From Leibniz integral rule to Reynolds' transport theorem

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Abstract. Evaluating the time derivative of an integral over a moving volume can be performed with the help of Reynolds' transport theorem, which is a generalisation of Leibniz integral rule. Most continuum or fluid mechanics textbooks provide a demonstration of Reynolds' transport theorem by having recourse to a blend of schematics and elementary calculus (e.g. Kundu et al. 2012). In this working note, a rather rigorous approach to the demonstration is presented, whose starting point is the introduction of a new coordinate system transforming the moving volume into a fixed one. The crux of the line of argument lies in the evaluation of the Jacobian of the coordinate transformation and the selection of the instant at which time derivatives are calculated. A graphical illustration and the treatment of a few special cases help grasp the significance of the theorem. Nothing is novel herein. In fact, some inspiration was sought in Aris (1962). It is noteworthy, however, that the style of the mathematical developments is intended to be familiar to those interested in fluid mechanics.

Motivation

The well-known *Leibniz integral rule* allows evaluating the derivative of an integral whose limits of integration are not constant:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} \varsigma(t,x)\,dx \;=\; \int_{a(t)}^{b(t)} \frac{\partial \varsigma}{\partial t}\,dx + \varsigma[t,b(t)]\frac{db}{dt} - \varsigma[t,a(t)]\frac{da}{dt} \;. \tag{1}$$

The generalisation thereof to the time derivative of an integral over a volume evolving in time is often referred to as *Reynolds' transport theorem*. The latter stipulates that the time derivative of the integral of any function of time and position $\varsigma(t,\mathbf{x})$ over the time-dependent control volume Ω delimited by surface Γ is the sum of a volume contribution and a surface term:

$$\frac{d}{dt}\int_{\Omega} \varsigma\, d\Omega \;=\; \int_{\Omega} \frac{\partial \varsigma}{\partial t}\, d\Omega + \int_{\Gamma} \varsigma \mathbf{v}^{\Gamma} \bullet \mathbf{n}\, d\Gamma \tag{2}$$

where t and $\mathbf{x}$ denote the time and the position vector, respectively; $\mathbf{v}^{\Gamma}$ is the velocity of a point belonging to the surface Γ and is the multi-dimensional counterpart of db/dt and $-da/dt$ in (1); unit vector $\mathbf{n}$ is normal to the boundary of control volume Ω.

Hereinafter, an attempt is made to prove in a relatively rigorous manner (i.e. without having recourse to graphics-based arguments) that (2) holds valid.

Coordinate transformation

Consider the following coordinate transformation

$$\begin{cases} t' = t \\ x' = x'(t,x,y,z)\;,\;\; y' = y'(t,x,y,z)\;,\;\; z' = z'(t,x,y,z) \end{cases} \tag{3}$$

and the related Jacobian

$$J(t',x',y',z') = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & \frac{\partial z}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & \frac{\partial z}{\partial y'} \\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'} \end{vmatrix} . \tag{4}$$

Then, the integral of $\varsigma(t,\mathbf{x})$ of over the control volume Ω transforms to

$$\int_{\Omega} \varsigma(t,\mathbf{x})\, d\Omega = \int_{\Omega'} J\varsigma(t',\mathbf{x}')\, d\Omega' , \tag{5}$$

where Ω' represents the control volume in the new coordinate space. Now assume that the coordinate transformation is such that Ω' appears to be motionless in the transformed space. As a consequence, the time derivative of the volume integral satisfies

$$\frac{d}{dt}\int_{\Omega} \varsigma\, d\Omega = \frac{d}{dt'}\int_{\hat{\Omega}} J\varsigma\, d\Omega' = \int_{\hat{\Omega}} \frac{\partial (J\varsigma)}{\partial t'}\, d\Omega' . \tag{6}$$

Without any loss of generality, it may be further assumed that the coordinates are similar at the instant $t = t_0 = t'$, i.e.

$$[x']_{t=t_0} = x , \quad [y']_{t=t_0} = y , \quad [z']_{t=t_0} = z . \tag{7}$$

This implies that, at this instant, the control volumes are similar in both the physical space and the transformed one (Figure 1),

$$\Omega(t_0) = \Omega'(t_0) \tag{8}$$

and that

$$\begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & \frac{\partial z}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & \frac{\partial z}{\partial y'} \\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'} \end{pmatrix}_{t'=t_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

so that the Jacobian satisfies

$$J(t_0,x',y',z') = 1 . \tag{10}$$

Next, the time derivative of the Jacobian may be manipulated as follows:

$$\left[\frac{\partial J}{\partial t'}\right]_{t'=t_0} = \begin{vmatrix} \frac{\partial}{\partial t'}\frac{\partial x}{\partial x'} & \frac{\partial}{\partial t'}\frac{\partial y}{\partial x'} & \frac{\partial}{\partial t'}\frac{\partial z}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & \frac{\partial z}{\partial y'} \\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'} \end{vmatrix}_{t'=t_0} + \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & \frac{\partial z}{\partial x'} \\ \frac{\partial}{\partial t'}\frac{\partial x}{\partial y'} & \frac{\partial}{\partial t'}\frac{\partial y}{\partial y'} & \frac{\partial}{\partial t'}\frac{\partial z}{\partial y'} \\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'} \end{vmatrix}_{t'=t_0}$$

$$
+\begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & \frac{\partial z}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & \frac{\partial z}{\partial y'} \\ \frac{\partial}{\partial t'}\frac{\partial x}{\partial z'} & \frac{\partial}{\partial t'}\frac{\partial y}{\partial z'} & \frac{\partial}{\partial t'}\frac{\partial z}{\partial z'} \end{vmatrix}_{t'=t_0}
$$

$$
= \begin{vmatrix} \frac{\partial}{\partial t'}\frac{\partial x}{\partial x'} & \frac{\partial}{\partial t'}\frac{\partial y}{\partial x'} & \frac{\partial}{\partial t'}\frac{\partial z}{\partial x'} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}_{t'=t_0} + \begin{vmatrix} 1 & 0 & 0 \\ \frac{\partial}{\partial t'}\frac{\partial x}{\partial y'} & \frac{\partial}{\partial t'}\frac{\partial y}{\partial y'} & \frac{\partial}{\partial t'}\frac{\partial z}{\partial y'} \\ 0 & 0 & 1 \end{vmatrix}_{t'=t_0} \tag{11}
$$

$$
+ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial}{\partial t'}\frac{\partial x}{\partial z'} & \frac{\partial}{\partial t'}\frac{\partial y}{\partial z'} & \frac{\partial}{\partial t'}\frac{\partial z}{\partial z'} \end{vmatrix}_{t'=t_0} = \left[\frac{\partial}{\partial t'}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial t'}\frac{\partial y}{\partial y'} + \frac{\partial}{\partial t'}\frac{\partial z}{\partial z'}\right]_{t'=t_0}
$$

As a consequence, at $t' = t_0$, the time derivative of the Jacobian transforms to

$$
\left[\frac{\partial J}{\partial t'}\right]_{t'=t_0} = \left[\frac{\partial}{\partial t'}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial t'}\frac{\partial y}{\partial y'} + \frac{\partial}{\partial t'}\frac{\partial z}{\partial z'}\right]_{t'=t_0} = \left[\frac{\partial}{\partial x'}\frac{\partial x}{\partial t'} + \frac{\partial}{\partial y'}\frac{\partial y}{\partial t'} + \frac{\partial}{\partial z'}\frac{\partial z}{\partial t'}\right]_{t'=t_0}
$$

$$
= \underbrace{\left[\frac{\partial}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t'}\right]_{t'=t_0}}_{\substack{\text{since } \frac{\partial}{\partial x'}=\frac{\partial}{\partial x},\ \frac{\partial}{\partial y'}=\frac{\partial}{\partial y},\ \frac{\partial}{\partial z'}=\frac{\partial}{\partial z} \text{ at } t'=t_0 \\ \text{see (3) and (7)}}} = \left[\nabla \bullet \frac{\partial \mathbf{x}}{\partial t'}\right]_{t'=t_0} \tag{12}
$$

On the other hand, coordinate transformation (3) is such that the time derivative operator obeys

$$
\frac{\partial}{\partial t'} = \underbrace{\frac{\partial t}{\partial t'}}_{=1}\frac{\partial}{\partial t} + \underbrace{\frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial t'}\frac{\partial}{\partial z}}_{=\frac{\partial \mathbf{x}}{\partial t'}\bullet\nabla} = \frac{\partial}{\partial t} + \frac{\partial \mathbf{x}}{\partial t'}\bullet\nabla\ . \tag{13}
$$

Time derivative of the volume integral

All of the results established above can now be used to evaluate the time rate of change of the volume integral under consideration, i.e. the left-hand side of (5).

Evaluating (6) at $t = t_0 = \widehat{t}$, using (8), (10), (12) and (13), one obtains

$$\left[\frac{d}{dt}\int_{\Omega}\varsigma\,d\Omega\right]_{t=t_0} = \left[\int_{\Omega'}\frac{\partial(J\varsigma)}{\partial t'}\,d\Omega'\right]_{t'=t_0} = \left[\int_{\Omega}\left(J\frac{\partial\varsigma}{\partial t'}+\varsigma\frac{\partial J}{\partial t'}\right)d\Omega'\right]_{t'=t_0}$$

$$= \left[\int_{\Omega'}\left(\frac{\partial\varsigma}{\partial t'}+\varsigma\nabla\bullet\frac{\partial\mathbf{x}}{\partial t'}\right)d\Omega'\right]_{t'=t_0} = \left[\int_{\Omega}\left(\frac{\partial\varsigma}{\partial t}+\frac{\partial\mathbf{x}}{\partial t'}\bullet\nabla\varsigma+\varsigma\nabla\bullet\frac{\partial\mathbf{x}}{\partial t'}\right)d\Omega\right]_{t=t_0} \tag{14}$$

$$= \left[\int_{\Omega}\left(\frac{\partial\varsigma}{\partial t}+\nabla\bullet\left(\varsigma\frac{\partial\mathbf{x}}{\partial t'}\right)\right)d\Omega\right]_{t=t_0}$$

Applying the divergence theorem to the last term in the right-hand side member of the expression above yields

$$\left[\frac{d}{dt}\int_{\Omega}\varsigma\,d\Omega\right]_{t=t_0} = \left[\int_{\Omega}\frac{\partial\varsigma}{\partial t}\,d\Omega+\int_{\Gamma}\varsigma\frac{\partial\mathbf{x}}{\partial t'}\bullet\mathbf{n}\,d\Gamma\right]_{t=t_0} . \tag{15}$$

The expression $\partial\mathbf{x}/\partial t'$ is the velocity in the physical space of a point that is at a standstill in the transformed space. Therefore, on the boundary of the control domain, this velocity is the velocity at which the points of this surface move. Upon denoting this velocity $\mathbf{v}^{\Gamma}$, relation (19) transforms to

$$\left[\frac{d}{dt}\int_{\Omega}\varsigma\,d\Omega\right]_{t=t_0} = \left[\int_{\Omega}\frac{\partial\varsigma}{\partial t}\,d\Omega+\int_{\Gamma}\varsigma\mathbf{v}^{\Gamma}\bullet\mathbf{n}\,d\Gamma\right]_{t=t_0} \tag{16}$$

This expression holds valid for any value of t_0. Therefore, (2) is correct. QED.

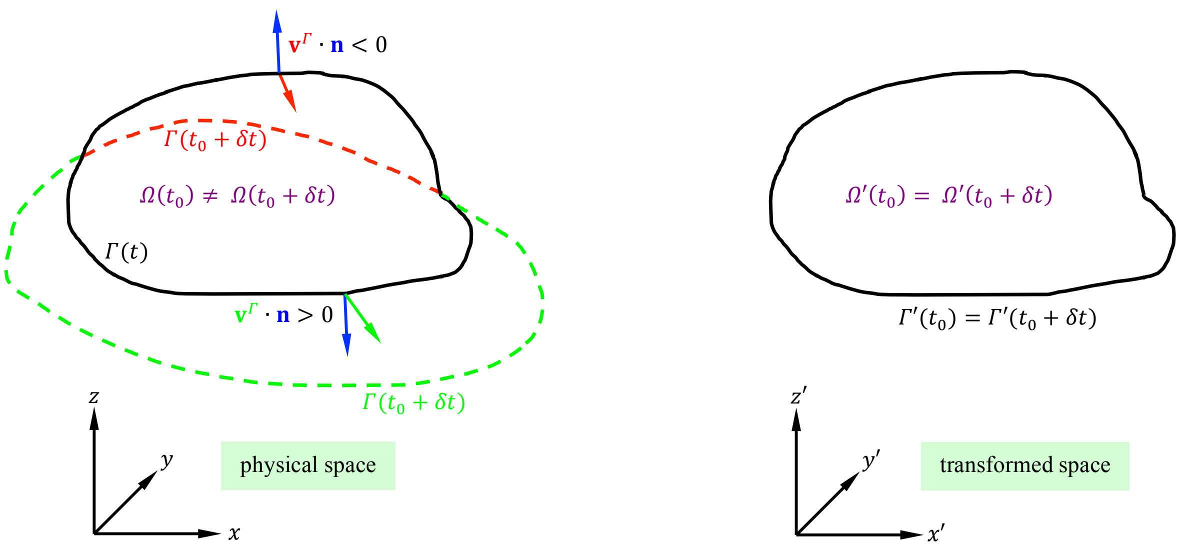

Figure 1. Illustration of the evolution of control volume Ω, which is limited by surface Γ, in the physical space (left panel) and the transformed space (right panel), in which it is fixed. Unit vector $\mathbf{n}$ is the outward normal to the boundary, whilst $\mathbf{v}^{\Gamma}$ denotes the velocity of a point of this surface.

Illustrations

Reynolds' transport theorem can be established in several manners. The demonstration provided above is but one of them. To help one grasp the significance of this theorem, it is worth considering two limit cases. First, if the function ς is a constant, expression (2) simplifies to

$$\frac{d}{dt}\int_{\Omega} \varsigma \, d\Omega = \varsigma \frac{d}{dt}\int_{\Omega} d\Omega = \varsigma \frac{dV}{dt} = \varsigma \int_{\Gamma} \mathbf{v}^{\Gamma} \cdot \mathbf{n} \, d\Gamma \ , \tag{17}$$

where V denotes the value of the volume of the control domain Ω. It is no surprise that the derivative of the integral is proportional to the expansion rate of the control volume and the latter is directly related to the velocity of its surface points, i.e.

$$\frac{dV}{dt} = \int_{\Gamma} \mathbf{v}^{\Gamma} \cdot \mathbf{n} \, d\Gamma \ . \tag{18}$$

On the other hand, if the control volume Ω is time-independent, then its surface points are motionless ($\mathbf{v}^{\Gamma} = 0$). As a consequence, relation (2) simplifies to

$$\frac{d}{dt}\int_{\Omega} \varsigma \, d\Omega = \int_{\Omega} \frac{\partial \varsigma}{\partial t} d\Omega \ , \tag{19}$$

as expected.

If Ω is a material volume, then the velocity of the surface points is equal to the continuous medium velocity ($\mathbf{v}^{\Gamma}(t,\mathbf{x}) = \mathbf{v}(t,\mathbf{x}), \ \mathbf{x} \in \Gamma$). In this particular case, Reynolds' transport theorem may be written as follows:

$$\frac{d}{dt}\int_{\Omega} \varsigma \, d\Omega = \int_{\Omega} \frac{\partial \varsigma}{\partial t} d\Omega + \int_{\Gamma} \varsigma \mathbf{v} \cdot \mathbf{n} \, d\Gamma \ . \tag{20}$$

Then, using the divergence theorem, (20) transforms to

$$\frac{d}{dt}\int_{\Omega} \varsigma \, d\Omega = \int_{\Omega} \left[\frac{\partial \varsigma}{\partial t} + \nabla \cdot (\varsigma \mathbf{v}) \right] d\Omega \ . \tag{21}$$

Setting $\varsigma = 1$, combining (17) and (21) leads to the following identity

$$\frac{d}{dt}\int_{\Omega} d\Omega = \frac{dV}{dt} = \int_{\Omega} \nabla \cdot \mathbf{v} \, d\Omega \ . \tag{22}$$

So, the time rate of change of the volume of a material volume depends on the divergence of the velocity. If, on the other hand, one sets $\varsigma(t,\mathbf{x}) = \rho(t,\mathbf{x})$, where $\rho(t,\mathbf{x})$ the density of a continuous medium, then (21) yields

$$\underbrace{\frac{d}{dt}\int_{\Omega} \rho \, d\Omega}_{=m(t)} = \int_{\Omega} \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right]}_{=0 \text{ (continuity equation)}} d\Omega = 0 \ . \tag{23}$$

Thus, $m(t)$, the mass contained in material volume Ω, remains constant as time progresses. This does not imply, however, that Ω always contains the same molecules. In fact, unresolved processes (e.g. diffusion) are likely to cause molecules to constantly cross the boundary of Ω (unless this surface is impermeable), but the incoming mass flux is equal to the outgoing one at any instant.

If the material volume is reduced to a fluid parcel, i.e. an elemental material volume, then the time rate of change of the value of its volume obeys

$$\frac{1}{\delta V(t)} \frac{d}{dt} \delta V(t) = \nabla \bullet \mathbf{v}(t,\mathbf{x}) \quad . \tag{24}$$

In addition, the (elemental) mass of a fluid parcel is time-independent.

Concluding remarks

Reynolds' transport theorem generally is instrumental in the derivation and manipulation of conservation equations in continuum mechanics as well as the construction of arbitrary Lagrangian-Eulerian (ALE) numerical methods (e.g. Delandmeter 2018). This theorem is also of use for establishing properties of the solutions of such equations (e.g. Deleersnijder 2019).

References

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