact convex set may be approximated by a symmetric convex polygon which is automatically a zonotope in $\mathbb{R}^2$.

For example, the usual Euclidean norm is represented by the uniform measure on the dual sphere $\mathcal{S}$ of V' since the uniform measure is the only (up to multiplicative constants) invariant measure under rotations

and thus, balls induced by its associated norm are Euclidean balls. This argument is also valid in higher dimensions.

Another example is given by the ℓ^1 -norm on $\mathbb{R}^d$ defined for $v \in \mathbb{R}^d$ by

$$|v|_\infty \triangleq \sum_{i=1}^d |v_i|,$$

which can be represented with $\sigma = \sum_{i=1}^d \delta_{\eta_i}$, where $\eta_i(v_1, \dots, v_d) = v_i$.

In higher dimensions $\dim V \geq 3$, there are some norms that do not have a zonal representation. If $v = (v_1, \dots, v_d) \in \mathbb{R}^d$, then

$$|v|_\infty = \max\{|v_1|, \dots, |v_d|\}.$$

has no zonal representation of the form definition 3.4.1 by any positive measure σ . Indeed, if such measure σ existed, then its dual ball, which is the ℓ^1 -ball, should be a zonotope. However, the 1-ball is a hyperoctahedron, which has some 2-dimensional that are triangles, and is thus not a zonotope [2, Theorem 3.3].

The above example is not related with the fact that the ℓ^1 and ℓ^∞ norms are not strictly convex. In fact, there are also strictly convex norms that do not have a zonal representation definition 3.4.1. Indeed, in $\mathbb{R}^3$, the p -norm does not have a zonal representation, if $p \geq \frac{\log 3}{\log \frac{3}{2}}$, and for every $p > 2$ it is known that the p -norm does not have a zonal representation is the dimension is large enough [2, (32); 23, Proposition 2].

Remark 3.4.3. If $\sigma' : \mathcal{B}(V') \rightarrow [0, +\infty)$ is a measure such that for every Borel cone $C \in \mathcal{B}(V')$, $\sigma'(C) = \sigma(C)$, then for every $v \in V$,

$$\int_{V'} |\langle \eta, v \rangle| d\sigma'(\eta) = \int_{V'} |\langle \eta, v \rangle| d\sigma(\eta)$$

and for every vector measure $\mu : \Sigma \rightarrow V$,

$$\int_{V'} |\langle \eta, \mu \rangle|(A) d\sigma'(\eta) = \int_{V'} |\langle \eta, \mu \rangle|(A) d\sigma(\eta).$$

In particular, the measure σ satisfying the assumption of lemma 3.4.4 is not unique; a canonical representative can be given by requiring sigma

to be an even measure supported on the unit sphere of V' .

The existence of zonal representation of the norm in V automatically implies a similar representation formula for the mass. It may be seen as an alternative way of representing the mass, without relying on layerwise decompositions.

Lemma 3.4.4 (Representation of the mass). *If $\sigma : \mathcal{B}(V') \rightarrow \mathbb{R}^+$ is a zonal representation of $|\cdot|_V$, then for every σ -algebra Σ on X , every vector measure $\mu : \Sigma \rightarrow V$ and every $A \in \Sigma$,*

$$|\mu|_V(A) = \int_{V'} |\langle \eta, \mu \rangle|(A) d\sigma(\eta).$$

Proof. If $E_1, \dots, E_m \in \Sigma$ are disjoint subsets of A , we have

$$\sum_{i=1}^m |\mu(E_i)|_V = \int_{V'} \sum_{i=1}^m |\langle \eta, \mu(E_i) \rangle| d\sigma(\eta) \leq \int_{V'} |\langle \eta, \mu \rangle|(A) d\sigma(\eta)$$

and thus by the definition of total variation of a vector measure (definition 3.2.1)

$$|\mu|_V(A) \leq \int_{V'} |\langle \eta, \mu \rangle|(A) d\sigma(\eta). \quad (3.4.1)$$

Conversely, in view of remark 3.4.3 we can assume that the measure σ is supported in a compact set $K \subset V' \setminus \{0\}$. In particular σ is a finite measure. Let $\varepsilon > 0$, there exists $\eta_1, \dots, \eta_m \in K$ such that

$$K \subset \bigcup_{i=1}^m \overline{B_{V'}(\eta_i, \varepsilon)}.$$

In particular, if $\eta \in \overline{B_{V'}(\eta_i, \varepsilon)}$ and $E \in \Sigma$, we have

$$||\langle \eta, \mu \rangle|(E) - |\langle \eta_i, \mu \rangle|(E)| \leq \varepsilon |\mu|_V(E). \quad (3.4.2)$$

We decompose $K = \bigcup_{i=1}^\ell K_i$, with $K_1, \dots, K_\ell \in \mathcal{B}(V')$ pairwise disjoint and $K_i \subset \overline{B_{V'}(\eta_i, \varepsilon)}$. We have then by (3.4.2)

$$\int_{V'} |\langle \eta, \mu \rangle|(A) d\sigma(\eta) \leq \sum_{i=1}^d \sigma(K_i) |\langle \eta_i, \mu \rangle|(A) + \varepsilon \sigma(K) |\mu|_V(A). \quad (3.4.3)$$

Since $\langle \eta_i, \mu \rangle$ is a signed measure, by the Hahn–Jordan decomposition theorem and a common refinement, there exists a partition $E_1, \dots, E_m \in \Sigma$ of A such that for each $i \in \{1, \dots, \ell\}$,

$$|\langle \eta_i, \mu \rangle|(A) = \sum_{n=1}^m |\langle \eta_i, \mu \rangle(E_n)|. \quad (3.4.4)$$

We obtain thus by (3.4.2) again combined with (3.4.3) and (3.4.4)

$$\begin{aligned} \int_{V'} |\langle \eta, \mu \rangle|(A) d\sigma(\eta) \\ \leq \sum_{n=1}^m \int_{V'} |\langle \eta, \mu \rangle(E_n)| d\sigma(\eta) + \varepsilon \sigma(K) \left(|\mu|_V(A) + \sum_{n=1}^m |\mu|_V(E_n) \right). \end{aligned}$$

By our assumption on the seminorm this implies in turn by our assumption and by additivity of the total mass (proposition 3.2.4) and by the definition of total variation (definition 3.2.1) that

$$\begin{aligned} \int_{V'} |\langle \eta, \mu \rangle|(A) d\sigma(\eta) &\leq \sum_{n=1}^m |\mu(E_n)|_V + 2\varepsilon \sigma(K) |\mu|_V(A) \\ &\leq |\mu|_V(A) + 2\varepsilon \sigma(K) |\mu|_V(A). \end{aligned}$$

We conclude by letting $\varepsilon \rightarrow 0$ that

$$\int_{V'} |\langle \eta, \mu \rangle|(A) d\sigma(\eta) \leq |\mu|_V(A). \quad (3.4.5)$$

The conclusion then follows from (3.4.1) and (3.4.5). $\square$

3.4.2 Application to monotonicity and convergence

When V is equipped with an Euclidean norm the monotony of the mass with respect to range inclusion, theorem 3.1.4 stated in the introduction, is a direct consequence of the zonal representation, lemma 3.4.4.

Because of the particular structure of two-dimensional vector spaces, the proof we give also holds for any norm on V whenever $\dim V = 2$.

Proof of theorem 3.1.4. Given two vector measures μ, ν such that

$$\operatorname{rg} \mu(X) \subseteq \operatorname{rg} \nu(X),$$

we clearly have, for all $\eta \in \mathcal{S}$:

$$\langle \eta, \operatorname{rg} \mu(X) \rangle \subseteq \langle \eta, \operatorname{rg} \nu(X) \rangle.$$

Since the range commutes with linear maps, we directly obtain, for all $\eta \in \mathcal{S}$:

$$\operatorname{rg} \langle \eta, \mu \rangle(X) \subseteq \operatorname{rg} \langle \eta, \nu \rangle(X).$$

Since the measures $\langle \eta, \mu \rangle$ and $\langle \eta, \nu \rangle$ are real-valued, we obtain

$$|\langle \eta, \mu \rangle| \leq |\langle \eta, \nu \rangle|,$$

for all $\eta \in \mathcal{S}$. Since V is Euclidean or $\dim V = 2$, this implies, by lemma 3.4.4, $|\mu|_V(X) \leq |\nu|_V(X)$. If the norm is strictly convex, then the inequality is strict. $\square$

Relying on the integral and perimeter formulae obtained above, we also propose an alternative proof of theorem 3.1.3.

We first draw a link between the support of a zonal representation and strict convexity. The above characterization will turn out to be particularly efficient in dimension 2.

Definition 3.4.5. A zonal representation $\sigma : \mathcal{B}(V') \rightarrow [0, +\infty]$ of a seminorm $|\cdot|_V$ is *non-degenerate* whenever for every non-empty double cone $C \subset V$, $\sigma(C) > 0$.

We recall that the set $C \subset V'$ is an *open cone* whenever for every $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in C$, $\lambda C = C$.

In dimension 2, the non-degeneracy of the zonal representation is equivalent to strict convexity. This will turn out to be false in higher dimensions.

Proposition 3.4.6. Let V be a finite-dimensional space. Assume that σ is a zonal representation of the seminorm $|\cdot|_V$. The norm $|\cdot|_V$ is strictly convex if and only if for every linearly independent vectors $v, w \in V$,

there exists $\eta \in \text{supp}(\sigma)$ such that $\langle \eta, v \rangle$ and $\langle \eta, w \rangle$ do not have the same sign.

In particular, if V is a 2-dimensional vector space, then the norm is strictly convex if, and only if, σ is non-degenerate.

Proof of proposition 3.4.6. We observe that for every $\eta \in V'$ and $v \in V$, we have

$$|\langle \eta, v \rangle + \langle \eta, w \rangle| \geq |\langle \eta, v \rangle| + |\langle \eta, w \rangle|,$$

with equality if and only if the numbers $\langle \eta, v \rangle$ and $\langle \eta, w \rangle$ have the same sign. We have

$$\begin{aligned} |v + w|_V &= \int_{V'} |\langle \eta, v \rangle + \langle \eta, w \rangle| d\sigma(\eta) \\ &\geq \int_{V'} |\langle \eta, v \rangle| d\sigma(\eta) + \int_{V'} |\langle \eta, w \rangle| d\sigma(\eta) = |v|_V + |w|_V, \end{aligned}$$

with equality if and only if $\langle \eta, v \rangle$ and $\langle \eta, w \rangle$ have the same sign for every $\eta \in \text{supp } \sigma$, since the map $z \in V' \mapsto \langle \eta, z \rangle$ is continuous.

In particular, if σ is a non-degenerate zonal representation, then the norm $|\cdot|_V$ is strictly convex. In the two-dimensional case $\dim V = 2$, the separation condition of proposition 3.4.6 on the support conversely implies the density condition on the support. Indeed, for every $\eta \in V'$ and every $\delta > 0$, there exists $v, w \in V$ such that if $\eta' \in V'$ and $\langle \eta', v \rangle \leq 0 \leq \langle \eta', w \rangle$, then $|\eta - \eta'|_{V'} \leq \delta$. $\square$

Density of the support for a zonal representation of a strictly convex norm in higher dimensions $\dim V \geq 3$ fails in general. As an example, one can consider $V = \mathbb{R}^3$ and the measure $\sigma = \sigma_{x,y} + \sigma_{x,z} + \sigma_{z,y}$, where $\sigma_{x,y}$ is the uniform measure on the circle orthogonal to the y -axis; and similarly for $\sigma_{x,z}$ and $\sigma_{z,y}$. Then the support $\text{supp}(\sigma)$ is far from being dense in $\mathcal{S}$, but the induced norm

$$\int_{\mathcal{S}} |\langle \eta, v \rangle| d\sigma(\eta) = \sqrt{v_1^2 + v_2^2} + \sqrt{v_1^2 + v_3^2} + \sqrt{v_2^2 + v_3^2}.$$

is strictly convex.

We now give another proof of theorem 3.1.3 by mean of zonal integral

representation. We restrict ourselves to the case of Euclidean norms in the case $\dim V \geq 3$, and strictly convex norms in the case $\dim V = 2$.

Second proof of theorem 3.1.3, by zonal representation. As the converse statement holds in all generality, we shall only need to prove that mass convergence implies convergence of the range.

For every $\eta \in V'$, the sequence $(\langle \eta, \mu_n \rangle)_{n \in \mathbb{N}}$ converges widely to $(\langle \eta, \mu \rangle)$ and by proposition 3.2.21, we have

$$\liminf_{n \rightarrow \infty} |\langle \eta, \mu_n \rangle|(X) \geq |\langle \eta, \mu \rangle|(X). \quad (3.4.6)$$

Since $|\cdot|_V$ is a Euclidean norm, there exists a constant $\kappa > 0$ such that for every $v \in V$,

$$|v|_V = \kappa \int_{\mathcal{S}} |\langle \eta, v \rangle| d\eta \quad (3.4.7)$$

where $\mathcal{S}$ is the unit sphere in $V \simeq V'$. By lemma 3.4.4 and by our assumption, we have

$$\limsup_{n \rightarrow \infty} \int_{\mathcal{S}} |\langle \eta, \mu_n \rangle|(X) d\eta \leq \int_{\mathcal{S}} |\langle \eta, \mu \rangle|(X) d\eta. \quad (3.4.8)$$

For every $\eta_1, \eta_2 \in V'$ and $n \in \mathbb{N}$, we have

$$||\langle \eta_2, \mu_n \rangle|(X) - |\langle \eta_1, \mu_n \rangle|(X)| \leq |\eta_2 - \eta_1|_{V'} |\mu_n|_V(X). \quad (3.4.9)$$

By (3.4.6), (3.4.8) and (3.4.9), we deduce that for every $\eta \in \mathbb{S}_{V'}$, we have

$$\lim_{n \rightarrow \infty} |\langle \eta, \mu_n \rangle|(X) = |\langle \eta, \mu \rangle|(X). \quad (3.4.10)$$

By homogeneity, this still holds for every $\eta \in V'$ and the conclusion follows then from corollary 3.3.2. $\square$

3.5 Perimeter representation of masses

3.5.1 Perimeter formula

So far, we have used zonal representation of norms in spaces of dimension 2 to approach the problem of mass convergence from a geometric point of view. In particular, we were able to characterize strict convexity of the

norm through density of the support for the measure representing the norm.

We now give further geometric insight of the problem, by interpreting the total mass of a vector valued measure as the perimeter of its range (up to some multiplicative constant).

We first define the perimeter of a polygon in a two-dimensional space.

Definition 3.5.1. If $\dim V = 2$ and $P \subset V$ if P is a polygon of vertices $v_1, \dots, v_\ell$, we define its perimeter as

$$\text{Per}_V(P) \triangleq \sum_{i=1}^{\ell} |v_\ell - v_{\ell-1}|_V,$$

with the convention that $v_0 = v_\ell$.

This notion of perimeter is analogous to the notion of perimeter in crystalline variational problems [9, 11, 22, 24].

The perimeter of convex polygons is monotone with respect to set inclusion.

Lemma 3.5.2. *If $\dim V = 2$ and $P_1 \subseteq P_2 \subset V$ are convex polygons, then*

$$\text{Per}_V(P_1) \leq \text{Per}_V(P_2).$$

Proof. Let $H \subset V$ be a half-plane such that $P_1 \subset H$ and ∂H contains one side of ∂P_1 . Then by the triangle inequality for the seminorm $\text{Per}_V(H \cap P_2) \leq \text{Per}_V(H)$. By repeating this process on each edge of the polygon P_1 , we reach the conclusion. $\square$

Lemma 3.5.2 suggests to extend the definition of perimeter to convex sets as the least upper bound of the perimeter of polygonal subsets.

Definition 3.5.3. If $C \subset V$ is convex, we set

$$\text{Per}_V(C) \triangleq \sup \{ \text{Per}_V(P) : P \subseteq C \text{ is a convex polygon} \}.$$

In view of lemma 3.5.2, lemma 3.5.2 and definition 3.5.3 coincide for polygons.

The monotonicity property of lemma 3.5.2 can be transferred to arbitrary subsets and the equality cases can be characterized for strictly convex norms.

Proposition 3.5.4. *If $C_1 \subseteq C_2 \subseteq V$ are compact convex sets, then*

$$\text{Per}_V(C_1) \leq \text{Per}_V(C_2),$$

If $|\cdot|_V$ is a strictly convex norm and if equality holds, then $C_1 = C_2$.

Proof. The inequality follows from lemma 3.5.2 and definition 3.5.3.

For the equality case, we assume by contradiction that $C_1 \subsetneq C_2$. There exists $v \in C_2 \setminus C_1$. Let $H \subset V$ be a half-plane such that $v \notin H$ and $C_1 \subset H$. In particular, for every $w_1, w_2 \in C_1$, we have since $|\cdot|_V$ is a strictly convex norm,

$$|w_2 - w_1|_V < |w_2 - v|_V + |v - w_1|_V.$$

We define

$$\eta \triangleq \inf\{|w_2 - v|_V + |v - w_1|_V - |w_2 - w_1|_V : w_1, w_2 \in C_1\}.$$

Since the set C_1 is compact and convex, we have $\eta > 0$. If $P_1 \subseteq C_1$ is a polygon, then the polygon P_2 defined as the convex hull of P_1 and v satisfies $P_2 \subseteq C_2$ and

$$\text{Per}_V(P_2) \geq \text{Per}_V(P_1) + \eta.$$

By taking the supremum, it follows from definition 3.5.3 that

$$\text{Per}_V(C_2) \geq \text{Per}_V(C_1) + \eta > \text{Per}_V(C_1). \quad \square$$

We establish theorem 3.1.6 which states that

$$|\mu|_V(X) = \frac{1}{2} \text{Per}(\text{rg } \mu(X)).$$

This formula together with lemma 3.4.4 yield a Cauchy–Crofton-like

integral formula for the perimeter: for any compact convex set $C \subseteq V$:

$$\text{Per}_V(C) = 2 \int_S \text{diam}\langle \eta, C \rangle \, d\sigma(\eta). \quad (3.5.1)$$

The proof of theorem 3.1.6 is based on the following property of polygons.

Lemma 3.5.5. *Let V be a two-dimensional vector space and $|\cdot|_V$ be a seminorm on V . For every $m \in \mathbb{N}$ and $v_1, \dots, v_m \in V$, one has*

$$\sum_{n=1}^m |v_n|_V = \frac{1}{2} \text{Per}_V([0, v_1] + \dots + [0, v_m]).$$

Lemma 3.5.5 implies in particular immediately that theorem 3.1.6 holds when μ has finite support.

Proof of lemma 3.5.5. We proceed by induction. The identity is trivial if $m = 0$. The induction step follows from the observation that

$$\frac{1}{2} \text{Per}_V([0, v_1] + \dots + [0, v_{m+1}]) = \frac{1}{2} \text{Per}_V([0, v_1] + \dots + [0, v_m]) + |v_{m+1}|_V. \quad \square$$

Proof of theorem 3.1.6. If $E_1, \dots, E_m \in \Sigma$ are disjoint subsets of A , then we have

$$\sum_{n=1}^m |\mu(E_n)|_V = \frac{1}{2} \text{Per}_V([0, \mu(E_1)] + \dots + [0, \mu(E_m)]).$$

Since $[0, \mu(E_1)] + \dots + [0, \mu(E_m)] \subset \text{rg } \mu(A)$, it follows from the definition of total variation (definition 3.2.1) that

$$|\mu|_V(A) \leq \frac{1}{2} \text{Per}_V(\text{rg } \mu(A)). \quad (3.5.2)$$

Conversely, if $P \subset \text{rg } \mu(A)$ is a polygon, there exist sets $F_1, \dots, F_k \in \Sigma$ such that $F_1, \dots, F_k \subseteq A$, $\mu(F_1), \dots, \mu(F_k)$ are the vertices of P . Hence there exists disjoint sets $E_1, \dots, E_m \in \Sigma$ such that $P \subseteq [0, \mu(E_1)] + \dots +$

$[0, \mu(E_m)]$. It follows thus by definition 3.5.3 that

$$\frac{1}{2} \text{Per}_V(P) \leq \frac{1}{2} \text{Per}_V([0, \mu(E_1)] + \cdots + [0, \mu(E_m)]) = \sum_{n=1}^m |\mu(E_n)|_V(A). \quad (3.5.3)$$

Hence by definition of total variation (definition 3.2.1),

$$\frac{1}{2} \text{Per}_V(P) \leq |\mu|_V(A)$$

and thus by definition of perimeter (definition 3.5.3)

$$\frac{1}{2} \text{Per}_V(\text{rg } \mu(A)) \leq |\mu|_V(A). \quad (3.5.4)$$

The conclusion comes from the combination of (3.5.2) and (3.5.4). $\square$

3.5.2 Application to monotonicity and convergence

Theorem 3.1.6 gives a proof of the monotonicity of the range with respect to the mass.

Second proof of theorem 3.1.4 when $\dim V = 2$. We apply theorem 3.1.6 and proposition 3.5.4. $\square$

As a final result, we give a proof of theorem 3.1.3 in the two-dimensional case, based on the perimeter formula, theorem 3.1.6.

Third proof of theorem 3.1.3 via perimeter formula. Given a compact and convex set $C \subset V$, we define the function $\gamma_C : V \rightarrow [0, +\infty)$ by

$$\gamma_C(v) \triangleq \text{Per}_V(\text{conv}(C + \{v\})). \quad (3.5.5)$$

We observe that, by the triangle inequality and the definitions, for every $v_1, v_2 \in V$,

$$|\gamma_C(v_2) - \gamma_C(v_1)| \leq 2|v_2 - v_1|_V \quad (3.5.6)$$

and that if $0 \in C$,

$$|v|_V \leq \gamma_C(v). \quad (3.5.7)$$

Let $\varepsilon > 0$. By the Weierstrass extreme value theorem and, since $|\cdot|_V$ is a strictly convex norm, by proposition 3.5.4 we have

$$\delta \triangleq \inf\{\gamma_{\text{rg } \mu(X)}(v) : \text{dist}(v, \text{rg } \mu(X)) \geq \varepsilon\} - \text{Per}_V(\text{rg } \mu(X)) > 0.$$

By (3.5.5) and definition 3.5.3 for every $v \in V$, there exists a polygon $P_v \subset \text{rg } \mu(X)$ such that

$$\gamma_{P_v}(v) \geq \gamma_{\text{rg } \mu(X)}(v) - \frac{\delta}{4}.$$

Thus if $|v' - v|_V \leq \frac{\delta}{16}$, we have by (3.5.6)

$$\gamma_{P_v}(v') \geq \gamma_{\text{rg } \mu(X)}(v') - \frac{\delta}{2}$$

and thus if $\gamma_{P_v}(v') \leq \text{Per}_V(\text{rg } \mu(X)) + \frac{\delta}{2}$, then $v' \in \text{rg } \mu(X) + \overline{B(0, \varepsilon)}$. In view of (3.5.7), by a compactness argument, there exists a polygon $P \subseteq \text{rg } \mu(X)$ such that if $\gamma_P(v) \leq \text{Per}_V(\text{rg } \mu(X)) + \frac{\delta}{2}$, then $v \in \text{rg } \mu(X) + \overline{B(0, \varepsilon)}$. By proposition 3.2.23 and definition 3.2.22, there exists a sequence of polygons $(P_n)_{n \in \mathbb{N}}$ whose vertices converge to those of P and such that for every $n \in \mathbb{N}$, $P_n \subset \text{rg } \mu_n(X)$. For every $v \in V$, we have $\lim_{n \rightarrow \infty} \gamma_{P_n}(v) = \gamma_P(v)$, and thus in view of (3.5.6), for $n \in \mathbb{N}$ large enough, if $v \in V$ satisfies $\gamma_{P_n}(v) \leq \text{Per}_V(\text{rg } \mu(X)) + \frac{\delta}{4}$, then $v \in \text{rg } \mu(X) + \overline{B(0, \varepsilon)}$.

Assume now that $v \in \text{rg } \mu_n(X)$. One has then $\text{conv } P_n + \{v\} \subset \text{rg } \mu_n(X)$, and thus if $n \in \mathbb{N}$ is large enough, we have

$$\gamma_{P_n}(v) \leq \text{Per}_V(\mu_n) \leq \text{Per}_V(\mu) + \frac{\delta}{4}.$$

By construction of P_n , this implies that $v \in \text{rg } \mu(X) + \overline{B(0, \varepsilon)}$ and thus

$$\text{rg } \mu_n(X) \subseteq \text{rg } \mu(X) + \overline{B(0, \varepsilon)}. \quad (3.5.8)$$

The conclusion then follows from (3.5.8) and proposition 3.2.23. $\square$

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Part II

Rearrangements and vortex motion in lakes

Chapter 4

Vortex motion for the lake equations

*The following chapter reflects a paper that was submitted,
available at arXiv:1901.01717.*

Collaborator: Jean Van Schaftingen.

4.1 Introduction

The *lake equations* model an incompressible inviscid flow of a fluid in a lake whose velocity varies on distances whose scale is large compared to the depth (shallow water) and is small compared to the speed of gravity waves (its *Froude number* is small: $\text{Fr} \ll 1$) [8, (1.1)]. Mathematically, the lake is represented by its projection of its volume on a horizontal planar open set $D \subseteq \mathbb{R}^2$ and by a positive depth function $b : D \rightarrow (0, +\infty)$; the *velocity field* $\mathbf{u} : \mathbb{R} \times D \rightarrow \mathbb{R}^2$ and the *surface height* $h : \mathbb{R} \times D \rightarrow \mathbb{R}$ are governed by the system of equations

$$\begin{cases} \nabla \cdot (b \mathbf{u}) = 0 & \text{on } \mathbb{R} \times D, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla h & \text{on } \mathbb{R} \times D, \\ \mathbf{u} \cdot \boldsymbol{\nu} = 0 & \text{on } \mathbb{R} \times \partial D, \end{cases} \quad (4.1.1)$$

where $\boldsymbol{\nu}$ denotes the outgoing normal vector at the boundary ∂D of the domain. The equations (4.1.1) express respectively the conservation

of mass, the conservation of momentum and the impermeability of the boundary ∂D . In particular when the depth b is constant, the lake equations (4.1.1) reduce to the two-dimensional Euler equations of inviscid incompressible flows. The lake equations (4.1.1) can be derived formally from the three-dimensional Euler equations [8] and have been justified mathematically in the periodic case [42]. They appear in the mean-field limit for the Gross–Pitaevskii equation, which is the Schrödinger flow for the Ginzburg–Landau energy, under forcing and pinning [19]. Weak solutions of the Cauchy problem for the lake equations (4.1.1) exist globally [24, 31, 33, 40, 41]; these solutions are unique [5, 31, 40] and as smooth as the data permits it ([24, 32, 41] and section B below).

The *vorticity* $\omega = \nabla \times \mathbf{u}$ of a flow governed by the lake equations (4.1.1) obeys the *vorticity equation*

$$\partial_t \omega + b \mathbf{u} \cdot \nabla \left(\frac{\omega}{b} \right) = 0 \quad \text{in } D. \quad (4.1.2)$$

For the *planar Euler equation*, corresponding to constant depth b , the vorticity equation (4.1.2) has been known since the works of Helmholtz, Kirchhoff and Routh to have singular vortex-point solutions whose vorticity is a linear combination of Dirac deltas whose position is governed by a dynamical system whose Hamiltonian is the *Kirchhoff–Routh stream function* [23, §5; 28, Zwanzigste Vörllesung, §2–3; 46, §23]. These vortex point solutions are merely distributional solutions of the Euler equations; since the works of Scheffer and Shirelman [47–49], the latter are known to exhibit unphysical behaviors in general. In a seminal work, Marchioro and Pulvirenti have proved mathematically that the singular vortex-point solutions are in fact the *limits of solutions* of the planar Euler equations whose initial data’s vorticity concentrates into Dirac masses [37].

For the lake equations (4.1.1), Richardson computed by *formal matched asymptotics* that the position $q : \mathbb{R} \rightarrow D$ of a vortex of vorticity Γ and its typical radius ε should evolve according to the law [45, (5.1)]

$$\dot{q}(t) \simeq \frac{\Gamma^2}{4\pi} \ln \frac{1}{\varepsilon} (\nabla^\perp \ln b)(q(t)), \quad (4.1.3)$$

where $\nabla^\perp \ln b = (\partial_2 \ln b, -\partial_1 \ln b)$. A similar law was derived from

axisymmetric Euler flow and verified *experimentally* for vortex dipoles moving towards a planar sloping beach [9, 43] and was tested *numerically* on barred beaches [6] in order to understand the *rip currents* which represent a hazard to swimmers. As a consequence of the law (4.1.3), vortex points should follow at the leading order the level sets of the bathymetry b . In comparison with the planar Euler equation, the velocity of a vortex depends on its radius and the dominant term is local: it interacts at the leading order neither with the boundary nor with vortices that remain at a positive distance. These formal, experimental and numerical results raise the question whether the evolution law (4.1.3) is mathematically the limiting behaviour of families of solutions to the lake equations (4.1.1).

In the *stationary case* for the lake equations, where the velocity $\mathbf{u}$ does not depend on the time t (4.1.1), there exist families of stationary solutions concentrated at a point of maximal depth or at a point where the irrotational flow generated by a boundary condition of order $\ln \frac{1}{\varepsilon}$ balances the diverging motion of (4.1.3) [14–16] (see also chapters 5, 6). (Corresponding results were already known for the planar Euler equations [3, 7, 50–52].) This approach also yields a rotating singular vortex pair in a rotation-invariant lake [14] (see chapter 5).

When $D = (0, +\infty) \times \mathbb{R}$ and $b(r, z) = r$, the lake equations (4.1.1) is in fact the axisymmetric three-dimensional Euler equation. A single vortex ring is known formally to evolve according to (4.1.3) since the work of Helmholtz and Kelvin [23, §6 and letter from Thompson], which is a particular case of Da Rios law of evolution of three-dimensional vortices by a binormal curvature flow rescaled by a factor $\ln \frac{1}{\varepsilon}$ (Da Rios law [12, 44], see [1, §2.1] for a derivation in modern formalism). Benedetto, Caglioti and Marchioro have proved that axisymmetric flows whose initial vorticity concentrates on a vortex ring satisfy asymptotically this law [2]. For arbitrary filaments, Jerrard and Seis have proved the asymptotic binormal curvature flow under some hypotheses on the solution of the three-dimensional equation [25].

In order to state our main result describing flows whose initial vorticity is concentrating by their bathymetry, we rely on two conserved integral

quantities of the flow: the *vortex circulation* of the flow at time $t \in \mathbb{R}$

$$\Gamma(t) \triangleq \int_D \omega(t), \quad (4.1.4)$$

and the *kinetic energy* at time $t \in \mathbb{R}$,

$$E(t) \triangleq \frac{1}{2} \int_D |\mathbf{u}(t)|^2 b, \quad (4.1.5)$$

which are independent of the time $t \in \mathbb{R}$ when $\mathbf{u}$ is a classical solution of the lake equation.

Our main result characterizes the asymptotic behavior of solutions when the vorticity of the initial data shrinks to a Dirac mass:

Theorem 4.1.1. *Let $D \subseteq \mathbb{R}^2$ be a bounded domain of class C^2 and $b \in C^2(\bar{D}, (0, +\infty))$. Assume that b is constant on each component of ∂D . If*

(a) $(\mathbf{u}^n, h^n)_{n>0} \in C^1(\mathbb{R} \times \bar{D}, \mathbb{R}^2) \times C(\mathbb{R} \times \bar{D})$ *is a family of classical solutions to the lake equations (4.1.1),*

(b) $\omega^n(0) \geq 0$ *everywhere in D ,*

(c) $\frac{1}{\Gamma^n} \omega^n \rightharpoonup \delta_{q_0}$ *narrowly as measures,*

(d) $\omega^n(0) \leq \Gamma^n \exp \frac{8\pi E^n(0)}{\Gamma^n(0) \Omega^n(0)}$ *everywhere in D ,*

then for every $s \in \mathbb{R}$,

$$\frac{1}{\Gamma^n} \omega^n \left(\frac{\Gamma^n(0)s}{E^n(0)}, \cdot \right) \rightharpoonup \delta_{q(s)} \quad \text{and} \quad \frac{|\mathbf{u}^n|^2}{E^n} \left(\frac{\Gamma^n(0)s}{E^n(0)}, \cdot \right) \rightharpoonup \delta_{q(s)}$$

narrowly as measures, where $q : \mathbb{R} \rightarrow D$ is the unique solution of the Cauchy problem

$$\begin{cases} q'(s) = -(\nabla^\perp b^{-1})(q(s)) & s \in \mathbb{R}, \\ q(0) = q_0. \end{cases}$$

The narrow convergence means explicitly that we assume that for

every test function $\varphi \in C(\bar{D})$:

$$\lim_{n \rightarrow \infty} \frac{1}{\Gamma^n} \int_D \omega^n \varphi = \varphi(q_0);$$

and we conclude that for every test function $\varphi \in C(\bar{D})$ and every $s \in \mathbb{R}$:

$$\lim_{n \rightarrow \infty} \frac{1}{\Gamma^n} \int_D \omega^n \left(\frac{\Gamma_n(0)s}{E_n(0)} \right) \varphi = \varphi(q(s)).$$

The narrow convergence of the energy density is similar to the result obtained by Dávila, del Pino, Musso and Wei [13].

Since the lake equations (4.1.1) admit classical solutions, examples of solutions satisfying the assumptions of theorem 4.1.1 are given by rescaling an initial boundary data. Given a nonnegative function $f \in C_c^\infty(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f = 1$, a sequence of positive numbers $(\varepsilon^n)_{n \in \mathbb{N}}$ converging to 0, $q_0 \in \mathbb{R}^2$ and a sequence $(\Gamma^n)_{n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$ $B(q_0, \varepsilon^n) \subset D$, we set for each $x \in D$

$$\omega^n(0, x) \triangleq \Gamma^n f\left(\frac{x - q_0}{\varepsilon^n}\right),$$

and it can be computed that as $n \rightarrow \infty$

$$\Gamma^n(0) = \Gamma^n, \quad E^n(0) = \frac{(\Gamma^n)^2}{4\pi} \ln \frac{1}{\varepsilon^n} + O(1), \quad \Omega^n(0) = \Gamma^n (1 + O(\varepsilon^n)),$$

and thus, by theorem 4.1.1,

$$\frac{1}{\Gamma^n} \omega^n \left(\frac{4\pi s}{\Gamma^n \ln \frac{1}{\varepsilon^n}}, \cdot \right) \rightharpoonup \delta_{q(s)},$$

narrowly as measures on $\bar{D}$, where the motion of q is governed by Richardson's law (4.1.3).

The assumption that the depth b is constant on each component of the boundary implies that the solution q of the Cauchy problem obtained in the conclusion of theorem 4.1.1 remains inside the domain D and is thus global; the assumption plays an important role in our method, but we have no reason to believe that it should be necessary for the convergence to hold on a time interval in which there is no collision with

the boundary.

The assumption that the domain D is simply-connected yields a slightly simpler statement; it will be removed in the sequel under an additional condition that the circulations are controlled by the vortex circulation (see theorem 4.5.4 below). Similarly, our proof also covers weak solutions of the lake equations in the vortex formulation.

When the depth b is a constant function, theorem 4.1.1 implies that the vortex is stationary at the time scale $\Gamma^n(0)/E^n(0)$; this does contradict the classical planar vortex motion which occurs at a time scale of $1/\Gamma^n(0)$, which is much larger in the regime $E^n(0)/\Gamma^n(0)^2 \rightarrow +\infty$.

The description of the motion of vortices in theorem 4.1.1 can be formally written as

$$\dot{q}(t) \simeq -\frac{E}{\Gamma} \left(\nabla^\perp \frac{1}{b} \right) (q(t)). \quad (4.1.6)$$

An advantage of the formulation (4.1.6) is that the typical radius ε of the vortex, which is not necessarily preserved or even well-defined a priori along the flow, is replaced by a conserved quantity.

A first step in the proof of theorem 4.1.1, is to prove that the vorticity of the solution $\mathbf{u}_n(t)$ at any time $t \in \mathbb{R}$ concentrates as $n \rightarrow +\infty$. In contrast to other works for the planar Euler equations [38] or cylindrically symmetric Euler equations in the space [2, 38] in which the geometry of the vorticity region is constrained through its diameter or area, we rely on a typical length scale

$$\rho_n(t) \triangleq \exp \left(-\frac{4\pi E_n(t)}{\Gamma_n(t) \Omega_n(t)} \right),$$

which is defined in terms of integral quantities related to the flow: the energy $E_n(t)$ defined in (4.1.5), the circulation $\Gamma_n(t)$ defined in (4.1.4) and the *total vorticity* $\Omega_n(t)$ defined by

$$\Omega(t) \triangleq \int_D \omega(t) b. \quad (4.1.7)$$

The first two are conserved by the evolution of the lake equations, and $\Omega_n(t) \geq (\inf_D b) \Gamma_n(t)$ so that $\rho_n(t) \rightarrow 0$ uniformly as $n \rightarrow +\infty$.

In order to show that the vorticity effectively concentrates on balls of radius of the order $\rho_n(t)$, we rely on the assumption (d) of theorem 4.1.1 and on the fact that although the total vorticity $\Omega_n(t)$ is not conserved, it has variations of the order Γ_n on time-scales of the order Γ_n/E_n ; the proof of this statement relies on the constancy of the depth b on connected components of the boundary ∂D (see proposition 4.5.3).

The strategy to obtain the equation of motion of the vortex, is to study the *center of vorticity*

$$q_n(t) \triangleq \frac{1}{\Gamma_n} \int_D x \omega_n(t, x) \, dx.$$

A formal derivation argument gives the formula

$$\dot{q}_n(t) \simeq \frac{1}{\Gamma_n(t)} \int_D \frac{\nabla^\perp b}{b^2} \psi_n(t) \omega_n(t);$$

a suitable study of the asymptotics of the vorticity $\omega_n(t)$ and of the stream function $\psi_n(t)$ shows that the right-hand side behaves asymptotically as $-(\nabla^\perp b^{-1})E_n(t)/\Gamma_n(t)$. Unfortunately the derivation formula for q_n would require the identity to be constant on the boundary ∂D ; we bypass this technical obstacle by considering a modified version of the center of vorticity which is close to the center of vorticity thanks to concentration estimates and some repulsion properties of the boundary.

The sequel of the present work is organized as follows. In section 4.2, we precise the notion of weak solution of the lake equations in the vorticity formulation that we are using in the present work and we explain how the velocity can be reconstructed from the vorticity and the circulation around the boundary components and why the circulation Γ and energy E are preserved for weak solutions. In section 4.3, we expand the velocity construction formula in terms of the depth b and the Green function G_D for the classical Dirichlet problem on D at a level of precision required by the proof of our main result. These asymptotics are used in section 4.4 to obtain various concentration estimates on the vorticity. In section 4.5, we prove our main asymptotic result, after having obtained an asymptotic representation of derivatives of quantities and of the total vorticity Ω .

Theorem 4.1.1, as a first result on the asymptotic behavior of vortices for the lake equations raises several open problems for future research that are presented in section 4.6.

In a first appendix, we state some variants of classical results for transport equations [17] for velocities preserving the density b . The second appendix is devoted to a classical derivation of regularity results for the lake equations (4.1.1); this implies in particular that the classical solutions of the lake equations (4.1.1) appearing in theorem 4.1.1 exist for any smooth initial data.

4.2 The lake model

4.2.1 Weak vortex formulation of the lake equation

A lake is represented by its projection on a bounded domain $D \subseteq \mathbb{R}^2$ of the horizontal plane endowed and a depth function $b : D \rightarrow (0, +\infty)$. We assume that the domain D can be written as

$$D = D_0 \setminus \bigcup_{i=1}^m I_i,$$

where the set $D_0 \subset \mathbb{R}^2$ is simply-connected and its boundary is of class C^2 and the islands $I_1, \dots, I_m \subseteq D_0$ are disjoint simply-connected compact sets whose boundary is of class C^2 . We assume that $b \in C^2(\bar{D}, (0, +\infty))$. In particular, the depth b remains bounded away from 0 on the domain D .

A weak solution of the vorticity formulation of the lake equations will satisfy weakly the following system

$$\left\{ \begin{array}{ll} \nabla \cdot (b \mathbf{u}) = 0 & \text{in } [0, +\infty) \times D, \\ \mathbf{u} \cdot \boldsymbol{\nu} = 0 & \text{on } [0, +\infty) \times \partial D, \\ \partial_t \omega + \nabla \cdot (\mathbf{u} \omega) = 0 & \text{in } [0, +\infty) \times D, \\ \nabla \times \mathbf{u} = \omega & \text{in } [0, +\infty) \times D, \\ \omega(0, \cdot) = \omega_0 & \text{on } D. \end{array} \right.$$

More precisely, it will fulfill the following definition (see [24, Definition

2.2; 31, Definition 1.2]):

Definition 4.2.1. Given an initial pair $(\omega_0, \mathbf{u}_0)$ with $\omega_0 \in L^\infty(D, \mathbb{R})$ and $\mathbf{u}_0 \in L^\infty(D, \mathbb{R}^2)$ that satisfies weakly

$$\begin{cases} \nabla \cdot (b \mathbf{u}_0) = 0 & \text{in } D, \\ \mathbf{u}_0 \cdot \boldsymbol{\nu} = 0 & \text{on } \partial D, \\ \nabla \times \mathbf{u}_0 = \omega_0 & \text{in } D, \end{cases}$$

a pair $(\omega, \mathbf{u}) \in L^\infty([0, +\infty) \times D, \mathbb{R}) \times L^\infty([0, +\infty), L^2(D, \mathbb{R}^2))$ is a *weak solution of the lake equations in the vorticity formulation* with initial condition $(\omega_0, \mathbf{u}_0)$ whenever

(i) for every test function $\varphi \in C_c^1([0, +\infty) \times \bar{D})$, one has

$$\int_0^{+\infty} \int_D b \mathbf{u} \cdot \nabla \varphi = 0,$$

(ii) for every test function $\varphi \in C_c^1([0, +\infty) \times \bar{D})$ such that for every $t \in [0, +\infty)$, $\varphi|_{\{t\} \times D_0} = 0$ and for every $i \in \{1, \dots, m\}$, $\varphi|_{\{t\} \times \partial I_i}$ is constant, one has

$$\int_0^{+\infty} \int_D (\mathbf{u} \cdot \nabla^\perp \varphi - \omega \varphi) = \int_0^{+\infty} \int_D (\mathbf{u}_0 \cdot \nabla^\perp \varphi - \omega_0 \varphi),$$

(iii) for every test function $\varphi \in C_c^1([0, +\infty) \times D)$, one has

$$\int_D \omega_0 \varphi(0, \cdot) + \int_0^{+\infty} \int_D \omega (\partial_t \varphi + \mathbf{u} \cdot \nabla \varphi) = 0.$$

A pair $(\omega, \mathbf{u}) \in L^\infty(\mathbb{R} \times D, \mathbb{R}) \times L^\infty(\mathbb{R}, L^2(D, \mathbb{R}^2))$ is a weak solution of the vorticity formulation of the lake equations with initial condition $(\omega_0, \mathbf{u}_0)$ whenever the functions $t \in [0, +\infty) \mapsto (\omega(t), \mathbf{u}(t))$ and $t \in [0, +\infty) \mapsto (-\omega(-t), -\mathbf{u}(-t))$ are both weak solution to the vorticity formulation with initial condition $(\omega_0, \mathbf{u}_0)$.

Here and in the sequel, $\boldsymbol{\tau}$ denotes the unit tangent vector to the boundary ∂D chosen so that $\det(\boldsymbol{\nu}, \boldsymbol{\tau}) = 1$. The set $C_c^1([0, +\infty) \times \bar{D})$ is the set of maps $\varphi \in C^1([0, +\infty) \times \bar{D})$ such that there exists $T > 0$

such that $\varphi = 0$ in $(T, +\infty) \times D$; $C_c^1([0, +\infty) \times D)$ is the set of maps $\varphi \in C^1([0, +\infty) \times \bar{D})$ such that there exists $T > 0$ and a compact set $K \subset D$ such that $\varphi = 0$ in $([0, +\infty) \times D) \setminus ([0, T] \times K)$.

Compared to [31, Definition 1.2], definition 4.2.1 considers less test functions in (iii) — this will not matter eventually (see proposition 4.2.7 below) — and incorporates the conservation of circulation around the components of the boundary. Indeed, it follows from (ii) in definition 4.2.1 that the circulation $\Gamma_i(t)$ along ∂I_i at time t can be defined by

$$\Gamma_i(t) \triangleq \int_D (\mathbf{u}(t) \cdot \nabla^\perp \varphi - \omega(t) \varphi),$$

for any function $\varphi \in C^1(\bar{D})$ such that $\varphi = 1$ on ∂I_i and for each $j \in \{1, \dots, m\} \setminus \{i\}$, $\varphi = 0$ on ∂I_j (see also [31, (2.12)]). In view of (ii) in definition 4.2.1, we have $\Gamma_i(t) = \Gamma_i(0)$ for almost every $t \in \mathbb{R}$.

The surface height h does not appear in the weak formulation of definition 4.2.1, in accordance with the fact that ∇h can be recovered in (4.1.1) from $\mathbf{u}$.

The lake equations have at least one global weak solution in the vorticity formulation [5, Theorem 2.2 ii); 31, Lemma 2.11 & Theorem 1.3; 33, Theorem 1]; this solution is unique [5, Theorem 2.2 iii); 31, §2.3; 40, Theorem 1.1]. If moreover $\omega_0 \in C^{k,\alpha}(\bar{D})$, one has $\omega \in C^{k,\alpha}([-T, T] \times D)$, $\mathbf{u} \in C^{k,\alpha}([-T, T] \times D, \mathbb{R}^2)$ and $\mathbf{u} \in L^\infty([-T, T], C^{k+1,\alpha}(\bar{D}, \mathbb{R}^2))$ ([24, Theorem 4.1] and proposition B.1 below).

An alternative to definition 4.2.1 is the notion of weak solution for the velocity formulation [31, Proposition 2.13], based on (4.1.1). Under regularity assumptions on the domain Ω , the depth function b and on the initial data $(\omega_0, \mathbf{u}_0)$, both notions are equivalent [31, Proposition A.1].

4.2.2 Velocity reconstruction

Given a vorticity $\omega \in L^1(D)$ and circulations $\Gamma_1, \dots, \Gamma_m$, the *velocity reconstruction problem* consists in finding a vector field $\mathbf{u} : D \rightarrow \mathbb{R}^2$ such

that

$$\left\{ \begin{array}{ll} \nabla \cdot (b \mathbf{u}) = 0 & \text{in } D, \\ \mathbf{u} \cdot \boldsymbol{\nu} = 0 & \text{on } \partial D, \\ \nabla \times \mathbf{u} = \omega & \text{in } D, \\ \oint_{\partial I_i} \frac{\mathbf{u} \cdot \boldsymbol{\tau}}{b} = \Gamma_i & \text{for } i \in \{1, \dots, m\}, \end{array} \right. \quad (4.2.1)$$

weakly. That is, for every test function $\varphi \in C^1(\bar{D})$,

$$\int_D b \mathbf{u} \cdot \nabla \varphi = 0,$$

and for every $\varphi \in C^1(\bar{D})$ such that $\varphi|_{\partial D_0} = 0$ and $\varphi|_{\partial I_i} = \lambda_i \in \mathbb{R}$,

$$\int_D (\mathbf{u} \cdot \nabla^\perp \varphi - \omega \varphi) = \sum_{i=1}^m \Gamma_i \lambda_i.$$

The system (4.2.1) corresponds thus for a fixed time t to the continuity equation (i) and the definition of vorticity (ii) in definition 4.2.1.

In view of the divergence-free condition $\nabla \cdot (b \mathbf{u}) = 0$ in D and of the boundary condition $\mathbf{u} \cdot \boldsymbol{\nu} = 0$ on ∂D , the solution can be written as $\mathbf{u} = (\nabla^\perp \psi)/b$, where $\psi : D \rightarrow \mathbb{R}$ is a scalar *stream function* of the velocity field $\mathbf{u}$ and the function ψ satisfies the elliptic problem

$$\left\{ \begin{array}{ll} -\nabla \cdot (b^{-1} \nabla \psi) = \omega & \text{in } D, \\ \psi = 0 & \text{on } \partial D_0, \\ \psi \text{ is constant} & \text{on } \partial I_i \text{ for each } i \in \{1, \dots, m\}, \\ \int_{\partial I_i} b^{-1} \frac{\partial \psi}{\partial \boldsymbol{\nu}} = \Gamma_i & \text{for each } i \in \{1, \dots, m\}. \end{array} \right. \quad (4.2.2)$$

Since the ansatz $\mathbf{u} = (\nabla^\perp \psi)/b$ only defines the stream function ψ/b up to an additive constant, the boundary condition on ∂D_0 fixes the choice of a particular stream function.

The problem (4.2.2) can be handled by first solving the corresponding

classical Dirichlet problem:

$$\begin{cases} -\nabla \cdot (b^{-1} \nabla \psi) = \omega & \text{in } D, \\ \psi = 0 & \text{on } \partial D. \end{cases} \quad (4.2.3)$$

Since the function b is smooth and bounded from above and from below on D , one has the classical result:

Proposition 4.2.2. *For every $p \in (1, +\infty)$, there exists a linear continuous operator $\mathcal{G}_b : L^p(D) \rightarrow W^{2,p}(D, \mathbb{R}^2)$ such that for every $\omega \in L^p(D)$, the function $\mathcal{G}_b[\omega]$ is a weak solution of the problem (4.2.3).*

Proof. See for example [21, theorem 9.15]. $\square$

We now describe the solution to (4.2.2) in terms of (4.2.3) (see [33, §3] in the case of the planar Euler equations where b is constant, see [29, §6; 35; 50, (45)]).

Proposition 4.2.3. *For every $i \in \{1, \dots, m\}$, there exists a unique function $\psi_i \in C^2(D)$ that solves (4.2.2) with $\omega = 0$, $\Gamma_i = 1$ and $\Gamma_j = 0$ when $i \neq j$.*

Proof. For every $i \in \{1, \dots, m\}$, let $\varphi_i \in C^2(\bar{D})$ be a classical solution to the Dirichlet problem

$$\begin{cases} -\nabla \cdot (b^{-1} \nabla \varphi_i) = 0 & \text{in } D, \\ \varphi_i = \delta_{ij} & \text{on } \partial I_j \text{ for each } j \in \{1, \dots, m\}, \end{cases} \quad (4.2.4)$$

where δ_{ij} is the Kronecker delta, that is, $\delta_{ij} = 1$ whenever $i = j$ and $\delta_{ij} = 0$ otherwise. Since the functions $\varphi_1, \dots, \varphi_m$ are by construction linearly independent and since the domain D is connected, the matrix $(D_{ij})_{1 \leq i, j \leq m}$ defined by

$$D_{ij} \triangleq \int_D \frac{\nabla \varphi_i \cdot \nabla \varphi_j}{b} \quad (4.2.5)$$

is positive-definite and thus invertible. Let $(D_{ij}^{-1})_{1 \leq i, j \leq m}$ denote the inverse of this matrix. For every $i \in \{1, \dots, m\}$, we define the function

$\psi_i : D \rightarrow \mathbb{R}$ by

$$\psi_i \triangleq \sum_{j=1}^m D_{ij}^{-1} \varphi_j.$$

The function ψ_i satisfies the equation $-\nabla \cdot b^{-1} \nabla \psi_i = 0$, the boundary condition $\psi_i = 0$ on ∂D_0 and ψ_i is constant on each ∂I_j . Finally, we compute, in view of the boundary conditions satisfied by φ_j and the definition of ψ_i :

$$\begin{aligned} \int_{\partial I_j} \frac{1}{b} \frac{\partial \psi_i}{\partial \boldsymbol{\nu}} &= \sum_{\ell=1}^m D_{i\ell}^{-1} \int_{\partial D} \frac{\varphi_j}{b} \frac{\partial \varphi_\ell}{\partial \boldsymbol{\nu}} \\ &= \sum_{\ell=1}^m D_{i\ell}^{-1} \int_D \frac{\nabla \varphi_j \cdot \nabla \varphi_\ell}{b} = \sum_{\ell=1}^m D_{i\ell}^{-1} D_{\ell j} = \delta_{ij}. \quad \square \end{aligned}$$

Proposition 4.2.4. *For every $p \in (1, +\infty)$, there exists a linear continuous operator $\mathcal{K}_b : L^p(D) \rightarrow W^{2,p}(D)$ such that for every $\omega \in L^p(D)$, the function $\mathcal{K}_b[\omega]$ satisfies problem (4.2.2) weakly with $\Gamma_1 = \dots = \Gamma_m = 0$. Moreover, there exists a smooth function $Q_b \in C^2(\bar{D} \times \bar{D}, \mathbb{R}^2)$ such that for each $\omega \in L^p(D)$,*

$$\mathcal{K}_b[\omega] = \mathcal{G}_b[\omega] + \int_D Q_b(\cdot, y) \omega(y) \, dy.$$

Proof of proposition 4.2.4. We define the function $Q_b : \bar{D} \times \bar{D} \rightarrow \mathbb{R}$ for every $x, y \in D$ by

$$Q_b(x, y) \triangleq - \sum_{i,j=1}^m \varphi_i(x) D_{ij}^{-1} \varphi_j(y),$$

where the functions φ_i and the matrix D_{ij}^{-1} were defined in (4.2.4) and (4.2.5) in the proof of proposition 4.2.3, and for each $\omega \in L^p(D)$,

$$\mathcal{K}_b[\omega] \triangleq \mathcal{G}_b[\omega] + \int_D Q_b(\cdot, y) \omega(y) \, dy.$$

By linearity, we have $-\nabla \cdot (b^{-1} \nabla \mathcal{K}_b[\omega]) = -\nabla \cdot (b^{-1} \nabla \mathcal{G}_b[\omega]) = \omega$, $\mathcal{K}_b[\omega] = \mathcal{G}_b[\omega] = 0$ on ∂D_0 and that $\mathcal{K}_b[\omega]$ is constant on each component of the

boundary. Finally, we have for each $i \in \{1, \dots, m\}$,

$$\begin{aligned} \int_{\partial I_i} \frac{1}{b} \frac{\partial \mathcal{K}_b[\omega]}{\partial \boldsymbol{\nu}} &= \int_{\partial D} \frac{1}{b} \frac{\partial \mathcal{G}_b[\omega]}{\partial \boldsymbol{\nu}} \varphi_i - \sum_{j,\ell=1}^m D_{j\ell}^{-1} \int_{\partial D} \frac{1}{b} \varphi_i \frac{\partial \varphi_j}{\partial \boldsymbol{\nu}} \int_D \varphi_\ell \omega \\ &= \int_D \omega \varphi_i - \sum_{j,\ell=1}^m D_{j\ell}^{-1} \int_D \frac{\nabla \varphi_i \cdot \nabla \varphi_j}{b} \int_D \omega \varphi_\ell = 0. \quad \square \end{aligned}$$

We deduce from propositions 4.2.3 and 4.2.4 that for every $\omega \in L^p(D)$ and every $\Gamma_1, \dots, \Gamma_m \in \mathbb{R}$, the solution $\psi : D \rightarrow \mathbb{R}$ to the problem (4.2.2) is given for each $x \in D$ by

$$\begin{aligned} \psi(x) &= \mathcal{K}_b[\omega](x) + \sum_{i=1}^m \Gamma_i \psi_i(x) \\ &= \mathcal{G}_b[\omega](x) + \int_D Q_b(x, y) \omega(y) dy + \sum_{i=1}^m \Gamma_i \psi_i(x). \end{aligned} \quad (4.2.6)$$

The associated velocity field $\mathbf{u} : D \rightarrow \mathbb{R}^2$ is then given for each $x \in D$ by (see also [31, Lemma 2.6; 33, Lemma 5])

$$\begin{aligned} \mathbf{u}(x) &= \nabla^\perp \mathcal{K}_b[\omega](x) + \sum_{i=1}^m \Gamma_i \nabla^\perp \psi_i(x) \\ &= \nabla^\perp \mathcal{G}_b[\omega](x) + \int_D \nabla^\perp Q_b(x, y) \omega(y) dy + \sum_{i=1}^m \Gamma_i \nabla^\perp \psi_i(x), \end{aligned} \quad (4.2.7)$$

where we adopt the convention that $\nabla^\perp$ only acts on the first two-dimensional variable of the function Q_b .

We conclude this section by showing how the kinetic energy defined by (4.1.5) at a fixed time can be computed in terms of the vorticity and the circulations.

Proposition 4.2.5. *If $\omega \in L^p(D)$ for some $p > 1$, then $\mathbf{u} \in L^2(D, \mathbb{R}^2)$ and*

$$\begin{aligned} E &= \frac{1}{2} \int_D |\mathbf{u}|^2 b \\ &= \frac{1}{2} \int_D \omega \mathcal{K}_b[\omega] + \sum_{i=1}^m \Gamma_i \int_D \omega \psi_i + \sum_{i,j=1}^m \frac{\Gamma_i \Gamma_j}{2} \int_{D \times D} \frac{\nabla \psi_i \cdot \nabla \psi_j}{b}. \end{aligned}$$

Proof. In view of the representation formula for the velocity field (4.2.7), we have, by integration by parts and by definition of $\mathcal{K}_b[\omega]$ in proposition 4.2.4

$$\begin{aligned} E &= \frac{1}{2} \int_D \frac{|\nabla \mathcal{K}_b[\omega] + \sum_{i=1}^m \Gamma_i \nabla \psi_i|^2}{b} \\ &= \frac{1}{2} \int_D \omega \mathcal{K}_b[\omega] + \sum_{i=1}^m \Gamma_i \int_D \omega \psi_i + \sum_{i,j=1}^m \frac{\Gamma_i \Gamma_j}{2} \int_{D \times D} \frac{\nabla \psi_i \cdot \nabla \psi_j}{b}. \end{aligned} \quad (4.2.8)$$

□

4.2.3 Additional regularity of weak solutions

We apply the previous results to the regularity of stream functions of weak solutions (see [5, Theorem 1 i]):

Proposition 4.2.6. *If $(\omega, \mathbf{u}) \in L^\infty(\mathbb{R} \times D, \mathbb{R}) \times L^\infty(\mathbb{R}, L^2(D, \mathbb{R}^2))$ is a weak solution to the vortex formulation of the lake equations, then for every $p \in [1, +\infty)$, we have*

$$\mathbf{u} \in L^\infty([0, +\infty), W^{1,p}(D))$$

and

$$\mathcal{K}_b[\omega] \in W^{1,\infty}([0, +\infty) \times D, \mathbb{R}).$$

Proof. We first observe that by proposition 4.2.4, for almost every $t \in [0, +\infty)$, we have

$$\begin{aligned} \|\mathcal{K}_b[\omega(t)]\|_{W^{2,p}(D)} &\leq C_1 \|\omega(t)\|_{L^p(D)} \\ &\leq C_2 \|\omega(t)\|_{L^\infty(D)} \leq C_2 \|\omega\|_{L^\infty([0, +\infty) \times D)}. \end{aligned} \quad (4.2.9)$$

In particular, by taking $p > 2$, we have by the supercritical Sobolev embedding theorem that $\mathcal{K}_b[\omega] \in L^\infty([0, +\infty), W^{1,\infty}(D))$. Next, we have in view of the definition of $\mathcal{K}_b$, for every $\varphi \in C_c^1([0, +\infty) \times \bar{D})$,

$$\int_0^{+\infty} \int_D \mathcal{K}_b[\omega(t)] \partial_t \varphi(t) = \int_0^{+\infty} \int_D \omega(t) \partial_t \mathcal{K}_b[\varphi(t)].$$

We also have for every $t \in [0, +\infty)$ by (4.2.7)

$$\int_D \omega(t) \mathbf{u}(t) \cdot \nabla \mathcal{K}_b[\varphi(t)] = \int_D \omega(t) \nabla^\perp \mathcal{K}_b[\omega(t)] \cdot \nabla \mathcal{K}_b[\varphi(t)]$$

and thus for every $t \in [0, +\infty)$

$$\begin{aligned} \left| \int_0^{+\infty} \int_D \mathcal{K}_b[\omega(t)] \partial_t \varphi(t) \right| &= \left| \int_D \omega(t) \mathbf{u}(t) \cdot \nabla \mathcal{K}_b[\varphi(t)] \right| \\ &\leq C_3 \|\omega(t)\|_{L^\infty(D)}^2 \int_D |\varphi(t)|. \end{aligned}$$

This implies thus by definition 4.2.1 that $\partial_t \mathcal{K}_b[\omega] \in L^\infty([0, +\infty) \times D, \mathbb{R})$. We conclude then that $\mathcal{K}_b[\omega] \in W^{1,\infty}([1, +\infty) \times D)$. $\square$

The regularity that we have obtained so far implies that in fact the spatial boundary conditions on the test functions in definition 4.2.1 (iii) can be completely relaxed.

Proposition 4.2.7. *If a pair $(\omega, \mathbf{u})$ with $\omega \in L^\infty([0, +\infty) \times D, \mathbb{R})$ and $\mathbf{u} \in L^\infty([0, +\infty), L^2(D, \mathbb{R}^2))$ is a weak solution of the vorticity formulation of the lake equations with initial conditions $\omega_0 \in L^\infty(D)$ and $\mathbf{u}_0 \in L^\infty(D, \mathbb{R}^2)$, then for every test function $\varphi \in C_c^1([0, +\infty) \times \bar{D})$, one has*

$$\int_D \omega_0 \varphi(0, \cdot) + \int_0^{+\infty} \int_D \omega (\partial_t \varphi + \mathbf{u} \cdot \nabla \varphi) = 0.$$

Proof. This follows from proposition 4.2.6, definition 4.2.1 (iii) and proposition A.1. $\square$

4.2.4 Transport of the potential vorticity

The vorticity equation (4.1.2) can be rewritten as

$$\partial_t \omega + \nabla \cdot (b \mathbf{u} \omega) = 0 \tag{4.2.10}$$

and implies that the vortex circulation $\Gamma(t)$ defined by (4.1.4) of classical solutions of the lake equations (4.1.1) is conserved. The next proposition shows that this is still the case for weak solutions of the vorticity formulation of the lake equations (definition 4.2.1).

Proposition 4.2.8. *If $(\omega, \mathbf{u}) \in L^\infty(\mathbb{R} \times D, \mathbb{R}) \times L^\infty(\mathbb{R}, L^2(D, \mathbb{R}^2))$ is a weak solution to the vortex formulation of the lake equations, then $\omega \in C(\mathbb{R}, L^1(D))$ and for every $t \in \mathbb{R}$,*

$$\Gamma(t) = \int_D \omega(t) = \int_D \omega_0 = \Gamma(0).$$

Proof. We follow [31, §2.3]. We observe that for every $t \in \mathbb{R}$, $\mathbf{u}(t) \in W_{\text{loc}}^{1,1}(D)$ and thus if $\varphi \in C_c^1([0, +\infty) \times D)$, we have

$$\begin{aligned} \int_0^{+\infty} \int_D \frac{\omega}{b} (\partial_t \varphi + \nabla \cdot (\mathbf{u} \varphi)) + \int_D \frac{\omega_0}{b} \varphi(0, \cdot) \\ = \int_0^{+\infty} \int_D \omega (\partial_t \frac{\varphi}{b} + \mathbf{u} \cdot \nabla \frac{\varphi}{b}) + \int_D \omega_0 \frac{\varphi(0, \cdot)}{b} = 0. \end{aligned} \quad (4.2.11)$$

By proposition 4.2.6, proposition A.2 is applicable to $f_0 = \omega_0/b$ and gives the conclusion. $\square$

4.2.5 Conservation of energy

We now consider the total kinetic energy defined by (4.1.5). For classical solutions, one can show that the energy equation

$$\partial_t (b \frac{|\mathbf{u}|^2}{2}) + \nabla \cdot (b \mathbf{u} \frac{|\mathbf{u}|^2}{2}) = -\nabla \cdot (b \mathbf{u} h) \quad (4.2.12)$$

holds, and consequently, since $b \mathbf{u} \cdot \boldsymbol{\nu} = 0$ on the boundary, we have conservation of the total kinetic energy for classical solutions. The total kinetic energy still remains constant for weak solutions of the vortex formulation of the lake equations (definition 4.2.1).

Proposition 4.2.9 (Conservation of energy). *If $(\omega, \mathbf{u}) \in L^\infty(\mathbb{R} \times D, \mathbb{R}) \times L^\infty(\mathbb{R}, L^2(D, \mathbb{R}^2))$ is a weak solution to the vortex formulation of the lake equations, then for almost every $t \in \mathbb{R}$,*

$$E(t) = E(0).$$

The proof of proposition 4.2.9 relies on the following derivation formula.

Lemma 4.2.10. *Given a weak solution of the vortex formulation of the lake equations $(\omega, \mathbf{u}) \in L^\infty([0, +\infty) \times D, \mathbb{R}) \times L^\infty([0, +\infty), L^2(D, \mathbb{R}^2))$, we have for every $\theta \in C_c^\infty([0, +\infty))$,*

$$\begin{aligned} & \frac{1}{2} \int_0^\infty \left(\int_D \omega(t) \mathcal{K}_b[\omega(t)] \right) \theta'(t) dt \\ &= -\frac{\theta(0)}{2} \int_D \omega_0 \mathcal{K}_b[\omega_0] - \int_0^\infty \left(\int_D \omega(t) \partial_t \mathcal{K}_b[\omega](t) \right) \theta(t) dt. \end{aligned} \quad (4.2.13)$$

Proof. For every $h \in (0, +\infty)$, we have by a change of variable

$$\begin{aligned} & \frac{1}{2} \int_0^{+\infty} \left(\int_D \omega(t) \mathcal{K}_b[\omega(t)] \right) \frac{\theta(t+h) - \theta(t)}{h} dt \\ &= \frac{1}{2} \int_0^{+\infty} \left(\int_D \frac{\omega(t-h) \mathcal{K}_b[\omega(t-h)] - \omega(t) \mathcal{K}_b[\omega(t)]}{h} \right) \theta(t) dt \\ &= -\frac{1}{2h} \int_0^h \left(\int_D \omega(t) \mathcal{K}_b[\omega(t)] \right) \theta(t) dt. \end{aligned} \quad (4.2.14)$$

By Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{1}{2} \int_0^{+\infty} \left(\int_D \omega(t) \mathcal{K}_b[\omega(t)] \right) \frac{\theta(t+h) - \theta(t)}{h} dt \\ &= \frac{1}{2} \int_0^{+\infty} \left(\int_D \omega(t) \mathcal{K}_b[\omega(t)] \right) \theta'(t) dt. \end{aligned} \quad (4.2.15)$$

Since $\omega \in C([0, +\infty), L^1(D))$ by proposition 4.2.8, we also have

$$\lim_{h \rightarrow 0} \int_0^h \left(\int_D \omega(t) \mathcal{K}_b[\omega(t)] \right) \theta(t) dt = \frac{1}{2} \int_D \omega_0 \mathcal{K}_b[\omega_0] \theta(0). \quad (4.2.16)$$

For every $t \in [0, +\infty)$, since

$$\int_D \omega(t) \mathcal{K}_b[\omega(t-h)] = \int_D \omega(t-h) \mathcal{K}_b[\omega(t)],$$

we have

$$\int_D \omega(t-h) \mathcal{K}_b[\omega(t-h)] - \omega(t) \mathcal{K}_b[\omega(t)]$$

$$= \int_D (\omega(t-h) + \omega(t)) (\mathcal{K}_b[\omega(t-h)] - \mathcal{K}_b[\omega(t)]),$$

and thus by the weak convergence of difference quotients to the weak derivative and by proposition 4.2.8, we obtain

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{2} \int_0^{+\infty} \left(\int_D \frac{\omega(t-h) \mathcal{K}_b[\omega(t-h)] - \omega(t) \mathcal{K}_b[\omega(t)]}{h} \right) \theta(t) dt \\ = \int_0^\infty \left(\int_D \omega(t) \partial_t \mathcal{K}_b[\omega](t) \right) \theta(t) dt. \end{aligned} \quad (4.2.17)$$

The conclusion follows from (4.2.14), (4.2.15), (4.2.16) and (4.2.17). $\square$

Proof of proposition 4.2.9. Consider a function $\theta \in C_c^\infty([0, +\infty))$. We have to prove that

$$E(0) \theta(0) + \int_0^\infty \theta'(t) E(t) dt = 0.$$

We rely on the energy formula of proposition 4.2.5. We first have by lemma 4.2.10,

$$\begin{aligned} \frac{1}{2} \int_0^{+\infty} \int_D \omega(t) \mathcal{K}_b[\omega(t)] \theta'(t) dt \\ = \frac{\theta(0)}{2} \int_D \omega_0 \mathcal{K}_b[\omega_0] + \int_0^{+\infty} \int_D \omega(t) \partial_t (\mathcal{K}_b[\omega] \theta)(t) dt, \end{aligned}$$

since by Leibniz's rule $\partial_t (\mathcal{K}_b[\omega] \theta)(t) = \partial_t (\mathcal{K}_b[\omega])(t) \theta(t) + \mathcal{K}_b[\omega(t)] \theta'(t)$. By definition of weak solution of the lake equations in the vorticity formulation (definition 4.2.1 (iii)), we have then

$$\begin{aligned} \frac{1}{2} \int_0^{+\infty} \int_D \omega(t) \mathcal{K}_b[\omega(t)] \theta'(t) dt \\ = -\frac{1}{2} \int_D \omega_0 \mathcal{K}_b[\omega_0] - \int_0^{+\infty} \int_D \omega(t) \mathbf{u}(t) \cdot \nabla \mathcal{K}_b[\omega(t)] \theta(t) dt, \end{aligned} \quad (4.2.18)$$

and, for every $i \in \{1, \dots, m\}$,

$$\int_0^{+\infty} \int_D \omega(t) \psi_i \theta'(t) dt$$

$$= -\theta(0) \int_D \omega_0 \psi_i - \int_0^{+\infty} \int_D \omega(t) \mathbf{u}(t) \cdot \nabla \psi_i(t) \theta(t) dt. \quad (4.2.19)$$

This implies thus, by combining proposition 4.2.5 with the identities (4.2.18) and (4.2.19) and the velocity reconstruction formula (4.2.7), that

$$E(0) \theta(0) + \int_0^\infty \theta'(t) E(t) dt = \int_0^{+\infty} \left(\int_D \omega(t) \mathbf{u}(t) \cdot \mathbf{u}(t)^\perp \right) \theta(t) dt = 0. \quad \square$$

4.3 Velocity reconstruction expansion

In the sequel, we will need to understand the behavior of the operator $\mathcal{K}_b$ appearing in the construction in proposition 4.2.4 of the stream function satisfying (4.2.2). In view of proposition 4.2.4, this can be done through the study of the operator $\mathcal{G}_b$ associated to the solution of the Dirichlet problem (4.2.3), whose existence was given in proposition 4.2.2.

4.3.1 Construction of the Green function

We represent the Green function of the Dirichlet problem (4.2.3) as a perturbation of the Green operator of the classical Laplacian on the same domain with Dirichlet boundary conditions.

Proposition 4.3.1. *There exists a function $S_b \in C^{0,1}(D \times D)$ such that for every $\omega \in L^p(D)$ and every $x \in D$,*

$$\mathcal{G}_b[\omega](x) = \int_D \left(G_D(x, y) \sqrt{b(x) b(y)} + S_b(x, y) \right) \omega(y) dy.$$

Here $G_D : D \times D \rightarrow \mathbb{R}$ is the Green function of the Laplacian $-\Delta$ with Dirichlet boundary conditions on the boundary ∂D , that is, if $f \in L^p(\Omega)$ and if

$$u(x) = \int_D G_D(x, y) f(y) dy,$$

then

$$\begin{cases} -\Delta u = f & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

In particular, proposition 4.3.1 implies that the weighted Dirichlet problem (4.2.3) has a Green function $G_b : D \times D \rightarrow \mathbb{R}$ defined for each $x, y \in D$ with $x \neq y$ by

$$G_b(x, y) \triangleq G_D(x, y) \sqrt{b(x)b(y)} + S_b(x, y),$$

and thus the stream function problem (4.2.2) also has a Green function in view of proposition 4.2.4.

The proof of proposition 4.3.1 will rely on the fundamental estimate, which is a classical consequence of the maximum principle for the Laplacian operator $-\Delta$.

Proposition 4.3.2. *For every $x, y \in D$*

$$0 \leq G_D(x, y) \leq \frac{1}{2\pi} \ln \frac{\text{diam}(D)}{|x - y|}.$$

Proof of proposition 4.3.1. For each $y \in D$, let $S_b(\cdot, y) \in W_0^{1,2}(D)$ be the unique weak solution to

$$\begin{cases} -\nabla \cdot (b^{-1} \nabla S_b(\cdot, y)) = -G_D(\cdot, y) \sqrt{b(y)} \left(\Delta \frac{1}{\sqrt{b}} \right) & \text{in } D, \\ S_b(\cdot, y) = 0 & \text{on } \partial D. \end{cases}$$

Since $b \in C^2(\bar{D}, (0, +\infty))$, by classical elliptic regularity estimates (see for example [21, theorem 9.15]), for every $y \in D$ we have $S_b(\cdot, y) \in W^{2,p}(D)$ for every $p \in (1, +\infty)$ and

$$\|S_b(\cdot, y)\|_{W^{2,p}(D)} \leq C_1 \|G_D(\cdot, y)\|_{L^p(D)}. \quad (4.3.1)$$

By proposition 4.3.2, we have

$$\|G_D(\cdot, y)\|_{L^p(D)} \leq \left(\int_{B(0, \text{diam}(D))} \left(\ln \frac{\text{diam } D}{|z|} \right)^p dz \right)^{\frac{1}{p}} \leq C_2.$$

It follows in particular by the classical Sobolev embedding theorem and by (4.3.1) that

$$\sup_{y \in D} \|\nabla S_b(\cdot, y)\|_{L^\infty(D)} < +\infty.$$

Finally, we observe that if $\omega_1, \omega_2 \in L^p(D)$, we have

$$\begin{aligned} \int_D \omega_1 \mathcal{G}_b[\omega_2] - \iint_{D \times D} G(x, y) \omega_1(x) \omega_2(y) \sqrt{b(x) b(y)} \, dx \, dy \\ = \int_D \omega_2 \mathcal{G}_b[\omega_1] - \iint_{D \times D} G(x, y) \omega_2(x) \omega_1(y) \sqrt{b(x) b(y)} \, dx \, dy \end{aligned}$$

and therefore

$$\begin{aligned} \iint_{D \times D} S_b(x, y) \omega_1(x) \omega_2(y) \, dx \, dy \\ = \iint_{D \times D} S_b(x, y) \omega_2(x) \omega_1(y) \, dx \, dy. \end{aligned}$$

It follows that for every $x, y \in D$, $S_b(x, y) = S_b(y, x)$, and thus the function S_b is Lipschitz-continuous on $D \times D$. $\square$

As a consequence of proposition 4.3.1, the velocity field $\mathbf{u}$ admits the integral representation

Proposition 4.3.3. *There exists a Lipschitz-continuous function $R_b \in C^{0,1}(D \times D)$ such that for every $\omega \in L^p(D)$, one has*

$$\mathcal{K}_b[\omega](x) = \int_D (G_D(x, y) \sqrt{b(x) b(y)} + R_b(x, y)) \omega(y) \, dy.$$

Proof of proposition 4.3.3. This follows from proposition 4.2.4 and proposition 4.3.1 with $S_b = R_b + Q_b$. $\square$

4.3.2 Estimate on the Green function

We will also need a version of proposition 4.3.2 which is sharper close to the boundary.

Proposition 4.3.4. *There exists a constant C such that for every $x, y \in D$,*

$$G_D(x, y) \leq \frac{1}{4\pi} \ln \left(1 + C \frac{\text{dist}(x, \partial D) \text{dist}(y, \partial D)}{|x - y|^2} \right).$$

Proof. This can be obtained by observing that for the unit disk $\mathbb{D}^2 \subset \mathbb{R}^2$,

one has for each $x, y \in \mathbb{D}^2$ such that $x \neq y$,

$$G_{\mathbb{D}^2}(x, y) = \frac{1}{4\pi} \ln \left(1 + \frac{(1 - |x|^2)(1 - |y|^2)}{|x - y|^2} \right),$$

and by applying conformal mapping techniques as in the proof of proposition 4.3.7. $\square$

4.3.3 Gradient estimates of Green function of the Laplacian

Our goal now is to obtain estimates on the derivative of the Green function G_D . A first classical estimate is available [4].

Proposition 4.3.5. *There exists a constant C such that for every $x, y \in D$, one has*

$$|\nabla G_D(x, y)| \leq \frac{C}{|x - y|}.$$

We will need a more refined directional information about the Green function of the Laplacian, we first observe that in view of the definition of the regular part $H_D : D \times D \rightarrow \mathbb{R}$ for $x, y \in D$ such that $x \neq y$ as

$$H_D(x, y) \triangleq G_D(x, y) - \frac{1}{2\pi} \ln \frac{1}{|x - y|}, \quad (4.3.2)$$

we have for every $x, y \in D$ such that $x \neq y$

$$\nabla G_D(x, y) + \nabla G_D(y, x) = \nabla H_D(x, y) + \nabla H_D(y, x). \quad (4.3.3)$$

Here above, ∇G_D denotes the gradient of $G_D(x, y)$ with respect to its first variable.

In view of the regularity properties of the regular part of the Green function, we get

Proposition 4.3.6 (Interior symmetric gradient estimate). *For every $\delta > 0$, there exist $C > 0$ such that if $\text{dist}(x, \partial D) + \text{dist}(y, \partial D) + |x - y| \geq \delta$,*

$$|\nabla G_D(x, y) + \nabla G_D(y, x)| \leq C.$$

Proof. This follows from (4.3.3) and the smoothness of the regular H_D part of the Green function defined in (4.3.2). $\square$

We now investigate what the estimate of proposition 4.3.6 becomes near the boundary ∂D . We start by observing the Green function of the Laplacian on the half-plane $\mathbb{R}_+^2 = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$, which is given for each $x = (x_1, x_2), y = (y_1, y_2) \in D$ by

$$G_{\mathbb{R}_+^2}(x, y) = \frac{1}{4\pi} \ln \left(1 + \frac{4x_2y_2}{|x - y|^2} \right).$$

The gradient of this function with respect to its first variable, is then given by

$$\nabla G_{\mathbb{R}_+^2}(x, y) = \frac{1}{\pi(|x - y|^2 + 4x_2y_2)} \left((0, y_2) - 2x_2y_2 \frac{x - y}{|x - y|^2} \right).$$

One computes then that

$$\nabla G_{\mathbb{R}_+^2}(x, y) + \nabla G_{\mathbb{R}_+^2}(y, x) = \frac{(0, x_2 + y_2)}{\pi(|x - y|^2 + 4x_2y_2)}. \quad (4.3.4)$$

A notable feature of (4.3.4) is the vanishing of the tangential component.

We are going to extend formula (4.3.4) to any bounded domain, simply or non simply connected. To do this, we are going to show that (4.3.4) holds in a disk. From there, we are going to show that we can conformally transform D_0 to a disk, and find back similar estimates near ∂D_0 , that is: near the boundary of D that does not correspond to islands. The estimate would also hold in near every connected component of ∂D , after transformation via a conformal map of the form $z \in \mathbb{C} \setminus \{0\} \mapsto 1/z$.

Proposition 4.3.7. *If $\delta > 0$, there exists a constant $C > 0$ such that if $\text{dist}(x, \partial D) + \text{dist}(y, \partial D) + |x - y| \leq \delta$, then*

$$\left| \nabla G_D(x, y) + \nabla G_D(y, x) - \frac{x - P_{\partial D}(x) + y - P_{\partial D}(y)}{\pi(|x - y|^2 + 4 \text{dist}(x, \partial D) \text{dist}(y, \partial D))} \right| \leq C.$$

Proof. We start by observing the Green function of the Laplacian on the disk $\mathbb{D}^2 \subset \mathbb{R}^2$, which is defined for each $\tilde{x}, \tilde{y} \in \mathbb{D}^2$ by

$$G_{\mathbb{D}^2}(\tilde{x}, \tilde{y}) = \frac{1}{4\pi} \ln \left(1 + \frac{(1 - |\tilde{x}|^2)(1 - |\tilde{y}|^2)}{|\tilde{x} - \tilde{y}|^2} \right).$$

The gradient of this function, for fixed $\tilde{y} \in \mathbb{D}^2$, is then given by

$$\begin{aligned} \nabla G_{\mathbb{D}^2}(\tilde{x}, \tilde{y}) = & \frac{-1}{2\pi(|\tilde{x} - \tilde{y}|^2 + (1 - |\tilde{x}|^2)(1 - |\tilde{y}|^2))} \left(\tilde{x}(1 - |\tilde{y}|^2) \right. \\ & \left. + \frac{(1 - |\tilde{y}|^2)(1 - |\tilde{x}|^2)}{|\tilde{x} - \tilde{y}|^2} (\tilde{x} - \tilde{y}) \right), \end{aligned}$$

and thus we have

$$\nabla G_{\mathbb{D}^2}(\tilde{x}, \tilde{y}) + \nabla G_{\mathbb{D}^2}(\tilde{y}, \tilde{x}) = -\frac{\tilde{x}(1 - |\tilde{y}|^2) + \tilde{y}(1 - |\tilde{x}|^2)}{2\pi(|\tilde{x} - \tilde{y}|^2 + (1 - |\tilde{x}|^2)(1 - |\tilde{y}|^2))}.$$

We observe that

$$\begin{aligned} (1 - |\tilde{x}|^2)(1 - |\tilde{y}|^2) &= (1 - |\tilde{x}|)(1 - |\tilde{y}|)(2 - (1 - |\tilde{x}|))(2 - (1 - |\tilde{y}|)) \\ &= 4(1 - |\tilde{x}|)(1 - |\tilde{y}|) + O((1 - |\tilde{x}|)^2 + (1 - |\tilde{y}|)^2) \end{aligned}$$

and

$$\begin{aligned} & \tilde{x}(1 - |\tilde{y}|^2) + \tilde{y}(1 - |\tilde{x}|^2) \\ &= (\tilde{x} - \tilde{y})(|\tilde{y}|^2 - |\tilde{x}|^2) + 2\left(\tilde{x} - \frac{\tilde{x}}{|\tilde{x}|} + \tilde{y} - \frac{\tilde{y}}{|\tilde{y}|}\right) \\ &+ \tilde{x}(|\tilde{x}| - 1)^2\left(1 + \frac{2}{|\tilde{x}|}\right) + \tilde{y}(|\tilde{y}| - 1)^2\left(1 + \frac{2}{|\tilde{y}|}\right) \\ &= 2\left(\tilde{x} - \frac{\tilde{x}}{|\tilde{x}|} + \tilde{y} - \frac{\tilde{y}}{|\tilde{y}|}\right) \\ &+ O(|x - y|^2 + (1 - |\tilde{x}|)^2 + (1 - |\tilde{y}|)^2). \end{aligned}$$

It thus follows that when $|x - y|^2 + (1 - |\tilde{x}|)^2 + (1 - |\tilde{y}|)^2$ is small enough,

$$\left| \nabla G_{\mathbb{D}^2}(\tilde{x}, \tilde{y}) + \nabla G_{\mathbb{D}^2}(\tilde{y}, \tilde{x}) - \frac{\tilde{x} - \frac{\tilde{x}}{|\tilde{x}|} + \tilde{y} - \frac{\tilde{y}}{|\tilde{y}|}}{\pi(|\tilde{x} - \tilde{y}|^2 + 4(1 - |\tilde{x}|)(1 - |\tilde{y}|))} \right| \leq C_1. \quad (4.3.5)$$

By the classical Riemann mapping theorem (see for example [30, theorems 4.0.1 and 5.2.1]), there exists a map $\Phi \in C^2(\bar{D}_0, \bar{\mathbb{D}}^2)$ which is a diffeomorphism up to the boundary and which is conformal map. For each $i \in \{1, \dots, m\}$, we have $\Phi(I_i) \cap \partial\mathbb{D}^2 = \emptyset$.

We define all $x, y \in D$:

$$\check{G}_D(x, y) \triangleq G_{\mathbb{D}^2}(\Phi(x), \Phi(y)).$$

We compute

$$\nabla \check{G}_D(x, y) = (D\Phi(x))^* [\nabla G_{\mathbb{D}^2}(\Phi(x), \Phi(y))],$$

and thus

$$\begin{aligned} \nabla \check{G}_D(x, y) + \nabla \check{G}_D(y, x) \\ = (D\Phi(x))^* [\nabla G_{\mathbb{D}^2}(\Phi(x), \Phi(y)) + \nabla G_{\mathbb{D}^2}(\Phi(y), \Phi(x))] \\ + (D\Phi(y) - D\Phi(x))^* [\nabla G_{\mathbb{D}^2}(\Phi(y), \Phi(x))]. \end{aligned}$$

We observe that when $|x - y| + \text{dist}(x, \partial D_0) + \text{dist}(y, \partial D_0) \rightarrow 0$, we have

$$\begin{aligned} \Phi(x) - \frac{\Phi(x)}{|\Phi(x)|} &= D\Phi(x)[x - P_{\partial D}(x)] + O(\text{dist}(x, \partial D)^2), \\ \Phi(y) - \frac{\Phi(y)}{|\Phi(y)|} &= D\Phi(x)[y - P_{\partial D}(y)] + O(\text{dist}(y, \partial D)^2 + |x - y|^2), \\ |\Phi(x) - \Phi(y)|^2 &= |D\Phi(x)|^2 |x - y|^2 + O(|x - y|^3), \\ (1 - |\Phi(x)|) &= \frac{\text{dist}(x, \partial D)}{|D\Phi(x)|} + O((\text{dist } x, \partial D)^2), \\ (1 - |\Phi(y)|) &= \frac{\text{dist}(y, \partial D)}{|D\Phi(x)|} + O((\text{dist } y, \partial D)^2 + |x - y|^2), \end{aligned}$$

from which we deduce that

$$\begin{aligned} \left| (D\Phi(x))^* [\nabla G_{\mathbb{D}^2}(\Phi(x), \Phi(y)) + \nabla G_{\mathbb{D}^2}(\Phi(y), \Phi(x))] \right. \\ \left. - \frac{x - P_{\partial D}(x) + y - P_{\partial D}(y)}{\pi(|x - y|^2 + 4 \text{dist}(x, \partial D) \text{dist}(y, \partial D))} \right| \leq C_2. \quad (4.3.6) \end{aligned}$$

We also have immediately

$$|(D\Phi(y) - D\Phi(x))^*[\nabla G_{\mathbb{D}^2}(\Phi(y), \Phi(x))]| \leq C_3. \quad (4.3.7)$$

Now we draw the link between the transport $\check{G}_D$ and the Green's function G_D we target.

Since the map Φ is conformal, the function $G_{\mathbb{D}^2}(\cdot, y)$ is harmonic in $\mathbb{D}^2 \setminus \{y\}$ and there exists $\delta > 0$ such that if $\text{dist}(y, \partial D_0) \leq \delta$, then $\Theta(\cdot, y) \triangleq G_D(\cdot, y) - \check{G}_D(\cdot, y)$ is bounded uniformly in a neighborhood of $\bigcup_{i=1}^m I_i$; this implies that for every $x, y \in D$ such that $\text{dist}(x, \partial D) + \text{dist}(y, \partial D) \leq \delta$, we have

$$|\nabla \Theta(x, y)| \leq C_4. \quad (4.3.8)$$

The conclusion in when $\text{dist}(x, \partial D_0) + \text{dist}(y, \partial D_0) + |x - y| \leq \delta$ follows by combining the estimates (4.3.6), (4.3.7) and (4.3.8). The other components $I_1, \dots, I_m$ of the boundary can be reduced to this case by a suitable adaptation of the conformal mapping $z \in \mathbb{C} \setminus \{0\} \mapsto 1/z$. $\square$

4.4 Vortex estimates

In this section we derive several estimates on the vorticity that govern the concentration of the vorticity.

In order to control the shape of the vortex, we will recurrently rely on the Lorentz norm [36] of a vorticity $\omega : D \rightarrow \mathbb{R}$ which will be defined as

$$\begin{aligned} \|\omega\|_{\Lambda_{(\ln)_+, 1}} &\triangleq \sup \left\{ \int_{\mathbb{R}^2} \left(\ln \frac{1}{|x|} \right)_+ \omega'(x) dx \right. \\ &\quad \left. | \omega' : \mathbb{R}^2 \rightarrow \mathbb{R} \text{ and for every } \lambda > 0, \right. \\ &\quad \left. |\{x \in D \mid |\omega(x)| > \lambda\}| = |\{x \in \mathbb{R}^2 \mid |\omega'(x)| > \lambda\}| \right\}. \end{aligned} \quad (4.4.1)$$

We use the Lebesgue measure in the definition, despite the fact that the flow transports the measure with density b of the potential vorticity ω/b .

By the Hardy–Littlewood rearrangement inequality (see for example [34, theorem 3.4]), the supremum in (4.4.1) is actually reached by the

radially symmetric nonincreasing rearrangement ω^* of ω , whose superlevel sets are balls centered on 0.

4.4.1 Stream function estimate

We first show how the Lorentz norm can be used to obtain a bound on the stream function.

Proposition 4.4.1 (Boundedness of the stream function). *There exists a constant $C > 0$ that depends only on D and b , such that for every nonnegative function $\omega \in L^p(\Omega)$ and every $\rho > 0$, we have*

$$\begin{aligned} \|\mathcal{K}_b[\omega]\|_{L^\infty(D)} &\leq \frac{1}{2\pi} \int_D \ln \frac{1}{\max(\rho, |x-y|)} \omega(y) b(y) \, dy \\ &\quad + \frac{(\sup_D b) \rho^2 \|\omega(\rho \cdot)\|_{\Lambda(\ln)_+, 1}}{2\pi} + C \int_D \omega. \end{aligned}$$

Proof. By writing the function $\mathcal{K}_b[\omega]$ in terms of integral kernels of proposition 4.3.3, we have for each $x \in D$,

$$\begin{aligned} 0 \leq \mathcal{K}_b[\omega](x) &= \int_D G_D(x, y) \omega(y) b(y) \, dy + \int_D R_b(x, y) \omega(y) \, dy \\ &\quad + \int_D G_D(x, y) \omega(y) \left(\sqrt{b(x)} - \sqrt{b(y)} \right) \sqrt{b(y)} \, dy. \end{aligned}$$

The function R_b is uniformly bounded, by proposition 4.3.3. Moreover, using the direct estimate of proposition 4.3.2, we obtain for every $x \in D$

$$\int_D G_D(x, y) \omega(y) b(y) \, dy \leq \frac{1}{2\pi} \int_D \ln \frac{\text{diam } D}{|x-y|} \omega(y) b(y) \, dy.$$

Since for every $x, y \in \mathbb{R}^2$ such that $x \neq y$,

$$\ln \frac{\text{diam } D}{|x-y|} = \ln \frac{\text{diam } D}{\max(\rho, |x-y|)} + \left(\ln \frac{\rho}{|y-x|} \right)_+,$$

we have then for every $x \in D$,

$$\mathcal{K}_b[\omega](x) \leq \frac{1}{2\pi} \int_D \ln \frac{1}{\max(\rho, |x-y|)} \omega(y) b(y) \, dy$$

$$+ \frac{(\sup_D b) \rho^2}{2\pi} \int_{x+\rho z \in D} \left(\ln \frac{1}{|z|} \right)_+ \omega(x + \rho z) dz + C \int_D \omega,$$

for some constant $C > 0$ that depends only on D and b . The conclusion now follows from the definition of the Lorentz norm (4.4.1). $\square$

4.4.2 Energy concentration estimate

The following proposition gives estimates on the kinetic energy outside a ball.

Proposition 4.4.2. *There exists a constant $C > 0$ that depends only on D and b , such that for every nonnegative function $\omega \in L^p(\Omega)$ and every $R > r > \rho > 0$, we have*

$$\begin{aligned} \int_{D \setminus B(a, R)} |\mathbf{u}|^2 &\leq C \left(|\Gamma| \ln \frac{1}{\rho} \int_{D \setminus B(a, r)} \omega \right. \\ &\quad \left. + |\Gamma|^2 \ln \frac{1}{R-r} + \rho^2 \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}} |\Gamma| + \|\Gamma\|^2 \right). \end{aligned} \quad (4.4.2)$$

Proof. We define the function $\phi \triangleq \mathcal{K}_b[\omega] + 2 \sum_{i=1}^m \Gamma_i \psi_i$.

By proposition 4.4.1, we have for every $x \in \Omega \setminus B(a, R)$ and since $r \geq \rho$,

$$\begin{aligned} \phi(x) &\leq \frac{1}{2\pi} \ln \frac{1}{R-r} \int_{B(a, r)} \omega b \\ &\quad + \frac{1}{2\pi} \ln \frac{1}{\rho} \int_{\Omega \setminus B(a, r)} \omega b + C_1 (\rho^2 \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}} + \|\Gamma\|) \\ &\leq \lambda \triangleq C_2 \left(|\Gamma| \ln \frac{1}{R-r} + \ln \frac{1}{\rho} \int_{\Omega \setminus B(a, r)} \omega + \rho^2 \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}} + \|\Gamma\| \right) \end{aligned} \quad (4.4.3)$$

We have by the representation formula for the velocity field (4.2.7), as in (4.2.8) in the proof of proposition 4.2.5,

$$\int_{D \setminus B(a, R)} |\mathbf{u}|^2 \leq \int_{\phi^{-1}((0, \lambda))} |\mathbf{u}|^2$$

$$= \frac{1}{2} \int_{\phi^{-1}((0,\lambda))} \frac{\nabla \mathcal{K}_b[\omega] \cdot \nabla \phi}{b} + \sum_{i,j=1}^m \frac{\Gamma_i \Gamma_j}{2} \int_{\phi^{-1}((0,\lambda))} \frac{\nabla \psi_i \cdot \nabla \psi_j}{b}. \quad (4.4.4)$$

We have then,

$$\begin{aligned} \frac{1}{2} \int_{\phi^{-1}((0,\lambda))} \frac{\nabla \mathcal{K}_b[\omega] \cdot \nabla \phi}{b} &= \frac{1}{2} \int_{\Omega} \frac{\nabla \mathcal{K}_b[\omega] \cdot \nabla \max(\phi, \lambda)}{b} \\ &= \frac{1}{2} \int_{\Omega} \omega \max(\phi, \lambda) \leq \frac{|\Gamma| \lambda}{2}. \end{aligned} \quad (4.4.5)$$

On the other hand, we have

$$\sum_{i,j=1}^m \frac{\Gamma_i \Gamma_j}{2} \int_{\phi^{-1}((0,\lambda))} \frac{\nabla \psi_i \cdot \nabla \psi_j}{b} \leq C_3 \|\Gamma\|^2. \quad (4.4.6)$$

The conclusion follows from the combination of (4.4.4), (4.4.5), (4.4.3) and (4.4.6). $\square$

4.4.3 Vortex concentration

The next estimate shows that there is a characteristic radius ρ defined in terms of conserved quantities such that if the Lorentz norm at the scale ρ remains bounded, then the vorticity is concentrated in a region of radius comparable to ρ .

Proposition 4.4.3 (Concentration estimate). *There exists a constant $C > 0$ that depends only on D, b , such that for all $R > 1$, we have*

$$\inf_{a \in D} \int_{D \setminus B(a, R\rho)} \omega \leq \frac{C}{\ln(R)} \left(\rho^2 \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}} + \frac{\|\Gamma\|^2}{|\Gamma|} \right),$$

where

$$\rho \triangleq \exp \left(- \frac{4\pi E}{\Gamma \Omega} \right). \quad (4.4.7)$$

Here and in the sequel, we use the notation

$$\|\Gamma\| = |\Gamma| + \sum_{i=1}^m |\Gamma_i|. \quad (4.4.8)$$

The proof of proposition 4.4.3, follows ideas introduced by Turking-

ton [53] and Turkington & Friedmann [20] for the Euler equations. It was also used in chapters 5 and 6 in the study of steady solutions of the lake equations by energy maximization [14, 15].

Proof of proposition 4.4.3. We define the set

$$A \triangleq \left\{ x \in D \mid \psi(x) \geq \frac{1}{\Gamma} \int_D \psi \omega \right\}, \quad (4.4.9)$$

in terms of the stream function $\psi \triangleq \mathcal{K}_b[\omega] + \sum_{i=1}^k \Gamma_i \psi_i$. We observe that by definition of the vortex circulation Γ in (4.1.4), the set A is not empty.

By proposition 4.3.3 and proposition 4.3.2, since by assumption the function b is Lipschitz-continuous, we have for every $x \in D$,

$$\begin{aligned} \mathcal{K}_b[\omega](x) &\leq \frac{1}{2\pi} \int_D \ln \frac{\text{diam } D}{|x-y|} \sqrt{b(x)b(y)} \, dy + \int_D R_b(x,y) \omega(y) \, dy \\ &\leq \frac{1}{2\pi} \ln \frac{1}{\rho} \int_D \omega b + \frac{1}{2\pi} \int_D \ln \frac{\rho}{|x-y|} \omega(y) b(y) \, dy + C_1 |\Gamma|, \end{aligned}$$

and thus by definition of ψ and of $\|\Gamma\|$ in (4.4.8),

$$\psi(x) \leq \frac{1}{2\pi} \ln \frac{1}{\rho} \int_D \omega b + \frac{1}{2\pi} \int_D \ln \frac{\rho}{|x-y|} \omega(y) b(y) \, dy + C_2 \|\Gamma\|. \quad (4.4.10)$$

On the other hand, setting ρ to the value given by (4.4.7), we obtain for each $x \in A$, in view of proposition 4.2.5 and the definition of the set A by (4.4.9)

$$\begin{aligned} \psi(x) &\geq \frac{2E}{\Gamma} - \sum_{i=1}^m \frac{\Gamma_i}{\Gamma} \int_D \omega \psi_i - \sum_{i,j=1}^m \frac{\Gamma_i \Gamma_j}{\Gamma} \int_D \nabla \psi_i \cdot \nabla \psi_j \\ &\geq \frac{1}{2\pi} \ln \frac{1}{\rho} \int_D \omega b - C_3 \frac{\|\Gamma\|^2}{|\Gamma|}. \end{aligned} \quad (4.4.11)$$

The combination of (4.4.10) and (4.4.11), shows that for each $x \in A$,

$$\frac{1}{2\pi} \int_D \ln \frac{|x-y|}{\rho} \omega(y) b(y) \, dy \leq C_4 \frac{\|\Gamma\|^2}{|\Gamma|}. \quad (4.4.12)$$

In order to conclude, we start from the inequality

$$\ln(R) \frac{\inf_D b}{4\pi} \int_{D \setminus B(x, R\rho)} \omega(y) \, dy \leq \frac{1}{4\pi} \int_{D \setminus B(x, R\rho)} \ln \frac{|x-y|}{\rho} \omega(y) b(y) \, dy. \quad (4.4.13)$$

We also observe that

$$\begin{aligned} \frac{1}{4\pi} \int_{D \cap B(x, R\rho)} \ln \frac{\rho}{|x-y|} \omega(y) b(y) \, dy \\ \leq \frac{\rho^2 \sup_D b}{4\pi} \int_D \left(\ln \frac{1}{|z|} \right)_+ \omega(x + \rho z) \, dz \\ \leq C_5 \rho^2 \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}}, \end{aligned} \quad (4.4.14)$$

in view of the definition of the Lorentz norm (4.4.1). Therefore, if $x \in A$, in view of (4.4.12), (4.4.13) and (4.4.14), we have for some constant $C_6 > 0$ and for each $x \in A$ and $R > 1$:

$$\int_{D \setminus B(x, R\rho)} \omega \leq \frac{C_6}{\ln(R)} \left(\rho^2 \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}} + \frac{\|\Gamma\|^2}{|\Gamma|} \right). \quad \square$$

4.4.4 Boundary repulsion

The next estimate shows that the vorticity ω cannot be concentrated too much in a neighborhood of the boundary when ρ is small.

Proposition 4.4.4. *One has*

$$\int_D \omega(x) \ln \left(1 + \frac{\text{diam } D}{\rho + \text{dist}(x, \partial D)} \right) \, dx \leq C \left(\rho^2 \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}} + \frac{\|\Gamma\|^2}{|\Gamma|} \right).$$

Proof. By the energy identity of proposition 4.2.5, the decomposition of the Green function of proposition 4.3.3 and the upper bound on the Green function of the Laplacian of proposition 4.3.4, we have

$$\begin{aligned} E = \frac{1}{2} \int_D \omega \mathcal{K}_b[\omega] + \sum_{i=1}^m \Gamma_i \int_D \psi_i \omega + \sum_{i,j=1}^m \frac{\Gamma_i \Gamma_j}{2} \int_D \nabla \psi_i \cdot \nabla \psi_j \\ \leq I + C_1 \|\Gamma\|^2, \end{aligned} \quad (4.4.15)$$

where

$$I = \frac{1}{8\pi} \iint_{D \times D} \ln \left(1 + C_2 \frac{\text{dist}(x, \partial D) \text{dist}(y, \partial D)}{|x - y|^2} \right) \omega(x) \omega(y) \sqrt{b(x) b(y)} \, dx \, dy$$

The first term on the right-hand side of I can be bounded by

$$\begin{aligned} I &\leq \frac{C}{8\pi} \iint_{D \times D} \ln \left(\rho^2 + C_2 \frac{\rho^2 \text{dist}(x, \partial D) \text{dist}(y, \partial D)}{|x - y|^2} \right) \omega(x) \omega(y) \, dx \, dy \\ &\quad + \frac{\ln \frac{1}{\rho}}{4\pi} \iint_{D \times D} \omega(x) \omega(y) \sqrt{b(x) b(y)} \, dx \, dy \quad (4.4.16) \end{aligned}$$

The integral of the first term on the right-hand side of (4.4.16) can be bounded by the Cauchy–Schwarz inequality, as

$$\begin{aligned} &\iint_{D \times D} \omega(x) \omega(y) \sqrt{b(x) b(y)} \, dx \, dy \\ &\leq \left(\iint_{D \times D} \omega(x) \omega(y) b(x) \, dx \, dy \right)^{\frac{1}{2}} \left(\iint_{D \times D} \omega(x) \omega(y) b(y) \, dx \, dy \right)^{\frac{1}{2}} \\ &= \Omega \Gamma. \quad (4.4.17) \end{aligned}$$

We split the second integral of the right-hand side of (4.4.16) in two regions, depending on whether $|x - y| \leq \rho$. For the first one we have, since the domain D is bounded and in view of the definition of the Lorentz norm in (4.4.1)

$$\begin{aligned} &\iint_{\substack{(x,y) \in D \times D \\ |x-y| \leq \rho}} \ln \left(\rho^2 + C_2 \frac{\rho^2 \text{dist}(x, \partial D) \text{dist}(y, \partial D)}{|x - y|^2} \right) \omega(x) \omega(y) \, dx \, dy \\ &\leq 2 \iint_{\substack{(x,y) \in D \times D \\ |x-y| \leq \rho}} \ln \frac{C_3 \rho}{|x - y|} \omega(x) \omega(y) \, dx \, dy \\ &\leq C_4 \left(\rho^2 \|\omega(\rho \cdot)\|_{\Lambda(\ln)_+, 1} + |\Gamma| \right) |\Gamma|; \quad (4.4.18) \end{aligned}$$

for the second part we have

$$\begin{aligned}
& \iint_{\substack{(x,y) \in D \times D \\ |x-y| \geq \rho}} \ln \left(\rho^2 + C_2 \frac{\text{dist}(x, \partial D) \text{dist}(y, \partial D)}{|x-y|^2} \right) \omega(x) \omega(y) \, dx \, dy \\
& \leq \iint_{D \times D} \ln(\rho^2 + C_2 \text{dist}(x, \partial D) \text{dist}(y, \partial D)) \omega(x) \omega(y) \, dx \, dy \\
& \leq 2 \iint_{D \times D} \ln(\rho + \sqrt{C_2} \text{dist}(x, \partial D)) \omega(x) \omega(y) \, dx \, dy + C_5 |\Gamma|^2 \\
& = 2 |\Gamma| \int_D \ln(\rho + \sqrt{C_2} \text{dist}(x, \partial D)) \omega(x) \, dx + C_6 |\Gamma|^2. \quad (4.4.19)
\end{aligned}$$

The conclusion then follows from the combination of (4.4.15), (4.4.16), (4.4.17), (4.4.18) and (4.4.19). $\square$

4.4.5 Center of vorticity

We define now the *center of vorticity*

$$q \triangleq \frac{1}{\Gamma} \int_D x \omega(x) \, dx$$

and we prove that concentration occurs in fact around the center of vorticity, as a consequence of proposition 4.4.3.

Proposition 4.4.5 (Concentration around the center of vorticity). *There exists constants $C, C' > 0$ such that, for all $R > 1$, we have*

$$\int_{D \setminus B(q, \rho_*(\rho, R))} \omega \leq \frac{C}{\ln(R)} \left(\rho^2 \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}} + \frac{\|\Gamma\|^2}{|\Gamma|} \right),$$

with ρ defined in (4.4.7) and

$$\rho_*(\rho, R) = R\rho + \frac{C'}{\ln(R)} \left(\frac{\rho^2 \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}}}{|\Gamma|} + \frac{\|\Gamma\|^2}{|\Gamma|^2} \right).$$

Proof. Let $R > 1$ be any number. By proposition 4.4.3, there exists a

constant C_1 (independent on $R > 1$) and some point $a \in D$ such that

$$\int_{D \setminus B(a, R\rho)} \omega \leq \frac{C_1}{\ln(R)} \left(\rho^2 \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}} + \frac{\|\Gamma\|^2}{|\Gamma|} \right). \quad (4.4.20)$$

We now compute

$$\begin{aligned} |q - a| &= \frac{1}{|\Gamma|} \left| \int_D (x - a) \omega(x) dx \right| \\ &\leq \frac{R\rho}{|\Gamma|} \left(\int_{B(a, R\rho)} \omega \right) + \frac{\text{diam } D}{|\Gamma|} \int_{D \setminus B(a, R\rho)} \omega \\ &\leq R\rho + \frac{C_2}{\ln(R)} \left(\frac{\rho^2 \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}}}{|\Gamma|} + \frac{\|\Gamma\|^2}{|\Gamma|^2} \right), \end{aligned} \quad (4.4.21)$$

for some other constant $C_2 > 0$ independent of $R > 1$ and on ω . The conclusion follows from (4.4.21) and (4.4.20). $\square$

As a consequence of proposition 4.4.4 we estimate the distance between the center of vorticity q to the boundary ∂D .

Proposition 4.4.6 (Confinement of the center of vorticity). *There exists a constant $C > 0$ such that for all $R > 1$,*

$$C \left(\frac{\rho^2 \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}}}{|\Gamma|} + \frac{\|\Gamma\|^2}{|\Gamma|^2} \right) \left(\frac{1}{\ln(1 + \frac{\text{diam } D}{\text{dist}(q, \partial D) + \rho + \rho_*(R, \rho)})} + \frac{1}{\ln R} \right) \geq 1,$$

where $\rho_*(R, \rho)$ is defined in proposition 4.4.5.

Proof of proposition 4.4.6. By proposition 4.4.5, we have

$$\int_{D \setminus B(q, \rho_*(R, \rho))} \omega \leq \frac{C_1}{\ln(R)} \left(\rho^2 \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}} + \frac{\|\Gamma\|^2}{|\Gamma|} \right). \quad (4.4.22)$$

On the other hand by proposition 4.4.4, we have

$$\begin{aligned} &\int_{B(q, \rho_*(R, \rho))} \omega \\ &\leq \frac{1}{\ln(1 + \frac{\text{diam } D}{\text{dist}(q, \partial D) + \rho + \rho_*(R, \rho)})} \int_D \omega(x) \ln \left(1 + \frac{\text{diam } D}{\rho + \text{dist}(x, \partial D)} \right) dx \end{aligned}$$

$$\leq C_2 \frac{\left(\frac{\|\Gamma\|^2}{|\Gamma|} + \|\omega(\rho \cdot)\|_{\Lambda_{(\ln)_+, 1}} \right)}{\ln \left(1 + \frac{\text{diam } D}{\text{dist}(q, \partial D) + \rho + \rho_*(R, \rho)} \right)}. \quad (4.4.23)$$

The conclusion follows from (4.4.22), (4.4.23) and the definition of Γ in (4.1.4). $\square$

4.4.6 Transport of Lorentz norms

The Lorentz norm that was defined in (4.4.1) and that appeared in all the concentration estimates of this section, is invariant under transformations that preserve the measure of level sets of ω , but by (4.1.2) the flow under the lake equations (4.1.1) preserves the measure with density b of level sets of ω/b instead. The next proposition shows that the Lorentz norms can be kept into control.

Proposition 4.4.7 (Transport of Lorentz norm). *Let $\omega, \omega' : D \rightarrow \mathbb{R}$. If for every $\lambda > 0$,*

$$\int_{\omega(x) > \lambda b(x)} b(x) \, dx = \int_{\omega'(x) > \lambda b(x)} b(x) \, dx,$$

then

$$\|\omega'\|_{\Lambda_{(\ln)_+, 1}} \leq \left(\frac{\sup_D b}{\inf_D b} \right)^2 \|\omega\|_{\Lambda_{(\ln)_+, 1}}.$$

The proof of proposition 4.4.7 is based on the following geometrical computation of the Lorentz norm.

Lemma 4.4.8. *For every $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}$, we have*

$$\|\omega\|_{\Lambda_{(\ln)_+, 1}} = 2\pi \int_0^\infty \int_0^{\sqrt{|\{\omega > \lambda\}|/\pi}} r \left(\ln \frac{1}{r} \right)_+ \, dr \, d\lambda.$$

Proof. By the Hardy–Littlewood rearrangement inequality, we have

$$\int_{\mathbb{R}^2} \left(\ln \frac{1}{|x|} \right)_+ |\omega|(x) \, dx \leq \int_{\mathbb{R}^2} \left(\ln \frac{1}{|x|} \right)_+ |\omega|^*(x) \, dx,$$

where $|\omega|^* : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a radial function such that for every $s > 0$,

$|\{x \in \mathbb{R}^2 \mid |\omega|^*(x) > \lambda\}| = |\{x \in \mathbb{R}^2 \mid |\omega|(x) > \lambda\}|$. We compute then

$$\begin{aligned} \int_{\mathbb{R}^2} \left(\ln \frac{1}{|x|} \right)_+ |\omega|^*(x) dx &= \int_0^{+\infty} \int_{|\omega|^*(x) > \lambda} \left(\ln \frac{1}{|x|} \right)_+ dx d\lambda \\ &= 2\pi \int_0^{+\infty} \int_0^{\sqrt{|\{\omega\}^* > \lambda\}|/\pi} r \left(\ln \frac{1}{r} \right)_+ dr \\ &= 2\pi \int_0^{+\infty} \int_0^{\sqrt{|\{\omega\} > \lambda\}|/\pi} r \left(\ln \frac{1}{r} \right)_+ dr. \quad \square \end{aligned}$$

Proof of proposition 4.4.7. We have for every $\lambda > 0$, by assumption

$$|\{\omega' > \lambda\}| \leq \frac{1}{m} \int_{\{|\omega'|/b > \lambda/M\}} b = \frac{1}{m} \int_{\{|\omega|/b > \lambda/M\}} b \leq \alpha |\{\omega > \lambda/\alpha\}|,$$

where $m = \inf_D b$, $M = \sup_D b$ and $\alpha = M/m$.

By lemma 4.4.8, we have now

$$\begin{aligned} \|\omega'\|_{\Lambda_{(\ln)_+,1}} &= 2\pi \int_0^{+\infty} \int_0^{\sqrt{|\{\omega' > \lambda\}|/\pi}} r \left(\ln \frac{1}{r} \right)_+ dr d\lambda \\ &\leq 2\pi \int_0^{+\infty} \int_0^{\sqrt{\alpha |\{\omega > \lambda/\alpha\}|/\pi}} r \left(\ln \frac{1}{r} \right)_+ dr d\lambda \\ &= \alpha^2 2\pi \int_0^{+\infty} \int_0^{\sqrt{|\{\omega > \lambda'\}|/\pi}} r' \left(\ln \frac{1}{\sqrt{\alpha} r'} \right)_+ dr' d\lambda' \\ &\leq \alpha^2 2\pi \int_0^{+\infty} \int_0^{\sqrt{|\{\omega > \lambda'\}|/\pi}} r' \left(\ln \frac{1}{r'} \right)_+ dr' d\lambda'. \quad \square \end{aligned}$$

Finally we estimate the behavior of the Lorentz norm under rescaling on the domain.

Proposition 4.4.9. *If $\sigma \in (0, +\infty)$ and $\omega : D \rightarrow \mathbb{R}$, then*

$$\|\omega(\sigma \cdot)\|_{\Lambda_{(\ln)_+,1}} \leq \frac{1}{\sigma^2} \left(\|\omega\|_{\Lambda_{(\ln)_+,1}} + \left(\ln \frac{1}{\sigma} \right)_+ \|\omega\|_{L^1} \right).$$

Proof. We have, by lemma 4.4.8,

$$\|\omega(\sigma \cdot)\|_{\Lambda_{(\ln)_+,1}} = 2\pi \int_0^\infty \int_0^{\sqrt{|\{\omega > \lambda\}|/(\sigma\pi)}} r \left(\ln \frac{1}{r} \right)_+ dr d\lambda$$

and by a change of variable:

$$\|\omega(\sigma \cdot)\|_{\Lambda_{(\ln)_+,1}} = \frac{2\pi}{\sigma^2} \int_0^\infty \int_0^{\sqrt{|\{\omega>\lambda\}|/\pi}} r' \left(\ln \frac{1}{\sigma r'} \right)_+ dr' d\lambda,$$

From this, we conclude

$$\begin{aligned} \|\omega(\sigma \cdot)\|_{\Lambda_{(\ln)_+,1}} &\leq \frac{2\pi}{\sigma^2} \left(\int_0^\infty \int_0^{\sqrt{|\{\omega>\lambda\}|/\pi}} r' \left(\ln \frac{1}{r'} \right)_+ dr' d\lambda \right. \\ &\quad \left. + \int_0^\infty \int_0^{\sqrt{|\{\omega>\lambda\}|/\pi}} r' \left(\ln \frac{1}{\sigma} \right)_+ dr' d\lambda \right) \end{aligned}$$

and the conclusion follows. $\square$

4.5 Asymptotics evolution of vortices

4.5.1 Asymptotic representation of derivatives

In order to study the evolution of vortices, we will need to differentiate several quantities of the form

$$\int_D \omega(t) \eta,$$

where $\eta \in C^1(\bar{D})$ is a given spatial test function.

Proposition 4.5.1. *If $(\omega, \mathbf{u}) \in L^\infty(\mathbb{R} \times D) \times L^\infty(\mathbb{R}, L^2(D))$ is a weak solution to the vorticity formulation lake equation and if $\eta \in C^1(\bar{D})$, then the function $t \in \mathbb{R} \mapsto \int_D \omega(t) \eta$ is weakly differentiable and for almost every $t \in \mathbb{R}$,*

$$\begin{aligned} \frac{d}{dt} \int_D \omega(t) \eta &= \int_D \omega(t) \mathbf{u}(t) \cdot \nabla \eta \\ &= - \int_D \omega(t) \frac{\nabla(\mathcal{K}_b[\omega(t)])}{b} \times \nabla \eta - \sum_{i=1}^m \Gamma_i \int_D \omega(t) \nabla \psi_i \times \nabla \eta. \end{aligned}$$

Proof. Given $\theta \in C_c^1([0, +\infty))$, we apply proposition 4.2.7 to the test

function $\theta\eta$ and we obtain

$$\theta(0) \int_D \omega_0 \eta + \int_0^{+\infty} \int_D \omega(t) (\theta'(t)\eta + \theta(t) \mathbf{u}(t) \cdot \nabla \eta) dt = 0.$$

The second identity follows then from (4.2.7). $\square$

Under the additional assumption that the function η is constant on each component of the boundary ∂D , we obtain a representation in which the gradient of the stream function is replaced by the gradient the depth function.

Proposition 4.5.2. *If $\eta \in C^2(\bar{D})$ is constant on each component of ∂D , then there exists a constant $C > 0$ such that for every $\omega \in L^\infty(D)$,*

$$\left| \int_D \omega \frac{\nabla(\mathcal{K}_b[\omega])}{b} \times \nabla \eta - \frac{1}{2} \int_D \omega \mathcal{K}_b[\omega] \frac{\nabla b}{b^2} \times \nabla \eta \right| \leq C |\Gamma|^2.$$

Proof of proposition 4.5.2. By proposition 4.3.3, we have the following identity:

$$\begin{aligned} & \int_D \omega \frac{\nabla(\mathcal{K}_b[\omega])}{b} \times \nabla \eta - \frac{1}{2} \int_D \omega \mathcal{K}_b[\omega] \frac{\nabla b}{b^2} \times \nabla \eta \\ &= \iint_{D \times D} \nabla G_D(x, y) \times \nabla \eta(x) \omega(x) \omega(y) \sqrt{\frac{b(y)}{b(x)}} dx dy \\ & \quad + \iint_{D \times D} \frac{\nabla R_b(x, y)}{b} \times \nabla \eta(x) \omega(x) \omega(y) dx dy. \end{aligned} \quad (4.5.1)$$

For the second term on the right-hand side of (4.5.1), we have by the boundedness properties of the derivatives of the function R_b ,

$$\left| \iint_{D \times D} \frac{\nabla R_b(x, y)}{b} \times \nabla \eta(x) \omega(x) \omega(y) dx dy \right| \leq C_1 |\Gamma|^2. \quad (4.5.2)$$

We now estimate the first term on the right-hand side of (4.5.1). By symmetry, we have

$$\begin{aligned}
& \iint_{D \times D} \sqrt{\frac{b(y)}{b(x)}} (\nabla G_D(x, y) \times \nabla \eta(x)) \omega(x) \omega(y) \, dx \, dy \\
&= \frac{1}{2} \iint_{D \times D} \sqrt{b(x)b(y)} \left(\nabla G_D(x, y) \times \frac{\nabla \eta(x)}{b(x)} \right. \\
&\quad \left. + \nabla G_D(y, x) \times \frac{\nabla \eta(y)}{b(y)} \right) \omega(x) \omega(y) \, dx \, dy \quad (4.5.3)
\end{aligned}$$

For every $x, y \in D$, we have

$$\begin{aligned}
& \nabla G_D(x, y) \times \frac{\nabla \eta(x)}{b(x)} + \nabla G_D(y, x) \times \frac{\nabla \eta(y)}{b(y)} \\
&= (\nabla G_D(x, y) + \nabla G_D(y, x)) \times \frac{\nabla \eta(x)}{b(x)} \\
&\quad + \nabla G_D(x, y) \times \left(\frac{\nabla \eta(y)}{b(y)} - \frac{\nabla \eta(x)}{b(x)} \right). \quad (4.5.4)
\end{aligned}$$

By proposition 4.3.5, since by assumption $\nabla \eta$ and b are both Lipschitz-continuous, we have

$$\left| \nabla G_D(x, y) \times \left(\frac{\nabla \eta(y)}{b(y)} - \frac{\nabla \eta(x)}{b(x)} \right) \right| \leq C_2. \quad (4.5.5)$$

Since the function η is constant on the boundary, its gradient $\nabla \eta$ is asymptotically parallel to the normal near the boundary, and thus by proposition 4.3.6 and (4.3.7), we have

$$\left| (\nabla G_D(x, y) + \nabla G_D(y, x)) \times \frac{\nabla \eta(x)}{b(x)} \right| \leq C_3. \quad (4.5.6)$$

By combining (4.5.3), (4.5.4), (4.5.6) and (4.5.5), we deduce that

$$\left| \iint_{D \times D} \sqrt{\frac{b(y)}{b(x)}} (\nabla G_D(x, y) \times \nabla \eta(x)) \omega(x) \omega(y) \, dx \, dy \right| \leq C_4 |\Gamma|^2. \quad (4.5.7)$$

The conclusion follows from the combination of the identity (4.5.1) with the inequalities (4.5.2) and (4.5.7) $\square$

4.5.2 Asymptotic conservation of the total vorticity

For the lake equations (4.1.1), the *total vorticity*, defined by (4.1.7) is not conserved in general. Indeed, by proposition 4.5.1, one has for almost every $t \in \mathbb{R}$

$$\dot{\Omega}(t) = \int_D \omega(t) \mathbf{u}(t) \cdot \nabla b,$$

and there is no reason for the right-hand side to vanish. On the other hand Richardson's formal law (4.1.3) suggests that vorticity should follow level lines of the depth and thus one can hope the total vorticity to be asymptotically preserved. The following result gives a bound on the variation of the total vorticity during the motion.

Proposition 4.5.3 (Asymptotic conservation of the total vorticity). *If the function $b \in C^2(\bar{D})$ is constant on each connected component of the boundary, then there exists a constant $C > 0$ that depends only on D and b , such that if $(\omega, \mathbf{u}) \in L^\infty(\mathbb{R} \times D) \times L^\infty(\mathbb{R}, L^2(D))$ is a weak solution to the vorticity formulation lake equation then we have for almost every $t \in \mathbb{R}$:*

$$|\Omega(t) - \Omega(0)| \leq C |\Gamma| \|\Gamma\| |t|.$$

The conclusion of proposition 4.5.3 can be rewritten as

$$\frac{|\Omega(t) - \Omega(0)|}{|\Gamma|} \leq C \frac{|\Gamma| \|\Gamma\|}{E} \frac{E|t|}{|\Gamma|},$$

in the regime where $\frac{E}{|\Gamma| \|\Gamma\|} \rightarrow +\infty$, the above estimate may be interpreted as stating that at the time scale Γ/E the variations of the total vorticity are much smaller than the total circulation.

Proof of proposition 4.5.3. By proposition 4.5.1 with $\eta = b$, the function Ω is weakly differentiable and for almost every $t \in \mathbb{R}$,

$$\dot{\Omega}(t) = - \int_D \frac{\nabla(\mathcal{K}_b[\omega(t)]) \times \nabla b}{b} \omega(t) - \sum_{i=1}^m \Gamma_i \int_D \frac{\nabla \psi_i \times \nabla b}{b} \omega(t). \quad (4.5.8)$$

Since we have assumed the bathymetry function b to be constant on the boundary, we apply proposition 4.5.2 with $\eta = b$ and we obtain, since

$$\nabla b \times \nabla b = 0,$$

$$\left| \int_D \frac{\nabla(\mathcal{K}_b[\omega(t)]) \times \nabla b}{b} \omega(t) \right| \leq C_1 |\Gamma|^2. \quad (4.5.9)$$

By the boundedness properties of the gradients of b and ψ_i and by definition of $\|\Gamma\|$ in (4.4.8), we also have

$$\left| \sum_{i=1}^m \Gamma_i \int_D \frac{\nabla \psi_i \times \nabla b}{b} \omega(t) \right| \leq C_2 |\Gamma| \|\Gamma\|. \quad (4.5.10)$$

By (4.5.8), (4.5.9) and (4.5.10) we deduce in view of proposition 4.5.1 that for almost every $t \in \mathbb{R}$,

$$|\dot{\Omega}(t)| \leq C_3 |\Gamma| \|\Gamma\|.$$

The conclusion then follows by integration. $\square$

4.5.3 Evolution of a singular vortex

We are now in position to state and prove the main result of the present work.

Theorem 4.5.4 (Evolution of the vortex core). *Let $D \subseteq \mathbb{R}^2$ be a bounded domain of class C^2 and $b \in C^2(\bar{D}, (0, +\infty))$. Assume that b is constant on each component of ∂D . If*

(a) $(\mathbf{u}^n)_{n>0}$ is family of weak solutions to the lake equations (4.1.1),

(b) $\omega^n(0) \geq 0$ almost everywhere on D ,

(c) there exists $q_0 \in D$ such that for every $\eta \in C(\Omega)$,

$$\lim_{n \rightarrow \infty} \frac{1}{\Gamma^n} \int_D \eta \omega^n(0) = \eta(q_0),$$

(d) $\sup_{n \in \mathbb{N}} \frac{1}{|\Gamma^n|} \|(\rho^n)^2 \omega^n(0, \rho^n \cdot)\|_{\Lambda_{(\ln)_+, 1}} < +\infty$ where $\rho^n = \exp\left(-\frac{4\pi E^n}{\Gamma^n \Omega^n}\right)$,

(e) $\sup_{n \in \mathbb{N}} \frac{\|\Gamma^n\|}{|\Gamma^n|} < +\infty$,

and let $q_* : \mathbb{R} \rightarrow D$ be the unique solution to the Cauchy problem

$$\begin{cases} \dot{q}_*(s) = -\nabla^\perp \left(\frac{1}{b} \right) (q_*(s)) & \text{if } s \in \mathbb{R}, \\ q_*(0) = q_0, \end{cases}$$

then one has, for every $\eta \in C^1(\bar{D})$, uniformly in $s \in \mathbb{R}$ over compact subsets,

$$\lim_{n \rightarrow \infty} \frac{1}{\Gamma^n} \int_D \eta \omega^n(\Gamma_n s / E_n) = \eta(q_*(s))$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{E^n} \int_D \eta |\mathbf{u}^n|^2(\Gamma_n s / E_n) = \eta(q_*(s)).$$

Theorem 4.5.4 implies immediately theorem 4.1.1. Indeed, it suffices to observe that by (4.4.1) the Lorentz norm is controlled by the L^∞ norm and that (e) is trivially satisfied since when D is simply connected, $m = 0$ and $\|\Gamma\| = |\Gamma|$ by definition in (4.4.8).

Proof of theorem 4.5.4. We first note that by proposition 4.3.3, and (4.3.2) and by proposition 4.2.5, we have for every $\delta > 0$ such that $\overline{B(q_0, \delta)} \subset D$,

$$\begin{aligned} & \frac{1}{4\pi} \ln \frac{1}{2\delta} \left(\int_{\overline{B(q_0, \delta)}} \sqrt{b} \omega^n(0) \right)^2 \\ & \leq \int_{B(q_0, \delta)} \frac{1}{4\pi} \ln \frac{1}{|y-x|} \sqrt{b(x)b(y)} \omega^n(0, x) \omega^n(0, y) dy dx \\ & \leq \frac{1}{2} \int_D \omega \mathcal{K}_b[\omega] + C_1 \|\Gamma^n\|^2 \leq E^n(0) + C_2 \|\Gamma^n\|^2. \end{aligned}$$

so that by the assumptions (c) and (e), for every $\delta > 0$ small enough we have

$$\liminf_{n \rightarrow \infty} \frac{E^n}{(\Gamma^n)^2} \geq \frac{1}{4\pi} \ln \frac{1}{2\delta}$$

and thus

$$\lim_{n \rightarrow \infty} \frac{E^n}{(\Gamma^n)^2} = +\infty. \quad (4.5.11)$$

By definition of $\rho^n(t)$ in (4.4.7), we observe that for each $n \in \mathbb{N}$ and

$t \in \mathbb{R}$,

$$\rho^n(t) = \exp\left(-\frac{4\pi E^n}{\Gamma^n \Omega^n(t)}\right) \leq \exp\left(-\frac{4\pi E^n}{|\Gamma^n|^2 \sup_D b}\right),$$

and thus by (4.5.11), we have $\rho^n \rightarrow 0$ uniformly on $\mathbb{R}$ as $n \rightarrow \infty$. Moreover, we have by definition of $\rho^n(t)$ and by proposition 4.5.3 for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\left| \ln \frac{\rho^n(t)}{\rho^n(0)} \right| = \frac{4\pi E^n |\Omega^n(t) - \Omega^n(0)|}{\Gamma^n \Omega^n(0) \Omega^n(t)} \leq \frac{C_3 E^n |t|}{\Gamma^n}.$$

By proposition 4.4.9 and then proposition 4.4.7, this implies that for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$

$$\begin{aligned} \rho^n(t)^2 \|\omega^n(t, \rho^n(t) \cdot)\|_{\Lambda_{(\ln)_+, 1}} &\leq \rho^n(0)^2 \left(\|\omega^n(t, \rho^n(0) \cdot)\|_{\Lambda_{(\ln)_+, 1}} + C_3 E^n |t| \right) \\ &\leq C_4 \rho^n(0)^2 \left(\|\omega^n(0, \rho^n(0) \cdot)\|_{\Lambda_{(\ln)_+, 1}} + E^n |t| \right) \\ &\leq C_5 |\Gamma^n| \left(1 + \frac{E^n |t|}{|\Gamma^n|} \right), \quad (4.5.12) \end{aligned}$$

in view of our assumption (e). Since $\rho^n(t)$ uniformly converges to 0 as $n \rightarrow \infty$, one can choose $R^n(t) \triangleq 1/\sqrt{\rho^n(t)}$ for sufficiently large $n \in \mathbb{N}$ in proposition 4.4.6. For every $S > 0$, there exists then $\delta > 0$ such that if $n \in \mathbb{N}$ is large enough and if $E^n |t|/\Gamma^n \leq S$, then $q^n(t) \in D$ and $\text{dist}(q^n(t), \partial D) \geq \delta$.

Let $\eta \in C^\infty(\mathbb{R}^2)$ be a positive function bounded by 1, such that $\eta(x) = 0$ if $\text{dist}(x, \partial D) \leq \delta/3$ and $\eta(x) = 1$ if $\text{dist}(x, \partial D) \geq 2\delta/3$. We define the truncated center of vorticity $\tilde{q}^n : \mathbb{R} \rightarrow \mathbb{R}^2$ by setting for $t \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\tilde{q}^n(t) \triangleq \frac{1}{\Gamma} \int_D \eta(x) x \omega^n(t, x) dx.$$

We observe that for every n large enough, $\rho_*^n(t)(R^n(t), \rho^n(t)) \leq \frac{\delta}{3}$ and

$$\int_{\text{dist}(x, \partial D) \leq \delta} \omega^n \leq \int_{D \setminus B(q(t), \rho_*^n(t)(R^n(t), \rho^n(t)))} \omega^n(t, x) dx.$$

By proposition 4.4.5, we obtain

$$\begin{aligned} |q^n(t) - \tilde{q}^n(t)| &\leq \frac{1}{\Gamma} \int_{D \setminus B(q(t), \rho_*^n(t)(R^n(t), \rho^n(t)))} \omega^n(t, x) \, dx \\ &\leq \frac{C_6}{\ln(R^n(t))} \left(\rho^n(t)^2 \frac{\|\omega^n(t, \rho^n(t) \cdot)\|_{\Lambda_{(\ln)_+, 1}}}{\Gamma^n} + 1 \right). \end{aligned} \quad (4.5.13)$$

By (4.5.12) and by the choice of $R^n(t)$ it follows that $(\tilde{q}^n(\Gamma^n \cdot / E_n) - q^n(\Gamma^n \cdot / E_n))_{n \in \mathbb{N}}$ converges uniformly to 0 over $[-S, S]$.

By proposition 4.5.1, we have for almost every $t \in \mathbb{R}$,

$$\dot{q}^n(t) = \frac{1}{\Gamma^n} \left(\int_D \frac{\nabla^\perp \mathcal{K}_b[\omega^n(t)] \cdot \nabla \xi}{b} \omega^n(t) + \sum_{i=1}^m \Gamma_i^n \int_D \frac{\nabla^\perp \psi \cdot \nabla \xi}{b} \omega^n(t) \right)$$

where the vector field $\xi \in C_c^\infty(D, \mathbb{R}^2)$ is defined for each $x \in D$ by $\xi(x) \triangleq x \eta(x)$. In view of proposition 4.5.2, we have

$$\left| \dot{q}^n(t) - \frac{1}{2\Gamma^n} \int_D \frac{\nabla^\perp b \cdot \nabla \xi}{b^2} \omega^n(t) \mathcal{K}_b[\omega^n(t)] \right| \leq C_7 \|\Gamma^n\| \quad (4.5.14)$$

Now we observe that since ∇b is Lipschitz-continuous and since $\xi(x) = x$ if $\text{dist}(x, \partial D) \geq \delta$, we have for every $x \in D$ and y such that $\text{dist}(y, \partial D) \geq \delta$,

$$\left| \frac{\nabla^\perp b(x) \cdot \nabla \xi(x)}{b(x)^2} - \frac{\nabla^\perp b(y)}{b^2(y)} \right| \leq C_8 |x - y|$$

and thus

$$\begin{aligned} &\left| \int_D \left(\frac{\nabla^\perp b \cdot \nabla \xi}{b^2} - \frac{\nabla^\perp b(q^n(t))}{b^2(q^n(t))} \right) \omega^n(t) \mathcal{K}_b[\omega^n(t)] \right| \\ &\leq C_8 \left(\int_D |x - q^n(t)| |\omega^n(t, x)| \, dx \right) \|\mathcal{K}_b[\omega^n](t)\|_{L^\infty(D)}. \end{aligned} \quad (4.5.15)$$

We have by a direct bound

$$\int_{D \cap B(q^n(t), \rho_*^n(t)(R^n(t), \rho^n(t)))} |x - q^n(t)| |\omega^n(t, x)| \, dx$$

$$\leq C_9 \rho_*^n(t)(R^n(t), \rho^n(t)) |\Gamma^n| \quad (4.5.16)$$

and by proposition 4.4.5,

$$\begin{aligned} & \int_{D \setminus B(q^n(t), \rho_*^n(t)(R^n(t), \rho^n(t)))} |x - q^n(t)| |\omega^n(t, x)| dx \\ & \leq \frac{C_{10}}{\ln R^n(t)} (\rho^n(t)^2 \|\omega(\rho^n(t) \cdot)\|_{\Lambda_{(\ln)_+, 1}} + |\Gamma^n|). \end{aligned} \quad (4.5.17)$$

Thus we have by (4.5.15), (4.5.16), (4.5.17) and by proposition 4.4.1,

$$\begin{aligned} & \left| \int_D \left(\frac{\nabla^\perp b \cdot \nabla \xi}{b^2} - \frac{\nabla^\perp b(q^n(t))}{b^2(q^n(t))} \right) \omega^n(t) \mathcal{K}_b[\omega^n(t)] \right| \\ & \leq C_{11} \left((\rho_*^n(t)(R^n(t), \rho^n(t))) |\Gamma^n| \right. \\ & \quad \left. + \frac{1}{\ln R^n(t)} (\rho^n(t)^2 \|\omega^n(t, \rho^n(t) \cdot)\|_{\Lambda_{(\ln)_+, 1}} + |\Gamma^n|) \right). \end{aligned} \quad (4.5.18)$$

Summarizing (4.5.14) and (4.5.18) we conclude that

$$\left| \frac{\Gamma^n}{E^n} \dot{q}^n(\Gamma^n s / E^n) - \frac{\nabla^\perp b(\tilde{q}^n(\Gamma^n s / E^n))}{b^n(\tilde{q}^n(\Gamma^n s / E^n))^2} \right| \rightarrow 0,$$

uniformly over $s \in [-S, S]$.

This implies in turn that $\tilde{q}^n(\Gamma^n \cdot / E^n)$ converges uniformly on compact subsets of $\mathbb{R}$ to q_* . Finally, we conclude that $q^n(\Gamma^n \cdot / E^n) \rightarrow q_*$ uniformly over compact subsets of $\mathbb{R}$. By (4.5.17) the narrow convergence of vorticity measures follows.

The convergence of energy density measures then follows from proposition 4.4.2. $\square$

4.6 Open problems

The present work has given a first description of the asymptotic vortex dynamics for the lake equations (4.1.1). The setting in which we have been working does not cover the whole spectrum of physically relevant situations and suggests for future research some problems that we could not tackle with the techniques that we have developed here.

A first problem would be to determine whether theorem 4.5.4 holds when shore of the lake is a beach rather than a cliff, that is when b goes smoothly to 0 on the boundary.

Open problem 1. Does a single vortex follow asymptotically the level lines of the depth b when $\inf_D b = 0$?

Whereas in our proof the assumption that $\inf_D b > 0$ plays a role in the construction of the Green function and in keeping control on the Lorentz norm, stationary results cover the case of where the depth b behaves like a power of the distance function close to the boundary [14; 15; 16, §3] (see also chapters 5, 6).

Open problem 2. Does a single vortex follow asymptotically the level lines of the depth b when the domain is unbounded?

The boundedness of the domain and of b is used mainly in the construction and estimates on the Green functions.

The probably most accessible case would be when $D = \mathbb{R}^2$ and b is constant outside a compact set; an interesting result would cover the case where $D = [0, +\infty) \times \mathbb{R}$, with $b(r, z) = r$, corresponding to the construction of vortex rings for the three-dimensional Euler equations (see [2]).

The solution of open problem 2 would also show that the evolution of clifford tori in the binormal curvature flow [27].

Another problem would be the case of non-constant boundary values of b .

Open problem 3. Does a single vortex follow asymptotically the level lines of the depth b when b is not constant on the boundary?

Currently, the constancy plays a crucial role in the proof and the application of proposition 4.5.2.

An issue with this setting is that the limiting equation would suggest vortices exiting the domain in finite time. This would not be consistent with the conservation of circulation. A possible solution to this paradox is that the interaction with the boundary at very short range perturbs strongly the asymptotics and makes the law of movement invalid.

When the lake has a flat bottom, that is when b is constant on some region, our results do not give an interesting description of the movement of the vortices, that occurs on a larger time scale. In analogy with the planar Euler equation, which corresponds to the case where b is constant on the whole domain, we expect this movement to occur at a time-scale of the order $1/\Gamma$.

Open problem 4. Describe the movement of a single vortex in a flat region of the lake at time scales of the order $1/\Gamma$.

We expect this to be described by some sort of Green function adapted to the problem. A similar second-order asymptotic description was already given for stationary vortex pairs [15] (see chapter 6). One question is whether the movement depends only on the shape of the set on which b is flat or whether it depends fully on b and on D . The first scenario would be consistent with results for an analogous Ginzburg–Landau problem with discontinuous pinning [18].

Finally, it would be natural to consider the problem where the vorticity concentrates in several regions.

Open problem 5. Do solutions whose initial vorticity concentrates at several points have these vortex patches following level lines of b ?

This situation is not accessible to our proofs because we characterize the size of the vortex region by some global integral quantities.

An issue raised by this problem would be possible collision of vortices moving on the same line. They would probably interact at a small scale and produce potentially a vortex pair whose movement might be governed by a different equation and might have a different characteristic timescale. A similar related problem would be the description of vortex pairs.

A Weak solutions of the transport equation

A first interesting fact is that for a transport equation with no flux through the boundary, it is equivalent to test the equation against compactly supported smooth functions or functions that are smooth up to the boundary.

Proposition A.1. *Assume that $\mathbf{u} \in L^\infty(W^{1,1}(D))$ and that $\mathbf{u} \cdot \boldsymbol{\nu} = 0$ on ∂D in the sense of traces. If $f_0 \in L^\infty(D)$ and $f \in L^\infty([0, +\infty) \times D)$ satisfy for every $\varphi \in C_c^1([0, +\infty) \times D)$ the identity*

$$\int_0^{+\infty} \int_D f (\partial_t \varphi + \mathbf{u} \cdot \nabla \varphi) + \int_D f_0 \varphi(0, \cdot) = 0,$$

then the identity holds for every $\varphi \in C_c^1([0, +\infty) \times \bar{D})$.

Proof. We consider a map $\theta \in C^1((0, +\infty))$ such that $\theta = 0$ on $(0, \frac{1}{2})$ and $\theta(t) = 1$ on $[1, +\infty)$ and we define for each $n \in \mathbb{N}$, the function $\chi_n : D \rightarrow \mathbb{R}$ for each $x \in D$ by $\chi_n(x) \triangleq \theta(n \operatorname{dist}(x, \partial D))$. By the smoothness assumption on D , $\chi_n \in C_c^1(D)$. Since $\mathbf{u} \cdot \boldsymbol{\nu} = 0$ in the sense of traces, we have for every $T \in [0, +\infty)$,

$$\int_0^T \int_D |\mathbf{u} \cdot \nabla \chi_n| \leq C_1 \int_0^T \int_{\substack{x \in D \\ \operatorname{dist}(x, \partial D) \leq \frac{1}{n}}} |\nabla \mathbf{u}|,$$

and thus by Lebesgue's dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^T \int_D |\mathbf{u} \cdot \nabla \chi_n| = 0. \quad (\text{A.1})$$

For each $\varphi \in C_c^1([0, +\infty) \times \bar{D})$ and every $n \in \mathbb{N}$, we take $\chi_n \varphi \in C_c^1([0, +\infty) \times D)$ as test function, and we obtain by assumption

$$\int_0^{+\infty} \int_D \chi_n f (\partial_t \varphi + \mathbf{u} \cdot \nabla \varphi) + \int_D \chi_n f_0 \varphi(0, \cdot) = - \int_0^{+\infty} \int_D \varphi f \mathbf{u} \cdot \nabla \chi_n;$$

the conclusion follows by letting $n \rightarrow \infty$, and using (A.1). $\square$

The flow $\mathbf{u}$ can be integrated following DiPerna and P.-L. Lions [17] in order to provide solutions to the corresponding transport problem.

Proposition A.2. *Let $b \in C^1(\bar{D}, (0, +\infty))$ and assume that the velocity field satisfies $\mathbf{u} \in L^1_{\text{loc}}([0, +\infty), W^{1,1}(D) \cap L^\infty(D))$. If $\mathbf{u} \cdot \boldsymbol{\nu} = 0$ in the sense of traces and if $\nabla \cdot (b\mathbf{u}) = 0$ in D almost everywhere, then there is a unique Borel-measurable function $X : [0, +\infty) \times [0, +\infty) \times D \rightarrow D$, such that*

(i) *the map $(s, t) \in [0, +\infty)^2 \mapsto X(s, t, \cdot)$ is continuous for the convergence in measure,*

(ii) *for every $r, s, t \in [0, +\infty)$ and almost every $x \in D$, one has $X(s, t, x) = X(s, r, X(r, t, x))$,*

(iii) *for every function $f_0 \in L^\infty(D)$ and every $s, t \in [0, +\infty)$, one has*

$$\int_D f_0(X(s, t, x)) b(x) \, dx = \int_D f_0(x) b(x) \, dx,$$

(iv) *for almost every $x \in D$,*

$$X(s, t, x) = x - \int_s^t \mathbf{u}(r, X(s, r, x)) \, dr.$$

Moreover, for every $f_0 \in L^\infty(D)$, $f_0 \circ X(0, \cdot)$ is the unique function $f \in L^\infty([0, +\infty) \times D)$ that satisfies for every $\varphi \in C_c^1([0, +\infty) \times D)$,

$$\int_0^{+\infty} \int_D f(\partial_t \varphi + \mathbf{u} \cdot \nabla \varphi) + \int_D f_0 \varphi(0, \cdot) = 0$$

and $f \in C([0, +\infty), L^1(D))$.

As a corollary of the above representation formula, the potential vorticity $\omega(t)/b$ at any time $t \geq 0$ is a rearrangement of the initial potential vorticity ω_0/b , in the sense of the weighted Lebesgue measure $d\mu(x) = b(x) \, dx$.

Note that, a priori, the statements only make sense when the function f_0 is Borel measurable; the proposition implies then that $X(s, t, \cdot)$ preserves Lebesgue null sets and thus allows one to extend the statement to Lebesgue-measurable functions.

Proof of proposition A.2. We first observe that $\nabla \cdot \mathbf{u} = \mathbf{u} \cdot \nabla(\ln b)$ almost everywhere on D and thus $\nabla \cdot \mathbf{u} \in L^\infty(D)$. The existence and the properties (i), (ii) and (iv) of X follow from the DiPerna–Lions theory [17, Theorem III.2], as does the characterization of solutions to the transport equations and the continuity of the latter [17, Corollary II.2]. By proposition A.1, the transport equation holds for each test function $\varphi \in C_c^1([0, +\infty) \times \bar{D})$.

Given $f_0 \in L^\infty(D)$ as an initial data, we observe that the function $f : [0, +\infty) \times D \rightarrow \mathbb{R}$ defined for each $(t, x) \in [0, +\infty) \times D$ by $f(t, x) \triangleq f_0(X(t, 0, x))$ satisfies the transport equation. By taking $\varphi \in C_c^1([0, +\infty) \times \bar{D})$ defined for each $t, x \in [0, +\infty) \times \bar{D}$ by $\varphi(t, x) \triangleq b(x)\theta(t)$, with $\theta \in C_c^1((0, +\infty))$ as test function we have

$$\begin{aligned} \int_0^{+\infty} \theta'(t) \left(\int_D f_0(X(t, 0, \cdot)) b \right) dt + \theta(0) \int_D f_0 b \\ = - \int_0^{+\infty} \left(\int_D f(X(t, 0, \cdot)) \nabla \cdot (b \mathbf{u}(t)) \right) dt = 0, \quad (\text{A.2}) \end{aligned}$$

since $\nabla \cdot (b \mathbf{u}(t)) = 0$ almost everywhere in D for almost every $t \in [0, +\infty)$ and the conclusion follows. $\square$

B Regularity of solutions with smooth initial data

We prove that when the initial vorticity is smooth enough, then weak solutions of the vorticity formulation of the lake equations have some regularity.

Proposition B.1. *Assume that $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, that D is of class C^{k+1} and that $b \in (C^2 \cap C^{k+1, \alpha})(D)$. If $\omega_0 \in C^{k, \alpha}(\bar{D}, \mathbb{R})$ and if $(\omega, \mathbf{u}) \in L^\infty([0, +\infty) \times D, \mathbb{R}) \times L^\infty([0, +\infty), L^2(D, \mathbb{R}^2))$ is a weak solution to the vortex formulation of the lake equations, then for every $T > 0$, $\omega \in C^{k, \alpha}([0, T] \times \bar{D})$ and $\mathbf{u} \in L^\infty([0, T], C^{k+1, \alpha}(\bar{D}, \mathbb{R}^2)) \cap C^{k, \alpha}([0, T] \times \bar{D}, \mathbb{R}^2)$.*

When $k = 0$, proposition B.1 is due to Huang [24, Theorem 4.1].

Our proof follows the same strategy as proofs of the regularity of solutions of the planar Euler equations [39, §2.4] (see also [26, §3.1]).

The first tool that we use is the fact that the velocity field generated by a bounded vorticity field satisfies a bound known as quasi-Lipschitz bound [26, Lemma 1.4; 39, Lemma 3.1] or logarithmically Lipschitz [22].

Lemma B.2. *There exists a constant $C > 0$ such that for every $\omega \in L^\infty(\Omega)$ and every $x, y \in D$, one has*

$$|\nabla \mathcal{K}_b[\omega](x) - \nabla \mathcal{K}_b[\omega](y)| \leq C |y - x| \ln \frac{2 \operatorname{diam} D}{|y - x|}.$$

Proof. By proposition 4.3.3, we have

$$\begin{aligned} \nabla \mathcal{K}_b[\omega](x) &= \int_D \nabla G_D(x, z) \omega(z) \sqrt{b(x) b(z)} \, dz \\ &+ \frac{1}{2} \int_D G_D(x, z) \omega(z) \sqrt{\frac{b(z)}{b(x)}} \nabla b(x) \, dz + \int_D \nabla R_b(x, z) \omega(z) \, dz. \end{aligned} \quad (\text{B.1})$$

We first have the estimate

$$\begin{aligned} &\left| \int_D \nabla G_D(y, z) \omega(z) \sqrt{b(y) b(z)} \, dz \right. \\ &\quad \left. - \int_D \nabla G_D(x, z) \omega(z) \sqrt{b(x) b(z)} \, dz \right| \\ &\leq C_1 \|\omega\|_{L^\infty(D)} |y - x| \ln \frac{2 \operatorname{diam}(D)}{|y - x|} \end{aligned} \quad (\text{B.2})$$

(see [39, Lemma 2.3.1 and Appendix 2.3]). Next, we have

$$\begin{aligned} &\left| \int_D G_D(y, z) \omega(z) \sqrt{\frac{b(z)}{b(x)}} \nabla b(y) \, dz \right. \\ &\quad \left. - \int_D G_D(x, z) \omega(z) \sqrt{\frac{b(z)}{b(x)}} \nabla b(x) \, dz \right| \\ &\leq \frac{|\nabla b(y) - \nabla b(x)|}{\sqrt{b(x)}} \int_D G_D(x, z) |\omega(z)| \sqrt{b(z)} \, dz \\ &+ C_2 \frac{|\nabla b(x)|}{\sqrt{b(x)}} \int_D |G_D(y, z) - G_D(x, z)| |\omega(z)| \sqrt{b(z)} \, dz. \end{aligned} \quad (\text{B.3})$$

We compute now by proposition 4.3.5,

$$\begin{aligned}
& \int_D |G_D(y, z) - G_D(x, z)| \, dz \\
& \leq \int_0^1 \int_D |\nabla G_D((1-s)x + sy, z)| |y - x| \, dz \, ds \\
& \leq \int_0^1 \int_D \frac{C_3 |y - x|}{|(1-s)x + sy - z|} \, dz \, ds \leq C_4 |y - x|. \quad (\text{B.4})
\end{aligned}$$

By (B.3) and (B.4), we deduce, since the derivative of ∇b is bounded, that the gap

$$\begin{aligned}
& \left| \int_D G_D(y, z) \omega(z) \sqrt{\frac{b(z)}{b(y)}} \nabla b(y) \, dz \right. \\
& \quad \left. - \int_D G_D(x, z) \omega(z) \sqrt{\frac{b(z)}{b(x)}} \nabla b(x) \, dz \right| \quad (\text{B.5})
\end{aligned}$$

is bounded by

$$C_5 \|\omega\|_{L^\infty(D)} |y - x|.$$

In order to control the variation of $\nabla R_b(\cdot, z)$, we recall that by the proof of proposition 4.3.3, $R_b = S_b + Q_b$ for some function $Q_b \in C^2(D \times D)$ defined in proposition 4.2.4 and for some function S_b constructed in the proof of proposition 4.3.1 in such a way that for each $z \in D$, the function $S_b(\cdot, z) \in W_0^{1,2}(D)$ is the unique solution of the elliptic problem

$$\begin{cases} -\nabla \cdot (b^{-1} \nabla S_b(\cdot, z)) = -G_D(\cdot, z) \sqrt{b(z)} \left(\Delta \frac{1}{\sqrt{b}} \right) & \text{in } D, \\ S_b(\cdot, z) = 0 & \text{on } \partial D. \end{cases}$$

Hence, in order to conclude the proof, it is sufficient to focus on the S_b -term. We recall that for all $z \in D$ the function S_b admits the integral representation

$$S_b(x, z) = -\sqrt{b(z)} \int_D G_b(y, x) G_D(y, z) \left(\Delta \frac{1}{\sqrt{b}} \right)(y) \, dy;$$

and moreover, S_b is continuous and symmetric on $D \times D$ (proposi-

tion 4.3.1). Therefore, we have for all $x, z \in D$:

$$S_b(x, z) = S_b(z, x) = -\sqrt{b(x)} \int_D G_b(y, z) G_D(y, x) \left(\Delta \frac{1}{\sqrt{b}} \right)(y) dy,$$

or equivalently, using the symmetry of the Green's function G_D :

$$S_b(x, z) = - \int_D G_b(y, z) \left(\sqrt{b} G_D(\cdot, y) \right)(x) \left(\Delta \frac{1}{\sqrt{b}} \right)(y) dy.$$

In particular, a direct application of Fubini's theorem shows that, for almost-every $x \in D$, we have

$$\begin{aligned} & \nabla \int_D S_b(x, y) \omega(y) dy \\ &= \int_D \omega(y) \left(\int_D G_b(z, y) \left(\Delta \frac{1}{\sqrt{b}} \right)(z) \nabla \left(\sqrt{b} G_D(\cdot, z) \right)(x) dz \right) dy. \end{aligned} \quad (\text{B.6})$$

Since the L^p -norms of Green's functions are uniformly bounded on as y varies in D [54], we may apply estimates (B.2) and (B.5) to obtain

$$\begin{aligned} & \left| \int_D \nabla S_b(x, z) \omega(z) dz - \int_D \nabla S_b(x, z) \omega(z) dz \right| \\ & \leq C_6 \|\omega\|_{L^\infty(D)} |y - x| \ln \frac{2 \operatorname{diam}(D)}{|y - x|}, \end{aligned}$$

and therefore

$$\begin{aligned} & \left| \int_D \nabla R_b(x, z) \omega(z) dz - \int_D \nabla R_b(x, z) \omega(z) dz \right| \\ & \leq C_7 \|\omega\|_{L^\infty(D)} |y - x| \ln \frac{2 \operatorname{diam}(D)}{|y - x|}. \end{aligned} \quad (\text{B.7})$$

The conclusion follows from (B.1), (B.2), (B.5) and (B.7). $\square$

The next tool is Grönwall type estimate for a logarithmic perturbation of linear growth.

Lemma B.3. *Let $A, B, C \in [0, +\infty)$, $A < C$ and let f be a continuous function from $[0, +\infty)$ to $(0, C)$, that is: $f \in C([0, +\infty), (0, C))$. If for*

every $t \in [0, +\infty)$,

$$f(t) \leq A + B \int_0^t f(s) \ln \frac{C}{f(s)} \, ds,$$

then for every $t \in [0, +\infty)$,

$$f(t) \leq C \exp\left(-\ln \frac{C}{A} e^{-Bt}\right).$$

Proof. We observe that if the function $u \in C^1(\mathbb{R}, (0, +C))$ satisfies for every $t \in I$ the equation

$$u'(t) = B u(t) \ln \frac{C}{u(t)},$$

then

$$u(t) = C \exp\left(-e^{-Bt} \ln \frac{C}{u(0)}\right).$$

and the conclusion follows then by comparison. $\square$

We finally rely on the next classical regularity property of Lagrangian flows.

Lemma B.4. *If $\mathbf{u} \in C^{k-1}([0, +\infty) \times \bar{D}, \mathbb{R}^2) \cap C([0, +\infty), C^k(\bar{D}, \mathbb{R}^2))$ and if the function $X \in C^1([0, +\infty), C(\bar{D}, \bar{D}))$ satisfies*

$$\begin{cases} \partial_t X(t, x) = \mathbf{u}(t, X(t, x)) & \text{if } t \in [0, +\infty) \text{ and } x \in D, \\ X(0, x) = x & \text{if } x \in D, \end{cases}$$

then $X \in C^k([0, +\infty) \times D)$. If moreover $\mathbf{u} \in C^{k-1, \alpha}([0, +\infty) \times \bar{D}, \mathbb{R}^2) \cap L^\infty([0, +\infty), C^{k, \alpha}(\bar{D}, \mathbb{R}^2))$, then $X \in C^{k, \alpha}([0, +\infty) \times D)$.

Proof. The first part is classical (see for example [10, §1.7]). For the second part, we first have by the first part $X \in C^k(\mathbb{R} \times \bar{D}, \bar{D})$ and thus by the chain rule for Hölder continuous functions (see for example [11, Theorem 16.31]) $\partial_t X \in C^{k-1, \alpha}(\mathbb{R} \times \bar{D}, \bar{D})$.

Next we observe that for every $T > 0$ and $t \in [0, T]$, we have

$$|D_x^k X(t, y) - D_x^k X(t, x)|$$

$$\begin{aligned}
&\leq \int_0^t |D^k f(X(s, y)) [D_x^k X(s, y)] - D^k f(X(s, x)) [D_x^k X(s, x)]| \, ds \\
&\quad + C_1 |y - x| \leq C_2 \int_0^t |D_x^k X(s, y) - D_x^k X(s, x)| \, ds + C_3 |y - x|^\alpha,
\end{aligned}$$

and it follows then from the classical Grönwall inequality that

$$|D_x^k X(t, y) - D_x^k X(t, x)| \leq C_4 |y - x|^\alpha. \quad \square$$

Proof of proposition B.1. By proposition 4.2.6, proposition A.2 is applicable to $f_0 = \omega_0/b$ and implies that for every $t \in \mathbb{R}$,

$$\omega(t, x) = \frac{b(x)}{b(X(t, x))} \omega_0(X(t, x)).$$

By proposition A.2 and lemma B.2, we have for every $x, y \in D$,

$$\begin{aligned}
|X(t, y) - X(t, x)| &\leq |y - x| + C_1 \left((\|\omega_0\|_{L^\infty(D)} \right. \\
&\quad \left. + \|\Gamma\|) \int_0^t |X(s, y) - X(s, x)| \ln \frac{2 \operatorname{diam} D}{|X(s, y) - X(s, x)|} \, ds \right).
\end{aligned}$$

It follows then by lemma B.3, that

$$\begin{aligned}
&|X(t, y) - X(t, x)| \\
&\leq C_2 \exp\left(-\alpha e^{-C_3 t} \ln \frac{2 \operatorname{diam} D}{|y - x|}\right) = C_4 \left(\frac{|y - x|}{2 \operatorname{diam} D}\right)^{\alpha \exp(-C_3 t)}.
\end{aligned}$$

This implies thus that

$$|\omega(t, y) - \omega(t, x)| \leq C_5 \left(\frac{|y - x|}{2 \operatorname{diam} D}\right)^{\alpha \exp(-C_3 t)}.$$

By the representation formula for the velocity (4.2.7) and classical regularity estimates [21, Theorem 6.8], it follows then that we have the inclusion $\mathbf{u} \in C([0, +\infty), C^1(D))$. By classical regularity theory of the Lagrangian flow, this implies that $X \in C^1([0, +\infty) \times \bar{D}, \bar{D})$ and thus by composition $\omega \in C^{0,\alpha}([0, T] \times D)$. By regularity estimates [21, Theorem 6.8] we have then $\mathbf{u} \in L^\infty([0, T], C^{1,\alpha}(\bar{D}))$ and $u \in C^{0,\alpha}([0, T] \times \bar{D})$.

We assume now that $\mathbf{u} \in L^\infty([0, T], C^{k,\alpha}(\bar{D})) \cap C^{k-1,\alpha}([0, T] \times \bar{D})$ and that $\omega_0 \in C^{k,\alpha}(\bar{D})$. By regularity of the Lagrangian flow (lemma B.4), we have $X \in C^{k,\alpha}(D)$ and thus $\omega \in C^{k,\alpha}([0, T] \times D)$. By classical regularity estimates, this implies that $\mathbf{u} \in L^\infty([0, T], C^{k+1,\alpha}(\bar{D})) \cap C^{k,\alpha}([0, T] \times \bar{D})$. $\square$

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Chapter 5

Desingularization of a steady vortex pair in the lake equation

*The following chapter reflects a paper that was submitted,
available at [arXiv:1711.06497](https://arxiv.org/abs/1711.06497).*

5.1 Introduction

Statement of the problem

The lake equations arise from the incompressible 3D Euler equations in a regime where the typical velocity magnitude is small in comparison to the magnitude of gravity waves (small Froude number regime $\text{Fr} \ll 1$, see also [10]). Mathematically, a lake can be modeled as a planar open set $D \subseteq \mathbb{R}^2$ together with a positive depth function b . The *velocity field* $v : \mathbb{R} \times D \rightarrow \mathbb{R}^2$ and the *pressure field* $p : \mathbb{R} \times D \rightarrow \mathbb{R}$ are governed by the system of equations

$$\begin{cases} \operatorname{div}(bv) = 0 & \text{on } \mathbb{R} \times D, \\ \partial_t v + (v \cdot \nabla)v + fv^\perp = -\nabla p & \text{on } \mathbb{R} \times D, \\ bv \cdot \hat{\eta} = \nu & \text{on } \mathbb{R} \times \partial D. \end{cases}$$

Here $\hat{\eta} : \partial D \rightarrow \mathbb{R}^2$ is the unit outward normal to ∂D , $(v_1, v_2)^\perp = (-v_2, v_1)$, $f : D \rightarrow \mathbb{R}$ is a Coriolis parameter, and $\nu : \partial D \rightarrow \mathbb{R}$ is a penetration condition. When the depth function is constant, $b \equiv 1$, the system reduces to the incompressible 2D Euler equations. The velocity field v in the lake model may be understood as the horizontal velocity of a column water, whose total mass may vary according to the depth of the lake [10]. Global well-posedness of the lake equations has been studied by Levermore & Oliver & Titi [17], Lacave & Nguyen & Pausader [18], Munteanu [23], Huang & Chaocheng [14].

The problem we study in this article is the asymptotic behavior of steady velocity fields that are constructed as maximizers of the kinetic energy

$$E(v_\epsilon) = \frac{1}{2} \int_D |v_\epsilon|^2 d\mu, \quad d\mu = b dm,$$

where dm is the Lebesgue measure in the plane $\mathbb{R}^2$; in the regime where the vorticity in the lake vanishes: $d\mu(\{\text{vortex exists}\}) \rightarrow 0$ as $\epsilon \rightarrow 0$. Before formulating more precisely this problem, let us give some more motivation.

Let first consider the easier case where the Coriolis force is null: $f = 0$; and the topography does not vary: $b \equiv 1$. Introducing the vorticity $\omega = \text{curl}(v)$ (and up to prescribe circulation conditions on ∂D), one may think of v as $v = \text{curl}^{-1}(\omega)$, which allows one to think off the energy as a function of ω only:

$$E(v) \equiv E(\omega).$$

Moreover, taking the curl of the evolution equation (second equation in the lake system), we get the transport equation

$$\partial_t \omega + (v | \nabla \omega) = 0.$$

These observations were the starting point for Arnold [1] and Benjamin [3] to suggest that any steady solution of the 2D Euler equations should be a critical point of the energy under the following constraint: if ω solves the steady transport equation: $(\text{curl}^{-1}(\omega) | \nabla \omega)_{\mathbb{R}^2} = 0$; then for every function $D : \mathbb{R} \rightarrow \text{Diff}_{dm}(D)$ valued in the set of dm -measure preserving

diffeomorphisms of D with $D(0) = \text{Id}_D$, the energy functional

$$E : \mathbb{R} \rightarrow \mathbb{R} : E(t) := E(\omega \circ D_t^{-1})$$

has a critical point at $t = 0$. On the other hand, any vortex constructed as $\omega_t = \omega \circ D_t^{-1}$ satisfies the infinite number of constraints:

$$\text{dm}(\{\omega_t \geq \lambda\}) = \text{dm}(\omega \geq \lambda), \quad \forall \lambda \in \mathbb{R}.$$

We say that such a field ω_t is a dm-rearrangement of ω . The notion of rearrangement is a relaxed condition in comparison to the identity $\omega_t = \omega \circ D_t^{-1}$, since the regularity of the transformation D becomes out of concern. From this motivation, the following energy maximization principle has been extensively investigated by Arnold [1], Benjamin [3] and Burton [7–9]: *Find a maximizer of the energy E restricted on the set of all dm-rearrangements of a given profile function ω^* .* The asymptotic of maximizers as $\text{dm}(\{\omega_\epsilon \neq 0\}) = \epsilon^2, \epsilon \rightarrow 0$, has been studied by Turkington [27–29], Turkington & Friedman [13], and Elcrat & Miller [30] when ω_ϵ is non sign changing: $\omega_\epsilon \geq 0$. The authors have shown that the family of maximizers $\{\omega_\epsilon : \epsilon > 0\}$ tends to a Dirac mass as $\epsilon \rightarrow 0$, and the concentration point is chosen according to the geometry of ∂D (see also [19, 20, 22, 29] for similar results for the time-dependent problem, and [11, 26] for similar results based on other approaches).

The situation appears to be different in the lake model, when b is not constant. In fact, Richardson [25] has shown that the motion of a singular non sign changing vortex is led by the depth function b , independently (at leading order) of the geometry of ∂D . More precisely, Richardson's formula states that the motion of the center of mass Z_ϵ of a non sign changing potential vortex ζ_ϵ with $\text{d}\mu(\{\zeta_\epsilon > 0\}) = \epsilon^2$, follows the evolution law

$$\frac{d}{dt} Z_\epsilon(t) = \frac{1}{2\pi} \nabla^\perp b(Z_\epsilon(t)) \log \frac{1}{\epsilon} \mathcal{S}_\epsilon + \mathcal{O}(\mathcal{S}_\epsilon),$$

with $\mathcal{S}_\epsilon = \int_D \zeta_\epsilon \text{d}\mu$. Note that an application of $b^{-1} \text{curl}$ on the lake

evolution equation yields the new transport equation

$$\partial_t \zeta + (v|\nabla \zeta) = 0,$$

where $\zeta = b^{-1}(\text{curl}(v) - f)$ is the *potential vortex field* associated with the velocity v , where the external contribution f has been removed. Following the lines of thought of Arnold and Benjamin, we expect a variational principle of the following form: *Steady solutions with prescribed constraints on the $d\mu$ -measure of their super level sets, can be constructed by maximization of the energy.*

Formal statement and results

A measurable function $\zeta : D \rightarrow \mathbb{R}$ may be decomposed in a positive part $(\zeta)_+$ and a negative part $(\zeta)_-$, both measurable. Let us fix a distribution function

$$D : \mathbb{R}^+ \rightarrow [0, d\mu(D)]$$

normalized as

$$\int_{\mathbb{R}^+} D(t) dt = 1,$$

and such that there exists $p > 1$ with

$$\int_{\mathbb{R}^+} t^p D(t) dt < +\infty.$$

We also fix $\tau \in [0, 1]$. We say that a family of measurable functions $\{\zeta_\epsilon : \epsilon > 0\}$ satisfies the constraint (D) if, for all $\epsilon > 0$, we have

$$\begin{aligned} d\mu(\{(\zeta_\epsilon)_+ \geq \lambda\}) &= \frac{\epsilon^2}{\delta} D\left(\frac{\epsilon^2 \lambda}{\delta \tau} \log \frac{1}{\epsilon}\right), \\ d\mu(\{(\zeta_\epsilon)_- \geq \lambda\}) &= \frac{\epsilon^2}{\delta} D\left(\frac{\epsilon^2 \lambda}{\delta(1-\tau)} \log \frac{1}{\epsilon}\right), \end{aligned} \tag{D}$$

where $\delta = \sup_{\lambda > 0} D(\lambda)$, so that we always have $d\mu(\{(\zeta_\epsilon)_+ > 0\}) \leq \epsilon^2$ and $d\mu(\{(\zeta_\epsilon)_- > 0\}) \leq \epsilon^2$. It is also easy to check that $\zeta_\epsilon \in L^p(D, d\mu)$ and

$$\int_D (\zeta_\epsilon)_+ d\mu = \frac{\tau}{\log \frac{1}{\epsilon}}, \quad \text{and} \quad \int_D (\zeta_\epsilon)_- d\mu = \frac{1-\tau}{\log \frac{1}{\epsilon}}.$$

The above growth factor $\log \frac{1}{\epsilon}$ is motivated by Richardson's formula. The energy E_ϵ we maximize is given by

$$E(\zeta) = \frac{1}{2} \int_D (\mathfrak{K}\zeta + \Psi)\zeta \, d\mu,$$

where $flow$ is the flow that induces the Coriolis vorticity f : thus $\text{curl}(b^{-1}\nabla^\perp\Psi) = bf$; and $\mathfrak{K}\zeta$ is the flow that induces the “internal vorticity”: $\text{curl}(b^{-1}\nabla^\perp\mathfrak{K}\zeta) = b\zeta$. We prove the following theorem:

Theorem 5.1.1. *Assume that $D \subseteq \mathbb{R}^2$ is a (simply connected) bounded domain of class C^1 and assume that $b \in C(\overline{D}) \cap W_{loc}^{1,\infty}(D)$ with $\inf_D b > 0$, or $b = \phi^\alpha$ for some $\alpha > 0$ and ϕ a regularization of the distance at the boundary ∂D . Let $\Psi \in C(\overline{D}) \cap W_{loc}^{1,\infty}(D)$. There exists $\lambda_0 > 0$ depending only on τ, Ψ and b such that, for all $\lambda \in (0, \lambda_0)$, there exists a family $\{\zeta_\epsilon \in L^p(D, d\mu) : \epsilon > 0\}$ of steady solutions for the lake equations, obtained by maximization of the energy E over the class of all functions that satisfy constraint (D).*

Furthermore, if $\tau > 0$, the only possible accumulation points of $\{(\zeta_\epsilon)_+ : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, in the sense of convergence of probability measures on D , are Dirac masses δ_{x^} with*

$$\frac{\tau b(x^*)}{4\pi} + \lambda \Psi(x^*) = \sup_D \left\{ \frac{\tau b}{4\pi} + \lambda \Psi \right\},$$

and if $\tau < 1$, the only possible accumulation points of $\{(\zeta_\epsilon)_- : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, in the sense of convergence of probability measures on D , are Dirac masses δ_{x_} with*

$$\frac{(1-\tau)b(x_*)}{4\pi} - \lambda \Psi(x_*) = \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \lambda \Psi \right\}.$$

We briefly mention that theorem 5.1.1 will actually be proved for non necessarily simply connected domains, provided coherent circulation conditions are prescribed. Also, theorem 5.1.1 will be proved under slightly more general regularity assumptions on D and b , and covers two situations encountered in the literature: the case of a non vanishing depth function b [14, 17], and the case of a degenerated depth function that vanishes as a polynomial of some regularized distance at the boundary [4,

18]. Although theorem 5.1.1 is stated for regular domains, we will be in position to deal with irregular domains as well.

In the case of a non perturbed lake ($\Psi = 0$), our energy maximizers look like singular vortex pairs as $\epsilon \rightarrow 0$, both located near a point of maximal depth b . If we add some small perturbation $\lambda\Psi$, the two parts of the pairs separate from each others, according to Ψ . The condition that the *vortex strength*

$$\int_D \zeta_\epsilon \, d\mu = (2\tau - 1) \frac{1}{\log \frac{1}{\epsilon}}$$

vanishes as $(\log \frac{1}{\epsilon})^{-1}$ is expected from Richardson's law. In the lake equations, this scale length is the typical scale length for the dynamic of the vortex to become physically relevant.

Taking advantage of radial symmetries, theorem 5.1.1 may be used to prove the following time-dependent result:

Theorem 5.1.2. *Assume that $D = B(0, 1)$ and $b \in C(\overline{D}) \cap W_{loc}^{1,\infty}(D)$, $b(x) = b(|x|^2)$, is a radial function with $\inf_D b > 0$, or $b = \phi^\alpha$ for some $\alpha > 0$ and ϕ a regularization of the distance at the boundary $\partial B(0, 1)$. There exists $\nu_0 > 0$ depending only on $\tau \in [0, 1]$ such that, for all $\nu \in (0, \nu_0)$, there exists a family of curves $\{\zeta_\epsilon \in C(\mathbb{R}, L^p(D, d\mu)) : \epsilon > 0\}$ solving the time-dependent lake equations with the additional property that for each $t \in \mathbb{R}$; and the vortex $\zeta_\epsilon(t)$ at time t is obtained from $\zeta_\epsilon(0)$ by a rotation of clockwise angle νt , and satisfies constraint (D).*

Furthermore, if $\tau > 0$, the only possible accumulation points of $\{(\zeta_\epsilon(0))_+ : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, in the sense of convergence of probability measures on D , are Dirac masses δ_{x^} with*

$$\frac{\tau b(x^*)}{4\pi} - \frac{\nu}{2} \int_0^{|x^*|^2} b(s) \, ds = \sup_{z \in D} \left\{ \frac{\tau b(z)}{4\pi} - \frac{\nu}{2} \int_0^{|z|^2} b(s) \, ds \right\},$$

and if $\tau < 1$, the only possible accumulation points of $\{(\zeta_\epsilon)_- : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, in the sense of convergence of probability measures on D , are Dirac masses δ_{x_} with*

$$\frac{(1 - \tau)b(x_*)}{4\pi} + \frac{\nu}{2} \int_0^{|x_*|^2} b(s) \, ds = \sup_{z \in D} \left\{ \frac{(1 - \tau)b(z)}{4\pi} + \frac{\nu}{2} \int_0^{|z|^2} b(s) \, ds \right\}.$$

Method and insights of the proof

The condition $\operatorname{div}(bv) = 0$ together with $\operatorname{curl}(v) = b\zeta$ motivates to construct v as $v = b^{-1}\nabla^\perp\psi$ with ψ such that

$$-\operatorname{div}(b^{-1}\nabla\psi) = b\zeta.$$

This is an elliptic equation which may be solved on D , provided circulation conditions are prescribed. The function framework for this equation is discussed in the *first section*. We prove that the solution ψ constructed above depends on ζ through an operator $\mathfrak{K}$. In *appendix*, we prove an integral expansion

$$\mathfrak{K}\zeta(x) = \int_D G_D(x, y)\zeta(y) \, d\mu(y) + \int_D F(x, y)\zeta(y) \, d\mu(y),$$

where G_D is the Green's function for $-\Delta$ in D with Dirichlet boundary conditions, and $F : D \times D \rightarrow \mathbb{R}$ is a bounded and continuous function. Such Green's function expansion is not known for degenerate elliptic equations (see [15, 21, 32] for existence of Green's function and related estimates in the case where b is non degenerated). We prove this expansion by regularity theory [31], under suitable assumptions on both D and b . We cover the case of non regular domains D with $b \in C(\overline{D}) \cap W_{\text{loc}}^{1,\infty}(D)$ satisfying $\inf_D b > 0$, or b satisfying the ad hoc condition

$$\inf_{\substack{q \in [1, +\infty) \\ \beta \in [0, 2)}} \limsup_{d(x, \partial D)} \left\{ \frac{1}{(\delta_q(x))^\beta} \frac{|\nabla b(x)|^2}{b(x)} \right\} < +\infty;$$

or in other words: there exists $\beta \in [0, 2)$, $q \geq 1$ and a neighborhood $\mathcal{U}$ of ∂D such that, for all $x \in \mathcal{U}$:

$$\frac{1}{(\delta_q(x))^\beta} \frac{|\nabla b(x)|^2}{b(x)} \leq C$$

for some $C > 0$. Here δ_q is the mean distance at the boundary of D of order q , as defined in [2]; and it is well defined even for non regular domains. In comparison with [23], we do not rely on a Muckenhoupt condition. In a *second section*, we recall some preliminaries in rearrangements theory, and we use the work of Burton [6–9] to construct maximizers of

the energy that are also solutions of the lake equations. In *section 3*, we exploit our integral representation to study asymptotic properties of energy maximizers as the vortex profile vanishes. The strategies of proof is to compare the maximal energy with the energy produces by very symmetric competitors, as one would do for the Euler equations. However, since the relevant measure is $d\mu(x) = b(x)dx$, we have to face some technical difficulties. We manage to prove a generalization of theorem 5.1.1. The proof of theorem 5.1.2 is done in *section 4*.

5.2 The stream function

5.2.1 Lakes

By a *lake* we mean a couple (D, b) where $D \subseteq \mathbb{R}^2$ is a bounded open set and $b \in C(\overline{D})$ is a non negative continuous function. We ask the following additional conditions:

1. $D = \overline{D_0} \setminus \bigcup_{i=0}^m \mathcal{C}_i$ for a bounded connected open set $D_0 \subseteq \mathbb{R}^2$ and disjoint compact simply connected subsets $\mathcal{C}_i \subset \overline{D_0}$, with the convention that $\mathcal{C}_0 = \partial D_0$;
2. for all $i = 0, \dots, m$, we have either $\text{cap}(\mathcal{C}_i) > 0$, or b admits a Dini continuous extension by 0 on $D \cup \mathcal{C}_i$;
3. for all compact subset $K \subset D$, we have $\inf_K b > 0$.

In the above definition, each $\partial \mathcal{C}_i$ represents a shore. We allow $\mathcal{C}_i$ to be the boundary of a closed connected compact set (in which case we represent an island), $\mathcal{C}_i$ a (non closed) curve, but we also allow $\mathcal{C}_i$ to be reduced to a point shore, provided the depth function b is sufficiently regular in a neighborhood of $\mathcal{C}_i$.

5.2.2 Basic functional framework

In this section we introduce a functional framework for the stream function. We write $W^{1,2}(D)$ the standard Sobolev space of square

integrable functions on D whose distributional derivatives exist as square integrable functions on D , endowed with its usual scalar product

$$(\phi|\psi)_{W^{1,2}} = \int_D \phi\psi \, dm + \int_D (\nabla\phi|\nabla\psi)_{\mathbb{R}^2} \, dm.$$

Here dm denotes the Lebesgue measure on D . We denote by $\mathcal{H}$ the set of those functions $\phi \in W^{1,2}(D)$ such that $b^{-1}|\nabla\phi|^2 \in L^1(D, dm)$, endowed with the scalar product

$$(\phi|\psi)_{\mathcal{H}} = (\phi|\psi)_{W^{1,2}} + \int_D (\nabla\phi|\nabla\psi)_{\mathbb{R}^2} \frac{dm}{b}.$$

Since b is assumed to be positive on compact subsets of D , we have the set inclusion $C_c^1(D) \subseteq \mathcal{H}$. We denote by $\mathcal{H}_0 = \overline{C_c^1(D)}$ the closure of $C_c^1(D)$ in $\mathcal{H}$.

Proposition 5.2.1. *The quadratic form*

$$(\phi, \psi) \in \mathcal{H}_0 \mapsto (\phi|\psi)_{\mathcal{H}_0} = \int_D (\nabla\phi|\nabla\psi)_{\mathbb{R}^2} \frac{dm}{b}$$

defines a scalar product on $\mathcal{H}_0$ whose induced norm is equivalent to the norm induced by $\mathcal{H}$.

Proof. The fact that it defines a scalar product is straightforward. For the equivalence between the norms, one first observe that

$$\|\phi\|_{\mathcal{H}_0}^2 \leq \|\phi\|_{\mathcal{H}}^2$$

for all $\phi \in \mathcal{H}_0$. For a converse inequality, one may write

$$\|\phi\|_{\mathcal{H}_0}^2 \geq \frac{1}{2} \|\phi\|_{\mathcal{H}_0}^2 + \frac{1}{2(\sup_D b)} \int_D |\nabla\phi|^2 \, dm.$$

Since D is a bounded set, we may apply the standard Poincaré's inequality to obtain some constant $C_1 > 0$ such that

$$C_1 \int_D |\nabla\phi|^2 \, dm \geq \|\phi\|_{W^{1,2}}^2,$$

and thus we have

$$\|\phi\|_{\mathcal{H}_0}^2 \geq C_2(\|\phi\|_{\mathcal{H}_0}^2 + \|\phi\|_{W^{1,2}}^2) = C_2\|\phi\|_{\mathcal{H}}^2. \quad \square$$

5.2.3 Circulations

In this section we show how we are going to solve a problem of the form

$$\begin{cases} -\operatorname{div}(b^{-1}\nabla\phi) = 0, \\ \oint_{\partial\mathcal{C}_i} (b^{-1}\nabla^\perp\phi|\tau_i)_{\mathbb{R}^2} = c_i. \end{cases} \quad (\text{C})$$

Here above, the real numbers $c_0, \dots, c_m \in \mathbb{R}$ are fixed, and τ_i denotes the tangent vector field associated with $\mathcal{C}_i$ with clockwise orientation for $i = 1, \dots, m$, counterclockwise orientation for $\mathcal{C}_0$; in other words: $\tau_i = \hat{\eta}_i^\perp$ where $\hat{\eta}_i$ is the unit normal vector field outward to the boundary of ∂D_i , and $(v_1, v_2)^\perp = (-v_2, v_1)$. Each quantity

$$\oint_{\partial\mathcal{C}_i} (b^{-1}\nabla^\perp\phi|\tau_i)_{\mathbb{R}^2}$$

represents the *circulation of the velocity field* $b^{-1}\nabla^\perp\phi$ along $\partial\mathcal{C}_i$. The first equation

$$\operatorname{curl}(b^{-1}\nabla^\perp\phi) = -\operatorname{div}(b^{-1}\nabla\phi) = 0$$

means that the curl of the vector field $b^{-1}\nabla^\perp\phi$ is null, that is, no vortex is produced by this velocity field. Observe that from the circulation conditions we obtain

$$\begin{aligned} \sum_{i=0}^m \oint_{\partial\mathcal{C}_i} (b^{-1}\nabla^\perp\phi|\tau_i)_{\mathbb{R}^2} &= \sum_{i=0}^m \int_{\partial\mathcal{C}_i} (b^{-1}\nabla\phi|-\eta_i)_{\mathbb{R}^2} \\ &= - \int_{\partial D} (b^{-1}\nabla\phi|\eta)_{\mathbb{R}^2} = \int_D -\operatorname{div}(b^{-1}\nabla\phi), \end{aligned}$$

where η is the unit normal outward vector on D , and η_i its restriction on $\partial\mathcal{C}_i$. For problem (C), this leads to the consistency condition

$$\sum_{i=0}^m c_i = \int_D \operatorname{curl}(b^{-1}\nabla^\perp\phi) \, \mathrm{d}m = 0.$$

Since we work in an a priori rough setting (the normal vector τ_i may not be defined), we understand problem (C) in the weak sense. Let us assume that, for all $i = 0, \dots, m$, we have constructed a function $\psi_i \in \mathcal{H}$ that solves

$$\begin{cases} -\operatorname{div}(b^{-1}\nabla\psi_i) = 0, \\ \psi_i = \delta_{ij} \quad \text{on } \partial\mathcal{C}_j, \forall j \in \{0, \dots, m\}. \end{cases} \quad (\text{T})$$

Here δ_{ij} denotes the Kronecher's symbol. Then we would have

$$\begin{aligned} c_j &= \int_{\partial\mathcal{C}_j} (b^{-1}\nabla\phi|-\eta_j)_{\mathbb{R}^2} = - \int_{\partial D} (\psi_j b^{-1}\nabla\phi|_{\eta})_{\mathbb{R}^2} \\ &= - \int_D \operatorname{div}(\psi_j b^{-1}\nabla\phi), \end{aligned}$$

and therefore, if $\phi \in \mathcal{H}$ is a solution of problem (C), we have

$$c_j = - \int_D (\nabla\psi_j|\nabla\phi)_{\mathbb{R}^2} \frac{dm}{b}.$$

It is therefore natural to look for a solution $\phi \in \mathcal{H}$ of problem (C) as a linear combination

$$\phi = \sum_{i=0}^m \alpha_i \psi_i,$$

where the coefficients $\alpha_0, \dots, \alpha_m \in \mathbb{R}$ should be chosen in such a way that

$$-c_j = \sum_{i=0}^m \alpha_i \int_D (\nabla\psi_j|\nabla\psi_i)_{\mathbb{R}^2} \frac{dm}{b}, \quad \text{for all } j \in \{0, \dots, m\}.$$

Such a family $\psi_1, \dots, \psi_m$ would thus be useful to construct weak solutions of the circulation problem. However, one should take into account that D may be irregular, so that the notion of trace needed to define $\psi_1, \dots, \psi_m$, may be hard to define. In this section, we begin by proving that problem (T) admits a unique solution in a certain weak sense, and this solution is bounded. Once we have proved the existence of the functions $\psi_1, \dots, \psi_m$, we focus on the existence of a solution for our weak formulation of problem (C).

Proposition 5.2.2. *For all $i \in \{0, \dots, m\}$, there exists a unique function*

$\psi_i \in \mathcal{H}$ that satisfies

$$\int_D (\nabla \psi_i | \nabla \phi)_{\mathbb{R}^2} \frac{dm}{b} = 0, \quad \text{for all } \phi \in \mathcal{H}_0,$$

and such that there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of functions from $C^1(\overline{D})$ converging in $\mathcal{H}$ to ψ_i ; with $\varphi_n = \delta_{ij}$ on a neighborhood of $\partial \mathcal{C}_j$, for all $j \in \{0, \dots, m\}$. Moreover, the solution ψ_i satisfies $0 \leq \psi_i \leq 1$ on D .

This kind of existence result is standard [17, Lemma 4] and it is usually proved by using Schauder's theory. Here we propose an original proof based on purely functional analysis methods, that may be applied for non regular domains.

Proof. Let us consider the set

$$\Gamma_i = \{\varphi \in C^1(\overline{D}) : \varphi = \delta_{ij} \text{ on a neighborhood of } \partial \mathcal{C}_j, \\ \text{for all } j \in \{0, \dots, m\}\}.$$

We are going to work in the quotient space $H = \mathcal{H}/\sim$, where $\sim$ is the equivalence relation

$$\phi_1 \sim \phi_2 \iff \int_D |\nabla(\phi_1 - \phi_2)|^2 \frac{dm}{b} = 0.$$

The equivalence class of a function $\phi \in \mathcal{H}$ is simply denoted by $[\phi]$. The quotient H is then endowed with the well-defined scalar product

$$([\phi_1] | [\phi_2])_H := \int_D (\nabla \phi_1 | \nabla \phi_2)_{\mathbb{R}^2} \frac{dm}{b}.$$

We claim that the set

$$\Pi_i = \{[\phi] : \phi \in \overline{\Gamma_i}\}$$

is a complete and convex subset of H . The set Γ_i itself is convex, and so is its closure $\overline{\Gamma_i}$ in $\mathcal{H}$. Thus Π_i is convex as well. For the completeness, the key point is to observe that the equivalence class $[\phi]$ of some $\phi \in \mathcal{H}_0$ only contains ϕ . Hence the equivalence class of some $[\phi]$ with $\phi \in \overline{\Gamma_i}$ only contains ϕ . Hence completeness of Π_i is consequence of completeness of $\mathcal{H}_0$. According to the convex projection theorem [5, Theorem 5.2], there

exists a unique $\psi_i \in \overline{\Gamma_i}$ such that $[\psi_i] \in \Pi_i$ is the projection of $[0] \in H$ onto Π_i . For all $t \in \mathbb{R}$ and for all $\varphi \in \mathcal{H}_0$, we have $\psi_i + t\varphi \in \overline{\Gamma_i}$ and in particular, we must have

$$\|[\psi_i + t\varphi]\|_H^2 \geq \|[\psi_i]\|_H^2,$$

which also reads as

$$|t| \int_D |\nabla \varphi|^2 \frac{dm}{b} + 2 \frac{t}{|t|} \int_D (\nabla \varphi | \nabla \psi_i)_{\mathbb{R}^2} \frac{dm}{b} \geq 0.$$

Letting $t \rightarrow 0$ yields the identity

$$\int_D (\nabla \varphi | \nabla \psi_i)_{\mathbb{R}^2} \frac{dm}{b} = 0,$$

for all $\varphi \in \mathcal{H}_0$.

Finally, the inequalities $0 \leq \psi_i \leq 1$ are proved by an adaptation of weak maximum principle techniques [31, Theorem 8.1]. Let us illustrate this by proving $\psi_i \leq 1$. The lower bound $\psi_i \geq 0$ will be proved using similar arguments. By construction of ψ_i , the positive part $(\psi_i - 1)_+$ belongs to $\mathcal{H}_0$. In particular, we have

$$0 = \int_D (\nabla \psi_i | \nabla (\psi_i - 1)_+)_{\mathbb{R}^2} \frac{dm}{b} = \int_D |\nabla (\psi_i - 1)_+|^2 \frac{dm}{b}.$$

Thus $\nabla (\psi_i - 1)_+ = 0$ on D . This is possible only if $(\psi_i - 1)_+ = 0$ on D , and thus $\psi_i \leq 1$. $\square$

Now we turn to the existence of solution of problem (C). We recall that by a weak solution of problem (C), we mean a linear combination

$$\phi = \sum_{i=0}^m \alpha_i \psi_i,$$

with $\alpha_0, \dots, \alpha_m \in \mathbb{R}$ such that

$$-c_j = \sum_{i=0}^m \alpha_i \int_D (\nabla \psi_j | \nabla \psi_i)_{\mathbb{R}^2} \frac{dm}{b}.$$

We require a first lemma:

Lemma 5.2.3. *A linear combination*

$$\phi = \sum_{i=0}^m \alpha_i \psi_i$$

with $\alpha_0, \dots, \alpha_m \in \mathbb{R}$ belongs to $\mathcal{H}_0$ if, and only if, we have $\alpha_i = 0$ for all $i = 0, \dots, m$. In particular,

$$\sum_{i=0}^m \psi_i = 1.$$

This lemma strongly uses the regularity assumptions on b near a point shore.

Proof. If we have $\alpha_i = 0$ for all $i \in \{0, \dots, m\}$, then the linear combination is null and belongs to $\mathcal{H}_0$. From this it is easy to see that $1 - \sum_{i=0}^m \psi_i$ belongs to $\mathcal{H}_0$ by construction, and therefore $\sum_{i=0}^m \psi_i = 1$. For the converse, let us assume that $\sum_{i=0}^m \alpha_i \psi_i$ belongs to $\mathcal{H}_0$. By construction of each ψ_i and by definition of $\mathcal{H}_0$ there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of smooth functions $\varphi_n \in C^1(\overline{D})$ and $\varphi_n = \alpha_i$ on a neighborhood of $\mathcal{C}_i$; and the sequence $(\varphi_n)_{n \in \mathbb{N}}$ converges in $\mathcal{H}$ to 0. Up to a subtraction by α_0 , we have found a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of $C_c^1(D)$ converging in $\mathcal{H}$ to $-\alpha_0$. By completeness of $W_0^{1,2}(\mathbb{R}^2)$ we have $\alpha_0 = 0$ and the convergence also occurs in $W_0^{1,2}(\mathbb{R}^2)$. Up to passing to a subsequence, we may assume that the convergence occurs quasi-everywhere on $\mathbb{R}^2$ [33, Section 7.2]. Let $i \in \{0, \dots, m\}$ be such that $\text{cap}(\mathcal{C}_i) > 0$. Then we must have $\alpha_i = -\alpha_0$. Assume that there exists $i \in \{0, \dots, m\}$ such that $\text{cap}(\mathcal{C}_i) = 0$. Since $\mathcal{C}_i$ is connected and has null capacity, Pólya-Szegő's inequality [33, Theorem 8.3.14] implies that $\mathcal{C}_i = \{p_i\}$. By assumption, the depth function b may be extended by 0 on $D \cup \{p_i\}$, and the extension (still denoted b) is Dini continuous at p_i . In particular, we have for all $\varphi \in C^1(\mathbb{R}^2)$ supported in D :

$$\begin{aligned} |\varphi(p_i)| &= \left| \int_D (\nabla \varphi(x) \cdot \frac{p_i - x}{|p_i - x|^2})_{\mathbb{R}^2} \frac{\text{d}m(x)}{b(x)} \right| \\ &\leq \left(\int_D \frac{b(x)}{|p_i - x|^2} \text{d}m(x) \right)^{\frac{1}{2}} \left(\int_D |\nabla \varphi|^2 \frac{\text{d}m}{b} \right)^{\frac{1}{2}}. \end{aligned}$$

There exists a constant $C_1(b, p_i) > 0$ such that for all $\varphi \in C^1(\mathbb{R}^2)$ supported in D , we have

$$|\varphi(p_i)| \leq C_1 \left(\int_D |\nabla \varphi|^2 \frac{dm}{b} \right)^{\frac{1}{2}} \leq C_1 \|\varphi\|_{\mathcal{H}}.$$

Applying this inequality on the sequence $(\varphi_n)_{n \in \mathbb{N}}$ yields

$$|\alpha_i| = \lim_{n \rightarrow +\infty} |\varphi_n(p_i)| = -\alpha_0 = 0.$$

□

Theorem 5.2.4. *The symmetric matrix $\mathcal{A} \in \mathbb{R}^{(m+1) \times (m+1)}$ defined by*

$$\mathcal{A}_{ij} = \int_D (\nabla \psi_i | \nabla \psi_j)_{\mathbb{R}^2} \frac{dm}{b}$$

has kernel spanned by $(1, \dots, 1) \in \mathbb{R}^{m+1}$ and non negative eigenvalues. In particular, for any choice $c_0, \dots, c_m \in \mathbb{R}$ such that

$$\sum_{i=0}^m c_i = 0,$$

there exists a unique weak solution $\phi \in \mathcal{H}$ for problem (C) of the form $\phi = \sum_{i=1}^m \alpha_i \psi_i$.

Proof. According to lemma 5.2.3, we already know that $(1, \dots, 1)$ belongs to the kernel $\text{Ker}(\mathcal{A})$ of $\mathcal{A}$. Let $(\alpha_0, \dots, \alpha_m)$ be another element of $\text{Ker}(\mathcal{A})$. We compute

$$\int_D \left| \nabla \sum_{i=0}^m \alpha_i \psi_i \right|^2 \frac{dm}{b} = (\alpha_0, \dots, \alpha_m) \mathcal{A} (\alpha_0, \dots, \alpha_m)^\dagger = 0,$$

where the $M^\dagger$ denotes the transpose of M . In particular, the linear combination $\sum_{i=0}^m \alpha_i \psi_i$ is a constant function. According to lemma 5.2.3, we have $\alpha_0 = \alpha_i$ for all $i \in \{0, \dots, m\}$, and thus $\text{Ker}(\mathcal{A})$ is spanned by $(1, \dots, 1)$. Now if $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathcal{A}$, with eigenvector

$\alpha = (\alpha_0, \dots, \alpha_m)$, the same computation shows that

$$0 \leq \int_D \left| \nabla \sum_{i=0}^m \alpha_i \psi_i \right|^2 \frac{dm}{b} = \lambda |\alpha|^2,$$

and thus $\lambda \geq 0$. Finally, since $\mathcal{A}$ is symmetric, its range is the orthogonal space of its kernel in $\mathbb{R}^{m+1}$, and the restriction of $\mathcal{A}$ to its range defines a linear isomorphism. The conclusion follows. $\square$

5.2.4 Construction of the stream function

In this section, we show how to construct a stream function $\psi \in \mathcal{H}$ that solves

$$\begin{cases} -\operatorname{div}(b^{-1} \nabla \psi) = \omega, \\ \oint_{\partial \mathcal{C}_i} (b^{-1} \nabla^\perp \psi | \tau_i)_{\mathbb{R}^2} = c_i, \end{cases}$$

for given circulations $c_0, \dots, c_m$ that satisfy

$$\sum_{i=0}^m c_i = \int_D \omega \, dm.$$

The function ω is the physical vortex associated to the velocity field $b^{-1} \nabla^\perp \psi$. As mentioned in the introduction, it will be suitable to work with the potential vortex $\zeta = b^{-1} \omega$ instead, so we change our notations and we solve in $\mathcal{H}$:

$$\begin{cases} -\operatorname{div}(b^{-1} \nabla \psi) = b \zeta, \\ \oint_{\partial \mathcal{C}_i} (b^{-1} \nabla^\perp \psi | \tau_i)_{\mathbb{R}^2} = c_i. \end{cases} \quad (\text{P})$$

The functional space of interest for the vortex data ζ is the Lebesgue space $L^p(D, d\mu)$, where $p > 1$ and the measure $d\mu(x) = b(x) \, dm(x)$ is the invariant measure associated to the lake (D, b) . Observe that since b is a bounded function, we automatically have $(b\zeta) \in L^p(D, dm)$.

Let us explain in what sense we understand the weak form of the problem (P). Assume that $\psi \in \mathcal{H}$ solves the above problem, and let $\phi \in C_c^1(D)$ be a smooth test function. By the divergence theorem we

have formally, if ψ is a solution of problem (P):

$$\int_D \phi \zeta \, d\mu = - \int_D \phi \operatorname{div} \left(b^{-1} \nabla \psi \right) \, d\mu = \int_D (\nabla \psi | \nabla \phi)_{\mathbb{R}^2} \frac{d\mu}{b}.$$

Since we have continuous embeddings

$$\begin{aligned} (\mathcal{H}, \|\cdot\|_{\mathcal{H}}) &\hookrightarrow (W^{1,2}(D), \|\cdot\|_{W^{1,2}}) \\ &\hookrightarrow (L^{p'}(D, d\mu), \|\cdot\|_{L^{p'}(D, d\mu)}) \hookrightarrow (L^{p'}(D, d\mu), \|\cdot\|_{L^{p'}(D, d\mu)}), \end{aligned}$$

a weak formulation of the elliptic equation in problem (P) would read as

$$\int_D \phi \zeta \, d\mu = (\psi | \phi)_{\mathcal{H}_0}, \quad \text{for all } \phi \in \mathcal{H}_0.$$

Let us now turn to the interpretation of the circulation conditions. As in the previous section, we compute for all $j \in \{0, \dots, m\}$, at least formally:

$$\begin{aligned} c_j &= - \int_{\mathcal{C}_j} (b^{-1} \nabla \psi | -\eta_j)_{\mathbb{R}^2} = - \int_{\partial D} (\psi_j b^{-1} \nabla \psi | -\eta)_{\mathbb{R}^2} \\ &= - \int_D \operatorname{div} \left(\psi_j b^{-1} \nabla \psi \right), \end{aligned}$$

so that

$$-c_j + \int_D \psi_j \zeta \, d\mu = \int_D (\nabla \psi | \nabla \psi_j)_{\mathbb{R}^2} \frac{d\mu}{b}.$$

Accordingly, we propose the following weak formulation of problem (P).

Definition 5.2.5. A function $\psi \in \mathcal{H}$ is a weak solution of problem (P) if $\psi = u + \mathfrak{H}(\zeta)$ for $u \in \mathcal{H}_0$ such that

$$(u | \phi)_{\mathcal{H}_0} = \int_D \phi \zeta \, d\mu, \quad \text{for all } \phi \in \mathcal{H}_0,$$

and

$$\mathfrak{H}(\zeta) = \sum_{i=0}^m \alpha_i (\psi_i - c_i)$$

for $\alpha_0, \dots, \alpha_m \in \mathbb{R}$ such that $\alpha_0 = 0$ and for all $j \in \{0, \dots, m\}$:

$$-c_j + \int_D \psi_j \zeta \, d\mu = \sum_{i=0}^m \alpha_i \int_D (\nabla \psi_i | \nabla \psi_j)_{\mathbb{R}^2} \frac{d\mu}{b}.$$

Theorem 5.2.6. *Given $\zeta \in L^p(D, d\mu)$ for $p > 1$ and prescribed circulations $c_0, \dots, c_m \in \mathbb{R}$ with*

$$\sum_{i=0}^m c_i = \int_D \zeta d\mu,$$

there exists a unique linear combination

$$\mathfrak{H}(\zeta) = \sum_{i=1}^m \alpha_i (\psi_i - c_i) \in \mathcal{H},$$

where ψ_i is given by proposition 5.2.2; and a unique $\mathfrak{K}(\zeta) \in \mathcal{H}_0$, such that the combined flow $\mathfrak{K}(\zeta) + \mathfrak{H}(\zeta) \in \mathcal{H}$ solves problem (P) in the sense of definition 5.2.5.

Proof. Let us consider the functional

$$\Phi : \mathcal{H}_0 \rightarrow \mathbb{R} : \phi \mapsto \frac{1}{2} \|\phi\|_{\mathcal{H}_0}^2 - \int_D \phi \zeta d\mu.$$

The functional Φ is strictly convex, hence it admits at most one minimizer. The existence of the minimizer follows from the lower semi-continuity of the norm and the compact embedding

$$(\mathcal{H}_0, \|\cdot\|_{\mathcal{H}_0}) \hookrightarrow (L^{p'}(D, d\mu), \|\cdot\|_{L^{p'}_{d\mu}}).$$

We define $\mathfrak{K}(\zeta)$ as the unique minimizer of Φ . From this variational principle we conclude that

$$(\mathfrak{K}(\zeta)|\phi)_{\mathcal{H}_0} = \int_D \phi \zeta d\mu, \quad \text{for all } \phi \in \mathcal{H}_0,$$

According to theorem 5.2.4, the required coefficients $\alpha_0, \dots, \alpha_m \in \mathbb{R}$ are uniquely determined if we impose $\alpha_0 = 0$. We define

$$\mathfrak{H}(\zeta) = \sum_{i=1}^m \alpha_i (\psi_i - c_i).$$

Now by construction of the static flows ψ_i for $i = 0, \dots, m$, the function $\mathfrak{K}(\zeta) + \mathfrak{H}(\zeta) \in \mathcal{H}$ is a weak solution of problem (P) in the sense of definition 5.2.5. The uniqueness follows from lemma 5.2.3. $\square$

The choice of α_0 in our definition of weak solution is motivated by the following result:

Corollary 5.2.7 (Energy equivalence). *For all $\zeta \in L^p(D, d\mu)$, we have*

$$\int_D (\mathfrak{K}(\zeta) + \mathfrak{H}(\zeta)) \zeta d\mu = \int_D \|\nabla(\mathfrak{K}(\zeta) + \mathfrak{H}(\zeta))\|_{\mathbb{R}^2}^2 \frac{dm}{b}.$$

Below are listed basic properties of the stream function.

Corollary 5.2.8 (Symmetry). *For all $\zeta_1, \zeta_2 \in L^p(D, d\mu)$, we have*

$$\int_D \zeta_1 \mathfrak{K}(\zeta_2) d\mu = \int_D \zeta_2 \mathfrak{K}(\zeta_1) d\mu$$

and similarly

$$\int_D \zeta_1 \mathfrak{H}(\zeta_2) d\mu = \int_D \zeta_2 \mathfrak{H}(\zeta_1) d\mu.$$

Corollary 5.2.9 (Linearity). *For all $\zeta_1, \zeta_2 \in L^p(D, d\mu)$ and for all $\alpha \in \mathbb{R}$, we have*

$$\mathfrak{K}(\zeta_1 + \alpha \zeta_2) = \mathfrak{K}(\zeta_1) + \alpha \mathfrak{K}(\zeta_2).$$

Corollary 5.2.10 (Positivity). *For all $\zeta \in L^p(D, d\mu)$ with $\zeta \geq 0$, we have $\mathfrak{K}(\zeta) \geq 0$.*

Corollary 5.2.11 (Boundedness). *There exists constants $C, C' > 0$ such that For all $\zeta \in L^p(D, d\mu)$ with $\zeta \geq 0$, we have*

$$\|\mathfrak{K}(\zeta)\|_{\mathcal{H}_0} \leq C \|\zeta\|_{L^p_{d\mu}},$$

and

$$\|\mathfrak{H}(\zeta)\|_{\mathcal{H}} \leq C' \|\zeta\|_{L^1_{d\mu}}.$$

Proposition 5.2.12. *If a sequence $(\zeta_n)_{n \in \mathbb{N}}$ of $L^p(D, d\mu)$ functions is bounded in $L^p(D, d\mu)$ and weakly converges to some $\zeta \in L^p(D, d\mu)$, then the sequences $(\mathfrak{K}(\zeta_n))_{n \in \mathbb{N}}$ and $(\mathfrak{H}(\zeta_n))_{n \in \mathbb{N}}$ both converge strongly, respectively to $\mathfrak{K}(\zeta)$ in $\mathcal{H}_0$ and $\mathfrak{H}(\zeta)$ in $\mathcal{H}$.*

Proof. We first prove the claim for the operator $\mathfrak{K}$. Since we have a compact embedding $(\mathcal{H}_0, \|\cdot\|_{\mathcal{H}_0}) \hookrightarrow (L^{p'}(D, d\mu), \|\cdot\|_{L^{p'}_{d\mu}})$, it is sufficient to prove that every accumulation point of the sequence $(\mathfrak{K}(\zeta_n))_{n \in \mathbb{N}}$ in

the sense of strong topology of $L^{p'}(D, d\mu)$ equals $\mathfrak{K}(\zeta)$, and that the convergence also occurs in $(\mathcal{H}_0, \|\cdot\|_{\mathcal{H}_0})$. Let then $u \in L^{p'}(D, d\mu)$ be an accumulation point of $(\mathfrak{K}(\zeta_n))_{n \in \mathbb{N}}$ in $L^{p'}(D, d\mu)$. There exists a subsequence $(\mathfrak{K}(\zeta_{n_k}))_{k \in \mathbb{N}}$ converging in $L^{p'}(D, d\mu)$ to $u \in L^{p'}(D, d\mu)$. In particular, for all test function $\phi \in C_c^1(D)$, we have

$$\begin{aligned} \int_D \phi u \, d\mu &= \lim_{k \rightarrow +\infty} \int_D \phi \mathfrak{K}(\zeta_{n_k}) \, d\mu \\ &= \lim_{k \rightarrow +\infty} \int_D \mathfrak{K}(\phi) \zeta_{n_k} \, d\mu = \int_D \mathfrak{K}(\phi) \zeta \, d\mu = \int_D \phi \mathfrak{K}(\zeta) \, d\mu. \end{aligned}$$

Therefore $u = \mathfrak{K}(\zeta)$ in $L^p(D, d\mu)$, that is, almost-everywhere on D . On the other hand, we also have

$$\|\mathfrak{K}(\zeta) - \mathfrak{K}(\zeta_{n_k})\|_{\mathcal{H}_0}^2 = \int_D \zeta \mathfrak{K}(\zeta) \, d\mu + \int_D \zeta_{n_k} \mathfrak{K}(\zeta_{n_k}) \, d\mu - 2 \int_D \zeta_{n_k} \mathfrak{K}(\zeta) \, d\mu.$$

The right side converges to 0, and therefore $(\mathfrak{K}(\zeta_{n_k}))_{k \in \mathbb{N}}$ converges strongly in $(\mathcal{H}_0, \|\cdot\|_{\mathcal{H}_0})$ to the function $\mathfrak{K}(\zeta)$.

Let us now prove the similar statement for the operator $\mathfrak{H}$. By construction, the coefficients $\alpha_0, \dots, \alpha_m$ given by theorem 5.2.4 depends on the vortex function ζ only through the perturbed circulations

$$-c_j + \int_D \psi_j \zeta \, d\mu.$$

In particular, weak convergence of the sequence $(\zeta_n)_{n \in \mathbb{N}}$ implies strong convergence in $\mathbb{R}$ of each perturbed circulations, and so strong convergence in $\mathbb{R}$ of each coefficient α_j . Therefore the sequence $(\mathfrak{H}(\zeta_n))_{n \in \mathbb{N}}$ strongly converges in $\mathcal{H}$ to some $u \in \mathcal{H}$ and by uniqueness, we must have the identity $u = \mathfrak{H}(\zeta)$. $\square$

5.2.5 Continuous lakes

Let us write $G_D : D \times D \rightarrow \mathbb{R}$ the Green's function for the Laplace's operator $-\Delta$ with Dirichlet's boundary conditions on ∂D . We extend G_D by 0 on $\overline{D} \times \overline{D}$. The following lemma ensures that this extension makes sense, at least in the sense of weak convergence in every $W_0^{1,q}(D)$, $q \in (1, 2)$:

Lemma 5.2.13. *The function $[y \in \overline{D} \mapsto G_D(\cdot, y)]$ is continuous in the sense of weak convergence in $W_0^{1,q}(D)$, for any $q \in (1, 2)$.*

Proof. Let $q \in (1, 2)$ be fixed. The fact that $G_D(\cdot, y)$ is uniformly bounded in $W_0^{1,q}(D)$ as y varies in $\overline{D}$ follows from standard argument. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence of points in D converging to some $y \in \overline{D}$. The sequence $(G_D(\cdot, y_n))_{n \in \mathbb{N}}$ admits at least one accumulation point $g \in W_0^{1,q}(D)$ in the sense of weak convergence in $W_0^{1,q}(D)$. We are done if we prove $g = G_D(\cdot, y)$, since the weak convergence on a bounded set defines a metrizable topology. Let $L : W_0^{1,q}(D) \rightarrow \mathbb{R}$ be any non null continuous linear functional, where the norm of $W_0^{1,q}(D)$ is understood as the gradient norm in $L^q(D)$. According to James' representation theorem [33, Proposition 5.2.3], there exists a (unique) $u \in W_0^{1,q}(D)$ such that $\|u\|_{W_0^{1,q}} = 1$ and

$$\begin{aligned} L(v) &= \|L\|_{(W_0^{1,q})^*} \int_D (\nabla v | \nabla u)_{\mathbb{R}^2} |\nabla u|^{q-2} \\ &= \|L\|_{(W_0^{1,q})^*} \int_D (\nabla v | \nabla \frac{|u|^{q-1}}{q-1})_{\mathbb{R}^2}. \end{aligned}$$

Now it is readily checked that there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of $C_c^1(D)$ functions converging in $W_0^{1, \frac{q}{q-1}}(D)$ to $|u|^{q-1}$, so that

$$L(G_D(\cdot, y) - g) = \lim_{n \rightarrow +\infty} \|L\|_{(W_0^{1,q})^*} \int_D (\nabla(G_D(\cdot, y) - g) | \nabla \varphi_n)_{\mathbb{R}^2} = 0.$$

In the last line we have used the definition of the Green's function together with the definition of the weak limit g . This shows that every element of $(W_0^{1,q}(D))^*$ vanishes on $G_D(\cdot, y) - g$. This forces $G_D(\cdot, y) = g$, and therefore the sequence $(G_D(\cdot, y_n))_{n \in \mathbb{N}}$ converges weakly to $G_D(\cdot, y)$. $\square$

As a matter of fact, we mention that the uniform boundedness argument for the Green's function that we rely on in the previous lemma may be proved through at least three different approaches: through regularity and a duality argument in the spirit of [21] where the characterization of the dual of $L^\infty(D)$ as a set of finitely additive measures; from complex analysis through the Riemann conformal mapping theorem; or from direct measure geometric like argument [32]. We do not enter into details.

Definition 5.2.14. A lake (D, b) is said to be *continuous* if

1. the function $b \in C(\overline{D})$ admits weak derivatives,
2. there exists $\ell > 2$ such that function

$$y \in D \mapsto \int_D g(\cdot, y)^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} dm$$

is bounded on $\overline{D}$,

3. for all $y_\star \in \overline{D}$, we have

$$\lim_{y \rightarrow y_\star} \int_D |g(\cdot, y) - g(\cdot, y_\star)|^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} dm = 0.$$

Note that the fact that b is positive on compact subsets implies that $b \in W_{\text{loc}}^{1, \ell}(D)$, for some $\ell > 2$. In particular we must have $C^{0, \alpha}(K)$ for all compact subsets $K \Subset D$, and for some $\alpha > 0$ depending on K . The main theorem we are going to use on continuous lakes is the following:

Theorem 5.2.15. *Let (D, b) be a continuous lake. There exists a bounded measurable function $R : D \times D \rightarrow \mathbb{R}$ such that, for all $\zeta \in L^p(D, d\mu)$, $p > 1$, we have almost-everywhere*

$$\mathfrak{K}\zeta(x) = b(x) \int_D G_D(x, y) \zeta(y) d\mu(y) + \int_D R(x, y) \zeta(y) d\mu(y).$$

Furthermore, we have $R(\cdot, y) \in \mathcal{H}_0$ for all $y \in D$, with for all $\varphi \in \mathcal{H}_0$:

$$\int_D (\nabla R(\cdot, y) | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b} = - \int_D (G_D(\cdot, y) \nabla b | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b}.$$

In particular, there exists a bounded measurable function $F : D \times D \rightarrow \mathbb{R}$ such that

$$\mathfrak{K}\zeta(x) + \mathfrak{H}\zeta(x) = b(x) \int_D G_D(x, y) \zeta(y) d\mu(y) + \int_D F(x, y) \zeta(y) d\mu(y).$$

The proof of theorem 5.2.15 is based on standard ideas from regularity theory. The so-constructed function R is measurable on the product space $D \times D$, which will allow us to manipulate it through Fubini's

theorem. In fact, we can prove that R is continuous on $D \times D$, but this extra assumption will not be used in the text. The details on the construction of R are done in the appendix of this chapter. Note that the existence and boundedness of the function F simply follows from the one of R and the construction of $\mathfrak{H}\zeta$.

Below we give examples of continuous lakes (D, b) .

Example 5.2.16. If $b \in W^{1,\infty}(D)$ with $\inf_D b > 0$, then (D, b) is a continuous lake. Here we rely on the property that $G_D(\cdot, y)$ converges weakly in $W_0^{1,q}(D)$ to $G_D(\cdot, y_*)$ as y converges to y_* ; then we use Rellich-Kondrashov's theorem to obtain strong convergence in $L^p(D, \text{dm})$ for all $p \in [1, +\infty)$.

Example 5.2.17. If $\delta : \overline{D} \rightarrow \mathbb{R}$ is a regularization of the distance at the boundary, then any function $b : D \rightarrow \mathbb{R}$ that satisfies

$$\limsup_{\delta(x) \rightarrow 0} \left\{ \frac{1}{\delta(x)^\beta} \frac{|\nabla b(x)|^2}{b(x)} \right\} < +\infty$$

for some small exponent $\beta \in [0, 2)$, yields to a continuous lake (D, b) . This follows from the Hardy's inequality for regular domains D . This family of examples covers the cases of $b = \delta^\alpha$, for sufficiently well behaving $\alpha \in C^1(\overline{D})$. Indeed we have

$$\frac{|\nabla b|^2}{b} \sim \delta^\alpha |\log \delta|^2 |\nabla \alpha|^2 + \delta^{\alpha-2} \alpha |\nabla \delta|^2.$$

If $\inf_D \alpha > 0$, then $\alpha - 2 > -2$ uniformly on D , and the Hardy's inequality may be applied. If, for example, $\alpha = \delta^p$ for some $p > 1$, then we would have

$$\frac{|\nabla b|^2}{b} \sim \delta^{\delta^p + 2(p-1)} |\log \delta|^2 + \delta^{\delta^p - 2 + p} |\nabla \delta|^2$$

and again $\delta^p - 2 + p > -1$ uniformly on D . Observe that the special case (D, b) with $D = B(0, 1)$ and $b(x) = 1 - |x|$, does not satisfy the $\mathcal{A}_2$ Muckenhoupt condition.

Example 5.2.18. Let us consider $D = [-1, 1]^2 \setminus \{(0, y) : y \leq 0\}$ the square of side length 2 centered on 0, without one half inner vertical

median. On this set we consider the function

$$b : D \rightarrow \mathbb{R}^+ : b(x_1, x_2) = |x_1| + x_2 \chi_{\{x_2 > 0\}}.$$

So the lake (D, b) has a line shore. We compute

$$\frac{|\nabla b(x_1, x_2)|^2}{b(x_1, x_2)} \leq \frac{2}{|x_1| + x_2 \chi_{\{x_2 > 0\}}}.$$

By linearity of the integral, we do not lose in generality in splitting D into three disjoint parts:

$$\begin{aligned} D_1 &:= \{(x_1, x_2) \in D : |x_1| < 1, 0 < x_2 < 1\}, \\ D_2 &:= \{(x_1, x_2) \in D : (x_1 \geq 1, x_2 > 0) \text{ or } (x_2 \geq 1)\}, \\ D_3 &:= \{(x_1, x_2) \in D : x_2 \leq 0\}. \end{aligned}$$

On D_2 the function $b^{-1}|\nabla b|^2$ is uniformly bounded, so one may rely on the first example to control the contribution from D_2 . On D_1 we always have

$$\frac{|\nabla b(x_1, x_2)|^2}{b(x_1, x_2)} \leq \frac{2}{\sqrt{x_1^2 + x_2^2}},$$

and therefore it belongs to $L^q(D_1)$ for all $q \in [1, 2)$. The conclusion on D_1 then follows as in the previous example. For the last part, an elementary computation shows that for all $(x_1, x_2) \in D_3$, we have

$$\frac{|\nabla b(x_1, x_2)|^2}{b(x_1, x_2)} \leq \frac{2}{|x_1|} \leq C_1 \frac{1}{d_p(x_1, x_2)},$$

where d_p is the mean distance at the boundary ∂D of order $p > 1$, defined for all $x \in D$ by

$$(d_p(x))^{-1} = 2 \left(\int_{\mathbb{P}_2} \frac{1}{(\rho_\nu(x))^p} d\sigma(\nu) \right)^{\frac{1}{p}},$$

where $\mathbb{P}_2$ denotes the 2-dimensional projective plane endowed with its Haar measure $d\sigma$, and $\rho_\nu(x)$ is the least distance at x to the boundary in the direction ν [2, Chapter 3]. One may then use the Hardy's inequality on general domains [2, Theorem 3.3.2] to obtain the conclusion on D_3 .

5.3 Maximization of the energy

Let us consider an energy functional of the form

$$E : L^p(D, d\mu) \rightarrow \mathbb{R} : E(\zeta) = \frac{1}{2} \int_D \zeta(\mathfrak{K} + \mathfrak{H})(\zeta) d\mu + \int_D \Psi \zeta d\mu.$$

The flow $\Psi \in \mathcal{H}$ is independent of the vortex ζ . The assumption $\Psi \in \mathcal{H}$ is of physical importance, since it means that the associated velocity $b^{-1}\nabla\Psi$ brings a finite contribution to the kinetic energy of the system.

5.3.1 Rearrangement of functions

Let $\zeta \in L^p(D, d\mu)$ be a given function. We recall that by a $d\mu$ -rearrangement of ζ , we mean a function $\tilde{\zeta} \in L^p(D, d\mu)$ such that, for all $\lambda \in \mathbb{R}$, we have

$$d\mu(\{\zeta \geq \lambda\}) = d\mu(\{\tilde{\zeta} \geq \lambda\}).$$

This defines an equivalence relation on $L^p(D, d\mu)$. The set of all $d\mu$ -rearrangements of a given function $\zeta \in L^p(D, d\mu)$ will be denoted by $\text{Rearg}(\zeta)$. Observe that for $\zeta \in L^p(D, d\mu)$ and $\xi \in \text{Rearg}(\zeta)$, we have $(\xi)_+ \in \text{Rearg}((\zeta)_+)$ and similarly $(\xi)_- \in \text{Rearg}((\zeta)_-)$. It then follows from the Cavalieri's principle (see also [24]) that for all $\tilde{\zeta} \in \text{Rearg}(\zeta)$, we have

$$\left(\int_D |\xi|^q d\mu \right)^{\frac{1}{q}} = \left(\int_D |\zeta|^q d\mu \right)^{\frac{1}{q}},$$

for any $q \in [1, +\infty)$. In particular, for $p \in (1, +\infty)$, the set $\text{Rearg}(\zeta)$ is closed with respect to the strong topology of $L^p(D, d\mu)$, and relatively compact with respect to the weak topology.

The following proposition provides a way to construct “well behaving” rearrangements. The proof is standard in symmetrization theory, but we recall it for completeness.

Proposition 5.3.1. *For all $x \in D$, there exists a function*

$$x^\sharp : L^1(D, d\mu) \rightarrow L^1(D, d\mu)$$

such that for all positive function $\zeta \in L^1(D, d\mu)$ we have

1. $x^\sharp \zeta \in \text{Rearg}(\zeta)$;

2. the superlevel sets of $x\sharp\zeta$ are balls in D centered on x .

Proof. For all $\lambda \in \mathbb{R}$, we define

$$r_\lambda = \sup \left\{ r \geq 0 : d\mu(D \cap \overline{B(x, r)}) \leq d\mu(\{\zeta \geq \lambda\}) \right\}.$$

We have $r_{\lambda_1} \leq r_{\lambda_2}$ as soon as $\lambda_1 \geq \lambda_2$. We then define the function

$$x\sharp\zeta : D \rightarrow \mathbb{R}^+ : x\sharp\zeta(y) = \sup \{ \lambda \in \mathbb{R} : y \in D \cap \overline{B(x, r_\lambda)} \}.$$

It then follows from the definition of $(x\sharp\zeta)$ that for all $\lambda \in \mathbb{R}$:

$$\{x\sharp\zeta \geq \lambda\} = D \cap \overline{B\left(x, \inf_{s < \lambda} r_s\right)} = D \cap \overline{B(x, r_\lambda)}.$$

On the other hand, we have

$$d\mu(D \cap \overline{B(x, r_\lambda)}) = d\mu(\{\zeta \geq \lambda\}),$$

and therefore $x\sharp\zeta$ is a $d\mu$ -rearrangement of ζ whose superlevel sets are balls centered on x . $\square$

Proposition 5.3.2. *The energy functional E is strictly convex and for all $\zeta \in L^p(D, d\mu)$, the function $(\mathfrak{K} + \mathfrak{H})(\zeta) + \Psi$ belongs to the subgradient of E at point ζ .*

Proof. The third contribution

$$\zeta \in L^p(D, d\mu) \mapsto \int_D \Psi \zeta \, d\mu$$

is linear, hence convex. The first contribution

$$\zeta \in L^p(D, d\mu) \mapsto \frac{1}{2} \int_D \zeta \mathfrak{K}(\zeta) \, d\mu = \frac{1}{2} \|\mathfrak{K}(\zeta)\|_{\mathcal{H}_0}^2$$

is strictly convex. It remains to prove that the second contribution

$$\zeta \in L^p(D, d\mu) \mapsto \frac{1}{2} \int_D \zeta \mathfrak{H}(\zeta) \, d\mu$$

is convex. Let $\zeta_1, \zeta_2 \in L^p(D, d\mu)$ and $t \in [0, 1]$. A computation shows

that, if $\zeta = \zeta_1 - \zeta_2$, we have

$$\int_D \zeta \mathfrak{H}(\zeta) \, d\mu = \sum_{i=0}^m \alpha_i \int_D \zeta(\psi_i - c_i) \, d\mu = (\alpha_0, \dots, \alpha_m) \mathcal{A}(\alpha_0, \dots, \alpha_m)^\dagger,$$

where $\mathcal{A}$ is the matrix constructed in theorem 5.2.4. According to theorem 5.2.4, the matrix $\mathcal{A}$ has non negative eigenvalues, and therefore the above quantity is non negative as well. In particular, we have

$$\begin{aligned} \frac{1}{2} \int_D (t\zeta_1 + (1-t)\zeta_2) \mathfrak{H}(t\zeta_1 + (1-t)\zeta_2) \, d\mu \\ \leq t \frac{1}{2} \int_D \zeta_1 \mathfrak{H}(\zeta_1) \, d\mu + (1-t) \frac{1}{2} \int_D \zeta_2 \mathfrak{H}(\zeta_2) \, d\mu. \end{aligned}$$

For the last claim, we have to prove that, for all $\tilde{\zeta} \in L^p(D, d\mu)$, we have

$$E(\tilde{\zeta}) - E(\zeta) \geq \int_D (\tilde{\zeta} - \zeta)(\mathfrak{K}(\zeta) + \mathfrak{H}(\zeta) + \Psi) \, d\mu.$$

By linearity, it is sufficient to check that

$$\int_D \tilde{\zeta}(\mathfrak{K} + \mathfrak{H})(\tilde{\zeta}) \, d\mu - \int_D \zeta(\mathfrak{K} + \mathfrak{H})(\zeta) \, d\mu \geq 2 \int_D (\tilde{\zeta} - \zeta)(\mathfrak{K} + \mathfrak{H})(\zeta) \, d\mu.$$

Here we have used the fact that the eigenvalues of $\mathcal{A}$ are non negative. $\square$

In this section we show that E admits a maximizer over $\text{Rearg}(\zeta)$, and that such maximizer leads to solution of the steady lake equations.

Proposition 5.3.3. *Let $\zeta \in L^p(D, d\mu)$, $p > 1$. The restriction of the energy E on $\text{Rearg}(\zeta)$ admits at least one maximizer $\tilde{\zeta}$. If $\tilde{\zeta}u \in L^1_{loc}(D, \mathbb{R}^2)$ is defined by*

$$u = b^{-1} \nabla^\perp (\mathfrak{K}(\tilde{\zeta}) + \mathfrak{H}(\tilde{\zeta}) + \Psi),$$

then for all $\phi \in C_c^1(D)$, we have

$$\int_D (\tilde{\zeta}u | \nabla \phi)_{\mathbb{R}^2} \, d\mu = 0.$$

The proof strongly relies on an existence theorem due to Burton [7, Theorem A], which has been proved in the case of sign changing vortices by an adaptation of the bathtub principle [16].

Proof. By [7, Theorem A], there exists a maximizer $\tilde{\zeta} \in \text{Rearg}(\zeta)$ of E over $\text{Rearg}(\zeta)$, and a non decreasing function $G : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\tilde{\zeta} = G\left(\mathfrak{R}(\tilde{\zeta}) + \mathfrak{H}(\tilde{\zeta}) + \Psi\right).$$

Let us write for short $\psi = \mathfrak{R}(\tilde{\zeta}) + \mathfrak{H}(\tilde{\zeta}) + \Psi$, and assume that

$$\tilde{\zeta} b^{-1} \nabla^\perp \psi \in L^1_{\text{loc}}(D, \mathbb{R}^2).$$

We define, for all $n \in \mathbb{N}$:

$$[G]_n : \mathbb{R} \rightarrow \mathbb{R} : [G]_n(t) = \min\{n, \max\{G(t), -n\}\}.$$

For all $n \in \mathbb{N}$, the function $[G]_n$ is non decreasing and bounded. We define similarly the truncation at $k \in \mathbb{N}$ of the function ψ , so that:

$$[\psi]_k \in \mathcal{H}, \quad \nabla [\psi]_k = \chi_{\{|\psi| \leq k\}} \nabla \psi.$$

The function

$$F_{n,k} : [-n, +\infty) \rightarrow \mathbb{R} : \int_{-k}^t [G]_n(s) ds$$

is Lipschitz continuous. The function $F_{n,k} \circ [\psi]_k$ then belongs to $W^{1,2}(D)$ with

$$\nabla(F_{n,k} \circ [\psi]_k) = ([G]_n \circ [\psi]_k) \nabla [\psi]_k.$$

As a result, it follows from the divergence theorem that for all $\phi \in C_c^1(D)$, we have

$$\int_D (\nabla^\perp [\psi]_k | \nabla \phi)_{\mathbb{R}^2} ([G]_n \circ [\psi]_k) dm = 0.$$

For all $n \in \mathbb{N}$ we have

$$\left| (\nabla^\perp [\psi]_k | \nabla \phi)_{\mathbb{R}^2} ([G]_n \circ [\psi]_k) \right| \leq \left| (\nabla^\perp \psi | \nabla \phi)_{\mathbb{R}^2} \tilde{\zeta} \right|,$$

and the latter belongs to $L^1(D, \mu)$ by Hölder's inequality. On the other hand, we have almost-everywhere on D :

$$\lim_{k \rightarrow +\infty} (\nabla^\perp [\psi]_k(x) | \nabla \phi(x))_{\mathbb{R}^2} [G]_n([\psi]_k(x))$$

$$= (\nabla^\perp \psi(x) | \nabla \phi(x))_{\mathbb{R}^2} [G]_n(\psi(x)),$$

because $\nabla \psi = 0$ almost-everywhere on a set of the form $\{\psi = \alpha\}$. Since G is monotone, the set of discontinuities of G is at most countable, hence we miss at most a countable union of negligible sets in D . According to the Lebesgue's dominated convergence theorem, we obtain

$$\int_D (\tilde{\zeta} \nabla^\perp \psi | \nabla \phi)_{\mathbb{R}^2} \, d\mu = 0,$$

which is equivalent to the conclusion. $\square$

5.4 Asymptotic behavior of maximizers

In this section, we turn our attention to the asymptotic behavior of a maximizing family $\{\zeta_\epsilon \in L^p(D, d\mu) : \epsilon > 0\}$ with related energies

$$E_\epsilon(\zeta) = \frac{1}{2} \int_D \zeta \mathfrak{R}(\zeta) + \zeta \mathfrak{H}_\epsilon(\zeta) \, d\mu + \int_D \lambda_\epsilon \Psi \zeta \, d\mu.$$

We assume that the vortex profiles obey the identities

$$\tau \mathcal{S}_\epsilon = \int_D (\zeta_\epsilon)_+ \, d\mu, \quad (1 - \tau) \mathcal{S}_\epsilon = \int_D (\zeta_\epsilon)_- \, d\mu$$

for some $\tau \in [0, 1]$ and $\mathcal{S}_\epsilon > 0$. We make the following ($d\mu$ -rearrangement invariant) extra assumption:

$$\sup_{\epsilon > 0} \left\{ \frac{\|(\zeta_\epsilon)_+\|_{L^p_{d\mu}} \epsilon^{2(1-\frac{1}{p})}}{\tau \mathcal{S}_\epsilon} + \frac{\|(\zeta_\epsilon)_-\|_{L^p_{d\mu}} \epsilon^{2(1-\frac{1}{p})}}{(1-\tau) \mathcal{S}_\epsilon} \right\} < +\infty.$$

(Here above, the convention $\frac{0}{0} = 0$ is used.) This extra condition is motivated by the fact that any family that satisfies constraint (D) (page 182) as in the introduction also satisfies the above control condition.

More precisely, if we are given a distribution function

$$D : \mathbb{R}^+ \rightarrow [0, d\mu(D)]$$

normalized as

$$\int_{\mathbb{R}^+} D(t) dt = 1,$$

and such that there exists $p > 1$ with

$$\int_{\mathbb{R}^+} t^p D(t) dt < +\infty,$$

and if we define, for all $\epsilon > 0$, a reference profile $\tilde{\zeta}_\epsilon$ such that

$$\begin{aligned} d\mu(\{\tilde{\zeta}_\epsilon\}_+ \geq \lambda) &= \frac{\epsilon^2}{\delta} D\left(\frac{\epsilon^2 \lambda}{\delta \tau} \log \frac{1}{\epsilon}\right), \\ d\mu(\{\tilde{\zeta}_\epsilon\}_- \geq \lambda) &= \frac{\epsilon^2}{\delta} D\left(\frac{\epsilon^2 \lambda}{\delta(1-\tau)} \log \frac{1}{\epsilon}\right), \end{aligned}$$

where $\delta = \sup_{\lambda > 0} D(\lambda)$, then using the axiom of choice a family of maximizers $\{\zeta_\epsilon : \epsilon > 0\}$ as above always exists. In other words, the condition we impose on the $L^p(D, d\mu)$ -norms is satisfied by every family constructed by some scaling process.

The circulations $c_{\epsilon;0}, \dots, c_{\epsilon;m} \in \mathbb{R}$ that come into play in the definition of operator $\mathfrak{H}_\epsilon$ are assumed to depend on $\epsilon > 0$ through

$$c_{\epsilon;i} = c_i(2\tau - 1)\mathcal{S}_\epsilon$$

for $c_0, \dots, c_m \in \mathbb{R}$ independent of $\epsilon > 0$, and

$$\sum_{i=0}^m c_i = 1.$$

We also assume that we have $\lambda_\epsilon \geq 0$ with

$$\lim_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} = 1.$$

On the lake (D, b) we make the assumption that (D, b) is continuous, in the sense of definition 5.2.14. We also make the following assumption that the external flow Ψ is continuous on $\overline{D}$, and sufficiently small so that:

1. there exists $\varsigma > 0$ such that, for all sufficiently small $\epsilon > 0$, if $\tau > 0$:

$$\sup_D \left\{ \frac{4\pi\lambda_\epsilon|\Psi|}{(\sup_D b)\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} \leq \frac{\tau}{2} - \varsigma;$$

and if $\tau < 1$:

$$\sup_D \left\{ \frac{4\pi\lambda_\epsilon|\Psi|}{(\sup_D b)\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} \leq \frac{(1-\tau)}{2} - \varsigma.$$

If $\tau > 0$ (resp. $\tau < 1$), then the above conditions ensure that the function

$$\phi : \overline{D} \rightarrow \mathbb{R} : \phi(x) = \frac{\tau b(x)}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi(x)$$

(resp. $b/(4\pi) - \phi$) never reaches a maximal value on a shore, that is: on some point $x_\star$ where $b(x_\star) = 0$. This qualitative information will be used several times, but we will also require the quantitative estimate to show that the positive and the negative parts of the vortex remain concentrated.

Finally, we write $d\zeta(x) = \frac{\zeta}{\|\zeta\|_{L^1_{d\mu}}} d\mu$ for all non null function $\zeta \in L^p(D, d\mu)$, and $d0 = 0$.

5.4.1 Leading partial flows

The leading partial flows induced by some $\zeta \in L^p(D, d\mu)$ are defined for all $x \in D$ by

$$\mathcal{T}_\zeta^+(x) = \frac{b(x)}{4\pi} \int_D \log \frac{\text{diam}(D)}{|x-y|} (\zeta)_+(y) d\mu(y) + \lambda_\epsilon \Psi(x),$$

and

$$\mathcal{T}_\zeta^-(x) = \frac{b(x)}{4\pi} \int_D \log \frac{\text{diam}(D)}{|x-y|} (\zeta)_-(y) d\mu(y) - \lambda_\epsilon \Psi(x).$$

Proposition 5.4.1. *Let*

$$G_D(x, y) = \frac{1}{2\pi} \log \frac{\text{diam}(D)}{|x-y|} - H(x, y)$$

be the Green's function for the Laplace's operator $-\Delta$ on D with Dirichlet

boundary conditions, let F be given as in theorem 5.2.15. Then we have for all $\zeta \in L^p(D, d\mu)$ and for all $\epsilon > 0$:

$$\begin{aligned} E_\epsilon(\zeta) &= \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_\zeta^+ d(\zeta)_+ + (1 - \tau) \mathcal{S}_\epsilon \int_D \mathcal{T}_\zeta^- d(\zeta)_- \\ &\quad + \frac{1}{2} \iint_{D \times D} (F(x, y) - b(x)H(x, y)) \zeta(x) \zeta(y) d\mu \otimes d\mu(x, y) \\ &\quad - \iint_{D \times D} \frac{b(x) + b(y)}{4\pi} \log \frac{\text{diam}(D)}{|x - y|} (\zeta)_+(x) (\zeta)_-(y) d\mu \otimes d\mu(x, y). \end{aligned}$$

In particular, there exists a constant $C > 0$ such that

$$E_\epsilon(\zeta) \leq \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_\zeta^+ d(\zeta)_+ + (1 - \tau) \mathcal{S}_\epsilon \int_D \mathcal{T}_\zeta^- d(\zeta)_- + C \mathcal{S}_\epsilon^2.$$

The proof is straightforward from the fact that (D, b) is a continuous lake, by using theorem 5.2.15. and the properties of H , R , $\mathfrak{H}$, and the positivity of $\mathfrak{K}$. We omit it.

Proposition 5.4.2. *There exists a constant $C > 0$ such that for all $\epsilon > 0$ and for all $\zeta \in \text{Rearg}(\zeta_\epsilon)$:*

$$\mathcal{T}_\zeta^+(x) \leq \frac{\tau b(x)}{4\pi} \log \frac{1}{\epsilon} \mathcal{S}_\epsilon + \lambda_\epsilon \Psi(x) + C \tau \mathcal{S}_\epsilon$$

and

$$\mathcal{T}_\zeta^-(x) \leq \frac{(1 - \tau)b(x)}{4\pi} \log \frac{1}{\epsilon} \mathcal{S}_\epsilon - \lambda_\epsilon \Psi(x) + C(1 - \tau) \mathcal{S}_\epsilon.$$

Proof. In both cases, it is sufficient to control the quantity

$$\int_D \log \frac{\text{diam}(D)}{|x - y|} (\zeta_\epsilon)_+ d\mu$$

by a suitable upper bound. Define

$$M = \sup_{\epsilon > 0} \left\{ \frac{\|(\zeta_\epsilon)_+\|_{L^p_{d\mu}} \epsilon^{2(1-\frac{1}{p})}}{\tau \mathcal{S}_\epsilon} + \frac{\|(\zeta_\epsilon)_-\|_{L^p_{d\mu}} \epsilon^{2(1-\frac{1}{p})}}{(1 - \tau) \mathcal{S}_\epsilon} \right\} < +\infty,$$

so that

$$\sup_{\epsilon > 0} \left\{ \frac{\|(\zeta_\epsilon)_+\|_{L^p_{d\mu}} \epsilon^{2(1-\frac{1}{p})}}{\mathcal{S}_\epsilon} \right\} \leq \tau M < +\infty.$$

In the above condition, one may replace ζ_ϵ by any of its $d\mu$ -rearrangement,

because $d\mu$ -rearrangements preserve every $L^q(D, d\mu)$ -norms, $q \in [1, +\infty]$. From this we compute, for all $\zeta \in \text{Rearg}(\zeta_\epsilon)$:

$$\begin{aligned} \int_D \log \frac{\epsilon}{|x-y|} (\zeta)_+(y) d\mu(y) &\leq \int_D \left(\log \frac{\epsilon}{|x-y|} \right)_+ (\zeta)_+(y) d\mu(y) \\ &\leq \|(\zeta)_+\|_{L^p_{d\mu}} \left(\sup_D b \right)^{\frac{1}{p}} \left(\int_D \left(\log \frac{\epsilon}{|x-y|} \right)_+^{p'} d\mu(y) \right)^{\frac{1}{p'}} \\ &\leq \|(\zeta)_+\|_{L^p_{d\mu}} \left(\sup_D b \right)^{\frac{1}{p}} \left(\int_{B(0,\epsilon)} \left(\log \frac{\epsilon}{|y|} \right)_+^{p'} d\mu(y) \right)^{\frac{1}{p'}} \\ &\leq \|(\zeta)_+\|_{L^p_{d\mu}} \left(\sup_D b \right)^{\frac{1}{p'}} \epsilon^{\frac{2}{p'}} \left(\int_{B(0,1)} \left(\log \frac{1}{|y|} \right)_+^{p'} d\mu(y) \right)^{\frac{1}{p'}}. \end{aligned}$$

Using the definition of M , there exists a constant $C_1 > 0$ such that

$$\int_D \log \frac{\epsilon}{|x-y|} (\zeta)_+(y) d\mu(y) \leq C_1 \mathcal{S}_\epsilon.$$

The conclusion now follows from the definitions of $\mathcal{T}_\zeta^+$ and $\mathcal{T}_\zeta^-$. $\square$

The following lemma will be used several times in the text.

Lemma 5.4.3. *Let $x_1, x_2 \in \overline{D}$ be such that $b(x_1) > 0$ and $b(x_2) > 0$, and let $\mathcal{U}_1, \mathcal{U}_2 \subseteq D$ be open sets containing respectively x_1 and x_2 . There exists approximation families $\{x_\epsilon^1 \in \mathcal{U}_1 : \epsilon > 0\}$ and $\{x_\epsilon^2 \in \mathcal{U}_2 : \epsilon > 0\}$ such that*

$$\lim_{\epsilon \rightarrow 0} d(x_1, x_\epsilon^1) = 0, \quad \lim_{\epsilon \rightarrow 0} d(x_2, x_\epsilon^2) = 0,$$

and the following conditions hold for sufficiently small $\epsilon > 0$:

- there exists $r_1 > 0$ such that $\epsilon^2 \leq d\mu(B(x_\epsilon^1, r_1\epsilon))$;
- there exists $r_2 > 0$ such that $\epsilon^2 \leq d\mu(B(x_\epsilon^2, r_2\epsilon))$;
- $B(x^1, r_1\epsilon) \subseteq \mathcal{U}_1$ and $B(x^2, r_2\epsilon) \subseteq \mathcal{U}_2$;
- $d(B(x^1, r_1\epsilon), \partial D) \geq \frac{1}{\log \frac{1}{\epsilon}}$ and $d(B(x^2, r_2\epsilon), \partial D) \geq \frac{1}{\log \frac{1}{\epsilon}}$;
- $d(B(x^1, r_1\epsilon), B(x^2, r_2\epsilon)) \geq \frac{1}{\log \frac{1}{\epsilon}}$.

We omit the proof of lemma 5.4.3, which may be done by geometric arguments. Note that since we do not claim anything on the regularity of ∂D at points x_1 and x_2 , the rate of convergence of the families $\{x_\epsilon^1 : \epsilon > 0\}$ and $\{x_\epsilon^2 : \epsilon > 0\}$ to their respective limit is not explicitly known.

Proposition 5.4.4. *For all $\epsilon > 0$ sufficiently small, there exists $\Theta_\epsilon \leq 0$ with $\lim_{\epsilon \rightarrow 0} \Theta_\epsilon = 0$ and such that there holds*

$$\begin{aligned} \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} & \left(\tau \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right. \\ & \left. + (1 - \tau) \sup_D \left\{ \frac{(1 - \tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right) \\ & \leq E_\epsilon(\zeta_\epsilon) - \Theta_\epsilon \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon}. \end{aligned}$$

Proof. Define the function

$$\phi : \overline{D} \rightarrow \mathbb{R} : \phi(z) = \frac{\tau b(z)}{4\pi} + \Psi(z).$$

The function ϕ reaches a maximal value at some point $x_1 \in \{b > 0\}$, and the function $\frac{b}{4\pi} - \phi$ reaches a maximal value at some point $x_2 \in \{b > 0\}$. There exists $\eta_1, \eta_2 > 0$ such that the sets $\mathcal{U}_1 = \{b > \eta_1\}$ and $\mathcal{U}_2 = \{b > \eta_2\}$ contain respectively x_1 and x_2 . Let $\{x_\epsilon^1 \in \mathcal{U}_1 : \epsilon > 0\}$ and $\{x_\epsilon^2 \in \mathcal{U}_2 : \epsilon > 0\}$ be given by lemma 5.4.3, and $r_1, r_2 > 0$ the associated radii. For all $\epsilon > 0$ sufficiently small, we compare the maximal energy with the energy produced by symmetrized pairs $x_\epsilon^{1\#}(\zeta_\epsilon)_+$ and $x_\epsilon^{2\#}(\zeta_\epsilon)_-$. For sufficiently small $\epsilon > 0$, the function $\xi_\epsilon = (x_\epsilon^{1\#}(\zeta_\epsilon)_+ - x_\epsilon^{2\#}(\zeta_\epsilon)_-)$ is a $d\mu$ -rearrangement of ζ_ϵ , and

$$\{x_\epsilon^{1\#}(\zeta_\epsilon)_+ > 0\} \subseteq B(x_\epsilon^1, r_1\epsilon), \quad \{x_\epsilon^{2\#}(\zeta_\epsilon)_- > 0\} \subseteq B(x_\epsilon^2, r_2\epsilon).$$

Now we compute the energy E_ϵ produced by the above competitor ξ_ϵ using proposition 5.4.1:

$$E_\epsilon(\xi_\epsilon) = \int_D \mathcal{T}_{x_\epsilon^{1\#}(\zeta_\epsilon)_+}^+ x_\epsilon^{1\#}(\zeta_\epsilon)_+ d\mu + \int_D \mathcal{T}_{x_\epsilon^{2\#}(\zeta_\epsilon)_-}^- x_\epsilon^{2\#}(\zeta_\epsilon)_- d\mu$$

$$\begin{aligned}
& + \frac{1}{2} \iint_{D \times D} (F(x, y) - b(x)H(x, y)) \xi_\epsilon(x) \xi_\epsilon(y) \, d\mu \otimes d\mu(x, y) \\
& - \iint_{D \times D} \frac{b(x) + b(y)}{4\pi} \log \frac{\text{diam}(D)}{|x - y|} x_\epsilon^1 \#(\zeta_\epsilon)_+(x) x_\epsilon^2 \#(\zeta_\epsilon)_-(y) \, d\mu \otimes d\mu(x, y).
\end{aligned}$$

We recall that F is bounded, and H is symmetric, positive, continuous and satisfies

$$H(x, y) \leq \frac{1}{2\pi} \log \frac{\text{diam}(D)}{d(y, \partial D)}.$$

Because the diameter of $\{x_\epsilon^1 \#(\zeta_\epsilon)_+ > 0\}$ is smaller than $r_1\epsilon$, we have

$$\begin{aligned}
& \iint_{D \times D} \frac{b(x)}{4\pi} \log \frac{\text{diam}(D)}{|x - y|} x_\epsilon^1 \#(\zeta_\epsilon)_+ x_\epsilon^1 \#(\zeta_\epsilon)_+ \, d\mu \, d\mu \\
& \geq \tau \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \int_D \frac{b}{4\pi} x_\epsilon^1 \#(\zeta_\epsilon)_+ \, d\mu - C_1 \mathcal{S}_\epsilon^2.
\end{aligned}$$

for some constant $C_1 > 0$. Similarly, there exists $C_2 > 0$ such that

$$\begin{aligned}
& \iint_{D \times D} \frac{b(x)}{4\pi} \log \frac{\text{diam}(D)}{|x - y|} x_\epsilon^2 \#(\zeta_\epsilon)_- x_\epsilon^2 \#(\zeta_\epsilon)_- \, d\mu \, d\mu \\
& \geq (1 - \tau) \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \int_D \frac{b}{4\pi} x_\epsilon^2 \#(\zeta_\epsilon)_- \, d\mu - C_3 \mathcal{S}_\epsilon^2.
\end{aligned}$$

By construction (lemma 5.4.3), we also have

$$d\left(B(x_\epsilon^1, r_1\epsilon), B(x_\epsilon^2, r_2\epsilon)\right) \geq \frac{1}{\log \frac{1}{\epsilon}},$$

so that there exists $C_4 > 0$ with

$$\begin{aligned}
& \iint_{D \times D} \frac{b(x)}{4\pi} \log \frac{\text{diam}(D)}{|x - y|} x_\epsilon^1 \#(\zeta_\epsilon)_+ x_\epsilon^2 \#(\zeta_\epsilon)_- \, d\mu \, d\mu \\
& \leq \tau(1 - \tau) C_4 \mathcal{S}_\epsilon^2 \left(1 + \log \log \frac{1}{\epsilon}\right).
\end{aligned}$$

Recalling the conditions (lemma 5.4.3) that

$$d\left(B(x_\epsilon^1, r_1\epsilon), \partial D\right) \geq \frac{1}{\log \frac{1}{\epsilon}}, \quad d\left(B(x_\epsilon^2, r_2\epsilon), \partial D\right) \geq \frac{1}{\log \frac{1}{\epsilon}},$$

the upper bound for H and the boundedness of R , we obtain

$$E_\epsilon(x_\epsilon^1 \sharp(\zeta_\epsilon)_+ - x_\epsilon^2 \sharp(\zeta_\epsilon)_-) \geq T_1 + T_2 + T_3,$$

with

$$T_1 = \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \int_D \frac{\tau b}{4\pi} x_\epsilon^1 \sharp(\zeta_\epsilon)_+ d\mu + \int_D x_\epsilon^1 \sharp(\zeta_\epsilon)_+ \lambda_\epsilon \Psi d\mu,$$

$$T_2 = \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \int_D \frac{(1-\tau)b}{4\pi} x_\epsilon^2 \sharp(\zeta_\epsilon)_- d\mu - \int_D x_\epsilon^2 \sharp(\zeta_\epsilon)_- \lambda_\epsilon \Psi d\mu$$

and

$$T_3 = -C_4 \mathcal{S}_\epsilon^2 \left(1 + \log \log \frac{1}{\epsilon} \right).$$

From this we conclude that

$$\begin{aligned} E_\epsilon(x_\epsilon^1 \sharp(\zeta_\epsilon)_+ - x_\epsilon^2 \sharp(\zeta_\epsilon)_-) &\geq \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \inf_{B(x_\epsilon^1, r_1 \epsilon)} \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \\ &\quad + (1-\tau) \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \inf_{B(x_\epsilon^2, r_2 \epsilon)} \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \\ &\quad - C_4 \mathcal{S}_\epsilon^2 \left(1 + \log \log \frac{1}{\epsilon} \right). \end{aligned}$$

By definition of the family $\{\zeta_\epsilon : \epsilon > 0\}$, we have

$$E_\epsilon(\xi_\epsilon) = E_\epsilon(x_\epsilon^1 \sharp(\zeta_\epsilon)_+ - x_\epsilon^2 \sharp(\zeta_\epsilon)_-) \leq E_\epsilon((\zeta_\epsilon)_+ - (\zeta_\epsilon)_-) = E_\epsilon(\zeta_\epsilon),$$

and therefore we have

$$\begin{aligned} &\mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\tau \inf_{B(x_\epsilon^1, r_1 \epsilon)} \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right. \\ &\quad \left. + (1-\tau) \inf_{B(x_\epsilon^2, r_2 \epsilon)} \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right) \\ &\leq E_\epsilon(\zeta_\epsilon) + C_4 \left(1 + \log \log \frac{1}{\epsilon} \right). \end{aligned}$$

We define the error term

$$\Theta_\epsilon = \tau E_1 + (1-\tau) E_2 - C_4 \frac{1 + \log \log \frac{1}{\epsilon}}{\log \frac{1}{\epsilon}},$$

where

$$E_1 = \inf_{B(x_\epsilon^1, r_1\epsilon)} \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} - \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\}$$

and

$$E_2 = \inf_{B(x_\epsilon^2, r_2\epsilon)} \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} - \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\}.$$

By definition, we directly have $\Theta_\epsilon \leq 0$, and

$$\begin{aligned} \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} & \left(\tau \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right. \\ & \quad \left. + (1-\tau) \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right) \\ & \leq E_\epsilon(\zeta_\epsilon) - \Theta_\epsilon \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon}. \end{aligned}$$

Finally, the uniform continuity of b and Ψ on the compact set $\overline{D}$, shows that we have $\lim_{\epsilon \rightarrow 0} \Theta_\epsilon = 0$. $\square$

In this section we show that the main part of the vortex core is located in an area of the domain D where the leading partial flows are large. The truncation process was already used by Turkington in the context of time-dependent solutions of the Euler equations [29] (see also [27, 28]).

Corollary 5.4.5. *Let $\{\Theta_\epsilon \in \mathbb{R} : \epsilon > 0\}$ be a family of real numbers satisfying the claim of proposition 5.4.4. There exists a constant $C > 0$ such that for all $\kappa > 0$, we have*

$$\tau \mathcal{S}_\epsilon \int_{D \setminus D_\epsilon^\kappa} d(\zeta_\epsilon)_+ \leq \frac{1}{\kappa} \frac{4\pi}{\sup_D b} \mathcal{S}_\epsilon \left(\Theta_\epsilon + \frac{C}{\log \frac{1}{\epsilon}} \right).$$

with

$$D_\epsilon^\kappa = \left\{ x \in D : \mathcal{T}_{\zeta_\epsilon}^+(x) \geq \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ - \kappa \frac{\sup_D b}{4\pi} \log \frac{1}{\epsilon} \mathcal{S}_\epsilon \right\};$$

and similarly

$$(1 - \tau) \mathcal{S}_\epsilon \int_{D \setminus U_\epsilon^\kappa} d(\zeta_\epsilon)_- \leq \frac{1}{\kappa} \frac{4\pi}{\sup_D b} \mathcal{S}_\epsilon \left(\Theta_\epsilon + \frac{C}{\log \frac{1}{\epsilon}} \right).$$

with

$$U_\epsilon^\kappa = \left\{ x \in D : \mathcal{T}_{\zeta_\epsilon}^-(x) \geq \int_D \mathcal{T}_{\zeta_\epsilon}^- d(\zeta_\epsilon)_- - \kappa \frac{\sup_D b}{4\pi} \log \frac{1}{\epsilon} \mathcal{S}_\epsilon \right\}.$$

Proof. It is sufficient to prove the claim for $\mathcal{T}_{\zeta_\epsilon}^+$ and $\tau > 0$. By assumption, we have for all sufficiently small $\epsilon > 0$:

$$\begin{aligned} \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} & \left(\tau \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right. \\ & \quad \left. + (1 - \tau) \sup_D \left\{ \frac{(1 - \tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right) \\ & \leq E_\epsilon(\zeta_\epsilon) - \Theta_\epsilon \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon}. \end{aligned}$$

According to proposition 5.4.1, there exists $C_1 > 0$ such that

$$E_\epsilon(\zeta_\epsilon) \leq \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ + (1 - \tau) \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^- d(\zeta_\epsilon)_- + C_1 \mathcal{S}_\epsilon^2.$$

From this we conclude the inequality

$$\begin{aligned} \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} & \left(\tau \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right. \\ & \quad \left. + (1 - \tau) \sup_D \left\{ \frac{(1 - \tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right) \\ & \leq \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ + (1 - \tau) \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^- d(\zeta_\epsilon)_- \\ & \quad - \Theta_\epsilon \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} + C_1 \mathcal{S}_\epsilon^2. \end{aligned}$$

On the other hand, an application of proposition 5.4.2 for $\mathcal{T}_{\zeta_\epsilon}^-$ yields

$$\begin{aligned} \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \tau \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \\ \leq \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ - \Theta_\epsilon \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} + C_2 \mathcal{S}_\epsilon^2. \end{aligned}$$

An application of proposition 5.4.2 for $\mathcal{T}_{\zeta_\epsilon}^+$ yields in turn

$$\mathcal{T}_{\zeta_\epsilon}^+ \leq \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} + C_3 \mathcal{S}_\epsilon.$$

On D we thus have

$$\tau \mathcal{S}_\epsilon \mathcal{T}_{\zeta_\epsilon}^+ - \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ \leq \Theta_\epsilon \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} + C_4 \mathcal{S}_\epsilon^2.$$

Now we compute

$$\begin{aligned} \int_{D \setminus D_\epsilon^\kappa} \left(\int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ - \mathcal{T}_{\zeta_\epsilon}^+ \right) d(\zeta_\epsilon)_+ \\ = \int_{D_\epsilon^\kappa} \left(\mathcal{T}_{\zeta_\epsilon}^+ - \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ \right) d(\zeta_\epsilon)_+. \end{aligned}$$

Using the definition of D_ϵ^κ we obtain

$$\tau \mathcal{S}_\epsilon \int_{D \setminus D_\epsilon^\kappa} d(\zeta_\epsilon)_+ \leq \frac{1}{\kappa} \frac{4\pi}{\sup_D b} \mathcal{S}_\epsilon \left(\Theta_\epsilon + \frac{C_4}{\log \frac{1}{\epsilon}} \right). \quad \square$$

5.4.2 Concentration of the truncated vortices

In this section we prove that most of the vortex is located in two small balls: one containing the positive part of the pair, and the other the negative part of the pair.

Theorem 5.4.6. *Let $\{\Theta_\epsilon : \epsilon > 0\}$ be a family of real numbers as in proposition 5.4.4, and let*

$$D_\epsilon^\kappa = \left\{ x \in D : \mathcal{T}_{\zeta_\epsilon}^+(x) \geq \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ - \kappa \frac{\sup_D b}{4\pi} \log \frac{1}{\epsilon} \mathcal{S}_\epsilon \right\}$$

and

$$U_\epsilon^\kappa = \left\{ x \in D : \mathcal{T}_{\zeta_\epsilon}^-(x) \geq \int_D \mathcal{T}_{\zeta_\epsilon}^- d(\zeta_\epsilon)_- - \kappa \frac{\sup_D b}{4\pi} \log \frac{1}{\epsilon} \mathcal{S}_\epsilon \right\}.$$

There exists $\sigma, \kappa > 0$ such that for all sufficiently small $\epsilon > 0$,

$$\text{diam}(D_\epsilon^\kappa) \leq \epsilon^\sigma$$

and

$$\text{diam}(U_\epsilon^\kappa) \leq \epsilon^\sigma.$$

Proof. It is sufficient to prove the claim for $(\zeta_\epsilon)_+$ and $\tau > 0$. Recall that we always assume that there exists $\varsigma > 0$ such that, for all sufficiently small $\epsilon > 0$:

$$\sup_D \left\{ \frac{4\pi\lambda_\epsilon|\Psi|}{(\sup_D b)\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} \leq \frac{\tau}{2} - \varsigma.$$

In particular, there exists $s, \delta, \eta \in (0, 1)$ close to 1 and $\kappa > 0$ close to 0 such that for all sufficiently small $\epsilon > 0$:

$$(1 - \delta)\tau + (1 - \eta)\frac{(1 - \tau)^2}{\tau} + \kappa + \sup_D \left\{ \frac{4\pi\lambda_\epsilon|\Psi|}{(\sup_D b)\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} < \frac{s\tau}{2}.$$

Let $\{\Theta_\epsilon \in \mathbb{R} : \epsilon > 0\}$ be a family of real numbers as in proposition 5.4.4. Let $C > 0$ and D_ϵ^+ be given as in corollary 5.4.5. First pick points $x^*, y^* \in D$ such that

$$\frac{\tau b(x^*)}{4\pi} > \frac{\delta + 1}{2} \sup_D \frac{\tau b}{4\pi}$$

and $b(y^*) > 0$, $y^* \neq x^*$ and

$$\frac{(1 - \tau)b(y^*)}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi(y^*) > \frac{\eta + 1}{2} \sup_D \left\{ \frac{(1 - \tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\}.$$

Such points exist by continuity. Let us compute the energy produced by the competitor $\tilde{\zeta}_\epsilon = x^* \sharp(\zeta_\epsilon)_+ - y^* \sharp(\zeta_\epsilon)_-$. For sufficiently small $\epsilon > 0$, the function $\tilde{\zeta}$ belongs to $\text{Rearg}(\zeta_\epsilon)$ because $y^* \neq x^*$. Thus for sufficiently small $\epsilon > 0$ we have

$$E_\epsilon(x^* \sharp(\zeta_\epsilon)_+ - y^* \sharp(\zeta_\epsilon)_-) \leq E_\epsilon(\zeta_\epsilon).$$

According to proposition 5.4.1 and proposition 5.4.2, we then have

$$\begin{aligned} E_\epsilon(\zeta_\epsilon) &\leq \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ \\ &\quad + (1 - \tau) \log \frac{1}{\epsilon} \mathcal{S}_\epsilon^2 \sup_D \left\{ \frac{(1 - \tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} + C_1 \mathcal{S}_\epsilon^2. \end{aligned}$$

On the other hand, because $x^\star \neq y^\star$ with $d(x^\star, \partial D) > 0$ and $d(y^\star, \partial D) > 0$, there exists $C_2 > 0$ such that for all sufficiently small $\epsilon > 0$, we have

$$\begin{aligned} \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\frac{\delta \tau (\sup_D b)}{4\pi} - \sup_D \left\{ \frac{\lambda_\epsilon |\Psi|}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} \right) \\ + (1 - \tau) \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \eta \sup_D \left\{ \frac{(1 - \tau)b}{4\pi} - \frac{\lambda_\epsilon \Psi}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} \\ \leq E_\epsilon(x^\star \# (\zeta_\epsilon)_+ - y^\star \# (\zeta_\epsilon)_-) + C_2 \mathcal{S}_\epsilon^2. \end{aligned}$$

From this we conclude that, for all sufficiently small $\epsilon > 0$:

$$\begin{aligned} \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ &\geq \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\frac{\delta \tau (\sup_D b)}{4\pi} - \sup_D \left\{ \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} \right) \\ &\quad + (\eta - 1)(1 - \tau) \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \sup_D \left\{ \frac{(1 - \tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} - C_3 \mathcal{S}_\epsilon^2 \end{aligned}$$

Recalling the definition of D_ϵ^κ in corollary 5.4.5, we have for all $x \in D_\epsilon^\kappa$:

$$\begin{aligned} \tau \mathcal{S}_\epsilon \mathcal{T}_{\zeta_\epsilon}^+(x) &\geq \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\frac{\delta \tau (\sup_D b)}{4\pi} - \sup_D \left\{ \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} \right) \\ &\quad + (\eta - 1)(1 - \tau) \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \sup_D \left\{ \frac{(1 - \tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} \\ &\quad - C_4 \mathcal{S}_\epsilon^2 - \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \kappa \frac{\sup_D b}{4\pi}. \end{aligned}$$

Now by definition of $\mathcal{T}_{\zeta_\epsilon}^+$, we have

$$\mathcal{T}_{\zeta_\epsilon}^+(x) \leq \frac{\sup_D b}{4\pi} \int_D \log \frac{\text{diam}(D)}{|x - y|} (\zeta_\epsilon)_+(y) d\mu(y) + \sup_D \{\lambda_\epsilon |\Psi|\},$$

and therefore, for all $x \in D_\epsilon^\kappa$ and sufficiently small $\epsilon > 0$:

$$\begin{aligned} & \tau^2 \mathcal{S}_\epsilon^2 \frac{\sup_D b}{4\pi} \int_D \log \frac{\epsilon}{|x-y|} \, d(\zeta_\epsilon)_+(y) \\ & \geq \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\frac{(\delta-1)\tau(\sup_D b)}{4\pi} - \sup_D \left\{ \frac{2\lambda_\epsilon |\Psi|}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} \right) \\ & \quad + (\eta-1)(1-\tau) \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} \\ & \quad - C_5 \mathcal{S}_\epsilon^2 - \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \kappa \frac{\sup_D b}{4\pi}. \end{aligned}$$

Regrouping terms, we obtain

$$\begin{aligned} & \tau^2 \mathcal{S}_\epsilon^2 \int_D \log \frac{\epsilon}{|x-y|} \, d(\zeta_\epsilon)_+(y) \\ & \geq -C_6 \mathcal{S}_\epsilon^2 - \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left((1-\delta)\tau^2 + (1-\eta)(1-\tau)^2 + \kappa\tau \right) \\ & \quad - \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \sup_D \left\{ \frac{8\pi\lambda_\epsilon}{(\sup_D b)\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} \end{aligned}$$

Since $\tau > 0$, we obtain in particular

$$\begin{aligned} & \tau \mathcal{S}_\epsilon \int_D \log \frac{\epsilon}{|x-y|} \, d(\zeta_\epsilon)_+(y) \\ & \geq -C_7 \mathcal{S}_\epsilon - \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \left((1-\delta)\tau + (1-\eta) \frac{(1-\tau)^2}{\tau} + \kappa \right) \\ & \quad - \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \sup_D \left\{ \frac{8\pi\lambda_\epsilon}{(\sup_D b)\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} \end{aligned}$$

Using the fact that

$$(1-\delta)\tau + (1-\eta) \frac{(1-\tau)^2}{\tau} + \kappa + \sup_D \left\{ \frac{8\pi\lambda_\epsilon}{(\sup_D b)\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} < \frac{s\tau}{2},$$

we obtain the strict lower bound

$$\tau \mathcal{S}_\epsilon \int_D \log \frac{\epsilon}{|x-y|} \, d(\zeta_\epsilon)_+(y) > -C_8 \mathcal{S}_\epsilon - \frac{s\tau}{2} \mathcal{S}_\epsilon \log \frac{1}{\epsilon}.$$

Fix $R \geq 1$. Then using the same computation than the one in the proof of proposition 5.4.2, we infer the existence of some constant $C_9 > 0$ such

that, for all $x \in D$:

$$\int_{B(x, R\epsilon)} \log \frac{\epsilon}{|x-y|} d(\zeta_\epsilon)_+(y) \leq C_9.$$

Injecting this upper bound in the above computation, we obtain for all $x \in D_\epsilon^\kappa$ and for all $R \geq 1$:

$$\tau \mathcal{S}_\epsilon \int_{D \setminus B(x, R\epsilon)} \log \frac{\epsilon}{|x-y|} d(\zeta_\epsilon)_+(y) > -C_{10} \mathcal{S}_\epsilon - \frac{s\tau}{2} \mathcal{S}_\epsilon \log \frac{1}{\epsilon}.$$

Thus for all $R > 1$ and for all $x \in D_\epsilon^\kappa$, we have

$$\tau \mathcal{S}_\epsilon \int_{D \setminus B(x, R\epsilon)} d(\zeta_\epsilon)_+(y) < \frac{C_{11} \mathcal{S}_\epsilon + \frac{s\tau}{2} \mathcal{S}_\epsilon \log \frac{1}{\epsilon}}{\log(R)}.$$

Choose $R = \epsilon^{-s}$. Then we have for all $x \in D_\epsilon^\kappa$, and for all sufficiently small $\epsilon > 0$:

$$\tau \mathcal{S}_\epsilon \int_{D \setminus B(x, R\epsilon)} d(\zeta_\epsilon)_+(y) < \frac{1}{2} \tau \mathcal{S}_\epsilon.$$

Choosing $x_1, x_2 \in D_\epsilon^\kappa$, we now observe that if $d(x_1, x_2) \geq R\epsilon = \epsilon^{1-s}$, then we would have

$$\tau \mathcal{S}_\epsilon \leq \tau \mathcal{S}_\epsilon \int_{D \setminus B(x_1, R\epsilon)} d(\zeta_\epsilon)_+(y) + \tau \mathcal{S}_\epsilon \int_{D \setminus B(x_2, R\epsilon)} d(\zeta_\epsilon)_+(y) < \tau \mathcal{S}_\epsilon,$$

which would be a contradiction. Thus any two point x_1, x_2 in D_ϵ^κ satisfy $d(x_1, x_2) \leq \epsilon^{1-s}$. The claim now follows from corollary 5.4.5. $\square$

From the following result, one may conclude theorem 5.1.1, page 183:

Theorem 5.4.7. *If $\tau > 0$, then every accumulation points as $\epsilon \rightarrow 0$ of the family of probability measures $\{d(\zeta_\epsilon)_+ : \epsilon > 0\}$, in the sense of vague convergence of measures, is a Dirac mass δ_{x^*} with*

$$\frac{\tau b(x^*)}{4\pi} + \Psi(x^*) = \sup_D \left\{ \frac{\tau b}{4\pi} + \Psi \right\}.$$

Similarly, if $\tau < 1$, then every accumulation points as $\epsilon \rightarrow 0$ of the family of probability measures $\{d(\zeta_\epsilon)_- : \epsilon > 0\}$, in the sense of vague

convergence of measures, is a Dirac mass $\delta_{x_\star}$ with

$$\frac{(1-\tau)b(x_\star)}{4\pi} - \Psi(x_\star) = \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \Psi \right\}.$$

Proof. It is sufficient to prove the claim for the positive parts $(\zeta_\epsilon)_+$ with $\tau > 0$. According to proposition 5.4.4, there exists $\{\Theta_\epsilon : \epsilon > 0\}$ such that, for sufficiently small $\epsilon > 0$:

$$\begin{aligned} \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\tau \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right. \\ \left. + (1-\tau) \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right) \\ \leq E_\epsilon(\zeta_\epsilon) - \Theta_\epsilon \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon}. \end{aligned}$$

According to theorem 5.4.6, there exists $\sigma, \kappa > 0$ such that for all sufficiently small $\epsilon > 0$, there exists a ball $B_\epsilon^+ = B(x_\epsilon, \epsilon^\sigma)$ with $x_\epsilon \in D$ and such that

$$\int_{B_\epsilon^+} (\zeta_\epsilon)_+ \, d\mu \leq \frac{1}{\kappa} \frac{4\pi}{\sup_D b} \mathcal{S}_\epsilon \left(\Theta_\epsilon + \frac{C}{\log \frac{1}{\epsilon}} \right).$$

Let $x^\star \in \overline{D}$ be an accumulation point for $\{x_\epsilon : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, and $(\epsilon_n)_{n \in \mathbb{N}}$ the corresponding sequence, decreasing to 0. For all $\varphi \in C(\overline{D})$, we have from the construction of x_ϵ :

$$\lim_{n \rightarrow +\infty} \left| \varphi(x^\star) - \int_D \varphi \, d(\zeta_\epsilon)_+ \right| = 0.$$

Therefore the sequence of probability measures $((\zeta_{\epsilon_n})_+)_{n \in \mathbb{N}}$ converges in the sense of vague convergence to $\delta_{x^\star}$. Now for the last part of the claim, we apply proposition 5.4.2 and the Green's function expansion to obtain

$$\begin{aligned} E_\epsilon(\zeta_\epsilon) \leq \tau \mathcal{S}_\epsilon \int_D \left(\frac{\tau b}{4\pi} \log \frac{1}{\epsilon} \mathcal{S}_\epsilon + \lambda_\epsilon \Psi \right) d(\zeta_\epsilon)_+ \\ + (1-\tau) \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\log \frac{1}{\epsilon} \mathcal{S}_\epsilon} \Psi \right\} + C_1 \mathcal{S}_\epsilon^2. \end{aligned}$$

This gives the inequality

$$\sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} - \int_D \left(\frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right) d(\zeta_\epsilon)_+ \leq \frac{1}{\tau} \left(|\Theta_\epsilon| + \frac{C_2}{\log \frac{1}{\epsilon}} \right).$$

Letting $\epsilon \rightarrow 0$ and relying on compactness of $\overline{D}$ and continuity of b and Ψ , we obtain

$$\sup_D \left\{ \frac{\tau b}{4\pi} + \Psi \right\} = \lim_{\epsilon \rightarrow 0} \int_D \left(\frac{\tau b}{4\pi} + \Psi \right) d(\zeta_\epsilon)_+. \quad \square$$

5.5 Construction of a rotating singular pair

In this section we apply theorem 5.4.7 to construct a rotating singular pair in a lake $(B(0, 1), b)$ invariant under rotations. On $D = B(0, 1)$ we will consider a continuous radial depth function $b : D \rightarrow \mathbb{R}$ with the property that (D, b) is continuous. Our computations may also be done in the case where D is an annulus, the only required change to bring would be the introduction of prescribed circulations on the inner boundary.

Let us fix a distribution function

$$D : \mathbb{R}^+ \rightarrow [0, d\mu(D)]$$

normalized as

$$\int_{\mathbb{R}^+} D(t) dt = 1,$$

and such that there exists $p > 1$ with

$$\int_{\mathbb{R}^+} t^p D(t) dt < +\infty.$$

We also fix $\tau \in [0, 1]$. We say that a family of measurable functions $\{\zeta_\epsilon : \epsilon > 0\}$ satisfies the constraints (D)

$$\begin{aligned} d\mu(\{(\zeta_\epsilon)_+ \geq \lambda\}) &= \frac{\epsilon^2}{\delta} D\left(\frac{\epsilon^2 \lambda}{\delta \tau} \log \frac{1}{\epsilon}\right), \\ d\mu(\{(\zeta_\epsilon)_- \geq \lambda\}) &= \frac{\epsilon^2}{\delta} D\left(\frac{\epsilon^2 \lambda}{\delta(1-\tau)} \log \frac{1}{\epsilon}\right), \end{aligned} \quad (\text{D})$$

where $\delta = \sup_{\lambda > 0} D(\lambda)$. An adaptation of the proof of proposition 5.3.1 shows that such family always exists. A straightforward computation also shows that such a family satisfies

$$\begin{aligned} \mu(\{(\zeta_\epsilon)_+ > 0\}) &\leq \epsilon^2, & \mu(\{(\zeta_\epsilon)_- > 0\}) &\leq \epsilon^2, \\ \int_D (\zeta_\epsilon)_+ \, d\mu &= \frac{\tau}{\log \frac{1}{\epsilon}}, & \int_D (\zeta_\epsilon)_- \, d\mu &= \frac{1-\tau}{\log \frac{1}{\epsilon}}, \end{aligned}$$

and finally

$$\sup_{\epsilon > 0} \left\{ \log \frac{1}{\epsilon} \epsilon^{2(1-\frac{1}{p})} \left(\|(\zeta_\epsilon)_+\|_{L^p_{d\mu}} + \|(\zeta_\epsilon)_-\|_{L^p_{d\mu}} \right) \right\} < +\infty.$$

We are thus in position to apply theorem 5.4.7. Exploiting the symmetry of the domain, we construct a rotating vortex pair around the origin $(0, 0)$:

Theorem 5.5.1 (Theorem 5.1.2, p. 184). *Assume that $D = B(0, 1)$ and $b \in C(\overline{D}) \cap W_{loc}^{1,\infty}(D)$ is a radial function: $b(x) = b(|x|^2)$, such that (D, b) is a continuous lake.*

There exists $\nu_0 > 0$ depending only on $\tau \in [0, 1]$ such that, for all $\nu \in [0, \nu_0)$, there exists a family $\{(\zeta_\epsilon, u_\epsilon) \in C(\mathbb{R}, L^p(D, d\mu)) \times C(\mathbb{R}, L^2(D, d\mu)) : \epsilon > 0\}$ that solves the weak form of

$$\begin{cases} \operatorname{div}(bu_\epsilon) = 0 \\ \operatorname{curl}(u_\epsilon) = b\zeta_\epsilon \\ \partial_t \zeta_\epsilon + (u_\epsilon | \nabla \zeta_\epsilon)_{\mathbb{R}^2} = 0; \end{cases}$$

the family $\{\zeta_\epsilon : \epsilon > 0\}$ satisfies constraint (D), and for each $t \in \mathbb{R}$, the potential vortex $\zeta_\epsilon(t)$ at time t is obtained from $\zeta_\epsilon(0)$ by a rotation of clockwise angle νt .

Furthermore, if $\tau > 0$, the only possible accumulation points of $\{(\zeta_\epsilon(0))_+ : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, in the sense of convergence of probability measures on D , are Dirac masses δ_{x^} with*

$$\frac{\tau b(x^*)}{4\pi} - \frac{\nu}{2} \int_0^{|x^*|^2} b(s) \, ds = \sup_{z \in D} \left\{ \frac{\tau b(z)}{4\pi} - \frac{\nu}{2} \int_0^{|z|^2} b(s) \, ds \right\},$$

and if $\tau < 1$, the only possible accumulation points of $\{(\zeta_\epsilon)_- : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, in the sense of convergence of probability measures on D , are Dirac masses $\delta_{x_\star}$ with

$$\frac{(1-\tau)b(x_\star)}{4\pi} + \frac{\nu}{2} \int_0^{|x^\star|^2} b(s) \, ds = \sup_{z \in D} \left\{ \frac{(1-\tau)b(z)}{4\pi} + \frac{\nu}{2} \int_0^{|z|^2} b(s) \, ds \right\}.$$

Since the boundary of $D = B(0, 1)$ is connected, the operator $\nabla \mathfrak{H}$ is null, so that a weak formulation of the time-dependent evolution equation

$$\partial \zeta_t + (u_t |\nabla \zeta_t|)_{\mathbb{R}^2} = 0$$

reads as

$$\int_{\mathbb{R}} \int_D \partial_t (\varphi(t, \cdot)) \zeta_\epsilon(t) \, d\mu \, dt + \int_{\mathbb{R}} \int_D (\nabla^\perp \mathfrak{K}(\zeta_\epsilon(t)) | \nabla \varphi(t, \cdot) |)_{\mathbb{R}^2} \zeta_\epsilon \, d\mu \, dt = 0,$$

for all $\varphi \in C_c^1(\mathbb{R} \times D)$.

Proof of theorem 5.5.1. Define for all $x \in \overline{B(0, 1)}$:

$$\Psi(x) = \frac{\nu}{2} \int_0^{|x|^2} b(s) \, ds,$$

with $\nu \in \mathbb{R}$ such that

$$|\nu| < \frac{\min\{\tau, (1-\tau)\}}{4\pi}.$$

In particular, this implies that

$$\sup_D \left\{ \frac{4\pi |\Psi|}{(\sup_D b)} \right\} < \frac{\min\{\tau, 1-\tau\}}{2}.$$

For all $\epsilon > 0$, consider the energy

$$E_\epsilon : L^p(D, d\mu) \rightarrow \mathbb{R} : E_\epsilon(\zeta) = \frac{1}{2} \int_D \zeta \mathfrak{K} \zeta \, d\mu + \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \int_D \Psi \zeta \, d\mu.$$

According to theorem 5.4.7, there exists a family $\{\zeta_\epsilon : \epsilon > 0\}$ of solutions of the steady lake equation with Coriolis external flow Ψ , that maximizes E_ϵ over their set of own $d\mu$ -rearrangements. Thus for all $\phi \in C_c^1(D)$, we

have

$$\int_D \zeta_\epsilon (\nabla^\perp (\mathfrak{K}\zeta_\epsilon + \Psi) | \nabla \phi)_{\mathbb{R}^2} \, dm = 0,$$

or also, using the definition of Ψ :

$$\int_D \zeta_\epsilon (\nabla^\perp \mathfrak{K}\zeta_\epsilon - \nu x^\perp b(x) | \nabla \phi)_{\mathbb{R}^2} \, dm = 0.$$

For all $\epsilon > 0$ and for all $t \in \mathbb{R}$, let us define $\tilde{\zeta}_\epsilon(t) = \zeta_\epsilon \circ R_{t\nu}$, where R_α is the notation of angle α in the plane $\mathbb{R}^2$. We have $\tilde{\zeta}_\epsilon \in C(\mathbb{R}, \text{Rearg}(\zeta_\epsilon))$, for all $\epsilon > 0$. Furthermore, we have $\mathfrak{K}\tilde{\zeta}_\epsilon(t) = \mathfrak{K}\zeta_\epsilon \circ R_{t\nu}$. A change of variables in space shows that

$$\begin{aligned} & \int_{\mathbb{R}} \int_D (\nabla^\perp \mathfrak{K}\tilde{\zeta}_\epsilon(t)(x) | \nabla \varphi(t, x))_{\mathbb{R}^2} \tilde{\zeta}_\epsilon(t, x) \, dm(x) \, dt \\ &= \int_{\mathbb{R}} \int_D (\nabla^\perp \mathfrak{K}\zeta_\epsilon(x) | \nabla_x (\varphi(t, \cdot) \circ R_{t\nu}^{-1})(x))_{\mathbb{R}^2} \zeta_\epsilon \, dm(x) \, dt \\ &= -\nu \int_{\mathbb{R}} \int_D (x^\perp | \nabla (\varphi(t, \cdot) \circ R_{t\nu}^{-1})(x))_{\mathbb{R}^2} \zeta_\epsilon(x) \, d\mu(x) \, dt. \end{aligned}$$

Another change of variable in space and an integration by parts in time also give

$$\begin{aligned} & \int_{\mathbb{R}} \int_D \partial_t \varphi(t, x) \tilde{\zeta}_\epsilon(t, x) \, d\mu(x) \, dt = \int_D \int_{\mathbb{R}} \zeta_\epsilon(x) (\partial_t \varphi)(t, R_{-t\nu}(x)) \, dt \, d\mu \\ &= \nu \int_D \int_{\mathbb{R}} \zeta_\epsilon(x) (x^\perp | \nabla_x (\varphi \circ R_{t\nu}^{-1})(x))_{\mathbb{R}^2} \, dt \, d\mu(x). \end{aligned}$$

Hence we have

$$\begin{aligned} & \int_{\mathbb{R}} \int_D \partial_t \varphi(t, x) \tilde{\zeta}_\epsilon(t, x) \, d\mu(x) \, dt \\ &+ \int_{\mathbb{R}} \int_D (\nabla^\perp \mathfrak{K}\tilde{\zeta}_\epsilon(t)(x) | \nabla \varphi(t, x))_{\mathbb{R}^2} \tilde{\zeta}_\epsilon(t, x) \, dm(x) \, dt = 0, \end{aligned}$$

for all test function $\varphi \in C_c^1(\mathbb{R} \times D)$ and for all $\epsilon > 0$. $\square$

A Green's function expansion

In this section we prove our Green's function expansion, theorem 5.2.15; that is: we would like to prove that the operator $\mathfrak{K}$ admits the integral

representation

$$\mathfrak{R}\zeta(x) = b(x) \int_D G_D(x, y) \zeta(y) \, d\mu(y) + \int_D R(x, y) \zeta(y) \, d\mu(y)$$

for some $R : D \times D \rightarrow \mathbb{R}$ bounded and measurable, and G_D the Green's function for the Laplace's operator $-\Delta$ on D with Dirichlet boundary conditions.

Before doing the proof, let us sketch the general idea. For fixed $y \in D$, consider the ansatz

$$g^\sharp : D \rightarrow \mathbb{R} : g^\sharp(x) = b(x)G_D(x, y).$$

Then we have at a formal level

$$\nabla g^\sharp = b \nabla G_D(\cdot, y) + G_D(\cdot, y) \nabla b,$$

and therefore

$$\begin{aligned} -\operatorname{div}(b^{-1} \nabla g^\sharp) &= -\Delta G_D(\cdot, y) - \operatorname{div}(G_D(\cdot, y) b^{-1} \nabla b) \\ &= \delta_y - \operatorname{div}(G_D(\cdot, y) b^{-1} \nabla b). \end{aligned}$$

It is then natural to look for $R(\cdot, y) \in \mathcal{H}_0$ such that

$$-\operatorname{div}(b^{-1} \nabla R(\cdot, y)) = \operatorname{div}(G_D(\cdot, y) b^{-1} \nabla b),$$

whose weak form is

$$\int_D (\nabla R(\cdot, y) | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b} = - \int_D (G_D(\cdot, y) \nabla b | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b}, \quad \forall \varphi \in \mathcal{H}_0.$$

Proposition A.1. *Let (D, b) be a continuous lake. For all $y \in D$, there exists a unique function $R(\cdot, y) \in \mathcal{H}_0$ such that*

$$\int_D (\nabla R(\cdot, y) | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b} = - \int_D (G_D(\cdot, y) \nabla b | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b}, \quad \forall \varphi \in \mathcal{H}_0.$$

Moreover, the function $R : D \times D \rightarrow \mathbb{R} : (x, y) \mapsto R(x, y)$ is measurable and bounded on $D \times D$.

We mimic the proof of [31, Theorem 8.8].

Proof. We recall that since (D, b) is continuous, by definition 5.2.14, there exists $\ell > 2$ such that function

$$y \in D \mapsto \int_D g(\cdot, y)^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} \mathrm{d}m$$

is bounded on D , and for all $y_\star \in D$, we have

$$\lim_{y \rightarrow y_\star} \int_D |g(\cdot, y) - g(\cdot, y_\star)|^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} \mathrm{d}m = 0.$$

First define

$$\kappa = 2 \sup_{y \in D} \left\{ \int_D G_D(\cdot, y)^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} \mathrm{d}m \right\}^{\frac{1}{\ell}}.$$

From the Hölder's inequality, since D is bounded by assumption that (D, b) is a lake, we have

$$\left| \int_D G_D(\cdot, y)^2 \frac{|\nabla b|^2}{b} \mathrm{d}m \right|^{\frac{1}{2}} \leq C_1 \kappa,$$

for some $C_1 > 0$, and therefore the linear functional

$$\varphi \in \mathcal{H}_0 \mapsto - \int_D (G_D(\cdot, y) \nabla b | \nabla \varphi)_{\mathbb{R}^2} \frac{\mathrm{d}m}{b}$$

is continuous on $\mathcal{H}_0$. According to Riesz's representation theorem, there exists a unique function $R(\cdot, y) \in \mathcal{H}_0$ in $\mathcal{H}_0$ such that

$$\int_D (\nabla R(\cdot, y) | \nabla \varphi)_{\mathbb{R}^2} \frac{\mathrm{d}m}{b} = - \int_D (G_D(\cdot, y) \nabla b | \nabla \varphi)_{\mathbb{R}^2} \frac{\mathrm{d}m}{b}, \quad \forall \varphi \in \mathcal{H}_0,$$

and $\|R(\cdot, y)\|_{\mathcal{H}_0} \leq C_1 \kappa$. Let us now prove that $R(\cdot, y)$ is bounded independently of $y \in D$. For all $\beta \geq 1$ and for all $N > \kappa$, define the auxiliary functions

$$F : \mathbb{R}^+ \rightarrow \mathbb{R} : F(t) = \max \{N, t^\beta\},$$

and

$$G : \mathbb{R} \rightarrow \mathbb{R} : G(t) = \int_{\kappa}^{\max\{t, \kappa\}} |F'(s)|^2 \mathrm{d}s,$$

so that for all $t \in \mathbb{R}$, we have

$$G'(t) = \chi_{t \geq \kappa} |F'(t)|^2 = \chi_{[\kappa, N]}(t) \beta^2 t^{2(\beta-1)}.$$

Thus G is derivable with bounded derivative, and $G(t) = 0$ for all $t \leq \kappa$. Define the positive function $u = \max \{R(\cdot, y), \kappa\}$. Then the function $G \circ u$ belongs to $\mathcal{H}_0$ with

$$\nabla(G \circ u) = \nabla u (G' \circ u) = \chi_{\{R(\cdot, y) \geq \kappa\}} \nabla R(\cdot, y) (G' \circ u).$$

From this it follows that

$$\int_D (\nabla u |\nabla(G \circ u)|)_{\mathbb{R}^2} \frac{dm}{b} = - \int_D (G_D(\cdot, y) \nabla b |\nabla u|_{\mathbb{R}^2} (G' \circ u) \frac{dm}{b}.$$

Since $(\nabla u |\nabla(G \circ u)|)_{\mathbb{R}^2} = |\nabla(F \circ u)|^2$ is positive, one may apply Schwartz' and Young's inequalities to get

$$\int_D (\nabla u |\nabla(G \circ u)|)_{\mathbb{R}^2} \frac{dm}{b} \leq 4 \int_D G_D(\cdot, y)^2 \frac{|\nabla b|^2}{b} (G' \circ u) dm.$$

Recalling that $u \geq \kappa \geq 0$ by construction, we have

$$\int_D (\nabla u |\nabla(G \circ u)|)_{\mathbb{R}^2} \frac{dm}{b} \leq \frac{4}{\kappa^2} \int_D G_D(\cdot, y)^2 \frac{|\nabla b|^2}{b} |u|^2 (G' \circ u) dm,$$

or also, in terms of the auxiliary function F :

$$\int_D |\nabla(F \circ u)|^2 \frac{dm}{b} \leq \frac{4}{\kappa^2} \int_D G_D(\cdot, y)^2 \frac{|\nabla b|^2}{b} |u(F' \circ u)|^2 dm.$$

By Hölder's inequality

$$\int_D |\nabla(F \circ u)|^2 \frac{dm}{b} \leq \frac{4}{\kappa^2} \left(\int_D G_D(\cdot, y)^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} dm \right)^{\frac{2}{\ell}} \|u(F' \circ u)\|_{L_{\frac{2\ell}{\ell-2}}^{\frac{2\ell}{\ell-2}}}^2.$$

By construction of κ , we have

$$\int_D |\nabla(F \circ u)|^2 \frac{dm}{b} \leq \|u(F' \circ u)\|_{L_{\frac{2\ell}{\ell-2}}^{\frac{2\ell}{\ell-2}}}^2.$$

Since b is bounded and D is bounded, one may use Hölder's inequality

and Sobolev's inequality (because $F \circ u \in W_0^{1,2}(D)$) to show that there exists some constant $C_2 > 0$ such that

$$\|F \circ u\|_{L_{\text{dm}}^{\frac{2}{2-\ell}}} \leq C_2 \|u(F' \circ u)\|_{L_{\text{dm}}^{\frac{2\ell}{\ell-2}}}.$$

Using the definition of F and letting $N \rightarrow +\infty$, we obtain for all $\beta \geq 1$:

$$\|u^\beta\|_{L_{\text{dm}}^{\frac{2}{2-\ell}}} \leq \beta C_2 \|u^\beta\|_{L_{\text{dm}}^{\frac{2\ell}{\ell-2}}}.$$

One may now use the technique of iteration of norms to prove that there exists some constant $C_3 > 0$ such that

$$\|u\|_{L^\infty} \leq C_3 \|u\|_{L_{\text{dm}}^{\frac{2}{2-\ell}}} \leq C_4 (\kappa + \|\mathbf{R}(\cdot, y)\|_{\mathcal{H}_0}) \leq C_5 \kappa.$$

This implies, by construction of u , that $|\mathbf{R}(\cdot, y)| \leq \max\{1, C_5\} \kappa$. Observe that by symmetry of the absolute value, this estimate also hold for $-\mathbf{R}(\cdot, y)$, so that

$$\|\mathbf{R}(\cdot, y)\|_{L^\infty} \leq C_6 \left(\int_D G_D(\cdot, y)^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} \text{dm} \right)^{\frac{1}{\ell}},$$

for all $y \in D$. Because the lake (D, b) is continuous, the function

$$\mathbf{R} : D \times D \rightarrow \mathbb{R} : (x, y) \mapsto \mathbf{R}(x, y)$$

is uniformly bounded. Furthermore, for fixed $y \in D$, the function $[x \in D \mapsto \mathbf{R}(x, y)]$ is measurable (by construction) and for fixed $x \in D$, the function $[y \in D \mapsto \mathbf{R}(x, y)]$ is continuous. Indeed, by linearity of the elliptic equation and according to the above estimate, we have for all points $y, y^* \in D$:

$$\begin{aligned} |\mathbf{R}(x, y) - \mathbf{R}(x, y^*)| &\leq \sup_{z \in D} |\mathbf{R}(z, y) - \mathbf{R}(z, y^*)| \\ &\leq C_7 \left(\int_D |G_D(\cdot, y) - G_D(\cdot, y^*)|^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} \text{dm} \right)^{\frac{1}{\ell}}, \end{aligned}$$

which goes to 0 as $y \rightarrow y^*$. In particular, the function $\mathbf{R}$ is measurable on $D \times D$. $\square$

Remark A.2. Let $\varphi \in C_c^1(D)$ and consider the solution of the problem

$$-\operatorname{div}(b^{-1}\nabla\psi) = -\operatorname{div}(\varphi b^{-1}\nabla b).$$

Then using standard regularity theory [31, Theorem 8.22], one can prove that the solution ψ_φ of this problem is continuous. In particular, if we let φ goes to $G_D(\cdot, y)$ for some fixed $y \in D$, in such a way that

$$\left(\int_D |\varphi - G_D(\cdot, y)|^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} \mathrm{d}m \right)^{\frac{1}{\ell}} \rightarrow 0, \quad \varphi \rightarrow G_D(\cdot, y),$$

then we obtain that $\psi_\varphi \rightarrow R(\cdot, y)$ uniformly, and the function $R(\cdot, y)$ is continuous on D . Now using the uniform continuity of the function $R(x, \cdot)$, for fixed $x \in D$, we obtain the continuity of the function R on the product $D \times D$.

We are now ready to prove theorem 5.2.15, which we recall now:

Theorem (Theorem 5.2.15, page 200). *Let (D, b) be a continuous lake. There exists a bounded measurable function $R : D \times D \rightarrow \mathbb{R}$ such that, for all $\zeta \in L^p(D, \mathrm{d}\mu)$, $p > 1$, we have almost-everywhere*

$$\mathfrak{R}\zeta(x) = b(x) \int_D G_D(x, y) \zeta(y) \mathrm{d}\mu(y) + \int_D R(x, y) \zeta(y) \mathrm{d}\mu(y).$$

Furthermore, we have $R(\cdot, y) \in \mathcal{H}_0$ for all $y \in D$, with for all $\varphi \in \mathcal{H}_0$:

$$\int_D (\nabla R(\cdot, y) | \nabla \varphi)_{\mathbb{R}^2} \frac{\mathrm{d}m}{b} = - \int_D (G_D(\cdot, y) \nabla b | \nabla \varphi)_{\mathbb{R}^2} \frac{\mathrm{d}m}{b}.$$

Proof. Let R be as given by proposition A.1. Then R is bounded and measurable. Without loss of generality, one may assume that $\zeta \in L^\infty(D)$.

Define

$$\psi : D \rightarrow \mathbb{R} : \psi(x) = \int_D G_D(x, y) \zeta(y) \mathrm{d}\mu(y).$$

The function ψ belongs to $W_0^{1,2}(D) \cap L^\infty(D)$. Let $\varphi \in C_c^1(D)$, and let $(\psi_n)_{n \in \mathbb{N}}$ be a sequence of $C_c^1(D)$ functions converging in $W_0^{1,2}(D)$ and uniformly to the function ψ . We have

$$\int_D b \psi_n \nabla \varphi \mathrm{d}m = - \int_D (\nabla b) \psi_n \varphi \mathrm{d}m - \int_D b \varphi \nabla \psi_n \mathrm{d}m,$$

and letting $n \rightarrow +\infty$ yields the chain rule formula

$$\nabla(b\psi) = b\nabla\psi + \psi\nabla b.$$

On the other hand, for all $y \in D$, the function

$$x \in D \mapsto \nabla_x R(x, y) = \begin{pmatrix} \limsup_{t \rightarrow 0} \frac{R(x + (t, 0), y) - R(x, y)}{t} \\ \limsup_{t \rightarrow 0} \frac{R(x + (0, t), y) - R(x, y)}{t} \end{pmatrix}$$

almost-everywhere equals $\nabla R(\cdot, y)$ [12, Theorem 2, section 4.9.2]. Hence the function $(x, y) \mapsto \nabla_x R(x, y)$ is Lebesgue-integrable on the product $D \times D$ (endowed with the Lebesgue's measure), by Tonelli's theorem and proposition A.1. From this it follows that the ansatz function

$$\phi : D \rightarrow \mathbb{R} : \phi(x) = b(x)\psi(x) + \int_D R(x, y)\zeta(y) \, d\mu(y)$$

is weakly differentiable with weak gradient given by

$$b\nabla\psi + \psi\nabla b + \int_D \nabla_x R(x, y)\zeta(y) \, d\mu(y).$$

Now we compute from the definitions

$$\int_D (\nabla\phi | \nabla\varphi)_{\mathbb{R}^2} \frac{dm}{b} = \int_D \varphi\zeta \, d\mu,$$

so that $\phi \in W^{1,2}(D)$ is a weak solution for the same problem as $\mathfrak{R}(\zeta)$. It remains to prove that ϕ belongs to $\mathcal{H}_0$. First, let us prove that it belongs to $\mathcal{H}$. Since it already belongs to $W^{1,2}(D)$, it remains to observe that

$$\frac{|\nabla\phi|^2}{b} \leq C_1 \left(b|\nabla\psi|^2 + |\psi|^2 \frac{|\nabla b|^2}{b} + b^{-1} \left| \int_D \nabla_x R(x, y)\zeta(y) \, d\mu(y) \right|^2 \right)$$

is integrable on D . The last term is integrable, according to proposition A.1. The first term is integrable, by definition of ψ . The second term is integrable, according to Hölder's inequality and because (D, b) is a continuous lake. Therefore $\phi \in \mathcal{H}$. Let us now prove that $b\psi \in \mathcal{H}$ belongs to $\mathcal{H}_0$. Taking back our sequence $(\varphi_n)_{n \in \mathbb{N}}$, we use Hölder's inequality

to observe that the sequence $(b\varphi_n)_{n \in \mathbb{N}}$ converges to $b\psi$ in $\mathcal{H}$. Since it belongs to $\mathcal{H}_0$, we have $b\psi \in \mathcal{H}_0$. Now ϕ belongs to $\mathcal{H}_0$. To see this, let us exploit the Hilbert structure of $\mathcal{H}$ and consider a function $u \in (\mathcal{H}_0)^\perp$ in the orthogonal space of $\mathcal{H}_0 \subseteq \mathcal{H}$. Then we have, because $R(\cdot, y) \in \mathcal{H}_0$ and by Fubini's theorem:

$$(\phi|u)_{\mathcal{H}} = \int_D (R(\cdot, y)|u)_{\mathcal{H}} \zeta(y) \, d\mu(y) = 0,$$

so that $\phi \in ((\mathcal{H}_0)^\perp)^\perp = \mathcal{H}_0$. This yields $\phi = \mathfrak{K}(\zeta)$ whenever $\zeta \in L^\infty(D)$. Now both $\mathfrak{K}$ and the integral representation for ϕ are stable under limits in $L^p(D, d\mu)$, hence the conclusion. $\square$

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Chapter 6

Asymptotic of steady vortex pair in the lake equation

*The following chapter reflects a paper that will be published in
SIAM Journal on Mathematical Analysis.*

6.1 Introduction

The aim of this chapter is to bring new results in the study of the asymptotic behavior of steady vortex pairs in the lake equations. The *steady* lake equations in their velocity-pressure formulation read as [6]

$$\begin{cases} \operatorname{div}(bu) = 0, & \text{on } D \subseteq \mathbb{R}^2, \\ (u \cdot \nabla)u = \nabla p, & \text{on } D, \\ \oint_{\mathcal{C}_i} (u|_{\tau_i})_{\mathbb{R}^2} = c_i, & \forall i \in \{0, \dots, m\}, \end{cases}$$

where $\mathcal{C}_0, \dots, \mathcal{C}_m$ are the connected components of ∂D and c_i is the circulation along $\mathcal{C}_i$. The function $b : D \rightarrow \mathbb{R}^+$ is referred as the *depth function*. Let $\zeta = b^{-1} \operatorname{curl}(u)$ be the *potential vortex field* and $\mathfrak{K}\zeta$ be the

stream function defined as the solution of the elliptic problem:

$$\begin{cases} -\operatorname{div}(b^{-1}\nabla\mathfrak{K}\zeta) = b\zeta, \\ \oint_{\mathcal{C}_i} (b^{-1}\nabla^\perp\mathfrak{K}\zeta|_{\tau_i})_{\mathbb{R}^2} = c_i, \quad \forall i \in \{0, \dots, m\}. \end{cases}$$

The velocity field u is linked to the stream function by mean of the identity $u := b^{-1}\nabla^\perp\mathfrak{K}\zeta$, while the potential vortex satisfies the transport equation

$$\partial_t\zeta + (u|\nabla\zeta)_{\mathbb{R}^2} = 0.$$

This suggests that the measure μ is the relevant invariant measure for the problem. In fact, it is known that *steady* weak solutions of the lake problem may be obtained by energy maximization over the set of all μ -rearrangements of some given potential vortex (see chapter 5 [20], and see [3–5] for similar questions for the 3D Euler equations with axis-symmetry). To state it in a more concrete fashion, let us consider a given distribution function

$$D : \mathbb{R}^+ \rightarrow [0, \mu(D)]$$

such that

$$\int_{\mathbb{R}^+} D(\lambda) \, d\lambda = 1$$

and, for some $p > 1$,

$$\int_{\mathbb{R}^+} \lambda^p D(\lambda) \, d\lambda < +\infty.$$

We are interested in potential vortex fields ζ_ϵ that satisfy the distributional conditions

$$\begin{aligned} \mu(\{(\zeta_\epsilon)_+ \geq \lambda\}) &= \frac{\epsilon^2}{\delta} D\left(\frac{\epsilon^2\lambda}{\delta\tau\mathcal{S}_\epsilon}\right), \\ \mu(\{(\zeta_\epsilon)_- \geq \lambda\}) &= \frac{\epsilon^2}{\delta} D\left(\frac{\epsilon^2\lambda}{\delta(1-\tau)\mathcal{S}_\epsilon}\right), \end{aligned} \tag{6.1.1}$$

where $\delta = \sup_{\lambda>0} D(\lambda)$, $\tau \in [0, 1]$ and $\mathcal{S}_\epsilon > 0$. In these settings, the positive part $(\zeta_\epsilon)_+$ and the negative part $(\zeta_\epsilon)_-$ of the vortex would satisfy

$$\mu(\{(\zeta_\epsilon)_+ > 0\}) \leq \epsilon^2, \quad \mu(\{(\zeta_\epsilon)_- > 0\}) \leq \epsilon^2,$$

and we would also have the vortex-strength prescriptions

$$\int_D (\zeta_\epsilon)_+ \, d\mu = \tau \mathcal{S}_\epsilon, \quad \int_D (\zeta_\epsilon)_- \, d\mu = (1 - \tau) \mathcal{S}_\epsilon.$$

The following result is the starting point of our analysis:

Theorem (†) ([20, Theorem 3.1]). *Let (D, b) be a lake with $D \subseteq \mathbb{R}^2$ bounded and of class C^1 , and $b \in C^1(\overline{D})$ with $\inf_D b > 0$, or $b = \phi^\alpha$ with $\alpha > 0$ and ϕ a regularization of the distance at the boundary. Let $\{\zeta_\epsilon \in L^p(D, \mu) : \epsilon > 0\}$ be a family of solutions of the steady lake equations obtained by energy maximization over constraint (6.1.1).*

The accumulation points of $\{(\zeta_\epsilon)_+ : \epsilon > 0\}$ and $\{(\zeta_\epsilon)_- : \epsilon > 0\}$ are Dirac masses both centered around a point of maximal depth.

In order to prove theorem (†), we used the following integral kernel representation of the stream function [20]:

$$\mathfrak{R}\zeta(x) = \frac{b(x)}{2\pi} \int_D \log \frac{\text{diam}(D)}{|x - y|} \zeta(y) \, d\mu(y) + \int_D F(x, y) \zeta(y) \, d\mu(y),$$

where the function F is a correction function depending on b and the circulation conditions. Relying on the integral kernel expansion of the stream function, the vortex pair was proved to be of the form $\zeta_\epsilon = \zeta_\epsilon \chi_{D_\epsilon} + \zeta_\epsilon \chi_{D \setminus D_\epsilon}$, where $\zeta_\epsilon \chi_{D_\epsilon}$ tends to a singular vortex pair in the limit $\epsilon \rightarrow 0$, while

$$\int_{D \setminus D_\epsilon} \zeta_\epsilon \, d\mu \sim \mathcal{S}_\epsilon \frac{1}{\log \frac{1}{\epsilon}}.$$

We recall that the leading term in the energy $E_\epsilon(\zeta_\epsilon)$ grows like

$$\|\mathfrak{R}\zeta_\epsilon\|_\infty \int_D \zeta_\epsilon \, d\mu \sim \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon},$$

while the relevant second order terms are of order $\mathcal{S}_\epsilon^2$. Thus, if one wants to obtain a more accurate picture of the asymptotic behavior of the pair by going beyond the leading term, one should try to obtain estimates for the $\mathcal{S}_\epsilon^2$ -order terms.

Unfortunately, such estimates were hard to derive with the purely integral comparison techniques previously used. Indeed, we could naively try to compare the optimal energy $E_\epsilon(\zeta_\epsilon)$ with the energy of some nearly

spherical competitor $(\zeta_\epsilon \chi_{D_\epsilon})^* + \zeta_\epsilon \chi_{D \setminus D_\epsilon}$ obtained by some symmetrization technique. Although such a competitor would be suitable for estimations, the error we commit is estimated as

$$E_\epsilon(\zeta_\epsilon) - E_\epsilon(\zeta_\epsilon^*) \lesssim \|\mathfrak{R}\zeta\|_\infty \int_{D \setminus D} \zeta \, d\mu \sim \left(\int_D \zeta \, d\mu \right)^2 \sim \mathcal{S}_\epsilon^2.$$

Since the error of such a comparison process could not be estimated of lower order than $\mathcal{S}_\epsilon^2$, and because the second order relevant terms in the energy expansion $E_\epsilon(\zeta_\epsilon)$ are of order $\mathcal{S}_\epsilon^2$, the analysis could not be performed further.

Statement of the results

In view of the above discussion, a natural strategy is to improve the concentration result obtained in chapter 5 [20]. This is the first step of the chapter:

Theorem 6.1.1. *In the settings of theorem (†), there exists $\varsigma \in (0, 1]$ such that*

$$\limsup_{\epsilon \rightarrow 0} \frac{\text{diam}(\{(\zeta_\epsilon)_+ > 0\})}{\epsilon^\varsigma} < +\infty,$$

and

$$\limsup_{\epsilon \rightarrow 0} \frac{\text{diam}(\{(\zeta_\epsilon)_- > 0\})}{\epsilon^\varsigma} < +\infty.$$

In comparison with theorem (†), theorem 6.1.1 claims that concentration occurs essentially in two large balls of decreasing radius. With this result, we are able to study the second order behavior of the vortex pair:

Theorem 6.1.2. *In the settings of theorem (†), let $F : D \times D \rightarrow \mathbb{R}$ be the function defined in the Green's function expansion, and let $G : D \times D \rightarrow \mathbb{R}$ be the function defined by*

$$G(x, y) = \frac{b(x) + b(y)}{2\pi} \log \frac{\text{diam}(D)}{|x - y|}.$$

Assume that b admits at least two maximizers X, Y in D such that

$$\liminf_{\epsilon \rightarrow 0} d(\{(\zeta_\epsilon)_+ > 0\}, X) = 0, \quad \liminf_{\epsilon \rightarrow 0} d(\{(\zeta_\epsilon)_- > 0\}, Y) = 0.$$

Then (X, Y) minimizes the function

$$(x, y) \in D \times D \mapsto \left[\tau(1 - \tau)G(x, y) - \tau^2 F(x, x) - (1 - \tau)^2 F(y, y) + \tau(1 - \tau)(F(x, y) + F(y, x)) \right]$$

over the set $(D \cap \{b = \sup_D b\}) \times (D \cap \{b = \sup_D b\})$.

Note that when $b \equiv 1$, every point in D is a point of maximal depth. Moreover, when $b \equiv 1$, we have $\nabla b = 0$ and the correction function F in our Green's function expansion turns out to be the Kirchoff-Routh function, which rules the motion of singular vortex pairs in the 2D Euler equations [12, 13]. Although the function R may be hard to compute in the general setting of a lake (D, b) , theorem 6.1.2 fills a conceptual gap between the lake model and the 2D Euler equations.

We are also in position to investigate the asymptotic shape of the optimal vortices. Relying on tools from standard potential theory and on an *asymptotic version of the Riesz-Sobolev rearrangement inequality*, we prove the following result:

Theorem 6.1.3. *In the setting of theorem (†), and if D satisfies an interior cone condition, then there exist translations $\{T_\epsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^2\}$ such that every accumulation point as $\epsilon \rightarrow 0$, in the sense of convergence in measure, of the rescaled positive parts*

$$\left\{ \frac{\epsilon}{\tau \mathcal{S}_\epsilon} (\zeta_\epsilon)_+ (\epsilon T_\epsilon) : \epsilon > 0 \right\}$$

is a radially symmetric function. Similarly, there exist translations $\{T'_\epsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^2\}$ such that every accumulation point as $\epsilon \rightarrow 0$, in the sense of convergence in measure, of the rescaled negative parts

$$\left\{ \frac{\epsilon}{(1 - \tau) \mathcal{S}_\epsilon} (\zeta_\epsilon)_- (\epsilon T'_\epsilon) : \epsilon > 0 \right\}$$

is a radially symmetric function.

A final topic we would like to focus on is a relaxed maximization

problem, where the distribution function D is not known a priori but L^∞ -constraints are provided. As a consequence of the previous results, we will be able to prove the following:

Theorem 6.1.4. *Let Γ_ϵ be the set of functions defined by*

$$\Gamma_\epsilon = \left\{ \zeta : 0 \leq (\zeta)_+ \leq \epsilon^{-2} \tau \mathcal{S}_\epsilon, 0 \leq (\zeta)_- \leq \epsilon^{-2} (1 - \tau) \mathcal{S}_\epsilon; \right. \\ \left. \int_D (\zeta)_+ \, d\mu = \tau \mathcal{S}_\epsilon, \int_D (\zeta)_- \, d\mu = (1 - \tau) \mathcal{S}_\epsilon \right\}.$$

Every energy E_ϵ maximizer ζ_ϵ over Γ_ϵ is necessarily of the form

$$\zeta_\epsilon = \epsilon^{-2} \tau \mathcal{S}_\epsilon \chi_{\{\zeta_\epsilon > 0\}} - \epsilon^{-2} (1 - \tau) \mathcal{S}_\epsilon \chi_{\{\zeta_\epsilon < 0\}}$$

and

$$\mu(\{\zeta_\epsilon \neq 0\}) \leq \epsilon^2.$$

Furthermore, the sets $\{\zeta_\epsilon > 0\}$ and $\{\zeta_\epsilon < 0\}$ are asymptotically close to balls.

A similar problem was studied by Turkington [16] and Turkington & Friedmann [9], with the maximization class

$$\Gamma_\epsilon = \left\{ \zeta : 0 \leq \zeta \leq \epsilon^{-2} \mathcal{S}_\epsilon; \int_D \zeta \, d\mu = \tau \mathcal{S}_\epsilon \right\}.$$

Relying on potential theory techniques and symmetrization arguments, the authors managed to prove that maximizers should be a vortex patch and, actually, a ball. Although these techniques are available only in a limit regime, we prove a similar optimal-distribution result, relying on the convex structures of rearrangements.

Organization of the chapter

We begin by recalling the framework we are going to work in. Results are mainly cited from chapter 5 [20].

In a second section, we prove theorem 6.1.1 relying on the differential structure of the problem, following ideas of Turkington [16] and Elcrat & Miller [18].

The third section is devoted to the study of repulsion effects acting on the vortex pairs. We prove that the vortex pairs cannot come close to each others too quickly, and their distance to the boundary ∂D remains small in comparison to relevant asymptotic orders. We also prove theorem 6.1.2. The techniques of proof are purely integral comparison arguments, based on a variant of the Sobolev-Riesz rearrangement inequality and our integral kernel expansion of the stream function.

In the fourth section, we study the family of rescaled vortices and we prove that every accumulation point of the rescaled versions, in the sense of convergence in measure, is a symmetric function. This section is based on a previous work of Burchard & Guo [2].

The last section is devoted to the relaxed maximization problem. We rely on the convex structure of sets of rearrangements, already studied by Ryff [14, 15] and Burton [3–5].

6.2 Framework

In this section we recall the framework used in chapter 5 [20] for the energy maximization problem, and we recall properties of maximizing vortex pairs. We say that two measurable functions $f, g : D \rightarrow \mathbb{R}$ are μ -rearrangements of each other if, for all $\lambda \in \mathbb{R}$, we have

$$\mu(\{f \geq \lambda\}) = \mu(\{g \geq \lambda\}).$$

Here, the measure $d\mu(x) = b(x) dx$ is the Lebesgue measure weighted by the depth function b of the lake. We recall that the depth function $b \in \bigcup_{\alpha > 0} C^{0,\alpha}(\overline{D})$ is assumed to be Hölder continuous up to the boundary, and positive on compact subsets of D . On D we assume that there exist disjoint connected compact sets $\mathcal{C}_1, \dots, \mathcal{C}_m$ such that

$$D = \overline{D_0} \setminus \bigcup_{i=0}^m \mathcal{C}_i,$$

with D_0 a bounded open connected satisfying $D_0 = \text{int}(\overline{D_0})$, $\mathcal{C}_0 = \partial D_0$.

We are going to consider (measurable) potential vortices ζ_ϵ parame-

trized by $\epsilon > 0$ through

$$\mu(\{\zeta_\epsilon \neq 0\}) = \epsilon^2,$$

and satisfying the integral identities

$$\int_D (\zeta_\epsilon)_+ \, d\mu = \tau \mathcal{S}_\epsilon, \quad \int_D (\zeta_\epsilon)_- \, d\mu = (1 - \tau) \mathcal{S}_\epsilon,$$

where $\mathcal{S}_\epsilon > 0$ and $\tau \in (0, 1)$. As in chapter 5 [20], we also make the assumption that there exists $p > 1$ such that

$$\sup_{\epsilon > 0} \left\{ \frac{\|(\zeta_\epsilon)_+\|_{L_\mu^p} \epsilon^{2(1-\frac{1}{p})}}{\tau \mathcal{S}_\epsilon} + \frac{\|(\zeta_\epsilon)_-\|_{L_\mu^p} \epsilon^{2(1-\frac{1}{p})}}{(1-\tau) \mathcal{S}_\epsilon} \right\} < +\infty. \quad (6.2.1)$$

The limit cases $\tau = 0$ and $\tau = 1$ are also possible, although they demand more cautions. These limit cases represent the cases of a single non sign changing vortex. Observe that by construction, vortices constructed according to (6.1.1) also satisfy $L^p(D, \mu)$ constraint (6.2.1).

The *stream function* associated with some potential vortex ζ_ϵ is the solution of the elliptic equation

$$\begin{cases} -\operatorname{div}(b^{-1} \nabla \psi) = b \zeta_\epsilon, \\ \oint_{\partial \mathcal{C}_i} (b^{-1} \nabla^\perp \phi | \tau_i)_{\mathbb{R}^2} = c_i (2\tau - 1) \mathcal{S}_\epsilon, \end{cases} \quad (6.2.2)$$

where $c_0, \dots, c_m$ are real numbers such that, in view of Kelvin's theorem,

$$\sum_{i=0}^m c_i = 1.$$

As proved in chapter 5 [20], this elliptic problem (6.2.2) has a unique solution in some Sobolev space. Indeed, let us define the vector space

$$\mathcal{H} = \left\{ f \in W^{1,2}(D) : b^{-1} |\nabla f|^2 \in L^1(D) \right\},$$

which we endow with the scalar product

$$(f|g)_\mathcal{H} = \int_D \frac{(\nabla f(x) | \nabla g(x))_{\mathbb{R}^2}}{b(x)} \, dx + (f|g)_{W^{1,2}}.$$

The pair $(\mathcal{H}, (\cdot|\cdot)_{\mathcal{H}})$ is a Hilbert space, whose elements induce a finite “lake-energy”

$$\int_D (b^{-1} \|\nabla^\perp \phi\|_{\mathbb{R}^2})^2 d\mu < +\infty.$$

Since b is positive on compact sets, the collection $\mathcal{C}_c$ of those functions in $C^1(\overline{D})$ that are constant on a neighborhood of ∂D , belongs to $\mathcal{H}$. We write $\mathcal{H}_0$ the closure of $C_c^1(D)$ in $\mathcal{H}$, and we denote by $\mathcal{H} \times \mathcal{H}$ the bilinear rule

$$(f, g) \in \mathcal{H} \mapsto (f|g)_{\mathcal{H}_0} = \int_D \frac{(\nabla f(x)|\nabla g(x))_{\mathbb{R}^2}}{b(x)} dx.$$

Since D is bounded, one may rely on the standard Poincaré inequality to see that the latter defines an equivalent scalar product on $\mathcal{H}_0$. Whenever the lake (D, b) enjoys sufficient regularity properties, the elliptic problem (6.2.2) admits a weak solution of the form [20]

$$\psi(x) = \frac{b(x)}{2\pi} \int_D \log \frac{\text{diam}(D)}{|x-y|} \zeta(y) d\mu(y) + \int_D F(x, y) \zeta(y) d\mu(y), \quad (6.2.3)$$

where the function $F : D \times D \rightarrow \mathbb{R}$ is defined for all $x, y \in D$ by

$$F(x, y) = R(x, y) - b(x)H(x, y) + \sum_{i=0}^m (\psi_i(x) - c_i) \left[\mathcal{A}^{-1}[\psi_j(y) - c_j]_{0 \leq j \leq m} \right]_i,$$

and the functions $R, H, \mathcal{A}$ and $\psi_0, \dots, \psi_m$ are defined as follows:

- the function H is defined for all $x, y \in D$ by

$$H(x, y) = \frac{1}{2\pi} \log \frac{\text{diam}(D)}{|x-y|} - G_D(x, y),$$

where G_D is the Green’s function associated to the Laplace’s operator $-\Delta$ in D with Dirichlet boundary conditions;

- for all $y \in D$, the function $R(\cdot, y)$ belongs to $\mathcal{H}_0$ and for all $\varphi \in \mathcal{H}_0$, we have

$$\int_D (\nabla R(\cdot, y) | \nabla \varphi)_{\mathbb{R}^2} \frac{dx}{b} = \int_D (G_D(\cdot, y) \nabla b | \nabla \varphi)_{\mathbb{R}^2} \frac{dx}{b},$$

the function

$$y \in D \mapsto \|R(\cdot, y)\|_{L^\infty}$$

admits a uniformly continuous extension to $\overline{D}$, and the function

$$(x, y) \in D \times D \mapsto R(x, y)$$

is measurable.

- for all $i \in \{0, \dots, m\}$, the function $\psi_i \in L^\infty(D) \cap C(D) \cap \overline{\mathcal{C}_c}$ is the limit in $\mathcal{H}$ of functions in $\mathcal{C}_c$ that equal δ_{ij} on a neighborhood of $\mathcal{C}_j$, with δ_{ij} the Kronecher symbol; and ψ_i is bounded by 1 and satisfies

$$(\psi_j | \varphi)_{\mathcal{H}_0} = 0, \quad \text{for all } \varphi \in \mathcal{H}_0.$$

- the functions $\psi_0, \dots, \psi_m$ are linearly independent, we have

$$\overline{\mathcal{C}_c} = \overline{\mathcal{H}_0 \oplus \text{Vec}\langle \psi_0, \dots, \psi_m \rangle},$$

and

$$\sum_{i=0}^m \psi_i = 1$$

on D . Moreover, the operator

$$\mathcal{A} : \left\{ (v_0, \dots, v_m) \in \mathbb{R}^{m+1} : \sum_{i=0}^m v_i = 0 \right\} \rightarrow \left\{ (0, \alpha_1, \dots, \alpha_m) \right\},$$

is a linear isomorphism.

The existence of such an integral representation (6.2.3) for the stream function ψ is proved in the appendix of chapter 5, provided the lake (D, b) is regular enough.

Definition 6.2.1. A lake (D, b) is said to be *continuous* if the operator $\mathfrak{K}$ admits the integral representation (6.2.3), with F as above.

The first great family of continuous lakes may be obtained under the nondegeneracy condition $\inf_D b > 0$. In the case where D is of class C^1 and b does not degenerate, the lake (D, b) is continuous as soon as $b \in W^{1,\infty}(D)$. The crucial point here is that we also have $b \in \bigcup_{\alpha>0} C^{0,\alpha}(\overline{D})$, by Sobolev's embedding theorems. The C^1 regularity of the boundary may be relaxed as soon as b remains Hölder continuous globally on $\overline{D}$.

Another great family of continuous lakes is obtained when the depth function b degenerates as $b = \phi^\alpha$, where ϕ is the distance at the boundary ∂D . This situation leads to a continuous lake for C^2 domains. Again, for less regular domains, one should consider instead regularization of the distance function [11], or even integral versions of the distance at the boundary [1].

Finally, mixed degeneracy conditions are also possible. For more details and detailed examples of continuous lakes, we refer to chapter 5 [20].

The energy E_ϵ for a smooth lake takes the equivalent form

$$E_\epsilon(\zeta_\epsilon) = \iint_{D \times D} b(x) \log \frac{\text{diam}(D)}{|x-y|} \zeta_\epsilon(x) \zeta_\epsilon(y) \, d\mu \, d\mu(x, y) \\ + \iint_{D \times D} F(x, y) \zeta_\epsilon(x) \zeta_\epsilon(y) \, d\mu \, d\mu(x, y).$$

6.3 Improvement of leading order asymptotic

We introduce two important quantities, referred as the *first order flows*. The first order flows associated with a vortex $\zeta \in L^p(D, \mu)$ are defined as

$$\mathcal{T}_\zeta^+(x) = \frac{b(x)}{4\pi} \int_D \log \frac{\text{diam}(D)}{|x-y|} (\zeta)_+(y) \, d\mu(y),$$

and

$$\mathcal{T}_\zeta^-(x) = \frac{b(x)}{4\pi} \int_D \log \frac{\text{diam}(D)}{|x-y|} (\zeta)_-(y) \, d\mu(y).$$

For $\zeta \in L^p(D, \mu)$, the positive first order flow $\mathcal{T}_\zeta^+$ induced by ζ satisfies

$$\mathcal{T}_\zeta^+(x) \leq \frac{\sup_D b}{2\pi} \log \frac{\text{diam}(D)}{\epsilon} \int_D (\zeta)_+ \, d\mu + C \|\zeta\|_{L^p(\mu)} \epsilon^{2(1-\frac{1}{p})}.$$

Indeed, we have since $(\zeta)_+$ is positive

$$\int_D \log \frac{\epsilon}{|x-y|} (\zeta)_+(y) \, d\mu(y) \leq \int_D \left(\log \frac{\epsilon}{|x-y|} \right)_+ (\zeta)_+(y) \, d\mu(y) \\ \leq \|(\zeta)_+\|_{L_\mu^p} \left(\int_D \left(\log \frac{\epsilon}{|x-y|} \right)_+^{\frac{p}{p-1}} \, d\mu(y) \right)^{1-\frac{1}{p}},$$

and a change of variable now yields

$$\begin{aligned} \int_D \log \frac{\epsilon}{|x-y|} (\zeta)_+(y) d\mu(y) \\ \leq C_2 \|(\zeta)_+\|_{L^p_\mu} \left(\epsilon^2 \int_{B(0,1)} \left(\log \frac{1}{|y|} \right)_+^{\frac{p}{p-1}} dy \right)^{1-\frac{1}{p}}. \end{aligned}$$

Combining the above a priori estimate with the control condition (6.2.1), we conclude the following upper estimates for the first order flows:

Lemma 6.3.1. *There exists a constant $C > 0$ such that for all $\epsilon > 0$ and for all $\zeta \in \text{Rearg}(\zeta_\epsilon)$, we have for all $x \in D$*

$$\mathcal{T}_\zeta^+(x) \leq \frac{\tau \sup_D b}{2\pi} \mathcal{S}_\epsilon \log \frac{1}{\epsilon} + C \mathcal{S}_\epsilon$$

and

$$\mathcal{T}_\zeta^-(x) \leq \frac{(1-\tau) \sup_D b}{2\pi} \mathcal{S}_\epsilon \log \frac{1}{\epsilon} + C \mathcal{S}_\epsilon.$$

Next we have the following concentration theorem:

Theorem (†) ([20, Corollary 3.1, Theorem 3.1, Theorem 3.2]). *Let (D, b) be a regular lake. Let $\{\zeta_\epsilon \in L^p(D, \mu) : \epsilon > 0\}$ be a family of solutions of the steady lake equations obtained by energy maximization over their set of μ -rearrangements, with constrained (6.2.1).*

There exist $\kappa > 0$, $\varsigma \in (0, 1]$ and a family $\{\Theta_\epsilon : \epsilon > 0\}$ with $\lim_{\epsilon \rightarrow 0} \Theta_\epsilon$, such that for all $\epsilon > 0$ sufficiently small, the sets

$$D_\epsilon^\kappa = \left\{ x \in D : \mathcal{T}_{\zeta_\epsilon}^+(x) \geq \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ - \kappa \frac{\sup_D b}{4\pi} \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \right\}$$

and

$$U_\epsilon^\kappa = \left\{ x \in D : \mathcal{T}_{\zeta_\epsilon}^-(x) \geq \int_D \mathcal{T}_{\zeta_\epsilon}^- d(\zeta_\epsilon)_- - \kappa \frac{\sup_D b}{4\pi} \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \right\},$$

have diameter smaller than ϵ^ς , and

$$\limsup_{\epsilon \rightarrow 0} \int_{D \setminus D_\epsilon^\kappa} (\zeta_\epsilon)_+ d\mu \leq \frac{1}{\kappa} \frac{4\pi}{\sup_D b} \mathcal{S}_\epsilon \left(\Theta_\epsilon + \frac{C}{\log \frac{1}{\epsilon}} \right)$$

and

$$\limsup_{\epsilon \rightarrow 0} \int_{D \setminus U_\epsilon^\kappa} (\zeta_\epsilon)_- \, d\mu \leq \frac{1}{\kappa} \frac{4\pi}{\sup_D b} \mathcal{S}_\epsilon \left(\Theta_\epsilon + \frac{C}{\log \frac{1}{\epsilon}} \right).$$

Furthermore, given $X_\epsilon \in D_\epsilon^\kappa$ and $Y_\epsilon \in U_\epsilon^\kappa$ for all $\epsilon > 0$ sufficiently small, we have

$$\lim_{\epsilon \rightarrow 0} b(X_\epsilon) = \sup_D b = \lim_{\epsilon \rightarrow 0} b(Y_\epsilon).$$

For any positive measurable function f , the notation df used in theorem (†) indicates the set function defined for all measurable sets $A \subseteq D$ as

$$df(A) := \frac{1}{\|f\|_{L_\mu^1}} \int_A f \, d\mu$$

if f is nonvanishing; and identically null otherwise.

6.3.1 Essential concentration result

Proposition 6.3.1. *There exists $\varsigma > 0$ such that, for all sufficiently small $\epsilon > 0$, and for all $X_\epsilon \in \{(\zeta_\epsilon)_+ > 0\}$, for all $Y_\epsilon \in \{(\zeta_\epsilon)_- > 0\}$, we have*

$$\int_{D \setminus B(X_\epsilon, \epsilon^\varsigma)} d(\zeta_\epsilon)_+ = 0$$

and

$$\int_{D \setminus B(Y_\epsilon, \epsilon^\varsigma)} d(\zeta_\epsilon)_- = 0.$$

The proof is an adaptation of techniques due to Turkington [16, 17] (see also Elcrat & Miller [18] for a similar problem). Our situation is very similar but the computations turn out to be a bit more involved due to the presence of islands in the lake (D may not be simply connected) and because the vortex changes sign.

Proof. Fix $\epsilon > 0$. We prove the claim for the positive part $(\zeta_\epsilon)_+$ only. The claim for the negative part follows by symmetry. According to theorem (†), there exist $\kappa > 0$ and $\varsigma > 0$ such that for all sufficiently small $\epsilon > 0$, the diameter of the set

$$D_\epsilon^\kappa = \left\{ x \in D : \mathcal{T}_{\zeta_\epsilon}^+(x) \geq \int_D \mathcal{T}_{\zeta_\epsilon}^+ \, d(\zeta_\epsilon)_+ - \kappa \frac{\sup_D b}{4\pi} \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \right\}$$

is smaller than ϵ^ς . Furthermore, we have

$$\int_{D \setminus D_\epsilon^\kappa} (\zeta_\epsilon)_+ \, d\mu \leq \frac{1}{\kappa} \frac{4\pi}{\sup_D b} \mathcal{S}_\epsilon \left(\Theta_\epsilon + \frac{C}{\log \frac{1}{\epsilon}} \right),$$

for some family of numbers $\{\Theta_\epsilon : \epsilon > 0\}$ with $\lim_{\epsilon \rightarrow 0} \Theta_\epsilon = 0$. We are going to prove that the latter estimate may be improved as

$$\int_{D \setminus D_\epsilon^\kappa} (\zeta_\epsilon)_+ \, d\mu = 0.$$

Let us first decompose the stream function ψ associated with ζ_ϵ through the elliptic problem (6.2.2), as the sum

$$\psi = \mathfrak{R}\zeta_\epsilon + \mathfrak{H}_\epsilon\zeta_\epsilon, \quad \mathfrak{H}_\epsilon\zeta_\epsilon = \sum_{i=0}^m \alpha_i^\epsilon (\psi_i - c_i).$$

Since ζ_ϵ is a maximizer of the strictly convex functional E_ϵ over $\text{Rearg}(\zeta_\epsilon)$, it is also a maximizer over the weak closure $\overline{\text{Rearg}(\zeta_\epsilon)}^w$, by weak continuity of E_ϵ (see chapter 5 [20, Section 2]). Since the latter compact set is also convex in $L^p(D, \mu)$ [3, Theorem 6], it is straightforward to check that ζ_ϵ is the only maximizer of the linear functional

$$L : L^p(D, \mu) \rightarrow \mathbb{R} : L(\zeta) = \int_D (\mathfrak{R} + \mathfrak{H}_\epsilon)(\zeta_\epsilon) \zeta \, d\mu$$

over the set $\text{Rearg}(\zeta_\epsilon)$. Indeed, observe that $(\mathfrak{R} + \mathfrak{H}_\epsilon)(\zeta_\epsilon)$ belongs to the subgradient of E_ϵ at point ζ_ϵ , so that by definition of the subgradient at ζ_ϵ we have, for all $\zeta \in L^p(D, \mu)$:

$$E_\epsilon(\zeta) - E_\epsilon(\zeta_\epsilon) \geq L(\zeta - \zeta_\epsilon).$$

Taking $\zeta \in \text{Rearg}(\zeta_\epsilon)$ and $\lambda \in (0, 1)$, we define $\zeta_\lambda = \lambda\zeta + (1 - \lambda)\zeta_\epsilon$. Hence we have

$$E_\epsilon(\zeta_\lambda) - E_\epsilon(\zeta_\epsilon) \geq L(\zeta_\lambda - \zeta_\epsilon).$$

By linearity of L and strict convexity of E_ϵ , we have

$$\lambda E_\epsilon(\zeta) + (1 - \lambda)E_\epsilon(\zeta_\epsilon) - E_\epsilon(\zeta_\epsilon) > \lambda L(\zeta) + (1 - \lambda)L(\zeta_\epsilon) - L(\zeta_\epsilon),$$

Since the weak closure $\overline{\text{Rearg}(\zeta_\epsilon)}^w$ is convex [3, Theorem 6], we obtain

$$0 \geq E_\epsilon(\zeta) - E_\epsilon(\zeta_\epsilon) > L(\zeta) - L(\zeta_\epsilon),$$

and ζ_ϵ strictly maximizes L over its set of μ -rearrangements $\text{Rearg}(\zeta_\epsilon)$.

In particular [3, Theorem 5], there exists $\gamma_\epsilon^+ > 0$ such that

$$\{\mathfrak{K}(\zeta_\epsilon) + \mathfrak{H}_\epsilon(\zeta_\epsilon) > \gamma_\epsilon^+\} \subseteq \{\zeta_\epsilon > 0\} \subseteq \{\mathfrak{K}(\zeta_\epsilon) + \mathfrak{H}_\epsilon(\zeta_\epsilon) \geq \gamma_\epsilon^+\}.$$

Let $\phi = (\mathfrak{K}(\zeta_\epsilon) + \mathfrak{H}_\epsilon(\zeta_\epsilon) - \gamma_\epsilon^+)_+$. It is direct to prove that

$$\phi = \phi^\sharp + \sum_{i=0}^m (\beta_{\epsilon;i} - \gamma_\epsilon^+)_+ \psi_i,$$

for some $\phi^\sharp \in \mathcal{H}_0$ and for

$$\beta_{\epsilon;i} = \alpha_{\epsilon;i} - \sum_{j=0}^m \alpha_{\epsilon;j} c_j.$$

Now we observe that

$$\begin{aligned} \int_D (\nabla \phi | \nabla \phi)_{\mathbb{R}^2} \frac{dx}{b} &= \int_D (\nabla \phi | \nabla (\mathfrak{K}(\zeta_\epsilon) + \mathfrak{H}_\epsilon(\zeta_\epsilon)))_{\mathbb{R}^2} \frac{dx}{b} \\ &= \int_D \phi^\sharp \zeta_\epsilon d\mu + \sum_{j=0}^m (\beta_{\epsilon;j} - \gamma_\epsilon^+)_+ \int_D (\nabla \mathfrak{H}_\epsilon(\zeta_\epsilon) | \nabla \psi_j)_{\mathbb{R}^2} \frac{dx}{b}. \end{aligned}$$

By construction of $\alpha_{\epsilon;0}, \dots, \alpha_{\epsilon,m}$, we have for all $j \in \{0, \dots, m\}$:

$$\int_D (\nabla \phi | \nabla \phi)_{\mathbb{R}^2} \frac{dx}{b} = \int_D \phi \zeta_\epsilon d\mu - \sum_{j=0}^m (\beta_{\epsilon;j} - \gamma_\epsilon^+)_+ c_j (2\tau - 1) \mathcal{S}_\epsilon. \quad (6.3.1)$$

This may also be written as

$$\begin{aligned} \int_D (\nabla \phi | \nabla \phi)_{\mathbb{R}^2} \frac{dx}{b} &= \int_D (\phi - (\beta_{\epsilon;0} - \gamma_\epsilon^+)_+) \zeta_\epsilon d\mu \\ &\quad + (2\tau - 1) \mathcal{S}_\epsilon \left((\beta_{\epsilon;0} - \gamma_\epsilon^+)_+ - \sum_{j=0}^m (\beta_{\epsilon;j} - \gamma_\epsilon^+)_+ c_j \right). \end{aligned}$$

Observe that $u = \phi - (\beta_{\epsilon;0} - \gamma_\epsilon^+)_+$ belongs to $W_0^{1,2}(D_0)$. In particular, if

$p \in (1, 2)$, one may choose $q > 2$ such that $q^* = 2q(2-q) = p' = p/(p-1)$, so that by Sobolev's inequality in $W_0^{1,q}(D_0)$, we obtain

$$\begin{aligned} \|u\|_{L^{p'}(D)} &\leq \|u\|_{L^{p'}(D_0)} \leq C_1 \|\nabla u\|_{L^q(D_0)} \\ &= C_1 \|\nabla u\|_{L^q(D)} \leq C_2 \mu(\{u > 0\})^{\frac{1}{p'}} \|\nabla u\|_{\mathcal{H}_0}, \end{aligned}$$

where in the last step we have used Hölder's inequality starting from the weighted Lebesgue's space $L^{2/q}(D, b^{-1}(x) dx)$. The same estimate holds whenever $p \geq 2$, because the control condition (6.5.4) holds with p replaced by $q \in (1, p]$, by Hölder's inequality. From this we conclude that

$$\int_D u \zeta_\epsilon d\mu \leq C_3 \mathcal{S}_\epsilon \|u\|_{\mathcal{H}_0}.$$

Hence we infer the estimate

$$\begin{aligned} \int_D (\nabla \phi | \nabla \phi)_{\mathbb{R}^2} \frac{dx}{b} &\leq C \mathcal{S}_\epsilon \left(\int_D (\nabla \phi | \nabla \phi)_{\mathbb{R}^2} \frac{dx}{b} \right)^{\frac{1}{2}} \\ &\quad + C \mathcal{S}_\epsilon \left((\beta_{\epsilon;0} - \gamma_\epsilon^+)_+ - \sum_{j=0}^m (\beta_{\epsilon;j} - \gamma_\epsilon^+)_+ c_j \right), \end{aligned}$$

from which we deduce that

$$\begin{aligned} &\left(\int_D (\nabla \phi | \nabla \phi)_{\mathbb{R}^2} \frac{dx}{b} \right)^{\frac{1}{2}} \\ &\leq \frac{C \mathcal{S}_\epsilon + \sqrt{C^2 \mathcal{S}_\epsilon^2 + 4C \mathcal{S}_\epsilon \left((\beta_{\epsilon;0} - \gamma_\epsilon^+)_+ - \sum_{j=0}^m (\beta_{\epsilon;j} - \gamma_\epsilon^+)_+ c_j \right)}}{2}. \end{aligned} \tag{6.3.2}$$

Although γ_ϵ^+ may be of great order in comparison with $\mathcal{S}_\epsilon$, the particular structure of the above estimate allows us to write, for all $\epsilon > 0$ sufficiently small:

$$\left(\int_D (\nabla \phi | \nabla \phi)_{\mathbb{R}^2} \frac{dx}{b} \right)^{\frac{1}{2}} \leq C_4 \mathcal{S}_\epsilon.$$

Injecting this estimate into (6.3.1), and using the definition of ϕ , we

obtain

$$\begin{aligned} C_5 \mathcal{S}_\epsilon^2 &\geq \int_D (\mathfrak{R}\zeta_\epsilon + \mathfrak{H}_\epsilon \zeta_\epsilon) (\zeta_\epsilon)_+ \, d\mu - \gamma_\epsilon^+ \int_D (\zeta_\epsilon)_+ \, d\mu \\ &\quad - \sum_{j=0}^m (\beta_{\epsilon;j} - \gamma_\epsilon^+)_+ c_j \int_D \zeta_\epsilon \, d\mu \geq 0. \end{aligned}$$

According to lemma 6.3.1, we have

$$\begin{aligned} \gamma_\epsilon^+ \tau \mathcal{S}_\epsilon + (2\tau - 1) \mathcal{S}_\epsilon \sum_{j=0}^m (\beta_{\epsilon;j} - \gamma_\epsilon^+)_+ c_j \\ \geq \int_D (\zeta_\epsilon)_+ \mathcal{T}_{\zeta_\epsilon}^+ \, d\mu - C_6 \mathcal{S}_\epsilon^2 \left(\frac{1}{\log \frac{1}{\epsilon}} + \Theta_\epsilon \right) \log \frac{1}{\epsilon}. \end{aligned}$$

As $\epsilon \rightarrow 0$, the right-hand side blows up as $\mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon}$ by the concentration result in theorem (†). This forces us to have $\gamma_\epsilon^+ \geq \max_{0 \leq i \leq m} \beta_{\epsilon;i}$ for all $\epsilon > 0$ sufficiently small. For all sufficiently small $\epsilon > 0$, we thus have

$$\gamma_\epsilon^+ \geq \int_D \mathcal{T}_{\zeta_\epsilon}^+ \, d(\zeta_\epsilon)_+ - C_7 \mathcal{S}_\epsilon \left(\frac{1}{\log \frac{1}{\epsilon}} + \Theta_\epsilon \right) \log \frac{1}{\epsilon}.$$

For all $x \in \{\zeta_\epsilon > 0\}$, we obtain by definition of γ_ϵ^+

$$\mathfrak{R}\zeta_\epsilon(x) + \mathfrak{H}_\epsilon \zeta_\epsilon(x) \geq \gamma_\epsilon^+ \geq \int_D \mathcal{T}_{\zeta_\epsilon}^+ \, d(\zeta_\epsilon)_+ - C_8 \mathcal{S}_\epsilon \left(\frac{1}{\log \frac{1}{\epsilon}} + \Theta_\epsilon \right) \log \frac{1}{\epsilon}.$$

Using the fact that the Green's function G_D is a positive function, and the uniform bounds on $\mathbf{R}$ and the flows $\psi_0, \dots, \psi_m$, we conclude the lower estimate, for all sufficiently small $\epsilon > 0$ and for all $x \in \{\zeta_\epsilon > 0\}$:

$$\mathcal{T}_{\zeta_\epsilon}^+(x) \geq \int_D \mathcal{T}_{\zeta_\epsilon}^+ \, d(\zeta_\epsilon)_+ - \kappa \mathcal{S}_\epsilon \log \frac{1}{\epsilon}.$$

In particular, we have

$$\{(\zeta_\epsilon)_+ > 0\} = \{\zeta_\epsilon > 0\} \subseteq \{\mathfrak{R}\zeta_\epsilon + \mathfrak{H}_\epsilon \zeta_\epsilon \geq \gamma_\epsilon^+\} \subseteq D_\epsilon^\kappa,$$

which proves the claim for the positive part of the vortex. $\square$

6.4 Repulsion effects

An important feature in the theory of standard symmetrization is the use of radial competitors together with geometric inequalities. In this direction, we are going to prove a variant of the well-known Riesz-Sobolev rearrangement inequality. One should pay attention that we do not work with the Lebesgue measure, but with the weighted measure $\mu(x) = b(x) dx$. This measure may not behave nicely with respect to geometric transformations. Rather than assuming geometric conditions on b , we propose an *asymptotic variant* of the Riesz-Sobolev inequality. This will be sufficient for our purposes.

We first require the following standard lemma:

Proposition 6.4.1. *For all $x \in D$, there exists a function*

$$x^\# \cdot : L^1(D, \mu) \rightarrow L^1(D, \mu)$$

such that for all positive functions $\zeta \in L^1(D, \mu)$ we have

1. $x^\# \zeta \in \text{Rearg}(\zeta)$;
2. *the superlevel sets of $x^\# \zeta$ are balls (in D) centered on x .*

Furthermore, for all $x \in D$ and for all positive functions $\zeta_1, \zeta_2 \in L^1(D, \mu)$, we have

$$\|x^\# \zeta_1 - x^\# \zeta_2\|_{L^1_\mu} \leq \|\zeta_1 - \zeta_2\|_{L^1_\mu}.$$

Proof. The first part of the claim was proved in chapter 5 [20, Proposition 2.1] and it is a standard construction in the field of symmetrizations. For the second claim, let $x \in D$ and $\zeta_1, \zeta_2 \in L^1(D, \mu)$. Fix $\lambda \in \mathbb{R}^+$. By construction, the sets $\{x^\# \zeta_1 \geq \lambda\}$ and $\{x^\# \zeta_2 \geq \lambda\}$ are balls centered on x . Without loss of generality, assume that

$$\mu(\{x^\# \zeta_1 \geq \lambda\}) \leq \mu(\{x^\# \zeta_2 \geq \lambda\}),$$

so that $\{x^\# \zeta_1 \geq \lambda\} \subseteq \{x^\# \zeta_2 \geq \lambda\}$. One then estimates through a direct computation

$$\mu(\{x^\# \zeta_1 \geq \lambda\} \Delta \{x^\# \zeta_2 \geq \lambda\}) \leq \mu(\{\zeta_1 \geq \lambda\} \Delta \{\zeta_2 \geq \lambda\}).$$

The conclusion for the $L^1(D, \mu)$ -norms follows from Cavalieri's principle. $\square$

Proposition 6.4.2 (Asymptotic Riesz-Sobolev rearrangement inequality). *Let $R > 0$ and for all $\epsilon > 0$, let $X_\epsilon^* \in D$ be such that $\mu(B(X_\epsilon^*, R)) \geq \epsilon^2$ with $\inf_{B(X_\epsilon^*, R)} b > 0$. There exists $C > 0$ such that, for all sufficiently small $\epsilon > 0$ and for all $X \in \{(\zeta_\epsilon)_+ > 0\}$, we have*

$$\begin{aligned} & \iint_{D \times D} \log \frac{\text{diam}(D)}{|x - y|} d(\zeta_\epsilon)_+ d(\zeta_\epsilon)_+(x, y) \\ & \leq \iint_{D \times D} \log \frac{\text{diam}(D)}{|x - y|} dX_\epsilon^* \# (\zeta_\epsilon)_+ dX_\epsilon^* \# (\zeta_\epsilon)_+(x, y) + \log \sqrt{\frac{b(X)}{b(X_\epsilon^*)}} \\ & \quad + C \left(\omega_b(\text{diam}(\{X_\epsilon^* \# (\zeta_\epsilon)_+ > 0\})) + \omega_b(\text{diam}(\{(\zeta_\epsilon)_+ > 0\})) \right). \end{aligned}$$

Proof. Let us fix $\epsilon > 0$, and write for short $X^* = X_\epsilon^*$, $\zeta = (\zeta_\epsilon)_+$ and $\zeta^* = X^* \# (\zeta_\epsilon)_+$, $\theta = \text{diam}(\{\zeta > 0\})$ and $\theta^* = \text{diam}(\{\zeta^* > 0\})$. Assuming $\epsilon > 0$ sufficiently small, we may assume that $\text{diam}(\{\zeta > 0\}) \leq R$ and

$$\inf_{\{\zeta > 0\}} b \geq \delta \sup_D b, \quad \inf_{\{\zeta^* > 0\}} b \geq \delta^* \sup_D b,$$

for $\delta, \delta^* > 0$. Let $X \in \{(\zeta_\epsilon)_+ > 0\}$. Let us write for short $r = \sqrt{\frac{b(X)}{b(X^*)}}$ and

$$\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : \phi(x) = X^* + r(x - X).$$

Finally define the following auxiliary functions: ζ^Δ is the Lebesgue nonincreasing Lebesgue rearrangement of ζ around the point X , and $\xi = \zeta^\Delta \circ \phi^{-1}$. From the definition it follows that every super level set of ξ is a ball centered on X^* , with

$$m(\{\xi \geq t\}) = r^2 m(\{\zeta \geq t\}),$$

where m denotes the usual Lebesgue measure. From this estimate we

have

$$\left| b(X^*)m(\{\xi \geq t\}) - \mu(\{\zeta \geq t\}) \right| \leq \frac{\omega_b(\theta)}{\delta \sup_D b} \mu(\{\zeta \geq t\}).$$

Similarly, one may also compute

$$\left| b(X^*)m(\{\zeta^* \geq t\}) - \mu(\{\zeta \geq t\}) \right| \leq \frac{\omega_b(\theta^*)}{\delta^* \sup_D b} \mu(\{\zeta \geq t\}).$$

According to the standard Riesz-Sobolev rearrangement inequality [7, 10] and using the change of variable formula, we have

$$\begin{aligned} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{r^{-1}\theta^*}{|x-y|} \zeta(x)\zeta(y) \, d(x, y) \\ \leq \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\log \frac{r^{-1}\theta^*}{|x-y|} \right)_+ \zeta^\Delta(x)\zeta^\Delta(y) \, d(x, y) \\ \leq \frac{1}{r^4} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\log \frac{\theta^*}{|x-y|} \right)_+ \xi(x)\xi(y) \, d(x, y). \end{aligned}$$

This yields

$$\begin{aligned} b(X)^2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{r^{-1}\theta^*}{|x-y|} \zeta(x)\zeta(y) \, d(x, y) \\ \leq b(X^*)^2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\log \frac{\theta^*}{|x-y|} \right)_+ \xi(x)\xi(y) \, d(x, y). \quad (6.4.1) \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\log \frac{\theta^*}{|x-y|} \right)_+ \zeta^*(x)\zeta^*(y) \, d\mu \, d\mu(x, y) \\ = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{\theta^*}{|x-y|} \zeta^*(x)\zeta^*(y) \, d\mu \, d\mu(x, y), \end{aligned}$$

and the difference

$$\begin{aligned}
b(X^\star)^2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\log \frac{\theta^\star}{|x-y|} \right)_+ \xi(x) \xi(y) \, d(x, y) \\
- \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\log \frac{\theta^\star}{|x-y|} \right)_+ \zeta^\star(x) \zeta^\star(y) \, d\mu \, d\mu(x, y)
\end{aligned}$$

equals the double integral

$$\begin{aligned}
\int_{\mathbb{R}^2} (b(X^\star) \xi(x) - b(x) \zeta^\star(x)) \\
\int_{\mathbb{R}^2} \left(\log \frac{\theta^\star}{|x-y|} \right)_+ (b(X^\star) \xi(y) + b(y) \zeta^\star(y)) \, d(x, y).
\end{aligned}$$

Let us estimate

$$\begin{aligned}
\int_{\mathbb{R}^2} |b(X^\star) \xi(x) - b(x) \zeta^\star(x)| \, dx \\
\leq \int_{\mathbb{R}^2} b(X^\star) |\zeta^\star(x) - \xi(x)| \, dx + \int_{\mathbb{R}^2} |b(X^\star) - b(x)| \zeta^\star(x) \, dx \\
\leq b(X^\star) \int_0^{+\infty} \mathbf{m}(\{\zeta^\star \geq t\} \Delta \{\xi \geq t\}) \, dt + \omega_b(\theta^\star) \int_{\mathbb{R}^2} \zeta^\star(x) \, dx.
\end{aligned}$$

For the second term, we have

$$\int_{\mathbb{R}^2} \zeta^\star(x) \, dx \leq \frac{1}{\delta^\star \sup_D b} \int_{\mathbb{R}^2} \zeta^\star(x) \, d\mu(x) = \frac{\tau \mathcal{S}_\epsilon}{\delta^\star \sup_D b}.$$

For the first term, we have

$$\begin{aligned}
b(X^\star) \mathbf{m}(\{\zeta^\star \geq t\} \Delta \{\xi \geq t\}) &= \left| b(X^\star) \mathbf{m}(\{\zeta^\star \geq t\}) - b(X^\star) \mathbf{m}(\{\xi \geq t\}) \right| \\
&\leq \left(\frac{\omega_b(\theta^\star)}{\delta^\star \sup_D b} + \frac{\omega_b(\theta)}{\delta \sup_D b} \right) \mu(\{\zeta \geq t\})
\end{aligned}$$

Integrating over $t \in [0, +\infty)$ yields

$$b(X^\star) \int_0^{+\infty} \mathbf{m}(\{\zeta^\star \geq t\} \Delta \{\xi \geq t\}) \, dt \leq \left(\frac{\omega_b(\theta^\star)}{\delta^\star \sup_D b} + \frac{\omega_b(\theta)}{\delta \sup_D b} \right) \tau \mathcal{S}_\epsilon.$$

Therefore we have

$$\int_{\mathbb{R}^2} |b(X^\star)\xi(x) - b(x)\zeta^\star(x)| dx \leq \left(\frac{\omega_b(\theta^\star)}{\delta^\star \sup_D b} + \frac{\omega_b(\theta)}{\delta \sup_D b} \right) \tau \mathcal{S}_\epsilon.$$

Using the fact that $\theta^\star$ is of order ϵ , we obtain some positive constant $C_1(\delta, \delta^\star, b) > 0$ such that

$$\begin{aligned} b(X^\star)^2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\log \frac{\theta^\star}{|x-y|} \right)_+ \xi(x)\xi(y) d(x, y) \\ - \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{\theta^\star}{|x-y|} \zeta^\star(x)\zeta^\star(y) d\mu d\mu(x, y) \\ \leq C_1 \left(\omega_b(\theta^\star) + \omega_b(\theta) \right) \mathcal{S}_\epsilon^2. \end{aligned}$$

Injecting the previous estimate into (6.4.1) yields

$$\begin{aligned} b(X)^2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\log \frac{r^{-1}\theta^\star}{|x-y|} \right)_+ \zeta(x)\zeta(y) d(x, y) \\ - \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{\theta^\star}{|x-y|} \zeta^\star(x)\zeta^\star(y) d\mu d\mu(x, y) \\ \leq C_1 \left(\omega_b(\theta^\star) + \omega_b(\theta) \right) \mathcal{S}_\epsilon^2. \end{aligned}$$

Since b is Hölder continuous and ζ_ϵ concentrates on points of maximal depth, this may be rewritten as

$$\begin{aligned} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{r^{-1}\theta^\star}{|x-y|} \zeta(x)\zeta(y) d\mu d\mu(x, y) \\ - \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{\theta^\star}{|x-y|} \zeta^\star(x)\zeta^\star(y) d\mu d\mu(x, y) \\ \leq C_2 \left(\omega_b(\theta^\star) + \omega_b(\theta) \right) \mathcal{S}_\epsilon^2, \end{aligned}$$

and therefore

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{\text{diam}(D)}{|x-y|} \zeta(x)\zeta(y) d\mu d\mu(x, y)$$

$$\begin{aligned}
&\leq \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{D(D)}{|x-y|} \zeta^\star(x) \zeta^\star(y) \, d\mu \, d\mu(x, y) \\
&\quad + \tau^2 \mathcal{S}_\epsilon^2 \log \sqrt{\frac{b(X)}{b(X^\star)}} + C_2 \left(\omega_b(\theta^\star) + \omega_b(\theta) \right) \mathcal{S}_\epsilon^2. \quad \square
\end{aligned}$$

6.4.1 A priori estimate for repulsion

The aim of this section is to prove the following a priori estimates for the repulsion of the pair.

Proposition 6.4.3. *Let $X \in \overline{D}$ be a maximizer of b , and assume that D satisfies an interior cone condition at X . There exist constants $C_1, C_2, C_3 > 0$ and exponents $\gamma_1, \gamma_2, \gamma_3 > 0$ such that, for all sufficiently small $\epsilon > 0$, we have*

$$\begin{aligned}
d(\{(\zeta_\epsilon)_+ > 0\}, \partial D) &\geq C_1 \left(\frac{1}{\log \frac{1}{\epsilon}} \right)^{\gamma_1}, \\
d(\{(\zeta_\epsilon)_- > 0\}, \partial D) &\geq C_2 \left(\frac{1}{\log \frac{1}{\epsilon}} \right)^{\gamma_2}, \\
d(\{(\zeta_\epsilon)_+ > 0\}, \{(\zeta_\epsilon)_- > 0\}) &\geq C_3 \left(\frac{1}{\log \frac{1}{\epsilon}} \right)^{\gamma_3}.
\end{aligned}$$

The cone condition we impose may be relaxed by a more general cusp-like condition, but we think that this improvement does not bring a better understanding of the general behavior.

Lemma 6.4.4 ([9, Lemma 2.2]). *Let $\mathcal{U} = \text{interior}(\overline{D})$. For all $x, y \in D$, we have*

$$\begin{aligned}
&\frac{1}{2\pi} \log \frac{\text{diam}(D)}{\max\{|x-y|, d(x, \partial D), d(y, \partial D)\}} \\
&\geq H(x, y) \geq \frac{1}{2\pi} \log \frac{\text{diam}(D)}{|x-y| + 2 \max\{d(x, \partial \mathcal{U}), d(y, \partial \mathcal{U})\}}.
\end{aligned}$$

Proof. The upper bound for H follows from the fact that the Green's function G_D is positive, and from the weak maximum principle for H . Since H is a symmetric function (because so is the Green's function G_D),

we may assume without loss of generality that

$$d(x, \partial\mathcal{U}) \leq d(y, \partial\mathcal{U}).$$

Let $B(z, r)$ be a ball in $\mathbb{R}^2 \setminus \overline{D}$, and consider the function

$$\tilde{G}_D : \mathbb{R}^2 \setminus \left\{ y, z + \frac{r^2(y-z)}{|y-z|^2} \right\} \rightarrow \mathbb{R} : \tilde{G}_D(w) = \frac{1}{2\pi} \log \frac{\left| \frac{|y-z|}{r}(w-z) - \frac{r(y-z)}{|y-z|} \right|}{|w-y|}.$$

This function is harmonic on $\mathbb{R}^2 \setminus \{y, z + \frac{r^2(y-z)}{|y-z|^2}\}$. It is also null on $\partial B(z, r)$, and it is nonnegative on ∂D . To see this, one may compute, for all $w \in \mathbb{R}^2 \setminus \{y, z + \frac{r^2(y-z)}{|y-z|^2}\}$:

$$4\pi\tilde{G}_D(w) = \log \frac{\frac{|y-z|^2}{r^2}|w-z|^2 + r^2 - 2(w-z) \cdot (y-z)}{|w-z|^2 + |y-z|^2 - 2(w-z) \cdot (y-z)},$$

which vanishes if $|w-z|=r$; and on ∂D we have $|w-z| \geq r$ and thus $\tilde{g}(w) \geq 0$. It follows from the weak maximum principle that for all $w \in D$, we have $G_D(w, y) \leq \tilde{G}_D(w)$. In particular we obtain

$$\begin{aligned} H(x, y) &\geq \frac{1}{2\pi} \log \frac{\text{diam}(D)}{\left| \frac{|y-z|}{r}(x-z) - \frac{r(y-z)}{|y-z|} \right|} \\ &= \frac{1}{2\pi} \log \frac{\text{diam}(D) \frac{|y-z|}{r}}{\left| (x-z) - \frac{r^2(y-z)}{|y-z|^2} \right|} \\ &= \frac{1}{2\pi} \log \frac{\text{diam}(D) \frac{|y-z|}{r}}{\left| (x-y) + \frac{|y-z|^2 - r^2}{|y-z|^2} (y-z) \right|} \\ &\geq \frac{1}{2\pi} \log \frac{\text{diam}(D) \frac{|y-z|}{r}}{|x-y| + \frac{|y-z|+r}{|y-z|} (|y-z|-r)}, \end{aligned}$$

where in the last line we have used the triangular inequality. Now using the fact that $y \notin B(z, r)$, we have

$$H(x, y) \geq \frac{1}{2\pi} \log \frac{\text{diam}(D)}{|x-y| + 2(|y-z|-r)}.$$

This inequality is true for all balls $B(z, r) \subseteq \mathbb{R}^2 \setminus \overline{D}$, so that

$$H(x, y) \geq \frac{1}{2\pi} \log \frac{\text{diam}(D)}{|x - y| + 2 \, d(y, \partial\mathcal{U})}.$$

As mentioned above, the conclusion follows by symmetry. $\square$

Proof of proposition 6.4.3. For all $\epsilon > 0$, let $X_\epsilon \in \{(\zeta_\epsilon)_+ > 0\}$ and $Y_\epsilon \in \{(\zeta_\epsilon)_- > 0\}$. Since b is assumed to be Hölder continuous on $\overline{D}$, let $\alpha > 0$ be such that $b \in C^{0,\alpha}(\overline{D})$. Let us also define $\mathcal{U} = \text{interior}(\overline{D})$, and consider two families of points $\{X_\epsilon^* : \epsilon > 0\}$ and $\{Y_\epsilon^* : \epsilon > 0\}$ such that there exists a constant $C > 0$ such that for all sufficiently small $\epsilon > 0$, we have

$$\frac{1}{C(\log \frac{1}{\epsilon})^{\frac{1}{\alpha}}} \leq d(X_\epsilon^*, \partial\mathcal{U}) \leq \frac{C}{(\log \frac{1}{\epsilon})^{\frac{1}{\alpha}}},$$

and the same estimates hold for both $d(Y_\epsilon^*, \partial\mathcal{U})$ and $d(X_\epsilon^*, Y_\epsilon^*)$. Such families exist, because there exists a point X in $\overline{D}$ that maximizes b and $\overline{D}$ satisfies an interior cone condition at X . For all sufficiently small $\epsilon > 0$, the function

$$\tilde{\zeta}_\epsilon = X_\epsilon \# (\zeta_\epsilon)_+ - Y_\epsilon \# (\zeta_\epsilon)_-$$

is a μ -rearrangement of ζ_ϵ , and in particular we have $E_\epsilon(\tilde{\zeta}_\epsilon) \leq E_\epsilon(\zeta_\epsilon)$. We are now going to estimate the above energies. Using the integral kernel representation (6.2.3), page 245, and relying on the boundedness of R and the boundary flows $\psi_0, \dots, \psi_m$, we first expand

$$\begin{aligned} E_\epsilon(\zeta_\epsilon) &\leq \int_D (\zeta_\epsilon)_+ \mathcal{T}_{\zeta_\epsilon}^+ d\mu + \int_D (\zeta_\epsilon)_- \mathcal{T}_{\zeta_\epsilon}^- d\mu \\ &\quad - \iint_{D \times D} (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_+(y) b(x) H(x, y) d\mu d\mu(x, y) \\ &\quad - \iint_{D \times D} (\zeta_\epsilon)_-(x) (\zeta_\epsilon)_-(y) b(x) H(x, y) d\mu d\mu(x, y) \\ &\quad - \iint_{D \times D} (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_-(y) (b(x) + b(y)) G_D(x, y) d\mu d\mu(x, y) + C_1 \mathcal{S}_\epsilon^2, \end{aligned}$$

Relying on the Riesz-Sobolev rearrangement inequality, proposition 6.4.2,

we also have

$$\int_D (\zeta_\epsilon)_+ \mathcal{T}_{\zeta_\epsilon}^+ d\mu \leq \int_D \frac{\sup_D b}{b} \left(\tilde{\zeta}_\epsilon \right)_+ \mathcal{T}_{\tilde{\zeta}_\epsilon}^+ d\mu.$$

By construction of the family $\{X_\epsilon^* : \epsilon > 0\}$, we have for sufficiently small $\epsilon > 0$ and for all $x \in \{(\tilde{\zeta}_\epsilon)_+ > 0\}$:

$$\left(\sup_D b \right) - b(x) \leq C_2 \left(d(X_\epsilon^*, \partial\mathcal{U}) \right)^\alpha \leq C_3 \frac{1}{\log \frac{1}{\epsilon}}.$$

From this we conclude, for all sufficiently small $\epsilon > 0$:

$$\int_D (\zeta_\epsilon)_+ \mathcal{T}_{\zeta_\epsilon}^+ d\mu \leq \int_D \left(\tilde{\zeta}_\epsilon \right)_+ \mathcal{T}_{\tilde{\zeta}_\epsilon}^+ d\mu + C_4 \mathcal{S}_\epsilon^2.$$

Similarly, we have

$$\int_D (\zeta_\epsilon)_- \mathcal{T}_{\zeta_\epsilon}^- d\mu \leq \int_D \left(\tilde{\zeta}_\epsilon \right)_- \mathcal{T}_{\tilde{\zeta}_\epsilon}^- d\mu + C_5 \mathcal{S}_\epsilon^2.$$

This yields the estimate

$$\begin{aligned} E_\epsilon(\zeta_\epsilon) &\leq \int_D \left(\tilde{\zeta}_\epsilon \right)_+ \mathcal{T}_{\tilde{\zeta}_\epsilon}^+ d\mu + \int_D \left(\tilde{\zeta}_\epsilon \right)_- \mathcal{T}_{\tilde{\zeta}_\epsilon}^- d\mu \\ &\quad - \iint_{D \times D} (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_+(y) b(x) H(x, y) d\mu d\mu(x, y) \\ &\quad - \iint_{D \times D} (\zeta_\epsilon)_-(x) (\zeta_\epsilon)_-(y) b(x) H(x, y) d\mu d\mu(x, y) \\ &\quad - \iint_{D \times D} (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_-(y) (b(x) + b(y)) G_D(x, y) d\mu d\mu(x, y) + C_6 \mathcal{S}_\epsilon^2. \end{aligned} \tag{6.4.2}$$

Now we estimate the energy $E_\epsilon(\tilde{\zeta}_\epsilon)$ from below. We first expand $E_\epsilon(\tilde{\zeta}_\epsilon)$ using the integral kernel representation (6.2.3), page 245,

$$\begin{aligned} E_\epsilon(\tilde{\zeta}_\epsilon) &\geq \int_D \left(\tilde{\zeta}_\epsilon \right)_+ \mathcal{T}_{\tilde{\zeta}_\epsilon}^+ d\mu + \int_D \left(\tilde{\zeta}_\epsilon \right)_- \mathcal{T}_{\tilde{\zeta}_\epsilon}^- d\mu \\ &\quad - \iint_{D \times D} \left(\tilde{\zeta}_\epsilon \right)_+(x) \left(\tilde{\zeta}_\epsilon \right)_+(y) b(x) H(x, y) d\mu d\mu(x, y) \end{aligned}$$

$$\begin{aligned}
& - \iint_{D \times D} \left(\tilde{\zeta}_\epsilon \right)_- (x) \left(\tilde{\zeta}_\epsilon \right)_- (y) b(x) H(x, y) \, d\mu \, d\mu(x, y) \\
& - \iint_{D \times D} \left(\tilde{\zeta}_\epsilon \right)_+ (x) \left(\tilde{\zeta}_\epsilon \right)_- (y) (b(x) + b(y)) G_D(x, y) \, d\mu \, d\mu(x, y) - C_7 \mathcal{S}_\epsilon^2.
\end{aligned}$$

Since H is a positive function, we also have

$$\begin{aligned}
E_\epsilon(\tilde{\zeta}_\epsilon) & \geq \int_D \left(\tilde{\zeta}_\epsilon \right)_+ \mathcal{T}_{\tilde{\zeta}_\epsilon}^+ \, d\mu + \int_D \left(\tilde{\zeta}_\epsilon \right)_- \mathcal{T}_{\tilde{\zeta}_\epsilon}^- \, d\mu \\
& - \iint_{D \times D} \left(\tilde{\zeta}_\epsilon \right)_+ (x) \left(\tilde{\zeta}_\epsilon \right)_+ (y) b(x) H(x, y) \, d\mu \, d\mu(x, y) \\
& - \iint_{D \times D} \left(\tilde{\zeta}_\epsilon \right)_- (x) \left(\tilde{\zeta}_\epsilon \right)_- (y) b(x) H(x, y) \, d\mu \, d\mu(x, y) \\
& - \iint_{D \times D} \left(\tilde{\zeta}_\epsilon \right)_+ (x) \left(\tilde{\zeta}_\epsilon \right)_- (y) \frac{b(x) + b(y)}{2\pi} \log \frac{\text{diam}(D)}{|x - y|} \, d\mu \, d\mu(x, y) - C_8 \mathcal{S}_\epsilon^2.
\end{aligned}$$

By constructions of $\{X_\epsilon^\star : \epsilon > 0\}$ and $\{Y_\epsilon^\star : \epsilon > 0\}$ together with lemma 6.4.4, we have, for all sufficiently small $\epsilon > 0$:

$$\begin{aligned}
& \iint_{D \times D} \left(\tilde{\zeta}_\epsilon \right)_+ (x) \left(\tilde{\zeta}_\epsilon \right)_+ (y) b(x) H(x, y) \, d\mu \, d\mu(x, y) \\
& \leq \frac{\sup_D b}{2\pi\alpha} \tau^2 \mathcal{S}_\epsilon^2 \log \log \frac{1}{\epsilon} + C_9 \mathcal{S}_\epsilon^2,
\end{aligned}$$

and similarly

$$\begin{aligned}
& \iint_{D \times D} \left(\tilde{\zeta}_\epsilon \right)_- (x) \left(\tilde{\zeta}_\epsilon \right)_- (y) b(x) H(x, y) \, d\mu \, d\mu(x, y) \\
& \leq \frac{\sup_D b}{2\pi\alpha} (1 - \tau)^2 \mathcal{S}_\epsilon^2 \log \log \frac{1}{\epsilon} + C_{10} \mathcal{S}_\epsilon^2
\end{aligned}$$

and

$$\begin{aligned}
& \iint_{D \times D} \left(\tilde{\zeta}_\epsilon \right)_+ (x) \left(\tilde{\zeta}_\epsilon \right)_- (y) \frac{b(x) + b(y)}{2\pi} \log \frac{\text{diam}(D)}{|x - y|} \, d\mu \, d\mu(x, y) \\
& \leq \frac{\sup_D b}{2\pi\alpha} (2\tau(1 - \tau)) \mathcal{S}_\epsilon^2 \log \log \frac{1}{\epsilon} + C_{11} \mathcal{S}_\epsilon^2.
\end{aligned}$$

From these estimates we conclude that for all sufficiently small $\epsilon > 0$, we have

$$E_\epsilon(\tilde{\zeta}_\epsilon) \geq \int_D \left(\tilde{\zeta}_\epsilon \right)_+ \mathcal{T}_{\tilde{\zeta}_\epsilon}^+ d\mu + \int_D \left(\tilde{\zeta}_\epsilon \right)_- \mathcal{T}_{\tilde{\zeta}_\epsilon}^- d\mu - \frac{\sup_D b}{2\pi\alpha} \mathcal{S}_\epsilon^2 \log \log \frac{1}{\epsilon} - C_{12} \mathcal{S}_\epsilon^2. \quad (6.4.3)$$

Combining estimates (6.4.2) and (6.4.3) yields

$$\begin{aligned} & \iint_{D \times D} (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_+(y) b(x) H(x, y) d\mu d\mu(x, y) \\ & + \iint_{D \times D} (\zeta_\epsilon)_-(x) (\zeta_\epsilon)_-(y) b(x) H(x, y) d\mu d\mu(x, y) \\ & + \iint_{D \times D} (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_-(y) (b(x) + b(y)) G_D(x, y) d\mu d\mu(x, y) \\ & \leq \frac{\sup_D b}{2\pi\alpha} \mathcal{S}_\epsilon^2 \log \log \frac{1}{\epsilon} + C_{13} \mathcal{S}_\epsilon^2. \end{aligned}$$

Let $\kappa \in (0, 1)$. Taking advantage of the positivity of both G_D and its regular part H , we have for all $\epsilon > 0$ sufficiently small, since ζ_ϵ concentrates on points of maximal depth:

$$\begin{aligned} & \iint_{D \times D} (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_+(y) H(x, y) d\mu d\mu(x, y) \\ & + \iint_{D \times D} (\zeta_\epsilon)_-(x) (\zeta_\epsilon)_-(y) H(x, y) d\mu d\mu(x, y) \\ & + 2 \iint_{D \times D} (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_-(y) G_D(x, y) d\mu d\mu(x, y) \\ & \leq \frac{\kappa^{-1}}{2\pi\alpha} \mathcal{S}_\epsilon^2 \log \log \frac{1}{\epsilon} + C_{14} \mathcal{S}_\epsilon^2. \quad (6.4.4) \end{aligned}$$

Estimate (6.4.4) gives us a bound for each of the three remaining terms. Using lemma 6.4.4 and the positivity of G_D and its regular part H , we first obtain for all sufficiently small $\epsilon > 0$:

$$\tau^2 \mathcal{S}_\epsilon^2 \log \frac{1}{d(X_\epsilon, \partial D)} \leq \frac{2 - \kappa}{\kappa\alpha} \mathcal{S}_\epsilon^2 \log \log \frac{1}{\epsilon}$$

and

$$(1 - \tau)^2 \mathcal{S}_\epsilon^2 \log \frac{1}{d(Y_\epsilon, \partial D)} \leq \frac{2 - \kappa}{\kappa \alpha} \mathcal{S}_\epsilon^2 \log \log \frac{1}{\epsilon}.$$

This proves that, for sufficiently small $\epsilon > 0$ depending on $\kappa \in (0, 1)$, we have:

$$d(X_\epsilon, \partial D) \gtrsim C_{15} \left(\frac{1}{\log \frac{1}{\epsilon}} \right)^{\frac{2 - \kappa}{\kappa \alpha \tau^2}}, \quad d(Y_\epsilon, \partial D) \gtrsim C_{16} \left(\frac{1}{\log \frac{1}{\epsilon}} \right)^{\frac{2 - \kappa}{\kappa \alpha (1 - \tau)^2}}.$$

The constants here may depend on κ, τ, α, b and D , but are independent of $\epsilon > 0$. Now to conclude the proof, one observes that we still have, from estimate (6.4.4) and the positivity of H :

$$2 \iint_{D \times D} (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_-(y) G_D(x, y) d\mu d\mu(x, y) \leq \frac{\kappa^{-1}}{2\pi\alpha} \mathcal{S}_\epsilon^2 \log \log \frac{1}{\epsilon} + C_{17} \mathcal{S}_\epsilon^2,$$

which may be rewritten as

$$\begin{aligned} 2 \iint_{D \times D} (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_-(y) \log \frac{\text{diam}(D)}{|x - y|} d\mu d\mu(x, y) \\ \leq 2 \iint_{D \times D} (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_-(y) H(x, y) d\mu d\mu(x, y) \\ + \frac{\kappa^{-1}}{\alpha} \mathcal{S}_\epsilon^2 \log \log \frac{1}{\epsilon} + C_{18} \mathcal{S}_\epsilon^2. \end{aligned}$$

Using lemma 6.4.4 and the previous estimates on the distance at the boundary, we obtain for sufficiently small $\epsilon > 0$:

$$\begin{aligned} \mathcal{S}_\epsilon^2 \log \frac{1}{d(X_\epsilon, Y_\epsilon)} \leq \mathcal{S}_\epsilon^2 \log \frac{1}{\max\{d(Y_\epsilon, \partial D), d(X_\epsilon, \partial D)\}} \\ + \frac{2 - \kappa}{2\tau(1 - \tau)\kappa\alpha} \mathcal{S}_\epsilon^2 \log \log \frac{1}{\epsilon}, \end{aligned}$$

and the latter is smaller than

$$\mathcal{S}_\epsilon^2 \log \left(C_{19} \left(\log \frac{1}{\epsilon} \right)^{\frac{2 - \kappa}{\kappa \alpha} \left(\frac{1}{\min\{\tau^2, (1 - \tau)^2\}} + \frac{1}{2\tau(1 - \tau)} \right)} \right),$$

for some constant $C_{19} > 0$. We thus have obtained the estimate

$$d(X_\epsilon, Y_\epsilon) \gtrsim C_{20} \left(\frac{1}{\log \frac{1}{\epsilon}} \right)^{\frac{2-\kappa}{\kappa\alpha} \left(\frac{1}{\min\{\tau^2, (1-\tau)^2\}} + \frac{1}{2\tau(1-\tau)} \right)}. \quad \square$$

6.4.2 Accurate localization rule

In this section we prove an accurate localization rule when b admits two or more maximizers inside D . In such a situation, we prove that the vortex pair always separates, and never reaches the boundary ∂D . We are not going to use the result of this section in the remaining part of the text. Rather, we present the results because we think they illustrate an interesting link to the localization of vortex pairs for the 2D Euler equations.

Lemma 6.4.5. *Assume that b admits at least two maximizers $X^\star, Y^\star$ in D . Then we have*

$$\liminf_{\epsilon \rightarrow 0} d(\{\zeta_\epsilon \neq 0\}, \partial D) > 0,$$

and

$$\liminf_{\epsilon \rightarrow 0} d(\{(\zeta_\epsilon)_+ > 0\}, \{(\zeta_\epsilon)_- > 0\}) > 0.$$

Proof. Let us consider, for sufficiently small $\epsilon > 0$, the function

$$\zeta_\epsilon^\star = X^\star \# (\zeta_\epsilon)_+ - Y^\star \# (\zeta_\epsilon)_-.$$

The function $\zeta_\epsilon^\star$ is a μ -rearrangement of ζ_ϵ , and in particular we must have $E_\epsilon(\zeta_\epsilon^\star) \leq E_\epsilon(\zeta_\epsilon)$. Now one should consider the same estimates as those used in the proof of proposition 6.4.3. The $\log \log \frac{1}{\epsilon}$ term from (6.4.3) would reduce to a constant error term, and the conclusion would directly follow. $\square$

Proposition 6.4.6. *Let $F : D \times D \rightarrow \mathbb{R}$ be the function defined by*

$$F(x, y) = -b(x)H(x, y) + \sum_{i=0}^m (\psi_i(x) - c_i) \left[\mathcal{A}^{-1}[\psi_j(y) - c_j]_{0 \leq j \leq m} \right]_i + R(x, y),$$

and let $G : D \times D$ be the function defined by

$$G(x, y) = \frac{b(x) + b(y)}{2\pi} \log \frac{\text{diam}(D)}{|x - y|}.$$

Assume that b admits at least two maximizers X, Y in D such that

$$\liminf_{\epsilon \rightarrow 0} d(\{(\zeta_\epsilon)_+ > 0\}, X) = 0, \quad \liminf_{\epsilon \rightarrow 0} d(\{(\zeta_\epsilon)_- > 0\}, Y) = 0.$$

Then (X, Y) minimizes the function

$$(x, y) \in D \times D \mapsto \left[\tau(1 - \tau)G(x, y) - \tau^2 F(x, x) - (1 - \tau)^2 F(y, y) + \tau(1 - \tau)(F(x, y) + F(y, x)) \right]$$

over the set $(D \cap \{b = \sup_D b\}) \times (D \cap \{b = \sup_D b\})$.

It would be interesting to derive an analogue for proposition 6.4.6 in the situation where the maximizers of b are not necessarily inside D . Since we always have

$$\tau^2 + (1 - \tau)^2 \geq 2\tau(1 - \tau),$$

we would expect that the pair first tries to remain far from ∂D before trying to be separated. However, a close inspection of the proofs shows that the precise modulus of continuity of b comes into play. We think that further analysis of the auxiliary function R would be needed to answer that question.

Proof of proposition 6.4.6. Let $X^*, Y^* \in D$ be two different maximizers of b over D . We consider $\epsilon > 0$ so small that the function

$$\zeta_\epsilon^* = X^* \# (\zeta_\epsilon)_+ - Y^* \# (\zeta_\epsilon)_-$$

is a μ -rearrangement of ζ_ϵ . In particular we have $E_\epsilon(\zeta_\epsilon^*) \leq E_\epsilon(\zeta_\epsilon)$. We are going to expand these energies. Let us write in all generality

$$E_\epsilon(\zeta_\epsilon^*) = \int_D (\zeta_\epsilon^*)_+ \mathcal{T}_{\zeta_\epsilon^*}^+ d\mu + \int_D (\zeta_\epsilon^*)_- \mathcal{T}_{\zeta_\epsilon^*}^- d\mu$$

$$- \iint_{D \times D} G (\zeta_\epsilon^*)_+ (\zeta_\epsilon^*)_- \, d\mu \, d\mu(x, y) + \iint_{D \times D} F \zeta_\epsilon^* \zeta_\epsilon^* \, d\mu \, d\mu,$$

where F and G are defined as in the statement, and recall that since (D, b) is a continuous lake, the function F is continuous on $D \times D$. Because both $\{\zeta_\epsilon^* : \epsilon > 0\}$ and $\{\zeta_\epsilon : \epsilon > 0\}$ concentrate, we have

$$\begin{aligned} & \liminf_{\epsilon \rightarrow 0} \frac{1}{\mathcal{S}_\epsilon^2} \iint_{D \times D} F \zeta_\epsilon \zeta_\epsilon \, d\mu \, d\mu \\ &= \tau^2 F(X, X) + (1 - \tau)^2 F(Y, Y) - \tau(1 - \tau)(F(X, Y) + F(Y, X)), \end{aligned}$$

while

$$\begin{aligned} & \liminf_{\epsilon \rightarrow 0} \frac{1}{\mathcal{S}_\epsilon^2} \iint_{D \times D} F \zeta_\epsilon^* \zeta_\epsilon^* \, d\mu \, d\mu \\ &= \tau^2 F(X^*, X^*) + (1 - \tau)^2 F(Y^*, Y^*) - \tau(1 - \tau)(F(X^*, Y^*) + F(Y^*, X^*)). \end{aligned}$$

On the other hand, because X^* and Y^* are exact maximizers of b , we may apply Riesz-Sobolev rearrangement inequality, proposition 6.4.2, to see that there exists $\Theta_\epsilon > 0$ with $\lim_{\epsilon \rightarrow 0} \Theta_\epsilon = 0$, and such that

$$\begin{aligned} E_\epsilon(\zeta_\epsilon^*) &= \left(\sup_D b \right) \int_D (\zeta_\epsilon^*)_+ b^{-1} \mathcal{T}_{\zeta_\epsilon^*}^+ \, d\mu + \left(\sup_D b \right) \int_D (\zeta_\epsilon^*)_- b^{-1} \mathcal{T}_{\zeta_\epsilon^*}^- \, d\mu \\ &+ \iint_{D \times D} (b(x) + b(y)) G_D(x, y) (\zeta_\epsilon^*)_+(x) (\zeta_\epsilon^*)_-(y) \, d\mu \, d\mu(x, y) \\ &+ \iint_{D \times D} F(x, y) \zeta_\epsilon^*(x) \zeta_\epsilon^*(y) \, d\mu \, d\mu(x, y) + \Theta_\epsilon \mathcal{S}_\epsilon^2. \end{aligned}$$

This yields the inequality

$$\begin{aligned} & \frac{1}{\mathcal{S}_\epsilon^2} \iint_{D \times D} G (\zeta_\epsilon)_+ (\zeta_\epsilon)_- \, d\mu \, d\mu - \frac{1}{\mathcal{S}_\epsilon^2} \iint_{D \times D} F \zeta_\epsilon \zeta_\epsilon \, d\mu \, d\mu \\ & \leq \frac{1}{\mathcal{S}_\epsilon^2} \iint_{D \times D} G (\zeta_\epsilon^*)_+ (\zeta_\epsilon^*)_- \, d\mu \, d\mu - \frac{1}{\mathcal{S}_\epsilon^2} \iint_{D \times D} F \zeta_\epsilon^* \zeta_\epsilon^* \, d\mu \, d\mu + \Theta_\epsilon \end{aligned}$$

This latter inequality holds for all couple of points $(X^*, Y^*) \in D \times D$ with $X^* \neq Y^*$ and both X^*, Y^* maximize b . Letting $\epsilon \rightarrow 0$ and applying Fatou's lemma on the left hand side, we obtain the desired conclusion. $\square$

6.5 Asymptotic shape

In this section we prove that the vortex pair asymptotically looks like two symmetric functions. We base our analysis on results obtained by Burchard & Guo [2, Lemma 3.2] on the asymptotic shape of asymptotic maximizers of singular integrals. More precisely, our goal is to prove that

$$\limsup_{\epsilon \rightarrow 0} \left\{ E_\epsilon(\zeta_\epsilon) - E_\epsilon(\zeta_\epsilon^*) \right\} = 0,$$

for some radially symmetric μ -rearrangement ζ_ϵ^* of the energy maximizer ζ_ϵ . Then we exploit this information to obtain convergence in shape of the rescaled maximizers.

6.5.1 Preliminary study of oscillations

The aim of this paragraph is to prove the following asymptotic limit estimate:

Proposition 6.5.1. *For all $X_\epsilon \in \{(\zeta_\epsilon)_+ > 0\}$, let $\zeta_\epsilon^* = X_{\epsilon\sharp}(\zeta_\epsilon)_+ - (\zeta_\epsilon)_-$. We have*

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} \left\{ \iint_{D \times D} F(x, y) \zeta_\epsilon(x) \zeta_\epsilon(y) \, d\mu \, d\mu(x, y) \right. \\ - \iint_{D \times D} G(x, y) (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_-(y) \, d\mu \, d\mu(x, y) \\ - \iint_{D \times D} F(x, y) \zeta_\epsilon^*(x) \zeta_\epsilon^*(y) \, d\mu \, d\mu(x, y) \\ \left. + \iint_{D \times D} G(x, y) (\zeta_\epsilon^*)_+(x) (\zeta_\epsilon^*)_-(y) \, d\mu \, d\mu(x, y) \right\} = 0. \end{aligned}$$

The following a priori estimate from elliptic regularity theory is useful:

Proposition 6.5.2 ([19, Theorem 8.22]). *Let $\mathcal{U} \subseteq D$ be such that $\inf_{\mathcal{U}} b > 0$. Let $f_1, f_2 \in L^q(\mathcal{U}, dx)$ for some $q > 2$, and $\psi \in W^{1,2}(\mathcal{U})$ be such that, for all $\varphi \in C_c^1(\mathcal{U})$, we have*

$$\int_{\mathcal{U}} (\nabla \psi(x) | \nabla \varphi(x))_{\mathbb{R}^2} \frac{dx}{b(x)} = \int_{\mathcal{U}} ((f_1(x), f_2(x)) | \nabla \varphi(x)) \, dx.$$

For all $\gamma, \varsigma > 0$, there exist $\sigma > 0$ and a constant $C > 0$ such that for all $\epsilon > 0$ and for all $y \in \mathcal{U}$ with $d(y, \partial D) \geq (\log \frac{1}{\epsilon})^\gamma$, we have

$$\sup_{x \in B(y, \epsilon^\varsigma)} \left\{ |\psi(x) - \psi(y)| \right\} \leq C \epsilon^\sigma \left(\|\psi\|_{L^\infty(\mathcal{U})} + \|(f_1, f_2)\|_{L^q} \right).$$

Since we already know that the maximizing vortex pair ζ_ϵ concentrates on a point of maximal depth, we may apply proposition 6.5.2 to control the oscillation of each R_y , on the vortices core. Also, proposition 6.5.2 for $b \equiv 1$ yields an estimate for the regular part of the Green's function H . Finally, we also recall that the function (see chapter 5 [20])

$$y \in \overline{D} \mapsto \|R(\cdot, y)\|_{L^\infty}$$

is continuous on $\overline{D}$, since (D, b) is a continuous lake by assumption. The vortex $(\zeta_\epsilon)_+$ and $(\zeta_\epsilon)_-$ both concentrate with diameter of a priori order ϵ^ς , while their distance from each other and from the boundary ∂D is of some order not smaller than $(\log \frac{1}{\epsilon})^\gamma$. From these a priori results and a direct application of proposition 6.5.2, one can easily deduce proposition 6.5.1. We skip the proof.

6.5.2 Scaling process and energy convergence

We fix a family of points $\{X_\epsilon \in \{(\zeta_\epsilon)_+ > 0\} : \epsilon > 0\}$, and we recall that

$$\{(\zeta_\epsilon)_+ > 0\} \subseteq B(X_\epsilon, \epsilon^\varsigma),$$

for some $\varsigma > 0$ independent of $\epsilon > 0$. Our analysis is made for the positive part of the vortex pair, but similar results hold for the negative part as well.

Given a positive and measurable function $f : D \rightarrow \mathbb{R}^+$ with finite μ -integral, we define its scaled version as the function

$$\mathfrak{s}f : \mathbb{R}^2 \rightarrow \mathbb{R}^+ : \mathfrak{s}f(x) = \frac{\epsilon}{\int_D f \, d\mu} f(\epsilon x + X_\epsilon).$$

The rescaled version of the measure μ is $d\mathfrak{s}\mu$ defined as

$$\int_{\mathbb{R}^2} g(x) d\mathfrak{s}\mu(x) = \frac{1}{\epsilon} \int_D g(\epsilon^{-1}(x - X_\epsilon)) d\mu(x).$$

By construction, we have

$$\int_{\mathbb{R}^2} \mathfrak{s}f d\mathfrak{s}\mu = 1,$$

while

$$\|\mathfrak{s}f\|_{L^p_{d\mathfrak{s}\mu}} = \frac{\|f\|_{L^p_\mu} \epsilon^{2(1-\frac{1}{p})}}{\int_D f d\mu}.$$

Observe that the scaling process depends on $\epsilon > 0$ and on X_ϵ , although we do not explicitly mention this dependence in our notations.

Lemma 6.5.3. *Let $f_\epsilon = \mathfrak{s}(\zeta_\epsilon)_+$ and $f_\epsilon^* = \mathfrak{s}(X_\epsilon \# (\zeta_\epsilon)_+)$. We have*

$$\lim_{\epsilon \rightarrow 0} \left\{ \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x - y|} f_\epsilon^*(x) f_\epsilon^*(y) d(x, y) - \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x - y|} f_\epsilon(x) f_\epsilon(y) d(x, y) \right\} = 0.$$

Proof. According to proposition 6.5.1 together with the energy maximization principle and the Riesz-Sobolev rearrangement inequality, proposition 6.4.2, one directly has the asymptotic behavior

$$\lim_{\epsilon \rightarrow 0} \left\{ \iint_{D \times D} \log \frac{\text{diam}(D)}{|x - y|} X_{\epsilon \# (\zeta_\epsilon)_+}(x) X_{\epsilon \# (\zeta_\epsilon)_+}(y) d\mu d\mu(x, y) - \iint_{D \times D} \log \frac{\text{diam}(D)}{|x - y|} (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_+(y) d\mu d\mu(x, y) \right\} = 0.$$

Since $(\zeta_\epsilon)_+$ and $X_{\epsilon \# (\zeta_\epsilon)_+}$ are μ -rearrangements of each other, we also get

$$\lim_{\epsilon \rightarrow 0} \left\{ \iint_{D \times D} \log \frac{\epsilon}{|x - y|} X_{\epsilon \# (\zeta_\epsilon)_+}(x) X_{\epsilon \# (\zeta_\epsilon)_+}(y) d\mu d\mu(x, y) - \iint_{D \times D} \log \frac{\epsilon}{|x - y|} (\zeta_\epsilon)_+(x) (\zeta_\epsilon)_+(y) d\mu d\mu(x, y) \right\} = 0.$$

The scaling process yields to

$$\lim_{\epsilon \rightarrow 0} \left\{ \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x-y|} f_\epsilon^*(x) f_\epsilon^*(y) d\mathbf{s}\mu d\mathbf{s}\mu(x, y) \right. \\ \left. - \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x-y|} f_\epsilon(x) f_\epsilon(y) d\mathbf{s}\mu d\mathbf{s}\mu(x, y) \right\} = 0.$$

Now observe that the diameter of $\{f_\epsilon > 0\}$ is of order $\epsilon^{\varsigma-1}$, while the diameter of $\{f_\epsilon^* > 0\}$ is of order 1. Hence, since $b \in C^{0,\alpha}(\overline{D})$ is uniformly bounded on $\{\zeta_\epsilon > 0\}$, independently of $\epsilon > 0$ sufficiently small, we obtain

$$\lim_{\epsilon \rightarrow 0} \left\{ \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x-y|} f_\epsilon^*(x) f_\epsilon^*(y) d(x, y) \right. \\ \left. - \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x-y|} f_\epsilon(x) f_\epsilon(y) d(x, y) \right\} = 0.$$

□

6.5.3 Criterion for convergence

We require the following sufficient criterion to prove convergence in measure of a sequence of functions. The distribution of a positive and measurable function $f : D \rightarrow \mathbb{R}$ is defined by

$$D[f] : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{+\infty\} : D[f](t) = \mu(\{f > t\}).$$

Lemma 6.5.4 (Criterion for convergence in measure). *Let $(f_n)_{n \in \mathbb{N}}$ be a family of measurable, positive and real valued functions, and f be a measurable positive and real valued function. We assume that*

$$D[f](0) < +\infty \quad \text{and} \quad \lim_{M \rightarrow +\infty} D[f](M) = 0.$$

If the sequence $(D[f_n])_{n \in \mathbb{N}}$ converges pointwisely to $D[f]$, and if for μ -almost-all every $t \geq 0$, we have

$$\lim_{n \rightarrow \infty} \mu(\{f_n > t\} \Delta \{f > t\}) = 0,$$

then $(f_n)_{n \in \mathbb{N}}$ converges in μ -measure to f .

This result is standard, but we have not found it in the literature.

Proof. Let us fix $s > 0$. If there exists $x \in \{f_n > f + s\}$, choose $M = M(x, s) > 0$ such that $f(x) \leq M$. For all $t > 0$, we define

$$A_t = \{f_n > t + s/2\} \setminus \{f > t\}$$

and we compute

$$\begin{aligned} \int_0^M \chi_{A_t}(x) dt &= \int_0^{f(x)} \chi_{A_t}(x) dt + \int_{f(x)}^{f_n(x) - \frac{s}{2}} \chi_{A_t}(x) dt \\ &\quad + \int_{f_n(x) - \frac{s}{2}}^M \chi_{A_t}(x) dt \\ &= \int_{f(x)}^{f_n(x) - \frac{s}{2}} 1 dt = f_n(x) - \frac{s}{2} - f(x) \\ &\geq \frac{s}{2} \chi_{\{f_n \geq f+s\}}(x), \end{aligned}$$

and therefore

$$\chi_{\{f_n \geq f+s\}}(x) \leq \frac{2}{s} \int_0^M \chi_{\{f_n > t+s/2\} \setminus \{f > t\}}(x) dt + \chi_{\{f > M\}}(x).$$

The last inequality extends for all $x \in D$ and for all $M > 0$. It also holds if $\{f_n > f + s\}$ is the empty set. By symmetry, we prove similarly that for all $s > 0$, for all $M > 0$ and for all $x \in D$:

$$\chi_{\{f \geq f_n+s\}}(x) \leq \frac{2}{s} \int_0^M \chi_{\{f > t+s/2\} \setminus \{f_n > t\}}(x) dt + \chi_{\{f_n > M\}}(x).$$

Since we have

$$\{f_n > t + s/2\} \setminus \{f > t\} \cup \{f > t + s/2\} \setminus \{f_n > t\} \subseteq \{f > t\} \Delta \{f_n > t\},$$

we obtain

$$\chi_{\{|f_n - f| \geq s\}}(x) \leq \frac{2}{s} \int_0^M \chi_{\{f > t\} \Delta \{f_n > t\}}(x) dt + \chi_{\{f > M\}}(x) + \chi_{\{f_n > M\}}(x).$$

In particular, one concludes by Tonelli's theorem that for all $s > 0$ and

for all $M > 0$, we have

$$\begin{aligned} \mu(\{|f_n - f| \geq s\}) &\leq \frac{2}{s} \int_0^M \mu(\{f > t\} \Delta \{f_n > t\}) dt + \mu(\{f > M\}) \\ &\quad + \mu(\{f_n > M\}). \end{aligned}$$

As a corollary of our assumptions, we have

$$\limsup_{n \rightarrow +\infty} \mu(\{f_n > 0\}) < +\infty,$$

and thus one may apply the Lebesgue dominated convergence theorem and the condition $\lim_{M \rightarrow +\infty} D[f](M) = 0$ to obtain the conclusion. $\square$

Remark 6.5.5. The condition is actually an equivalent condition, but we are not going to use the counterpart in this text.

6.5.4 Asymptotic shape

Lemma 6.5.6. *Let $f_\epsilon = \mathfrak{s}(\zeta_\epsilon)_+$ and let us denote by f_ϵ^Δ the symmetric radially nonincreasing Lebesgue-rearrangement of f_ϵ centered on 0. We have*

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} \left\{ \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x - y|} f_\epsilon^\Delta(x) f_\epsilon^\Delta(y) dx dy \right. \\ \left. - \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x - y|} f_\epsilon(x) f_\epsilon(y) dx dy \right\} = 0. \end{aligned}$$

Proof. Let $f_\epsilon^\star = \mathfrak{s}(X_{\epsilon\sharp}(\zeta_\epsilon)_+)$. For all $\epsilon > 0$ sufficiently small, the functions $f_\epsilon^\star$ and f_ϵ^Δ are both supported in some ball $B(0, R)$ with $R > 0$ independent of $\epsilon > 0$. Furthermore, they belong to $L^p(B(0, R), dx)$. By the scaling process, it is straightforward to see that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \left\{ m(\{x \in B(0, R) : f_\epsilon^\star(x) > t\}) \right. \\ \left. - m(\{x \in B(0, R) : f_\epsilon^\Delta(x) > t\}) \right\} = 0, \end{aligned}$$

where m denotes the usual Lebesgue measure. According to lemma 6.5.4,

we conclude that a function g is an accumulation point in the sense of convergence in Lebesgue measure for the family $\{f_\epsilon^\Delta : \epsilon > 0\}$ if, and only if, it is also an accumulation point in the sense of convergence in Lebesgue measure for the family $\{f_\epsilon^* : \epsilon > 0\}$.

According to Helly's selection principle, the family $\{f_\epsilon^\Delta : \epsilon > 0\}$ admits at least one accumulation point (in the sense of convergence in Lebesgue measure) $g \in L^p(B(0, R), dx)$ that is symmetric radially nonincreasing. Up to taking a new subsequence, and because the set $\{f_\epsilon^\Delta : \epsilon > 0\}$ is bounded in $L^p(B(0, R), dx)$, one may assume that the convergence occurs weakly in $L^p(B(0, R), dx)$. To avoid heavy notations, we assume that $f_\epsilon^\Delta \rightarrow g$ weakly and in convergence in Lebesgue measure as $\epsilon \rightarrow 0$. We deduce that $f_\epsilon^* \rightarrow g$ weakly and in convergence in Lebesgue measure as $\epsilon \rightarrow 0$. Let us write, for all $u \in L^p(\mathbb{R}^2, dx)$,

$$-\log * u : \mathbb{R}^2 \rightarrow \mathbb{R} : -\log * u(y) = \int_{\mathbb{R}^2} u(x) \log \frac{1}{|x - y|} dx.$$

We have $-\log * u \in W^{1,p}(B(0, R))$ whenever $u \in L^p(\mathbb{R}^2, dx)$ [19, Theorem 9.9]. Relying on Rellich-Kondrashov theorem, and up to taking a subsequence, we may abuse notation and assume that

$$-\log * f_\epsilon^\Delta \rightarrow -\log * g$$

weakly in $W^{1,p}(B(0, R))$ and strongly in $L^p(B(0, R), dx)$; and

$$-\log * f_\epsilon^* \rightarrow -\log * g$$

weakly in $W^{1,p}(B(0, R))$ and strongly in $L^p(B(0, R), dx)$. The conclusion now follows directly by a strong-weak convergence argument. $\square$

Using a theory developed by Buchard & Guo [2, Lemma 3.2], one may now assert the following optimal shape theorem:

Theorem 6.5.7. *Every accumulation point of the family $\{\mathfrak{s}(\zeta_\epsilon)_+ : \epsilon > 0\}$ in the sense of Lebesgue measure in $\mathbb{R}^2$ may be written as $f^* \circ T$, where T is a translation in $\mathbb{R}^2$ and f^* is a radially nonincreasing function. In particular, there exist $R > 0$ independent of $\epsilon > 0$, and a family*

$\{X_\epsilon \in D : \epsilon > 0\}$, such that

$$\lim_{\epsilon \rightarrow 0} \left\{ \frac{1}{\tau \mathcal{S}_\epsilon} \int_{D \setminus B(X_\epsilon, R\epsilon)} (\zeta_\epsilon)_+ \, d\mu \right\} = 0.$$

We skip the proof of this result, since it is a direct copy of the proof of [2, Lemma 3.2]. Since we are working with the Newtonian potential kernel K , the proof applies in $L^p(\mathbb{R}^2, dx)$ instead of $L^2(\mathbb{R}^2, dx)$.

6.6 Application to a relaxed maximization problem

In this last section, we apply the result previously obtained for a more common problem that already appeared in the work of Turkington [9, 16, 17] for non sign changing vortex. Let us consider the following energy maximization problem: Maximize the energy E_ϵ over the class

$$\Gamma_\epsilon = \left\{ f : D \rightarrow \mathbb{R} : |f| \leq \epsilon^{-2} \mathcal{S}_\epsilon, \right. \\ \left. \int_D (f)_+ \, d\mu = \tau \mathcal{S}_\epsilon, \int_D (f)_- \, d\mu = (1 - \tau) \mathcal{S}_\epsilon \right\}.$$

This class is physically motivated by the fact that the integral constraints are constraints on the *vortex strength*, while the $L^\infty(D)$ constraint is natural, since the vortex is a kinematic quantity. Without too much effort, we show that the results we have previously obtained may be applied to this problem.

Theorem 6.6.1. *There exists at least one maximizer of E_ϵ over the class Γ_ϵ . Moreover, every maximizer f_ϵ^* is a function of the form*

$$f_\epsilon^* = \epsilon^{-2} \mathcal{S}_\epsilon (\chi_{A_\epsilon^+} - \chi_{A_\epsilon^-})$$

with $A_\epsilon^+ \cap A_\epsilon^- = \emptyset$ two distinct measurable sets such that

$$\mu(A_\epsilon^+) = \tau \epsilon^2, \quad \mu(A_\epsilon^-) = (1 - \tau) \epsilon^2,$$

and there exist $X_\epsilon \in A_\epsilon^+$, $Y_\epsilon \in A_\epsilon^-$, such that

$$\lim_{\epsilon \rightarrow 0} \frac{\mu(A_\epsilon^+ \Delta B(X_\epsilon, R\sqrt{\tau}\epsilon))}{\epsilon^2} = 0, \quad \lim_{\epsilon \rightarrow 0} \frac{\mu(A_\epsilon^- \Delta B(Y_\epsilon, R\sqrt{1-\tau}\epsilon))}{\epsilon^2} = 0;$$

where $R^{-1} = \sqrt{\pi \sup_D b}$.

Proof. The existence part follow from usual functional analysis techniques. More precisely, the class Γ_ϵ is weakly compact in $L^p(D, \mu)$, for every $p > 1$, while the energy E_ϵ is weakly continuous on bounded set in $L^p(D, \mu)$, for every $p > 1$. The last assumptions of the theorem directly follows from our preliminary work. All that remains to be proved is that every maximizer f^* of E_ϵ over Γ_ϵ is of the form

$$f^* = \epsilon^{-2} \mathcal{S}_\epsilon(\chi_A - \chi_B)$$

for some measurable disjoint sets $A, B \subseteq D$. By symmetry, it is sufficient to prove that we have $(f^*)_+ = \epsilon^{-2} \mathcal{S}_\epsilon \chi_A$. We define the measure μ_ϵ as

$$\mu_\epsilon(A) = \mu(A \cap \{f^* \geq 0\}),$$

for all measurable sets $A \subseteq D$. We define also the energy

$$\tilde{E}_\epsilon(f) = E_\epsilon(f - (f^*)_+),$$

and the new class

$$\tilde{\Gamma}_\epsilon = \left\{ f : 0 \leq f \leq \epsilon^{-2} \mathcal{S}_\epsilon : \{f > 0\} \cap \{f^* < 0\} = \emptyset, \int_D f \, d\mu_\epsilon = \tau \mathcal{S}_\epsilon \right\}.$$

Then clearly every function $f \in \tilde{\Gamma}_\epsilon$ induces a function $f - (f^*)_+ \in \Gamma_\epsilon$, so that

$$\tilde{E}_\epsilon(f) = E_\epsilon(f - (f^*)_+) \leq E_\epsilon(f^*) = \tilde{E}_\epsilon((f^*)_+).$$

Therefore, the function $(f^*)_+ \in \tilde{\Gamma}_\epsilon$ is a maximizer of $\tilde{E}_\epsilon$ on $\tilde{\Gamma}_\epsilon$.

Furthermore, one observes that a function f belongs to $\tilde{\Gamma}_\epsilon$ if, and only if we have

$$\int_D f \, d\mu_\epsilon = \tau \mathcal{S}_\epsilon$$

and the additional property that, for all $t \geq 0$:

$$\int_0^t f^{\mu_\epsilon}(s) \, ds \leq \int_0^t \epsilon^{-2} \mathcal{S}_\epsilon \chi_{[0, \tau\epsilon)}(s) \, ds,$$

where $f^{\mu_\epsilon} \in L^\infty(\mathbb{R}^+)$ denotes the nondecreasing μ_ϵ -rearrangement of f in the positive half line $\mathbb{R}^+$. In other words, we have the set identity

$$\tilde{\Gamma}_\epsilon = \left\{ f \in L^\infty(D) : 0 \leq f, \{f > 0\} \subseteq D \setminus \{f^* < 0\}, \right. \\ \left. \int_D f \, d\mu_\epsilon = \tau \mathcal{S}_\epsilon, \forall t \geq 0 : \int_0^t f^{\mu_\epsilon}(s) \, ds \leq \int_0^t \epsilon^{-2} \mathcal{S}_\epsilon \chi_{[0, \tau\epsilon)}(s) \, ds \right\}.$$

Since we also have

$$\tau \mathcal{S}_\epsilon = \int_D (f^*)_+ \, d\mu_\epsilon \leq \epsilon^{-2} \mathcal{S}_\epsilon \, d\mu_\epsilon(\{f^* > 0\}),$$

there exists at least one measurable set $A_\epsilon \subseteq D \setminus \{f^* < 0\}$ such that

$$\mu(A_\epsilon) = \tau \epsilon^2.$$

In particular, we have $(\epsilon^{-2} \mathcal{S}_\epsilon \chi_{A_\epsilon})^{\mu_\epsilon} = \epsilon^{-2} \mathcal{S}_\epsilon \chi_{[0, \tau\epsilon)}$, the function $\epsilon^{-2} \mathcal{S}_\epsilon \chi_{A_\epsilon}$ belongs to $\tilde{\Gamma}_\epsilon$, and we have

$$\tilde{\Gamma}_\epsilon = \left\{ f \in L^\infty(D) : 0 \leq f \leq \epsilon^{-2} \mathcal{S}_\epsilon, \{f > 0\} \subseteq D \setminus \{f^* < 0\}, \right. \\ \left. \int_D f \, d\mu_\epsilon = \tau \mathcal{S}_\epsilon, \forall t \geq 0 : \int_0^t f^{\mu_\epsilon}(s) \, ds \leq \int_0^t (\epsilon^{-2} \mathcal{S}_\epsilon \chi_{A_\epsilon})^{\mu_\epsilon}(s) \, ds \right\}.$$

The class $\tilde{\Gamma}_\epsilon$ equals the weak closure in $L^p(D, \mu_\epsilon)$ of the indicator function $\epsilon^{-2} \mathcal{S}_\epsilon \chi_{A_\epsilon}$ [8, 14, 15]. Since $\tilde{E}_\epsilon$ is a strictly convex functional, every maximizer of $\tilde{E}_\epsilon$ over the convex set $\tilde{\Gamma}_\epsilon = \overline{\text{Rearg}(\epsilon^{-2} \mathcal{S}_\epsilon \chi_{A_\epsilon})}^W$ must be an extreme point of this set. On the other hand, the set of extreme points of $\overline{\text{Rearg}(\epsilon^{-2} \mathcal{S}_\epsilon \chi_{A_\epsilon})}^W$ is the set $\text{Rearg}(\epsilon^{-2} \mathcal{S}_\epsilon \chi_{A_\epsilon})$ [8, 14, 15]. From this it follows that $(f^*)_+$ is a μ_ϵ -rearrangement of $\epsilon^{-2} \mathcal{S}_\epsilon \chi_{A_\epsilon}$. $\square$

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