Beyond frames: Semi-frames and reproducing pairs

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Abstract

Frames are nowadays a standard tool in many areas of mathematics, physics and engineering. However, there are situations where it is difficult, even impossible, to design an appropriate frame. Thus there is room for generalizations, obtained by relaxing the constraints. A first case is that of semi-frames, in which one frame bound only is satisfied. Accordingly, on has to distinguish between upper and lower semi-frames. We will summarize this construction. Even more, one may get rid of both bounds, but then one needs two basic functions and one is led to the notion of reproducing pair. It turns out that every reproducing pair generates two Hilbert spaces, conjugate dual of each other. We will discuss in detail their construction and provide a number of examples, both discrete and continuous. Next, we notice that, by their very definition, the natural environment of a reproducing pair is a partial inner product space (PIP-space) with an L^2 central Hilbert space. A first possibility is to work in a rigged Hilbert space. Then, after describing the general construction, we will discuss two characteristic examples, namely, we take for the PIP-space a Hilbert scale or a lattice of L^p spaces.

1 Introduction

Representing functions in terms of simple ones, preferably with a small number of them, is a recurrent problem in analysis. It is particularly acute in signal and image processing, where transmission imposes severe constraints. Such signals are usually taken as square integrable functions on some manifold, hence they constitute a Hilbert space.

More generally, given a separable Hilbert space $\mathcal{H}$, one seeks to expand an arbitrary element $f \in \mathcal{H}$ in a sequence of simple, basic elements (atoms) $\Psi = (\psi_k)$, $k \in \Gamma$, with Γ a countable index set:

$$f = \sum_{k\in\Gamma} c_k \psi_k, \tag{1.1}$$

where the sum converges in an adequate fashion (e.g. in norm or unconditionally) and the coefficients c_k are (preferably) unique and easy to compute. There are several possibilities for obtaining that result. Namely, we can require that Ψ be:

(i) an orthonormal basis: the coefficients are unique, namely, $c_k = \langle \psi_k | f \rangle$, the convergence is unconditional;

(ii) a Riesz basis, *i.e.,* $\psi_k = V e_k$, where V is bounded bijective operator; the coefficients are unique, namely, $c_k = \langle \phi_k | f \rangle$, where (ϕ_k) is a unique Riesz basis dual to $(V e_k)$; the convergence is unconditional.

These two notions solve the problem, but they are very rigid and often not very manageable, leading mostly to infinite expansions. Thus frames were introduced for ensuring a better flexibility, originally in 1952 by Duffin and Schaeffer [19] in the context of nonharmonic analysis. The notion was revived by Daubechies, Grossmann and Meyer [17] in the early stages of wavelet theory and then became a very popular topic, in particular in Gabor and wavelet analysis [14, 16, 18, 23]. The reason is that a good frame in a Hilbert space is almost as good as an orthonormal basis for expanding arbitrary elements (albeit non-uniquely) and is often easier to construct. In order to put the present work in perspective, we recall that a sequence $\Psi = (\psi_k)$ is a frame for a Hilbert space $\mathcal{H}$ if there exist constants $0 < \mathsf{m} \leq \mathsf{M} < \infty$ (the frame bounds) such that

$$\mathsf{m} \left\|f\right\|^2 \leq \sum_{k\in\Gamma} |\langle\psi_k|f\rangle|^2 \leq \mathsf{M} \left\|f\right\|^2, \forall\, f \in \mathcal{H}. \tag{1.2}$$

Actually frames are most often considered in the discrete case, for instance in signal processing [16]. However, continuous frames have also been studied and offer interesting mathematical problems. They have been introduced originally by Ali, Gazeau and one of us [1, 2] and also, independently, by Kaiser [24]. Since then, several papers dealt with various aspects of the concept, see for instance [13, 20, 21] or [26]. The next step towards numerical applications will be, of course, discretization, but this is not our purpose in this article.

However, there may occur situations where it is impossible to satisfy both frame bounds at the same time. Therefore, several generalizations of frames have been introduced. Semi-frames [8, 9], for example, are obtained when functions only satisfy one of the two frame bounds. It turns out that a large portion of frame theory can be extended to this larger framework, in particular the notion of duality.

More recently, a new generalization of frames was introduced by Balazs and Speckbacher [29], namely, reproducing pairs. Here, given a measure space (X, μ), one considers a couple of weakly measurable functions (ψ, ϕ), instead of a single mapping, and one studies the correlation between the two (a precise definition is given below). This definition also includes the original definition of a continuous frame [1, 2] to which it reduces when $\psi = \phi$. The increase of freedom in choosing the mappings ψ and ϕ, however, leads to the problem of characterizing the range of the analysis operators, which in general need no more be contained in $L^2(X, \mathrm{d}\mu)$, as in the frame case. Therefore, it is natural to extend the theory to the case where the weakly measurable functions take their values in a partial inner product space (PIP-space), for instance, a rigged Hilbert space or a Hilbert scale.

The paper is organized as follows. In Section 2, we review the notions of frames and semi-frames and recall their salient properties. In Section 3 we introduce reproducing pairs, in particular their duality properties. Then, in Section 4, we discuss briefly the existence and uniqueness of reproducing partners. In Section 6, we motivate the link between reproducing pairs and PIP-spaces, first a Rigged Hilbert space (RHS) in Section 7, then a general PIP-space, more precisely a lattice of Banach spaces (LBS) or a lattice of Hilbert spaces (LHS), in Section 8. Finally, in Sections 9 and 10, respectively, we examine two particular cases, namely, a Hilbert scale and a lattice of L^p spaces.

2 Preliminaries: Frames and semi-frames

2.1 Frames

Before proceeding, we list our definitions and conventions. The framework is a (separable) Hilbert space $\mathcal{H}$, with the inner product $\langle\cdot|\cdot\rangle$ linear in the first factor. Given an operator A on $\mathcal{H}$, we denote its domain by $D(A)$, its range by $\mathsf{Ran}\,(A)$ and its kernel by $\mathsf{Ker}\,(A)$. $GL(\mathcal{H})$ denotes the set of all invertible bounded operators on $\mathcal{H}$ with bounded inverse. Throughout the paper, we will consider weakly measurable functions $\psi : X \to \mathcal{H}$, where (X, μ) is a locally compact space with a Radon measure μ, that is, $\langle f|\psi_x\rangle$ is $\mu-$measurable for every $f \in \mathcal{H}$.

The weakly measurable function ψ is a *continuous frame* if there exist constants $\mathsf{m} > 0$ and $\mathsf{M} < \infty$ (the frame bounds) such that

$$\mathsf{m} \left\|f\right\|^2 \leq \int_X |\langle f|\psi_x\rangle|^2 \,\mathrm{d}\mu(x) \leq \mathsf{M} \left\|f\right\|^2, \forall\, f \in \mathcal{H}. \tag{2.1}$$

Given the continuous frame ψ, the *analysis* operator $C_\psi : \mathcal{H} \to L^2(X, \mathrm{d}\mu)$ is defined [1] as

$$(C_\psi f)(x) = \langle f|\psi_x\rangle,\ f \in \mathcal{H}, \tag{2.2}$$

and the corresponding *synthesis operator* $C_\psi^* : L^2(X, \mathrm{d}\mu) \to \mathcal{H}$ as (the integral being understood in the weak sense, as usual)

$$C_\psi^* \xi = \int_X \xi(x)\, \psi_x \ \mathrm{d}\mu(x),\ \text{for}\ \ \xi \in L^2(X, \mathrm{d}\mu). \tag{2.3}$$

We set $S_\psi := C_\psi^* C_\psi$, *i.e.,*

$$\langle f|S_\psi f\rangle = \int_X |\langle f|\psi_x\rangle|^2 \ \mathrm{d}\mu(x). \tag{2.4}$$

Thus the so-called *frame or resolution operator* S_ψ is self-adjoint, invertible, bounded with bounded inverse S_ψ^{-1}, that is, $S_\psi \in GL(\mathcal{H})$.

In particular, if X is a discrete set with μ being a counting measure, we recover the standard definition (1.2) of a (discrete) frame [14, 16, 19].

An important concept in frame theory is that of duality. Given a frame $\Psi = \{\psi_x\}$, one says that a frame $\Phi = \{\phi_x\}$ is *dual* to the frame $\{\psi_x\}$ if one has

$$\langle f|g\rangle = \int_X \langle f|\phi_x\rangle\, \langle \psi_x|g\rangle \ \mathrm{d}\mu(x),\ \forall\, f, g \in \mathcal{H}. \tag{2.5}$$

Then Ψ is dual to Φ as well. The dual of a given frame Ψ is not unique in general, but one of them is distinguished, namely, the canonical dual $\widetilde{\psi_x} := S^{-1}\psi_x$.

2.2 Semi-frames

In practice, there are situations where the notion of frame is too restrictive, in the sense that one cannot satisfy *both* frame bounds simultaneously. Thus there is room for two natural generalizations, namely, we say that a family Ψ is an *upper (resp. lower) semi-frame*, if

(i) Ψ is total in $\mathcal{H}$;

(ii) Ψ satisfies the upper (resp. lower) frame inequality in (2.1).

Note that the lower frame inequality automatically implies that the family is total, *i.e.,* (ii) $\Rightarrow$ (i) for a lower semi-frame.

Let first Ψ be a (continuous) *upper semi-frame*, that is, there exists a constant $0 < \mathsf{M} < \infty$ such that

$$0 < \int_X |\langle f|\psi_x\rangle|^2 \ \mathrm{d}\mu(x) \le \mathsf{M}\, \|f\|^2\,,\ \forall\, f \in \mathcal{H},\ f \neq 0. \tag{2.6}$$

In this case, Ψ is a total set in $\mathcal{H}$, the operators C_ψ and S_ψ are bounded, S_ψ is injective and self-adjoint. Therefore $\mathsf{Ran}\,(S_\psi)$ is dense in $\mathcal{H}$ and S_ψ^{-1} is also self-adjoint. Thus, if Ψ is an upper semi-frame and not a frame, S_ψ is bounded and S_ψ^{-1} is unbounded, as follows immediately from (2.6).

Note that, if a family Ψ verifies the upper frame bound only, the map $x \mapsto \psi_x$ is often called a Bessel mapping. More precisely, Bessel maps are those for which $\int_X |\langle f|\psi_x\rangle|^2 \,\mathrm{d}\mu(x) < \infty$. By a standard argument based on the closed graph theorem, one gets the inequality on the right of (2.6).

We notice that an upper semi-frame Ψ is a frame if and only if there exists another upper semi-frame Φ which is dual to Ψ, in the sense of (2.5) [21].

Next, we say that a family $\Phi = \{\phi_x\}$ is a *lower semi-frame* if it satisfies the lower frame condition, that is, there exists a constant $\mathsf{m} > 0$ such that

$$\mathsf{m}\, \|f\|^2 \le \int_X |\langle \phi_x|f\rangle|^2 \ \mathrm{d}\mu(x),\ \ \forall\, f \in \mathcal{H}. \tag{2.7}$$

Clearly, (2.7) implies that the family Φ is total in $\mathcal{H}$.

[1] As usual, we identify a function f with its residue class in $L^2(X, \mathrm{d}\mu)$.

Following the terminology of Young [32] in the discrete case, we may call *moment space* of a measurable function ψ the range of its analysis operator, $C_\psi(\mathcal{H})$. Then one may say that a measurable function ψ is Bessel or an upper semi-frame if its moment space is contained in $L^2(X, \mathrm{d}\mu)$. On the contrary, a measurable function ϕ is a lower semi-frame if its moment space contains $L^2(X, \mathrm{d}\mu)$.

In the lower case, the definition of S_ϕ must be changed, since C_ϕ need not be densely defined, so that C_ϕ^* may not be well-defined. Instead, following [8, Sec.2] one defines the analysis operator (2.2) on the domain

$$D(C_\phi) = \{f \in \mathcal{H} : \int_X |\langle f|\phi_x\rangle|^2 \, \mathrm{d}\mu(x) < \infty\},$$

which need not be dense. As for the synthesis operator, we put

$$D_\phi F = \int_X F(x)\, \phi_x \, \mathrm{d}\mu(x), \;\; F \in L^2(X, \mathrm{d}\mu), \tag{2.8}$$

on the domain of all elements F for which the integral in (2.8) converges weakly in $\mathcal{H}$. Defining $S_\phi := D_\phi C_\phi$, it is shown in [8, Sec.2] that S_ϕ is unbounded and S_ϕ^{-1} is bounded.

With these definitions, we obtain a nice duality property between upper and lower semi-frames. In the discrete case, the role of upper, resp. lower semi-frame, is played by Bessel, resp. Riesz-Fischer sequences [32]. A Riesz-Fischer sequence is a sequence for which, for every sequence $\{a_n\} \in \ell^2$, there is a solution of the equation $\langle f|\phi_n\rangle = a_n$. One knows that every total Riesz-Fischer sequence satisfies the lower frame condition, which is equivalent to the existence of a Bessel sequence dual to it [15]. The same result holds here.

Proposition 2.1 *(i) Let $\Psi = \{\psi_x\}$ be an upper semi-frame, with upper frame bound M and let $\Phi = \{\phi_x\}$ be a total family dual to Ψ. Then Φ is a lower semi-frame, with lower frame bound M^{-1}.*

(ii) Conversely, if $\Phi = \{\phi_x\}$ is a lower semi-frame, there exists an upper semi-frame $\Psi = \{\psi_x\}$ dual to Φ, that is, one has, in the weak sense,

$$f = \int_X \langle f|\phi_x\rangle\, \psi_x \, \mathrm{d}\mu(x), \;\; \forall\, f \in D(C_\Phi).$$

A proof may be found in [8, Lemma 2.5 and Proposition 2.6]. However, the latter is slightly incomplete. Here is a corrected proof [30].

Proof of (ii) : The 'if' part is Lemma 2.5 of [8]. Let Φ be a lower semi-frame. Then $\mathsf{Ran}\,(C_\phi)$ is a closed subspace of $L^2(X, \mathrm{d}\mu)$, in virtue of the lower frame bound condition, and it is a reproducing kernel Hilbert space (RKHS), since one has, for $F = C_\phi f \in \mathsf{Ran}\,(C_\phi)$,

$$|F(x)| = |C_\phi f(x)| \leq \|f\|\, \|\phi_x\| \leq \sqrt{1/\mathsf{m}}\, \|\phi_x\|\, \|C_\phi f\|_2 = C_x\, \|F\|_2\,,$$

where m is the lower frame bound of Φ, *i.e.,* the evaluation functional is bounded. Let P be the orthogonal projection on $\mathsf{Ran}\,(C_\phi)$ and let $\{e_n\}_{n\in\mathbb{N}}$ be an arbitrary orthonormal basis of $L^2(X, \mathrm{d}\mu)$.

Define a linear operator $V : L^2(X, \mathrm{d}\mu) \to \mathcal{H}$ by $V = C_\phi^{-1}$ on $\mathsf{Ran}\,(C_\phi)$, by $V = 0$ on $\mathsf{Ran}\,(C_\phi)^\perp$ and extending by linearity. Then V is bounded, since $C_\phi^{-1} : \mathsf{Ran}\,(C_\phi) \to \mathcal{H}$ is bounded. Then, for all $f \in D(C_\phi),\; g \in \mathcal{H}$, we have

$$\begin{aligned}\langle f|g\rangle &= \langle VC_\phi f|g\rangle = \langle C_\phi f|V^*g\rangle_2 = \langle C_\phi f|V^*(\sum_{n\in\mathbb{N}}\langle g|e_n\rangle e_n)\rangle_2 \\ &= \langle C_\phi f|\sum_{n\in\mathbb{N}}\langle g|e_n\rangle V^*e_n\rangle_2 = \langle C_\phi f|\sum_{n\in\mathbb{N}}\langle g|e_n\rangle PV^*e_n\rangle_2 = \langle C_\phi f|C_\psi g\rangle_2,\end{aligned}$$

where we have put $\psi_x := \sum_{n\in\mathbb{N}} e_n \overline{(PV^*e_n)}(x)$. It remains to show that ψ_x is well defined for every $x \in X$. This is the case if and only if

$$\sum_{n\in\mathbb{N}} |(PV^*e_n)(x)|^2 < \infty, \;\forall\, x \in X.$$

But this follows from the fact that $\{PV^* e_n\}_{n\in\mathbb{N}}$ is a Bessel sequence in the RKHS $\mathsf{Ran}\,(C_\phi)$ [30]. One has indeed, for any $F \in \mathsf{Ran}\,(C_\phi)$,

$$\sum_{n\in\mathbb{N}} |\langle F|PV^* e_n\rangle_2|^2 = \sum_{n\in\mathbb{N}} |\langle VPF|e_n\rangle_2|^2 = \|VF\|^2 \leq C\,\|F\|_2^2\,,$$

since V is bounded and $PF = F$. □

In the same paper [8], concrete examples are presented, namely an upper semi-frame of affine coherent states and a lower semi-frame of wavelets on the 2-sphere. We will come back to these examples in Section 5.2.

In conclusion, if two semi-frames are in duality, either they are both frames, or else at least one of them is a lower semi-frame.

3 Reproducing pairs

Quite recently, a new generalization of frames was introduced by Balazs and Speckbacher [29], namely, reproducing pairs. Here, given a measure space (X, μ), one considers a couple of weakly measurable functions (ψ, ϕ), instead of a single mapping. The advantage is that no further conditions are imposed on these functions, which results in an increased flexibility.

More precisely, the couple of weakly measurable functions (ψ, ϕ) is called a *reproducing pair* if [10]

(a) The sesquilinear form

$$\Omega_{\psi,\phi}(f, g) = \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle\,\mathrm{d}\mu(x) \tag{3.1}$$

is well-defined and bounded on $\mathcal{H}\times\mathcal{H}$, that is, $|\Omega_{\psi,\phi}(f, g)| \leq c\,\|f\|\,\|g\|$, for some $c > 0$.

(b) The corresponding bounded (resolution) operator $S_{\psi,\phi}$ belongs to $GL(\mathcal{H})$.

Under these hypotheses, one has

$$S_{\psi,\phi} f = \int_X \langle f|\psi_x\rangle\phi_x\,\mathrm{d}\mu(x),\ \forall\, f \in \mathcal{H}, \tag{3.2}$$

the integral on the r.h.s. being defined in weak sense. If $\psi = \phi$, we recover the notion of continuous frame, as introduced in [1, 2], so that we have indeed a genuine generalization of the latter.

Notice that $S_{\psi,\phi}$ is in general neither positive, nor self-adjoint, since $S^*_{\psi,\phi} = S_{\phi,\psi}$. However, if (ψ, ϕ) is a reproducing pair, then $(\psi, S^{-1}_{\psi,\phi}\phi)$ is also a reproducing pair, for which the corresponding resolution operator is the identity, that is, ψ and ϕ are in duality. Therefore, there is no restriction of generality to assume that $S_{\phi,\psi} = I$ [29]. The worst that can happen is to replace some norms by equivalent ones.

3.1 The Hilbert spaces generated by a reproducing pair

It has been shown in [10] that each weakly measurable function ϕ generates an intrinsic pre-Hilbert space $V_\phi(X, \mu)$ and, moreover, a reproducing pair (ψ, ϕ) generates two Hilbert spaces, $V_\psi(X, \mu)$ and $V_\phi(X, \mu)$, conjugate dual of each other with respect to the $L^2(X, \mu)$ inner product. Let us sketch that construction, following closely [10]. Further generalizations will follow.

Given a weakly measurable function ϕ, let us denote by $\mathcal{V}_\phi(X, \mu)$ the space of all measurable functions $\xi : X \to \mathbb{C}$ such that the integral $\int_X \xi(x)\langle\phi_x|g\rangle\,\mathrm{d}\mu(x)$ exists for every $g \in \mathcal{H}$ (in the sense that $\xi\langle\phi.|g\rangle \in L^1(X, \mathrm{d}\mu)$) and defines a bounded conjugate linear functional on $\mathcal{H}$, *i.e.*, $\exists\, c > 0$ such that

$$\left|\int_X \xi(x)\langle\phi_x|g\rangle\,\mathrm{d}\mu(x)\right| \leq c\,\|g\|\,,\ \forall\, g \in \mathcal{H}. \tag{3.3}$$

Clearly, if (ψ, ϕ) is a reproducing pair, all functions $\xi(x) = \langle f|\psi_x\rangle$ belong to $\mathcal{V}_\phi(X, \mu)$.

By the Riesz lemma, we can define a linear map $T_\phi : \mathcal{V}_\phi(X,\mu) \to \mathcal{H}$ by the following weak relation

$$\langle T_\phi \xi | g\rangle = \int_X \xi(x)\langle \phi_x | g\rangle \,\mathrm{d}\mu(x), \ \forall \xi \in \mathcal{V}_\phi(X,\mu), g \in \mathcal{H}. \tag{3.4}$$

Next, we define the vector space

$$V_\phi(X,\mu) = \mathcal{V}_\phi(X,\mu)/\mathsf{Ker}\ T_\phi$$

and equip it with the norm

$$\|[\xi]_\phi\|_\phi := \sup_{\|g\|\le 1} \left| \int_X \xi(x)\langle \phi_x | g\rangle \,\mathrm{d}\mu(x)\right| = \sup_{\|g\|\le 1} |\langle T_\phi \xi | g\rangle|\,, \tag{3.5}$$

where we have put $[\xi]_\phi = \xi + \mathsf{Ker}\ T_\phi$ for $\xi \in \mathcal{V}_\phi(X,\mu)$. Clearly, $V_\phi(X,\mu)$ is a normed space. However, the norm $\|\cdot\|_\phi$ is in fact Hilbertian, that is, it derives from an inner product, as can be seen as follows. First, it turns out that the map $\widehat{T}_\phi : V_\phi(X,\mu) \to \mathcal{H}$, $\widehat{T}_\phi[\xi]_\phi := T_\phi \xi$ is a well-defined isometry of $V_\phi(X,\mu)$ into $\mathcal{H}$. Next, one may define on $V_\phi(X,\mu)$ an inner product by setting

$$\langle [\xi]_\phi | [\eta]_\phi\rangle_{(\phi)} := \langle \widehat{T}_\phi[\xi]_\phi | \widehat{T}_\phi[\eta]_\phi\rangle, \ [\xi]_\phi, [\eta]_\phi \in V_\phi(X,\mu),$$

and one shows that the norm defined by $\langle\cdot|\cdot\rangle_{(\phi)}$ coincides with the norm $\|\cdot\|_\phi$ defined in (3.5). One has indeed

$$\|[\xi]_\phi\|_{(\phi)} = \left\|\widehat{T}_\phi[\xi]_\phi\right\| = \|T_\phi \xi\| = \sup_{\|g\|\le 1} |\langle T_\phi \xi | g\rangle| = \|[\xi]_\phi\|_\phi\,.$$

Thus we may state:

Proposition 3.1 *Let ϕ be a weakly measurable function. Then $V_\phi(X,\mu)$ is a pre-Hilbert space with respect to the norm $\|\cdot\|_\phi$ and the map $\widehat{T}_\phi : V_\phi(X,\mu) \to \mathcal{H}$, $\widehat{T}_\phi[\xi]_\phi := T_\phi\xi$ is a well-defined isometry of $V_\phi(X,\mu)$ into $\mathcal{H}$.*

Let us denote by $V_\phi(X,\mu)^*$ the Hilbert dual space of $V_\phi(X,\mu)$, that is, the set of continuous linear functionals on $V_\phi(X,\mu)$. The norm $\|\cdot\|_{\phi^*}$ of $V_\phi(X,\mu)^*$ is defined, as usual, by

$$\|F\|_{\phi^*} = \sup_{\|[\xi]_\phi\|_\phi \le 1} |F([\xi]_\phi)|, \ F \in V_\phi(X,\mu)^*.$$

Now we define a conjugate linear map $\mathsf{C}_\phi : \mathcal{H} \to V_\phi(X,\mu)^*$ by

$$(\mathsf{C}_\phi f)([\xi]_\phi) := \int_X \xi(x)\langle \phi_x | f\rangle \,\mathrm{d}\mu(x), \ f \in \mathcal{H}, \tag{3.6}$$

which will take the role of the analysis operator C_ϕ of Section 2. Notice that C_ϕ is a *linear* map, whereas C_ϕ is conjugate linear. The discrepancy is explained in Remark 3.11 below.

Of course, (3.6) means that $(\mathsf{C}_\phi f)([\xi]_\phi) = \langle T_\phi \xi | f\rangle = \langle \widehat{T}_\phi[\xi]_\phi | f\rangle$, for every $f \in \mathcal{H}$. Thus $\mathsf{C}_\phi = \widehat{T}_\phi^*$, the adjoint map of $\widehat{T}_\phi$. By (3.3) it follows that C_ϕ is continuous . This implies that

$$\mathcal{H} = [\mathsf{Ran}\, \widehat{T}_\phi]^{\sim} \oplus \mathsf{Ker}\, \mathsf{C}_\phi, \tag{3.7}$$

where the first summand denotes the closure of $\mathsf{Ran}\, \widehat{T}_\phi$. Hence $\mathsf{C}_\phi^* = \widehat{T}_\phi^{**} = \widehat{T}_\phi$, if $V_\phi(X,\mu)$ is complete.

By modifying in an obvious way the definition given in Section 2, we say that ϕ is *μ-total* if $\mathsf{Ker}\, \mathsf{C}_\phi = \{0\}$, that is $\mathsf{Ran}\, \widehat{T}_\phi = \mathcal{H}$.

Remark 3.2 Whenever no confusion may arise, we will omit the explicit indication of residues classes and write simply, for instance, $\xi \in V_\phi(X,\mu)$ instead of $[\xi]_\phi \in V_\phi(X,\mu)$. Similarly, for the operator C_ϕ introduced in (3.6), we will often identify $\mathsf{C}_\phi f$, $f \in \mathcal{H}$, with $\langle \phi_x | f\rangle$, as a shortcut to $(\mathsf{C}_\phi f)(\xi) = \int_X \xi(x)\langle \phi_x | f\rangle \,\mathrm{d}\mu(x)$.

It is easy to see that $V_\phi(X,\mu)[\langle\cdot|\cdot\rangle_{(\phi)}]$ is complete, *i.e.,* it is a Hilbert space if and only if $\widehat{T}_\phi$ has closed range [10].

As a consequence of (3.7) we get

Corollary 3.3 *The following statements hold.*

(i) A weakly measurable function ϕ is μ-total if and only if $\mathsf{Ran}\,\widehat{T}_\phi$ is dense in $\mathcal{H}$.

(ii) If $V_\phi(X,\mu)$ is a Hilbert space, $\mathsf{Ran}\,\widehat{T}_\phi$ is equal to $\mathcal{H}$ if and only if ϕ is μ-total.

Now, things get simpler in the case of a reproducing pair. Namely,

Lemma 3.4 *If (ψ,ϕ) is a reproducing pair, then $\mathsf{Ran}\,\widehat{T}_\phi = \mathcal{H}$.*

Proof. Since $S_{\psi,\phi} \in GL(\mathcal{H})$, for every $h \in \mathcal{H}$, there exists a unique $f \in \mathcal{H}$ such that $S_{\psi,\phi} f = h$. But, by (3.2), we get

$$\langle h|g\rangle = \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle\,\mathrm{d}\mu(x),\ \forall\, f,g \in \mathcal{H},$$

that is, $h = \widehat{T}_\phi \overline{[\mathsf{C}_\psi f]}_\phi$, where the overbar denotes complex conjugation, as usual. □

Notice that, if (ψ,ϕ) is a reproducing pair, both functions are necessarily μ-total.

3.2 Duality properties of the spaces $V_\phi(X,\mu)$

When the space $V_\phi(X,\mu)$ is a Hilbert space, it is conjugate isomorphic to its dual, via the Riesz operator. In addition, if (ψ,ϕ) is a reproducing pair, the dual of $V_\phi(X,\mu)$ can be identified with $V_\psi(X,\mu)$ as we shall prove below. We emphasize that the duality is taken with respect to the sesquilinear form

$$\langle\!\langle \xi|\eta\rangle\!\rangle_\mu := \int_X \xi(x)\overline{\eta(x)}\,\mathrm{d}\mu(x), \tag{3.8}$$

which coincides with the inner product of $L^2(X,\mu)$ whenever the latter makes sense.

Theorem 3.5 *Let ϕ be a weakly measurable function. If F is a continuous linear functional on $V_\phi(X,\mu)$, then there exists a unique $g \in [\mathcal{M}_\phi]^\sim$, the closure of the range of $\widehat{T}_\phi$, such that*

$$F([\xi]_\phi) = \int_X \xi(x)\langle\phi_x|g\rangle\,\mathrm{d}\mu(x),\ \forall\,\xi \in \mathcal{V}_\phi(X,\mu) \tag{3.9}$$

and $\|F\|_{\phi^*} = \|g\|$, where $\|\cdot\|_{\phi^*}$ denotes the (dual) norm on $V_\phi(X,\mu)^*$. Moreover, every $g \in \mathcal{H}$ defines a continuous linear functional F on $V_\phi(X,\mu)$ with $\|F\|_{\phi^*} \le \|g\|$, by (3.9). In particular, if $g \in \mathsf{Ran}\,\widehat{T}_\phi$, then $\|F\|_{\phi^*} = \|g\|$.

Proof. Let $F \in V_\phi(X,\mu)^*$. Then, there exists $c > 0$ such that

$$|F([\xi]_\phi)| \le c\,\|[\xi]_\phi\|_\phi = c\,\|T_\phi\xi\|\,,\ \forall\,\xi \in \mathcal{V}_\phi(X,\mu).$$

Let $\mathcal{M}_\phi := \{T_\phi\xi : \xi \in \mathcal{V}_\phi(X,\mu)\} = \mathsf{Ran}\,\widehat{T}_\phi$. Then $\mathcal{M}_\phi$ is a vector subspace of $\mathcal{H}$, with closure $[\mathcal{M}_\phi]^\sim$.

Let $\widetilde{F}$ be the linear functional defined on $\mathcal{M}_\phi$ by

$$\widetilde{F}(T_\phi\xi) := F([\xi]_\phi),\ \xi \in \mathcal{V}_\phi(X,\mu).$$

We notice that $\widetilde{F}$ is well-defined. Indeed, if $T_\phi\xi = T_\phi\xi'$, then $\xi - \xi' \in \mathsf{Ker}\,T_\phi$. Hence, $[\xi]_\phi = [\xi']_\phi$ and $F([\xi]_\phi) = F([\xi']_\phi)$

Hence, $\widetilde{F}$ is a continuous linear functional on $\mathcal{M}_\phi$. Thus there exists a unique $g \in [\mathcal{M}_\phi]^\sim$such that

$$\widetilde{F}(T_\phi\xi) = \langle\widehat{T}_\phi[\xi]_\phi|g\rangle = \int_X \xi(x)\langle\phi_x|g\rangle\,\mathrm{d}\mu(x)$$

and $\|g\| = \|\widetilde{F}\|$.

In conclusion,

$$F([\xi]_\phi) = \int_X \xi(x)\langle \phi_x | g\rangle \,\mathrm{d}\mu(x), \ \forall\, \xi \in \mathcal{V}_\phi(X,\mu).$$

and $\|F\|_{\phi^*} = \|g\|$.

Moreover, every $g \in \mathcal{H}$ obviously defines a continuous linear functional F by (3.9) as $|F([\xi]_\phi)| \leq \|g\| \, \|[\xi]_\phi\|_\phi$. This inequality implies that $\|F\|_{\phi^*} \leq \|g\|$. In particular, if $g \in \mathsf{Ran}\, \widehat{T}_\phi$, then there exists $[\xi]_\phi \in \mathcal{V}_\phi(X,\mu)$, $\|[\xi]_\phi\|_\phi = 1$, such that $\widehat{T}_\phi[\xi]_\phi = g\|g\|^{-1}$. Hence $F([\xi]_\phi) = \langle \widehat{T}_\phi[\xi]_\phi | g\rangle = \|g\|$. □

Corollary 3.6 *Let ϕ be a μ-total weakly measurable function, then $\mathsf{C}_\phi : \mathcal{H} \to V_\phi(X,\mu)^*$ is a conjugate linear isometric isomorphism.*

Proof. C_ϕ is surjective by Theorem 3.5. As ϕ is μ-total, it follows by Corollary 3.3 that $\mathsf{Ran}\, \widehat{T}_\phi$ is dense in $\mathcal{H}$. Consequently, for $f \in \mathcal{H}$ it follows that

$$\|\mathsf{C}_\phi f\|_{\phi^*} = \sup_{\|[\xi]_\phi\|_\phi = 1} \left| \int_X \xi(x)\langle \phi_x | f\rangle \,\mathrm{d}\mu(x)\right| = \sup_{\|[\xi]_\phi\|_\phi = 1} |\langle \widehat{T}_\phi \xi | f\rangle| = \sup_{\|g\|=1,\ g\in \mathsf{Ran}\,\widehat{T}_\phi} |\langle g|f\rangle| = \|f\|\,. \qquad \square$$

Theorem 3.7 *If (ψ,ϕ) is a reproducing pair, then every continuous linear functional F on $V_\phi(X,\mu)$, i.e., $F \in V_\phi(X,\mu)^*$, can be represented as*

$$F([\xi]_\phi) = \int_X \xi(x)\overline{\eta(x)} \,\mathrm{d}\mu(x), \ \forall\, [\xi]_\phi \in V_\phi(X,\mu), \tag{3.10}$$

with $\eta \in \mathcal{V}_\psi(X,\mu)$. The residue class $[\eta]_\psi \in V_\psi(X,\mu)$ is uniquely determined.

Proof. By Theorem 3.5, we have the representation

$$F(\xi) = \int_X \xi(x)\langle \phi_x | g\rangle \,\mathrm{d}\mu(x).$$

It is easily seen that $\eta(x) = \langle g|\phi_x\rangle \in \mathcal{V}_\psi(X,\mu)$. Uniqueness is easy. □

The lesson of the previous statements is that the map

$$j : F \in V_\phi(X,\mu)^* \mapsto [\eta]_\psi \in V_\psi(X,\mu) \tag{3.11}$$

is well-defined and conjugate linear. On the other hand, $j(F) = j(F')$ implies easily $F = F'$. Therefore $V_\phi(X,\mu)^*$ can be identified with a closed subspace of $\overline{V_\psi}(X,\mu) := \{\overline{[\xi]_\psi} : \xi \in \mathcal{V}_\psi(X,\mu)\}$.

Let (ψ,ϕ) be a reproducing pair. We want to prove that the spaces $V_\phi(X,\mu)^*$ and $\overline{V_\psi}(X,\mu)$ can be identified. This is the first essential result of [10], to which we refer for a proof, see Lemmas 3.11 and 3.12. Corresponding to $\widehat{T}_\phi$, we introduce the linear operator $\widehat{C}_{\psi,\phi} : \mathcal{H} \to V_\phi(X,\mu)$ by $\widehat{C}_{\psi,\phi} f := [C_\psi f]_\phi$. We note that $\widehat{C}_{\psi,\phi} f = \widehat{C}_{\psi,\phi} f'$ implies $f = f'$, as can be seen easily.

Thus we state:

Theorem 3.8 *If (ψ,ϕ) is a reproducing pair, the map j defined in* (3.11) *is surjective. Hence $V_\phi(X,\mu)^* \simeq \overline{V_\psi}(X,\mu)$, where $\simeq$ denotes a bounded isomorphism and the norm $\|\cdot\|_\psi$ is the dual norm of $\|\cdot\|_\phi$. Moreover, $\mathsf{Ran}\,\widehat{C}_{\psi,\phi}[\|\cdot\|_\phi] = V_\phi(X,\mu)[\|\cdot\|_\phi]$ and $\mathsf{Ran}\,\widehat{C}_{\phi,\psi}[\|\cdot\|_\psi] = V_\psi(X,\mu)[\|\cdot\|_\psi]$.*

Proof. Let (ψ,ϕ) be a reproducing pair. First one shows that $\mathsf{Ran}\,\widehat{C}_{\psi,\phi}$ is closed in $V_\phi(X,\mu)[\|\cdot\|_\phi]$. Moreover, every $[\eta]_\psi \in V_\psi(X,\mu)$ defines a continuous linear functional on the closed subspace $\mathsf{Ran}\,\widehat{\mathsf{C}}_{\psi,\phi}[\|\cdot\|_\phi]$, which implies that the map j is surjective. Next, it turns out that $\mathsf{Ran}\,\widehat{C}_{\psi,\phi}$ is dense in $V_\phi(X,\mu)$. Hence, $\mathsf{Ran}\,\widehat{C}_{\psi,\phi}[\|\cdot\|_\phi]$ and $V_\phi(X,\mu)[\|\cdot\|_\phi]$ coincide, and similarly for the other pair. □

The first statement of the theorem implies that there exist $0 < \mathsf{m} \leq \mathsf{M} < \infty$ such that

$$\mathsf{m} \, \|f\| \leq \left\| \widehat{C}_{\psi,\phi} f \right\|_{\phi} \leq \mathsf{M} \, \|f\| \, , \; \forall \, f \in \mathcal{H}, \tag{3.12}$$

a relation that may have an independent interest. The inequalities (3.12) are, of course, very similar to the ones defining a frame, viz. (1.2) or (2.1). Yet they are more general, since they are satisfied for any reproducing pair, be it a frame or not.

By Theorems 3.7 and 3.8, it follows that, if (ψ, ϕ) is a reproducing pair, then for every $\eta \in V_\psi(X, \mu)$, there exists $g \in \mathcal{H}$ such that $\eta = \langle \phi.|g\rangle$.

In conclusion, we may state

Theorem 3.9 *If (ψ, ϕ) is a reproducing pair, the spaces $V_\phi(X, \mu)$ and $V_\psi(X, \mu)$ are both Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form* (3.8)*.*

Corollary 3.10 *If (ψ, ϕ) is a reproducing pair and $\phi = \psi$, then ψ is a continuous frame and $V_\psi(X, \mu)$ is a closed subspace of $L^2(X, \mu)$.*

Proof. Since the duality takes place with respect to the L^2 inner product, $V_\psi(X, \mu)$ is a subspace of $L^2(X, \mu)$. The equality $\mathsf{Ran}\, \widehat{C}_{\psi,\psi} = V_\psi(X, \mu)$ and the fact that $\widehat{C}_{\psi,\psi}$ is bounded from below with respect to the L^2-norm, by (3.12), imply that it is closed. □

Remark 3.11 The operator C_ϕ defined by (2.2) is linear, but the operator C_ϕ given in (3.6) is conjugate linear. However the latter maps $\mathcal{H}$ into $V_\phi(X, \mu)^*$, which is identified with $\overline{V_\psi}(X, \mu)$, thus C_ϕ maps $\mathcal{H}$ *linearly* into $V_\psi(X, \mu)$.

Actually Theorem 3.9 has an inverse. Indeed:

Theorem 3.12 *Let ϕ and ψ be weakly measurable and μ-total. Then, the couple (ψ, ϕ) is a reproducing pair if and only if $V_\phi(X, \mu)$ and $V_\psi(X, \mu)$ are Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form* (3.8)*.*

Proof. The 'if' part is Theorem 3.9. Let now $V_\phi(X, \mu)$ and $V_\psi(X, \mu)$ be Hilbert spaces in conjugate duality. Consider the sesquilinear form

$$\Omega_{\psi,\phi}(f, g) = \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle \, \mathrm{d}\mu(x), \;\; f, g \in \mathcal{H}.$$

By the definition of the norms $\|\cdot\|_\phi \, , \|\cdot\|_\psi$ and the duality condition, we have, for every $f, g \in \mathcal{H}$, the two inequalities

$$|\Omega_{\psi,\phi}(f, g)| \leq \|[\langle f|\psi.\rangle]_\phi\|_\phi \, \|g\| \, ,$$
$$|\Omega_{\psi,\phi}(f, g)| \leq \|[\langle g|\phi.\rangle]_\psi\|_\psi \, \|f\| \, .$$

This means, the form $\Omega_{\psi,\phi}$ is separately continuous, hence jointly continuous. Therefore there exists a bounded operator $S_{\psi,\phi}$ such that $\Omega_{\psi,\phi}(f, g) = \langle S_{\psi,\phi} f|g\rangle$. First the operator $S_{\psi,\phi}$ is injective. Indeed, since $\mathsf{C}^*_\phi = \widehat{T}_\phi$, we have

$$\langle S_{\psi,\phi} f|g\rangle = \langle C_\psi f|C_\phi g\rangle = \langle \widehat{C}_{\psi,\phi} f|C_\phi g\rangle = \langle \widehat{T}_\phi \widehat{C}_{\psi,\phi} f|g\rangle, \; \forall f, g \in \mathcal{H}.$$

Now $\widehat{T}_\phi$ is isometric and $\widehat{C}_{\psi,\phi}$ is injective, hence $\widehat{T}_\phi \widehat{C}_{\psi,\phi} f = 0$ implies $f = 0$. Next, $S_{\psi,\phi}$ is also surjective, by Corollary 3.3. Hence $S_{\psi,\phi}$ belongs to $GL(\mathcal{H})$. □

In addition to Lemma 3.8, there is another characterization of the space $V_\psi(X, \mu)$, in terms of an eigenvalue equation, based on the fact that $\langle S^{-1}_{\psi,\phi}\phi_y|\psi_x\rangle$ is a reproducing kernel [29, Prop.3].

Proposition 3.13 *Let (ψ, ϕ) be a reproducing pair. Let $\xi \in \mathcal{V}_\psi(X, \mu)$ and consider the eigenvalue equation*

$$\int_X \xi(y)\langle S^{-1}_{\psi,\phi}\phi_y|\psi_x\rangle \, \mathrm{d}\mu(y) = \lambda\xi(x). \tag{3.13}$$

Then $\xi \in \mathsf{Ran}\, \mathsf{C}_\phi$ if and only if $\lambda = 1$, and $\xi \in \mathsf{Ker}\, T_\psi$ if and only if $\lambda = 0$. Moreover, there are no other eigenvalues.

4 Existence and nonuniqueness of reproducing partners

Given a weakly measurable function ψ, it is not obvious that there exists another function ϕ such that (ψ, ϕ) is a reproducing pair. Here is a criterion towards the existence of a specific dual partner.

Theorem 4.1 *Let ϕ be a weakly measurable function and $e = \{e_n\}_{n\in\mathbb{N}}$ an orthonormal basis of $\mathcal{H}$. There exists another measurable function ψ, such that (ψ, ϕ) is a reproducing pair if and only if $\mathsf{Ran}\,\widehat{T}_\phi = \mathcal{H}$ and there exists a family $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{V}_\phi(X,\mu)$ such that*

$$[\xi_n]_\phi = [\widehat{T}_\phi^{-1} e_n]_\phi, \ \forall n \in \mathbb{N}, \quad \text{and} \quad \sum_{n\in\mathbb{N}} |\xi_n(x)|^2 < \infty, \ \text{for a.e. } x \in X. \tag{4.1}$$

The proof of this theorem is quite technical and may be found in [10, Sec.4]. If ϕ is in fact a frame, then the reproducing partner ψ given by the proof of Theorem 4.1 is also a frame. However, in principle there may exist another reproducing partner ψ which is not Bessel, although we have no example so far.

Given the weakly measurable function ϕ, the fact that (ψ, ϕ) is a reproducing pair does not determine the function ψ uniquely. Indeed we have :

Theorem 4.2 *Let (ψ, ϕ) be a reproducing pair,. Then (θ, ϕ) is a reproducing pair if and only if $\theta = A\psi + \theta_0$, where $A \in GL(\mathcal{H})$ and $[\langle f|\theta_0(\cdot)\rangle]_\phi = [0]_\phi, \ \forall f \in \mathcal{H}$, i.e., $\widehat{C}_{\theta_0,\phi} f = 0, \forall\, f \in \mathcal{H}$.*

Proof. If $\theta = A\psi + \theta_0$ as above, then $S_{\theta,\phi} f = \widehat{T}_\phi(\widehat{C}_{A\psi,\phi} + \widehat{C}_{\theta_0,\phi}) f = \widehat{T}_\phi(\widehat{C}_{A\psi,\phi} f) = \widehat{T}_\phi \widehat{C}_{\psi,\phi} A^* f = S_{\psi,\phi} A^* f$, hence $S_{\theta,\phi} = S_{\psi,\phi} A^* \in GL(\mathcal{H})$.

Conversely, assume that (θ, ϕ) is a reproducing pair. By Theorem 3.8, we have $V_\phi(X,\mu) = \mathsf{Ran}\, C_\psi / \mathsf{Ker}\, T_\phi = \mathsf{Ran}\, C_\theta / \mathsf{Ker}\, T_\phi$, *i.e.*, for every $f \in \mathcal{H}$ there exists $g \in \mathcal{H}$ such that $[C_\theta f]_\phi = [C_\psi g]_\phi$. Then, using successively the definition of $S_{\phi,\theta}$, the relation above and the reproducing kernel (3.13), we obtain

$$\begin{aligned}
\langle f | S_{\phi,\theta} (S_{\psi,\phi}^{-1})^* \psi(\cdot) \rangle &= \int_X \langle f|\theta_x\rangle \langle \phi_x | (S_{\psi,\phi}^{-1})^* \psi.\rangle \,\mathrm{d}\mu(x) \\
&= \int_X \langle g|\psi_x\rangle \langle \phi_x | (S_{\psi,\phi}^{-1})^* \psi.\rangle \,\mathrm{d}\mu(x) = \langle g|\psi.\rangle = \langle f|\theta.\rangle\,, \ \forall\, f \in \mathcal{H}.
\end{aligned}$$

This means that, for all $f \in \mathcal{H}$, we have $[C_\theta f]_\phi = [C_{A\psi} f]_\phi$ or, equivalently, $\widehat{C}_{\theta,\phi} = \widehat{C}_{A\psi,\phi}$, where $A := S_{\phi,\theta}(S_{\psi,\phi}^{-1})^* \in GL(\mathcal{H})$. Moreover, $C_\theta f(x) = C_{A\psi} f(x) + F(f,x)$ for a.e. $x \in X$ and every $f \in \mathcal{H}$, where $F(f,\cdot) \in \mathsf{Ker}\, T_\phi$, *i.e.*, $F(f,x) = \langle f|(\theta - A\psi)_x\rangle =: \langle f|\theta_{0x}\rangle$. □

In addition, the existence of a reproducing partner to a given function ϕ is preserved if one replaces $V_\phi(X,\mu)$ by an isomorphic space $V'_\phi(X,\mu)$. Indeed:

Corollary 4.3 *Let ϕ, ϕ' be weakly measurable functions and $V_\phi(X,\mu) \simeq V_{\phi'}(X,\mu)$, where $\simeq$ denotes a bounded isomorphism. There exists ψ such that (ψ, ϕ) is a reproducing pair if and only if there exists ψ' such that (ψ', ϕ') is a reproducing pair.*

Proof. Suppose that (ψ, ϕ) is a reproducing pair. Let j be the bounded isomorphism $j : V_\phi(X,\mu) \to V_{\phi'}(X,\mu)$. Then $j^* : V_{\phi'}(X,\mu)^* \to V_\phi(X,\mu)^*$ is also a bounded isomorphism. Since $C_{\phi'}$ is bounded by (3.3), in order to verify that there exists ψ' such that $S_{\psi',\phi'} \in GL(\mathcal{H})$, we only need to check that $C_{\phi'}^{-1}$ is bounded. For every $f \in \mathcal{H}$, there exists $g \in \mathcal{H}$ such that $C_\phi f = j^* C_{\phi'} g$. Hence, $g \mapsto C_\phi^{-1} j^* C_{\phi'} g$ is surjective and bounded. It is moreover injective since μ-totality of ϕ' implies injectivity of $C_{\phi'}$. Thus, the bounded inverse theorem implies that $(C_\phi^{-1} j^* C_{\phi'})^{-1}$ and consequently $C_{\phi'}^{-1}$ are bounded. Since the statement is symmetric in (ψ, ϕ) and (ψ', ϕ'), the converse implication holds as well. □

5 Examples of reproducing pairs

In this section, we present a few concrete examples of the construction of Section 3. More details may be found in [10]. We begin with discrete examples, that is, $X = \mathbb{N}$ with the counting measure.

5.1 Discrete examples

5.1.1 Orthonormal basis

Let $e = \{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis, then e is a frame and $\mathcal{V}_e(\mathbb{N}) = V_e(\mathbb{N}) = \ell^2(\mathbb{N})$. Indeed, for $\xi \in \mathcal{V}_e(\mathbb{N})$, we have

$$\Big|\sum_{n\in\mathbb{N}} \xi_n \langle e_n|g\rangle\Big| = \Big|\sum_{n\in\mathbb{N}} \xi_n \overline{g_n}\Big| \leq c\,\|g\| = c\,\|\{g_n\}_{n\in\mathbb{N}}\|_{\ell^2}\,,\ \forall g \in \mathcal{H},$$

where $g_n := \langle g|e_n\rangle$. Hence, $\xi \in \ell^2(\mathbb{N})^* = \ell^2(\mathbb{N})$. Moreover, since $\mathsf{Ker}\, T_e = \{0\}$, it follows that $\mathcal{V}_e(\mathbb{N}) = V_e(\mathbb{N})$ and $\|\cdot\|_{\ell^2} = \|\cdot\|_e$.

5.1.2 Riesz basis

Now consider a Riesz basis $r = \{r_n\}_{n\in\mathbb{N}}$. Then $r_n = Ae_n$ for some $A \in GL(\mathcal{H})$ [16]. Therefore $\mathcal{V}_r(\mathbb{N}) = V_r(\mathbb{N}) = \ell^2(\mathbb{N})$ as sets, but with equivalent (not necessary equal) norms, as can be seen easily. Hence r is a frame.

5.1.3 Discrete upper and lower-semi frames

Let $\theta = \{\theta_n\}_{n\in\mathbb{N}}$ be a discrete frame, $m = \{m_n\}_{n\in\mathbb{N}} \subset \mathbb{C}\backslash\{0\}$ and define $\phi := \{m_n\theta_n\}_{n\in\mathbb{N}}$. If $\{|m_n|\}_{n\in\mathbb{N}} \in c_0$, then ϕ is an upper semi-frame and if $\{|m_n|^{-1}\}_{n\in\mathbb{N}} \in c_0$, then ϕ is a lower semi-frame. Observe that in both cases ϕ is not a frame.

It can easily be seen that $V_\phi(\mathbb{N}) = M_{1/m}(V_\theta(\mathbb{N})) = M_{1/m}(\mathsf{Ran}\, C_\theta)$ as sets, where M_m is the multiplication operator defined by $(M_m\xi)_n = m_n\xi_n$. Moreover, $\|\cdot\|_\phi \asymp \|\cdot\|_{\ell^2_m}$, where $\|\xi\|_{\ell^2_m} := \sum_{n\in\mathbb{N}} |\xi_n m_n|^2$. Also, $V_\phi(\mathbb{N})^* = M_m(V_\theta(\mathbb{N})^*) = M_m(V_\theta(\mathbb{N})) = M_m(\mathsf{Ran}\, C_\theta)$ as sets.

Using Theorem 4.1, one may show that there exists ψ such that (ψ, ϕ) is a reproducing pair [10]. A natural choice of a reproducing partner is $\psi := \{(1/\overline{m_n})\theta_n\}_{n\in\mathbb{N}}$ as $S_{\psi,\phi} = S_\theta \in GL(\mathcal{H})$.

5.2 Continuous examples

5.2.1 Continuous frames

If ϕ is a continuous frame, Corollary 3.10 implies that $V_\phi(X, \mu)$ is a closed subspace of $L^2(X, \mu)$. Now, since $L^2(X, \mu) = \mathsf{Ran}\, C_\phi \oplus \mathsf{Ker}\, D_\phi$, it follows that $V_\phi(X, \mu)[\|\cdot\|_\phi] \simeq \mathsf{Ran}\, C_\phi[\|\cdot\|_{L^2}]$.

5.2.2 1D continuous wavelets

Let $\psi, \phi \in L^2(\mathbb{R}, \mathrm{d}x)$ and consider the continuous wavelet systems $\phi_{x,a} = T_x D_a \phi$, where, as usual, T_x denotes the translation operator and D_a the dilation operator. If

$$\int_{\mathbb{R}} |\widehat{\psi}(\omega)\widehat{\phi}(\omega)| \frac{\mathrm{d}\omega}{|\omega|} < \infty \tag{5.1}$$

then (ψ, ϕ) is a reproducing pair for $L^2(\mathbb{R}, dx)$ with $S_{\psi,\phi} = c_{\psi,\phi} I$ [23, Theorem 10.1], where

$$c_{\psi,\phi} := \int_{\mathbb{R}} \overline{\widehat{\psi}(\omega)}\widehat{\phi}(\omega) \frac{\mathrm{d}\omega}{|\omega|}.$$

Actually the relation $S_{\psi,\phi} = c_{\psi,\phi} I$ simply expresses the well-known orthogonality relations of wavelet transforms. A similar result holds true for D-dimensional continuous wavelets [3] and, more generally, for all coherent states associated to square integrable group representations [3, Chaps. 8 and 12].

For $\psi = \phi$, the cross-admissibility condition (5.1) reduces to the classical admissibility condition

$$c_\phi := \int_{\mathbb{R}} |\widehat{\phi}(\omega)|^2 \frac{\mathrm{d}\omega}{|\omega|} < \infty. \tag{5.2}$$

Considering the obvious inequalities

$$|c_{\psi,\phi}| \leq \int_{\mathbb{R}} |\widehat{\psi}(\omega)\widehat{\phi}(\omega)| \frac{\mathrm{d}\omega}{|\omega|} \leq c_\phi^{1/2} c_\psi^{1/2},$$

we see that condition (5.1) is automatically satisfied whenever ϕ and ψ are both admissible. However, it is possible to choose a mother wavelet ϕ that does not satisfy the admissibility condition (5.2) and still obtain a reproducing pair (ψ, ϕ). Consider for example the Gaussian window $\phi(x) = e^{-\pi x^2}$, then $c_\phi = \infty$, which implies that ϕ is not a continuous wavelet frame. However, if one defines $\psi \in L^2(\mathbb{R}, dx)$ in the Fourier domain via $\widehat{\psi}(\omega) = |\omega|\widehat{\phi}(\omega)$, it follows that $0 < c_{\psi,\phi} = \|\phi\|_2^2 < \infty$. Hence (ψ, ϕ) is a reproducing pair. This example clearly shows the increasing flexibility obtained when replacing continuous frames by reproducing pairs.

5.2.3 *A continuous upper semi-frame: affine coherent states*

In [8, Sec. 2.6] the following example of an upper semi-frame is investigated. Define $\mathcal{H}^{(n)} := L^2(\mathbb{R}^+, r^{n-1}\, \mathrm{d}r)$, $n \in \mathbb{N}$, and the measure space $(X, \mu) = (\mathbb{R}, \mathrm{d}x)$. Let $\psi \in \mathcal{H}^{(n)}$ and define the affine coherent state

$$\psi_x(r) = e^{-ixr}\psi(r), \quad r \in \mathbb{R}^+.$$

Then ψ is admissible if $\sup_{r\in\mathbb{R}^+} \mathfrak{s}(r) = 1$, where $\mathfrak{s}(r) := 2\pi r^{n-1}|\psi(r)|^2$, and $|\psi(r)| \neq 0$, for a.e. $r \in \mathbb{R}^+$. The frame operator is given by the multiplication operator on $\mathcal{H}^{(n)}$

$$(S_\psi f)(r) = \mathfrak{s}(r) f(r),$$

and, more generally,

$$(S_\psi^m f)(r) = [\mathfrak{s}(r)]^m f(r), \ \forall\, m \in \mathbb{Z}.$$

Hence S_ψ is bounded and S_ψ^{-1} is unbounded.

The function ψ enjoys the interesting property that we can characterize the space $V_\psi(\mathbb{R}, \mathrm{d}x)$ and its norm. First, we show that $T_\phi\xi = \widehat{\xi}\psi$, which in turn implies that $\widehat{\xi}$ has to be given by an almost everywhere defined function which satisfies $\widehat{\xi}\psi \in \mathcal{H}^{(n)}$. Hence $\xi \in V_\psi(\mathbb{R}, \mathrm{d}x)$ provided $\widehat{\xi}\psi \in \mathcal{H}^{(n)}$ and then $\|\xi\|_\psi = \left\|\widehat{\xi}\psi\right\|$.

When looking for a reproducing partner for ψ, we first ask whether there exists an affine coherent state $\phi_x(r) = e^{-ixr}\phi(r),\ r \in \mathbb{R}^+,\ \phi \in \mathcal{H}^{(n)}$, such that (ψ, ϕ) forms a reproducing pair. The answer is negative. Indeed, since ψ is Bessel and not a frame, its dual ϕ is by necessity a lower semi-frame, whereas an affine coherent state must be Bessel, but can never satisfy the lower frame bound. Hence, there is no pair of affine coherent states forming a reproducing pair. This fact can also be proven by an explicit calculation.

More generally, we may look for a reproducing partner which is not an affine coherent state. First C_ψ is an isometry by Corollary 3.6, but $\mathsf{Ran}\, \widehat{T}_\psi \neq \mathcal{H}$. Indeed, if we had $\mathsf{Ran}\, \widehat{T}_\psi = \mathcal{H}$, an arbitrary element $h \in \mathcal{H}^{(n)} = L^2(\mathbb{R}^+, r^{n-1}\, \mathrm{d}r)$ could be written as $h = \widehat{T}_\psi\xi = \widehat{\xi}\psi$ for some $\xi \in V_\psi(\mathbb{R}, \mathrm{d}x)$. This applies, in particular, to ψ itself, which also belongs to $\mathcal{H}^{(n)}$. This in turn implies that there exists ξ, such that $\widehat{\xi}(r) = 1$ for a.e. $r \geq 0$. But there is no function that satisfies this condition (however the δ-distribution does the job).

This discussion has two major consequences. First, it shows that $V_\psi(\mathbb{R}, \mathrm{d}x)$ is *not* a Hilbert space, since it is not complete. Second, ψ has *no* reproducing partner at all.

5.2.4 *Continuous wavelets on the sphere*

Next we consider the continuous wavelet transform on the 2-sphere $\mathbb{S}^2$ [3, 5]. For a mother wavelet $\phi \in \mathcal{H} = L^2(\mathbb{S}^2, \mathrm{d}\mu)$, define the family of spherical wavelets

$$\phi_{\varrho,a} := R_\varrho D_a \phi, \ \text{where } (\varrho, a) \in X := SO(3) \times \mathbb{R}^+.$$

Here, D_a denotes the stereographic dilation operator and R_ϱ the unitary rotation on $\mathbb{S}^2$.

It has been shown in [5, Theorem 3.3] that the operator S_ϕ is diagonal in Fourier space (harmonic analysis on the 2-sphere reduces to expansions in spherical harmonics Y_l^m, $l \in \mathbb{N}, m = -l, \ldots, l$), thus it is a Fourier multiplier $\widehat{S_\phi f}(l,m) = s_\phi(l)\widehat{f}(l,m)$ with the symbol s_ϕ given by

$$s_\phi(l) := \frac{8\pi^2}{2l+1} \sum_{|m \leq l} \int_0^\infty \left|\widehat{D_a\phi}(l,m)\right|^2 \frac{\mathrm{d}a}{a^3},\ l \in \{0\} \cup \mathbb{N},$$

where $\widehat{D_a\phi}(l,m) := \langle Y_l^m | D_a\phi\rangle$ is the Fourier coefficient of $D_a\phi$.

The result of the analysis is twofold. First, the wavelet $\phi \in L^2(\mathbb{S}^2,\, \mathrm{d}\mu)$ is admissible if and only if there exists a constant $\mathsf{c} > 0$ such that $s_\phi(l) \leq \mathsf{c},\ \forall\, l \in \mathbb{N}$, equivalently, if the frame operator S_ϕ is bounded. In addition, for any admissible axisymmetric wavelet ϕ, there exists a constant $\mathsf{d} > 0$ such that $\mathsf{d} \leq s_\phi(l) \leq \mathsf{c},\ \forall\, l \in \mathbb{N}$. Equivalently, S_ϕ and S_ϕ^{-1} are both bounded, *i.e.,* the family of spherical wavelets $\{\phi_{a,\varrho},\ (\varrho, a) \in X = \mathrm{SO}(3) \times \mathbb{R}_+^*\}$ is a continuous frame. One notices, however, that the upper frame bound, which is implied by the constant c, does depend on ϕ, whereas the lower frame bound, which derives from d, does not, it follows from the asymptotic behavior of the function Y_l^m for large l.

However, it turns out [31] that the reconstruction formula converges if $\mathsf{d} \leq s_\phi(l) < \infty$ for all $l \in \{0\} \cup \mathbb{N}$, and this implies that ϕ (which is *not* admissible) is in fact a lower semi-frame and S_ϕ is unbounded, but densely defined.

We may now apply Theorem 4.1 to investigate the existence of a reproducing partner for ϕ. First, we show that $\mathsf{Ran}\,\widehat{T}_\phi = \mathcal{H}$. The operator M_ϕ defined by $\widehat{M_\phi f}(l,m) = s_\phi(l)^{-1}\widehat{f}(l,m)$ is bounded and constitutes a right inverse to S_ϕ. Hence, for every $f \in \mathcal{H}$, it holds

$$f = S_\phi M_\phi f = \widehat{T}_\phi [C_\phi M_\phi f]_\phi \in \mathsf{Ran}\,\widehat{T}_\phi.$$

Choosing $\xi_{l,m}(\varrho, a) := C_\phi(S_\phi^{-1} Y_l^m)(\varrho, a) = \langle S_\phi^{-1} Y_l^m | \phi_{\varrho,a}\rangle$ as a representative of $[\widehat{T}_\phi^{-1} Y_l^m]_\phi$ yields for every $(\varrho, a) \in \mathbb{R} \times \mathbb{R}^+$:

$$\begin{aligned}\sum_{l=0}^\infty \sum_{|m|\leq l} |\xi_{l,m}(\varrho,a)|^2 &= \sum_{l=0}^\infty \sum_{|m|\leq l} |\langle S_\phi^{-1} Y_l^n | \phi_{\varrho,a}\rangle|^2 = \sum_{l=0}^\infty \sum_{|m|\leq l} |s_\phi(l)^{-1} \widehat{\phi}_{\varrho,a}(l,m)|^2 \\ &\leq \frac{1}{\mathsf{d}} \sum_{l=0}^\infty \sum_{|m|\leq l} |\widehat{\phi}_{\varrho,a}(l,m)|^2 = \frac{1}{\mathsf{d}} \left\|\phi_{\varrho,a}\right\|^2 < \infty.\end{aligned}$$

Thus there exists (at least one) function $\psi \in L^2(\mathbb{S}^2,\, \mathrm{d}\mu)$ such that (ψ, ϕ) is a reproducing pair.

Moreover, as for the wavelets on $\mathbb{R}^d$, it is possible to choose another continuous wavelet system $\psi_{\varrho,a}$ as reproducing partner if the symbol $s_{\psi,\phi}$, defined by

$$s_{\psi,\phi}(l) := \frac{8\pi^2}{2l+1} \sum_{|m \leq l} \int_0^\infty \widehat{D_a\psi}(l,m)\, \overline{\widehat{D_a\phi}(l,m)} \frac{\mathrm{d}a}{a^3}.$$

satisfies $\mathsf{m} \leq |s_{\psi,\phi}(l)| \leq \mathsf{M}$ for all $l \in \{0\} \cup \mathbb{N}$.

6 Interlude: Reproducing pairs and PIP-spaces

Let (ψ, ϕ) be a reproducing pair. By definition,

$$\langle S_{\psi,\phi} f | g\rangle = \int_X \langle f | \psi_x\rangle \langle \phi_x | g\rangle \,\mathrm{d}\mu(x) = \int_X C_\psi f(x)\, \overline{C_\phi g(x)}\, \mathrm{d}\mu(x) \tag{6.1}$$

is well defined for all $f, g \in \mathcal{H}$ (here we revert to the *linear* maps C_ψ, C_ϕ defined in (2.2)). The r.h.s. coincides with the sesquilinear form (3.8), that is, the L^2 inner product, but generalized, since in general $C_\psi f, C_\phi g$ need not belong to $L^2(X,\, \mathrm{d}\mu)$.

This fact clearly indicates that the analysis should be performed in the framework of a partial inner product space (PIP-space) of measurable functions on X [6]. The question is, how to embed $\mathsf{Ran}\,(C_\psi)$ and $\mathsf{Ran}\,(C_\phi)$ into the corresponding assaying subspaces. Next we have to determine how the Hilbert spaces V_ψ and V_ϕ are related to the latter. Following [11], we will examine successively the cases of a rigged Hilbert space (RHS) and a genuine PIP-space. Then we particularize the results to a Hilbert scale and to a PIP-space of L^p spaces. The motivation for the last case is the following. If, following [29], we make the innocuous assumption that the map $x \mapsto \psi_x$ is bounded, *i.e.,* $\sup_{x\in X} \|\psi_x\|_{\mathcal{H}} \leq c$ for some $c > 0$ (often $\|\psi_x\|_{\mathcal{H}} =$ const., e.g. for wavelets or coherent states), then $(C_\psi f)(x) = \langle f|\psi_x\rangle \in L^\infty(X, \mathrm{d}\mu)$ so that a PIP-space based on the lattice generated by the family $\{L^p(X, \mathrm{d}\mu), 1 \leq p \leq \infty,\}$ may be a good solution.

7 Reproducing pairs and RHS

We begin with the simplest example of a PIP-space, namely, a rigged Hilbert space (RHS). Let indeed $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ be a RHS, where $\mathcal{D}^\times[t^\times]$ denotes the space of continuous conjugate linear functionals on $\mathcal{D}$, equipped with the strong dual topology $t^\times$. We assume that $\mathcal{D}[t]$ is reflexive, so that t and $t^\times$ coincide with the respective Mackey topologies [28]. Given a measure space (X, μ), we denote by $\langle\cdot,\cdot\rangle$ the sesquilinear form expressing the duality between $\mathcal{D}^\times$ and $\mathcal{D}$. As usual, we suppose that this sesquilinear form extends the inner product of $\mathcal{D}$ (and $\mathcal{H}$). This allows to build the triplet above.

Let $x \in X \mapsto \psi_x$, $x \in X \mapsto \phi_x$ be weakly measurable functions from X into $\mathcal{D}^\times$. Instead of (3.1), we consider the sesquilinear form

$$\Omega^{\mathcal{D}}_{\psi,\phi}(f, g) = \int_X \langle f, \psi_x\rangle\langle\phi_x, g\rangle \,\mathrm{d}\mu(x),\ f, g \in \mathcal{D}. \tag{7.1}$$

For short, we put $\Omega^{\mathcal{D}} := \Omega^{\mathcal{D}}_{\psi,\phi}$ and we assume that $\Omega^{\mathcal{D}}$ is jointly continuous on $\mathcal{D}\times\mathcal{D}$, that is , $\Omega^{\mathcal{D}} \in \mathsf{B}(\mathcal{D},\mathcal{D})$ in the notation of [4, Sec.10.2]. Then the relation

$$\langle S_{\psi,\phi} f, g\rangle := \int_X \langle f, \psi_x\rangle\langle\phi_x, g\rangle \,\mathrm{d}\mu(x),\ \forall\, f, g \in \mathcal{D}, \tag{7.2}$$

tells us that the operator $S_{\psi,\phi}$ belongs to $\mathcal{L}(\mathcal{D},\mathcal{D}^\times)$, the space of all continuous linear maps from $\mathcal{D}$ into $\mathcal{D}^\times$.

7.1 A Hilbertian approach

We first assume that the sesquilinear form $\Omega^{\mathcal{D}}$ is well-defined and bounded on $\mathcal{D} \times \mathcal{D}$ in the topology of $\mathcal{H}$. Then $\Omega^{\mathcal{D}}$ extends to a bounded sesquilinear form on $\mathcal{H} \times \mathcal{H}$, denoted by the same symbol.

The definition of the space $\mathcal{V}_\phi(X, \mu)$ must be modified as follows. Instead of (3.3), we suppose that the integral below exists and defines a conjugate linear functional on $\mathcal{D}$, bounded in the topology of $\mathcal{H}$, *i.e.,*

$$\left|\int_X \xi(x)\langle\phi_x, g\rangle \,\mathrm{d}\mu(x)\right| \leq c\,\|g\|\ ,\ \forall\, g \in \mathcal{D}. \tag{7.3}$$

Then the functional extends to a bounded conjugate linear functional on $\mathcal{H}$, since $\mathcal{D}$ is dense in $\mathcal{H}$. Hence, for every $\xi \in \mathcal{V}_\phi(X, \mu)$, there exists a unique vector $h_{\phi,\xi} \in \mathcal{H}$ such that

$$\int_X \xi(x)\langle\phi_x, g\rangle \,\mathrm{d}\mu(x) = \langle h_{\phi,\xi}|g\rangle, \quad \forall g \in \mathcal{D}.$$

Next, we define a linear map $T_\phi : \mathcal{V}_\phi(X, \mu) \to \mathcal{H}$ by

$$T_\phi\xi = h_{\phi,\xi} \in \mathcal{H},\ \forall\, \xi \in \mathcal{V}_\phi(X, \mu), \tag{7.4}$$

in the following weak sense

$$\langle T_\phi\xi|g\rangle = \langle h_{\phi,\xi}|g\rangle = \int_X \xi(x)\langle\phi_x, g\rangle \,\mathrm{d}\mu(x),\ g \in \mathcal{D}, \xi \in \mathcal{V}_\phi(X, \mu).$$

The rest proceeds as before. We consider the space $V_\phi(X,\mu) = \mathcal{V}_\phi(X,\mu)/\mathsf{Ker}\, T_\phi$, with the norm $\|[\xi]_\phi\|_\phi = \|T_\phi \xi\|$, where, for $\xi \in V_\phi(X,\mu)$, we have put $[\xi]_\phi = \xi + \mathsf{Ker}\, T_\phi$. Then $V_\phi(X,\mu)$ is a pre-Hilbert space for that norm.

Assume, in addition, that the corresponding bounded operator $S_{\psi,\phi}$ is an element of $GL(\mathcal{H})$. Then (ψ, ϕ) is a reproducing pair and Theorem 3.9 remains true, that is,

Theorem 7.1 *If (ψ, ϕ) is a reproducing pair, the spaces $V_\phi(X,\mu)$ and $V_\psi(X,\mu)$ are both Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form* (3.8) *or* (3.10)*, namely,*

$$\langle [\xi]_\phi | [\eta]_\psi \rangle = \langle\!\langle \xi | \eta \rangle\!\rangle_\mu = \int_X \xi(x)\overline{\eta(x)}\,\mathrm{d}\mu(x),\ \forall\, \xi \in \mathcal{V}_\phi(X,\mu),\ \eta \in \mathcal{V}_\psi(X,\mu). \tag{7.5}$$

Example 7.2 To give a trivial example, consider the Schwartz rigged Hilbert space $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}, \mathrm{d}x) \subset \mathcal{S}^\times(\mathbb{R})$, $(X,\mu) = (\mathbb{R}, \mathrm{d}x)$, $\psi_x(t) = \phi_x(t) = \frac{1}{\sqrt{2\pi}} e^{ixt}$. Then $C_\phi f = \widehat{f}$, the Fourier transform, so that $\langle f | \phi(\cdot)\rangle \in L^2(\mathbb{R}, \mathrm{d}x)$. In this case

$$\Omega^{\mathcal{D}}_{\psi,\phi}(f,g) = \int_{\mathbb{R}} \langle f, \psi_x\rangle \langle \phi_x, g\rangle\, \mathrm{d}x = \langle \widehat{f} | \widehat{g} \rangle = \langle f | g\rangle,\ \forall f, g \in \mathcal{S}(\mathbb{R}),$$

and $V_\psi(\mathbb{R}, \mathrm{d}x) = V_\phi(\mathbb{R}, \mathrm{d}x) = L^2(\mathbb{R}, \mathrm{d}x)$.

7.2 The general case

In the general case, we only assume that the form $\Omega^{\mathcal{D}}$ is jointly continuous on $\mathcal{D}\times\mathcal{D}$, with no other regularity requirement. In that case, the vector space $\mathcal{V}_\phi(X,\mu)$ must be defined differently. Let the topology of $\mathcal{D}$ be given by a directed family $\mathfrak{P}$ of seminorms. Given a weakly measurable function ϕ, we denote again by $\mathcal{V}_\phi(X,\mu)$ the space of all measurable functions $\xi : X \to \mathbb{C}$ such that the integral $\int_X \xi(x)\langle \phi_x, g\rangle\, \mathrm{d}\mu(x)$ exists for every $g \in \mathcal{D}$ and defines a continuous conjugate linear functional on $\mathcal{D}$, that is, there exists a constant $c > 0$ and a seminorm $\mathsf{p} \in \mathfrak{P}$ such that

$$\left| \int_X \xi(x)\langle \phi_x, g\rangle\, \mathrm{d}\mu(x) \right| \leq c\, \mathsf{p}(g),\ \forall\, g \in \mathcal{D}.$$

This in turn determines a linear map $T_\phi : \mathcal{V}_\phi(X,\mu) \to \mathcal{D}^\times$ by the following relation

$$\langle T_\phi \xi, g\rangle = \int_X \xi(x)\langle \phi_x, g\rangle\, \mathrm{d}\mu(x),\ \forall\, \xi \in \mathcal{V}_\phi(X,\mu), g \in \mathcal{D}. \tag{7.6}$$

Next, we define as before the vector space

$$V_\phi(X,\mu) = \mathcal{V}_\phi(X,\mu)/\mathsf{Ker}\, T_\phi,$$

and we put again $[\xi]_\phi = \xi + \mathsf{Ker}\, T_\phi$ for $\xi \in \mathcal{V}_\phi(X,\mu)$.

Now we define the topology of $V_\phi(X,\mu)$ by means of the strong dual topology $t^\times$ of $\mathcal{D}^\times$, which we recall is defined by the seminorms

$$\|F\|_{\mathcal{Q}} = \sup_{g\in\mathcal{Q}} |\langle F | g\rangle, \quad F \in \mathcal{D}^\times,$$

where $\mathcal{Q}$ runs over the family of bounded subsets of $\mathcal{D}[t]$. As said above, the reflexivity of $\mathcal{D}$ entails that $t^\times$ is equal to the Mackey topology $\tau(\mathcal{D}^\times, \mathcal{D})$. Define the following seminorm on $V_\phi(X,\mu)$:

$$\widehat{\mathsf{p}}_{\mathcal{Q}}([\xi]_\phi) := \sup_{g\in\mathcal{Q}} |\langle T_\phi \xi, g\rangle|\,, \tag{7.7}$$

where $\mathcal{Q}$ is a bounded subset of $\mathcal{D}[t]$. Then we may state

Lemma 7.3 *The map $\widehat{T}_\phi : V_\phi(X,\mu) \to \mathcal{D}^\times$, $\widehat{T}_\phi[\xi]_\phi := T_\phi \xi$ is a well-defined linear map of $V_\phi(X,\mu)$ into $\mathcal{D}^\times$ and, for every bounded subset $\mathcal{Q}$ of $\mathcal{D}[t]$, one has*

$$\widehat{\mathsf{p}}_{\mathcal{Q}}([\xi]_\phi) = \|T_\phi \xi\|_{\mathcal{Q}}, \quad \forall \xi \in \mathcal{V}_\phi(X,\mu)$$

The latter equality obviously implies the continuity of T_ϕ.

Next we investigate the dual $V_\phi(X,\mu)^*$ of the space $V_\phi(X,\mu)$, that is, the set of continuous linear functionals on $V_\phi(X,\mu)$. We equip $V_\phi(X,\mu)^*$ with the strong dual topology, which is defined by the family of seminorms

$$\mathsf{q}_{\mathcal{R}}(F) := \sup_{[\xi]_\phi \in \mathcal{R}} |F([\xi]_\phi)|,$$

where $\mathcal{R}$ runs over the bounded subsets of $V_\phi(X,\mu)$.

Theorem 7.4 *Assume that $\mathcal{D}[t]$ is a reflexive space and let ϕ be a weakly measurable function. If F is a continuous linear functional on $V_\phi(X,\mu)$, then there exists a unique $g \in \mathcal{D}$ such that*

$$F([\xi]_\phi) = \int_X \xi(x)\langle \phi_x, g\rangle \,\mathrm{d}\mu(x),\ \forall\, \xi \in V_\phi(X,\mu) \tag{7.8}$$

Moreover, every $g \in \mathcal{H}$ defines a continuous linear functional F on $V_\phi(X,\mu)$ with $\|F\|_{\phi^*} \leq \|g\|$, by (7.8).

The proof of this theorem follows closely that of Theorem 3.5, replacing Hilbertian norms by appropriate seminorms and using the reflexivity of $\mathcal{D}$. Details may be found in [11].

In the present context, the analysis operator C_ϕ is defined in the usual way, given in (2.2). Then, particularizing the discussion of Theorem 3.5 to the functional $\langle \cdot, C_\phi g\rangle$, one can interpret the analysis operator C_ϕ as a continuous operator from $\mathcal{D}$ to $V_\phi(X,\mu)^*$. As in the case of frames or semi-frames, one may characterize the synthesis operator in terms of the analysis operator.

Proposition 7.5 *For a weakly measurable function ϕ , $\widehat{T}_\phi \subseteq C_\phi^*$. If, in addition, $V_\phi(X,\mu)$ is reflexive, then $\widehat{T}_\phi^* = C_\phi$. Moreover, ϕ is μ-total (i.e., $\mathsf{Ker}\, C_\phi = \{0\}$) if and only if $\mathsf{Ran}\, \widehat{T}_\phi$ is dense in $\mathcal{D}^\times$.*

Proof. As $C_\phi : \mathcal{D} \to V_\phi(X,\mu)^*$ is a continuous operator, it has a continuous adjoint $C_\phi^* : V_\phi(X,\mu)^{**} \to \mathcal{H}$ [28, Sec.IV.7.4]. Let $C_\phi^\sharp := C_\phi^* \restriction V_\phi(X,\mu)$. Then $C_\phi^\sharp = \widehat{T}_\phi$ since, for every $f \in \mathcal{D}$, $[\xi]_\phi \in V_\phi(X,\mu)$,

$$\langle C_\phi f, [\xi]_\phi\rangle = \int_X \langle f, \phi_x\rangle \overline{\xi(x)}\,\mathrm{d}\mu(x) = \langle f, \widehat{T}_\phi[\xi]_\phi\rangle. \tag{7.9}$$

If $V_\phi(X,\mu)$ is reflexive, we have, of course, $C_\phi^\sharp = C_\phi^* = \widehat{T}_\phi$.

If ϕ is not μ-total, then there exists $f \in \mathcal{D}$, $f \neq 0$ such that $(C_\phi f)(x) = 0$ for a.e. $x \in X$. Hence, $f \in (\mathsf{Ran}\, \widehat{T}_\phi)^\perp := \{f \in \mathcal{D} : \langle F, f\rangle = 0, \forall F \in \mathsf{Ran}\, \widehat{T}_\phi\}$ by (7.9). Conversely, if ϕ is μ-total, as $(\mathsf{Ran}\, \widehat{T}_\phi)^\perp = \mathsf{Ker}\, C_\phi = \{0\}$, by the reflexivity of $\mathcal{D}$ and $\mathcal{D}^\times$, it follows that $\mathsf{Ran}\, \widehat{T}_\phi$ is dense in $\mathcal{D}^\times$. □

In a way similar to what we have done above, we can define the space $V_\psi(X,\mu)$, its topology, the residue classes $[\eta]_\psi$, the operator T_ψ, etc, replacing ϕ by ψ. Then, $V_\psi(X,\mu)$ is a locally convex space.

Theorem 7.6 *Assume that the form* (7.1) *is jointly continuous on $\mathcal{D} \times \mathcal{D}$. Then, every continuous linear functional F on $V_\phi(X,\mu)$, i.e., $F \in V_\phi(X,\mu)^*$, can be represented as*

$$F([\xi]_\phi) = \int_X \xi(x)\overline{\eta(x)}\,\mathrm{d}\mu(x),\ \forall\, [\xi]_\phi \in V_\phi(X,\mu), \tag{7.10}$$

with $\eta \in \mathcal{V}_\psi(X,\mu)$. The residue class $[\eta]_\psi \in V_\psi(X,\mu)$ is uniquely determined.

Proof. By Theorem 7.4, we have the representation

$$F(\xi) = \int_X \xi(x)\langle \phi_x, g\rangle \,\mathrm{d}\mu(x).$$

It is easily seen that $\eta(x) := \langle g, \phi_x\rangle \in \mathcal{V}_\psi(X,\mu)$.

It remains to prove uniqueness. Suppose that

$$F(\xi) = \int_X \xi(x)\overline{\eta'(x)}\,\mathrm{d}\mu(x).$$

Then

$$\int_X \xi(x)(\overline{\eta'(x)} - \overline{\eta(x)})\,\mathrm{d}\mu(x) = 0.$$

Now the function $\xi(x)$ is arbitrary. Hence, taking in particular for $\xi(x)$ the functions $\langle\psi_x, f\rangle, f \in \mathcal{D}$, we get $[\eta]_\psi = [\eta']_\psi$. □

The lesson of the previous statements is that the map

$$j : F \in V_\phi(X,\mu)^* \mapsto [\eta]_\psi \in V_\psi(X,\mu) \tag{7.11}$$

is well-defined and conjugate linear. On the other hand, $j(F) = j(F')$ implies easily $F = F'$. Therefore $V_\phi(X,\mu)^*$ can be identified with a closed subspace of $\overline{V_\psi}(X,\mu) := \{\overline{[\xi]_\psi} : \xi \in \mathcal{V}_\psi(X,\mu)\}$, the conjugate space of $V_\psi(X,\mu)$.

Working in the framework of Hilbert spaces, as in Section 7.1, we proved in Theorem 3.9 that the spaces $V_\phi(X,\mu)^*$ and $\overline{V_\psi}(X,\mu)$ can be identified. The conclusion was that if (ψ,ϕ) is a reproducing pair, the spaces $V_\phi(X,\mu)$ and $V_\psi(X,\mu)$ are both Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form (3.10). And if ϕ and ψ are also μ-total, then the converse statement holds true.

In the present situation, however, a result of this kind cannot be proved with techniques similar to those of Section 3.2 , which are specific of Hilbert spaces. In particular, the condition (b), $S_{\psi,\phi} \in GL(\mathcal{H})$, which was essential in the proof of [10, Lemma 3.11], is now missing, and it is not clear by what regularity condition it should replaced.

8 Reproducing pairs and genuine PIP-spaces

In this section, we will consider the case where our measurable functions take their values in a genuine PIP-space. However, for simplicity, we will restrict ourselves to a lattice of Banach spaces (LBS) or a lattice of Hilbert spaces (LHS). For the convenience of the reader, we have summarized in the Appendix the basic notions concerning LBSs and LHSs. Further information may be found in our monograph [6].

Let (X,μ) be a locally compact, σ-compact measure space. Let $V_J = \{V_p, p \in J\}$ be a LBS or a LHS of measurable functions on X. Thus the central Hilbert space is $\mathcal{H} := V_o = L^2(X,\mu)$ and the spaces $V_p, V_{\overline{p}}$ are reflexive Banach spaces or Hilbert spaces, conjugate dual of each other with respect to the L^2 inner product, as follows from (A.2). The partial inner product, which extends that of $L^2(X,\mu)$, is denoted again by $\langle\cdot|\cdot\rangle$. As usual we put $V = \sum_{p\in J} V_p$ and $V^\# = \bigcap_{p\in J} V_p$. Thus $\psi : X \to V$ really means that $\psi : X \to V_p$ for some $p \in J$, since V is the algebraic inductive limit of $\{V_p, p \in J\}$ [28] (see the Appendix).

Example 8.1 A typical example is the lattice generated by the Lebesgue spaces $L^p(\mathbb{R}, \mathrm{d}x)$, $1 \le p \le \infty$, with $\frac{1}{p} + \frac{1}{\overline{p}} = 1$ [6]. We shall discuss it in detail in Section 10.

Two approaches are possible, depending whether the functions ψ_x themselves belong to V or rather the scalar functions $C_\psi f$. However, the first possibility is the exact generalization of the one used in the RHS case in Section 7. Hence it does not exploit the PIP-space structure, only the RHS $V^\# \subset \mathcal{H} \subset V$! Thus we turn to the second strategy.

Let ψ, ϕ be weakly measurable functions from X into $\mathcal{H}$. In view of (6.1), (A.2) and the definition of V, we assume that the following condition holds:

(p) $\exists\, p \in J$ such that $C_\psi f = \langle f|\psi_\cdot\rangle \in V_p$ and $C_\phi g = \langle g|\phi_\cdot\rangle \in V_{\overline{p}}, \forall\, f, g \in \mathcal{H}$.

Notice that, in Condition (p), the index p cannot depend on f, g. We need some uniformity, in the form $C_\psi(\mathcal{H}) \subset V_p$ and $C_\phi(\mathcal{H}) \subset V_{\overline{p}}$. This is fully in line with the philosophy of PIP-spaces: the building blocks are the (assaying) subspaces V_p, not individual vectors.

Since $V_{\overline{p}}$ is the conjugate dual of V_p, the relation

$$\Omega_{\psi,\phi}(f,g) := \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle \,\mathrm{d}\mu(x),\ f,g \in \mathcal{H},$$

defines a sesquilinear form on $\mathcal{H}\times\mathcal{H}$ and one has

$$|\Omega_{\psi,\phi}(f,g)| \leq \|C_\psi f\|_p \ \|C_\phi g\|_{\overline{p}}\,,\ \forall\, f,g \in \mathcal{H}. \tag{8.1}$$

If $\Omega_{\psi,\phi}$ is bounded as a form on $\mathcal{H}\times\mathcal{H}$ (this is not automatic, see Proposition 8.2), there exists a bounded operator $S_{\psi,\phi}$ in $\mathcal{H}$ such that

$$\int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle \,\mathrm{d}\mu(x) = \langle S_{\psi,\phi} f|g\rangle,\ \forall\, f,g \in \mathcal{H}. \tag{8.2}$$

Then (ψ,ϕ) is a *reproducing pair* if $S_{\psi,\phi} \in GL(\mathcal{H})$.

Let us suppose that the spaces V_p have the following property:

(k) If $\xi_n \to \xi$ in V_p, then, for every compact subset $K \subset X$, there exists a subsequence $\{\xi_n^K\}$ of $\{\xi_n\}$ which converges to ξ almost everywhere in K.

We note that condition (k) is satisfied by the L^p-spaces [27].

As seen before, $C_\psi : \mathcal{H} \to V$, in general. This means, given $f \in \mathcal{H}$, there exists $p \in J$ such that $C_\psi f = \langle f|\psi.\rangle \in V_p$. We define

$$D_r(C_\psi) = \{f \in \mathcal{H} :\ C_\psi f \in V_r\},\ r \in J.$$

In particular, $D_r(C_\psi) = \mathcal{H}$ is equivalent to $C_\psi(\mathcal{H}) \subset V_r$.

Proposition 8.2 *Assume that* (k) *holds. Then*
(i) $C_\psi : D_r(C_\psi) \to V_r$ is a closed linear map.
(ii) If, for some $r \in J$, $C_\psi(\mathcal{H}) \subset V_r$, then $C_\psi : \mathcal{H} \to V_r$ is continuous.

Proof. (i) Let $f_n \to f$ in $\mathcal{H}$ and $\{C_\psi f_n\}$ be Cauchy in V_r. Since V_r is complete, there exists $\xi \in V_r$ such that $\|C_\psi f_n - \xi\|_r \to 0$. By (k), for every compact subset $K \subset X$, there exists a subsequence $\{f_n^K\}$ of $\{f_n\}$ such that $(C_\psi f_n^K)(x) \to \xi(x)$ a.e. in K. On the other hand, since $f_n \to f$ in $\mathcal{H}$, we get

$$\langle f_n|\psi_x\rangle \to \langle f|\psi_x\rangle, \quad \forall x \in X,$$

and the same holds true, of course, for $\{f_n^K\}$. From this we conclude that $\xi(x) = \langle f|\psi_x\rangle$ almost everywhere. Thus, $f \in D_r(C_\psi)$ and $\xi = C_\psi f$.

(ii) As for the continuity of $C_\psi : \mathcal{H} \to V_r$ it follows from (i) and the closed graph theorem. □

Combining Proposition 8.2(ii) with (8.1), we get

Corollary 8.3 *Assume that* (k) *holds. If $C_\psi(\mathcal{H}) \subset V_p$ and $C_\phi(\mathcal{H}) \subset V_{\overline{p}}$, the form Ω is bounded on $\mathcal{H}\times\mathcal{H}$, that is, $|\Omega_{\psi,\phi}(f,g)| \leq c\,\|f\|\ \|g\|$.*

If $C_\psi(\mathcal{H}) \subset V_r$, we will assume that $C_\psi : \mathcal{H} \to V_r$ is continuous. According to Proposition 8.2, this is automatic if condition (k) holds.

If $C_\psi : \mathcal{H} \to V_r$ continuously, then $C_\psi^* : V_{\overline{r}} \to \mathcal{H}$ exists and it is continuous. By definition, if $\xi \in V_{\overline{r}}$,

$$\langle C_\psi f|\xi\rangle = \int_X \langle f|\psi_x\rangle\overline{\xi(x)}\,\mathrm{d}\mu(x) = \langle f|\int_X \psi_x\,\xi(x)\,\mathrm{d}\mu(x)\rangle,\ \forall\, f \in \mathcal{H}. \tag{8.3}$$

Thus,

$$C_\psi^*\xi = \int_X \psi_x\,\xi(x)\,\mathrm{d}\mu(x).$$

Assume now that for some $p \in J$, $C_\psi : \mathcal{H} \to V_p$ and $C_\phi : \mathcal{H} \to V_{\overline{p}}$ continuously. Then, $C_\phi^* : V_p \to \mathcal{H}$ so that $C_\phi^* C_\psi$ is a well-defined bounded operator in $\mathcal{H}$. As before, we have

$$C_\phi^* \eta = \int_X \eta(x) \phi_x \, \mathrm{d}\mu(x), \ \forall \eta \in V_p.$$

Hence,

$$C_\phi^* C_\psi f = \int_X \langle f | \psi_x \rangle \phi_x \, \mathrm{d}\mu(x) = S_{\psi,\phi} f, \quad \forall f \in \mathcal{H},$$

the last equality following also from (8.2) and Corollary 8.3. Of course, this does not yet imply that $S_{\psi,\phi} \in GL(\mathcal{H})$, thus we don't know whether (ψ, ϕ) is a reproducing pair.

According to (3.3), the pre-Hilbert space $\mathcal{V}_\phi(X,\mu)$ consists of the measurable functions ξ such that

$$\left| \int_X \xi(x) \overline{(C_\phi g)(x)} \, \mathrm{d}\mu(x) \right| \leq c \, \|g\| \, , \ \forall g \in \mathcal{H}. \tag{8.4}$$

Since $C_\phi : \mathcal{H} \to V_{\overline{p}}$, the integral is well defined for all $\xi \in V_p$. This means, the inner product on the l.h.s. is in fact the partial inner product of V, which coincides with the L^2 inner product whenever the latter makes sense. Thus we may rewrite the relation (8.4) as

$$|\langle \xi | C_\phi g \rangle| \leq c \, \|g\| \, , \forall g \in \mathcal{H}, \ \xi \in V_p \, ,$$

where $\langle \cdot | \cdot \rangle$ denotes the partial inner product. Next, by (A.2), one has, for $\xi \in V_p, g \in \mathcal{H}$,

$$|\langle \xi | C_\phi g \rangle| \leq \|\xi\|_p \, \|C_\phi g\|_{\overline{p}} \leq c \, \|\xi\|_p \, \|g\| \, ,$$

where the last inequality follows from Proposition 8.2 or the assumption of continuity of C_ϕ. Hence $\xi \in \mathcal{V}_\phi(X,\mu)$, so that $V_p \subset \mathcal{V}_\phi(X,\mu)$.

As for the adjoint operator, we have $C_\phi^* : V_p \to \mathcal{H}$. Then we may write, for $\xi \in V_p, g \in \mathcal{H}$, $\langle \xi | C_\phi g \rangle = \langle T_\phi \xi | g \rangle$, thus C_ϕ^* is the restriction from $\mathcal{V}_\phi(X,\mu)$ to V_p of the operator $T_\phi : \mathcal{V}_\phi \to \mathcal{H}$ introduced in Section 2, which reads now as

$$\langle T_\phi \xi | g \rangle = \int_X \xi(x) \langle \phi_x | g \rangle \, \mathrm{d}\mu(x), \ \forall \xi \in V_p, g \in \mathcal{H}. \tag{8.5}$$

Thus $C_\phi^* \subset T_\phi$.

From now on, the construction proceeds as in Section 7. The space $V_\phi(X,\mu) = \mathcal{V}_\phi(X,\mu)/\mathsf{Ker}\; T_\phi$, with the norm $\|[\xi]_\phi\|_\phi = \|T_\phi \xi\|$, is a pre-Hilbert space. Then Theorem 3.9 and the other results from Section 3.2 remain true. In particular, we have:

Theorem 8.4 *If (ψ, ϕ) is a reproducing pair, the spaces $V_\phi(X,\mu)$ and $V_\psi(X,\mu)$ are both Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form* (3.8)*, namely,*

$$\langle\!\langle \xi | \eta \rangle\!\rangle_\mu := \int_X \xi(x) \overline{\eta(x)} \, \mathrm{d}\mu(x).$$

Note the form (3.8) coincides with the inner product of $L^2(X,\mu)$ whenever the latter makes sense.

Let (ψ, ϕ) is a reproducing pair. Assume again that $C_\phi : \mathcal{H} \to V_{\overline{p}}$ continuously, which me may write $\widehat{C}_{\phi,\psi} : \mathcal{H} \to V_{\overline{p}}/\mathsf{Ker}\; T_\psi$, where $\widehat{C}_{\phi,\psi} : \mathcal{H} \to V_\psi(X,\mu)$ is the operator defined by $\widehat{C}_{\phi,\psi} f := [C_\phi f]_\psi$, already introduced in Section 3.2. In addition, by Theorem 3.8, one has $\mathsf{Ran}\, \widehat{C}_{\psi,\phi}[\|\cdot\|_\phi] = V_\phi(X,\mu)[\| \cdot \|_\phi]$ and $\mathsf{Ran}\, \widehat{C}_{\phi,\psi}[\|\cdot\|_\psi] = V_\psi(X,\mu)[\| \cdot \|_\psi]$.

Putting everything together, we get

Corollary 8.5 *Let (ψ, ϕ) be a reproducing pair. Then, if $C_\psi : \mathcal{H} \to V_p$ and $C_\phi : \mathcal{H} \to V_{\overline{p}}$ continuously, one has*

$$\widehat{C}_{\phi,\psi} : \mathcal{H} \to V_{\overline{p}}/\mathsf{Ker}\; T_\psi = V_\psi(X,\mu) \simeq \overline{V_\phi}(X,\mu)^*, \tag{8.6}$$

$$\widehat{C}_{\psi,\phi} : \mathcal{H} \to V_p/\mathsf{Ker}\; T_\phi = V_\phi(X,\mu) \simeq \overline{V_\psi}(X,\mu)^*. \tag{8.7}$$

In these relations, the equality sign means an isomorphism of vector spaces, whereas $\simeq$ denotes an isomorphism of Hilbert spaces.

Proof. On one hand, we have $\mathsf{Ran}\,\widehat{C}_{\phi,\psi} = V_\psi(X,\mu)$. On the other hand, under the assumption $C_\phi(\mathcal{H}) \subset V_{\overline{p}}$, one has $V_{\overline{p}} \subset \mathcal{V}_\psi(X,\mu)$, hence $V_{\overline{p}}/\mathsf{Ker}\ T_\psi = \{\xi + \mathsf{Ker}\,T_\psi,\ \xi \in V_{\overline{p}}\} \subset V_\psi(X,\mu)$. Thus we get $V_\psi(X,\mu) = V_{\overline{p}}/\mathsf{Ker}\ T_\psi$ as vector spaces. Similarly $V_\phi(X,\mu) = V_p/\mathsf{Ker}\ T_\phi$. □

9 The case of a Hilbert triplet or a Hilbert scale

9.1 The general construction

We have derived in the previous section the relations $V_p \subset \mathcal{V}_\phi(X,\mu), V_{\overline{p}} \subset \mathcal{V}_\psi(X,\mu)$, and their equivalent ones (8.6)-(8.7). Then, since $V_\psi(X,\mu)$ and $V_\phi(X,\mu)$ are both Hilbert spaces, it seems natural to take for $V_p, V_{\overline{p}}$ Hilbert spaces as well, that is, take for V a LHS. The simplest case is then a Hilbert chain, for instance, the scale (A.4) $\{\mathcal{H}_k, k \in \mathbb{Z}\}$ built on the powers of a self-adjoint operator $A > I$. This situation is quite interesting, since in that case one may get results about spectral properties of symmetric operators (in the sense of PIP-space operators) [6,12].

Thus, let (ψ,ϕ) be a reproducing pair. For simplicity, we assume that $S_{\psi,\phi} = I$, that is, ψ,ϕ are dual to each other.

If ψ and ϕ are both frames, there is nothing to say, since then $C_\psi(\mathcal{H}), C_\phi(\mathcal{H}) \subset L^2(X,\mu) = \mathcal{H}_o$, so that there is no need for a Hilbert scale. Thus we assume that ψ is an upper semi-frame and ϕ is a lower semi-frame, dual to each other. It follows that $C_\psi(\mathcal{H}) \subset L^2(X,\mu)$. Hence Condition (p) becomes: There is an index $k \geq 1$ such that $C_\psi : \mathcal{H} \to \mathcal{H}_k$ and $C_\phi : \mathcal{H} \to \mathcal{H}_{\overline{k}}$ continuously, thus $V_p \equiv \mathcal{H}_k$ and $V_{\overline{p}} \equiv \mathcal{H}_{\overline{k}}$. This means we are working in the Hilbert triplet

$$V_p \equiv \mathcal{H}_k \subset \mathcal{H}_o = L^2(X,\mu) \subset \mathcal{H}_{\overline{k}} \equiv V_{\overline{p}}\,. \tag{9.1}$$

Next, according to Corollary 8.5, we have $V_\psi(X,\mu) = \mathcal{H}_{\overline{k}}/\mathsf{Ker}\ T_\psi$ and $V_\phi(X,\mu) = \mathcal{H}_k/\mathsf{Ker}\ T_\phi$, as vector spaces.

In addition, since ϕ is a lower semi-frame, we know that C_ϕ has closed range in $L^2(X,\mu)$ and is injective [8, Lemma 2.1]. However its domain

$$D(C_\phi) := \{f \in \mathcal{H} : \int_X |\langle f|\phi_x\rangle|^2 \,\mathrm{d}\mu(x) < \infty\}$$

need not be dense, it could be $\{0\}$. Thus C_ϕ maps its domain $D(C_\phi)$ onto a closed subspace of $L^2(X,\mu)$, possibly trivial, and the whole of $\mathcal{H}$ into the larger space $\mathcal{H}_{\overline{k}}$.

9.2 Examples

As for concrete examples of such Hilbert scales, we might mention two. First the Sobolev spaces $H^k(\mathbb{R})$, $k \in \mathbb{Z}$, in $\mathcal{H}_0 = L^2(\mathbb{R}, dx)$, which is the scale generated by the powers of the positive self-adjoint operator $A^{1/2}$, where $A := 1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$. The other one corresponds to the quantum harmonic oscillator, with Hamiltonian $A_{\rm osc} := x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$. The spectrum of $A_{\rm osc}$ is $\{2n+1, n = 0,1,2,\ldots\}$ and it gets diagonalized on the basis of Hermite functions. It follows that $A_{\rm osc}^{-1}$, which maps every $\mathcal{H}_k$ onto $\mathcal{H}_{k-1}$, is a Hilbert-Schmidt operator. Therefore, the end space of the scale $\mathcal{D}^\infty(A_{\rm osc}) := \bigcap_k \mathcal{H}_k$, which is simply Schwartz' space $\mathcal{S}$ of C^∞ functions of fast decrease, is a nuclear space.

Actually one may give an explicit example, using a Sobolev-type scale. Let $\mathcal{H}_K$ be a reproducing kernel Hilbert space (RKHS) of (nice) functions on a measure space (X,μ), with kernel function $k_x, x \in X$, that is, $f(x) = \langle f|k_x\rangle_K, \forall f \in \mathcal{H}_K$. The corresponding reproducing kernel is $K(x,y) = k_y(x) = \langle k_y|k_x\rangle_K$. Choose the weight function $m(x) > 1$, the analog of the weight $(1+|x|^2)$ considered in the Sobolev case. Define the Hilbert scale $\mathcal{H}_l$, $l \in \mathbb{Z}$, determined by the multiplication operator $Af(x) = m(x)f(x)$, $\forall x \in X$. Hence, for each $l \geq 1$,

$$\mathcal{H}_l \subset \mathcal{H}_0 \equiv \mathcal{H}_K \subset \mathcal{H}_{\overline{l}}\,.$$

Then, for some $n \geq 1$, define the measurable functions $\phi_x = k_x m^n(x), \psi_x = k_x m^{-n}(x)$, so that $C_\psi : \mathcal{H}_K \to \mathcal{H}_n$, $C_\phi : \mathcal{H}_K \to \mathcal{H}_{\overline{n}}$ continuously, where $\mathcal{H}_n \subset \mathcal{H}_K \subset \mathcal{H}_{\overline{n}}$, and ψ,ϕ are dual of each other. One has indeed

$\langle\phi_x|g\rangle_K = \langle k_x m^n(x)|g\rangle_K = \langle k_x|g\, m^n(x)\rangle_K = \overline{g(x)}\, m^n(x) \in \mathcal{H}_{\overline{n}}$ and $\langle\psi_x|g\rangle_K = \overline{g(x)}\, m^{-n}(x) \in \mathcal{H}_n$, which implies duality. Thus (ψ, ϕ) is a reproducing pair with $S_{\psi,\phi} = I$, ψ is an upper semi-frame and ϕ a lower semi-frame.

In this case, one can compute the operators T_ψ, T_ϕ explicitly. The definition (8.5) reads as

$$\begin{aligned}\langle T_\phi \xi|g\rangle_K &= \int_X \xi(x)\langle\phi_x|g\rangle_K \,\mathrm{d}\mu(x),\ \forall\, \xi \in \mathcal{H}_n, g \in \mathcal{H}_K,\\ &= \int_X \xi(x)\,\overline{g(x)}\, m^n(x)\,\mathrm{d}\mu,\end{aligned}$$

that is, $(T_\phi\xi)(x) = \xi(x)\, m^n(x)$ or $T_\phi\xi = \xi\, m^n$. However, since the weight $m(x) > 1$ is invertible, $\overline{g}\, m^n$ runs over the whole of $\mathcal{H}_{\overline{n}}$ whenever g runs over H_K. Hence $\xi \in \mathsf{Ker}\ T_\phi \subset \mathcal{H}_n$ means that $\langle T_\phi\xi|g\rangle_K = 0$, $\forall g \in H_K$, which implies $\xi = 0$, since the duality between $\mathcal{H}_n$ and $\mathcal{H}_{\overline{n}}$ is separating. The same reasoning yields $\mathsf{Ker}\ T_\psi = \{0\}$. Therefore $V_\phi(X, \mu) \simeq \mathsf{Ran}\, C_\psi = \mathcal{H}_n$ and $V_\psi(X, \mu) \simeq \mathsf{Ran}\, C_\phi = \mathcal{H}_{\overline{n}}$.

A more general situation may be derived from the discrete example of Section 6.1.3 of [10]. Take a sequence of weights $m := \{|m_n|\}_{n\in\mathbb{N}} \in c_0, m_n \neq 0$, and consider the space ℓ^2_m with norm $\|\xi\|_{\ell^2_m} := \sum_{n\in\mathbb{N}} |m_n\xi_n|^2$. Then we have the following triplet replacing (9.1)

$$\ell^2_{1/m} \subset \ell^2 \subset \ell^2_m. \tag{9.2}$$

Next, for each $n \in \mathbb{N}$, define $\psi_n = m_n\theta_n$, where θ is a frame or an orthonormal basis in ℓ^2. Then ψ is an upper semi-frame. Moreover, $\phi := \{(1/\overline{m_n})\theta_n\}_{n\in\mathbb{N}}$ is a lower semi-frame, dual to ψ, thus (ψ, ϕ) is a reproducing pair. Hence, by [10, Theorem 3.13] (see also Section 5.1.3), $V_\psi \simeq \mathsf{Ran}\, C_\phi = M_{1/m}(V_\theta(\mathbb{N})) = \ell^2_m$ and $V_\phi \simeq \mathsf{Ran}\, C_\psi = M_m(V_\theta(\mathbb{N})) = \ell^2_{1/m}$ (here we take for granted that $\mathsf{Ker}\ T_\psi = \mathsf{Ker}\ T_\phi = \{0\}$).

For making contact with the situation of (9.1), consider in ℓ^2 the diagonal operator $A := \mathrm{diag}[n], n \in \mathbb{N}$ (the number operator), that is $(A\xi)_n = n\,\xi_n, n \in \mathbb{N}$, which is obviously self-adjoint and larger than 1. Then $\mathcal{H}_k = D(A^k)$ with norm $\|\xi\|_k = \left\|A^k\xi\right\| \equiv \ell^2_{r^{(k)}}$, where $(r^{(k)})_n = n^k$ (note that $1/r^{(k)} \in c_0$). Hence we have

$$\mathcal{H}_k = \ell^2_{r^{(k)}} \subset \mathcal{H}_o = \ell^2 \subset \mathcal{H}_{\overline{k}} = \ell^2_{1/r^{(k)}}\,, \tag{9.3}$$

where $(1/r^{(k)})_n = n^{-k}$. In addition, as in the continuous case discussed above, the end space of the scale, $\mathcal{D}^\infty(A) := \bigcap_k \mathcal{H}_k$, is simply Schwartz's space s of fast decreasing sequences, with dual $\mathcal{D}_{\overline{\infty}}(A) := \bigcup_k \mathcal{H}_k = s'$, the space of slowly increasing sequences. Here too, this construction shows that the space s is nuclear, since every embedding $A^{-1} : \mathcal{H}_{k+1} \to \mathcal{H}_k$ is a Hilbert-Schmidt operator.

However, the construction described above yields a much more general family of examples, since the weight sequences m are not ordered.

10 The case of L^p spaces

Following the suggestion made at the end of Section 2, we present now several possibilities of taking $\mathsf{Ran}\, C_\psi$ in the context of the Lebesgue spaces $L^p(\mathbb{R}, \mathrm{d}x)$.

As it is well-known, these spaces don't form a chain, since two of them are never comparable. We have only

$$L^p \cap L^q \subset L^s, \text{ for all } s \text{ such that } \ p < s < q.$$

Take the lattice $\mathcal{J}$ generated by $\mathcal{I} = \{L^p(\mathbb{R}, \mathrm{d}x), 1 \le p \le \infty\}$, with lattice operations [6, Sec.4.1.2]:

- $L^p \wedge L^q = L^p \cap L^q$ is a Banach space for the projective norm $\|f\|_{p\wedge q} = \|f\|_p + \|f\|_q$
- $L^p \vee L^q = L^p + L^q$ is a Banach space for the inductive norm $\|f\|_{p\vee q} = \inf_{f=g+h} \{\|g\|_p + \|h\|_q ; g \in L^p,\, h \in L^q\}$
- For $1 < p, q < \infty$, both spaces $L^p \wedge L^q$ and $L^p \vee L^q$ are reflexive and $(L^p \wedge L^q)^\times = L^{\overline{p}} \vee L^{\overline{q}}$.

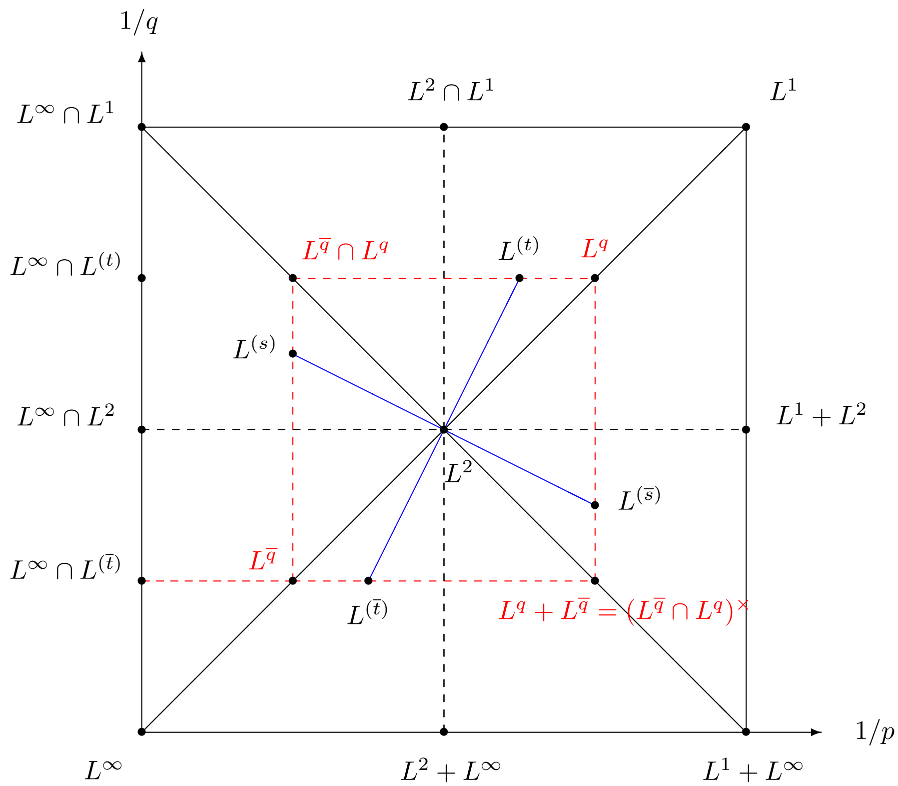

Figure 1: (i) The three full chains (black); (ii) The pair $L^{(s)}, L^{(\overline{s})}$ for s in the second quadrant (blue); (iii) The pair $L^{(t)}, L^{(\overline{t})}$ for t in the first quadrant (blue) (from [11]).

Moreover, no additional spaces are obtained by iterating the lattice operations to any finite order. Thus we obtain an involutive lattice and a LBS, denoted by $V_{\rm J}$ with elements denoted generically by $L^{(s)}, s = (p, q)$.

Following [6, Sec.4.1.2], we represent the space $L^{(p,q)}$ by the point $(1/p, 1/q)$ of the unit square $\mathrm{J} = [0, 1] \times [0, 1]$. In this representation, the spaces L^p are on the main diagonal, intersections $L^p \cap L^q$ above it and sums $L^p + L^q$ below, the duality is $[L^{(s}]^\times = L^{(\overline{s})}$, where $s = (p, q)$ and $\overline{s} = (\overline{p}, \overline{q})$, that is, symmetry with respect to L^2. Hence, $L^{(p,q} \subset L^{(p',q')}$ if $(1/p, 1/q)$ is on the left and/or above $(1/p', 1/q')$ The extreme spaces are

$$V_{\rm J}^{\#} = L^\infty \cap L^1 \quad \text{ and } \quad V_{\rm J} = L^1 + L^\infty.$$

Note that the space $L^1 + L^\infty$ has been considered by Gould [22]. For a full picture, see [6, Fig.4.1].

There are three possibilities for using the L^p lattice for controlling reproducing pairs

(1) Exploit the *full lattice* $\mathcal{J}$, that is, find (p, q) such that, $\forall f, g \in \mathcal{H}$, $C_\psi f \,\#\, C_\phi g$ in the PIP-space $V_{\rm J}$, that is, $C_\psi f \in L^{(p,q)}$ and $C_\phi g \in L^{(\overline{p},\overline{q})}$.

(2) Select in $V_{\rm J}$ a self-dual *Banach chain* $V_{\rm I}$, centered around L^2, symbolically.

$$\ldots L^{(s)} \subset \ldots \subset L^2 \subset \ldots \subset L^{(\overline{s})} \ldots, \tag{10.1}$$

such that $C_\psi f \in L^{(s)}$ and $C_\phi g \in L^{(\overline{s})}$ (or vice-versa). Here are three examples of such Banach chains. Of course, in each case, one may also select a symmetric subset of the chain.

- The *anti-diagonal* chain : $q = \overline{p}$

$$L^\infty \cap L^1 \subset \ldots \subset L^{\overline{q}} \cap L^q \subset \ldots \subset L^2 \subset \ldots \subset L^q + L^{\overline{q}} = (L^{\overline{q}} \cap L^q)^\times \subset \ldots \subset L^1 + L^\infty.$$

- The *horizontal* chain $q = 2$:

$$L^\infty \cap L^2 \subset \ldots \subset L^2 \subset \ldots \subset L^1 + L^2.$$

- The *vertical* chain $p = 2$:
$$L^2 \cap L^1 \subset \ldots \subset L^2 \subset \ldots \subset L^2 + L^\infty.$$

All three chains are presented in Figure 1. In each case, the full chain belongs to the second and fourth quadrants (top left and bottom right). A typical point is then $s = (p, q)$ with, $2 \leq p \leq \infty, 1 \leq q \leq 2$, so that one has the situation depicted in (10.1), that is, the spaces $L^{(s)}, L^{(\overline{s})}$ to which $C_\psi f$, resp. $C_\phi g$, belong, are necessarily comparable to each other and to L^2. In particular, one of them is necessarily contained in L^2 (see Figure 1).

(3) Choose a dual pair in the first and third quadrant (top right, bottom left). A typical point is then $t = (p', q')$, with $1 < p', q' < 2$, so that the spaces $L^{(t)}, L^{(\overline{t})}$ are never comparable to each other, nor to L^2.

Let us now add the boundedness condition already mentioned in Section 2, $\sup_{x\in X} \|\psi_x\|_{\mathcal{H}} \leq c$ and $\sup_{x\in X} \|\phi_x\|_{\mathcal{H}} \leq c'$ for some $c, c' > 0$. Then $C_\psi f(x) = \langle f|\psi_x\rangle \in L^\infty(X, \mathrm{d}\mu)$ and $C_\phi f(x) = \langle f|\phi_x\rangle \in L^\infty(X, \mathrm{d}\mu)$. Therefore, the third case reduces to the second one, since we have now (in the situation of Figure 1).
$$L^\infty \cap L^{(t)} \subset L^\infty \cap L^2 \subset L^\infty \cap L^{(\overline{t})}. \tag{10.2}$$
Note that none of these spaces is reflexive.

Following the pattern of Hilbert scales, we choose a (Gel'fand) triplet of Banach spaces. One could have, for instance, a triplet of reflexive Banach spaces such as
$$L^{(s)} \subset L^2 \subset L^{(\overline{s})}, \tag{10.3}$$
corresponding to a point s inside of the second quadrant, as shown in Figure 1. In this case, according to (8.6) and (8.7), $V_\psi = L^{(\overline{s})}/\mathsf{Ker}\, T_\psi$ and $V_\phi = L^{(s)}/\mathsf{Ker}\, T_\phi$.

On the contrary, if we choose a point t in the second quadrant, case (3) above, it seems that no triplet arises. However, if (ψ, ϕ) is a nontrivial reproducing pair, with $S_{\psi,\phi} = I$, that is, ψ, ϕ are dual to each other, one of them, say ψ, is an upper semi-frame and then necessarily ϕ is a lower semi-frame (see the remark at the end of Section 2.2). Therefore $C_\psi(\mathcal{H}) \subset L^2(X, \mu)$, that is, case (3) cannot be realized.

In conclusion, the only acceptable solution is a triplet of the type (10.3), with s strictly inside of the second quadrant, that is, $s = (p, q)$ with, $2 \leq p < \infty, 1 < q \leq 2$.

A word of explanation is in order here, concerning the relations $V_\psi = L^{(\overline{s})}/\mathsf{Ker}\, T_\psi$ and $V_\phi = L^{(s)}/\mathsf{Ker}\, T_\phi$. Both $L^{(s)}$ and $L^{(\overline{s})}$ are reflexive Banach spaces, with their usual norm, and so are the quotients by T_ψ, resp. T_ϕ. On the other hand, $V_\psi(X, \mu)[\|\cdot\|_\psi]$ and $V_\phi(X, \mu)[\|\cdot\|_\phi]$ are Hilbert spaces. However, there is no contradiction, since the equality sign = denotes an isomorphism of vector spaces only, without reference to any topology. Moreover, the two norms, Banach and Hilbert, *cannot* be comparable, lest they are equivalent [25, Coroll. 1.6.8], which is impossible in the case of $L^p, p \neq 2$. The same is true for any LBS where the spaces V_p are not Hilbert spaces.

Although we don't have an explicit example of a reproducing pair, we indicate a possible construction towards one. Let $\theta^{(1)} : \mathbb{R} \to L^2$ be a measurable function such that $\langle h|\theta_x^{(1)}\rangle \in L^q$, $1 < q < 2$, $\forall\, h \in L^2$, and let $\theta^{(2)} : \mathbb{R} \to L^2$ be a measurable function such that $\langle h|\theta_x^{(2)}\rangle \in L^{\overline{q}}$, $\forall\, h \in L^2$. Define $\psi_x := \min(\theta_x^{(1)}, \theta_x^{(2)}) \equiv \theta_x^{(1)} \wedge \theta_x^{(2)}$ and $\phi_x := \max(\theta_x^{(1)}, \theta_x^{(2)}) \equiv \theta_x^{(1)} \vee \theta_x^{(2)}$. Then we have $\psi_x \leq \phi_x$ for almost all $x \in \mathbb{R}$, and
$$(C_\psi h)(x) = \langle h|\psi_x\rangle \in L^q \cap L^{\overline{q}},\ \forall\, h \in L^2$$
$$(C_\phi h)(x) = \langle h|\phi_x\rangle \in L^q + L^{\overline{q}},\ \forall\, h \in L^2$$
and we have indeed $L^q \cap L^{\overline{q}} \subset L^2 \subset L^q + L^{\overline{q}}$. It remains to guarantee that ψ and ϕ are dual to each other, that is,
$$\int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle \,\mathrm{d}\mu(x) = \int_X C_\psi f(x)\, \overline{C_\phi g(x)}\, \mathrm{d}\mu(x) = \langle f|g\rangle,\ \forall\, f, g \in L^2.$$

11 Concluding remarks

Starting with the well-known notion of frame, both discrete and continuous, we have introduced a first natural generalization, namely, semi-frames, both upper and lower ones. The main result is that the two types are dual of each other. Indeed, if two semi-frames are in duality, either they are both frames, or else at least one of them is a lower semi-frame. Take for example $\phi = \{ne_n\} \cup \{\frac{1}{k}e_k\}_k$ and $\psi = \{\frac{1}{2n}e_n\}_n \cup \{\frac{k}{2}e_k\}_k$ with e_n an orthonormal basis, thenψ, ϕ are in duality and both are lower semi-frames but not Bessel [30].

Then the next step is to drop the restriction imposed by the frame bounds on the two measurable functions in duality, and this leads to the notion of reproducing pair. We have seen that the latter is quite rich. It generates a whole mathematical structure, which ultimately leads to a pair of Hilbert spaces, conjugate dual to each other with respect to the $L^2(X, \mu)$ inner product. We have given several concrete examples in Section 5. These, and additional ones, should allow one to better specify the best assumptions to be made on the measurable functions or, more precisely, on the nature of the range of the analysis operators C_ψ, C_ϕ.

This is clearly seen in the definition (6.1), which immediately suggests to perform the analysis in the context of PIP-spaces [6]. In particular, a natural choice is a scale, or simply a triplet, of Hilbert spaces, the two extreme spaces being conjugate duals of each other with respect to the $L^2(X, \mu)$ inner product. Another possibility consists of exploiting the lattice of all $L^p(\mathbb{R}, \mathrm{d}x)$ spaces, or a subset thereof, in particular a (Gel'fand) triplet of Banach spaces. Some examples have been described above, but clearly more work along these lines is in order.

Another interesting direction consists in considering a whole family $\mathcal{G}$ of μ-total, weakly measurable functions $\phi : X \to \mathcal{H}$, instead of only one. To each $\phi \in \mathcal{G}$ we can associate the pre-Hilbert space $V_\phi(X, \mu)[\|\cdot\|_\phi]$ and take its completion $\widetilde{V_\phi}(X, \mu)[\|\cdot\|_\phi]$. If ϕ has a partner $\psi \in \mathcal{G}$ such that (ψ, ϕ) is a reproducing pair, both spaces $V_\phi(X, \mu) = \widetilde{V_\phi}(X, \mu)[\|\cdot\|_\phi]$ and $V_\psi(X, \mu) = \widetilde{V_\psi}(X, \mu)[\|\cdot\|_\phi]$ are Hilbert spaces, conjugate dual to each other. In the general case, however, the question of completeness of $V_\phi(X, \mu)[\|\cdot\|_\phi]$ is open. Can one find conditions under which it holds? Also once might study the relationship between different pre-Hilbert spaces $V_\phi(X, \mu)$. When is one contained in another one?

Appendix. Lattices of Banach or Hilbert spaces

For the convenience of the reader, we summarize in this Appendix the basic facts concerning PIP-spaces and operators on them. However, we will restrict the discussion to the simpler case of a lattice of Banach (LBS) or Hilbert spaces (LHS). Further information may be found in our monograph [6] or our review paper [7].

Let thus $\mathcal{J} = \{V_p, \, p \in I\}$ be a family of Hilbert spaces or reflexive Banach spaces, partially ordered by inclusion. Then $\mathcal{I}$ generates an involutive lattice $\mathcal{J}$, indexed by J, through the operations $(p, q, r \in I)$:

. involution: $V_r \leftrightarrow V_{\overline{r}} = V_r^\times$, the conjugate dual of V_r
. infimum: $V_{p\wedge q} := V_p \wedge V_q = V_p \cap V_q$
. supremum: $V_{p\vee q} := V_p \vee V_q = V_p + V_q$.

It turns out that both $V_{p\wedge q}$ and $V_{p\vee q}$ are Hilbert spaces, resp. reflexive Banach spaces, under appropriate norms (the so-called projective, resp. inductive norms). Assume that the following conditions are satisfied:

(i) $\mathcal{I}$ contains a unique self-dual, Hilbert subspace $V_o = V_{\overline{o}}$.

(ii) for every $V_r \in \mathcal{I}$, the norm $\|\cdot\|_{\overline{r}}$ on $V_{\overline{r}} = V_r^\times$ is the conjugate of the norm $\|\cdot\|_r$ on V_r.

In addition to the family $\mathcal{J} = \{V_r, \, r \in J\}$, it is convenient to consider the two spaces $V^\#$ and V defined as

$$V = \sum_{q \in I} V_q, \quad V^\# = \bigcap_{q \in I} V_q. \tag{A.1}$$

These two spaces themselves usually do *not* belong to $\mathcal{I}$. According to the general theory of PIP-spaces [6], V is the algebraic inductive imit of the V_p's and $V^\#$ is the projective limit of the V_p's .

We say that two vectors $f, g \in V$ are *compatible* if there exists $r \in J$ such that $f \in V_r, g \in V_{\overline{r}}$. Then a *partial inner product* on V is a Hermitian form $\langle\cdot|\cdot\rangle$ defined exactly on compatible pairs of vectors. In particular, the partial inner product $\langle\cdot|\cdot\rangle$ coincides with the inner product of V_o on the latter. A *partial* *inner product space* (PIP-space) is a vector space V equipped with a partial inner product. Clearly LBSs and LHSs are particular cases of PIP-spaces.

We will assume that our PIP-space $(V, \langle\cdot|\cdot\rangle)$ is *nondegenerate*, that is, $\langle f|g\rangle = 0$ for all $f \in V^{\#}$ implies $g = 0$. As a consequence, $(V^{\#}, V)$ and every couple $(V_r, V_{\overline{r}})$, $r \in J$, are a dual pair in the sense of topological vector spaces [28]. In particular, the original norm topology on V_r coincides with its Mackey topology $\tau(V_r, V_{\overline{r}})$, so that indeed its conjugate dual is $(V_r)^{\times} = V_{\overline{r}}$, $\forall\, r \in J$. Then, $r < s$ implies $V_r \subset V_s$, and the embedding operator $E_{sr} : V_r \to V_s$ is continuous and has dense range. In particular, $V^{\#}$ is dense in every V_r. In the sequel, we also assume the partial inner product to be positive definite, $\langle f|f\rangle > 0$ whenever $f \neq 0$.

Then we have the familiar (Schwarz) inequality

$$\xi \in V_p,\ \eta \in V_{\overline{p}} \quad \text{implies} \quad \xi\overline{\eta} \in L^1(X, \mu) \quad \text{and} \quad \left|\int_X \xi(x)\overline{\eta(x)}\, \mathrm{d}\mu(x)\right| \leq \|\xi\|_p\, \|\eta\|_{\overline{p}}. \tag{A.2}$$

A standard, albeit trivial, example is that of a rigged Hilbert space (RHS) $\Phi \subset \mathcal{H} \subset \Phi^{\#}$ (it is trivial because the lattice $\mathcal{I}$ contains only three elements).

Familiar concrete examples of PIP-spaces are sequence spaces, with $V = \omega$ the space of *all* complex sequences $x = (x_n)$, and spaces of locally integrable functions with $V = L^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{d}x)$, the space of Lebesgue measurable functions, integrable over compact subsets.

Among LBSs, the simplest example is that of a chain of reflexive Banach spaces. The prototype is the chain $\mathcal{I} = \{L^p := L^p([0,1]; dx),\ 1 < p < \infty\}$ of Lebesgue spaces over the interval [0, 1].

$$L^{\infty} \subset \ldots \subset L^{\overline{q}} \subset L^{\overline{r}} \subset \ldots \subset L^2 \subset \ldots \subset L^r \subset L^q \subset \ldots \subset L^1, \tag{A.3}$$

where $1 < q < r < 2$ (of course, L^{∞} and L^1 are not reflexive). Here L^q and $L^{\overline{q}}$ are dual to each other $(1/q + 1/\overline{q} = 1)$, and similarly $L^r, L^{\overline{r}}$ $(1/r + 1/\overline{r} = 1)$.

As for a LHS, the simplest example is the Hilbert scale generated by a self-adjoint operator $A > I$ in a Hilbert space $\mathcal{H}_o$. Let $\mathcal{H}_n$ be $D(A^n)$, the domain of A^n, equipped with the graph norm $\|f\|_n = \|A^n f\|$, $f \in D(A^n)$, for $n \in \mathbb{N}$ or $n \in \mathbb{R}^+$, and $\mathcal{H}_{\overline{n}} := \mathcal{H}_{-n} = \mathcal{H}_n^{\times}$ (conjugate dual):

$$\mathcal{D}^{\infty}(A) := \bigcap_n \mathcal{H}_n \subset \ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_0 \ \subset \mathcal{H}_{\overline{1}} \subset \mathcal{H}_{\overline{2}} \ldots \subset \mathcal{D}_{\overline{\infty}}(A) := \bigcup_n \mathcal{H}_n. \tag{A.4}$$

Note that here the index n may be integer or real, the link between the two cases being established by the spectral theorem for self-adjoint operators. Here again the inner product of $\mathcal{H}_0$ extends to each pair $\mathcal{H}_n, \mathcal{H}_{-n}$, but on $\mathcal{D}_{\overline{\infty}}(A)$ it yields only a *partial* inner product. A standard example is the scale of Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{Z}$, in $\mathcal{H}_0 = L^2(\mathbb{R}, dx)$.

Acknowledgments

This work was partly supported by the Istituto Nazionale di Alta Matematica (GNAMPA project “Proprietà spettrali di quasi *-algebre di operatori”). JPA acknowledges gratefully the hospitality of the Dipartimento di Matematica e Informatica, Università di Palermo, whereas CT acknowledges that of the Institut de Recherche en Mathématique et Physique, Université catholique de Louvain. We also thank Michael Speckbacher for supplying the new proof of Proposition 2.1.

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