BOUSSINESQ'S EQUATION FOR WATER WAVES: THE SOLITON RESOLUTION CONJECTURE FOR SECTOR IV

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Abstract. We consider the Boussinesq equation on the line for a broad class of Schwartz initial data relevant for water waves. In a recent work, we identified ten main sectors describing the asymptotic behavior of the solution, and for each of these sectors we gave an exact expression for the leading asymptotic term in the case when no solitons are present. In this paper, we derive an asymptotic formula in Sector IV, characterized by $\frac{x}{t} \in (\frac{1}{\sqrt{3}}, 1)$, in the case when solitons are present. In particular, our results provide an exact expression for the soliton-radiation interaction to leading order and a verification of the soliton resolution conjecture for the Boussinesq equation in Sector IV.

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1. Introduction

The Boussinesq equation [4]

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx}, \tag{1.1}$$

where $u(x,t)$ is a real-valued function and subscripts denote partial derivatives, models small-amplitude dispersive waves in shallow water propagating in both the right and left directions, see e.g. [21]. Equation (1.1) supports solitons [1, 2, 16] and admits a 3×3 Lax pair [29]. Until recently [5], the determination of the long-time asymptotics for the solution of (1.1) was still an open problem, see e.g. Deift's list of open problems in [14].

In [5], we studied the direct and inverse scattering problems for (1.1). Among other things, we established that for a wide class of initial data relevant for water waves, the solution is unique and exists globally. We identified ten main asymptotic sectors and for each of these sectors, we computed the leading asymptotics of the solution. The proofs of some of these asymptotic formulas were omitted in [5] for conciseness. Moreover, only solitonless solutions were considered.

The purpose of this paper is to derive an asymptotic formula for the solution in the sector $\frac{x}{t} \in (\frac{1}{\sqrt{3}}, 1)$ in the case when solitons are present. This sector was referred to as Sector IV in [5]. In the special case when solitons are absent, our formula reduces to the formula announced in [5], thus providing a proof of this formula. In the case when solitons are present, our formula quantifies the asymptotic effect of solitons on the solution in Sector IV.

The soliton resolution conjecture for an integrable equation expresses the expectation that a solution of the equation with generic initial data eventually evolves into a set of solitons superimposed on a dispersive radiative background. Results on soliton resolution

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for equations with 2×2 Lax pairs can be found in [3, 10–13, 17, 19, 20, 23, 24, 26, 27], see also [9] for a probabilistic approach. Since $x/t < 1$ in Sector IV and all one-soliton solutions of (1.1) travel with speeds greater than 1 [16], the soliton resolution conjecture for (1.1) suggests that there should be no order 1 contributions to the asymptotics in Sector IV stemming from solitons. This is indeed what we find: our formula shows that the solution of (1.1) in Sector IV is $O(t^{-1/2})$ as $t \to \infty$. Our results therefore verify the soliton resolution conjecture for the Boussinesq equation in this sector. We stress that although the solitons themselves are not directly observable in Sector IV for large times, they do have a nontrivial effect on the radiation in Sector IV; we compute the leading order contribution of this effect exactly.

It is important to note that we restrict ourselves to initial data which contain no high-frequency modes (in a sense made precise in Assumption (*iii*) below). This assumption is consistent with the derivation of the Boussinesq equation as a model for water waves, which assumes that high-frequency Fourier modes are absent, or at least highly suppressed. In fact, if high-frequency modes are present, the solution will contain components that grow exponentially in time and the soliton resolution conjecture will automatically fail.

Our main result is presented in Section 2. Its proof relies on a Deift–Zhou [15] steepest descent analysis of a row-vector Riemann–Hilbert (RH) problem, and this RH problem is stated in Section 3. Section 4 contains a brief overview of the proof. In Sections 5 and 6, we carry out two transformations of the RH problem. After introducing a global parametrix in Section 7, we perform two further transformations in Sections 8 and 9. Using two local parametrices constructed in Sections 10 and 11, respectively, we arrive at a small-norm RH problem whose large t behavior is analyzed in Section 12. Finally, the long-time asymptotics of the solution $u(x,t)$ of (1.1) is obtained in Section 13.

1.1. **Notation.** We use the following notation throughout the paper.

- $C > 0$ and $c > 0$ denote generic constants that may change within a computation.
- $[A]_1$, $[A]_2$, and $[A]_3$ denote the first, second, and third columns of a 3×3 matrix A.
- If A is an $n \times m$ matrix, we define $|A| \geq 0$ by $|A|^2 = \Sigma_{i,j}|A_{ij}|^2$. For a piecewise smooth contour $\gamma \subset \mathbb{C}$ and $1 \leq p \leq \infty$, we define $\|A\|_{L^p(\gamma)} := \||A|\|_{L^p(\gamma)}$.
- $\mathbb{D} = \{k \in \mathbb{C} \,|\, |k| < 1\}$ denotes the open unit disk and $\partial\mathbb{D} = \{k \in \mathbb{C} \,|\, |k| = 1\}$ denotes the unit circle.
- $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.
- $\mathcal{Q} := \{\kappa_j\}_{j=1}^6$ and $\hat{\mathcal{Q}} := \mathcal{Q} \cup \{0\}$, where $\{\kappa_j = e^{\frac{\pi i(j-1)}{3}}\}_1^6$ are the sixth roots of unity.
- $\{D_n\}_{n=1}^6$ denote the open subsets of the complex plane shown in Figure 2.
- $\Gamma = \cup_{j=1}^9 \Gamma_j$ denotes the contour shown and oriented as in Figure 2.
- $\hat{\Gamma}_j = \Gamma_j \cup \partial\mathbb{D}$ denotes the union of Γ_j and the unit circle.

2. Main results

We consider the initial value problem for (1.1) with initial data

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \tag{2.1}$$

where $u_0, u_1 \in \mathcal{S}(\mathbb{R})$ are real-valued functions in the Schwartz class. We assume that u_1 obeys the condition $\int_{\mathbb{R}} u_1(x)dx = 0$; this assumption is natural because it ensures that the total mass $\int_{\mathbb{R}} u(x,t)dx$ is conserved in time. A result of [5] states that the solution to the initial value problem (1.1)–(2.1) can be expressed in terms of the solution n of a row-vector RH problem. The jump matrix of this RH problem is expressed in terms of two scalar reflection coefficients, $r_1(k)$ and $r_2(k)$, which are defined as follows.

2.1. **Definition of r_1 and r_2.** Let $\omega := e^{\frac{2\pi i}{3}}$ and define $\{l_j(k), z_j(k)\}_{j=1}^3$ by

$$l_j(k) = i\frac{\omega^j k + (\omega^j k)^{-1}}{2\sqrt{3}}, \qquad z_j(k) = i\frac{(\omega^j k)^2 + (\omega^j k)^{-2}}{4\sqrt{3}}, \qquad k \in \mathbb{C}\setminus\{0\}. \tag{2.2}$$

Let $P(k)$ and $\mathsf{U}(x,k)$ be given by

$$P(k) = \begin{pmatrix} 1 & 1 & 1 \\ l_1(k) & l_2(k) & l_3(k) \\ l_1(k)^2 & l_2(k)^2 & l_3(k)^2 \end{pmatrix}, \quad \mathsf{U}(x,k) = P(k)^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{u_{0x}}{4} - \frac{iv_0}{4\sqrt{3}} & -\frac{u_0}{2} & 0 \end{pmatrix} P(k),$$

where $v_0(x) = \int_{-\infty}^x u_1(x')dx'$. Let X, X^A, Y, Y^A be the unique solutions of the Volterra integral equations

$$\begin{aligned}
X(x,k) &= I - \int_x^\infty e^{(x-x')\widehat{\mathcal{L}(k)}}(\mathsf{U}X)(x',k)dx', \\
X^A(x,k) &= I + \int_x^\infty e^{-(x-x')\widehat{\mathcal{L}(k)}}(\mathsf{U}^T X^A)(x',k)dx', \\
Y(x,k) &= I + \int_{-\infty}^x e^{(x-x')\widehat{\mathcal{L}(k)}}(\mathsf{U}Y)(x',k)dx', \\
Y^A(x,k) &= I - \int_{-\infty}^x e^{-(x-x')\widehat{\mathcal{L}(k)}}(\mathsf{U}^T Y^A)(x',k)dx',
\end{aligned}$$

where $\mathcal{L} = \operatorname{diag}(l_1, l_2, l_3)$, $(\cdot)^T$ denotes the transpose operation, and $e^{\hat{\mathcal{L}}}$ acts on a 3×3 matrix M by $e^{\hat{\mathcal{L}}}M = e^{\mathcal{L}}Me^{-\mathcal{L}}$. Define $s(k)$ and $s^A(k)$ by

$$s(k) = I - \int_{\mathbb{R}} e^{-x\widehat{\mathcal{L}(k)}}(\mathsf{U}X)(x,k)dx, \qquad s^A(k) = I + \int_{\mathbb{R}} e^{x\widehat{\mathcal{L}(k)}}(\mathsf{U}^T X^A)(x,k)dx.$$

The two spectral functions $\{r_j(k)\}_1^2$ are defined by

$$\begin{cases} r_1(k) = \frac{(s(k))_{12}}{(s(k))_{11}}, & k \in \hat{\Gamma}_1 \setminus \hat{\mathcal{Q}}, \\ r_2(k) = \frac{(s^A(k))_{12}}{(s^A(k))_{11}}, & k \in \hat{\Gamma}_4 \setminus \hat{\mathcal{Q}}. \end{cases} \tag{2.3}$$

2.2. **Solitons.** In [7], we extended the approach of [5] to the case when solitons are present. In this case, in addition to $r_1(k)$ and $r_2(k)$, the formulation of the RH problem also involves a set of poles Z and a set of corresponding residue constants $\{c_{k_0}\}_{k_0\in\mathsf{Z}} \subset \mathbb{C}$. In what follows, we describe how Z and $\{c_{k_0}\}_{k_0\in\mathsf{Z}} \subset \mathbb{C}$ are defined.

Solitons are generated by the possible zeros of the functions $s_{11}(k)$ and $s_{11}^A(k)$. As shown in [5], s_{11} and s_{11}^A have analytic extensions to the interiors of $\bar{D}_1 \cup \bar{D}_2$ and $\bar{D}_4 \cup \bar{D}_5$, respectively. We will restrict ourselves to the generic case when s_{11} and $s_{11}^A(k)$ have no zeros on the contour Γ. The symmetries $s_{11}(k) = s_{11}(\omega/k)$ and $s_{11}^A(k) = \overline{s_{11}(\bar{k}^{-1})}$ (also established in [5]) then imply that it is enough to consider the zeros of s_{11} in D_2. We decompose the open set D_2 into three parts: $D_2 = D_{\text{reg}} \sqcup D_{\text{sing}} \sqcup (D_2 \cap \mathbb{R})$, where

$$\begin{aligned}
D_{\text{reg}} &:= D_2 \cap \big(\{k \,|\, |k| > 1, \operatorname{Im} k > 0\} \cup \{k \,|\, |k| < 1, \operatorname{Im} k < 0\}\big), \\
D_{\text{sing}} &:= D_2 \cap \big(\{k \,|\, |k| > 1, \operatorname{Im} k < 0\} \cup \{k \,|\, |k| < 1, \operatorname{Im} k > 0\}\big),
\end{aligned}$$

and $D_2 \cap \mathbb{R} = (-1, 0) \cup (1, \infty)$. We denote the right and left parts of D_{reg} by D_{reg}^R and D_{reg}^L, respectively, and similarly for D_{sing}, see Figure 1.

We showed in [7] that zeros in D_{sing} give rise to solitons with singularities. We will therefore assume that the zero-set Z of $s_{11}(k)$ is contained in $D_{\text{reg}} \cup (-1, 0) \cup (1, \infty)$. We

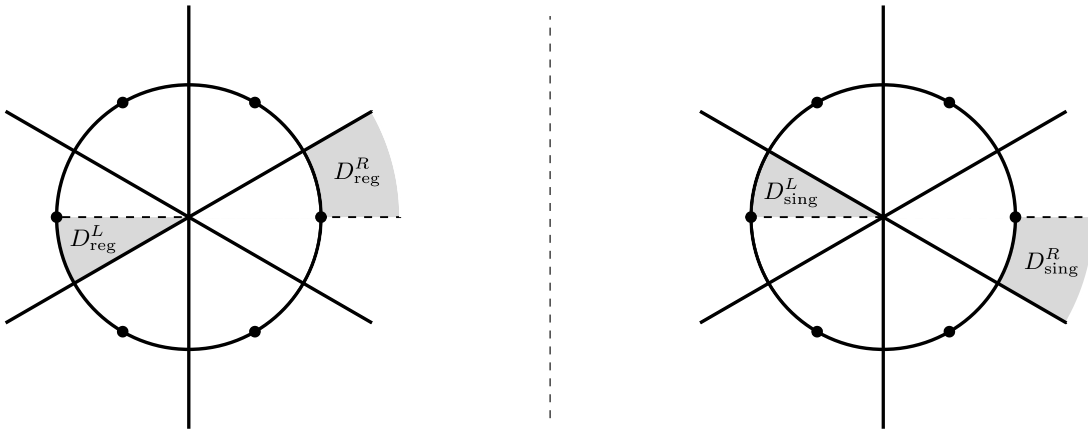

Figure 1. *The open sets* $D_{\text{reg}}^R, D_{\text{reg}}^L, D_{\text{sing}}^R$*, and* D_{sing}^L*.*

will consider the generic case of a finite number of simple zeros. In the case when the initial data u_0 and u_1 have compact support, the associated residue constants are then defined by

$$c_{k_0} := \begin{cases} -\frac{s_{13}(k_0)}{\dot{s}_{11}(k_0)}, & k_0 \in \mathsf{Z} \setminus \mathbb{R}, \\ -\frac{s_{12}(k_0)}{\dot{s}_{11}(k_0)}, & k_0 \in \mathsf{Z} \cap \mathbb{R}. \end{cases} \tag{2.4}$$

If u_0 and v_0 do not have compact support, then the expressions in (2.4) are in general not well-defined and the following more involved definition is required: Define the vector-valued function $w(x,k)$ by

$$w = \begin{pmatrix} Y_{21}^A X_{32}^A - Y_{31}^A X_{22}^A \\ Y_{31}^A X_{12}^A - Y_{11}^A X_{32}^A \\ Y_{11}^A X_{22}^A - Y_{21}^A X_{12}^A \end{pmatrix}. \tag{2.5}$$

If $k_0 \in \mathsf{Z} \setminus \mathbb{R}$, then c_{k_0} is defined as the unique complex constant such that

$$\frac{w(x,k_0)}{\dot{s}_{11}(k_0)} = c_{k_0} e^{(l_1(k_0)-l_3(k_0))x}[X(x,k_0)]_1 \qquad \text{for all } x \in \mathbb{R}. \tag{2.6a}$$

If $k_0 \in \mathsf{Z} \cap \mathbb{R}$, then c_{k_0} is defined as the unique complex constant such that

$$\frac{[Y(x,k_0)]_2}{\dot{s}_{22}^A(k_0)} = c_{k_0} e^{(l_1(k_0)-l_2(k_0))x}[X(x,k_0)]_1 \qquad \text{for all } x \in \mathbb{R}. \tag{2.6b}$$

It turns out that simple zeros of $s_{11}(k)$ in D_{reg}^R and D_{reg}^L generate right- and left-moving breather solitons, respectively. Similarly, simple zeros of $s_{11}(k)$ in $(1,\infty)$ and $(-1,0)$ generate right- and left-moving solitons, respectively, and it was shown in [7] that a soliton generated by a zero in $(-1,0) \cup (1,\infty)$ is non-singular if and only if the associated residue constant c_{k_0} satisfies $i(\omega^2 k_0^2 - \omega)c_{k_0} \notin (-\infty, 0)$. This motivates the following assumptions on Z and $\{c_{k_0}\}_{k_0 \in \mathsf{Z}}$.

2.3. **Assumptions.** As in [7, Theorem 2.11], our results will be valid under the following assumptions:

(i) Finite number of non-singular solitons: suppose that $s_{11}(k)$ has a simple zero at each point in Z, where $\mathsf{Z} \subset D_{\text{reg}} \cup (-1,0) \cup (1,\infty)$ is a set of finite cardinality, and that $s_{11}(k)$ has no other zeros in $k \in (\bar{D}_2 \cup \partial\mathbb{D}) \setminus \hat{\mathcal{Q}}$. If $k_0 \in \mathsf{Z} \setminus \mathbb{R}$, then we also assume that $s_{22}^A(k_0) \neq 0$. If $k_0 \in \mathsf{Z} \cap \mathbb{R}$, then we also assume that $i(\omega^2 k_0^2 - \omega)c_{k_0} \notin (-\infty, 0)$.

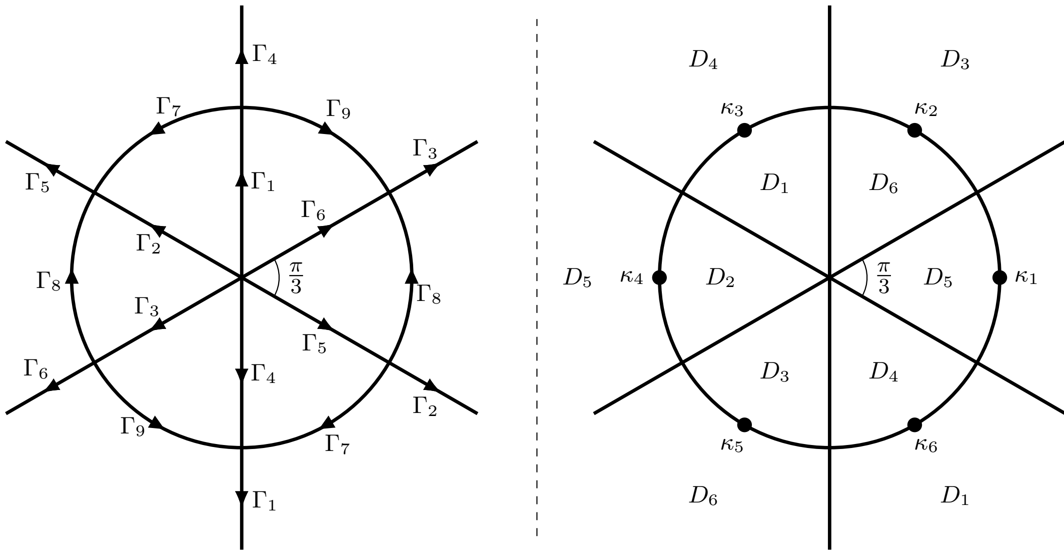

Figure 2. *The contour $\Gamma = \cup_{j=1}^{9}\Gamma_j$ in the complex k-plane (left) and the open sets D_n, $n = 1, \dots, 6$, together with the sixth roots of unity κ_j, $j = 1, \dots, 6$ (right).*

(ii) The spectral functions s and s^A have generic behavior near $k = 1$ and $k = -1$: we suppose for $k_\star = 1$ and $k_\star = -1$ that

$$\lim_{k\to k_\star}(k-k_\star)s(k)_{11} \neq 0, \quad \lim_{k\to k_\star}(k-k_\star)s(k)_{13} \neq 0, \quad \lim_{k\to k_\star} s(k)_{31} \neq 0, \quad \lim_{k\to k_\star} s(k)_{33} \neq 0,$$
$$\lim_{k\to k_\star}(k-k_\star)s^A(k)_{11} \neq 0, \quad \lim_{k\to k_\star}(k-k_\star)s^A(k)_{31} \neq 0, \quad \lim_{k\to k_\star} s^A(k)_{13} \neq 0, \quad \lim_{k\to k_\star} s^A(k)_{33} \neq 0.$$

(iii) No high-frequency modes: we suppose that $r_1(k) = 0$ for all $k \in [0, i]$, where $[0, i]$ is the vertical segment from 0 to i.

We emphasize that Assumption (ii) is generic. Assumption (iii) ensures that the solution belongs to the physically relevant class of solutions which do not contain exponentially growing high-frequency modes. If Assumptions (i)–(iii) hold true, then the solution $u(x,t)$ of the initial value problem (1.1)–(2.1) exists globally [7].

2.4. **Statement of main result.** The formulation of our main theorem involves a number of quantities that we now introduce. Let $\zeta := x/t$ and assume that $\zeta \in (\frac{1}{\sqrt{3}}, 1)$. For $k \in \mathbb{C} \setminus \{0\}$ and $1 \le j < i \le 3$, define $\Phi_{ij}(\zeta, k)$ by

$$\Phi_{ij}(\zeta,k) = (l_i(k) - l_j(k))\zeta + (z_i(k) - z_j(k)), \tag{2.7}$$

where $l_j(k)$ and $z_j(k)$ are as in (2.2). The function $k \mapsto \Phi_{21}(\zeta, k)$ has four saddle points $\{k_j = k_j(\zeta)\}_{j=1}^4$ given by

$$k_2 = \frac{1}{4}\left(\zeta - \sqrt{8+\zeta^2} - i\sqrt{2}\sqrt{4 - \zeta^2 + \zeta\sqrt{8+\zeta^2}}\right), \tag{2.8a}$$

$$k_4 = \frac{1}{4}\left(\zeta + \sqrt{8+\zeta^2} - i\sqrt{2}\sqrt{4 - \zeta^2 - \zeta\sqrt{8+\zeta^2}}\right), \tag{2.8b}$$

$k_1 = \bar{k}_2$, and $k_3 = \bar{k}_4$. Note that $|k_2| = |k_4| = 1$, $\arg k_2 \in (-\frac{3\pi}{4}, -\frac{2\pi}{3})$, and $\arg k_4 \in (-\frac{\pi}{6}, 0)$. Define also $\delta_j(\zeta, k)$, $j = 1, \dots, 5$, by

$$\delta_1(\zeta,k) = \exp\left\{\frac{-1}{2\pi i}\int_i^{\omega k_4} \frac{\ln(1 + r_1(s) r_2(s))}{s-k} ds\right\}, \; \delta_4(\zeta,k) = \exp\left\{\frac{1}{2\pi i}\int_{\omega^2 k_2}^{\omega} \frac{\ln f(s)}{s-k} ds\right\},$$

$$\delta_2(\zeta,k) = \exp\left\{\frac{1}{2\pi i}\int_{\omega k_4}^{\omega^2 k_2} \frac{\ln(1+r_1(s)r_2(s))}{s-k}ds\right\}, \ \delta_5(\zeta,k) = \exp\left\{\frac{1}{2\pi i}\int_{\omega^2 k_2}^{\omega} \frac{\ln f(\omega^2 s)}{s-k}ds\right\},$$
$$\delta_3(\zeta,k) = \exp\left\{\frac{1}{2\pi i}\int_{\omega k_4}^{\omega^2 k_2} \frac{\ln f(s)}{s-k}ds\right\}, \tag{2.9}$$

where the paths follow the unit circle in the counterclockwise direction, the principal branch is used for the logarithms, and

$$f(k) := 1 + r_1(k)r_2(k) + r_1(\tfrac{1}{\omega^2 k})r_2(\tfrac{1}{\omega^2 k}), \qquad k \in \partial\mathbb{D}. \tag{2.10}$$

By [5, Theorem 2.3 and Lemma 2.13], r_1, r_2, f are well-defined on $\partial\mathbb{D}\setminus\mathcal{Q}$ and the arguments of all logarithms appearing in the above definitions of $\{\delta_j(\zeta,k)\}_{j=1}^5$ are >0. Define

$$\mathcal{P}(k) = \prod_{k_0\in \mathsf{Z}\cap D_{\mathrm{reg}}^R} \frac{(k-k_0)(k-k_0^{-1})(k-\omega^2\bar{k}_0)(k-\omega\bar{k}_0^{-1})}{(k-\bar{k}_0)(k-\bar{k}_0^{-1})(k-\omega k_0)(k-\omega^2 k_0^{-1})} \cdot \prod_{k_0\in\mathsf{Z}\cap(1,\infty)} \frac{(k-\omega^2 k_0)(k-\omega k_0^{-1})}{(k-\omega k_0)(k-\omega^2 k_0^{-1})},$$
$$\mathcal{D}_1(k) = \frac{\delta_1(\omega k)\delta_1(\frac{1}{\omega^2 k})^2}{\delta_1(\omega^2 k)^2\delta_1(\frac{1}{\omega k})\delta_1(\frac{1}{k})} \frac{\delta_2(\omega^2 k)\delta_2(\frac{1}{k})^2}{\delta_2(\omega k)^2\delta_2(\frac{1}{\omega^2 k})\delta_2(\frac{1}{\omega k})} \frac{\delta_3(\omega k)\delta_3(\omega^2 k)\delta_3(\frac{1}{\omega k})^2}{\delta_3(\frac{1}{k})\delta_3(\frac{1}{\omega^2 k})}$$
$$\times \frac{\delta_4(\omega^2 k)^2\delta_4(\frac{1}{k})\delta_4(\frac{1}{\omega k})}{\delta_4(k)\delta_4(\omega k)\delta_4(\frac{1}{\omega^2 k})^2} \frac{\delta_5(\omega k)^2\delta_5(\frac{1}{\omega k})\delta_5(\frac{1}{\omega^2 k})}{\delta_5(k)\delta_5(\frac{1}{k})^2\delta_5(\omega^2 k)},$$
$$\mathcal{D}_2(k) = \frac{\delta_1(\omega k)^2\delta_1(\frac{1}{\omega^2 k})\delta_1(\frac{1}{k})}{\delta_1(\omega^2 k)\delta_1(\frac{1}{\omega k})^2\delta_1(k)} \frac{\delta_2(\frac{1}{k})\delta_2(\frac{1}{\omega k})}{\delta_2(\omega k)\delta_2(\frac{1}{\omega^2 k})^2\delta_2(\omega^2 k)} \frac{\delta_3(\omega^2 k)^2\delta_3(\frac{1}{\omega k})\delta_3(\frac{1}{\omega^2 k})}{\delta_3(\frac{1}{k})^2\delta_3(\omega k)}$$
$$\times \frac{\delta_4(\omega^2 k)\delta_4(\frac{1}{\omega k})^2}{\delta_4(\omega k)^2\delta_4(\frac{1}{\omega^2 k})\delta_4(\frac{1}{k})} \frac{\delta_5(\omega k)\delta_5(\frac{1}{\omega^2 k})^2\delta_5(\omega^2 k)}{\delta_5(\frac{1}{k})\delta_5(\frac{1}{\omega k})}. \tag{2.11}$$

Define $\chi_j(\zeta,k)$, $j=1,2,3$, and $\tilde{\chi}_j(\zeta,k)$, $j=2,3,4,5$, by

$$\chi_1(\zeta,k) = -\frac{1}{2\pi i}\int_i^{\omega k_4} \ln_s(k-s)\, d\ln(1+r_1(s)r_2(s)),$$
$$\chi_2(\zeta,k) = \frac{1}{2\pi i}\int_{\omega k_4}^{\omega^2 k_2} \ln_s(k-s)\, d\ln(1+r_1(s)r_2(s)),$$
$$\chi_3(\zeta,k) = \frac{1}{2\pi i}\int_{\omega k_4}^{\omega^2 k_2} \ln_s(k-s)\, d\ln(f(s)),$$
$$\tilde{\chi}_2(\zeta,k) = \frac{1}{2\pi i}\int_{\omega k_4}^{\omega^2 k_2} \tilde{\ln}_s(k-s)\, d\ln(1+r_1(s)r_2(s)),$$
$$\tilde{\chi}_3(\zeta,k) = \frac{1}{2\pi i}\int_{\omega k_4}^{\omega^2 k_2} \tilde{\ln}_s(k-s)\, d\ln(f(s)),$$
$$\tilde{\chi}_4(\zeta,k) = \frac{1}{2\pi i}\fint_{\omega^2 k_2}^{\omega} \tilde{\ln}_s(k-s)\, d\ln(f(s))$$
$$:= \frac{1}{2\pi i}\lim_{\epsilon\to 0_+}\left(\int_{\omega^2 k_2}^{e^{i(\frac{2\pi}{3}-\epsilon)}} \tilde{\ln}_s(k-s)\, d\ln(f(s)) - \tilde{\ln}_\omega(k-\omega)\ln(f(e^{i(\frac{2\pi}{3}-\epsilon)}))\right),$$
$$\tilde{\chi}_5(\zeta,k) = \frac{1}{2\pi i}\fint_{\omega^2 k_2}^{\omega} \tilde{\ln}_s(k-s)\, d\ln(f(\omega^2 s))$$
$$:= \frac{1}{2\pi i}\lim_{\epsilon\to 0_+}\left(\int_{\omega^2 k_2}^{e^{i(\frac{2\pi}{3}-\epsilon)}} \tilde{\ln}_s(k-s)\, d\ln(f(\omega^2 s)) - \tilde{\ln}_\omega(k-\omega)\ln(f(e^{-i\epsilon}))\right), \tag{2.12}$$

where the paths follow $\partial\mathbb{D}$ in the counterclockwise direction. For $s \in \{e^{i\theta} \,|\, \theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]\}$, $k \mapsto \ln_s(k-s) = \ln|k-s| + i\arg_s(k-s)$ has a cut along $\{e^{i\theta} \,|\, \theta \in [\frac{\pi}{2}, \arg s]\} \cup (i, i\infty)$ and the branch is such that $\arg_s(1) = 2\pi$. Also, for $s \in \{e^{i\theta} \,|\, \theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]\}$, $k \mapsto \tilde{\ln}_s(k-s) := \ln|k-s| + i\tilde{\arg}_s(k-s)$ has a cut along $\{e^{i\theta} \,|\, \theta \in [\arg s, \pi]\} \cup (-\infty, 0)$, and satisfies $\tilde{\arg}_s(1) = 0$. Regularized integrals are needed in the definitions of $\tilde{\chi}_4, \tilde{\chi}_5$ because $f(1) = f(\omega) = 0$ (see [5, Lemma 2.13 (i)]). Define $\{\nu_j\}_{j=1}^4$, $\hat{\nu}_1$, and $\hat{\nu}_2$ by

$$\begin{aligned}
&\nu_1(k) = -\frac{1}{2\pi}\ln(1 + r_1(\omega k) r_2(\omega k)), \qquad \nu_2(k) = -\frac{1}{2\pi}\ln(1 + r_1(\omega^2 k) r_2(\omega^2 k)), \\
&\nu_3(k) = -\frac{1}{2\pi}\ln(f(\omega k)), \qquad \nu_4(k) = -\frac{1}{2\pi}\ln(f(\omega^2 k)), \\
&\hat{\nu}_1(k) := \nu_3(k) - \nu_1(k), \qquad \hat{\nu}_2(k) = \nu_2(k) + \nu_3(k) - \nu_4(k).
\end{aligned}$$

The following functions $d_{1,0}$ and $d_{2,0}$ appear in our final result:

$$\begin{aligned}
d_{1,0}(\zeta, t) &:= e^{-\chi_1(\zeta,\omega k_4) - \tilde{\chi}_2(\zeta,\omega k_4) + 2\tilde{\chi}_3(\zeta,\omega k_4)} \\
&\quad \times e^{i(\nu_2(k_2) - 2\nu_4(k_2))\tilde{\ln}_{\omega^2 k_2}(\omega k_4 - \omega^2 k_2)} t^{-i\hat{\nu}_1(k_4)} z_{1,\star}^{-2i\hat{\nu}_1(k_4)} \mathcal{D}_1(\omega k_4), && (2.13) \\
d_{2,0}(\zeta, t) &:= e^{-2\chi_2(\zeta,\omega^2 k_2) + \chi_3(\zeta,\omega^2 k_2) - \tilde{\chi}_4(\zeta,\omega^2 k_2) + 2\tilde{\chi}_5(\zeta,\omega^2 k_2)} \\
&\quad \times e^{i(\nu_3(k_4) - 2\nu_1(k_4))\ln_{\omega k_4}(\omega^2 k_2 - \omega k_4)} t^{-i\hat{\nu}_2(k_2)} z_{2,\star}^{-2i\hat{\nu}_2(k_2)} \mathcal{D}_2(\omega^2 k_2), && (2.14)
\end{aligned}$$

where $z_{1,\star}$, $z_{2,\star}$ are given by

$$\begin{aligned}
z_{1,\star} = z_{1,\star}(\zeta) &:= \sqrt{2} e^{\frac{\pi i}{4}} \sqrt{\omega \frac{4 - 3k_4\zeta - k_4^3\zeta}{4k_4^4}}, \qquad -i\omega k_4 z_{1,\star} > 0, \\
z_{2,\star} = z_{2,\star}(\zeta) &:= \sqrt{2} e^{\frac{\pi i}{4}} \sqrt{-\omega^2 \frac{4 - 3k_2\zeta - k_2^3\zeta}{4k_2^4}}, \qquad -i\omega^2 k_2 z_{2,\star} > 0,
\end{aligned}$$

and the branches for the complex powers $z_{j,\star}^{\alpha} = e^{\alpha \ln z_{j,\star}}$ are fixed by

$$\begin{aligned}
\ln z_{1,\star} &= \ln|z_{1,\star}| + i\arg z_{1,\star} = \ln|z_{1,\star}| + i\big(\tfrac{\pi}{2} - \arg(\omega k_4)\big), \\
\ln z_{2,\star} &= \ln|z_{2,\star}| + i\arg z_{2,\star} = \ln|z_{2,\star}| + i\big(\tfrac{\pi}{2} - \arg(\omega^2 k_2)\big),
\end{aligned}$$

with $\arg(\omega k_4) \in (\frac{\pi}{2}, \frac{2\pi}{3})$ and $\arg(\omega^2 k_2) \in (\frac{\pi}{2}, \frac{2\pi}{3})$. Define also

$$q_3 = |\tilde{r}(\tfrac{1}{k_4})|^{\frac{1}{2}} r_1(\tfrac{1}{k_4}), \quad q_2 = \tilde{r}(\omega^2 k_2)^{\frac{1}{2}} r_1(\omega^2 k_2), \quad q_5 = |\tilde{r}(\omega k_2)|^{\frac{1}{2}} r_1(\omega k_2), \quad q_6 = |\tilde{r}(\tfrac{1}{k_2})|^{\frac{1}{2}} r_1(\tfrac{1}{k_2}),$$

where

$$\tilde{r}(k) := \frac{\omega^2 - k^2}{1 - \omega^2 k^2}, \qquad k \in \mathbb{C} \setminus \{\omega^2, -\omega^2\}. \tag{2.15}$$

We now state our main result. The Gamma function $\Gamma(k)$ appears in the statement, as well as the square roots of $\hat{\nu}_1(k_4)$ and $\hat{\nu}_2(k_2)$. These square roots are well-defined and ≥ 0 thanks to the inequalities $\hat{\nu}_1(k_4) \geq 0$ and $\hat{\nu}_2(k_2) \geq 0$, which follow from [5, Lemma 2.13] and the fact that $\arg k_2 \in (-\frac{3\pi}{4}, -\frac{2\pi}{3})$ and $\arg k_4 \in (-\frac{\pi}{6}, 0)$.

Theorem 2.1 (Asymptotics in Sector IV)**.** *Let $u_0, u_1 \in \mathcal{S}(\mathbb{R})$ be real-valued and such that $\int_{\mathbb{R}} u_1 dx = 0$. Let $v_0(x) = \int_{-\infty}^{x} u_1(x')dx'$ and suppose Assumptions (i)–(iii) are fulfilled. Let $\mathcal{I}$ be a fixed compact subset of $(\frac{1}{\sqrt{3}}, 1)$. Then the global solution $u(x,t)$ of the initial value problem for (1.1) with initial data u_0, u_1 enjoys the following asymptotics as $t \to \infty$:*

$$u(x,t) = \frac{A_1(\zeta)}{\sqrt{t}}\cos\alpha_1(\zeta,t) + \frac{A_2(\zeta)}{\sqrt{t}}\cos\alpha_2(\zeta,t) + O\bigg(\frac{\ln t}{t}\bigg), \tag{2.16}$$

uniformly for $\zeta := \frac{x}{t} \in \mathcal{I}$*, where*

$$A_1(\zeta) := \frac{4\sqrt{3}\sqrt{\hat{\nu}_1(k_4)}\mathrm{Im}\,k_4}{-i\omega k_4 z_{1,\star}|\tilde{r}(\frac{1}{k_4})|^{\frac{1}{2}}}\sin(\arg(\omega k_4)),$$

$$A_2(\zeta) := \frac{-4\sqrt{3}\sqrt{\hat{\nu}_2(k_2)}|\tilde{r}(\frac{1}{k_2})|^{\frac{1}{2}}\,\mathrm{Im}\,k_2}{-i\omega^2 k_2 z_{2,\star}}\sin(\arg(\omega^2 k_2)),$$

$$\alpha_1(\zeta,t) := \frac{3\pi}{4} + \arg q_3 + \arg\Gamma(i\hat{\nu}_1(k_4)) + \arg d_{1,0} + \arg\tfrac{\mathcal{P}(\omega k_4)}{\mathcal{P}(\omega^2 k_4)} - t\,\mathrm{Im}\,\Phi_{31}(\zeta,\omega k_4),$$

$$\alpha_2(\zeta,t) := \frac{3\pi}{4} - \arg(q_6 - q_2 q_5) + \arg\Gamma(i\hat{\nu}_2(k_2)) + \arg d_{2,0} + \arg\tfrac{\mathcal{P}(\omega^2 k_2)}{\mathcal{P}(\omega k_2)} - t\,\mathrm{Im}\,\Phi_{32}(\zeta,\omega^2 k_2).$$

The effect of the solitons on the asymptotic behavior (2.16) of $u(x,t)$ is encoded in the phase shifts $\arg\frac{\mathcal{P}(\omega k_4)}{\mathcal{P}(\omega^2 k_4)}$ and $\arg\frac{\mathcal{P}(\omega^2 k_2)}{\mathcal{P}(\omega k_2)}$ that appear in the definitions of α_1 and α_2, respectively. If there are no solitons (i.e., if the set Z is empty), then these phase shifts are absent, and the formula (2.16) reduces to the formula for Sector IV given in [5]. Indeed, the only place where the set Z appears in the above definitions is in the definition (2.11) of $\mathcal{P}(k)$, and if Z is empty, then $\mathcal{P}(k)$ is identically equal to 1. Thus, these phase shifts describe the leading order soliton-radiation interaction in Sector IV. We observe that the products in the definition of $\mathcal{P}$ only run over the zeros in Z with positive real part; this corresponds to the fact that only right-moving solitons generate a phase shift in Sector IV.

3. The RH problem for n

We will prove Theorem 2.1 by analyzing a row-vector RH problem, whose solution is denoted by n. In what follows, we recall this RH problem from [7]. Let $\theta_{ij}(x,t,k) = t\,\Phi_{ij}(\zeta,k)$. For $j = 1,\dots,6$, we write $\Gamma_j = \Gamma_{j'} \cup \Gamma_{j''}$, where $\Gamma_{j'} = \Gamma_j \setminus \mathbb{D}$ and $\Gamma_{j''} := \Gamma_j \setminus \Gamma_{j'}$ with Γ_j as in Figure 2. The jump matrix $v(x,t,k)$ is defined for $k \in \Gamma$ by

$$v_{1'} = \begin{pmatrix} 1 & -r_1(k)e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ v_{1''} = \begin{pmatrix} 1 & 0 & 0 \\ r_1(\frac{1}{k})e^{\theta_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ v_{2'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -r_2(\frac{1}{\omega k})e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{2''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_2(\omega k)e^{\theta_{32}} & 1 \end{pmatrix},\ v_{3'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -r_1(\omega^2 k)e^{\theta_{31}} & 0 & 1 \end{pmatrix},\ v_{3''} = \begin{pmatrix} 1 & 0 & r_1(\frac{1}{\omega^2 k})e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{4'} = \begin{pmatrix} 1 & -r_2(\frac{1}{k})e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ v_{4''} = \begin{pmatrix} 1 & 0 & 0 \\ r_2(k)e^{\theta_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ v_{5'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -r_1(\omega k)e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{5''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r_1(\frac{1}{\omega k})e^{\theta_{32}} & 1 \end{pmatrix},\ v_{6'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -r_2(\frac{1}{\omega^2 k})e^{\theta_{31}} & 0 & 1 \end{pmatrix},\ v_{6''} = \begin{pmatrix} 1 & 0 & r_2(\omega^2 k)e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_7 = \begin{pmatrix} 1 & -r_1(k)e^{-\theta_{21}} & r_2(\omega^2 k)e^{-\theta_{31}} \\ -r_2(k)e^{\theta_{21}} & 1 + r_1(k)r_2(k) & \big(r_2(\frac{1}{\omega k}) - r_2(k)r_2(\omega^2 k)\big)e^{-\theta_{32}} \\ r_1(\omega^2 k)e^{\theta_{31}} & \big(r_1(\frac{1}{\omega k}) - r_1(k)r_1(\omega^2 k)\big)e^{\theta_{32}} & f(\omega^2 k) \end{pmatrix},$$

$$v_8 = \begin{pmatrix} f(k) & r_1(k)e^{-\theta_{21}} & \big(r_1(\frac{1}{\omega^2 k}) - r_1(k)r_1(\omega k)\big)e^{-\theta_{31}} \\ r_2(k)e^{\theta_{21}} & 1 & -r_1(\omega k)e^{-\theta_{32}} \\ \big(r_2(\frac{1}{\omega^2 k}) - r_2(\omega k)r_2(k)\big)e^{\theta_{31}} & -r_2(\omega k)e^{\theta_{32}} & 1 + r_1(\omega k)r_2(\omega k) \end{pmatrix},$$

$$v_9 = \begin{pmatrix} 1 + r_1(\omega^2 k) r_2(\omega^2 k) & \big(r_2(\frac{1}{k}) - r_2(\omega k) r_2(\omega^2 k)\big) e^{-\theta_{21}} & -r_2(\omega^2 k) e^{-\theta_{31}} \\ \big(r_1(\frac{1}{k}) - r_1(\omega k) r_1(\omega^2 k)\big) e^{\theta_{21}} & f(\omega k) & r_1(\omega k) e^{-\theta_{32}} \\ -r_1(\omega^2 k) e^{\theta_{31}} & r_2(\omega k) e^{\theta_{32}} & 1 \end{pmatrix}, \tag{3.1}$$

where $v_j, v_{j'}, v_{j''}$ are the restrictions of v to Γ_j, $\Gamma_{j'}$, and $\Gamma_{j''}$, respectively. Let $\Gamma_\star = \{i\kappa_j\}_{j=1}^6 \cup \{0\}$ be the set of intersection points of Γ. Let Z be as in Assumption (i) and let

$$\hat{\mathsf{Z}} := \mathsf{Z} \cup \mathsf{Z}^* \cup \omega\mathsf{Z} \cup \omega\mathsf{Z}^* \cup \omega^2\mathsf{Z} \cup \omega^2\mathsf{Z}^* \cup \mathsf{Z}^{-1} \cup \mathsf{Z}^{-*} \cup \omega\mathsf{Z}^{-1} \cup \omega\mathsf{Z}^{-*} \cup \omega^2\mathsf{Z}^{-1} \cup \omega^2\mathsf{Z}^{-*},$$

with $\mathsf{Z}^{-1} = \{k_0^{-1} \,|\, k_0 \in \mathsf{Z}\}$, $\mathsf{Z}^* = \{\bar{k}_0 \,|\, k_0 \in \mathsf{Z}\}$, and $\mathsf{Z}^{-*} = \{\bar{k}_0^{-1} \,|\, k_0 \in \mathsf{Z}\}$.

RH problem 3.1 (RH problem for n)**.** *Find a 1×3-row-vector valued function $n(x,t,k)$ with the following properties:*

(a) *$n(x,t,\cdot) : \mathbb{C} \setminus (\Gamma \cup \hat{\mathsf{Z}}) \to \mathbb{C}^{1\times 3}$ is analytic.*

(b) *The limits of $n(x,t,k)$ as k approaches $\Gamma \setminus \Gamma_\star$ from the left and right exist, are continuous on $\Gamma \setminus \Gamma_\star$, and are denoted by n_+ and n_-, respectively. Furthermore, they are related by*

$$n_+(x,t,k) = n_-(x,t,k) v(x,t,k) \qquad \text{for} \quad k \in \Gamma \setminus \Gamma_\star. \tag{3.2}$$

(c) *$n(x,t,k) = O(1)$ as $k \to k_\star \in \Gamma_\star$.*

(d) *For $k \in \mathbb{C} \setminus (\Gamma \cup \hat{\mathsf{Z}})$, n obeys the symmetries*

$$n(x,t,k) = n(x,t,\omega k)\mathcal{A}^{-1} = n(x,t,k^{-1})\mathcal{B}, \tag{3.3}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the matrices defined by

$$\mathcal{A} := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{B} := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.4}$$

(e) *$n(x,t,k) = (1,1,1) + O(k^{-1})$ as $k \to \infty$.*

(f) *At each point of $\hat{\mathsf{Z}}$, one entry of n has (at most) a simple pole while two entries are analytic. The following residue conditions hold at the points in $\hat{\mathsf{Z}}$: for each $k_0 \in \mathsf{Z} \setminus \mathbb{R}$,*

$$\begin{aligned}
&\operatorname*{Res}_{k=k_0} n_3(k) = c_{k_0} e^{-\theta_{31}(k_0)} n_1(k_0), && \operatorname*{Res}_{k=\omega k_0} n_2(k) = \omega c_{k_0} e^{-\theta_{31}(k_0)} n_3(\omega k_0), \\
&\operatorname*{Res}_{k=\omega^2 k_0} n_1(k) = \omega^2 c_{k_0} e^{-\theta_{31}(k_0)} n_2(\omega^2 k_0), && \operatorname*{Res}_{k=k_0^{-1}} n_3(k) = -k_0^{-2} c_{k_0} e^{-\theta_{31}(k_0)} n_2(k_0^{-1}), \\
&\operatorname*{Res}_{k=\omega^2 k_0^{-1}} n_1(k) = -\frac{\omega^2}{k_0^2} c_{k_0} e^{-\theta_{31}(k_0)} n_3(\omega^2 k_0^{-1}), && \operatorname*{Res}_{k=\omega k_0^{-1}} n_2(k) = -\frac{\omega}{k_0^2} c_{k_0} e^{-\theta_{31}(k_0)} n_1(\omega k_0^{-1}), \\
&\operatorname*{Res}_{k=\bar{k}_0} n_2(k) = d_{k_0} e^{\theta_{32}(\bar{k}_0)} n_3(\bar{k}_0), && \operatorname*{Res}_{k=\omega \bar{k}_0} n_1(k) = \omega d_{k_0} e^{\theta_{32}(\bar{k}_0)} n_2(\omega \bar{k}_0), \\
&\operatorname*{Res}_{k=\omega^2 \bar{k}_0} n_3(k) = \omega^2 d_{k_0} e^{\theta_{32}(\bar{k}_0)} n_1(\omega^2 \bar{k}_0), && \operatorname*{Res}_{k=\bar{k}_0^{-1}} n_1(k) = -\bar{k}_0^{-2} d_{k_0} e^{\theta_{32}(\bar{k}_0)} n_3(\bar{k}_0^{-1}), \\
&\operatorname*{Res}_{k=\omega^2 \bar{k}_0^{-1}} n_2(k) = -\frac{\omega^2}{\bar{k}_0^2} d_{k_0} e^{\theta_{32}(\bar{k}_0)} n_1(\omega^2 \bar{k}_0^{-1}), && \operatorname*{Res}_{k=\omega \bar{k}_0^{-1}} n_3(k) = -\frac{\omega}{\bar{k}_0^2} d_{k_0} e^{\theta_{32}(\bar{k}_0)} n_2(\omega \bar{k}_0^{-1}),
\end{aligned} \tag{3.5}$$

and, for each $k_0 \in \mathsf{Z} \cap \mathbb{R}$,

$$\begin{aligned}
&\operatorname*{Res}_{k=k_0} n_2(k) = c_{k_0} e^{-\theta_{21}(k_0)} n_1(k_0), && \operatorname*{Res}_{k=\omega k_0} n_1(k) = \omega c_{k_0} e^{-\theta_{21}(k_0)} n_3(\omega k_0), \\
&\operatorname*{Res}_{k=\omega^2 k_0} n_3(k) = \omega^2 c_{k_0} e^{-\theta_{21}(k_0)} n_2(\omega^2 k_0), && \operatorname*{Res}_{k=k_0^{-1}} n_1(k) = -k_0^{-2} c_{k_0} e^{-\theta_{21}(k_0)} n_2(k_0^{-1}),
\end{aligned}$$

$$\operatorname*{Res}_{k=\omega^2 k_0^{-1}} n_2(k) = -\tfrac{\omega^2}{k_0^2} c_{k_0} e^{-\theta_{21}(k_0)} n_3(\omega^2 k_0^{-1}), \qquad \operatorname*{Res}_{k=\omega k_0^{-1}} n_3(k) = -\tfrac{\omega}{k_0^2} c_{k_0} e^{-\theta_{21}(k_0)} n_1(\omega k_0^{-1}), \tag{3.6}$$

where the (x,t)-dependence of n and of θ_{ij} has been omitted for conciseness and the complex constants d_{k_0} are defined by

$$d_{k_0} := \frac{\bar{k}_0^2 - 1}{\omega^2(\omega^2 - \bar{k}_0^2)} \bar{c}_{k_0}, \qquad k_0 \in \mathsf{Z} \setminus \mathbb{R}. \tag{3.7}$$

If $k_0 \in \mathsf{Z}$ is such that $c_{k_0} = 0$, then the pole of n at k_0 is removable so k_0 can be removed from Z without affecting RH problem 3.1. In the rest of this paper, we will therefore assume that $c_{k_0} \neq 0$ for all $k_0 \in \mathsf{Z}$.

4. Brief overview of the proof

Under the assumptions of Theorem 2.1, RH problem 3.1 has a unique solution $n(x,t,k)$ for each $(x,t) \in \mathbb{R} \times [0,\infty)$, the function

$$n_3^{(1)}(x,t) := \lim_{k\to\infty} k(n_3(x,t,k) - 1)$$

is well-defined and smooth for $(x,t) \in \mathbb{R} \times [0,\infty)$, and

$$u(x,t) := -i\sqrt{3}\frac{\partial}{\partial x} n_3^{(1)}(x,t) \tag{4.1}$$

is a Schwartz class solution of (1.1) on $\mathbb{R} \times [0,\infty)$ with initial data u_0, u_1 [7]. By using Deift–Zhou steepest descent arguments, we will derive an asymptotic formula for $n_3^{(1)}$. Substituting this formula into (4.1), we will obtain the formula of Theorem 2.1.

The steepest descent analysis of RH problem 3.1 involves the saddle points of the three phase functions Φ_{21}, Φ_{31}, and Φ_{32} whose signature tables are displayed in Figure 3. The saddle points $\{k_j\}_{j=1}^4$ of Φ_{21} are given in (2.8). Using the relations

$$\Phi_{31}(\zeta,k) = -\Phi_{21}(\zeta,\omega^2 k), \qquad \Phi_{32}(\zeta,k) = \Phi_{21}(\zeta,\omega k), \tag{4.2}$$

we see that $\{\omega k_j\}_{j=1}^4$ are the saddle points of Φ_{31} and that $\{\omega^2 k_j\}_{j=1}^4$ are the saddle points of Φ_{32}. It turns out that the main contributions to the long-time asymptotics of $u(x,t)$ in Sector IV are generated by a global parametrix Δ^{-1} and from twelve local parametrices near the saddle points $\{k_j, \omega k_j, \omega^2 k_j\}_{j=1}^4$.

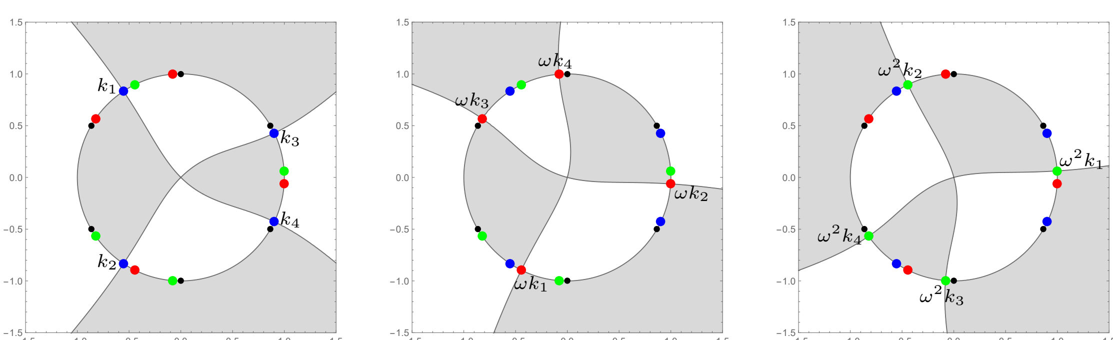

Figure 3. *From left to right: The signature tables for Φ_{21}, Φ_{31}, and Φ_{32} in Sector* IV. *In all images, the shaded regions correspond to $\{k \,|\, \operatorname{Re}\Phi_{ij} > 0\}$ and the white regions to $\{k \,|\, \operatorname{Re}\Phi_{ij} < 0\}$. The points k_1, k_2, k_3, k_4 are represented in blue, $\omega k_1, \omega k_2, \omega k_3, \omega k_4$ in red, and $\omega^2 k_1, \omega^2 k_2, \omega^2 k_3, \omega^2 k_4$ in green. The smaller black dots are the points $i\kappa_j$, $j = 1, \dots, 6$.*

Our proof uses a series of transformations $n \to n^{(1)} \to n^{(2)} \to n^{(3)} \to n^{(4)} \to \hat{n}$ to bring RH problem 3.1 to a small-norm RH problem. These transformations are invertible, implying that the RH problems satisfied by $\{n^{(j)}\}_1^4$ and $\hat{n}$ are equivalent to RH problem 3.1. The jump contours for the RH problems for $\{n^{(j)}\}_1^4$ and $\hat{n}$ will be denoted by $\{\Gamma^{(j)}\}_1^4$ and $\hat{\Gamma}$ and the associated jump matrices by $\{v^{(j)}\}_1^4$ and $\hat{v}$, respectively.

The jump matrix v in (3.1) obeys the following symmetries in accordance with (3.3):

$$v(x,t,k) = \mathcal{A}v(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}v(x,t,k^{-1})^{-1}\mathcal{B}, \qquad k \in \Gamma. \tag{4.3}$$

We will preserve the symmetries (3.3) and (4.3) at each step, so that, for $j = 1,2,3,4$,

$$v^{(j)}(x,t,k) = \mathcal{A}v^{(j)}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}v^{(j)}(x,t,k^{-1})^{-1}\mathcal{B}, \qquad k \in \Gamma^{(j)}, \tag{4.4}$$

$$n^{(j)}(x,t,k) = n^{(j)}(x,t,\omega k)\mathcal{A}^{-1} = n^{(j)}(x,t,k^{-1})\mathcal{B}, \qquad k \in \mathbb{C}\setminus\Gamma^{(j)}, \tag{4.5}$$

and similarly for $\hat{n}$ and $\hat{v}$. These symmetries imply that we only need to construct the local parametrices near ωk_4 and $\omega^2 k_2$, and that we only need to define the transformations $n \to n^{(1)}$, $n^{(j)} \to n^{(j+1)}$, and $n^{(4)} \to \hat{n}$ in the sector $\mathsf{S} := \{k \in \mathbb{C} | \arg k \in [\frac{\pi}{3}, \frac{2\pi}{3}]\}$. In $\mathbb{C}\setminus\mathsf{S}$, we will define the local parametrices and the transformations using the $\mathcal{A}$- and $\mathcal{B}$-symmetries.

We know from [5, Theorem 2.3] that the functions r_1 and r_2 satisfy the following properties: $r_1 \in C^\infty(\hat{\Gamma}_1)$, $r_2 \in C^\infty(\hat{\Gamma}_4 \setminus \{\omega^2, -\omega^2\})$, $r_1(\kappa_j) \neq 0$ for $j = 1,\dots,6$, $r_2(k)$ has simple poles at $k = \pm\omega^2$, and simple zeros at $k = \pm\omega$, and r_1, r_2 are rapidly decreasing as $|k| \to \infty$. Moreover,

$$r_1(\tfrac{1}{\omega k}) + r_2(\omega k) + r_1(\omega^2 k) r_2(\tfrac{1}{k}) = 0, \qquad k \in \partial\mathbb{D}\setminus\{\omega, -\omega\}, \tag{4.6}$$

$$r_2(k) = \tilde{r}(k)\overline{r_1(\bar{k}^{-1})}, \qquad \tilde{r}(k) := \frac{\omega^2 - k^2}{1 - \omega^2 k^2}, \qquad k \in \hat{\Gamma}_4 \setminus \{0, \omega^2, -\omega^2\}, \tag{4.7}$$

$$r_1(1) = r_1(-1) = 1, \qquad r_2(1) = r_2(-1) = -1. \tag{4.8}$$

These properties will be used repeatedly throughout the proof.

The residue conditions (3.5) involve the exponentials $e^{-\theta_{31}(x,t,k_0)}$ and $e^{\theta_{32}(x,t,\bar{k}_0)}$. We see from Figure 3 that these exponentials are small as $t \to \infty$ for $k_0 \in \mathsf{Z}\setminus\mathbb{R}$ if $\operatorname{Re} k_0 < 0$ (recall that $\mathsf{Z}\setminus\mathbb{R} \subset D_{\text{reg}}$). Similarly, the residue conditions (3.6) involve the exponential $e^{-\theta_{21}(x,t,k_0)}$, and we see from Figure 3 that this exponential is small as $t \to \infty$ for $k_0 \in \mathsf{Z}\cap\mathbb{R}$ if $\operatorname{Re} k_0 < 0$ (recall that $\mathsf{Z}\cap\mathbb{R} \subset (-1,0)\cup(1,\infty)$). This suggests that points k_0 in Z with $\operatorname{Re} k_0 < 0$ (i.e., points corresponding to left-moving solitons) will not contribute to the asymptotics in Sector IV at any finite order, and this is indeed what we will find. On the other hand, for $k_0 \in \mathsf{Z}$ with $\operatorname{Re} k_0 > 0$, the above exponentials are large as $t \to \infty$. By including an appropriate factor in the global parametrix, we can flip the signs of the exponents in these exponentials, thereby making them small for large t. The factor included in the global parametrix gives rise to the phase shifts $\arg \frac{\mathcal{P}(\omega k_4)}{\mathcal{P}(\omega^2 k_4)}$ and $\arg \frac{\mathcal{P}(\omega^2 k_2)}{\mathcal{P}(\omega k_2)}$ generated by right-moving solitons seen in Theorem 2.1.

5. The $n \to n^{(1)}$ transformation

We will open lenses around $\partial\mathbb{D}\setminus\mathcal{Q}$. Define

$$\hat{r}_j(k) := \frac{r_j(k)}{1 + r_1(k) r_2(k)}, \qquad j = 1,2,$$

and let

$$r_{j,1}(k) := r_j(k), \quad r_{j,2}(k) := \hat{r}_j(k), \quad r_{j,3}(k) := \frac{r_j(k) - r_j(\frac{1}{\omega k}) r_j(\frac{1}{\omega^2 k})}{1 + r_1(\frac{1}{\omega k}) r_2(\frac{1}{\omega k})},$$

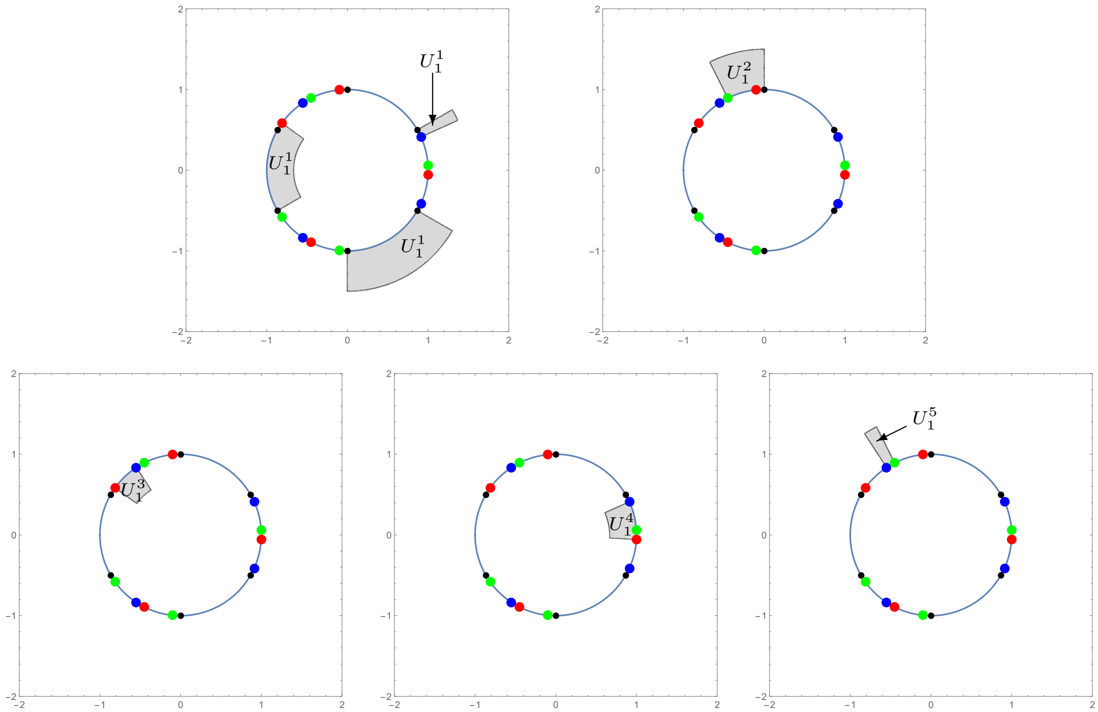

Figure 4. *The open subsets $\{U_1^\ell\}_{\ell=1}^5$ of the complex k-plane.*

$$r_{j,4}(k) := \frac{r_j(k)}{f(k)}, \quad r_{j,5}(k) = \frac{r_j(k) - r_j(\frac{1}{\omega k})r_j(\frac{1}{\omega^2 k})}{f(k)}.$$

The domains of definition of these spectral functions are (in general) only subsets of Γ. To open lenses around $\partial\mathbb{D}$, it is therefore necessary to decompose them into an analytic part and a small remainder. Let $M > 1$ and define open sets $\{U_1^\ell = U_1^\ell(\zeta, M)\}_{\ell=1}^5$ by (see Figure 4)

$$\begin{aligned}
U_1^1 &= \{k | \mathrm{Re}\,\Phi_{21} > 0\} \cap \big(\{k | \arg k \in (\arg(\omega k_3), \pi] \cup [-\pi, -\tfrac{5\pi}{6}),\ M^{-1} < |k| < 1\} \\
&\qquad \cup \{k | \arg k \in (-\tfrac{\pi}{2}, -\tfrac{\pi}{6}) \cup (\arg k_3, \tfrac{\pi}{6}),\ 1 < |k| < M\}\big), \\
U_1^2 &= \{k | \mathrm{Re}\,\Phi_{21} > 0\} \cap \{k | \arg k \in (\tfrac{\pi}{2}, \arg(\omega^2 k_2)),\ 1 < |k| < M\}, \\
U_1^3 &= \{k | \mathrm{Re}\,\Phi_{21} > 0\} \cap \{k | \arg k \in (\arg k_1, \arg(\omega k_3)),\ M^{-1} < |k| < 1\}, \\
U_1^4 &= \{k | \mathrm{Re}\,\Phi_{21} > 0\} \cap \{k | \arg k \in (\arg(\omega k_2), \arg k_3),\ M^{-1} < |k| < 1\}, \\
U_1^5 &= \{k | \mathrm{Re}\,\Phi_{21} > 0\} \cap \{k | \arg k \in (\arg(\omega^2 k_2), \arg k_1),\ 1 < |k| < M)\}.
\end{aligned}$$

It follows from [5, Lemma 2.13] that $f(k) = 0$ if and only if $k \in \{\pm 1, \pm\omega\}$, and that $1 + r_1(k)r_2(k) > 0$ for all $k \in \partial\mathbb{D}$ with $\arg k \in (\pi/3, \pi) \cup (-2\pi/3, 0)$. Therefore, for each $\ell \in \{1, 2, 3\}$, $r_{1,\ell}$ is well-defined for $k \in \partial U_1^\ell \cap \partial\mathbb{D}$; $r_{1,4}$ is well-defined for $k \in (\partial U_1^4 \cap \partial\mathbb{D}) \setminus \{1\}$; and $r_{1,5}$ is well-defined for $k \in (\partial U_1^5 \cap \partial\mathbb{D}) \setminus \{\omega\}$. Let $n_1 \geq 0$ denote the smallest integer such that $(k-1)^{n_1} r_{1,4}(k) = O(1)$ as $k \to 1$, $k \in \partial\mathbb{D}$, and let $n_\omega \geq 0$ denote the smallest integer such that $(k-\omega)^{n_\omega} r_{1,5}(k) = O(1)$ as $k \to \omega$, $k \in \partial\mathbb{D}$.

We now find decompositions of $\{r_{1,\ell}\}_{\ell=1}^5$; the decompositions of $\{r_{2,\ell}\}_{\ell=1}^5$ will then be obtained using the symmetry (4.7). Let $N \geq 1$ be an integer.

Lemma 5.1 (Decomposition lemma). *There exist $M>1$ and decompositions*

$$r_{1,\ell}(k) = r_{1,\ell,a}(x,t,k) + r_{1,\ell,r}(x,t,k), \qquad k \in \partial U_1^\ell \cap \partial\mathbb{D},\ \ell = 1,\dots,5, \tag{5.1}$$

such that $\{r_{1,\ell,a}, r_{1,\ell,r}\}_{\ell=1}^5$ satisfy the following properties:

(a) *For each $\zeta \in \mathcal{I}$, $t\geq 1$, and $\ell \in \{1,\dots,5\}$, $r_{1,\ell,a}(x,t,k)$ is defined and continuous for $k \in \bar{U}_1^\ell$ and analytic for $k \in U_1^\ell$.*

(b) *For each $\zeta \in \mathcal{I}$, $t \geq 1$, and $\ell \in \{1,\dots,5\}$, $r_{1,\ell,a}$ obeys*

$$\Big|r_{1,\ell,a}(x,t,k) - \sum_{j=0}^{N} \frac{r_{1,\ell}^{(j)}(k_\star)}{j!}(k-k_\star)^j\Big| \leq C|k-k_\star|^{N+1} e^{\frac{t}{4}|\operatorname{Re}\Phi_{21}(\zeta,k)|},\ k \in \bar{U}_1^\ell,\ k_\star \in \mathcal{R}_\ell,$$

where $\mathcal{R}_1 = \{\omega k_3, e^{\pm 5\pi i/6}, -1, -i, -\omega, e^{\pm\frac{\pi i}{6}}, k_1\}$, $\mathcal{R}_2 = \{i, \omega k_4, \omega^2 k_2\}$, $\mathcal{R}_3 = \{k_1, \omega k_3\}$, $\mathcal{R}_4 = \{\omega k_2, \omega^2 k_1, k_3\}$, $\mathcal{R}_5 = \{\omega^2 k_2, k_1\}$, and the constant C is independent of ζ, t, k. Moreover, for $\zeta \in \mathcal{I}$ and $t \geq 1$,

$$\Big|r_{1,4,a}(x,t,k) - \sum_{j=0}^{N+n_1} \frac{[(\cdot-1)^{n_1} r_{1,4}(\cdot)]^{(j)}(1)}{j!}(k-1)^{j-n_1}\Big| \leq C|k-1|^{N+1} e^{\frac{t}{4}|\operatorname{Re}\Phi_{21}(\zeta,k)|},$$

$$\Big|r_{1,5,a}(x,t,k) - \sum_{j=0}^{N+n_\omega} \frac{[(\cdot-\omega)^{n_\omega} r_{1,5}(\cdot)]^{(j)}(\omega)}{j!}(k-\omega)^{j-n_\omega}\Big| \leq C|k-\omega|^{N+1} e^{\frac{t}{4}|\operatorname{Re}\Phi_{21}(\zeta,k)|},$$

and these estimates hold for $k \in \bar{U}_1^4$ and $k \in \bar{U}_1^5$, respectively.

(c) *For each $1 \leq p \leq \infty$ and $\ell \in \{1,\dots,5\}$, the L^p-norm of $r_{1,\ell,r}(x,t,\cdot)$ on $\partial\bar{U}_1^\ell \cap \partial\mathbb{D}$ is $O(t^{-N})$ uniformly for $\zeta \in \mathcal{I}$ as $t \to \infty$.*

Proof. The function $\theta \mapsto -i\Phi_{21}(\zeta, e^{i\theta}) = (\zeta - \cos\theta)\sin\theta$ is real-valued and bijective on each connected component of $\partial U_1^\ell \cap \partial\mathbb{D}$, $\ell = 1,\dots,5$. Thus the statement can be proved using similar arguments as in [15]. Since these arguments are rather standard by now, we omit details. $\square$

In what follows, shrinking $M>1$ if necessary, we assume that $|k_0| > M$ for all $k_0 \in \mathsf{Z} \cap \big(D_{\mathrm{reg}}^R \cup (1,\infty)\big)$ and that $|k_0| < M^{-1}$ for all $k_0 \in \mathsf{Z} \cap \big(D_{\mathrm{reg}}^L \cup (-1,0)\big)$.

It is easy to check (using the second equation in (4.7)) that $\tilde{r}(k) \in \mathbb{R}$ for $k \in \partial\mathbb{D} \setminus \{-\omega^2, \omega^2\}$ and that $\tilde{r}(k) = \tilde{r}(\frac{1}{\omega k})\tilde{r}(\frac{1}{\omega^2 k})$. Using also the first equation in (4.7), we infer that $r_{2,\ell}(k) = \tilde{r}(k)\overline{r_{1,\ell}(\bar{k}^{-1})}$, $k \in \partial\mathbb{D} \cap U_1^\ell$, $\ell = 1,\dots,5$. Hence, for each $\ell \in \{1,\dots,5\}$ we let $U_2^\ell := \{k | \bar{k}^{-1} \in U_1^\ell\}$ and define a decomposition $r_{2,\ell} = r_{2,\ell,a} + r_{2,\ell,r}$ by

$$r_{2,\ell,a}(k) := \tilde{r}(k)\overline{r_{1,\ell,a}(\bar{k}^{-1})}, \quad k \in U_2^\ell, \qquad r_{2,\ell,r}(k) := \tilde{r}(k)\overline{r_{1,\ell,r}(\bar{k}^{-1})}, \quad k \in \partial U_2^\ell \cap \partial\mathbb{D}.$$

We now define the first transformation $n \to n^{(1)}$, which consists in opening lenses around $\partial\mathbb{D} \setminus \mathcal{Q}$. As mentioned in Section 4, we only need to define it explicitly in the sector S. We introduce a new contour $\Gamma^{(0)}$, which coincides as a set with $\Gamma \cap \mathsf{S}$, but is oriented and labeled differently, see Figure 5. We let $\Gamma_j^{(0)}$ denote the subcontour of $\Gamma^{(0)}$ labeled by j in Figure 5. For example,

$$\Gamma_8^{(0)} := \{e^{i\theta} \,|\, \theta \in (\arg(\omega^2 k_2), \tfrac{2\pi}{3})\}, \qquad \Gamma_{9_r}^{(0)} := \{e^{i\theta} \,|\, \theta \in (\tfrac{\pi}{3}, \tfrac{\pi}{2})\}.$$

We will open lenses differently on the four parts of $\Gamma^{(0)}$. On $\Gamma_{9_r}^{(0)}$, we will use the factorization

$$v_{9_r} = v_{9_L}^{(1)} v_{9_r}^{(1)} v_{9_R}^{(1)},$$

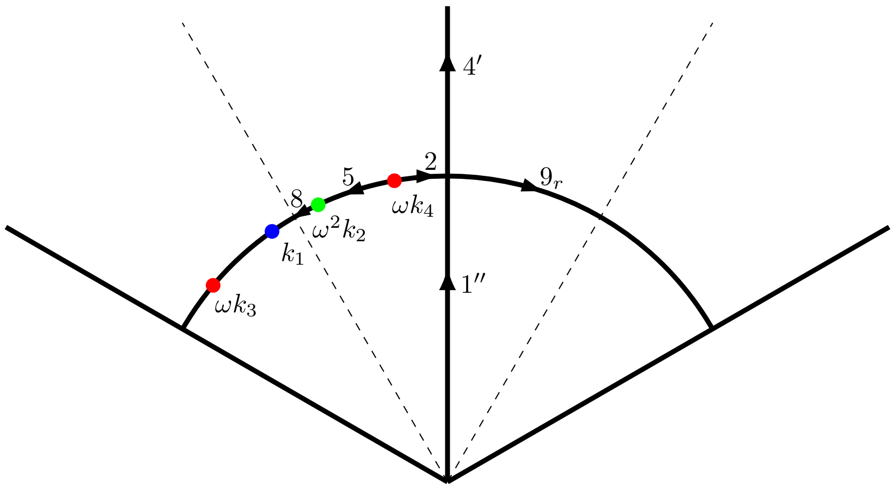

Figure 5. *The contour $\Gamma^{(0)}$ (solid), the boundary of S (dashed), and the saddle points ωk_4 (red), $\omega^2 k_2$ (green), k_1 (blue), and ωk_3 (red).*

where

$$v_{9_L}^{(1)} = \begin{pmatrix} 1 & 0 & -r_{2,a}(\omega^2 k)e^{-\theta_{31}} \\ r_{1,a}(\frac{1}{k})e^{\theta_{21}} & 1 & r_{1,a}(\omega k)e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \quad v_{9_R}^{(1)} = \begin{pmatrix} 1 & r_{2,a}(\frac{1}{k})e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ -r_{1,a}(\omega^2 k)e^{\theta_{31}} & r_{2,a}(\omega k)e^{\theta_{32}} & 1 \end{pmatrix},$$

$$v_{9_r}^{(1)} = I + \begin{pmatrix} r_{1,r}(\omega^2 k) r_{2,r}(\omega^2 k) & g_2(\omega k)e^{-\theta_{21}} & -r_{2,r}(\omega^2 k)e^{-\theta_{31}} \\ g_1(\omega k)e^{\theta_{21}} & g(\omega k) & h_1(\omega k)e^{-\theta_{32}} \\ -r_{1,r}(\omega^2 k)e^{\theta_{31}} & h_2(\omega k)e^{\theta_{32}} & 0 \end{pmatrix} \tag{5.2}$$

and

$$\begin{aligned} h_1(k) &= r_{1,r}(k) + r_{1,a}(\tfrac{1}{\omega^2 k}) r_{2,r}(\omega k), \qquad h_2(k) = r_{2,r}(k) + r_{2,a}(\tfrac{1}{\omega^2 k}) r_{1,r}(\omega k), \\ g_1(k) &= r_{1,r}(\tfrac{1}{\omega^2 k}) - r_{1,r}(\omega k)\big(r_{1,r}(k) + r_{1,a}(\tfrac{1}{\omega^2 k}) r_{2,r}(\omega k)\big), \\ g_2(k) &= r_{2,r}(\tfrac{1}{\omega^2 k}) - r_{2,r}(\omega k)\big(r_{2,r}(k) + r_{2,a}(\tfrac{1}{\omega^2 k}) r_{1,r}(\omega k)\big), \\ g(k) &= r_{1,r}(k)\big(r_{1,r}(\omega k) r_{2,a}(\tfrac{1}{\omega^2 k}) + r_{2,r}(k)\big) \\ &\quad + r_{1,a}(\tfrac{1}{\omega^2 k}) r_{2,r}(\omega k)\big(r_{1,r}(\omega k) r_{2,a}(\tfrac{1}{\omega^2 k}) + r_{2,r}(k)\big) + r_{1,r}(\tfrac{1}{\omega^2 k}) r_{2,r}(\tfrac{1}{\omega^2 k}). \end{aligned}$$

On $\Gamma_2^{(0)}$, we will use the factorization

$$v_7^{-1} = v_3^{(1)} v_2^{(1)} v_1^{(1)}, \qquad v_2^{(1)} = \begin{pmatrix} 1 + r_1(k) r_2(k) & 0 & 0 \\ 0 & \frac{1}{1+r_1(k)r_2(k)} & 0 \\ 0 & 0 & 1 \end{pmatrix} + v_{2,r}^{(1)}, \tag{5.3}$$

$$v_3^{(1)} = \begin{pmatrix} 1 & 0 & r_{1,a}(\frac{1}{\omega^2 k})e^{-\theta_{31}} \\ \hat{r}_{2,a}(k)e^{\theta_{21}} & 1 & -r_{2,a}(\frac{1}{\omega k})e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \quad v_1^{(1)} = \begin{pmatrix} 1 & \hat{r}_{1,a}(k)e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ r_{2,a}(\frac{1}{\omega^2 k})e^{\theta_{31}} & -r_{1,a}(\frac{1}{\omega k})e^{\theta_{32}} & 1 \end{pmatrix}.$$

On $\Gamma_5^{(0)}$, we will use the factorization

$$v_7 = v_4^{(1)} v_5^{(1)} v_6^{(1)}, \qquad v_5^{(1)} = \begin{pmatrix} \frac{1}{f(k)} & 0 & 0 \\ 0 & 1 + r_1(k) r_2(k) & 0 \\ 0 & 0 & \frac{f(k)}{1+r_1(k)r_2(k)} \end{pmatrix} + v_{5,r}^{(1)},$$

$$v_4^{(1)} = \begin{pmatrix} 1 & a_{12,a}e^{-\theta_{21}} & a_{13,a}e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & a_{32,a}e^{\theta_{32}} & 1 \end{pmatrix}, \qquad v_6^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ a_{21,a}e^{\theta_{21}} & 1 & a_{23,a}e^{-\theta_{32}} \\ a_{31,a}e^{\theta_{31}} & 0 & 1 \end{pmatrix}, \tag{5.4}$$

where

$$a_{12} = -\hat{r}_1(k), \qquad a_{13} = -\frac{r_1(\frac{1}{\omega^2 k})}{f(k)}, \qquad a_{23} = \frac{r_2(\frac{1}{\omega k}) - r_2(k)r_2(\omega^2 k)}{1 + r_1(k)r_2(k)},$$
$$a_{21} = -\hat{r}_2(k), \qquad a_{31} = -\frac{r_2(\frac{1}{\omega^2 k})}{f(k)}, \qquad a_{32} = \frac{r_1(\frac{1}{\omega k}) - r_1(k)r_1(\omega^2 k)}{1 + r_1(k)r_2(k)}.$$

and $a_{ij,a}, a_{ij,r}$ are the analytic continuation and the remainder of a_{ij} from Lemma 5.1, e.g., $a_{13,a} = -r_{1,4,a}(\frac{1}{\omega^2 k})$ and $a_{23,a} = r_{2,3,a}(\frac{1}{\omega k})$.

Finally, on $\Gamma_8^{(0)}$, we will use the factorization

$$v_7 = v_7^{(1)} v_8^{(1)} v_9^{(1)}, \qquad v_8^{(1)} = \begin{pmatrix} \frac{1}{f(k)} & 0 & 0 \\ 0 & \frac{f(k)}{f(\omega^2 k)} & 0 \\ 0 & 0 & f(\omega^2 k) \end{pmatrix} + v_{8,r}^{(1)},$$
$$v_7^{(1)} = \begin{pmatrix} 1 & b_{12,a}e^{-\theta_{21}} & b_{13,a}e^{-\theta_{31}} \\ 0 & 1 & b_{23,a}e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \qquad v_9^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ b_{21,a}e^{\theta_{21}} & 1 & 0 \\ b_{31,a}e^{\theta_{31}} & b_{32,a}e^{\theta_{32}} & 1 \end{pmatrix}, \tag{5.5}$$

where

$$b_{12} = -\frac{r_1(k) - r_1(\frac{1}{\omega k})r_1(\frac{1}{\omega^2 k})}{f(k)}, \qquad b_{13} = \frac{r_2(\omega^2 k)}{f(\omega^2 k)}, \qquad b_{23} = \frac{r_2(\frac{1}{\omega k}) - r_2(k)r_2(\omega^2 k)}{f(\omega^2 k)},$$
$$b_{21} = -\frac{r_2(k) - r_2(\frac{1}{\omega k})r_2(\frac{1}{\omega^2 k})}{f(k)}, \qquad b_{31} = \frac{r_1(\omega^2 k)}{f(\omega^2 k)}, \qquad b_{32} = \frac{r_1(\frac{1}{\omega k}) - r_1(k)r_1(\omega^2 k)}{f(\omega^2 k)},$$

and $b_{ij,a}, b_{ij,r}$ are the analytic continuation and the remainder of b_{ij} from Lemma 5.1, e.g., $b_{12,a} = -r_{1,5,a}(k)$ and $b_{23,a} = r_{2,5,a}(\frac{1}{\omega k})$. We do not write down the long expressions for $v_{2,r}^{(1)}, v_{5,r}^{(1)}$ and $v_{8,r}^{(1)}$ which are similar to the expression for $v_{2,r}^{(1)}$ given in [5, Section 8]; the only property of these matrices that is important for us is that

$$\|v_{j,r}^{(1)}\|_{(L^1\cap L^\infty)(\Gamma_j^{(0)})} = O(t^{-1}), \qquad \text{as } t \to \infty,\ j = 2, 5, 8,$$

as a consequence of Lemma 5.1.

Let $\Gamma^{(1)}$ be the contour shown in Figure 6, and define $G^{(1)}$ for $k \in \mathsf{S}$ by

$$G^{(1)} = \begin{cases} v_{9_L}^{(1)}, & k \text{ on the } - \text{ side of } \Gamma_{9_r}^{(1)}, \\ (v_{9_R}^{(1)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_{9_r}^{(1)}, \\ v_3^{(1)}, & k \text{ on the } - \text{ side of } \Gamma_2^{(1)}, \\ (v_1^{(1)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_2^{(1)}, \end{cases} \quad G^{(1)} = \begin{cases} v_4^{(1)}, & k \text{ on the } - \text{ side of } \Gamma_5^{(1)}, \\ (v_6^{(1)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_5^{(1)}, \\ v_7^{(1)}, & k \text{ on the } - \text{ side of } \Gamma_8^{(1)}, \\ (v_9^{(1)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_8^{(1)}, \\ I, & \text{otherwise}. \end{cases} \tag{5.6}$$

We now appeal to the $\mathcal{A}$- and $\mathcal{B}$-symmetries to extend the definition of $G^{(1)}$ to the whole complex plane:

$$G^{(1)}(x,t,k) = \mathcal{A}G^{(1)}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}G^{(1)}(x,t,k^{-1})\mathcal{B}, \qquad k \in \mathbb{C} \setminus \Gamma^{(1)}. \tag{5.7}$$

Define the sectionally meromorphic function $n^{(1)}$ by

$$n^{(1)}(x,t,k) = n(x,t,k)G^{(1)}(x,t,k), \qquad k \in \mathbb{C} \setminus (\Gamma^{(1)} \cup \hat{\mathsf{Z}}). \tag{5.8}$$

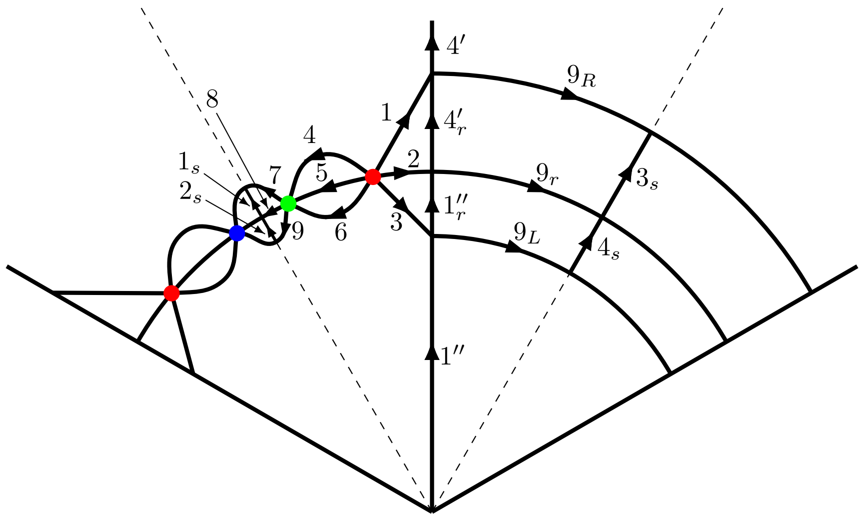

Figure 6. *The contour $\Gamma^{(1)}$ (solid) and the boundary of S (dashed). From right to left, the dots are ωk_4 (red), $\omega^2 k_2$ (green), k_1 (blue), and ωk_3 (red).*

The functions $G^{(1)}$ and $n^{(1)}$ are analytic on $\mathbb{C} \setminus \Gamma^{(1)}$. Indeed, let us look for example at the region on the $-$ side of $\Gamma_8^{(1)}$ that is inside the lens; in this region $G^{(1)} = v_7^{(1)}$ is given in terms of $r_{1,5,a}(k)$, $r_{2,5,a}(\frac{1}{\omega k})$, and $r_{2,4,a}(\omega^2 k)$, and it follows from Lemma 5.1 and the definitions of U_1^5, U_2^5, and U_2^4 that these functions are all analytic in this region.

The following lemma is a direct consequence of Figure 3. In this lemma, the disks $D_\epsilon(\omega^j)$ have been excluded; this is because $v_7^{(1)}, v_8^{(1)}, v_9^{(1)}$ (and hence also $n^{(1)}(x,t,k)$) are singular at $k = \omega$, due to the fact that $f(\omega) = f(1) = 0$.

Lemma 5.2. *For any $\epsilon > 0$, $G^{(1)}(x,t,k)$ and $G^{(1)}(x,t,k)^{-1}$ are uniformly bounded for $k \in \mathbb{C} \setminus (\Gamma^{(1)} \cup \cup_{j=0}^2 D_\epsilon(\omega^j))$, $t \geq 1$, and $\zeta \in \mathcal{I}$. Furthermore, $G^{(1)}(x,t,k) = I$ for all large enough $|k|$.*

Let $\Gamma_\star^{(1)}$ be the set of self-intersection points of $\Gamma^{(1)}$. The jump matrix $v^{(1)}$ for $n^{(1)}$ is given on $\Gamma^{(1)} \setminus \Gamma_\star^{(1)}$ as follows: For $j = 9_L, 9_r, 9_R, 1, \dots, 9$, the matrix $v_j^{(1)}$ is given by (5.2)–(5.5); for $j = 1'', 1_r'', 4_r', 4'$ it is given by

$$v_{1''}^{(1)} = v_{1''}, \quad v_{1_r''}^{(1)} = (v_{9_L}^{(1)})^{-1} v_{1''} v_3^{(1)}, \quad v_{4_r'}^{(1)} = v_{9_R}^{(1)} v_{4'} (v_1^{(1)})^{-1}, \quad v_{4'}^{(1)} = v_{4'}; \tag{5.9}$$

for $j = 1_s$ it is given by

$$\begin{aligned} v_{1_s}^{(1)} &= G_-^{(1)}(k)^{-1} G_+^{(1)}(k) = v_7^{(1)}(k)^{-1} \mathcal{A}\mathcal{B} v_9^{(1)}(\tfrac{1}{\omega k})^{-1} \mathcal{B}\mathcal{A}^{-1} \\ &= \begin{pmatrix} 1 & -(b_{12,a}(k) + b_{32,a}(\frac{1}{\omega k}))e^{-t\Phi_{21}} & (v_{1_s}^{(1)})_{13} \\ 0 & 1 & -(b_{21,a}(\frac{1}{\omega k}) + b_{23,a}(k))e^{-t\Phi_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{5.10}$$

$$(v_{1_s}^{(1)})_{13} := \big(b_{12,a}(k)(b_{21,a}(\tfrac{1}{\omega k}) + b_{23,a}(k)) + b_{21,a}(\tfrac{1}{\omega k}) b_{32,a}(\tfrac{1}{\omega k}) - b_{13,a}(k) - b_{31,a}(\tfrac{1}{\omega k})\big) e^{-t\Phi_{31}},$$

and similar expressions are valid for $j = 2_s, 3_s, 4_s$. Using the symmetries in (4.4) we extend $v^{(1)}$ to all of $\Gamma^{(1)} \setminus \Gamma_\star^{(1)}$.

Since $n^{(1)} = n$ near all the points of $\hat{\mathsf{Z}}$, the residue conditions (3.5)–(3.6) are not affected by the transformation $n \to n^{(1)}$. The function $n^{(1)}$ therefore satisfies the following RH problem for $j = 1$.

RH problem 5.3 (RH problem for $n^{(j)}$)**.** *Find $n^{(j)}(x,t,k)$ with the following properties:*

(a) $n^{(j)}(x,t,\cdot):\mathbb{C}\setminus(\Gamma^{(j)}\cup\hat{\mathsf{Z}})\to\mathbb{C}^{1\times 3}$ *is analytic.*

(b) *On $\Gamma^{(j)}\setminus\Gamma_\star^{(j)}$, the boundary values of $n^{(j)}$ exist, are continuous, and satisfy $n_+^{(j)}=n_-^{(j)}v^{(j)}$.*

(c) $n^{(j)}(x,t,k)=O(1)$ *as* $k\to k_\star\in\Gamma_\star^{(j)}\setminus\{1,\omega,\omega^2\}$.

(d) $n^{(j)}$ *obeys the symmetries* $n^{(j)}(x,t,k)=n^{(j)}(x,t,\omega k)\mathcal{A}^{-1}=n^{(j)}(x,t,k^{-1})\mathcal{B}$ *for* $k\in\mathbb{C}\setminus\Gamma^{(j)}$.

(e) $n^{(j)}(x,t,k)=(1,1,1)+O(k^{-1})$ *as* $k\to\infty$.

(f) *At each point of $\hat{\mathsf{Z}}$, one entry of $n^{(j)}$ has (at most) a simple pole while two entries are analytic. Moreover, $n^{(j)}$ satisfies the residue conditions (3.5)–(3.6) with n replaced by $n^{(j)}$.*

Note that $n^{(1)}$ is singular at the points $1,\omega,\omega^2$ in general. The behavior of $n^{(1)}$ near these singularities is as follows: As $k\to\omega$ from the right side of $\Gamma_8^{(1)}$ (i.e. $k\in D_4$ and $|k|>1$),

$$n^{(1)}(x,t,k)=\begin{pmatrix}O(1) & O(1) & O(1)\end{pmatrix}v_7^{(1)}=\begin{pmatrix}O(1) & O(1) & O(1)\end{pmatrix}\begin{pmatrix}1 & O(b_{12,a}) & O(\frac{1}{f(\omega^2k)})\\ 0 & 1 & O(\frac{k-\omega}{f(\omega^2k)})\\ 0 & 0 & 1\end{pmatrix},$$

and as $k\to\omega$ from the left side of $\Gamma_8^{(1)}$ (i.e. $k\in D_1$ and $|k|<1$),

$$\begin{aligned}n^{(1)}(x,t,k)&=\big(O(1)\,O(1)\,O(1)\big)\,(v_9^{(1)})^{-1}\\ &=\big(O(1)\,O(1)\,O(1)\big)\begin{pmatrix}1 & 0 & 0\\ O(\frac{k-\omega}{f(k)}) & 1 & 0\\ O(b_{21,a}b_{32,a}-b_{31,a}) & O(b_{32,a}) & 1\end{pmatrix}.\end{aligned}$$

We know from [5, Lemma 2.13] that $f(\pm1)=f(\pm\omega)=0$. Since $f\geq 0$ on $\partial\mathbb{D}$, these zeros must be at least double zeros. One could try to make the solution of this singular RH problem unique by specifying the pole structure at the points $1,\omega,\omega^2$ in further detail, but since the singularities at $1,\omega,\omega^2$ will be removed by the global parametrix below, we do not need to do this.

6. The $n^{(1)}\to n^{(2)}$ transformation

The jump matrix $v^{(1)}$ admits the following factorizations on $\Gamma^{(1)}\cap\mathsf{S}$ (the subscripts u and d indicate that the corresponding matrix will be deformed up or down):

$$\begin{aligned}&v_1^{(1)}=v_1^{(2)}v_{1u}, && v_3^{(1)}=v_{3d}v_3^{(2)}, && v_4^{(1)}=v_{4u}v_4^{(2)}=v_{7u}v_7^{(2)},\\ &v_6^{(1)}=v_6^{(2)}v_{6d}=v_8^{(2)}v_{8d}, && v_7^{(1)}=v_{9u}v_9^{(2)}, && v_9^{(1)}=v_{11}^{(2)}v_{11d},\end{aligned}$$

where

$$v_1^{(2)}:=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ r_{2,a}(\frac{1}{\omega^2k})e^{\theta_{31}} & 0 & 1\end{pmatrix},\qquad v_{1u}:=\begin{pmatrix}1 & \hat{r}_{1,a}e^{-\theta_{21}} & 0\\ 0 & 1 & 0\\ 0 & (-r_{1,a}(\frac{1}{\omega k})-\hat{r}_{1,a}(k)r_{2,a}(\frac{1}{\omega^2k}))e^{\theta_{32}} & 1\end{pmatrix},$$

$$v_3^{(2)}:=\begin{pmatrix}1 & 0 & r_{1,a}(\frac{1}{\omega^2k})e^{-\theta_{31}}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad v_{3d}:=\begin{pmatrix}1 & 0 & 0\\ \hat{r}_{2,a}e^{\theta_{21}} & 1 & (-r_{2,a}(\frac{1}{\omega k})-r_{1,a}(\frac{1}{\omega^2k})\hat{r}_{2,a}(k))e^{-\theta_{32}}\\ 0 & 0 & 1\end{pmatrix},$$

$$v_4^{(2)}:=\begin{pmatrix}1 & 0 & a_{13,a}e^{-\theta_{31}}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\qquad v_{4u}:=\begin{pmatrix}1 & a_{12,a}e^{-\theta_{21}} & 0\\ 0 & 1 & 0\\ 0 & a_{32,a}e^{\theta_{32}} & 1\end{pmatrix},$$

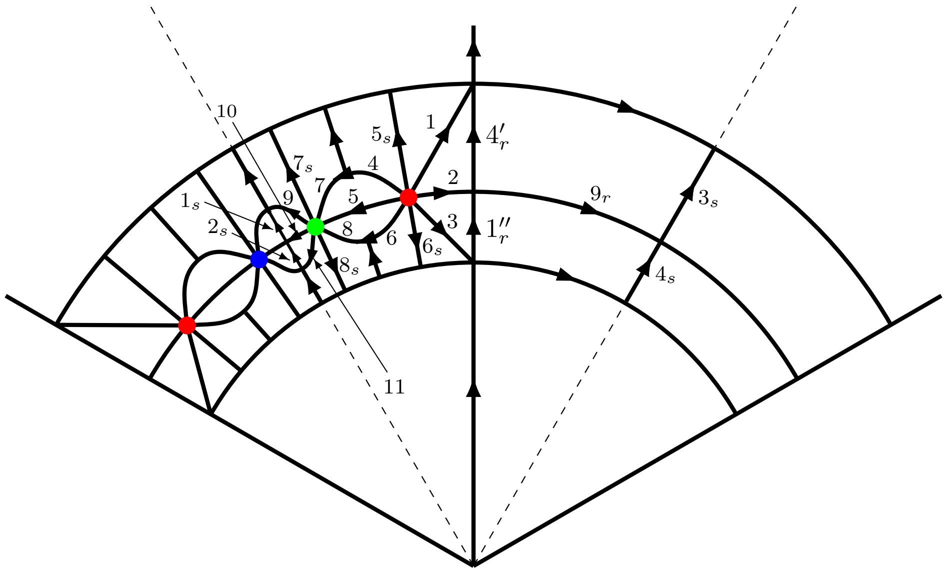

Figure 7. *The contour $\Gamma^{(2)}$ (solid) and the boundary of S (dashed). From right to left, the dots are ωk_4 (red), $\omega^2 k_2$ (green), k_1 (blue), and ωk_3 (red).*

$$v_6^{(2)} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31,a}e^{\theta_{31}} & 0 & 1 \end{pmatrix}, \qquad v_{6d} := \begin{pmatrix} 1 & 0 & 0 \\ a_{21,a}e^{\theta_{21}} & 1 & a_{23,a}e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix},$$
$$v_7^{(2)} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32,a}e^{\theta_{32}} & 1 \end{pmatrix}, \qquad v_{7u} := \begin{pmatrix} 1 & (a_{12,a} - a_{13,a}a_{32,a})e^{-\theta_{21}} & a_{13,a}e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$v_8^{(2)} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a_{23,a}e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \qquad v_{8d} := \begin{pmatrix} 1 & 0 & 0 \\ (a_{21,a} - a_{23,a}a_{31,a})e^{\theta_{21}} & 1 & 0 \\ a_{31,a}e^{\theta_{31}} & 0 & 1 \end{pmatrix},$$
$$v_9^{(2)} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b_{23,a}e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \qquad v_{9u} := \begin{pmatrix} 1 & b_{12,a}e^{-\theta_{21}} & (b_{13,a} - b_{12,a}b_{23,a})e^{-\theta_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$v_{11}^{(2)} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b_{32,a}e^{\theta_{32}} & 1 \end{pmatrix}, \qquad v_{11d} := \begin{pmatrix} 1 & 0 & 0 \\ b_{21,a}e^{\theta_{21}} & 1 & 0 \\ (b_{31,a} - b_{21,a}b_{32,a})e^{\theta_{31}} & 0 & 1 \end{pmatrix}. \tag{6.1}$$

Let $\Gamma^{(2)}$ be the contour shown in Figure 7, and let $\Gamma_j^{(2)}$ be the subcontour of $\Gamma^{(2)}$ labeled by j in Figure 7. We emphasize that $\Gamma_{10}^{(2)} := \{e^{i\theta} \,|\, \theta \in (\arg(\omega^2 k_2), \frac{2\pi}{3})\}$ ends at ω. Define the sectionally meromorphic function $n^{(2)}$ by

$$n^{(2)}(x,t,k) = n^{(1)}(x,t,k)G^{(2)}(x,t,k), \qquad k \in \mathbb{C} \setminus (\Gamma^{(2)} \cup \hat{\mathsf{Z}}), \tag{6.2}$$

where $G^{(2)}$ is defined for $k \in \mathsf{S}$ by

$$G^{(2)}(x,t,k) = \begin{cases} v_{1u}^{-1}, & k \text{ above } \Gamma_1^{(2)}, \\ v_{3d}, & k \text{ below } \Gamma_3^{(2)}, \\ v_{4u}, & k \text{ above } \Gamma_4^{(2)}, \\ v_{6d}^{-1}, & k \text{ below } \Gamma_6^{(2)}, \end{cases} \qquad G^{(2)}(x,t,k) = \begin{cases} v_{7u}, & k \text{ above } \Gamma_7^{(2)}, \\ v_{8d}^{-1}, & k \text{ below } \Gamma_8^{(2)}, \\ v_{9u}, & k \text{ above } \Gamma_9^{(2)}, \\ v_{11d}^{-1}, & k \text{ below } \Gamma_{11}^{(2)}, \\ I, & \text{otherwise}, \end{cases} \tag{6.3}$$

and $G^{(2)}$ is extended to all of $\mathbb{C} \setminus \Gamma^{(2)}$ using the $\mathcal{A}$- and $\mathcal{B}$-symmetries (as in (5.7)). Using Lemma 5.1 and Figure 3, we infer that the following holds.

Lemma 6.1. *For any $\epsilon > 0$, $G^{(2)}(x,t,k)$ and $G^{(2)}(x,t,k)^{-1}$ are uniformly bounded for $k \in \mathbb{C} \setminus (\Gamma^{(2)} \cup \cup_{j=0}^2 D_\epsilon(\omega^j))$, $t > 0$, and $\zeta \in \mathcal{I}$. Furthermore, $G^{(2)}(x,t,k) = I$ whenever $|k|$ is large enough.*

The jumps $v_j^{(2)}$ of $n^{(2)}$ are given for $j = 1, \dots, 11$ by (6.1) and

$$v_2^{(2)} := v_2^{(1)}, \quad v_5^{(2)} := v_5^{(1)}, \quad v_{10}^{(2)} := v_8^{(1)}. \tag{6.4}$$

All other jumps on $\Gamma^{(2)} \cap \mathsf{S}$ are small as $t \to \infty$. Indeed, this follows from the signature tables in Figure 3 together with the following observations. On $\Gamma_{9_r}^{(2)}$, $v^{(2)} - I$ is small because it involves small remainders. On $\Gamma_{4_r'}^{(2)}$ and $\Gamma_{1_r''}^{(2)}$, $v^{(2)} - I$ is small as a consequence of the following lemma which follows in the same way as [5, Lemma 8.5].

Lemma 6.2. *The L^∞-norm of $v^{(2)} - I$ on $\Gamma_{4_r'}^{(2)} \cup \Gamma_{1_r''}^{(2)}$ is $O(t^{-N-1})$ as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}$.*

We finally consider the jumps $v_j^{(2)} = v_j^{(1)}$, $j = 1_s, \dots, 4_s$ and

$$v_{5_s}^{(2)} = v_{1u} v_{4u}, \quad v_{6_s}^{(2)} = v_{6d} v_{3d}, \quad v_{7_s}^{(2)} = v_{7u}^{-1} v_{9u}, \quad v_{8_s}^{(2)} = v_{11d} v_{8d}^{-1}.$$

If the initial data u_0, v_0 have compact support so that all the spectral functions have analytic continuations and we can choose $b_{ij,a} = b_{ij}$, then a straightforward calculation shows that these jumps are absent (i.e., $v_j^{(2)} = I$, $j = 1_s, \dots, 8_s$) as a consequence of the relation (4.6) and the definition (2.10) of $f(k)$. In general, it seems difficult to construct analytic approximations which preserve the nonlinear relation (4.6). Therefore, the jump matrices $v_j^{(2)}$, $j = 1_s, \dots, 8_s$, will generally be nontrivial. However, as the next lemma demonstrates, it is not difficult to choose the analytic approximations so that these jumps are uniformly small for large t.

Lemma 6.3. *It is possible to choose the analytic approximations so that the L^∞-norm of $v_j^{(2)} - I$ on $\Gamma_j^{(2)}$, $j = 1_s, \dots, 8_s$, is $O(t^{-N})$ as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}$.*

Proof. Let us consider $v_{1_s}^{(2)}$; the other jumps are handled similarly. We see from (5.10) that $v_{1_s}^{(2)} = v_{1_s}^{(1)}$ has ones along the diagonal and that each of its off-diagonal entries is suppressed by an exponential of the form $e^{-t|\text{Re}\,\Phi_{ij}|}$ where $|\text{Re}\,\Phi_{ij}(\zeta,k)| \geq c|k - \omega|$ on $\Gamma_{1_s}^{(2)}$. Thus, by Lemma 5.1, $v_{1_s}^{(2)} - I$ is small as $t \to \infty$ everywhere on $\Gamma_{1_s}^{(2)}$ except possibly near ω. On the other hand, we know that the matrix $v_{1_s}^{(2)} - I$ vanishes identically on $\Gamma_{1_s}^{(2)}$ if all the spectral functions have analytic continuations thanks to (4.6). This means that if we choose all the analytic approximations such that they agree with the functions they are approximating to sufficiently high order at each of the sixth roots of unity κ_j (in the sense of (6.5) below), then we can ensure that

$$\|v_{1_s}^{(2)} - I\|_{L^\infty(\Gamma_{1_s}^{(2)})} \le C \sup_{k\in\Gamma_{1_s}^{(2)}} |k-\omega|^N e^{-\frac{c}{2}t|k-\omega|} \le Ct^{-N},$$

uniformly for $\zeta \in \mathcal{I}$, and the lemma follows.

To construct analytic approximations of the above type, we proceed as follows. For definiteness, let us consider b_{13}. The analytic approximation $b_{13,a}$ of b_{13} is needed in a region $\mathcal{U} := \{k \,|\, \arg k \in (\arg(\omega^2 k_2), \frac{2\pi}{3}),\ 1 < |k| < M\}$ to the right of $\Gamma_8^{(1)} = \Gamma_{10}^{(2)}$. Since $f(1) = 0$ and $r_2(1) = -1$, b_{13} has a singularity at ω. Thus, we first choose $m \ge 1$ such that $B(k) := (k-\omega)^m b_{13}(k)$ is regular at ω. Applying the method of [15], we construct a decomposition $B = B_a + B_r$ such that

$$\left|B_a(x,t,k) - \sum_{j=0}^{N_1} \frac{B^{(j)}(\omega)}{j!}(k-\omega)^j\right| \le C|k-\omega|^{N_1} e^{\frac{t}{8}|\operatorname{Re}\Phi_{31}(\zeta,k)|}, \qquad k \in \mathcal{U},\ \ \zeta \in \mathcal{I}.$$

This implies in particular that B_r has a zero of order at least N_1 at $k = \omega$. Then we define $b_{13,a} := (k-\omega)^{-m} B_a$ and $b_{13,r} := (k-\omega)^{-m} B_r$. By choosing N_1 large enough, we can ensure that the L^∞-norm of $b_{13,r}$ is uniformly small for large t and that

$$\left|b_{13,a}(x,t,k) - \sum_{j=-m}^{N} b_j(k-\omega)^j\right| \le C|k-\omega|^N e^{\frac{t}{8}|\operatorname{Re}\Phi_{31}(\zeta,k)|}, \qquad k \in \mathcal{U},\ \ \zeta \in \mathcal{I}, \tag{6.5}$$

where $\sum_{j=-m}^{N} b_j(k-\omega)^j$ denotes the expansion of b_{13} to order N at ω, i.e., the expansion such that $b_{13}(k) - \sum_{j=-m}^{N} b_j(k-\omega)^j = O((k-\omega)^{N+1})$ as $k \to \omega$. This means that $b_{13,a}$ approximates b_{13} to order N at ω. In a similar way, we construct analytic approximations which agree with the functions they are approximating to order N at each κ_j that lies in their domain of definition. ☐

The jumps on $\Gamma^{(2)} \setminus \mathsf{S}$ can be obtained using the symmetries (4.4).

The function $n^{(2)}$ satisfies RH problem 5.3 with $j = 2$. The next transformation $n^{(2)} \to n^{(3)}$ will make $v^{(3)} - I$ small everywhere on a new contour $\Gamma^{(3)}$, except on twelve small crosses centered at the saddle points $\{k_j, \omega k_j, \omega^2 k_j\}_{j=1}^4$. It will also change the residue conditions (3.5)–(3.6). To define this transformation, we first need to construct a global parametrix.

7. Global parametrix

For each $\zeta \in \mathcal{I}$, define the analytic functions

$$\delta_1(\zeta,\cdot) : \mathbb{C} \setminus \Gamma_2^{(2)} \to \mathbb{C}, \quad \delta_2(\zeta,\cdot), \delta_3(\zeta,\cdot) : \mathbb{C} \setminus \Gamma_5^{(2)} \to \mathbb{C}, \quad \delta_4(\zeta,\cdot), \delta_5(\zeta,\cdot) : \mathbb{C} \setminus \Gamma_{10}^{(2)} \to \mathbb{C}$$

by (2.9). The functions $\{\delta_j\}_1^5$ obey the jump relations

$$\begin{aligned}
&\delta_{1,+}(\zeta,k) = \delta_{1,-}(\zeta,k)(1 + r_1(k) r_2(k)), && k \in \Gamma_2^{(2)}, \\
&\delta_{2,+}(\zeta,k) = \delta_{2,-}(\zeta,k)(1 + r_1(k) r_2(k)), && k \in \Gamma_5^{(2)}, \\
&\delta_{3,+}(\zeta,k) = \delta_{3,-}(\zeta,k) f(k), && k \in \Gamma_5^{(2)}, \\
&\delta_{4,+}(\zeta,k) = \delta_{4,-}(\zeta,k) f(k), && k \in \Gamma_{10}^{(2)}, \\
&\delta_{5,+}(\zeta,k) = \delta_{5,-}(\zeta,k) f(\omega^2 k), && k \in \Gamma_{10}^{(2)},
\end{aligned}$$

where the contours $\Gamma_2^{(2)}$, $\Gamma_5^{(2)}$, and $\Gamma_{10}^{(2)}$ are oriented as in Figure 7, and

$$\delta_j(\zeta,k) = 1 + O(k^{-1}), \qquad k \to \infty,\ j = 1,\dots,5. \tag{7.1}$$

Further properties of the functions δ_j are collected in the following lemma.

Lemma 7.1. *The functions $\delta_j(\zeta,k)$, $j=1,\dots,5$, have the following properties:*

(a) *The functions $\{\delta_j(\zeta,k)\}_{j=1}^5$ can be written as*

$$\begin{aligned}
\delta_1(\zeta,k) &= \exp\Big(- i\nu_1 \ln_{\omega k_4}(k-\omega k_4) - \chi_1(\zeta,k)\Big),\\
\delta_2(\zeta,k) &= \exp\Big(- i\nu_1 \ln_{\omega k_4}(k-\omega k_4) + i\nu_2 \ln_{\omega^2 k_2}(k-\omega^2 k_2) - \chi_2(\zeta,k)\Big),\\
\delta_3(\zeta,k) &= \exp\Big(- i\nu_3 \ln_{\omega k_4}(k-\omega k_4) + i\nu_4 \ln_{\omega^2 k_2}(k-\omega^2 k_2) - \chi_3(\zeta,k)\Big),\\
\delta_4(\zeta,k) &= \exp\Big(- i\nu_4 \ln_{\omega^2 k_2}(k-\omega^2 k_2) - \chi_4(\zeta,k)\Big),\\
\delta_5(\zeta,k) &= \exp\Big(- i\nu_5 \ln_{\omega^2 k_2}(k-\omega^2 k_2) - \chi_5(\zeta,k)\Big),
\end{aligned}$$

where for $s \in \{e^{i\theta} : \theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]\}$, $k \mapsto \ln_s(k-s) := \ln|k-s| + i\arg_s(k-s)$ has a cut along $\{e^{i\theta} : \theta \in [\frac{\pi}{2}, \arg s]\} \cup (i, i\infty)$, and satisfies $\arg_s(1) = 2\pi$. The ν_j are defined by

$$\begin{aligned}
&\nu_1 = -\frac{1}{2\pi}\ln(1 + r_1(\omega k_4) r_2(\omega k_4)), \qquad \nu_2 = -\frac{1}{2\pi}\ln(1 + r_1(\omega^2 k_2) r_2(\omega^2 k_2)),\\
&\nu_3 = -\frac{1}{2\pi}\ln(f(\omega k_4)), \qquad \nu_4 = -\frac{1}{2\pi}\ln(f(\omega^2 k_2)), \qquad \nu_5 = -\frac{1}{2\pi}\ln(f(\omega k_2)),
\end{aligned}$$

and the functions χ_j are defined by

$$\begin{aligned}
\chi_1(\zeta,k) &= \frac{1}{2\pi i}\int_{\Gamma_2^{(2)}} \ln_s(k-s)\, d\ln(1 + r_1(s) r_2(s)),\\
\chi_2(\zeta,k) &= \frac{1}{2\pi i}\int_{\Gamma_5^{(2)}} \ln_s(k-s)\, d\ln(1 + r_1(s) r_2(s)),\\
\chi_3(\zeta,k) &= \frac{1}{2\pi i}\int_{\Gamma_5^{(2)}} \ln_s(k-s)\, d\ln(f(s)),\\
\chi_4(\zeta,k) &= \frac{1}{2\pi i}\fint_{\omega^2 k_2}^{\omega} \ln_s(k-s)\, d\ln(f(s))\\
&:= \frac{1}{2\pi i}\lim_{\epsilon \to 0_+}\bigg(\int_{\omega^2 k_2}^{e^{i(\frac{2\pi}{3}-\epsilon)}} \ln_s(k-s)\, d\ln(f(s)) - \ln_\omega(k-\omega)\ln(f(e^{i(\frac{2\pi}{3}-\epsilon)}))\bigg),\\
\chi_5(\zeta,k) &= \frac{1}{2\pi i}\fint_{\omega^2 k_2}^{\omega} \ln_s(k-s)\, d\ln(f(\omega^2 s))\\
&:= \frac{1}{2\pi i}\lim_{\epsilon \to 0_+}\bigg(\int_{\omega^2 k_2}^{e^{i(\frac{2\pi}{3}-\epsilon)}} \ln_s(k-s)\, d\ln(f(\omega^2 s)) - \ln_\omega(k-\omega)\ln(f(e^{-i\epsilon}))\bigg),
\end{aligned}$$

and the integration paths starting at $\omega^2 k_2$ are subsets of $\partial\mathbb{D}$ oriented in the counter-clockwise direction.

(b) *The functions $\{\delta_j(\zeta,k)\}_{j=1}^5$ can be written as*

$$\begin{aligned}
\delta_1(\zeta,k) &= \exp\big(- i\nu_1 \tilde{\ln}_{\omega k_4}(k-\omega k_4) - \tilde{\chi}_1(\zeta,k)\big),\\
\delta_2(\zeta,k) &= \exp\big(- i\nu_1 \tilde{\ln}_{\omega k_4}(k-\omega k_4) + i\nu_2 \tilde{\ln}_{\omega^2 k_2}(k-\omega^2 k_2) - \tilde{\chi}_2(\zeta,k)\big),\\
\delta_3(\zeta,k) &= \exp\big(- i\nu_3 \tilde{\ln}_{\omega k_4}(k-\omega k_4) + i\nu_4 \tilde{\ln}_{\omega^2 k_2}(k-\omega^2 k_2) - \tilde{\chi}_3(\zeta,k)\big),
\end{aligned}$$

$$\delta_4(\zeta,k) = \exp\big(-i\nu_4 \tilde{\ln}_{\omega^2 k_2}(k-\omega^2 k_2) - \tilde{\chi}_4(\zeta,k)\big),$$
$$\delta_5(\zeta,k) = \exp\big(-i\nu_5 \tilde{\ln}_{\omega^2 k_2}(k-\omega^2 k_2) - \tilde{\chi}_5(\zeta,k)\big),$$

where for $s \in \{e^{i\theta} : \theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]\}$, $k \mapsto \tilde{\ln}_s(k-s) := \ln|k-s| + i\tilde{\arg}_s(k-s)$ *has a cut along* $\{e^{i\theta} : \theta \in [\arg s, \pi]\} \cup (-\infty, 0)$, *satisfies* $\tilde{\arg}_s(1) = 0$, *and* $\tilde{\chi}_j(\zeta,k)$ *is defined in the same way as* $\chi_j(\zeta,k)$ *except that* $\ln_s$ *is replaced by* $\tilde{\ln}_s$ *and* $\ln_\omega$ *is replaced by* $\tilde{\ln}_\omega$.

(*c*) *For each* $\zeta \in \mathcal{I}$ *and* $j \in \{1,\dots,5\}$, $\delta_j(\zeta,k)$ *and* $\delta_j(\zeta,k)^{-1}$ *are analytic functions of* k *in their respective domains of definition. Moreover, for any* $\epsilon > 0$,

$$\sup_{\zeta\in\mathcal{I}} \sup_{k\in\mathbb{C}\setminus\Gamma_2^{(2)}} |\delta_1(\zeta,k)^{\pm1}| < \infty, \quad \sup_{\zeta\in\mathcal{I}} \sup_{k\in\mathbb{C}\setminus\Gamma_5^{(2)}} |\delta_2(\zeta,k)^{\pm1}| < \infty, \quad \sup_{\zeta\in\mathcal{I}} \sup_{k\in\mathbb{C}\setminus\Gamma_5^{(2)}} |\delta_3(\zeta,k)^{\pm1}| < \infty,$$
$$\sup_{\zeta\in\mathcal{I}} \sup_{\substack{k\in\mathbb{C}\setminus\Gamma_{10}^{(2)} \\ |k-\omega|>\epsilon}} |\delta_4(\zeta,k)^{\pm1}| < \infty, \quad \sup_{\zeta\in\mathcal{I}} \sup_{\substack{k\in\mathbb{C}\setminus\Gamma_{10}^{(2)} \\ |k-\omega|>\epsilon}} |\delta_5(\zeta,k)^{\pm1}| < \infty. \tag{7.2}$$

(*d*) *As* $k \to \omega k_4$ *along a path which is nontangential to* $\partial\mathbb{D}$, *we have*

$$|\chi_j(\zeta,k) - \chi_j(\zeta,\omega k_4)| \le C|k-\omega k_4|(1+|\ln|k-\omega k_4||), \qquad j=1,2, \tag{7.3}$$

and as $k \to \omega^2 k_2$ *along a path which is nontangential to* $\partial\mathbb{D}$, *we have*

$$|\chi_j(\zeta,k) - \chi_j(\zeta,\omega^2 k_2)| \le C|k-\omega^2 k_2|(1+|\ln|k-\omega^2 k_2||), \qquad j=2,3,4,5, \tag{7.4}$$

where C *is independent of* $\zeta \in \mathcal{I}$.

Proof. All assertions follow from the definitions (2.9) of the functions δ_j. $\square$

We will also need the following result.

Lemma 7.2. *For all* $\zeta \in (\frac{1}{\sqrt{3}}, 1)$, *the following inequalities hold:*

$$\hat{\nu}_1 := \nu_3 - \nu_1 \ge 0, \qquad \hat{\nu}_2 := \nu_2 + \nu_5 - \nu_4 \ge 0.$$

Proof. For $\zeta \in (\frac{1}{\sqrt{3}}, 1)$, we have $-\frac{\pi}{6} < \arg k_4 < 0$ and $-\frac{3\pi}{4} < \arg k_2 < -\frac{2\pi}{3}$. Thus the claim follows from [5, Lemma 2.13]. $\square$

In what follows, we often write $\delta_j(k)$ for $\delta_j(\zeta,k)$ for conciseness. For $k \in \mathbb{C}\setminus(\partial\mathbb{D}\cup\hat{\mathsf{Z}})$, we define

$$\Delta_{33}(\zeta,k) = \frac{\delta_1(\omega k)\delta_1(\frac{1}{\omega^2 k})}{\delta_1(\omega^2 k)\delta_1(\frac{1}{\omega k})} \frac{\delta_2(k)\delta_2(\frac{1}{k})}{\delta_2(\omega k)\delta_2(\frac{1}{\omega^2 k})} \frac{\delta_3(\omega^2 k)\delta_3(\frac{1}{\omega k})}{\delta_3(k)\delta_3(\frac{1}{k})} \frac{\delta_4(\omega^2 k)\delta_4(\frac{1}{\omega k})}{\delta_4(\omega k)\delta_4(\frac{1}{\omega^2 k})} \frac{\delta_5(\omega k)\delta_5(\frac{1}{\omega^2 k})}{\delta_5(k)\delta_5(\frac{1}{k})} \mathcal{P}(k),$$

where $\mathcal{P}$ is given by (2.11). The function Δ_{33} satisfies $\Delta_{33}(\zeta,\frac{1}{k}) = \Delta_{33}(\zeta,k)$, and

$$\Delta_{33,+}(\zeta,k) = \Delta_{33,-}(\zeta,k), \qquad k \in \Gamma_2^{(2)}, \tag{7.5a}$$
$$\Delta_{33,+}(\zeta,k) = \Delta_{33,-}(\zeta,k)\frac{1+r_1(k)r_2(k)}{f(k)}, \qquad k \in \Gamma_5^{(2)}, \tag{7.5b}$$
$$\Delta_{33,+}(\zeta,k) = \Delta_{33,-}(\zeta,k)\frac{1}{f(\omega^2 k)}, \qquad k \in \Gamma_{10}^{(2)}. \tag{7.5c}$$

Define $\Delta_{11}(\zeta,k) = \Delta_{33}(\zeta,\omega k)$ and $\Delta_{22}(\zeta,k) = \Delta_{33}(\zeta,\omega^2 k)$, and define

$$\Delta(\zeta,k) = \begin{pmatrix} \Delta_{11}(\zeta,k) & 0 & 0 \\ 0 & \Delta_{22}(\zeta,k) & 0 \\ 0 & 0 & \Delta_{33}(\zeta,k) \end{pmatrix}, \quad \zeta \in \mathcal{I},\ k \in \mathbb{C}\setminus(\partial\mathbb{D}\cup\hat{\mathsf{Z}}). \tag{7.6}$$

Then Δ obeys the symmetries

$$\Delta(\zeta,k) = \mathcal{A}\Delta(\zeta,\omega k)\mathcal{A}^{-1} = \mathcal{B}\Delta(\zeta,\tfrac{1}{k})\mathcal{B},$$

and satisfies the jump relations

$$\Delta_{11,+}(\zeta,k) = \Delta_{11,-}(\zeta,k)\frac{1}{1+r_1(k)r_2(k)}, \qquad k \in \Gamma_2^{(2)}, \tag{7.7a}$$

$$\Delta_{11,+}(\zeta,k) = \Delta_{11,-}(\zeta,k)f(k), \qquad k \in \Gamma_5^{(2)}, \tag{7.7b}$$

$$\Delta_{11,+}(\zeta,k) = \Delta_{11,-}(\zeta,k)f(k), \qquad k \in \Gamma_{10}^{(2)}, \tag{7.7c}$$

and

$$\Delta_{22,+}(\zeta,k) = \Delta_{22,-}(\zeta,k)(1+r_1(k)r_2(k)), \qquad k \in \Gamma_2^{(2)}, \tag{7.8a}$$

$$\Delta_{22,+}(\zeta,k) = \Delta_{22,-}(\zeta,k)\frac{1}{1+r_1(k)r_2(k)}, \qquad k \in \Gamma_5^{(2)}, \tag{7.8b}$$

$$\Delta_{22,+}(\zeta,k) = \Delta_{22,-}(\zeta,k)\frac{f(\omega^2 k)}{f(k)}, \qquad k \in \Gamma_{10}^{(2)}. \tag{7.8c}$$

The following lemma is a direct consequence of (7.6) and of Lemma 7.1.

Lemma 7.3. *For any $\epsilon > 0$, $\Delta(\zeta,k)$ and $\Delta(\zeta,k)^{-1}$ are uniformly bounded for $k \in \mathbb{C} \setminus (\cup_{j=0}^2 D_\epsilon(\omega^j) \cup \partial\mathbb{D} \cup \cup_{k_0\in\hat{\mathsf{Z}}} D_\epsilon(k_0))$ and $\zeta \in \mathcal{I}$. Furthermore, $\Delta(\zeta,k) = I + O(k^{-1})$ as $k \to \infty$.*

8. The $n^{(2)} \to n^{(3)}$ transformation

Define the sectionally meromorphic function $n^{(3)}$ by

$$n^{(3)}(x,t,k) = n^{(2)}(x,t,k)\Delta(\zeta,k), \qquad k \in \mathbb{C} \setminus (\Gamma^{(3)} \cup \hat{\mathsf{Z}}), \tag{8.1}$$

where $\Gamma^{(3)} = \Gamma^{(2)}$. The jumps for $n^{(3)}$ on $\Gamma^{(3)}$ are given by $v^{(3)} = \Delta_-^{-1} v^{(2)} \Delta_+$.

Lemma 8.1. *The jump matrix $v^{(3)}$ converges to the identity matrix I as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}$ and $k \in \Gamma^{(3)}$ except near the twelve saddle points $\{k_j, \omega k_j, \omega^2 k_j\}_{j=1}^4$, i.e., for $\zeta \in \mathcal{I}$,*

$$\|v^{(3)} - I\|_{(L^1\cap L^\infty)(\Gamma'^{(3)})} \le Ct^{-1}, \quad \text{where} \quad \Gamma'^{(3)} := \Gamma^{(3)} \setminus \cup_{\substack{j=1,\dots,4 \\ l=0,1,2}} D_\epsilon(\omega^l k_j). \tag{8.2}$$

Proof. From the expressions for $v_j^{(1)}$, $j = 2,5,8$, in (5.3)–(5.5) together with (6.4), (7.5), (7.7), (7.8), we deduce that

$$v_j^{(3)} = \Delta_-^{-1} v_j^{(1)} \Delta_+ = I + \Delta_-^{-1} v_{j,r}^{(1)} \Delta_+, \quad j = 2,5; \quad v_{10}^{(3)} = \Delta_-^{-1} v_8^{(1)} \Delta_+ = I + \Delta_-^{-1} v_{8,r}^{(1)} \Delta_+. \tag{8.3}$$

Using that the matrices $\Delta_-^{-1} v_{j,r}^{(1)} \Delta_+$, $j = 2,5,8$, are small as $t \to \infty$, and recalling the symmetries (4.4) and Lemmas 6.2 and 6.3, we infer that $v^{(3)} - I$ tends to 0 as $t \to \infty$, uniformly for $k \in \Gamma'^{(3)}$. □

We next consider the residue conditions at the points in $\hat{\mathsf{Z}}$.

Lemma 8.2 (Residue conditions for $n^{(3)}$)**.** *The function $n^{(3)}$ obeys the following residue conditions at the points in $\hat{\mathsf{Z}}$:*

(a) for each $k_0 \in \mathsf{Z} \setminus \mathbb{R}$ with $\operatorname{Re} k_0 > 0$,

$$\operatorname*{Res}_{k=k_0} n_1^{(3)}(k) = \frac{c_{k_0}^{-1} e^{\theta_{31}(k_0)} n_3^{(3)}(k_0)}{\Delta_{33}'(k_0)(\Delta_{11}^{-1})'(k_0)}, \qquad \operatorname*{Res}_{k=\omega k_0} n_3^{(3)}(k) = \frac{\omega^2 c_{k_0}^{-1} e^{\theta_{31}(k_0)} n_2^{(3)}(\omega k_0)}{\Delta_{22}'(\omega k_0)(\Delta_{33}^{-1})'(\omega k_0)},$$

$$\operatorname*{Res}_{k=\omega^2 k_0} n_2^{(3)}(k) = \frac{\omega c_{k_0}^{-1} e^{\theta_{31}(k_0)} n_1^{(3)}(\omega^2 k_0)}{\Delta_{11}'(\omega^2 k_0)(\Delta_{22}^{-1})'(\omega^2 k_0)}, \qquad \operatorname*{Res}_{k=k_0^{-1}} n_2^{(3)}(k) = \frac{-k_0^2 c_{k_0}^{-1} e^{\theta_{31}(k_0)} n_3^{(3)}(k_0^{-1})}{\Delta_{33}'(k_0^{-1})(\Delta_{22}^{-1})'(k_0^{-1})},$$

$$\begin{aligned}
&\operatorname*{Res}_{k=\omega^2 k_0^{-1}} n_3^{(3)}(k) = \frac{-\omega k_0^2 c_{k_0}^{-1} e^{\theta_{31}(k_0)} n_1^{(3)}(\omega^2 k_0^{-1})}{\Delta_{11}'(\omega^2 k_0^{-1})(\Delta_{33}^{-1})'(\omega^2 k_0^{-1})}, && \operatorname*{Res}_{k=\omega k_0^{-1}} n_1^{(3)}(k) = \frac{-\omega^2 k_0^2 c_{k_0}^{-1} e^{\theta_{31}(k_0)} n_2^{(3)}(\omega k_0^{-1})}{\Delta_{22}'(\omega k_0^{-1})(\Delta_{11}^{-1})'(\omega k_0^{-1})},\\
&\operatorname*{Res}_{k=\bar{k}_0} n_3^{(3)}(k) = \frac{d_{k_0}^{-1} e^{-\theta_{32}(\bar{k}_0)} n_2^{(3)}(\bar{k}_0)}{\Delta_{22}'(\bar{k}_0)(\Delta_{33}^{-1})'(\bar{k}_0)}, && \operatorname*{Res}_{k=\omega\bar{k}_0} n_2^{(3)}(k) = \frac{\omega^2 d_{k_0}^{-1} e^{-\theta_{32}(\bar{k}_0)} n_1^{(3)}(\omega\bar{k}_0)}{\Delta_{11}'(\omega\bar{k}_0)(\Delta_{22}^{-1})'(\omega\bar{k}_0)},\\
&\operatorname*{Res}_{k=\omega^2\bar{k}_0} n_1^{(3)}(k) = \frac{\omega d_{k_0}^{-1} e^{-\theta_{32}(\bar{k}_0)} n_3^{(3)}(\omega^2\bar{k}_0)}{\Delta_{33}'(\omega^2\bar{k}_0)(\Delta_{11}^{-1})'(\omega^2\bar{k}_0)}, && \operatorname*{Res}_{k=\bar{k}_0^{-1}} n_3^{(3)}(k) = \frac{-\bar{k}_0^2 d_{k_0}^{-1} e^{-\theta_{32}(\bar{k}_0)} n_1^{(3)}(\bar{k}_0^{-1})}{\Delta_{11}'(\bar{k}_0^{-1})(\Delta_{33}^{-1})'(\bar{k}_0^{-1})},\\
&\operatorname*{Res}_{k=\omega^2\bar{k}_0^{-1}} n_1^{(3)}(k) = \frac{\omega \bar{k}_0^2 d_{k_0}^{-1} e^{-\theta_{32}(\bar{k}_0)} n_2^{(3)}(\omega^2\bar{k}_0^{-1})}{-\Delta_{22}'(\omega^2\bar{k}_0^{-1})(\Delta_{11}^{-1})'(\omega^2\bar{k}_0^{-1})}, && \operatorname*{Res}_{k=\omega\bar{k}_0^{-1}} n_2^{(3)}(k) = \frac{\omega^2 \bar{k}_0^2 d_{k_0}^{-1} e^{-\theta_{32}(\bar{k}_0)} n_3^{(3)}(\omega\bar{k}_0^{-1})}{-\Delta_{33}'(\omega\bar{k}_0^{-1})(\Delta_{22}^{-1})'(\omega\bar{k}_0^{-1})};
\end{aligned} \tag{8.4}$$

(*b*) *for each* $k_0 \in \mathsf{Z} \cap \mathbb{R}$ *with* $\operatorname{Re} k_0 > 0$,

$$\begin{aligned}
&\operatorname*{Res}_{k=k_0} n_1^{(3)}(k) = \frac{c_{k_0}^{-1} e^{\theta_{21}(k_0)} n_2^{(3)}(k_0)}{\Delta_{22}'(k_0)(\Delta_{11}^{-1})'(k_0)}, && \operatorname*{Res}_{k=\omega k_0} n_3^{(3)}(k) = \frac{\omega^2 c_{k_0}^{-1} e^{\theta_{21}(k_0)} n_1^{(3)}(\omega k_0)}{\Delta_{11}'(\omega k_0)(\Delta_{33}^{-1})'(\omega k_0)},\\
&\operatorname*{Res}_{k=\omega^2 k_0} n_2^{(3)}(k) = \frac{\omega c_{k_0}^{-1} e^{\theta_{21}(k_0)} n_3^{(3)}(\omega^2 k_0)}{\Delta_{33}'(\omega^2 k_0)(\Delta_{22}^{-1})'(\omega^2 k_0)}, && \operatorname*{Res}_{k=k_0^{-1}} n_2^{(3)}(k) = \frac{-k_0^2 c_{k_0}^{-1} e^{\theta_{21}(k_0)} n_1^{(3)}(k_0^{-1})}{\Delta_{11}'(k_0^{-1})(\Delta_{22}^{-1})'(k_0^{-1})},\\
&\operatorname*{Res}_{k=\omega^2 k_0^{-1}} n_3^{(3)}(k) = \frac{-\omega k_0^2 c_{k_0}^{-1} e^{\theta_{21}(k_0)} n_2^{(3)}(\omega^2 k_0^{-1})}{\Delta_{22}'(\omega^2 k_0^{-1})(\Delta_{33}^{-1})'(\omega^2 k_0^{-1})}, && \operatorname*{Res}_{k=\omega k_0^{-1}} n_1^{(3)}(k) = \frac{\omega^2 k_0^2 c_{k_0}^{-1} e^{\theta_{21}(k_0)} n_3^{(3)}(\omega k_0^{-1})}{-\Delta_{33}'(\omega k_0^{-1})(\Delta_{11}^{-1})'(\omega k_0^{-1})};
\end{aligned} \tag{8.5}$$

(*c*) *for each* $k_0 \in \mathsf{Z} \setminus \mathbb{R}$ *with* $\operatorname{Re} k_0 < 0$,

$$\begin{aligned}
&\operatorname*{Res}_{k=k_0} n_3^{(3)}(k) = \frac{c_{k_0} e^{-\theta_{31}(k_0)} n_1^{(3)}(k_0)}{\Delta_{11}(k_0)\Delta_{33}^{-1}(k_0)}, && \operatorname*{Res}_{k=\omega k_0} n_2^{(3)}(k) = \frac{\omega c_{k_0} e^{-\theta_{31}(k_0)} n_3^{(3)}(\omega k_0)}{\Delta_{33}(\omega k_0)\Delta_{22}^{-1}(\omega k_0)},\\
&\operatorname*{Res}_{k=\omega^2 k_0} n_1^{(3)}(k) = \frac{\omega^2 c_{k_0} e^{-\theta_{31}(k_0)} n_2^{(3)}(\omega^2 k_0)}{\Delta_{22}(\omega^2 k_0)\Delta_{11}^{-1}(\omega^2 k_0)}, && \operatorname*{Res}_{k=k_0^{-1}} n_3^{(3)}(k) = \frac{-k_0^{-2} c_{k_0} e^{-\theta_{31}(k_0)} n_2^{(3)}(k_0^{-1})}{\Delta_{22}(k_0^{-1})\Delta_{33}^{-1}(k_0^{-1})},\\
&\operatorname*{Res}_{k=\omega^2 k_0^{-1}} n_1^{(3)}(k) = \frac{-\frac{\omega^2}{k_0^2} c_{k_0} e^{-\theta_{31}(k_0)} n_3^{(3)}(\omega^2 k_0^{-1})}{\Delta_{33}(\omega^2 k_0^{-1})\Delta_{11}^{-1}(\omega^2 k_0^{-1})}, && \operatorname*{Res}_{k=\omega k_0^{-1}} n_2^{(3)}(k) = \frac{-\frac{\omega}{k_0^2} c_{k_0} e^{-\theta_{31}(k_0)} n_1^{(3)}(\omega k_0^{-1})}{\Delta_{11}(\omega k_0^{-1})\Delta_{22}^{-1}(\omega k_0^{-1})},\\
&\operatorname*{Res}_{k=\bar{k}_0} n_2^{(3)}(k) = \frac{d_{k_0} e^{\theta_{32}(\bar{k}_0)} n_3^{(3)}(\bar{k}_0)}{\Delta_{33}(\bar{k}_0)\Delta_{22}^{-1}(\bar{k}_0)}, && \operatorname*{Res}_{k=\omega\bar{k}_0} n_1^{(3)}(k) = \frac{\omega d_{k_0} e^{\theta_{32}(\bar{k}_0)} n_2^{(3)}(\omega\bar{k}_0)}{\Delta_{22}(\omega\bar{k}_0)\Delta_{11}^{-1}(\omega\bar{k}_0)},\\
&\operatorname*{Res}_{k=\omega^2\bar{k}_0} n_3^{(3)}(k) = \frac{\omega^2 d_{k_0} e^{\theta_{32}(\bar{k}_0)} n_1^{(3)}(\omega^2\bar{k}_0)}{\Delta_{11}(\omega^2\bar{k}_0)\Delta_{33}^{-1}(\omega^2\bar{k}_0)}, && \operatorname*{Res}_{k=\bar{k}_0^{-1}} n_1^{(3)}(k) = \frac{-\bar{k}_0^{-2} d_{k_0} e^{\theta_{32}(\bar{k}_0)} n_3^{(3)}(\bar{k}_0^{-1})}{\Delta_{33}(\bar{k}_0^{-1})\Delta_{11}^{-1}(\bar{k}_0^{-1})},\\
&\operatorname*{Res}_{k=\omega^2\bar{k}_0^{-1}} n_2^{(3)}(k) = \frac{-\frac{\omega^2}{\bar{k}_0^2} d_{k_0} e^{\theta_{32}(\bar{k}_0)} n_1^{(3)}(\omega^2\bar{k}_0^{-1})}{\Delta_{11}(\omega^2\bar{k}_0^{-1})\Delta_{22}^{-1}(\omega^2\bar{k}_0^{-1})}, && \operatorname*{Res}_{k=\omega\bar{k}_0^{-1}} n_3^{(3)}(k) = \frac{-\frac{\omega}{\bar{k}_0^2} d_{k_0} e^{\theta_{32}(\bar{k}_0)} n_2^{(3)}(\omega\bar{k}_0^{-1})}{\Delta_{22}(\omega\bar{k}_0^{-1})\Delta_{33}^{-1}(\omega\bar{k}_0^{-1})};
\end{aligned} \tag{8.6}$$

(*d*) *for each* $k_0 \in \mathsf{Z} \cap \mathbb{R}$ *with* $\operatorname{Re} k_0 < 0$,

$$\begin{aligned}
&\operatorname*{Res}_{k=k_0} n_2^{(3)}(k) = \frac{c_{k_0} e^{-\theta_{21}(k_0)} n_1^{(3)}(k_0)}{\Delta_{11}(k_0)\Delta_{22}^{-1}(k_0)}, && \operatorname*{Res}_{k=\omega k_0} n_1^{(3)}(k) = \frac{\omega c_{k_0} e^{-\theta_{21}(k_0)} n_3^{(3)}(\omega k_0)}{\Delta_{33}(\omega k_0)\Delta_{11}^{-1}(\omega k_0)},\\
&\operatorname*{Res}_{k=\omega^2 k_0} n_3^{(3)}(k) = \frac{\omega^2 c_{k_0} e^{-\theta_{21}(k_0)} n_2^{(3)}(\omega^2 k_0)}{\Delta_{22}(\omega^2 k_0)\Delta_{33}^{-1}(\omega^2 k_0)}, && \operatorname*{Res}_{k=k_0^{-1}} n_1^{(3)}(k) = \frac{-k_0^{-2} c_{k_0} e^{-\theta_{21}(k_0)} n_2^{(3)}(k_0^{-1})}{\Delta_{22}(k_0^{-1})\Delta_{11}^{-1}(k_0^{-1})},
\end{aligned}$$

$$\underset{k=\omega^2 k_0^{-1}}{\operatorname{Res}} n_2^{(3)}(k) = \frac{-\frac{\omega^2}{k_0^2} c_{k_0} e^{-\theta_{21}(k_0)} n_3^{(3)}(\omega^2 k_0^{-1})}{\Delta_{33}(\omega^2 k_0^{-1}) \Delta_{22}^{-1}(\omega^2 k_0^{-1})}, \quad \underset{k=\omega k_0^{-1}}{\operatorname{Res}} n_3^{(3)}(k) = \frac{-\frac{\omega}{k_0^2} c_{k_0} e^{-\theta_{21}(k_0)} n_1^{(3)}(\omega k_0^{-1})}{\Delta_{11}(\omega k_0^{-1}) \Delta_{33}^{-1}(\omega k_0^{-1})}, \tag{8.7}$$

where the (x,t)-dependence has been suppressed for clarity. Moreover, at each point of $\hat{\mathsf{Z}}$, two entries of $n^{(3)}$ are analytic while one entry has (at most) a simple pole.

Proof. Since $\Delta(k) = \Delta(\zeta, k)$ is analytic at each point of $k_0 \in \mathsf{Z} \cap \{\operatorname{Re} k_0 < 0\}$, the claims (c) and (d) regarding points $k_0 \in \mathsf{Z}$ with $\operatorname{Re} k_0 < 0$ follow easily from RH problem 3.1 and the fact that $n^{(3)} = n\Delta$ near every point of Z. It remains to consider the points $k_0 \in \mathsf{Z}$ with $\operatorname{Re} k_0 > 0$. We prove the first residue condition in (8.4); the other conditions are proved in a similar way. Let $k_0 \in \mathsf{Z} \setminus \mathbb{R}$ be such that $\operatorname{Re} k_0 > 0$. Recall that $n^{(2)}$ obeys the same residue conditions as n; in particular, by (3.5),

$$\underset{k=k_0}{\operatorname{Res}} n_3^{(2)}(k) = c_{k_0} e^{-\theta_{31}(k_0)} n_1^{(2)}(k_0). \tag{8.8}$$

Since Δ_{33} has a simple zero at k_0 and $\Delta_{11}(k) = \Delta_{33}(\omega k)$ has a simple pole at k_0, $n_3^{(3)}$ has a removable singularity at k_0 and $n_1^{(3)}$ has a simple pole at k_0. Using (8.8), we obtain

$$n_3^{(3)}(k_0) = \lim_{k\to k_0} n_3^{(2)}(k)\Delta_{33}(k) = \underset{k=k_0}{\operatorname{Res}} n_3^{(2)}(k) \cdot \Delta_{33}'(k_0) = c_{k_0} e^{-\theta_{31}(k_0)} n_1^{(2)}(k_0)\Delta_{33}'(k_0),$$

and hence

$$\underset{k=k_0}{\operatorname{Res}} n_1^{(3)}(k) = \underset{k=k_0}{\operatorname{Res}} n_1^{(2)}(k)\Delta_{11}(k) = \frac{n_1^{(2)}(k_0)}{(\Delta_{11}^{-1})'(k_0)} = \frac{c_{k_0}^{-1} e^{\theta_{31}(k_0)} n_3^{(3)}(k_0)}{\Delta_{33}'(k_0)(\Delta_{11}^{-1})'(k_0)}.$$

This shows that only the first entry of $n^{(3)}$ is singular at k_0 and that the corresponding residue is as stated in (8.4). ☐

Using Lemma 8.2, we infer that $n^{(3)}$ satisfies RH problem 5.3 for $j = 3$ except that item (f) describing the behavior near points in $\hat{\mathsf{Z}}$ must be replaced by the following:

(f') At each point of $\hat{\mathsf{Z}}$, two entries of $n^{(3)}$ are analytic while one entry has (at most) a simple pole. Moreover, for $k_0 \in \mathsf{Z}$ such that $\operatorname{Re} k_0 > 0$, $n^{(3)}$ satisfies the residue conditions (8.4)–(8.5), while for $k_0 \in \mathsf{Z}$ such that $\operatorname{Re} k_0 < 0$, $n^{(3)}$ satisfies the residue conditions (8.6)–(8.7).

The residue conditions in (8.4)–(8.7) involve the following exponentials: (a) $e^{t\Phi_{31}(x,t,k_0)}$ and $e^{-t\Phi_{32}(x,t,\bar{k}_0)}$ for $k_0 \in D_{\mathrm{reg}}^R$, (b) $e^{t\Phi_{21}(x,t,k_0)}$ for $k_0 \in (1, \infty)$, (c) $e^{-t\Phi_{31}(x,t,k_0)}$ and $e^{t\Phi_{32}(x,t,\bar{k}_0)}$ for $k_0 \in D_{\mathrm{reg}}^L$, and (d) $e^{-t\Phi_{21}(x,t,k_0)}$ for $k_0 \in (-1, 0)$. It follows from the signature tables of Φ_{21}, Φ_{31}, and Φ_{32} (see Figure 3) that the real parts of all the exponents in these exponentials are negative. Thus, all the residues of $n^{(3)}$ at points in $\hat{\mathsf{Z}}$ are suppressed by exponentially small factor as $t \to \infty$. This suggests that the poles in $\hat{\mathsf{Z}}$ have no effect on the leading large t asymptotics of $n^{(3)}$. Our next transformation will help us make this claim precise.

9. The $n^{(3)} \to n^{(4)}$ transformation

The fourth transformation $n^{(3)} \to n^{(4)}$ replaces the residue conditions (8.4)–(8.7) by jumps on small circles. For each $k_0 \in \mathsf{Z}$, we let $D_\epsilon(k_0)$ be a small open disk centered at k_0 of radius $\epsilon > 0$. We let $D_\epsilon(k_0)^{-1}$ and $D_\epsilon(k_0)^*$ be the images of $D_\epsilon(k_0)$ under the maps

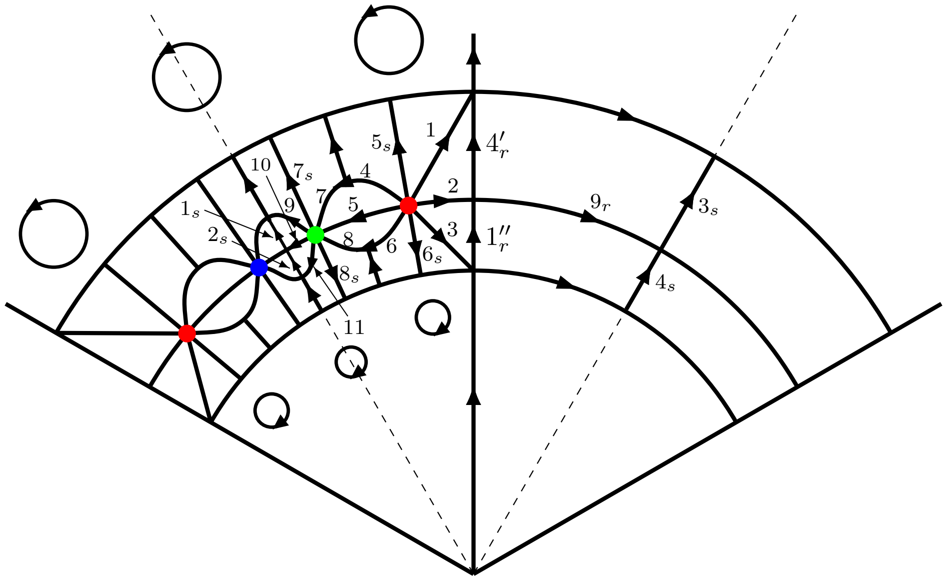

Figure 8. *The contour $\Gamma^{(4)}$ (solid) in a case when Z contains one point in D_{reg}^R, one point in $(1,\infty)$, and no other points; and the boundary of S (dashed). From right to left, the dots are ωk_4 (red), $\omega^2 k_2$ (green), k_1 (blue), and ωk_3 (red).*

$k \mapsto k^{-1}$ and $k \mapsto \bar{k}$, respectively. If $k_0 \in \mathbb{R}$, then $D_\epsilon(k_0)^* = D_\epsilon(k_0)$. Let $\partial D_\epsilon(k_0)$ and $\partial D_\epsilon(k_0)^*$ be oriented counterclockwise, and let $\partial D_\epsilon(k_0)^{-1}$ be oriented clockwise. Let

$$\mathcal{D}_{\text{sol}} = \bigcup_{k_0 \in \mathsf{Z}} \bigcup_{j=0,1,2} \Big(\omega^j D_\epsilon(k_0) \cup \omega^j D_\epsilon(k_0)^{-1} \cup \omega^j D_\epsilon(k_0)^* \cup \omega^j D_\epsilon(k_0)^{-*}\Big),$$

$$\partial\mathcal{D}_{\text{sol}} = \bigcup_{k_0 \in \mathsf{Z}} \bigcup_{j=0,1,2} \Big(\omega^j \partial D_\epsilon(k_0) \cup \omega^j \partial D_\epsilon(k_0)^{-1} \cup \omega^j \partial D_\epsilon(k_0)^* \cup \omega^j \partial D_\epsilon(k_0)^{-*}\Big),$$

so that $\partial\mathcal{D}_{\text{sol}}$ is the union of $6|\mathsf{Z} \cap \mathbb{R}| + 12|\mathsf{Z} \setminus \mathbb{R}|$ small circles. By shrinking $\epsilon > 0$ if necessary, we may assume that these circles do not intersect each other, and that they do not intersect $\Gamma^{(3)}$. We let $\Gamma^{(4)} := \Gamma^{(3)} \cup \partial\mathcal{D}_{\text{sol}}$ be the union of $\Gamma^{(3)}$ and the small circles $\partial\mathcal{D}_{\text{sol}}$, see Figure 8. Let $\Gamma_\star^{(4)}$ be the set of intersection points of $\Gamma^{(4)}$. Clearly, $\Gamma_\star^{(4)} = \Gamma_\star^{(3)}$.

The function $n^{(4)}$ is defined to equal $n^{(3)}$ except in $\mathcal{D}_{\text{sol}}$ where we define $n^{(4)}(x,t,k)$ as follows. If $k_0 \in \mathsf{Z} \setminus \mathbb{R}$, we define $n^{(4)}$ for $k \in D_\epsilon(k_0) \cup D_\epsilon(k_0)^*$ by

$$n^{(4)}(x,t,k) = \begin{cases} n^{(3)}(x,t,k) Q_1^R(x,t,k), & k \in D_\epsilon(k_0),\ k_0 \in \mathsf{Z} \cap D_{\text{reg}}^R, \\ n^{(3)}(x,t,k) Q_7^R(x,t,k), & k \in D_\epsilon(k_0)^*,\ k_0 \in \mathsf{Z} \cap D_{\text{reg}}^R, \\ n^{(3)}(x,t,k) Q_1^L(x,t,k), & k \in D_\epsilon(k_0),\ k_0 \in \mathsf{Z} \cap D_{\text{reg}}^L, \\ n^{(3)}(x,t,k) Q_7^L(x,t,k), & k \in D_\epsilon(k_0)^*,\ k_0 \in \mathsf{Z} \cap D_{\text{reg}}^L, \end{cases} \tag{9.1}$$

where

$$Q_1^R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-c_{k_0}^{-1} e^{\theta_{31}(x,t,k_0)}}{\Delta'_{33}(k_0)(\Delta_{11}^{-1})'(k_0)(k-k_0)} & 0 & 1 \end{pmatrix}, \qquad Q_7^R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{-d_{k_0}^{-1} e^{-\theta_{32}(x,t,\bar{k}_0)}}{\Delta'_{22}(\bar{k}_0)(\Delta_{33}^{-1})'(\bar{k}_0)(k-\bar{k}_0)} \\ 0 & 0 & 1 \end{pmatrix},$$

$$Q_1^L = \begin{pmatrix} 1 & 0 & \frac{-c_{k_0} e^{-\theta_{31}(x,t,k_0)}}{\Delta_{11}(k_0)\Delta_{33}^{-1}(k_0)(k-k_0)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q_7^L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-d_{k_0} e^{\theta_{32}(x,t,\bar{k}_0)}}{\Delta_{33}(\bar{k}_0)\Delta_{22}^{-1}(\bar{k}_0)(k-\bar{k}_0)} & 1 \end{pmatrix},$$

and if $k_0 \in \mathsf{Z} \cap \mathbb{R}$, we define $n^{(4)}$ for $k \in D_\epsilon(k_0)$ by

$$n^{(4)}(x,t,k) = \begin{cases} n^{(3)}(x,t,k) P_1^R(x,t,k), & k \in D_\epsilon(k_0),\ k_0 \in \mathsf{Z} \cap (1,\infty), \\ n^{(3)}(x,t,k) P_1^L(x,t,k), & k \in D_\epsilon(k_0),\ k_0 \in \mathsf{Z} \cap (-1,0), \end{cases} \tag{9.2}$$

where

$$P_1^R = \begin{pmatrix} 1 & 0 & 0 \\ \frac{-c_{k_0}^{-1} e^{\theta_{21}(x,t,k_0)}}{\Delta_{22}'(k_0)(\Delta_{11}^{-1})'(k_0)(k-k_0)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad P_1^L = \begin{pmatrix} 1 & \frac{-c_{k_0} e^{-\theta_{21}(x,t,k_0)}}{\Delta_{11}(k_0)\Delta_{22}^{-1}(k_0)(k-k_0)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then extend $n^{(4)}$ to all of $\mathcal{D}_{\rm sol}$ by means of the $\mathcal{A}$- and $\mathcal{B}$-symmetries (4.5).

It follows from the above definition of $n^{(4)}$ and Lemma 8.2 that $n^{(4)}$ has no poles at the points in $\hat{\mathsf{Z}}$. Moreover, on $\Gamma^{(4)} \setminus \Gamma_\star^{(4)}$, the boundary values of $n^{(4)}$ exist, are continuous, and satisfy $n_+^{(4)} = n_-^{(4)} v^{(4)}$, where $v^{(4)}$ equals $v^{(3)}$ on Γ and $v^{(4)}$ is defined on $\partial\mathcal{D}_{\rm sol}$ by setting

$$v(x,t,k) = \begin{cases} Q_1^R(x,t,k), & k \in \partial D_\epsilon(k_0),\ k_0 \in \mathsf{Z} \cap D_{\rm reg}^R, \\ Q_7^R(x,t,k), & k \in \partial D_\epsilon(k_0)^*,\ k_0 \in \mathsf{Z} \cap D_{\rm reg}^R, \\ Q_1^L(x,t,k), & k \in \partial D_\epsilon(k_0),\ k_0 \in \mathsf{Z} \cap D_{\rm reg}^L, \\ Q_7^L(x,t,k), & k \in \partial D_\epsilon(k_0)^*,\ k_0 \in \mathsf{Z} \cap D_{\rm reg}^L, \\ P_1^R(x,t,k), & k \in \partial D_\epsilon(k_0),\ k_0 \in \mathsf{Z} \cap (1,\infty), \\ P_1^L(x,t,k), & k \in \partial D_\epsilon(k_0),\ k_0 \in \mathsf{Z} \cap (-1,0), \end{cases} \tag{9.3}$$

and then extending it to all of $\partial\mathcal{D}_{\rm sol}$ by means of the $\mathcal{A}$- and $\mathcal{B}$-symmetries (4.4).

Lemma 9.1. *The jump matrix $v^{(4)}$ converges to the identity matrix I as $t \to \infty$ uniformly for $k \in \partial\mathcal{D}_{\rm sol}$ and $\zeta \in \mathcal{I}$. More precisely, for $\zeta \in \mathcal{I}$,*

$$\|v^{(4)} - I\|_{(L^1 \cap L^\infty)(\partial\mathcal{D}_{\rm sol})} \le C t^{-N}.$$

Proof. The statement is a direct consequence of the uniform exponential decay of the exponentials appearing in the definitions of $Q_1^R, Q_7^R, Q_1^L, Q_7^L, P_1^R, P_1^L$, together with the $\mathcal{A}$- and $\mathcal{B}$-symmetries. □

Due to Lemma 8.1 and the symmetries (4.5), it is sufficient to construct local parametrices approximating $n^{(4)}$ near ωk_4 and $\omega^2 k_2$.

10. Local parametrix near ωk_4

As $k \to \omega k_4$, we have

$$-(\Phi_{31}(\zeta,k) - \Phi_{31}(\zeta,\omega k_4)) = \Phi_{31,\omega k_4}(k-\omega k_4)^2 + O((k-\omega k_4)^3), \quad \Phi_{31,\omega k_4} := \omega \frac{4 - 3k_4\zeta - k_4^3\zeta}{4k_4^4}.$$

We define $z_1 = z_1(\zeta,t,k)$ by

$$z_1 = z_{1,\star}\sqrt{t}(k-\omega k_4)\hat{z}_1, \qquad \hat{z}_1 := \sqrt{\frac{2i(\Phi_{31}(\zeta,\omega k_4) - \Phi_{31}(\zeta,k))}{z_{1,\star}^2 (k-\omega k_4)^2}},$$

where the principal branch is chosen for $\hat{z}_1 = \hat{z}_1(\zeta,k)$, and

$$z_{1,\star} := \sqrt{2} e^{\frac{\pi i}{4}} \sqrt{\Phi_{31,\omega k_4}}, \qquad -i\omega k_4 z_{1,\star} > 0.$$

Note that $\hat{z}_1(\zeta,\omega k_4)=1$, and $t(\Phi_{31}(\zeta,k)-\Phi_{31}(\zeta,\omega k_4))=\frac{iz_1^2}{2}$. Let $\epsilon>0$ be small and independent of t. The map z_1 is conformal from $D_\epsilon(\omega k_4)$ to a neighborhood of 0 and its expansion as $k\to\omega k_4$ is given by

$$z_1=z_{1,\star}\sqrt{t}(k-\omega k_4)(1+O(k-\omega k_4)).$$

For all $k\in D_\epsilon(\omega k_4)$, we have

$$\begin{aligned}
&\ln_{\omega k_4}(k-\omega k_4)=\ln_0[z_{1,\star}(k-\omega k_4)\hat{z}_1]-\ln\hat{z}_1-\ln z_{1,\star},\\
&\tilde{\ln}_{\omega k_4}(k-\omega k_4)=\ln[z_{1,\star}(k-\omega k_4)\hat{z}_1]-\ln\hat{z}_1-\ln z_{1,\star},
\end{aligned}$$

where $\ln_0(k):=\ln|k|+i\arg_0 k$, $\arg_0(k)\in(0,2\pi)$, and ln is the principal logarithm. Shrinking $\epsilon>0$ if necessary, $k\mapsto\ln\hat{z}_1$ is analytic in $D_\epsilon(\omega k_4)$, $\ln\hat{z}_1=O(k-\omega k_4)$ as $k\to\omega k_4$, and

$$\ln z_{1,\star}=\ln|z_{1,\star}|+i\arg z_{1,\star}=\ln|z_{1,\star}|+i\big(\tfrac{\pi}{2}-\arg(\omega k_4)\big),$$

where $\arg(\omega k_4)\in(\frac{\pi}{2},\frac{2\pi}{3})$. By Lemma 7.1, we have

$$\begin{aligned}
&\delta_1(\zeta,k)=e^{-i\nu_1\ln_{\omega k_4}(k-\omega k_4)}e^{-\chi_1(\zeta,k)}=e^{-i\nu_1\ln_0 z_1}t^{\frac{i\nu_1}{2}}e^{i\nu_1\ln\hat{z}_1}e^{i\nu_1\ln z_{1,\star}}e^{-\chi_1(\zeta,k)},\\
&\delta_2(\zeta,k)=e^{-i\nu_1\ln z_1}t^{\frac{i\nu_1}{2}}e^{i\nu_1\ln\hat{z}_1}e^{i\nu_1\ln z_{1,\star}}e^{i\nu_2\tilde{\ln}_{\omega^2k_2}(k-\omega^2k_2)-\tilde{\chi}_2(\zeta,k)},\\
&\delta_3(\zeta,k)=e^{-i\nu_3\ln z_1}t^{\frac{i\nu_3}{2}}e^{i\nu_3\ln\hat{z}_1}e^{i\nu_3\ln z_{1,\star}}e^{i\nu_4\tilde{\ln}_{\omega^2k_2}(k-\omega^2k_2)-\tilde{\chi}_3(\zeta,k)}.
\end{aligned}$$

For simplicity, for $a\in\mathbb{C}$, we will write $z_{1,(0)}^a:=e^{a\ln_0 z_1}$, $z_{1,\star}^a:=e^{a\ln z_{1,\star}}$, $\hat{z}_1^a:=e^{a\ln\hat{z}_1}$, and $z_1^a:=e^{a\ln z_1}$. Using the above expressions for $\{\delta_j(\zeta,k)\}_{j=1}^3$, we obtain

$$\frac{\Delta_{33}(\zeta,k)}{\Delta_{11}(\zeta,k)}=\frac{\delta_1(\zeta,k)\delta_2(\zeta,k)}{\delta_3(\zeta,k)^2}\mathcal{D}_1(k)\frac{\mathcal{P}(k)}{\mathcal{P}(\omega k)}=d_{1,0}(\zeta,t)d_{1,1}(\zeta,k)z_1^{2i\nu_3}z_{1,(0)}^{-i\nu_1}z_1^{-i\nu_1},\tag{10.1}$$

where $d_{1,0}(\zeta,t)$ is defined in (2.13) and

$$\begin{aligned}
d_{1,1}(\zeta,k):=\;&e^{-\chi_1(\zeta,k)+\chi_1(\zeta,\omega k_4)-\tilde{\chi}_2(\zeta,k)+\tilde{\chi}_2(\zeta,\omega k_4)+2\tilde{\chi}_3(\zeta,k)-2\tilde{\chi}_3(\zeta,\omega k_4)}\\
&\times e^{i(\nu_2-2\nu_4)\left[\tilde{\ln}_{\omega^2k_2}(k-\omega^2k_2)-\tilde{\ln}_{\omega^2k_2}(\omega k_4-\omega^2k_2)\right]}\hat{z}_1^{2i(\nu_1-\nu_3)}\frac{\mathcal{D}_1(k)}{\mathcal{D}_1(\omega k_4)}\frac{\mathcal{P}(k)}{\mathcal{P}(\omega k)},
\end{aligned}$$

We summarize in Lemma 10.1 below some properties of $d_{1,0}(\zeta,t)$ and $d_{1,1}(\zeta,k)$. Define

$$Y_1(\zeta,t)=d_{1,0}(\zeta,t)^{\frac{\sigma}{2}}e^{-\frac{t}{2}\Phi_{31}(\zeta,\omega k_4)\sigma}\lambda_1^{\sigma},$$

where $\sigma=\operatorname{diag}(1,0,-1)$ and $\lambda_1\in\mathbb{C}$ is a parameter that will be chosen later. For $k\in D_\epsilon(\omega k_4)$, define

$$\tilde{n}(x,t,k)=n^{(4)}(x,t,k)Y_1(\zeta,t).\tag{10.2}$$

Let $\tilde{v}$ be the jump matrix of $\tilde{n}$, and let $\tilde{v}_j$ denote the restriction of $\tilde{v}$ to $\Gamma_j^{(4)}\cap D_\epsilon(\omega k_4)$. In view of (6.1), (8.1) and (10.1), the matrices $\tilde{v}_j$ can be written as

$$\tilde{v}_1=\begin{pmatrix}1&0&0\\0&1&0\\d_{1,1}^{-1}\lambda_1^2r_{2,a}(\frac{1}{\omega^2k})z_1^{-2i\nu_3}z_{1,(0)}^{i\nu_1}z_1^{i\nu_1}e^{\frac{iz_1^2}{2}}&0&1\end{pmatrix},\tag{10.3a}$$

$$\tilde{v}_4=\begin{pmatrix}1&0&-d_{1,1}\lambda_1^{-2}\big(\frac{r_1(\frac{1}{\omega^2k})}{f(k)}\big)_a z_1^{2i\nu_3}z_{1,(0)}^{-i\nu_1}z_1^{-i\nu_1}e^{-\frac{iz_1^2}{2}}\\0&1&0\\0&0&1\end{pmatrix},\tag{10.3b}$$

$$\tilde{v}_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -d_{1,1}^{-1}\lambda_1^2\big(\frac{r_2(\frac{1}{\omega^2 k})}{f(k)}\big)_a z_1^{-2i\nu_3} z_{1,(0)}^{i\nu_1} z_1^{i\nu_1} e^{\frac{iz_1^2}{2}} & 0 & 1 \end{pmatrix}, \tag{10.3c}$$

$$\tilde{v}_3 = \begin{pmatrix} 1 & 0 & d_{1,1}\lambda_1^{-2} r_{1,a}(\frac{1}{\omega^2 k}) z_1^{2i\nu_3} z_{1,(0)}^{-i\nu_1} z_1^{-i\nu_1} e^{-\frac{iz_1^2}{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10.3d}$$

Let $\mathcal{X}_1^\epsilon := \cup_{j=1}^4 \mathcal{X}_{1,j}^\epsilon$, where

$$\mathcal{X}_{1,1}^\epsilon = \Gamma_1^{(4)} \cap D_\epsilon(\omega k_4), \quad \mathcal{X}_{1,2}^\epsilon = \Gamma_4^{(4)} \cap D_\epsilon(\omega k_4), \quad \mathcal{X}_{1,3}^\epsilon = \Gamma_6^{(4)} \cap D_\epsilon(\omega k_4), \quad \mathcal{X}_{1,4}^\epsilon = \Gamma_3^{(4)} \cap D_\epsilon(\omega k_4).$$

Let $k_\star = \omega k_4$. The above expressions for $\tilde{v}_j$ suggest that we approximate $\tilde{n}$ by $(1,1,1)Y_1\tilde{m}^{\omega k_4}$, where $\tilde{m}^{\omega k_4}(x,t,k)$ is analytic for $k \in D_\epsilon(\omega k_4) \setminus \mathcal{X}_1$, obeys the jump relation $\tilde{m}_+^{\omega k_4} = \tilde{m}_-^{\omega k_4}\tilde{v}_{\mathcal{X}_1^\epsilon}^{\omega k_4}$ on $\mathcal{X}_1^\epsilon$ where

$$\tilde{v}_{\mathcal{X}_{1,1}^\epsilon}^{\omega k_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\mathcal{P}(\omega^2 k_4)}{\mathcal{P}(\omega k_4)}\lambda_1^2 r_2(\frac{1}{\omega^2 k_\star}) z_1^{-2i\nu_3} z_{1,(0)}^{i\nu_1} z_1^{i\nu_1} e^{\frac{iz_1^2}{2}} & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_{\mathcal{X}_{1,2}^\epsilon}^{\omega k_4} = \begin{pmatrix} 1 & 0 & -\frac{\mathcal{P}(\omega k_4)}{\mathcal{P}(\omega^2 k_4)}\lambda_1^{-2}\frac{r_1(\frac{1}{\omega^2 k_\star})}{f(k_\star)} z_1^{2i\nu_3} z_{1,(0)}^{-i\nu_1} z_1^{-i\nu_1} e^{-\frac{iz_1^2}{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_{\mathcal{X}_{1,3}^\epsilon}^{\omega k_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\mathcal{P}(\omega^2 k_4)}{\mathcal{P}(\omega k_4)}\lambda_1^2\frac{r_2(\frac{1}{\omega^2 k_\star})}{f(k_\star)} z_1^{-2i\nu_3} z_{1,(0)}^{i\nu_1} z_1^{i\nu_1} e^{\frac{iz_1^2}{2}} & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_{\mathcal{X}_{1,4}^\epsilon}^{\omega k_4} = \begin{pmatrix} 1 & 0 & \frac{\mathcal{P}(\omega k_4)}{\mathcal{P}(\omega^2 k_4)}\lambda_1^{-2} r_1(\frac{1}{\omega^2 k_\star}) z_1^{2i\nu_3} z_{1,(0)}^{-i\nu_1} z_1^{-i\nu_1} e^{-\frac{iz_1^2}{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and satisfies $\tilde{m}^{\omega k_4} \to I$ as $t \to \infty$ uniformly for $k \in \partial D_\epsilon(\omega k_4)$. We choose our free parameter λ_1 as follows:

$$\lambda_1 := \big(\tfrac{\mathcal{P}(\omega k_4)}{\mathcal{P}(\omega^2 k_4)}\big)^{\frac{1}{2}} |\tilde{r}(\tfrac{1}{\omega^2 k_\star})|^{-\frac{1}{4}} = \big(\tfrac{\mathcal{P}(\omega k_4)}{\mathcal{P}(\omega^2 k_4)}\big)^{\frac{1}{2}} |\tilde{r}(\tfrac{1}{k_4})|^{-\frac{1}{4}}, \tag{10.4}$$

and define the local parametrix $m^{\omega k_4}$ by

$$m^{\omega k_4}(x,t,k) = Y_1(\zeta,t) m^{X,(1)}(q_1,q_3,z_1(\zeta,t,k)) Y_1(\zeta,t)^{-1}, \qquad k \in D_\epsilon(\omega k_4), \tag{10.5}$$

where $m^{X,(1)}(\cdot) = m^{X,(1)}(q_1,q_3,\cdot)$ is the solution of the model RH problem of Lemma A.1 with

$$q_1 = \tilde{r}(\omega k_4)^{\frac{1}{2}} r_1(\omega k_4), \qquad \bar{q}_1 = \tilde{r}(\omega k_4)^{-\frac{1}{2}} r_2(\omega k_4) = \tilde{r}(\omega k_4)^{\frac{1}{2}}\overline{r_1(\omega k_4)},$$
$$q_3 = |\tilde{r}(\tfrac{1}{k_4})|^{\frac{1}{2}} r_1(\tfrac{1}{k_4}), \qquad \bar{q}_3 = -|\tilde{r}(\tfrac{1}{k_4})|^{-\frac{1}{2}} r_2(\tfrac{1}{k_4}) = |\tilde{r}(\tfrac{1}{k_4})|^{\frac{1}{2}}\overline{r_1(\tfrac{1}{k_4})}.$$

Lemma 10.1. *The function $Y_1(\zeta,t)$ is uniformly bounded:*

$$\sup_{\zeta\in\mathcal{I}}\sup_{t\geq 2} |Y_1(\zeta,t)^{\pm 1}| < C. \tag{10.6}$$

Moreover, the functions $d_{1,0}(\zeta,t)$ and $d_{1,1}(\zeta,k)$ satisfy

$$|d_{1,0}(\zeta,t)| = e^{-\pi\nu_1}, \qquad \zeta \in \mathcal{I},\ t \geq 2, \tag{10.7}$$

$$|d_{1,1}(\zeta,k) - \tfrac{\mathcal{P}(\omega k_4)}{\mathcal{P}(\omega^2 k_4)}| \leq C|k-\omega k_4|(1+|\ln|k-\omega k_4||), \qquad \zeta \in \mathcal{I},\ k \in \mathcal{X}_1^\epsilon. \tag{10.8}$$

Proof. Standard estimates give (10.6) and (10.8). To prove (10.7), we first note that

$$\begin{aligned}|d_{1,0}(\zeta,t)| &:= e^{-\mathrm{Re}\,\chi_1(\zeta,\omega k_4) - \mathrm{Re}\,\tilde{\chi}_2(\zeta,\omega k_4) + 2\mathrm{Re}\,\tilde{\chi}_3(\zeta,\omega k_4)} \\ &\quad \times e^{-(\nu_2-2\nu_4)\tilde{\arg}_{\omega^2 k_2}(\omega k_4 - \omega^2 k_2)} e^{-2(\nu_1-\nu_3)\arg z_{1,\star}} |\mathcal{D}_1(\omega k_4)|. \end{aligned} \tag{10.9}$$

We deduce from Lemma 7.1 that if $k \in \partial\mathbb{D} \setminus \{\arg k \in (\pi/2, 2\pi/3)\}$ then

$$\begin{aligned}
&|\delta_1(\zeta,k)| = e^{\nu_1 \frac{\arg_i(k)+\arg(\omega k_4)+\pi}{2} - \mathrm{Re}\,\chi_1(\zeta,k)}, \\
&|\delta_2(\zeta,k)| = e^{\nu_1 \frac{\arg_i(k)+\arg(\omega k_4)+\pi}{2} - \nu_2 \frac{\arg_i(k)+\arg(\omega^2 k_2)+\pi}{2} - \mathrm{Re}\,\chi_2(\zeta,k)}, \\
&|\delta_3(\zeta,k)| = e^{\nu_3 \frac{\arg_i(k)+\arg(\omega k_4)+\pi}{2} - \nu_4 \frac{\arg_i(k)+\arg(\omega^2 k_2)+\pi}{2} - \mathrm{Re}\,\chi_3(\zeta,k)}, \\
&|\delta_4(\zeta,k)| = e^{\nu_4 \frac{\arg_i(k)+\arg(\omega^2 k_2)+\pi}{2} - \mathrm{Re}\,\chi_4(\zeta,k)}, \\
&|\delta_5(\zeta,k)| = e^{\nu_5 \frac{\arg_i(k)+\arg(\omega^2 k_2)+\pi}{2} - \mathrm{Re}\,\chi_5(\zeta,k)},
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Re}\,\chi_1(\zeta,k) = \frac{1}{2\pi}\int_{\Gamma_2^{(2)}} \frac{\arg s}{2} d\ln(1+r_1(s)r_2(s)) + \frac{\arg_i k + \pi}{2}\nu_1, \\
&\mathrm{Re}\,\chi_2(\zeta,k) = \frac{1}{2\pi}\int_{\Gamma_5^{(2)}} \frac{\arg s}{2} d\ln(1+r_1(s)r_2(s)) + \frac{\arg_i k + \pi}{2}(\nu_1-\nu_2), \\
&\mathrm{Re}\,\chi_3(\zeta,k) = \frac{1}{2\pi}\int_{\Gamma_5^{(2)}} \frac{\arg s}{2} d\ln(f(s)) + \frac{\arg_i k + \pi}{2}(\nu_3-\nu_4), \\
&\mathrm{Re}\,\chi_4(\zeta,k) = \frac{1}{2\pi}\lim_{\epsilon\to 0_+}\bigg(\int_{\omega^2 k_2}^{e^{i(\frac{2\pi}{3}-\epsilon)}} \frac{\arg s}{2} d\ln(f(s)) - \frac{\arg\omega}{2}\ln(f(e^{i(\frac{2\pi}{3}-\epsilon)}))\bigg) + \frac{\arg_i k + \pi}{2}\nu_4, \\
&\mathrm{Re}\,\chi_5(\zeta,k) = \frac{1}{2\pi}\lim_{\epsilon\to 0_+}\bigg(\int_{\omega^2 k_2}^{e^{i(\frac{2\pi}{3}-\epsilon)}} \frac{\arg s}{2} d\ln(f(\omega^2 s)) - \frac{\arg\omega}{2}\ln(f(e^{-i\epsilon}))\bigg) + \frac{\arg_i k + \pi}{2}\nu_5,
\end{aligned}$$

where $\arg_i(k) \in (\pi/2, 5\pi/2)$. Hence, for $k \in \partial\mathbb{D} \setminus \{\arg k \in (\pi/2, 2\pi/3)\}$,

$$\begin{aligned}
&|\delta_1(\zeta,k)| = e^{\nu_1 \frac{\arg(\omega k_4)}{2} - \frac{1}{2\pi}\int_{\Gamma_2^{(2)}} \frac{\arg s}{2} d\ln(1+r_1(s)r_2(s))}, \\
&|\delta_2(\zeta,k)| = e^{\nu_1 \frac{\arg(\omega k_4)}{2} - \nu_2 \frac{\arg(\omega^2 k_2)}{2} - \frac{1}{2\pi}\int_{\Gamma_5^{(2)}} \frac{\arg s}{2} d\ln(1+r_1(s)r_2(s))}, \\
&|\delta_3(\zeta,k)| = e^{\nu_3 \frac{\arg(\omega k_4)}{2} - \nu_4 \frac{\arg(\omega^2 k_2)}{2} - \frac{1}{2\pi}\int_{\Gamma_5^{(2)}} \frac{\arg s}{2} d\ln(f(s))}, \\
&|\delta_4(\zeta,k)| = e^{\nu_4 \frac{\arg(\omega^2 k_2)}{2} - \frac{1}{2\pi}\lim_{\epsilon\to 0_+}\big(\int_{\omega^2 k_2}^{e^{i(\frac{2\pi}{3}-\epsilon)}} \frac{\arg s}{2} d\ln(f(s)) - \frac{\arg\omega}{2}\ln(f(e^{i(\frac{2\pi}{3}-\epsilon)}))\big)}, \\
&|\delta_5(\zeta,k)| = e^{\nu_5 \frac{\arg(\omega^2 k_2)}{2} - \frac{1}{2\pi}\lim_{\epsilon\to 0_+}\big(\int_{\omega^2 k_2}^{e^{i(\frac{2\pi}{3}-\epsilon)}} \frac{\arg s}{2} d\ln(f(\omega^2 s)) - \frac{\arg\omega}{2}\ln(f(e^{-i\epsilon}))\big)}.
\end{aligned} \tag{10.10}$$

Note that the right-hand sides in (10.10) are independent of k. Since the points $k_4, \omega^2 k_4$, $\frac{1}{k_4}$, $\frac{1}{\omega k_4}$, $\frac{1}{\omega^2 k_4}$ all lie in $\partial\mathbb{D} \setminus \{\arg k \in (\pi/2, 2\pi/3)\}$, we obtain

$$|\mathcal{D}_1(\omega k_4)| = \frac{e^{\frac{1}{2\pi}\int_{\Gamma_2^{(2)}} \frac{\arg s}{2} d\ln(1+r_1(s)r_2(s)) - \nu_1 \frac{\arg(\omega k_4)}{2}} e^{2\nu_3 \frac{\arg(\omega k_4)}{2} - 2\nu_4 \frac{\arg(\omega^2 k_2)}{2} - 2\frac{1}{2\pi}\int_{\Gamma_5^{(2)}} \frac{\arg s}{2} d\ln(f(s))}}{e^{\nu_1 \frac{\arg(\omega k_4)}{2} - \nu_2 \frac{\arg(\omega^2 k_2)}{2} - \frac{1}{2\pi}\int_{\Gamma_5^{(2)}} \frac{\arg s}{2} d\ln(1+r_1(s)r_2(s))}}.$$

Similar arguments show that

$$\operatorname{Re}\chi_1(\zeta,\omega k_4)=\frac{1}{2\pi}\int_{\Gamma_2^{(2)}}\frac{\arg s}{2}d\ln(1+r_1(s)r_2(s))+\frac{\arg(\omega k_4)+\pi}{2}\nu_1,$$
$$\operatorname{Re}\tilde{\chi}_2(\zeta,\omega k_4)=\frac{1}{2\pi}\int_{\Gamma_5^{(2)}}\frac{\arg s}{2}d\ln(1+r_1(s)r_2(s))+\frac{\arg(\omega k_4)-\pi}{2}(\nu_1-\nu_2),$$
$$\operatorname{Re}\tilde{\chi}_3(\zeta,\omega k_4)=\frac{1}{2\pi}\int_{\Gamma_5^{(2)}}\frac{\arg s}{2}d\ln(f(s))+\frac{\arg(\omega k_4)-\pi}{2}(\nu_3-\nu_4).$$

Plugging the above expressions into (10.9) and using that

$$\tilde{\arg}_{\omega^2 k_2}(\omega k_4-\omega^2 k_2)=\frac{\arg(\omega k_4)+\arg(\omega^2 k_2)-\pi}{2}\qquad\text{and}\qquad \arg z_{1,\star}=\tfrac{\pi}{2}-\arg(\omega k_4),$$

straightforward calculations show that $|d_{1,0}(\zeta,t)|=e^{-\pi\nu_1}$. □

Lemma 10.2. *The function $m^{\omega k_4}(x,t,k)$ defined in (10.5) is an analytic function of $k\in D_\epsilon(\omega k_4)\setminus\mathcal{X}_1^\epsilon$ which is uniformly bounded for $t\geq 2$, $\zeta\in\mathcal{I}$, and $k\in D_\epsilon(\omega k_4)\setminus\mathcal{X}_1^\epsilon$. Across $\mathcal{X}_1^\epsilon$, $m^{\omega k_4}$ obeys the jump condition $m_+^{\omega k_4}=m_-^{\omega k_4}v^{\omega k_4}$, where $v^{\omega k_4}$ satisfies*

$$\begin{cases}\|v^{(4)}-v^{\omega k_4}\|_{L^1(\mathcal{X}_1^\epsilon)}\leq Ct^{-1}\ln t,\\ \|v^{(4)}-v^{\omega k_4}\|_{L^\infty(\mathcal{X}_1^\epsilon)}\leq Ct^{-1/2}\ln t,\end{cases}\qquad \zeta\in\mathcal{I},\ t\geq 2. \tag{10.11}$$

Furthermore, as $t\to\infty$,

$$\|m^{\omega k_4}(x,t,\cdot)^{-1}-I\|_{L^\infty(\partial D_\epsilon(\omega k_4))}=O(t^{-1/2}), \tag{10.12}$$

$$m^{\omega k_4}-I=\frac{Y_1(\zeta,t)m_1^{X,(1)}Y_1(\zeta,t)^{-1}}{z_{1,\star}\sqrt{t}(k-\omega k_4)\hat{z}_1(\zeta,k)}+O(t^{-1}) \tag{10.13}$$

uniformly for $\zeta\in\mathcal{I}$, and $m_1^{X,(1)}=m_1^{X,(1)}(q_1,q_3)$ is given by (A.3).

Proof. By (10.2) and (10.5), we have

$$v^{(4)}-v^{\omega k_4}=Y_1(\zeta,t)(\tilde{v}-v^{X,(1)})Y_1(\zeta,t)^{-1},$$

where we recall that $\tilde{v}$ is given by (10.3). Thus, recalling (10.6), the bounds (10.11) follow if we can show that

$$\|\tilde{v}(x,t,\cdot)-v^{X,(1)}(x,t,z_1(\zeta,t,\cdot))\|_{L^1(\mathcal{X}_{1,j}^\epsilon)}\leq Ct^{-1}\ln t, \tag{10.14a}$$

$$\|\tilde{v}(x,t,\cdot)-v^{X,(1)}(x,t,z_1(\zeta,t,\cdot))\|_{L^\infty(\mathcal{X}_{1,j}^\epsilon)}\leq Ct^{-1/2}\ln t, \tag{10.14b}$$

for $j=1,\dots,4$. We give the proof of (10.14) for $j=1$; similar arguments apply when $j=2,3,4$.

For $k\in\mathcal{X}_{1,1}^\epsilon$, only the (31) element of the matrix $\tilde{v}-v^{X,1}$ is nonzero. Since $-\bar{q}_3=\frac{\mathcal{P}(\omega^2 k_4)}{\mathcal{P}(\omega k_4)}\lambda_1^2 r_2(\frac{1}{k_4})$ and

$$\big|e^{\frac{iz_1^2}{2}}\big|=e^{\operatorname{Re}(\frac{iz_1^2}{2})}=e^{-\frac{|z_1|^2}{2}},$$

we can estimate $|(\tilde{v}-v^{X,(1)})_{31}|$ as follows:

$$\begin{aligned}|(\tilde{v}-v^{X,(1)})_{31}|&=\left|d_{1,1}^{-1}\lambda_1^2 r_{2,a}(\tfrac{1}{\omega^2 k})z_1^{-2i\nu_3}z_{1,(0)}^{i\nu_1}z_1^{i\nu_1}e^{\frac{iz_1^2}{2}}-(-\bar{q}_3 z_1^{-2i\nu_3}z_{1,(0)}^{i\nu_1}z_1^{i\nu_1}e^{\frac{iz_1^2}{2}})\right|\\
&=\big|z_1^{-2i\nu_3}z_{1,(0)}^{i\nu_1}z_1^{i\nu_1}\big|\Big|(d_{1,1}^{-1}-\tfrac{\mathcal{P}(\omega^2 k_4)}{\mathcal{P}(\omega k_4)})\lambda_1^2 r_{2,a}(\tfrac{1}{\omega^2 k})+\big(\tfrac{\mathcal{P}(\omega^2 k_4)}{\mathcal{P}(\omega k_4)}\lambda_1^2 r_{2,a}(\tfrac{1}{\omega^2 k})-(-\bar{q}_3)\big)\Big|\big|e^{\frac{iz_1^2}{2}}\big|\\
&\leq C\bigg(|d_{1,1}^{-1}-\tfrac{\mathcal{P}(\omega^2 k_4)}{\mathcal{P}(\omega k_4)}||\lambda_1^2 r_{2,a}(\tfrac{1}{\omega^2 k})|+|\tfrac{\mathcal{P}(\omega^2 k_4)}{\mathcal{P}(\omega k_4)}\lambda_1^2 r_{2,a}(\tfrac{1}{\omega^2 k})-(-\bar{q}_3)|\bigg)e^{-\frac{|z_1|^2}{2}}\end{aligned}$$

$$\leq C\bigg(|d_{1,1}^{-1} - \tfrac{\mathcal{P}(\omega^2 k_4)}{\mathcal{P}(\omega k_4)}| + |k-\omega k_4|\bigg)e^{\frac{t}{4}|\mathrm{Re}\,\Phi_{31}(\zeta,k)|}e^{-\frac{|z_1|^2}{2}}$$
$$\leq C\bigg(|d_{1,1}^{-1} - \tfrac{\mathcal{P}(\omega^2 k_4)}{\mathcal{P}(\omega k_4)}| + |k-\omega k_4|\bigg)e^{-ct|k-\omega k_4|^2}, \qquad k \in \mathcal{X}_{1,1}^{\epsilon}.$$

Using (10.8), this gives

$$|(\tilde{v} - v^{X,(1)})_{31}| \leq C|k-\omega k_4|(1+|\ln|k-\omega k_4||)e^{-ct|k-\omega k_4|^2}, \qquad k \in \mathcal{X}_{1,1}^{\epsilon}.$$

Hence

$$\|(\tilde{v} - v^{X,(1)})_{31}\|_{L^1(\mathcal{X}_{1,1}^{\epsilon})} \leq C\int_0^\infty u(1+|\ln u|)e^{-ctu^2}du \leq Ct^{-1}\ln t,$$
$$\|(\tilde{v} - v^{X,(1)})_{31}\|_{L^\infty(\mathcal{X}_{1,1}^{\epsilon})} \leq C\sup_{u\geq 0} u(1+|\ln u|)e^{-ctu^2} \leq Ct^{-1/2}\ln t.$$

The variable z_1 tends to infinity as $t\to\infty$ if $k \in \partial D_\epsilon(\omega k_4)$, because

$$|z_1| = |z_{1,\star}|\sqrt{t}|k-\omega k_4|\,|\hat{z}_1| \geq c\sqrt{t}.$$

Thus equation (A.3) yields

$$m^{X,(1)}(q_1,q_3,z_1(\zeta,t,k)) = I + \frac{m_1^{X,(1)}(q_1,q_3)}{z_{1,\star}\sqrt{t}(k-\omega k_4)\hat{z}_1(\zeta,k)} + O(t^{-1}), \qquad t\to\infty,$$

uniformly with respect to $k \in \partial D_\epsilon(\omega k_4)$ and $\zeta\in\mathcal{I}$. Recalling the definition (10.5) of $m^{\omega k_4}$, this gives

$$m^{\omega k_4} - I = \frac{Y_1(\zeta,t)m_1^{X,(1)}(q_1,q_3)Y_1(\zeta,t)^{-1}}{z_{1,\star}\sqrt{t}(k-\omega k_4)\hat{z}_1(\zeta,k)} + O(t^{-1}), \qquad t\to\infty, \tag{10.15}$$

uniformly for $k\in\partial D_\epsilon(\omega k_4)$ and $\zeta\in\mathcal{I}$. This proves (10.12) and (10.13). ∎

11. Local parametrix near $\omega^2 k_2$

As $k\to\omega^2 k_2$, we have

$$-(\Phi_{32}(\zeta,k) - \Phi_{32}(\zeta,\omega^2 k_2)) = \Phi_{32,\omega^2 k_2}(k-\omega^2 k_2)^2 + O((k-\omega^2 k_2)^3),$$
$$\Phi_{32,\omega^2 k_2} := -\omega^2\frac{4-3k_2\zeta - k_2^3\zeta}{4k_2^4}.$$

We define $z_2 = z_2(\zeta,t,k)$ by

$$z_2 = z_{2,\star}\sqrt{t}(k-\omega^2 k_2)\hat{z}_2, \qquad \hat{z}_2 := \sqrt{\frac{2i(\Phi_{32}(\zeta,\omega^2 k_2) - \Phi_{32}(\zeta,k))}{z_{2,\star}^2(k-\omega^2 k_2)^2}},$$

where the principal branch is chosen for $\hat{z}_2 = \hat{z}_2(\zeta,k)$, and

$$z_{2,\star} := \sqrt{2}e^{\frac{\pi i}{4}}\sqrt{\Phi_{32,\omega^2 k_2}}, \qquad -i\omega^2 k_2 z_{2,\star} > 0.$$

Note that $\hat{z}_2(\zeta,\omega^2 k_2) = 1$, and $t(\Phi_{32}(\zeta,k) - \Phi_{32}(\zeta,\omega^2 k_2)) = \frac{iz_2^2}{2}$. Provided $\epsilon>0$ is chosen small enough, the map z_2 is conformal from $D_\epsilon(\omega^2 k_2)$ to a neighborhood of 0 and its expansion as $k\to\omega^2 k_2$ is given by

$$z_2 = z_{2,\star}\sqrt{t}(k-\omega^2 k_2)(1+O(k-\omega^2 k_2)).$$

For all $k\in D_\epsilon(\omega^2 k_2)$, we have

$$\ln_{\omega^2 k_2}(k-\omega^2 k_2) = \ln_0[z_{2,\star}(k-\omega^2 k_2)\hat{z}_2] - \ln\hat{z}_2 - \ln z_{2,\star},$$
$$\tilde{\ln}_{\omega^2 k_2}(k-\omega^2 k_2) = \ln[z_{2,\star}(k-\omega^2 k_2)\hat{z}_2] - \ln\hat{z}_2 - \ln z_{2,\star},$$

where $\ln_0(k) := \ln|k| + i\arg_0 k$, $\arg_0(k) \in (0, 2\pi)$, and ln is the principal logarithm. Shrinking $\epsilon > 0$ if necessary, $k \mapsto \ln \hat{z}_2$ is analytic in $D_\epsilon(\omega^2 k_2)$, $\ln \hat{z}_2 = O(k - \omega^2 k_2)$ as $k \to \omega^2 k_2$, and

$$\ln z_{2,\star} = \ln|z_{2,\star}| + i\arg z_{2,\star} = \ln|z_{2,\star}| + i\big(\tfrac{\pi}{2} - \arg(\omega^2 k_2)\big),$$

where $\arg(\omega^2 k_2) \in (\frac{\pi}{2}, \frac{2\pi}{3})$. By Lemma 7.1, we have

$$\begin{aligned}
\delta_2(\zeta,k) &= e^{-i\nu_1 \ln_{\omega k_4}(k-\omega k_4)} e^{-\chi_2(\zeta,k)} t^{-\frac{i\nu_2}{2}} z_{2,(0)}^{i\nu_2} \hat{z}_2^{-i\nu_2} z_{2,\star}^{-i\nu_2},\\
\delta_3(\zeta,k) &= e^{-i\nu_3 \ln_{\omega k_4}(k-\omega k_4)} e^{-\chi_3(\zeta,k)} t^{-\frac{i\nu_4}{2}} z_{2,(0)}^{i\nu_4} \hat{z}_2^{-i\nu_4} z_{2,\star}^{-i\nu_4},\\
\delta_4(\zeta,k) &= t^{\frac{i\nu_4}{2}} z_2^{-i\nu_4} \hat{z}_2^{i\nu_4} z_{2,\star}^{i\nu_4} e^{-\tilde{\chi}_4(\zeta,k)},\\
\delta_5(\zeta,k) &= t^{\frac{i\nu_5}{2}} z_2^{-i\nu_5} \hat{z}_2^{i\nu_5} z_{2,\star}^{i\nu_5} e^{-\tilde{\chi}_5(\zeta,k)},
\end{aligned}$$

where we have used the notation $z_{2,(0)}^a := e^{a\ln_0 z_2}$, $\hat{z}_2^a := e^{a\ln \hat{z}_2}$, $z_{2,\star}^a := e^{a\ln z_{2,\star}}$, and $z_2^a := e^{a\ln z_2}$ for $a \in \mathbb{C}$. We obtain

$$\frac{\Delta_{33}(\zeta,k)}{\Delta_{22}(\zeta,k)} = \frac{\delta_2(\zeta,k)^2\delta_4(\zeta,k)}{\delta_3(\zeta,k)\delta_5(\zeta,k)^2}\mathcal{D}_2(k)\frac{\mathcal{P}(k)}{\mathcal{P}(\omega^2 k)} = d_{2,0}(\zeta,t)d_{2,1}(\zeta,k) z_2^{i(2\nu_5-\nu_4)} z_{2,(0)}^{i(2\nu_2-\nu_4)}, \quad (11.1)$$

where $d_{2,0}(\zeta,t)$ is defined in (2.14) and

$$\begin{aligned}
d_{2,1}(\zeta,k) := {}& e^{-2\chi_2(\zeta,k)+2\chi_2(\zeta,\omega^2 k_2)+\chi_3(\zeta,k)-\chi_3(\zeta,\omega^2 k_2)-\tilde{\chi}_4(\zeta,k)+\tilde{\chi}_4(\zeta,\omega^2 k_2)+2\tilde{\chi}_5(\zeta,k)-2\tilde{\chi}_5(\zeta,\omega^2 k_2)}\\
&\times e^{i(\nu_3-2\nu_1)\left[\ln_{\omega k_4}(k-\omega k_4)-\ln_{\omega k_4}(\omega^2 k_2-\omega k_4)\right]} \hat{z}_2^{2i(\nu_4-\nu_5-\nu_2)} \frac{\mathcal{D}_2(k)}{\mathcal{D}_2(\omega^2 k_2)} \frac{\mathcal{P}(k)}{\mathcal{P}(\omega^2 k)}. \quad (11.2)
\end{aligned}$$

We state in Lemma 11.1 below some properties of $d_{2,0}(\zeta,t)$ and $d_{2,1}(\zeta,k)$. Define

$$Y_2(\zeta,t) = d_{2,0}(\zeta,t)^{\frac{\tilde{\sigma}}{2}} e^{-\frac{t}{2}\Phi_{32}(\zeta,\omega^2 k_2)\tilde{\sigma}} \lambda_2^{\tilde{\sigma}},$$

where $\tilde{\sigma} := \operatorname{diag}(0,1,-1)$ and $\lambda_2 \in \mathbb{C}$ is a parameter that will be chosen below. For $k \in D_\epsilon(\omega^2 k_2)$, define

$$\tilde{n}(x,t,k) = n^{(4)}(x,t,k) Y_2(\zeta,t).$$

Let $\tilde{v}$ be the jump matrix of $\tilde{n}$, and let $\tilde{v}_j$ be $\tilde{v}$ on $\Gamma_j^{(4)} \cap D_\epsilon(\omega^2 k_2)$. By (6.1), (8.1) and (11.1), the matrices $\tilde{v}_j$ can be written as

$$\begin{aligned}
\tilde{v}_7^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -d_{2,1}^{-1}\lambda_2^2\big(\frac{r_1(\frac{1}{\omega k})-r_1(k)r_1(\omega^2 k)}{1+r_1(k)r_2(k)}\big)_a z_2^{-i(2\nu_5-\nu_4)} z_{2,(0)}^{-i(2\nu_2-\nu_4)} e^{\frac{iz_2^2}{2}} & 1 \end{pmatrix},\\
\tilde{v}_9 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & d_{2,1}\lambda_2^{-2}\big(\frac{r_2(\frac{1}{\omega k})-r_2(k)r_2(\omega^2 k)}{f(\omega^2 k)}\big)_a z_2^{i(2\nu_5-\nu_4)} z_{2,(0)}^{i(2\nu_2-\nu_4)} e^{-\frac{iz_2^2}{2}} \\ 0 & 0 & 1 \end{pmatrix},\\
\tilde{v}_{11} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & d_{2,1}^{-1}\lambda_2^2\big(\frac{r_1(\frac{1}{\omega k})-r_1(k)r_1(\omega^2 k)}{f(\omega^2 k)}\big)_a z_2^{-i(2\nu_5-\nu_4)} z_{2,(0)}^{-i(2\nu_2-\nu_4)} e^{\frac{iz_2^2}{2}} & 1 \end{pmatrix},\\
\tilde{v}_8^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -d_{2,1}\lambda_2^{-2}\big(\frac{r_2(\frac{1}{\omega k})-r_2(k)r_2(\omega^2 k)}{1+r_1(k)r_2(k)}\big)_a z_2^{i(2\nu_5-\nu_4)} z_{2,(0)}^{i(2\nu_2-\nu_4)} e^{-\frac{iz_2^2}{2}} \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Let $\mathcal{X}_2^\epsilon := \cup_{j=1}^4 \mathcal{X}_{2,j}^\epsilon$, where

$$\mathcal{X}_{2,1}^\epsilon = \Gamma_7^{(4)} \cap D_\epsilon(\omega^2 k_2), \qquad \mathcal{X}_{2,2}^\epsilon = \Gamma_9^{(4)} \cap D_\epsilon(\omega^2 k_2),$$
$$\mathcal{X}_{2,3}^\epsilon = \Gamma_{11}^{(4)} \cap D_\epsilon(\omega^2 k_2), \qquad \mathcal{X}_{2,4}^\epsilon = \Gamma_8^{(4)} \cap D_\epsilon(\omega^2 k_2),$$

are oriented outwards from $\omega^2 k_2$. Let $k_\star = \omega^2 k_2$. The above expressions for $\tilde{v}_j$ suggest that we approximate $\tilde{n}$ by $(1,1,1)Y_2\tilde{m}^{\omega^2 k_2}$, where $\tilde{m}^{\omega^2 k_2}(x,t,k)$ is analytic for $k \in D_\epsilon(\omega^2 k_2)\setminus\mathcal{X}_2$, obeys the jump relation $\tilde{m}_+^{\omega^2 k_2} = \tilde{m}_-^{\omega^2 k_2}\tilde{v}_{\mathcal{X}_2^\epsilon}^{\omega^2 k_2}$ on $\mathcal{X}_2^\epsilon$ where

$$\tilde{v}_{\mathcal{X}_{2,1}^\epsilon}^{\omega^2 k_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{\mathcal{P}(\omega^2 k_\star)}{\mathcal{P}(k_\star)}\lambda_2^2 \frac{r_1(\frac{1}{\omega k_\star}) - r_1(k_\star)r_1(\omega^2 k_\star)}{1+r_1(k_\star)r_2(k_\star)} z_2^{-i(2\nu_5-\nu_4)} z_{2,(0)}^{-i(2\nu_2-\nu_4)} e^{\frac{iz_2^2}{2}} & 1 \end{pmatrix},$$

$$\tilde{v}_{\mathcal{X}_{2,2}^\epsilon}^{\omega^2 k_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\mathcal{P}(k_\star)}{\mathcal{P}(\omega^2 k_\star)}\lambda_2^{-2} \frac{r_2(\frac{1}{\omega k_\star}) - r_2(k_\star)r_2(\omega^2 k_\star)}{f(\omega^2 k_\star)} z_2^{i(2\nu_5-\nu_4)} z_{2,(0)}^{i(2\nu_2-\nu_4)} e^{-\frac{iz_2^2}{2}} \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_{\mathcal{X}_{2,3}^\epsilon}^{\omega^2 k_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{\mathcal{P}(\omega^2 k_\star)}{\mathcal{P}(k_\star)}\lambda_2^2 \frac{r_1(\frac{1}{\omega k_\star}) - r_1(k_\star)r_1(\omega^2 k_\star)}{f(\omega^2 k_\star)} z_2^{-i(2\nu_5-\nu_4)} z_{2,(0)}^{-i(2\nu_2-\nu_4)} e^{\frac{iz_2^2}{2}} & 1 \end{pmatrix},$$

$$\tilde{v}_{\mathcal{X}_{2,4}^\epsilon}^{\omega^2 k_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{\mathcal{P}(k_\star)}{\mathcal{P}(\omega^2 k_\star)}\lambda_2^{-2} \frac{r_2(\frac{1}{\omega k_\star}) - r_2(k_\star)r_2(\omega^2 k_\star)}{1+r_1(k_\star)r_2(k_\star)} z_2^{i(2\nu_5-\nu_4)} z_{2,(0)}^{i(2\nu_2-\nu_4)} e^{-\frac{iz_2^2}{2}} \\ 0 & 0 & 1 \end{pmatrix},$$

and satisfies $\tilde{m}^{\omega^2 k_2} \to I$ as $k \to \infty$. We choose our free parameter λ_2 as follows:

$$\lambda_2 = \Big(\frac{\mathcal{P}(k_\star)}{\mathcal{P}(\omega^2 k_\star)}\Big)^{\frac{1}{2}} |\tilde{r}(\tfrac{1}{\omega k_\star})|^{\frac{1}{4}} = \Big(\frac{\mathcal{P}(\omega^2 k_2)}{\mathcal{P}(\omega k_2)}\Big)^{\frac{1}{2}} |\tilde{r}(\tfrac{1}{k_2})|^{\frac{1}{4}}, \tag{11.3}$$

and define the local parametrix $m^{\omega^2 k_2}$ by

$$m^{\omega^2 k_2}(x,t,k) = Y_2(\zeta,t) m^{X,(2)}(q_2,q_4,q_5,q_6,z_2(\zeta,t,k)) Y_2(\zeta,t)^{-1}, \qquad k \in D_\epsilon(\omega^2 k_2), \tag{11.4}$$

where $m^{X,(2)}(\cdot) = m^{X,(2)}(q_2,q_4,q_5,q_6,\cdot)$ is the solution of the model RH problem of Lemma A.2 with

$$q_2 = \tilde{r}(k_\star)^{\frac{1}{2}} r_1(k_\star), \qquad \bar{q}_2 = \tilde{r}(k_\star)^{-\frac{1}{2}} r_2(k_\star) = \tilde{r}(k_\star)^{\frac{1}{2}} \overline{r_1(k_\star)},$$
$$q_4 = |\tilde{r}(\tfrac{1}{\omega^2 k_\star})|^{\frac{1}{2}} r_1(\tfrac{1}{\omega^2 k_\star}), \qquad \bar{q}_4 = -|\tilde{r}(\tfrac{1}{\omega^2 k_\star})|^{-\frac{1}{2}} r_2(\tfrac{1}{\omega^2 k_\star}) = |\tilde{r}(\tfrac{1}{\omega^2 k_\star})|^{\frac{1}{2}} \overline{r_1(\tfrac{1}{\omega^2 k_\star})},$$
$$q_5 = |\tilde{r}(\omega^2 k_\star)|^{\frac{1}{2}} r_1(\omega^2 k_\star), \qquad \bar{q}_5 = -|\tilde{r}(\omega^2 k_\star)|^{-\frac{1}{2}} r_2(\omega^2 k_\star) = |\tilde{r}(\omega^2 k_\star)|^{\frac{1}{2}} \overline{r_1(\omega^2 k_\star)},$$
$$q_6 = |\tilde{r}(\tfrac{1}{\omega k_\star})|^{\frac{1}{2}} r_1(\tfrac{1}{\omega k_\star}), \qquad \bar{q}_6 = -|\tilde{r}(\tfrac{1}{\omega k_\star})|^{-\frac{1}{2}} r_2(\tfrac{1}{\omega k_\star}) = |\tilde{r}(\tfrac{1}{\omega k_\star})|^{\frac{1}{2}} \overline{r_1(\tfrac{1}{\omega k_\star})}.$$

Using the relation (4.6) together with the fact that

$$\big|\tilde{r}(\tfrac{1}{\omega^2 k})\big|^{-\frac{1}{2}} = \big|\tilde{r}(\omega^2 k)\big|^{\frac{1}{2}} = |\tilde{r}(k)|^{-\frac{1}{2}}\big|\tilde{r}(\tfrac{1}{\omega k})\big|^{\frac{1}{2}}, \qquad k \in \mathbb{C}\setminus\{-1,1\},$$

we infer that $q_4 - \bar{q}_5 - q_2\bar{q}_6 = 0$, so that (A.5) is fulfilled as it must. The proof of the following lemma is similar to the proof of Lemma 10.1.

Lemma 11.1. *The function $Y_2(\zeta,t)$ is uniformly bounded, i.e., $\sup_{\zeta\in\mathcal{I}}\sup_{t\geq 2}|Y_2(\zeta,t)^{\pm 1}| < C$. Moreover, the functions $d_{2,0}(\zeta,t)$ and $d_{2,1}(\zeta,k)$ satisfy*

$$|d_{2,0}(\zeta,t)| = e^{\pi(2\nu_2-\nu_4)}, \qquad \zeta\in\mathcal{I},\ t\geq 2, \tag{11.5}$$

$$\Big|d_{2,1}(\zeta,k) - \frac{\mathcal{P}(\omega^2 k_2)}{\mathcal{P}(\omega k_2)}\Big| \leq C|k-\omega^2 k_2|(1+|\ln|k-\omega^2 k_2||), \qquad \zeta\in\mathcal{I},\ k\in\mathcal{X}_2^\epsilon. \tag{11.6}$$

The next lemma follows from the same kind of arguments used to prove Lemma 10.2.

Lemma 11.2. *The function $m^{\omega^2 k_2}(x,t,k)$ defined in (11.4) is an analytic function of $k \in D_\epsilon(\omega^2 k_2) \setminus \mathcal{X}_2^\epsilon$ which is uniformly bounded for $t \geq 2$, $\zeta \in \mathcal{I}$, and $k \in D_\epsilon(\omega^2 k_2) \setminus \mathcal{X}_2^\epsilon$. Across $\mathcal{X}_2^\epsilon$, $m^{\omega^2 k_2}$ obeys the jump condition $m_+^{\omega^2 k_2} = m_-^{\omega^2 k_2} v^{\omega^2 k_2}$, where $v^{\omega^2 k_2}$ satisfies*

$$\begin{cases} \|v^{(4)} - v^{\omega^2 k_2}\|_{L^1(\mathcal{X}_2^\epsilon)} \leq Ct^{-1}\ln t, \\ \|v^{(4)} - v^{\omega^2 k_2}\|_{L^\infty(\mathcal{X}_2^\epsilon)} \leq Ct^{-1/2}\ln t, \end{cases} \qquad \zeta \in \mathcal{I},\ t \geq 2. \tag{11.7}$$

Furthermore, as $t \to \infty$,

$$\|m^{\omega^2 k_2}(x,t,\cdot)^{-1} - I\|_{L^\infty(\partial D_\epsilon(\omega^2 k_2))} = O(t^{-1/2}), \tag{11.8}$$

$$m^{\omega^2 k_2} - I = \frac{Y_2(\zeta,t) m_1^{X,(2)}(q_2,q_4,q_5,q_6) Y_2(\zeta,t)^{-1}}{z_{2,\star}\sqrt{t}(k-\omega^2 k_2)\hat{z}_2(\zeta,k)} + O(t^{-1}) \tag{11.9}$$

uniformly for $\zeta \in \mathcal{I}$, where $\partial D_\epsilon(\omega^2 k_2)$ is oriented clockwise, and $m_1^{X,(2)}$ is given by (A.7).

12. The $n^{(4)} \to \hat{n}$ transformation

We use the symmetries

$$m^{\omega k_4}(x,t,k) = \mathcal{A} m^{\omega k_4}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B} m^{\omega k_4}(x,t,k^{-1})\mathcal{B},$$
$$m^{\omega^2 k_2}(x,t,k) = \mathcal{A} m^{\omega^2 k_2}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B} m^{\omega^2 k_2}(x,t,k^{-1})\mathcal{B},$$

to extend the domains of definition of $m^{\omega k_4}$ and $m^{\omega^2 k_2}$ from $D_\epsilon(\omega k_4)$ and $D_\epsilon(\omega^2 k_2)$ to $\mathcal{D}_{\omega k_4}$ and $\mathcal{D}_{\omega^2 k_2}$, respectively, where

$$\mathcal{D}_{\omega k_4} := D_\epsilon(\omega k_4) \cup \omega D_\epsilon(\omega k_4) \cup \omega^2 D_\epsilon(\omega k_4) \cup D_\epsilon(\tfrac{1}{\omega k_4}) \cup \omega D_\epsilon(\tfrac{1}{\omega k_4}) \cup \omega^2 D_\epsilon(\tfrac{1}{\omega k_4}),$$
$$\mathcal{D}_{\omega^2 k_2} := D_\epsilon(\omega^2 k_2) \cup \omega D_\epsilon(\omega^2 k_2) \cup \omega^2 D_\epsilon(\omega^2 k_2) \cup D_\epsilon(\tfrac{1}{\omega^2 k_2}) \cup \omega D_\epsilon(\tfrac{1}{\omega^2 k_2}) \cup \omega^2 D_\epsilon(\tfrac{1}{\omega^2 k_2}).$$

We will show that the solution $\hat{n}(x,t,k)$ defined by

$$\hat{n} = \begin{cases} n^{(4)}(m^{\omega k_4})^{-1}, & k \in \mathcal{D}_{\omega k_4}, \\ n^{(4)}(m^{\omega^2 k_2})^{-1}, & k \in \mathcal{D}_{\omega^2 k_2}, \\ n^{(4)}, & \text{elsewhere}, \end{cases} \tag{12.1}$$

is small for large t. Let $\hat{\Gamma} = \Gamma^{(4)} \cup \partial\mathcal{D}$ with $\mathcal{D} := \mathcal{D}_{\omega k_4} \cup \mathcal{D}_{\omega^2 k_2}$ be the contour displayed in Figure 9, where each circle that is part of $\partial\mathcal{D}$ is oriented negatively, and define the jump matrix $\hat{v}$ by

$$\hat{v} = \begin{cases} v^{(4)}, & k \in \hat{\Gamma} \setminus \bar{\mathcal{D}}, \\ m^{\omega k_4}, & k \in \partial\mathcal{D}_{\omega k_4}, \\ m^{\omega^2 k_2}, & k \in \partial\mathcal{D}_{\omega^2 k_2}, \\ m_-^{\omega k_4} v^{(4)} (m_+^{\omega k_4})^{-1}, & k \in \hat{\Gamma} \cap \mathcal{D}_{\omega k_4}, \\ m_-^{\omega^2 k_2} v^{(4)} (m_+^{\omega^2 k_2})^{-1}, & k \in \hat{\Gamma} \cap \mathcal{D}_{\omega^2 k_2}. \end{cases} \tag{12.2}$$

The function $\hat{n}$ is analytic on $\mathbb{C} \setminus \hat{\Gamma}$ and satisfies $\hat{n}_+ = \hat{n}_- \hat{v}$ for $k \in \hat{\Gamma} \setminus \hat{\Gamma}_\star$, where $\hat{\Gamma}_\star$ denotes the points of self-intersection of $\hat{\Gamma}$. As $k \to \infty$, $\hat{n}(x,t,k) = (1,1,1) + O(k^{-1})$.

Let $\hat{\mathcal{X}}^\epsilon = \bigcup_{j=1}^2 \left(\mathcal{X}_j^\epsilon \cup \omega\mathcal{X}_j^\epsilon \cup \omega^2\mathcal{X}_j^\epsilon \cup (\mathcal{X}_j^\epsilon)^{-1} \cup (\omega\mathcal{X}_j^\epsilon)^{-1} \cup (\omega^2\mathcal{X}_j^\epsilon)^{-1}\right)$.

Lemma 12.1. *Let $\hat{w} = \hat{v} - I$. The following estimates hold uniformly for $t \geq 2$ and $\zeta \in \mathcal{I}$:*

$$\|\hat{w}\|_{(L^1 \cap L^\infty)(\hat{\Gamma} \setminus (\partial\mathcal{D} \cup \hat{\mathcal{X}}^\epsilon))} \leq Ct^{-1}, \tag{12.3a}$$

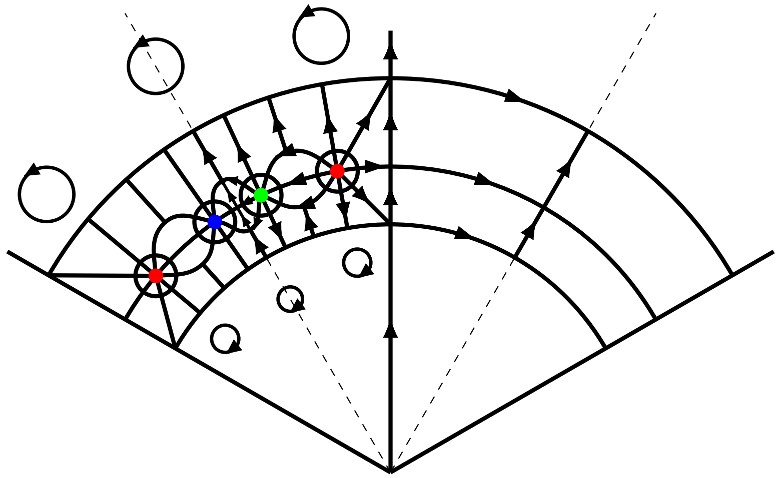

Figure 9. *The contour* $\hat{\Gamma} = \Gamma^{(4)} \cup \partial\mathcal{D}$ *for* $\arg k \in [\frac{\pi}{6}, \frac{5\pi}{6}]$ *(solid) and the boundary of* S *(dashed).*

$$\|\hat{w}\|_{(L^1\cap L^\infty)(\partial\mathcal{D})} \le Ct^{-1/2}, \tag{12.3b}$$

$$\|\hat{w}\|_{L^1(\hat{\mathcal{X}}^\epsilon)} \le Ct^{-1}\ln t, \tag{12.3c}$$

$$\|\hat{w}\|_{L^\infty(\hat{\mathcal{X}}^\epsilon)} \le Ct^{-1/2}\ln t. \tag{12.3d}$$

Proof. On $\hat{\Gamma} \setminus \bar{\mathcal{D}}$, the estimate (12.3a) follows from (12.2) and Lemmas 8.1 and 9.1; on $(\hat{\Gamma} \cap \mathcal{D}) \setminus \hat{\mathcal{X}}^\epsilon$, it follows from (12.2) and Lemmas 6.3, 7.3, 10.2, and 11.2. The estimate (12.3b) follows from (12.2), (10.12), and (11.8). Finally, (12.3c) and (12.3d) follow from (12.2), (10.11), and (11.7). $\square$

The estimates in Lemma 12.1 show that

$$\begin{cases} \|\hat{w}\|_{L^1(\hat{\Gamma})} \le Ct^{-1/2}, \\ \|\hat{w}\|_{L^\infty(\hat{\Gamma})} \le Ct^{-1/2}\ln t, \end{cases} \qquad t \ge 2,\ \zeta \in \mathcal{I}, \tag{12.4}$$

and hence, employing the general identity $\|f\|_{L^p} \le \|f\|_{L^1}^{1/p}\|f\|_{L^\infty}^{(p-1)/p}$,

$$\|\hat{w}\|_{L^p(\hat{\Gamma})} \le Ct^{-\frac{1}{2}}(\ln t)^{\frac{p-1}{p}}, \qquad t \ge 2,\ \zeta \in \mathcal{I}, \tag{12.5}$$

for each $1 \le p \le \infty$. The estimates (12.5) imply that $\hat{w} \in (L^2 \cap L^\infty)(\hat{\Gamma})$ and that there is a T such that $I - \hat{\mathcal{C}}_{\hat{w}(x,t,\cdot)} \in \mathcal{B}(L^2(\hat{\Gamma}))$ is invertible whenever $\zeta \in \mathcal{I}$ and $t \ge T$. So standard RH theory gives

$$\hat{n}(x,t,k) = (1,1,1) + \hat{\mathcal{C}}(\hat{\mu}\hat{w}) = (1,1,1) + \frac{1}{2\pi i}\int_{\hat{\Gamma}} \hat{\mu}(x,t,s)\hat{w}(x,t,s)\frac{ds}{s-k} \tag{12.6}$$

for $\zeta \in \mathcal{I}$ and $t \ge T$, where

$$\hat{\mu} := (1,1,1) + (I - \hat{\mathcal{C}}_{\hat{w}})^{-1}\hat{\mathcal{C}}_{\hat{w}}(1,1,1) \in (1,1,1) + L^2(\hat{\Gamma}). \tag{12.7}$$

Lemma 12.2. *Let* $1 < p < \infty$. *For all sufficiently large* t, *we have*

$$\|\hat{\mu} - (1,1,1)\|_{L^p(\hat{\Gamma})} \le Ct^{-\frac{1}{2}}(\ln t)^{\frac{p-1}{p}}, \qquad \zeta \in \mathcal{I}.$$

Proof. Suppose t is sufficiently large so that $\|\hat{w}\|_{L^\infty(\hat{\Gamma})} < K_p^{-1}$, where $K_p := \|\hat{\mathcal{C}}_-\|_{\mathcal{B}(L^p(\hat{\Gamma}))}$ is finite because $p > 1$. The estimate (12.5) implies that $\hat{w} \in L^p(\hat{\Gamma})$, and so $\hat{\mathcal{C}}_{\hat{w}}(1,1,1) \in L^p(\hat{\Gamma})$.

Thus

$$\begin{aligned}\|\hat{\mu}-(1,1,1)\|_{L^p(\hat{\Gamma})} &\leq \sum_{j=1}^{\infty}\|\hat{\mathcal{C}}_{\hat{w}}\|_{\mathcal{B}(L^p(\hat{\Gamma}))}^{j-1}\|\hat{\mathcal{C}}_{\hat{w}}(1,1,1)\|_{L^p(\hat{\Gamma})} \leq \sum_{j=1}^{\infty} K_p^j\|\hat{w}\|_{L^\infty(\hat{\Gamma})}^{j-1}\|\hat{w}\|_{L^p(\hat{\Gamma})} \\ &= \frac{K_p\|\hat{w}\|_{L^p(\hat{\Gamma})}}{1-K_p\|\hat{w}\|_{L^\infty(\hat{\Gamma})}}.\end{aligned}$$

The claim is now a consequence of (12.4) and (12.5). □

We now turn to the asymptotics of $\hat{n}$. The following limit exists as $k\to\infty$:

$$\hat{n}^{(1)}(x,t) := \lim_{k\to\infty} k(\hat{n}(x,t,k)-(1,1,1)) = -\frac{1}{2\pi i}\int_{\hat{\Gamma}}\hat{\mu}(x,t,k)\hat{w}(x,t,k)dk.$$

Lemma 12.3. *As $t\to\infty$,*

$$\hat{n}^{(1)}(x,t) = -\frac{(1,1,1)}{2\pi i}\int_{\partial\mathcal{D}}\hat{w}(x,t,k)dk + O(t^{-1}\ln t). \tag{12.8}$$

Proof. Since

$$\begin{aligned}\hat{n}^{(1)}(x,t) = &-\frac{(1,1,1)}{2\pi i}\int_{\partial\mathcal{D}}\hat{w}(x,t,k)dk - \frac{(1,1,1)}{2\pi i}\int_{\hat{\Gamma}\setminus\partial\mathcal{D}}\hat{w}(x,t,k)dk \\ &- \frac{1}{2\pi i}\int_{\hat{\Gamma}}(\hat{\mu}(x,t,k)-(1,1,1))\hat{w}(x,t,k)dk,\end{aligned}$$

the lemma follows from Lemmas 12.1 and 12.2 and straightforward estimates. □

We define the functions $\{F_1^{(l)}, F_2^{(l)}\}_{l\in\mathbb{Z}}$ by

$$\begin{aligned}F_1^{(l)}(\zeta,t) &= -\frac{1}{2\pi i}\int_{\partial D_\epsilon(\omega k_4)} k^{l-1}\hat{w}(x,t,k)dk = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(\omega k_4)} k^{l-1}(m^{\omega k_4}-I)dk, \\ F_2^{(l)}(\zeta,t) &= -\frac{1}{2\pi i}\int_{\partial D_\epsilon(\omega^2 k_2)} k^{l-1}\hat{w}(x,t,k)dk = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(\omega^2 k_2)} k^{l-1}(m^{\omega^2 k_2}-I)dk.\end{aligned}$$

We infer from (10.13) and (11.9) that $F_1^{(l)}$ and $F_2^{(l)}$ satisfy (recall that $\hat{z}_1(\zeta,\omega k_4)=1=\hat{z}_2(\zeta,\omega^2k_2)$ and that $\partial D_\epsilon(\omega k_4)$ and $\partial D_\epsilon(\omega^2 k_2)$ are oriented negatively)

$$\begin{aligned}F_1^{(l)}(\zeta,t) &= (\omega k_4)^{l-1}\frac{Y_1(\zeta,t)m_1^{X,(1)}(q_1,q_3)Y_1(\zeta,t)^{-1}}{z_{1,\star}\sqrt{t}} + O(t^{-1}) \\ &= -i(\omega k_4)^l Z_1(\zeta,t) + O(t^{-1}) \qquad \text{as } t\to\infty, \end{aligned} \tag{12.9}$$

$$\begin{aligned}F_2^{(l)}(\zeta,t) &= (\omega^2 k_2)^{l-1}\frac{Y_2(\zeta,t)m_1^{X,(2)}(q_2,q_4,q_5,q_6)Y_2(\zeta,t)^{-1}}{z_{2,\star}\sqrt{t}} + O(t^{-1}) \\ &= -i(\omega^2 k_2)^l Z_2(\zeta,t) + O(t^{-1}) \qquad \text{as } t\to\infty,\end{aligned} \tag{12.10}$$

where

$$Z_1(\zeta,t) := \frac{Y_1 m_1^{X,(1)}Y_1^{-1}}{-i\omega k_4 z_{1,\star}\sqrt{t}} = \frac{1}{-i\omega k_4 z_{1,\star}\sqrt{t}}\begin{pmatrix} 0 & 0 & \frac{\beta_{12}^{(1)}d_{1,0}\lambda_1^2}{e^{t\Phi_{31}(\zeta,\omega k_4)}} \\ 0 & 0 & 0 \\ \frac{\beta_{21}^{(1)}e^{t\Phi_{31}(\zeta,\omega k_4)}}{d_{1,0}\lambda_1^2} & 0 & 0\end{pmatrix}, \tag{12.11a}$$

$$Z_2(\zeta,t) := \frac{Y_2 m_1^{X,(2)}Y_2^{-1}}{-i\omega^2 k_2 z_{2,\star}\sqrt{t}} = \frac{1}{-i\omega^2 k_2 z_{2,\star}\sqrt{t}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\beta_{12}^{(2)}d_{2,0}\lambda_2^2}{e^{t\Phi_{32}(\zeta,\omega^2 k_2)}} \\ 0 & \frac{\beta_{21}^{(2)}e^{t\Phi_{32}(\zeta,\omega^2 k_2)}}{d_{2,0}\lambda_2^2} & 0\end{pmatrix}, \tag{12.11b}$$

and

$$\beta_{12}^{(1)} = \frac{e^{\frac{3\pi i}{4}} e^{\frac{\pi\hat{\nu}_1}{2}} e^{2\pi\nu_1}\sqrt{2\pi} q_3}{(e^{\pi\hat{\nu}_1} - e^{-\pi\hat{\nu}_1})\Gamma(-i\hat{\nu}_1)}, \qquad \beta_{21}^{(1)} = \frac{e^{-\frac{3\pi i}{4}} e^{\frac{\pi\hat{\nu}_1}{2}} \sqrt{2\pi}\bar{q}_3}{(e^{\pi\hat{\nu}_1} - e^{-\pi\hat{\nu}_1})\Gamma(i\hat{\nu}_1)}, \tag{12.12a}$$

$$\beta_{12}^{(2)} = \frac{e^{\frac{3\pi i}{4}} e^{\frac{\pi\hat{\nu}_2}{2}} e^{2\pi(\nu_4-\nu_2)}\sqrt{2\pi}(\bar{q}_6 - \bar{q}_2\bar{q}_5)}{(e^{\pi\hat{\nu}_2} - e^{-\pi\hat{\nu}_2})\Gamma(-i\hat{\nu}_2)}, \qquad \beta_{21}^{(2)} = \frac{e^{-\frac{3\pi i}{4}} e^{\frac{\pi\hat{\nu}_2}{2}} e^{2\pi\nu_2}\sqrt{2\pi}(q_6 - q_2 q_5)}{(e^{\pi\hat{\nu}_2} - e^{-\pi\hat{\nu}_2})\Gamma(i\hat{\nu}_2)}, \tag{12.12b}$$

and ν_1, $\hat{\nu}_1$, ν_2, $\hat{\nu}_2$, ν_4 have been defined in Lemmas 7.1 and 7.2.

Lemma 12.4. *For $l \in \mathbb{Z}$ and $j = 0, 1, 2$, we have*

$$-\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon(\omega k_4)} k^l \hat{w}(x,t,k)dk = \omega^{j(l+1)}\mathcal{A}^{-j}F_1^{(l+1)}(\zeta,t)\mathcal{A}^j, \tag{12.13}$$

$$-\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon((\omega k_4)^{-1})} k^l \hat{w}(x,t,k)dk = -\omega^{j(l+1)}\mathcal{A}^{-j}\mathcal{B}F_1^{(-l-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j, \tag{12.14}$$

$$-\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon(\omega^2 k_2)} k^l \hat{w}(x,t,k)dk = \omega^{j(l+1)}\mathcal{A}^{-j}F_2^{(l+1)}(\zeta,t)\mathcal{A}^j, \tag{12.15}$$

$$-\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon((\omega^2 k_2)^{-1})} k^l \hat{w}(x,t,k)dk = -\omega^{j(l+1)}\mathcal{A}^{-j}\mathcal{B}F_2^{(-l-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j. \tag{12.16}$$

Proof. The symmetry properties of $\hat{v}$ imply that

$$\hat{w}(x,t,k) = \mathcal{A}\hat{w}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}\hat{w}(x,t,k^{-1})\mathcal{B}, \qquad k \in \partial\mathcal{D}.$$

Therefore, for $j = 0, 1, 2$ we have

$$\begin{aligned}-\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon(\omega k_4)} k^l \hat{w}(x,t,k)dk &= -\frac{\omega^{j(l+1)}}{2\pi i}\int_{\partial D_\epsilon(\omega k_4)} k^l \hat{w}(x,t,\omega^j k)dk \\ &= \omega^{j(l+1)}\mathcal{A}^{-j}F_1^{(l+1)}(\zeta,t)\mathcal{A}^j,\end{aligned}$$

and

$$\begin{aligned}&-\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon((\omega k_4)^{-1})} k^l \hat{w}(x,t,k)dk = -\frac{\omega^{j(l+1)}}{2\pi i}\int_{\partial D_\epsilon((\omega k_4)^{-1})} k^l \hat{w}(x,t,\omega^j k)dk \\ &= -\frac{\omega^{j(l+1)}\mathcal{A}^{-j}}{2\pi i}\int_{\partial D_\epsilon((\omega k_4)^{-1})} k^l \hat{w}(x,t,k)dk\mathcal{A}^j \\ &= -\frac{\omega^{j(l+1)}\mathcal{A}^{-j}}{2\pi i}\int_{\partial D_\epsilon(\omega k_4)} k^{-l}\hat{w}(x,t,k^{-1})\frac{dk}{-k^2}\mathcal{A}^j \\ &= \frac{\omega^{j(l+1)}\mathcal{A}^{-j}\mathcal{B}}{2\pi i}\int_{\partial D_\epsilon(\omega k_4)} k^{-l-2}\hat{w}(x,t,k)dk\mathcal{B}\mathcal{A}^j = -\omega^{j(l+1)}\mathcal{A}^{-j}\mathcal{B}F_1^{(-l-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j.\end{aligned}$$

This proves (12.13) and (12.14). The proofs of (12.15) and (12.16) are similar. ∎

Using Lemma 12.4, we find

$$\begin{aligned}-\frac{1}{2\pi i}\int_{\partial\mathcal{D}} \hat{w}(x,t,k)dk = &-\sum_{j=0}^{2}\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon(\omega k_4)} \hat{w}(x,t,k)dk - \sum_{j=0}^{2}\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon((\omega k_4)^{-1})} \hat{w}(x,t,k)dk \\ &-\sum_{j=0}^{2}\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon(\omega^2 k_2)} \hat{w}(x,t,k)dk - \sum_{j=0}^{2}\frac{1}{2\pi i}\int_{\omega^j \partial D_\epsilon((\omega^2 k_2)^{-1})} \hat{w}(x,t,k)dk\end{aligned}$$

$$= \sum_{j=0}^{2} \omega^j \mathcal{A}^{-j} F_1^{(1)}(\zeta,t)\mathcal{A}^j - \sum_{j=0}^{2} \omega^j \mathcal{A}^{-j} \mathcal{B} F_1^{(-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j$$
$$+ \sum_{j=0}^{2} \omega^j \mathcal{A}^{-j} F_2^{(1)}(\zeta,t)\mathcal{A}^j - \sum_{j=0}^{2} \omega^j \mathcal{A}^{-j} \mathcal{B} F_2^{(-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j.$$

Therefore, (12.8), (12.9), and (12.10) imply that $\hat{n}^{(1)} = (1,1,1)\hat{m}^{(1)}$, where

$$\begin{aligned}\hat{m}^{(1)}(x,t) = & \sum_{j=0}^{2} \omega^j \mathcal{A}^{-j} F_1^{(1)}(\zeta,t)\mathcal{A}^j - \sum_{j=0}^{2} \omega^j \mathcal{A}^{-j} \mathcal{B} F_1^{(-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j \\
& + \sum_{j=0}^{2} \omega^j \mathcal{A}^{-j} F_2^{(1)}(\zeta,t)\mathcal{A}^j - \sum_{j=0}^{2} \omega^j \mathcal{A}^{-j} \mathcal{B} F_2^{(-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j + O(t^{-1}\ln t) \\
& = -i(\omega k_4)^1 \sum_{j=0}^{2} \omega^j \mathcal{A}^{-j} Z_1(\zeta,t)\mathcal{A}^j + i(\omega k_4)^{-1} \sum_{j=0}^{2} \omega^j \mathcal{A}^{-j} \mathcal{B} Z_1(\zeta,t)\mathcal{B}\mathcal{A}^j \\
& - i(\omega^2 k_2)^1 \sum_{j=0}^{2} \omega^j \mathcal{A}^{-j} Z_2(\zeta,t)\mathcal{A}^j + i(\omega^2 k_2)^{-1} \sum_{j=0}^{2} \omega^j \mathcal{A}^{-j} \mathcal{B} Z_2(\zeta,t)\mathcal{B}\mathcal{A}^j + O(t^{-1}\ln t)\end{aligned} \tag{12.17}$$

as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}$.

13. Asymptotics of u

Recall from (4.1) that $u(x,t) = -i\sqrt{3}\frac{\partial}{\partial x} n_3^{(1)}(x,t)$, where $n_3(x,t,k) = 1 + n_3^{(1)}(x,t)k^{-1} + O(k^{-2})$ as $k \to \infty$. Recall also that $n^{(4)}$ is defined to equal $n^{(3)}$ except in $\mathcal{D}_{\mathrm{sol}}$. Taking also the transformations (5.8), (6.2), (8.1), and (12.1) into account, we obtain

$$n = \hat{n}\Delta^{-1}(G^{(2)})^{-1}(G^{(1)})^{-1}$$

for $k \in \mathbb{C} \setminus (\hat{\Gamma} \cup \bar{\mathcal{D}} \cup \bar{\mathcal{D}}_{\mathrm{sol}})$, where $G^{(1)}$, $G^{(2)}$, Δ are defined in (5.6), (6.3), and (7.6), respectively. Thus

$$u(x,t) = -i\sqrt{3}\frac{\partial}{\partial x}\Big(\hat{n}_3^{(1)}(x,t) + \lim_{k\to\infty} k(\Delta_{33}(\zeta,k)^{-1} - 1)\Big) = -i\sqrt{3}\frac{\partial}{\partial x}\hat{n}_3^{(1)}(x,t) + O(t^{-1}) \tag{13.1}$$

as $t \to \infty$. By (12.17), as $t \to \infty$,

$$\begin{aligned}\hat{n}_3^{(1)} = & Z_{1,13}(i(\omega k_4)^{-1} - i(\omega k_4)^1) + Z_{1,31}(i\omega^2(\omega k_4)^{-1} - i\omega^1(\omega k_4)^1) \\
& + Z_{2,23}(i(\omega^2 k_2)^{-1} - i(\omega^2 k_2)^1) + Z_{2,32}(i\omega^1(\omega^2 k_2)^{-1} - i\omega^2(\omega^2 k_2)^1) + O(t^{-1}\ln t).\end{aligned}$$

Recall from Lemma 7.2 that $\hat{\nu}_1 = \nu_3 - \nu_1 \geq 0$ and $\hat{\nu}_2 = \nu_2 + \nu_5 - \nu_4 \geq 0$. From (10.7), (11.5), (12.11), and (12.12), we see that $\overline{Z_{1,13}} = \lambda_1^4 Z_{1,31}$ and $\overline{Z_{2,23}} = \lambda_2^4 Z_{2,32}$. Moreover, by (10.4) and (11.3), we have $\lambda_1^4 = -\tilde{r}(k_4^{-1})^{-1}\big(\frac{\mathcal{P}(\omega k_4)}{\mathcal{P}(\omega^2 k_4)}\big)^2$ and $\lambda_2^4 = -\tilde{r}(k_2^{-1})\big(\frac{\mathcal{P}(\omega^2 k_2)}{\mathcal{P}(\omega k_2)}\big)^2$. It follows that, as $t \to \infty$,

$$\begin{aligned}\hat{n}_3^{(1)} &= 2i\,\mathrm{Im}\Big(Z_{1,13}(i(\omega k_4)^{-1} - i(\omega k_4)^1) + Z_{2,23}(i(\omega^2 k_2)^{-1} - i(\omega^2 k_2)^1)\Big) + O(t^{-1}\ln t) \\
&= 2i\,\mathrm{Im}\Bigg[\frac{\beta_{12}^{(1)} d_{1,0}(i(\omega k_4)^{-1} - i(\omega k_4)^1)\mathcal{P}(\omega k_4)}{-i\omega k_4 z_{1,\star}\sqrt{t}e^{t\Phi_{31}(\zeta,\omega k_4)}\big|\tilde{r}(\frac{1}{k_4})\big|^{\frac{1}{2}}\mathcal{P}(\omega^2 k_4)}\end{aligned}$$

$$+\frac{\beta_{12}^{(2)}d_{2,0}(i(\omega^2k_2)^{-1}-i(\omega^2k_2)^1)\mathcal{P}(\omega^2k_2)}{-i\omega^2k_2z_{2,\star}\sqrt{t}e^{t\Phi_{32}(\zeta,\omega^2k_2)}\big|\tilde{r}(\frac{1}{k_2})\big|^{-\frac{1}{2}}\mathcal{P}(\omega k_2)}\Bigg]+O(t^{-1}\ln t).$$

The function $\mathcal{P}(k)$ obeys the symmetries $\mathcal{P}(k)=\mathcal{P}(k^{-1})=\overline{\mathcal{P}(\bar{k})}^{-1}$. It follows that $|\mathcal{P}(k)|=1$ for $k\in\partial\mathbb{D}$, and thus

$$\big|\tfrac{\mathcal{P}(\omega k_4)}{\mathcal{P}(\omega^2k_4)}\big|=\big|\tfrac{\mathcal{P}(\omega^2k_2)}{\mathcal{P}(\omega k_2)}\big|=1. \tag{13.2}$$

Using (12.12), (13.2), and the identities

$$d_{1,0}(\zeta,t)=e^{-\pi\nu_1}e^{i\arg d_{1,0}(\zeta,t)}, \qquad d_{2,0}(\zeta,t)=e^{\pi(2\nu_2-\nu_4)}e^{i\arg d_{2,0}(\zeta,t)},$$
$$i(\omega k_4)^{-1}-i(\omega k_4)^1=2\sin(\arg(\omega k_4(\zeta))), \quad i(\omega^2k_2)^{-1}-i(\omega^2k_2)^1=2\sin(\arg(\omega^2k_2(\zeta))),$$
$$|\Gamma(i\nu)|=\frac{\sqrt{2\pi}}{\sqrt{\nu(e^{\pi\nu}-e^{-\pi\nu})}}, \quad \nu\in\mathbb{R}, \qquad q_4-\bar{q}_5-q_2\bar{q}_6=0,$$

we obtain

$$\begin{aligned}\hat{n}_3^{(1)}&=\frac{4i\sqrt{\hat{\nu}_1}}{-i\omega k_4z_{1,\star}\sqrt{t}|\tilde{r}(\frac{1}{k_4})|^{\frac{1}{2}}}\sin(\arg(\omega k_4))\\&\times\sin(\tfrac{3\pi}{4}+\arg q_3+\arg\Gamma(i\hat{\nu}_1)+\arg d_{1,0}+\arg\tfrac{\mathcal{P}(\omega k_4)}{\mathcal{P}(\omega^2k_4)}-t\mathrm{Im}\,\Phi_{31}(\zeta,\omega k_4(\zeta)))\\&+\frac{4i\sqrt{\hat{\nu}_2}|\tilde{r}(\frac{1}{k_2})|^{\frac{1}{2}}}{-i\omega^2k_2z_{2,\star}\sqrt{t}}\sin(\arg(\omega^2k_2))\\&\times\sin(\tfrac{3\pi}{4}-\arg(q_6-q_2q_5)+\arg\Gamma(i\hat{\nu}_2)+\arg d_{2,0}+\arg\tfrac{\mathcal{P}(\omega^2k_2)}{\mathcal{P}(\omega k_2)}-t\mathrm{Im}\,\Phi_{32}(\zeta,\omega^2k_2(\zeta)))\\&+O(t^{-1}\ln t)\quad\text{as } t\to\infty.\end{aligned}$$

As in [8], it is possible to show that the above asymptotics can be differentiated with respect to x without increasing the error term. Therefore, as $t\to\infty$,

$$\begin{aligned}u(x,t)&=-i\sqrt{3}\frac{\partial}{\partial x}\hat{n}_3^{(1)}(x,t)+O(t^{-1})=\frac{-4\sqrt{3}\sqrt{\hat{\nu}_1}\frac{d}{d\zeta}\mathrm{Im}\,\Phi_{31}(\zeta,\omega k_4(\zeta))}{-i\omega k_4z_{1,\star}\sqrt{t}|\tilde{r}(\frac{1}{k_4})|^{\frac{1}{2}}}\sin(\arg(\omega k_4))\\&\times\cos(\tfrac{3\pi}{4}+\arg q_3+\arg\Gamma(i\hat{\nu}_1)+\arg d_{1,0}+\arg\tfrac{\mathcal{P}(\omega k_4)}{\mathcal{P}(\omega^2k_4)}-t\mathrm{Im}\,\Phi_{31}(\zeta,\omega k_4(\zeta)))\\&+\frac{-4\sqrt{3}\sqrt{\hat{\nu}_2}|\tilde{r}(\frac{1}{k_2})|^{\frac{1}{2}}\frac{d}{d\zeta}\mathrm{Im}\,\Phi_{32}(\zeta,\omega^2k_2(\zeta))}{-i\omega^2k_2z_{2,\star}\sqrt{t}}\sin(\arg(\omega^2k_2))\\&\times\cos(\tfrac{3\pi}{4}-\arg(q_6-q_2q_5)+\arg\Gamma(i\hat{\nu}_2)+\arg d_{2,0}+\arg\tfrac{\mathcal{P}(\omega^2k_2)}{\mathcal{P}(\omega k_2)}-t\mathrm{Im}\,\Phi_{32}(\zeta,\omega^2k_2(\zeta)))\\&+O(t^{-1}\ln t).\end{aligned}$$

In view of the relations

$$\frac{d}{d\zeta}[\mathrm{Im}\,\Phi_{31}(\zeta,\omega k_4(\zeta))]=-\mathrm{Im}\,k_4(\zeta), \qquad \frac{d}{d\zeta}[\mathrm{Im}\,\Phi_{32}(\zeta,\omega^2k_2(\zeta))]=\mathrm{Im}\,k_2(\zeta),$$

this completes the proof of Theorem 2.1.

Appendix A. Model RH problems

Let $X\subset\mathbb{C}$ be the cross defined by $X=X_1\cup\cdots\cup X_4$, where the rays

$$\begin{aligned}&X_1:=\big\{\rho e^{\frac{i\pi}{4}}\,\big|\,0\le\rho<\infty\big\}, && X_2:=\big\{\rho e^{\frac{3i\pi}{4}}\,\big|\,0\le\rho<\infty\big\},\\&X_3:=\big\{\rho e^{-\frac{3i\pi}{4}}\,\big|\,0\le\rho<\infty\big\}, && X_4:=\big\{\rho e^{-\frac{i\pi}{4}}\,\big|\,0\le\rho<\infty\big\},\end{aligned} \tag{A.1}$$

are oriented away from the origin. The proofs of the following two lemmas are similar to (but more complicated than) the proof for the parabolic cylinder model problem that appears in [15, 18]. A detailed proof of Lemma A.2 can be found in [6].

Lemma A.1 (Model RH problem needed near $k = \omega k_4$ for Sector IV)**.** *Let $q_1, q_3 \in \mathbb{C}$ be such that $1 + |q_1|^2 - |q_3| > 0$, define*

$$\nu_1 = -\tfrac{1}{2\pi}\ln(1+|q_1|^2), \qquad \nu_3 = -\tfrac{1}{2\pi}\ln(1+|q_1|^2-|q_3|^2),$$

and define the jump matrix $v^{X,(1)}(z)$ for $z \in X$ by

$$v^{X,(1)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\bar{q}_3 z^{-2i\nu_3} z_{(0)}^{i\nu_1} z^{i\nu_1} e^{\frac{iz^2}{2}} & 0 & 1 \end{pmatrix}, & z \in X_1, \\ \begin{pmatrix} 1 & 0 & \frac{-q_3}{1+|q_1|^2-|q_3|^2} z^{2i\nu_3} z_{(0)}^{-i\nu_1} z^{-i\nu_1} e^{-\frac{iz^2}{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_2, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\bar{q}_3}{1+|q_1|^2-|q_3|^2} z^{-2i\nu_3} z_{(0)}^{i\nu_1} z^{i\nu_1} e^{\frac{iz^2}{2}} & 0 & 1 \end{pmatrix}, & z \in X_3, \\ \begin{pmatrix} 1 & 0 & q_3 z^{2i\nu_3} z_{(0)}^{-i\nu_1} z^{-i\nu_1} e^{-\frac{iz^2}{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_4, \end{cases} \tag{A.2}$$

where the branches for $z^{i\nu_3}, z^{i\nu_1}$ lie on the negative real axis and the branch for $z_{(0)}^{i\nu_1}$ lies on the positive real axis, such that $z^{i\nu} = |z|^{i\nu} e^{-\nu \arg(z)}$, $\arg(z) \in (-\pi, \pi)$ and $z_{(0)}^{i\nu} = |z|^{i\nu} e^{-\nu \arg_0(z)}$, $\arg_0(z) \in (0, 2\pi)$. Then the RH problem

(a) $m^{X,(1)}(q_1, q_3, \cdot) : \mathbb{C} \setminus X \to \mathbb{C}^{3\times 3}$ is analytic;

(b) on $X \setminus \{0\}$, the boundary values of $m^{X,(1)}$ exist, are continuous, and satisfy $m_+^{X,(1)} = m_-^{X,(1)} v^{X,(1)}$;

(c) $m^{X,(1)}(q_1, q_3, z) = I + O(z^{-1})$ as $z \to \infty$, and $m^{X,(1)}(q_1, q_3, z) = O(1)$ as $z \to 0$;

has a unique solution $m^{X,(1)}(q_1, q_3, z)$. This solution satisfies

$$m^{X,(1)}(z) = I + \frac{m_1^{X,(1)}}{z} + O\left(\frac{1}{z^2}\right), \ z \to \infty, \qquad m_1^{X,(1)} := \begin{pmatrix} 0 & 0 & \beta_{12}^{(1)} \\ 0 & 0 & 0 \\ \beta_{21}^{(1)} & 0 & 0 \end{pmatrix}, \tag{A.3}$$

where the error term is uniform with respect to $\arg z \in [0, 2\pi]$ and q_1, q_3 in compact subsets of $\{q_1, q_3 \in \mathbb{C} \,|\, 1 + |q_1|^2 - |q_3| > 0\}$, and $\beta_{12}^{(1)}, \beta_{21}^{(1)}$ are defined by

$$\beta_{12}^{(1)} = \frac{e^{\frac{3\pi i}{4}} e^{\frac{3\pi \hat{\nu}_1}{2}} e^{2\pi\nu_1} \sqrt{2\pi} q_3}{(e^{2\pi\hat{\nu}_1} - 1)\Gamma(-i\hat{\nu}_1)}, \qquad \beta_{21}^{(1)} = \frac{e^{-\frac{3\pi i}{4}} e^{\frac{3\pi \hat{\nu}_1}{2}} \sqrt{2\pi} \bar{q}_3}{(e^{2\pi\hat{\nu}_1} - 1)\Gamma(i\hat{\nu}_1)}, \tag{A.4}$$

where $\hat{\nu}_1 := \nu_3 - \nu_1 \geq 0$. (Note that $\beta_{12}^{(1)}\beta_{21}^{(1)} = \hat{\nu}_1$.)

Lemma A.2 (Model RH problem needed near $k = \omega^2 k_2$)**.** *Let $q_2, q_4, q_5, q_6 \in \mathbb{C}$ be such that $1 + |q_2|^2 - |q_4|^2 > 0$, $1 - |q_5|^2 - |q_6|^2 > 0$, and*

$$q_4 - \bar{q}_5 - q_2\bar{q}_6 = 0. \tag{A.5}$$

Define $\nu_2, \nu_4, \nu_5 \in \mathbb{R}$ *by*

$$\nu_2 = -\tfrac{1}{2\pi}\ln(1+|q_2|^2), \quad \nu_4 = -\tfrac{1}{2\pi}\ln(1+|q_2|^2-|q_4|^2), \quad \nu_5 = -\tfrac{1}{2\pi}\ln(1-|q_5|^2-|q_6|^2).$$

Define the jump matrix $v^{X,(2)}(z)$ *for* $z \in X$ *by*

$$v^{X,(2)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{q_6-q_2q_5}{1+|q_2|^2} z^{-i(2\nu_5-\nu_4)} z_{(0)}^{-i(2\nu_2-\nu_4)} e^{\frac{iz^2}{2}} & 1 \end{pmatrix}, & z \in X_1, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{\bar{q}_6-\bar{q}_2\bar{q}_5}{1-|q_5|^2-|q_6|^2} z^{i(2\nu_5-\nu_4)} z_{(0)}^{i(2\nu_2-\nu_4)} e^{-\frac{iz^2}{2}} \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_2, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{q_6-q_2q_5}{1-|q_5|^2-|q_6|^2} z^{-i(2\nu_5-\nu_4)} z_{(0)}^{-i(2\nu_2-\nu_4)} e^{\frac{iz^2}{2}} & 1 \end{pmatrix}, & z \in X_3, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\bar{q}_6-\bar{q}_2\bar{q}_5}{1+|q_2|^2} z^{i(2\nu_5-\nu_4)} z_{(0)}^{i(2\nu_2-\nu_4)} e^{-\frac{iz^2}{2}} \\ 0 & 0 & 1 \end{pmatrix}, & z \in X_4, \end{cases} \tag{A.6}$$

where $z^{i\nu}$ *has a branch cut along* $(-\infty, 0]$ *and* $z_{(0)}^{i\nu}$ *has a branch cut along* $[0, +\infty)$ *such that* $z^{i\nu} = |z|^{i\nu}e^{-\nu \arg(z)}$, $\arg(z) \in (-\pi, \pi)$, *and* $z_{(0)}^{i\nu} = |z|^{i\nu}e^{-\nu \arg_0(z)}$, $\arg_0(z) \in (0, 2\pi)$. *Then the RH problem*

(*a*) $m^{X,(2)}(\cdot) = m^{X,(2)}(q_2, q_4, q_5, q_6, \cdot) : \mathbb{C} \setminus X \to \mathbb{C}^{3\times 3}$ *is analytic;*

(*b*) *on* $X \setminus \{0\}$, *the boundary values of* $m^{X,(2)}$ *exist, are continuous, and satisfy* $m_+^{X,(2)} = m_-^{X,(2)} v^{X,(2)}$;

(*c*) $m^{X,(2)}(z) = I + O(z^{-1})$ *as* $z \to \infty$, *and* $m^{X,(2)}(z) = O(1)$ *as* $z \to 0$;

has a unique solution $m^{X,(2)}(z)$. *This solution satisfies*

$$m^{X,(2)}(z) = I + \frac{m_1^{X,(2)}}{z} + O\bigg(\frac{1}{z^2}\bigg), \quad z \to \infty, \quad m_1^{X,(2)} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta_{12}^{(2)} \\ 0 & \beta_{21}^{(2)} & 0 \end{pmatrix}, \tag{A.7}$$

where the error term is uniform with respect to $\arg z \in [-\pi, \pi]$ *and* q_2, q_4, q_5, q_6 *in compact subsets of* $\{q_2, q_4, q_5, q_6 \in \mathbb{C} \,|\, 1 + |q_2|^2 - |q_4|^2 > 0, 1 - |q_5|^2 - |q_6|^2 > 0, q_4 - \bar{q}_5 - q_2\bar{q}_6 = 0\}$, *and*

$$\beta_{12}^{(2)} := \frac{e^{\frac{3\pi i}{4}} e^{\frac{\pi\hat{\nu}_2}{2}} e^{2\pi(\nu_4-\nu_2)}\sqrt{2\pi}(\bar{q}_6 - \bar{q}_2\bar{q}_5)}{(e^{\pi\hat{\nu}_2} - e^{-\pi\hat{\nu}_2})\Gamma(-i\hat{\nu}_2)}, \qquad \beta_{21}^{(2)} := \frac{e^{-\frac{3\pi i}{4}} e^{\frac{\pi\hat{\nu}_2}{2}} e^{2\pi\nu_2}\sqrt{2\pi}(q_6 - q_2q_5)}{(e^{\pi\hat{\nu}_2} - e^{-\pi\hat{\nu}_2})\Gamma(i\hat{\nu}_2)}, \tag{A.8}$$

where $\hat{\nu}_2 := \nu_2 + \nu_5 - \nu_4$. *(Note that* $\beta_{12}^{(2)}\beta_{21}^{(2)} = \hat{\nu}_2$.*)*

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