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Thermodynamically consistent noise modeling in nonlinear circuits

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Traditional attempts to extend the Nyquist-Johnson formula for thermal fluctuations to nonlinear dissipative elements have led to thermodynamically inconsistent models. In this work, we present a Langevin model for thermal noise in nonlinear dissipative elements, that is fully consistent with the main requirements of thermodynamics. The model accurately predicts the Gibbs (Maxwell-Boltzmann) distribution at thermal equilibrium and ensures zero expected voltages and currents, thereby resolving the well-known Brillouin's paradox and confirming compliance with the second law of thermodynamics.

1. Introduction

The theoretical foundation of thermal noise in electronic systems can be traced back to Einstein's 1905 work [1], where he described the Brownian motion of suspended particles as a diffusion process driven by random thermal agitation. Shortly thereafter, Langevin, in 1908, introduced a dynamic formulation that extended Newton's second law by incorporating a stochastic force term [2]. The resulting Langevin equation provided a time-domain description of Brownian motion by accounting for both the dissipative viscous drag and a rapidly fluctuating random force representing thermal agitation. This stochastic differential equation not only reproduced Einstein's diffusive behavior in the long-time limit but also enabled the analysis of system responses across various time and frequency scales.

Langevin's formalism became a powerful tool for connecting microscopic thermal fluctuations to macroscopic observables—an idea that would later become central to the fluctuation–dissipation theorem. This framework directly influenced the work of Johnson and Nyquist in the late 1920s, who investigated the electrical analogue of Brownian motion in linear resistive conductors. Johnson experimentally observed random voltage fluctuations across a linear resistor at thermal equilibrium [3], while Nyquist developed a theoretical explanation by modeling the resistor as a dissipative element coupled to a thermal reservoir [4]. Based on the same statistical principles underlying Langevin's approach, Nyquist derived an expression for the thermal noise spectral density using the equipartition theorem and linear response theory. The resulting formula[a],

$$S(\omega) = 2\,k\,T\,R$$

where $S(\omega)$ is the voltage power spectral density (in V^2/Hz), k is Boltzmann's constant, T is the absolute temperature, and R is the resistance, quantitatively describing the electrical manifestation of thermal agitation—now known as Johnson–Nyquist noise.

The Johnson–Nyquist theorem can be derived solely through thermodynamic arguments, e.g. from the equipartition theorem or as a special case of the fluctuation-dissipation theorem, making it fundamentally a thermodynamic statement. The only device-specific parameter it requires is the macroscopic resistance R, which characterizes the linear resistor.

Generalizing the Johnson–Nyquist theory to nonlinear resistors has proven to be particularly challenging. A direct application of the theory to nonlinear components leads to thermodynamic inconsistencies, particularly with respect to energy conservation and entropy production [5,6]. The most promising generalizations remain confined to specific classes of nonlinear networks [7,8], with recent works based on stochastic formulations of thermodynamics, to include networks out of equilibrium [9,10,11].

Over time, a belief has emerged that thermodynamically consistent Gaussian white noise models for nonlinear resistors do not exist [12,13,14]. In contrast, white noise models for shot noise in diodes, tunnel junctions, and MOS transistors, are standard and can yield thermodynamically acceptable behavior under certain constraints. Notably, in [15], the authors introduce a set of criteria to evaluate whether a given model complies with the principles of thermodynamics. The striking conclusion is that conventional Gaussian noise models for nonlinear devices predict circuit behavior that violates thermodynamic laws and, as such, should be discarded. This conclusion, however, relies on the tacit assumption that internal noise generation is independent of the load connected across the resistor. In reality, noise generation in a dissipative system depends on the constraints under which the system is maintained—of which the load is a key example. Consequently, a load-independent, i.e.

[a]This formula is defined for both negative and positive frequencies ω. The power spectral density becomes $S(\omega) = 4\,k\,T\,R$ when only positive frequencies are considered.

a fully modular description of thermal noise cannot, in general, be developed for nonlinear systems [7,16].

In this paper, we present an extension of the Johnson–Nyquist theory to nonlinear, passive resistors at thermal equilibrium with an external bath. The derivation is based on stochastic calculus and on the assumption that noise generation can be modulated by the load connected across the resistor. We demonstrate that this model is consistent with thermodynamic principles and, in particular, resolves Brillouin's paradox. Furthermore, we show that in the limiting case of a linear resistor, the model reduces to the classical Johnson–Nyquist theorem.

2. Thermodynamic requirements

We recall two of the thermodynamic criteria introduced in [15], which are used to evaluate whether a given model is consistent with the fundamental laws of thermodynamics. A third criterion, related to entropy production during transient dynamics, must also be satisfied but is not considered in this work, that focuses on devices operating in stationary conditions (time-constant bias).

Requirement No. 1: No isothermal conversion of heat into work

The second law of thermodynamics states that an isothermal system cannot convert heat into work as its sole effect.

Accordingly, a noisy dissipative device maintained at a fixed temperature T must not, on average, supply power to an external circuit. In particular, the average short-circuit current of such a device must be zero. A circuit that, at equilibrium, is capable of converting heat into work without any external power source would constitute a *perpetual motion machine of the first kind*, violating the principle of energy conservation. Similarly, a system that converts heat entirely into work while interacting with only a single heat reservoir would represent a *perpetual motion machine of the second kind*, in violation of the second law of thermodynamics.

Requirement No. 2: Gibbs (Maxwell-Boltzmann) distribution at equilibrium

In a circuit composed by lossless elements (capacitor and inductors) and dissipative devices at constant temperature, the equilibrium distribution for inductor fluxes φ and capacitor charges q must have the Gibbs (also called Maxwell-Boltzmann) form

$$\rho_{st}(\varphi, q) = A\, e^{-\frac{1}{k\,T}\, E(\varphi,q)} \tag{1}$$

where $E(\varphi, q)$ is the total energy stored in the inductors and capacitors, and A is a normalization constant ensuring that the integral of ρ_{st} over all the possible states (φ, q) sums to one.

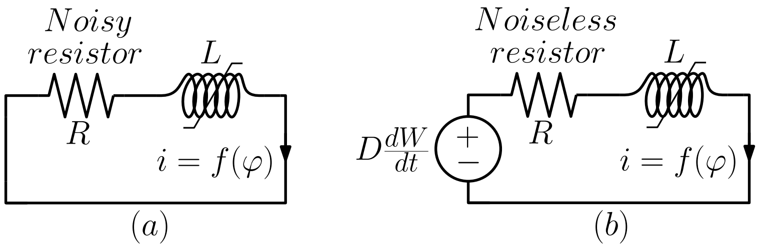

Fig. 1. **(a)** First-order circuit with noisy linear resistor. **(b)** The Thevenin equivalent circuit for the linear resistor, with a series-voltage white Gaussian noise source added.

3. First-order circuit with linear resistor

To illustrate the application of concepts and methods of stochastic calculus used in this work, we consider the first-order circuit shown in Fig. 1(a). The circuit is composed of a noisy linear resistor connected in series to a nonlinear, flux-controlled inductor with characteristic $i = f(\varphi)$. The state equation is obtained applying the Kirchhoff voltage law (KVL) and using each element's constitutive equation. Following Langevin approach, the random "force" $D\,dW_t$ (in this case a random voltage) is added to the equation, thereby obtaining the stochastic differential equation (SDE):

$$d\varphi = -R\,f(\varphi)dt + D\,dW_t \tag{2}$$

where the random voltage is proportional to a white Gaussian noise dW_t, the "formal" derivative of a Wiener process, characterized by $\mathbb{E}[dW_t] = 0$ and $\mathbb{E}[dW_t dW_s] = \delta(t-s)$ ($\mathbb{E}[\cdot]$ denotes the expectation operator). The SDE describes the equivalent circuit shown in Fig. 1(b), where the noisy resistor has been replaced by its Thevenin equivalent: a noiseless resistor in series with the random voltage source.

SDEs can be interpreted following different rules, depending on the definition adopted for the stochastic integrations. The two most common interpretations are the Itô and Stratonovich formulations [17,18]. More recently, particularly within the statistical physics community, a third interpretation—known as the anti-Itô or Hänggi-Klimontovich interpretation—has attracted attention [19]. The anti-Itô interpretation offers certain advantages, which will be discussed elsewhere.

For SDEs with additive (i.e., unmodulated) noise, such as Eq. (2), the solution remains the same regardless of the chosen interpretation. However, in the presence of multiplicative (i.e., modulated) noise, the solution depends on the chosen interpretation. In their fundamental work [15], the authors argued that the thermodynamic consistency requirements cannot be satisfied by a Gaussian noise model for any nonlinear element—regardless of the operating-point-dependent noise amplitude—within either the Itô or Stratonovich interpretations. Therefore, to challenge this conclusion, we shall adopt Itô interpretation for all SDEs[b].

[b]It can be shown that a thermodynamically consistent model can also be derived under the

Instead of solving the SDE (2), we consider the associated Fokker-Planck equation (FPE) for the probability density function (PDF) $\rho(t,\varphi)$:

$$\frac{\partial \rho}{\partial t} = \frac{\partial J(t,\varphi)}{\partial \varphi} = \frac{\partial}{\partial \varphi}\left[-R\, f(\varphi)\,\rho + \frac{D^2}{2}\frac{\partial \rho}{\partial \varphi}\right] \tag{3}$$

where $J(t,\varphi)$ is the probability current.

At thermal equilibrium, the circuit is expected to satisfy local detailed balance [20], which in turns implies the zero-flux condition[c] $J(t,\varphi)=0$. The corresponding solution for the FPE is the stationary PDF, which is found by integration after separation of variables, obtaining:

$$\rho_{st}(\varphi) = A\, e^{-\frac{2R}{D^2}\, F(\varphi)} \tag{4}$$

The constant A is determined by the normalization condition:

$$\int_{-\infty}^{+\infty} \rho_{st}(\varphi)\, d\varphi = 1 \tag{5}$$

while $F(\varphi) = \int f(\varphi)\, d\varphi$ is the energy stored in the nonlinear inductor[d]. In fact:

$$F(\varphi) = \int f(\varphi)\, d\varphi = \int f(\varphi)\, \frac{d\varphi}{dt} dt = \int p(t)\, dt \tag{6}$$

where $p(t) = i(t)v(t)$ is the instantaneous power in the nonlinear inductor.

According to the thermodynamic requirement no. 2, at thermal equilibrium the stationary PDF (4) must coincide with the Gibbs distribution (1), which implies:

$$D = \sqrt{2\, k\, T\, R} \tag{7}$$

which is the famous Johnson-Nyquist formula.

Note that, in general, even if the stationary PDF (4) is the Gibbs distribution, it is not Gaussian, unless the inductor is linear.

4. First-order circuit with nonlinear resistor

We now consider the case of the first-order circuit with a noisy nonlinear resistor shown in Fig. 2(a). It may be tempting to apply the same approach used for the linear resistor case, by introducing a white Gaussian noise source that is independent of the circuit state, as illustrated in Fig. 2(b).

The state equation for the circuit now reads

$$d\varphi = -r(f(\varphi))dt + D\, dW_t \tag{8}$$

Stratonovich interpretation.

[c]It is important to note that the inverse implication, i.e. zero-flux condition implies detailed balance, is not necessarily true.

[d]Like potential energy, the energy stored in a reactive element is only defined up to an additive constant. This is represented by the arbitrary integration constant, that is fixed to zero for simplicity.

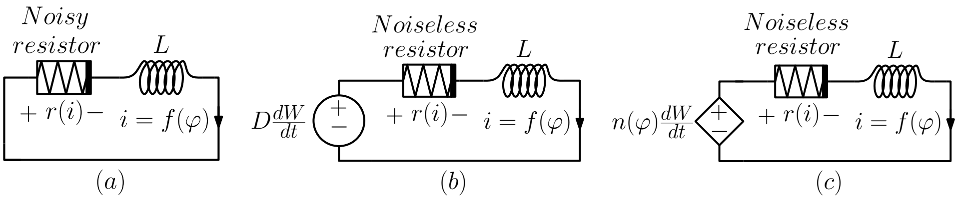

Fig. 2. **(a)** First-order circuit with noisy nonlinear resistor. **(b)** Thevenin equivalent circuit for the nonlinear resistor with a state independent random source. **(c)** Thevenin equivalent circuit for the nonlinear resistor with a state dependent random source.

Repeating the procedure of Sec. 3, we obtain the stationary PDF:

$$\rho_{st}(q) = A\, e^{-\frac{2}{D^2} R(\varphi)} \tag{9}$$

where

$$R(\varphi) = \int r(f(\varphi))\, d\varphi \tag{10}$$

Comparing Eq. (9) with the Gibbs measure it must be

$$D^2 = 2k\,T \frac{R(\varphi)}{F(\varphi)} \tag{11}$$

Since the left-hand side is constant, the equality holds if and only if $R(\varphi) \propto F(\varphi)$, e.g. for a restricted class of two-terminal elements, that include linear ones. In fact, for a linear inductor $F(\varphi) \propto \varphi^2$, and then $r(\varphi) \propto \varphi$, i.e. the resistor must also be linear.

Since the stationary PDF is not the Gibbs distribution, requirement no.1 is not satisfied. The expected current in the circuit is given by

$$\mathbb{E}[f(\varphi)] = \int_{-\infty}^{+\infty} f(\varphi) A\, e^{-\frac{2}{D^2} R(\varphi)}\, d\varphi \tag{12}$$

which, for general choices of $f(\varphi)$ and $R(\varphi)$ is not zero.

As an illustrative example, we consider a linear inductor in series with a nonlinear resistor characterized by a quadratic current-voltage relationship, given by $R(\varphi) = \frac{D^2}{2}(\varphi + \varphi^2)$. A resistor with a similar constitutive equation is discussed in [5]. This example is chosen primarily for its analytical tractability rather than for its physical realism. In particular, the resistor is active, as it produces a positive voltage for a sufficiently large negative current. Moreover, the constitutive equation should include parameters—omitted here for simplicity—to ensure dimensional coherence.

A straightforward calculation shows that

$$\mathbb{E}[i] = \mathbb{E}\left[\frac{\varphi}{L}\right] = \frac{A}{L} \int_{-\infty}^{+\infty} \varphi\, e^{-(\varphi+\varphi^2)}\, d\varphi = -\frac{A}{2L} \sqrt{\pi}\, e^{1/4} \tag{13}$$

Thus, even at thermal equilibrium, the nonlinear resistor rectifies its own fluctuations, resulting in a nonzero average current, and the inductor accumulates a finite internal flux. One could, in principle, imagine an idealized device that periodically disconnects the inductor from the nonlinear resistor and connects it instead to a linear resistor. In doing so, the inductor would release the stored energy, which is then dissipated by the linear resistor. Repeating this cycle indefinitely would result in continuous energy extraction from a single thermal reservoir—effectively realizing a Maxwell's demon. This paradoxical scenario is essentially what is known as Brillouin's paradox [5,20].

Therefore, we conclude that for nonlinear resistors, a thermodynamically consistent white Gaussian model with internal noise independent of the circuit state is not possible.

To solve the problem, we assume that the noise intensity is modulated by the load connected to the resistor, introducing a state-dependent noise modulating function $n(\varphi)$. Then we look whether there exists a function $n(\varphi)$ such that the stationary PDF corresponds to the Gibbs distribution. The equivalent two-terminal element for the noisy nonlinear resistor is shown in Fig. 2(c). The state equation reads:

$$d\varphi = -r\left(f(\varphi)\right)dt + n(\varphi)\,dW_t \tag{14}$$

where $n(\varphi)$ is to be determined imposing thermodynamic constrains. The associated FPE is:

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \varphi}\left[r\left(f(\varphi)\right)\rho + \frac{1}{2}\frac{\partial}{\partial \varphi}\left(n^2(\varphi)\rho\right)\right] \tag{15}$$

Again, the stationary solution is found imposing zero flux condition, obtaining

$$\frac{d}{d\varphi}\left(n^2(\varphi)\rho_{st}\right) = -2\,r\left(f(\varphi)\right)\,\rho_{st} \tag{16}$$

Integrating both sides and imposing that at equilibrium the stationary PDF is the Gibbs measure we find

$$n^2(\varphi) = -2\,e^{\frac{1}{k\,T}F(\varphi)}\int r\left(f(\varphi)\right)\,e^{-\frac{1}{k\,T}F(\varphi)}\,d\varphi \tag{17}$$

which is the main result of this paper.

One can obtain that for a linear resistor, (17) reduces to the celebrated Johnson-Nyquist formula. In fact, for a linear resistor connected in series with a nonlinear inductor we have $r(f(\varphi)) = Rf(\varphi)$. a Substituting this into equation (17), and assuming the arbitrary integration constant is zero, yields:

$$n^2(\varphi) = -2\,e^{\frac{1}{k\,T}F(\varphi)}(-k\,T)\int R\left(-\frac{f(\varphi)}{k\,T}\right)\,e^{-\frac{1}{k\,T}F(\varphi)}\,d\varphi = 2k\,T\,R \tag{18}$$

as required.

Moreover, the model (17) correctly predicts zero expected current through the elements effectively solving the Brillouin's paradox. In fact, because the stationary PDF is the Gibbs distribution we have

$$\mathbb{E}[i] = \mathbb{E}[f(\varphi)] = \int_{-\infty}^{+\infty} f(\varphi)A\,e^{-\frac{1}{kT}F(\varphi)}\,d\varphi = -k\,T\,\rho_{st}(\varphi)\Big|_{-\infty}^{+\infty} = 0 \tag{19}$$

where we used the fact that, for a well defined system, the Gibbs distribution goes to zero for $\varphi \to \pm\infty$.

Equation (17) demonstrates that it is indeed feasible to construct a white Gaussian noise model for thermal fluctuations in a nonlinear resistor within the Itô calculus framework, while preserving thermodynamic consistency—provided the noise intensity is allowed to depend on the external load connected to the resistor. Moreover, Eq. (17) implies a necessary condition for the existence of such a model: the right-hand side must remain strictly positive, thereby imposing a constraint on the class of nonlinear resistors that can be represented within this formalism. This condition merits further investigation and is currently the subject of analysis.

5. Conclusions

In this work, we present a thermodynamically consistent Gaussian white noise model for nonlinear, dissipative two-terminal elements. Unlike previous approaches, we relax the conventional assumption that thermal noise intensity is independent of the connected load. Instead, we propose that the noise intensity is modulated by a state-dependent function. By enforcing fundamental thermodynamic principles, we derive the specific form that this modulation function must take.

We develop our model for the case in which the nonlinear resistor is connected to an inductor, resulting in a Langevin equation interpreted in the Itô sense. We note, however, that the very same approach can be applied to more general circuits, and under the Stratonovich interpretation.

To represent the noisy nonlinear resistor, we construct an equivalent circuit composed of a noiseless nonlinear resistor in series with a controlled random voltage source. The proposed model satisfies two fundamental thermodynamic consistency conditions: it correctly reproduces the Gibbs distribution at thermal equilibrium and prohibits isothermal conversion of heat into work, thereby resolving the well-known Brillouin's paradox. Moreover, in the limiting case of a linear resistor, the model reduces to the classical Nyquist formula for thermal noise.

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