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Axiomatizing the Harsanyi Value, the Symmetric Egalitarian Solution and the Consistent Shapley Value

Geoffroy de Clippel[1]

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Abstract

The validity of Hart (1985)'s axiomatization of the Harsanyi value is shown to depend on the regularity conditions that are imposed on the games. Following this observation, we propose two related axiomatic characterizations, one of the symmetric egalitarian solution (cf. Kalai and Samet (1985)) and one of the consistent Shapley value (cf. Maschler and Owen (1992)). The three axiomatic results are studied, evaluated and compared in details.

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[1]Aspirant FNRS; CORE, Université catholique de Louvain, Belgium; e-mail: declippel@core.ucl.ac.be.

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Introduction

Various normative solutions have been proposed for cooperative games in coalitional form. The main ones are the Harsanyi value (cf. Harsanyi (1963)), the Shapley NTU value (cf. Shapley (1969)), the symmetric egalitarian solution (cf. Kalai and Samet (1985)), and the consistent Shapley value (cf. Maschler and Owen (1992)). For simplicity, we call them respectively the H-solution, the S-solution, the KS-solution, and the MO-solution. All these four solutions coincide with the Shapley TU-value on the class of TU-games and all, except the KS-solution, coincide with the Nash bargaining solution (cf. Nash (1950)) on the class of pure bargaining problems. The KS-solution coincides with the symmetric proportional solution (cf. Kalai (1977)) on the class of pure bargaining problems. The S-solution (resp. H-solution, resp. KS-solution) has already been axiomatized by Aumann (1985) (resp. Hart (1985); resp. Kalai and Samet (1985)).

The starting point of the present paper is the axiomatization of the H-solution proposed by Hart (1985). We show that its validity depends crucially on the regularity conditions that are imposed on the feasible set of the grand coalition. More precisely, we consider mainly three domains:

1. games for which the feasible set of each coalition is positively smooth,

2. games for which only the feasible set of the grand coalition is required to be positively smooth,

3. games for which no smoothness assumption is made.

Hart's axioms characterize the H-solution on the second domain, but are indeterminate on the first one, and incompatible on the third one. We refine this observation as follows. First, the KS-solution is the only solution that satisfies all the axioms proposed by Hart, except scale covariance, on the third domain. This provides an alternative axiomatization of the KS-solution. Second, the MO-solution is the only solution that satisfies a natural strengthening of Hart's axioms, on the first domain. This is the main result of the present paper, as there didn't exist any published[1] axiomatization[2] of the MO-solution for general[3] NTU games.

[1] As noted by Hart (1994) in his remark 5.3 (ii), the marginalist approach he follows in order to axiomatize the MO-solution on the class of hyperplane games, can be extended in order to include the case of more general NTU games, provided the concept of marginal contribution is appropriately defined.

[2] Nevertheless, Hart and Mas-Collel (1996) already provided a strong non-cooperative support to the MO-solution.

[3] The MO-solution has already been axiomatized in various ways on the class of hyperplane games (cf. Maschler and Owen (1989) and Hart (1994)).

In order to complete the picture, we refer the reader to Hart (1985) for a comparison between the S- and the H- solutions through their axiomatizations. Our results are closer to Hart's axiomatization of the H-solution (working with payoff configurations), than to Aumann's axiomatization of the S-solution (working with allocations for the grand coalition).

Our ideas are structured as follows. In the three first sections, we introduce notations and definitions. In particular, we recall the definitions of the KS-, H-, and MO-solutions in section 3. In section 4, we re-state the main theorem of Hart (1985). In section 5, we discuss Hart's result, by focusing on the importance of the regularity conditions he imposes on the games. The related two axiomatic characterizations of the KS- and the MO-solutions are given in section 6. In section 7, we propose bald reformulations of the three axiomatic results in order to better understand the techniques of the proofs and the specific role of each axiom. On the other hand, this perspicuous relecture of the results, allows us to compare them instantaneously and to explain precisely why there is such a dependence with respect to the domain of definition of the solutions. Finally, in section 8, we adapt the localization argument developed by Nash (1953) (cf. also Thomson (1981)) for justifying his IIA axiom, in order to further support our axiomatization of the MO-solution, with respect to the two related axiomatizations of the KS- and the H-solutions. Proofs are given in section 9.

1 Notations

Let n be a strictly positive integer, let $N := \{1, \ldots, n\}$ be the set of players, and let $P(N)$ be the set of coalitions, that is the set of nonempty subsets of N. The cardinality of a coalition S is denoted s. $\mathbb{R}^S$ is seen as a subset of $\mathbb{R}^N$: $\mathbb{R}^S := \{x \in \mathbb{R}^N \,|\, (\forall i \in N \backslash S) : x_i = 0\}$. Elements of the cartesian product $\times_{S \in P(N)} \mathbb{R}^S$ are called *payoff configurations*, as they will represent profiles of payoff allocations that are contingent on the coalition that forms (more about this interpretation in section 5). For each pair of vectors (λ, x) in $\mathbb{R}^N$, λx denotes the vector $(\lambda_1 x_1, \ldots, \lambda_n x_n)$. This expression should be distinguished from the inner product $\lambda.x := \sum_{i \in N} \lambda_i x_i$. Let S be a coalition. Inequalities between vectors in $\mathbb{R}^S$ are of three kinds: $\geq$, $>$, and $>>_S$ (usual definitions). Δ_+^S (resp. Δ_{++}^S) denotes the set of vectors in $\mathbb{R}^S$ with nonnegative (resp. strictly positive) components (in S) that sum up to 1. Let A be a convex subset of $\mathbb{R}^S$. Then the Pareto-frontier of A (denoted $\partial_S A$), is defined as follows:

$$\partial_S A := \{a \in A | \neg[(\exists a' \in A) : a' >>_S a]\}.$$

On the other hand, A is said to be *positively smooth* (in $\mathbb{R}^S$) if it admits a unique supporting hyperplane with a strictly positive normalized orthogonal vector, at each point of its Pareto-frontier.

2 Games

As usual, a *game* is a function that associates a nonempty, closed, comprehensive, and convex strict subset of $\mathbb{R}^S$, to each coalition S. The class of all games is denoted $\mathcal{G}$. A game V is *partially positively smooth* if the set $V(N)$ is positively smooth. The class of all partially positively smooth games is denoted $\mathcal{G}'_{PSm}$. Both Aumann (1985) and Hart (1985) essentially worked with solutions defined on $\mathcal{G}'_{PSm}$. A game V is *positively smooth* if the set $V(S)$ is positively smooth, for each coalition S. The class of all positively smooth games is denoted $\mathcal{G}_{PSm}$. Positive smoothness naturally imposes the same regularity condition on the feasible set of each coalition. A similar class of games was already used in Hart and Mas-Collel (1996). We have: $\mathcal{G}_{PSm} \subseteq \mathcal{G}'_{PSm} \subseteq \mathcal{G}$. Operations on games are defined from operations on sets, by applying them coalition by coalition. For example, $(V+W)(S) := V(S) + W(S)$ (resp. $(\partial V)(S) := \partial_S(V(S))$; resp. $\ldots$), for each $S \in P(N)$. Similarly, $V \subseteq W$ means that $V(S) \subseteq W(S)$, for each $S \in P(N)$.

A *TU-game* is a function $v : P(N) \to \mathbb{R}$. A game V is *equivalent to the TU-game* v if $V(S) = \{x \in \mathbb{R}^S \,|\, \sum_{i \in S} x_i \leq v(S)\}$, for each coalition S.

An *hyperplane game* is a game for which there exists a (unique, of course) pair $(\lambda, v) \in (\times_{S \in P(N)} \Delta^S_{++}) \times (\mathbb{R}^{P(N)})$ such that $V(S) = \{x \in \mathbb{R}^S | \lambda_S . x \leq v(S)\}$, for each coalition S. Any game that is equivalent to a TU-game, is an hyperplane game. Any hyperplane game is positively smooth.

3 Values and Solutions

Let G be a class of games. A *value defined on* G is a function σ that associates a payoff configuration, to each game belonging to G. The vector $\sigma_S(V) \in \mathbb{R}^S$ is called the *value of* V *for* S, for each game V in G and each coalition S. A *solution defined on* G is a function Σ that associates a (possibly empty) set of payoff configurations, to each game belonging to G.

We now define three solutions on $\mathcal{G}$: the KS-solution (cf. the symmetric egalitarian solution of Kalai and Samet (1985)), the H-solution (cf. the solution proposed by Harsanyi (1963)), and the MO-solution (cf. the consistent Shapley value of Maschler and Owen (1992)). To this end, we fix $V \in \mathcal{G}$.

The *KS-value* of V (denoted $\sigma_{KS}(V)$) is the unique payoff configuration $\sigma \in \partial V$ for which there exists a vector $\xi \in \mathbb{R}^{P(N)}$ such that $\sigma_{S,i} = \sum_{T \in P(S) \text{ s.t. } i \in T} \xi_T$, for each $i \in S$ and each $S \in P(N)$. It can be checked that σ_{KS} is well-defined. The *KS-solution* of V (denoted $\Sigma_{KS}(V)$) is simply the singleton containing $\sigma_{KS}(V)$.

The KS-value can be understood as expressing a conjunction of efficiency (cf. $\sigma_{KS}(V) \in \partial V$) and of equity (the real number ξ_T being understood as a dividend distributed by coalition T to each of its members).

The *H-solution* of V (denoted $\Sigma_H(V)$) is the set of payoff configurations $\sigma \in \partial V$ for which there exists a pair of vectors $(\lambda, \xi) \in \Delta_{++}^N \times \mathbb{R}^{P(N)}$ such that $\lambda.\sigma_N = \max_{y \in V(N)} \lambda.y$ and $\lambda_i \sigma_{S,i} = \sum_{T \in P(S) \text{ s.t. } i \in T} \xi_T$, for each $i \in S$ and each $S \in P(N)$. Each element σ of the H-solution of V fulfills at the same time an objective of efficiency (cf. $\sigma \in \partial V$), of utilitarianism in weighted utilities (cf. $\lambda.\sigma_N = \max_{y \in V(N)} \lambda.y$), and of equity in weighted utilities (the real number ξ_T being understood as a dividend in weighted utility distributed by coalition T to each of its members), for some vector λ of strictly positive weights. The endogeneous determination of the weights in order to obtain the conjunction of different objectives is reminiscent of the procedure proposed by Shapley (1969) in order to extend TU-solution concepts to NTU games.

Maschler and Owen (1989) defined an interesting value on the class of hyperplane games. Let us assume for the moment that V is an hyperplane game and let S be a coalition. For each enumeration π of S, we may define a *marginal worth vector* $MW(V, \pi) \in \mathbb{R}^S$ as follows:

$$\begin{aligned} MW_{\pi(i)}(V, \pi) := \max\{x_{\pi(i)} | x \in V(\{\pi(1), \ldots, \pi(i)\}) \\ \wedge (\forall j \in \{1, \ldots, i-1\}) : x_{\pi(j)} = MW_{\pi(j)}(V, \pi)\}, \end{aligned}$$

for each $i \in \{1, \ldots, s\}$. The real number $MW_{\pi(i)}(V, \pi)$ determines the maximal payoff that player $\pi(i)$ can get if he is the i^{th} player to enter the cooperation room, after players $\pi(1), \ldots, \pi(i-1)$ successively entered the room and were paid according to $MW(V, \pi)$. In this context, the value of V for S is the vector of expected payoffs, if all the enumerations of the players in S are equally likely, that is:

$$\sigma_S^*(V) := \frac{\sum_{\pi \in E(S)} MW(V, \pi)}{s!},$$

where $E(S)$ denotes the set of all the possible enumerations of the players in S. Maschler and Owen (1992) extended this value into a solution defined on $\mathcal{G}$. Let us consider an arbitrary game V in $\mathcal{G}$. The *MO-solution* of V (denoted $\Sigma_{MO}(V)$) is the set of payoff configurations $\sigma \in \partial V$ for which there exists a vector $\lambda \in \times_{S \in P(N)} \Delta_{++}^S$ such that λ_S is orthogonal to $V(S)$ at σ_S, for each $S \in P(N)$, and $\sigma = \sigma^*(V^\lambda)$, where V^λ is the hyperplane game that is characterized by the pair $(\lambda, (\max_{y \in V(S)} \lambda_S.y)_{S \in P(N)})$. Each element σ of $\Sigma_{MO}(V)$ (if any) thus specifies a vector of optimal allocations (one for each coalition) that coincides with the value (as defined above) of some hyperplane game that supports V at σ.

4 Hart's Axiomatization of the Harsanyi Solution

Here is the list of axioms proposed by Hart (1985) in order to characterize the Harsanyi solution. These axioms are expressed for solutions defined on a class G of games.

Axiom 1 (Efficiency) $\Sigma(V) \subseteq \partial V$, for each game $V \in G$.
Axiom 2 (Scale Covariance)[4] $\Sigma(\lambda V) = \lambda\Sigma(V)$, for each game $V \in G$ and each $\lambda \in \mathbb{R}^N_{++}$.
Axiom 3 (Independence Over Irrelevant Alternatives) $\Sigma(W) \cap V \subseteq \Sigma(V)$, for each pair of games $(V, W) \in G \times G$ such that $V \subseteq W$.
Axiom 4 (Conditional Super-Additivity) $[\Sigma(V) + \Sigma(W)] \cap \partial U \subseteq \Sigma(U)$, for each triple of games $(U, V, W) \in G \times G \times G$ such that $U = V + W$.
Axiom 5 (Unanimity Games)[5] For each coalition S and each real number c, $\Sigma(U_c^S) = \{\sigma\}$, where $\sigma_T := c\frac{1_S}{s}$ if $S \subseteq T$ and $\sigma_T := 0$ otherwise.
Axiom 6 (Zero-Inessential Games) $0 \in \Sigma(V)$, for each game $V \in G$ such that $0 \in \partial V$.

The axioms can easily be interpreted. Axiom 1 imposes on σ_S to be Pareto-optimal in $V(S)$, for each coalition S and each $\sigma \in \Sigma(V)$. Axiom 2 requires the covariance of the solution with respect to positive linear transformations of the games. As far as axiom 3 is concerned, let us consider a payoff configuration σ that belongs to the solution of W. If V is included in W (which means that each coalition faces less cooperative opportunities in V than in W), and if, at the same time, σ is still feasible in V, then σ should belong to the solution of V as well. If the payoff configurations σ and σ' belong to the solutions of V and W respectively, and if $\sigma + \sigma'$ is optimal in the game $V + W$, then $\sigma + \sigma'$ should belong to the solution of $V + W$, according to axiom 4. When a coalition S is a carrier in a unanimity game, axiom 5 mainly imposes an equal split of the value created (or lost) by S among its members, in each coalition containing S. Axiom 6 requires $0 \in \times_{S \in P(N)} \mathbb{R}^S$ to belong to the solution of V, when $0 \in \mathbb{R}^S$ is an optimal outcome in $V(S)$, for each $S \in P(N)$.

[4] We assume that $\lambda V \in G$, for each game $V \in G$ and each $\lambda \in \mathbb{R}^N_{++}$. The sets $\mathcal{G}$, $\mathcal{G}'_{PSm}$, and $\mathcal{G}_{PSm}$ all satisfy this property.

[5] For each coalition S and each real number c, U_c^S denotes the unanimity game on S defined by: $U_c^S(T) := \{x \in \mathbb{R}^T \mid \sum_{i \in T} x_i \leq c\}$ if $S \subseteq T$, and $U_c^S(T) := \{x \in \mathbb{R}^T \mid \sum_{i \in T} x_i \leq 0\}$ otherwise. For such a game, S is called a carrier. Each unanimity game is equivalent to a TU-game. We assume that G includes all the unanimity games. The sets $\mathcal{G}$, $\mathcal{G}'_{PSm}$, and $\mathcal{G}_{PSm}$ all satisfy this property.

The following proposition re-states the main theorem of Hart (1985).

Proposition 1 (Hart (1985)) The H-solution (restricted to $\mathcal{G}'_{PSm}$) is the only solution on $\mathcal{G}'_{PSm}$ that satisfies axioms 1 through 6.

5 Observations about Hart's Result

The validity of Hart's axiomatization of the H-solution depends crucially on the set of games over which solutions are assumed to be defined. Indeed, Σ_{MO} as a solution defined on $\mathcal{G}_{PSm}$, satisfies[6] axioms 1 through 6 as well. On the other hand, axioms 1 through 6 are incompatible when applied to solutions defined on $\mathcal{G}$. Indeed, let us assume that there exists a solution Σ defined on $\mathcal{G}$ that satisfies axioms 2, 3, 4, and 5. Let $\lambda := (2, 1, \ldots, 1) \in \mathbb{R}^N$ and let $V := \lambda U_n^N$. By axioms 2 and 5, $\Sigma(V) = \{(\lambda\sigma_S)_{S \in P(N)}\}$, where $\sigma_N := (1, \ldots, 1)$ and $\sigma_S := 0$, for each $S \in P(N) \setminus \{N\}$. Let W be the game defined as follows:

$$W(S) := \{x \in \mathbb{R}^S | x \leq \lambda\sigma_S\},$$

for each $S \in P(N)$. By axiom 3, $(\lambda\sigma_S)_{S \in P(N)} \in \Sigma(W)$. By axioms 4 and 5, $(\lambda\sigma_S)_{S \in P(N)} \in \Sigma(W + U_0^N) = \Sigma(U_{n+1}^N)$. But, on the other hand, axiom 5 implies that $\Sigma(U_{n+1}^N) = \{\sigma'\}$, where $\sigma'_N := \frac{n+1}{n}(1, \ldots, 1)$ and $\sigma'_S := 0$, for each $S \in P(N) \setminus \{N\}$. This is absurd.

In view of the importance of the regularity conditions that are imposed on the games, we may wonder why to consider a domain such as $\mathcal{G}'_{PSm}$ that treats asymmetrically the grand coalition and its strict subsets. Such a choice mirrors some lack of constistency of the H-solution. In order to develop this point, we clarify our interpretation of a solution.

As usual in cooperative game theory, we assume *a priori* that the grand coalition is being formed. The problem is then to specify the payoff allocation in $\mathbb{R}^N$ that could be agreed upon by the players in N. We adopt a normative approach (as opposed to a positive approach, illustrated by concepts such as the core, the von-Neumann/Morgenstern stable sets, ...), so that the objective is to express what could be a *fair* agreement, fairness being understood as being a combination of equity and efficiency (in some sense to be specified). We consider that the fairness of a potential agreement for the grand coalition, depends on what is feasible for that coalition, and on what is thought to happen if coalition S was formed instead of N, for each $S \in P(N) \setminus \{N\}$. We call "*threat from S*" the allocation that is

[6] We refer the reader to lemma 7 in section 9.3 for the proof of the fact that the MO-solution satisfies axiom 4 on $\mathcal{G}_{PSm}$. The five other axioms are easy to check.

expected to prevail if coalition S had to form. We focus on *credible* threats (similar to the idea of subgame perfection in extensive form games), as opposed to the usual rational threats of Harsanyi (1963). In other words, what is expected to happen if S had to form (for each $S \in P(N) \setminus \{N\}$) (although it is known that N indeed forms; counterfactual reasoning), is exactly what (fairly[7]) happens when S indeed forms. The S-component of a payoff configuration σ belonging to the solution of a given game, could thus be understood equally well as being what would be a fair agreement if S indeed forms, given the credible threats $(\sigma_T)_{T \in P(S) \setminus \{S\}}$, or as a credible threat influencing the fair agreement σ_T, if it is understood that T forms, for any fixed coalition T that is larger than S.
This interpretation goes in the sense of the first interpretation proposed by Hart (1985) on page 1299. Nevertheless, the H-solution doesn't[8] fit very well this interpretation, at least with respect to footnote 7. The following example was already treated by Hart and Mas-Collel (1996). Let V be the three-players hyperplane game characterized by the pair (λ, v), where $\lambda_{\{1\}} := (1,0,0)$, $\lambda_{\{2\}} := (0,1,0)$, $\lambda_{\{3\}} := (0,0,1)$, $\lambda_{\{1,2\}} := (\frac{1}{5}, \frac{4}{5}, 0)$, $\lambda_{\{1,3\}} := (\frac{1}{2}, 0, \frac{1}{2})$, $\lambda_{\{2,3\}} := (0, \frac{1}{2}, \frac{1}{2})$, $\lambda_{\{1,2,3\}} := (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $v(S) := 0$, if S equals $\{1\}$, $\{2\}$, $\{3\}$, $\{1,3\}$, or $\{2,3\}$, $v(\{1,2\}) := 20$, and $v(\{1,2,3\}) := 100/3$. V corresponds to (a slight variation[9] of) Owen (1972)'s banker game. The interpretation goes as follows. Player 1 can get 100\$ with the help of player 2. Players 1 and 2 can transfer money to each others, but only according to the exchange rate 4 : 1, without the help of player 3 (the banker). When the three players cooperate, any split of the 100\$ created through the cooperation of players 1 and 2 is feasible, thanks to the banker (players 1 and 2 could well decide to give some money to player 3 in exchange of his services). We have: $\Sigma_H(V) = \{\sigma\}$, where $\sigma_S = (0,0,0)$ if S equals $\{1\}$, $\{2\}$, $\{3\}$, $\{1,3\}$, or $\{2,3\}$, $\sigma_{\{1,2\}} = (20,20,0)$, and $\sigma_{\{1,2,3\}} = (40,40,20)$. The threat $(20,20,0)$ from coalition $\{1,2\}$ that justifies in part the fairness of the payoff allocation $(40,40,20)$ for the grand coalition, is not credible since the H-solution itself predicts $(50, 12.5, 0)$ as the fair outcome for coalition $\{1,2\}$ if it indeed forms (thus focusing on the restriction of V on $\{1,2\}$).

[7]We assume that the same criteria of fairness are applied, whatever the coalition that forms.

[8]Contrarily to the feeling one could have after reading example 5.6 of Hart (1985).

[9]Indeed, weighted transfers are not required to be bounded.

6 An Axiomatization of the KS- and the MO-Solutions

Following the observations made in the previous section, we propose two axiomatic characterizations, one of the KS-solution and one of the MO-solution, that are technically very close to Hart's axiomatization of the H-solution. The two axiomatic systems we propose, treat perfectly symmetrically all the coalitions (including as far as the domain of definition of the solutions is concerned).

We noted in the last section that axioms 1 through 6 are incompatible when applied to solutions defined on $\mathcal{G}$. Nevertheless, dropping axiom 2, allows us to go out of this impossibility result and to characterize exactly the KS-solution.

Proposition 2 The KS-solution is the only solution on $\mathcal{G}$ that satisfies axioms 1, 3, 4, 5, and 6.

Of course, the KS-solution violates axiom 2. Nevertheless, it satisfies a weaker version requiring covariance only with respect to common positive rescaling of the utilities. The KS-solution has already been axiomatized in a different way by Kalai and Samet (1985).

We also noted in the last section that axioms 1 through 6 don't specify a unique solution when we focus on solutions defined on $\mathcal{G}_{PSm}$. Indeed, both the H-solution and the MO-solution satisfy them on this domain. In fact, strengthening (in some sense[10]) axiom 5, allows us to exactly characterize the MO-solution.

<u>Axiom 7</u> (Conditional Random Dictatorship) For each game $V \in G$, each $\sigma \in \Sigma(V)$, and each $S \in P(N)$, if

$$x^i(V(S), \sigma_{S-i}) := \arg \max_{x \in V(S) \text{ s.t. } x_{S-i}=\sigma_{S-i}} x_i$$

exists[11], for each $i \in S$, and $\frac{\sum_{i\in S} x^i(V(S),\sigma_{S-i})}{s} \in \partial_S V(S)$, then $\sigma_S = \frac{\sum_{i\in S} x^i(V(S),\sigma_{S-i})}{s}$.

Axiom 7 is interpreted as follows. The payoff allocation $x^i(V(S), \sigma_{S-i})$ specifies the choice of player $i \in S$, when he has the dictatorial power to choose for S, having though the constraint to ascertain the participation of the other players in S, who have the outside option σ_{S-i} at their disposal. The vector $\frac{\sum_{i\in S} x^i(V(S),\sigma_{S-i})}{s}$ then represents the expected payoff allocation for the players in S if decisions in coalition

[10] Cf. section 7 in order to be able to better compare the philosophy behind propositions 1 and 3.

[11] Which means that $\{x \in V(S) | x_{S-i} = \sigma_{S-i}\}$ is nonempty and bounded.

S are taken as follows: choose at random, with equal probabilities, a player in S and give him the power of dictatorship. If this procedure determines an optimal decision for coalition S, then $\sigma_S = \frac{\sum_{i \in S} x^i(V(S), \sigma_{S-i})}{s}$.

On the other hand, axiom 7 is intuitively justified as follows. The procedure "pick up a dictator at random with equal probability" could be considered as determining the fair outcome when it is optimal. Moreover, axiom 7 specifies the kind of threats that could be made. When a player i is chosen as a dictator for a coalition S, he receives the power to choose for S. Nevertheless, he has to deal with the threats from the players in $S-i$, each one of them having the veto power to refuse player i's proposition in order to enforce the *status quo* option, which is assumed to be the formation of coalition $S-i$. Credibility of the threats then completely determines the outside option available to each player in $S-i$. We thus refer the reader to our interpretation of a solution (developed in section 5) in order to conclude the motivation of axiom 7. In fact, axiom 7 could replace the symmetry axiom (together with the scale covariance axiom) in the classical axiomatic characterization of the Nash bargaining solution. For instance, Professor Myerson uses a very similar axiom in order to characterize his generalization to incomplete information frameworks of the Nash bargaining solution (cf. section 5 of Myerson (1984)). As a final remark, notice that axiom 7 is relatively weak, since a relatively strong assumption (i.e. the optimality requirement) needs to be met in order to apply.

Proposition 3 The MO-solution (restricted to $\mathcal{G}_{PSm}$) is the only solution on $\mathcal{G}_{PSm}$ that satisfies axioms 1, 3, 4, and 7.

Proposition 3 is the main original result of the present paper. Indeed, there didn't exist any published[12] axiomatization[13] of the MO-solution for general[14] NTU games. Notice that the class of hyperplane games and the profile of normalized vectors that are orthogonal to the game at an element of the solution, that were used in the constructive definition of the MO-solution, make no explicit appearance in the axioms that characterize it.

The MO-solution satisfies axioms 2 and 6 as well. Nevertheless, they are not

[12]As noted by Hart (1994) in his remark 5.3 (ii), the marginalist approach he follows in order to axiomatize the MO-solution on the class of hyperplane games, can be extended in order to include the case of more general NTU games, provided the concept of marginal contribution is appropriately defined.

[13]Nevertheless, Hart and Mas-Collel (1996) already provided a strong non-cooperative support to the MO-solution.

[14]The MO-solution has already been axiomatized in various ways on the class of hyperplane games (cf. Maschler and Owen (1989) and Hart (1994)).

required for establishing the characterization result, because of the bite of axiom 7.

Of course, the H-solution doesn't satisfy axiom 7. Let us consider the banker game V described in section 5. Any solution Σ that satisfies axiom 7, must give $\Sigma(V) = \{\sigma^*(V)\}$. It is easy to check that $\sigma^*_S(V) = (0,0,0)$ if S equals $\{1\}$, $\{2\}$, $\{3\}$, $\{1,3\}$, or $\{2,3\}$, $\sigma^*_{\{1,2\}}(V) = (50, 12.5, 0)$, and $\sigma^*_{\{1,2,3\}}(V) = (50, 37.5, 12.5)$. In view of the computation of the H-solution in section 5, we conclude that it violates axiom 7.

7 A Perspicuous Re-Lecture of the three Axiomatic Results

We now propose bald reformulations of the three first propositions. The objectives are twofold. First, each axiom plays a more clear-cut role in the alternative statements than in the original ones. Second, the three results are then instantaneously comparable.

We introduce new axioms, stated for solutions Σ defined on a class G of games (that includes the class of hyperplane games).

When v is a TU-game, $Sh(S,v) \in \mathbb{R}^S$ denotes the Shapley TU-value of the game v restricted to S. In other words, $Sh(S,v)$ is defined by:

$$Sh_i(S,v) := \sum_{T \in P(S) \text{ s.t. } i \in T} \frac{(t-1)!(s-t)!}{s!}(v(T) - v(T \setminus \{i\})),$$

for each $i \in S$ and each $S \in P(N)$.

<u>Axiom 4'</u> (Independence Over Undominating Alternatives) $\Sigma(V) \cap \partial W \subseteq \Sigma(W)$, for each pair of games $(V,W) \in G \times G$ such that $V \subseteq W$.
<u>Axiom 7'</u> $\Sigma(V) = \{\sigma^*(V)\}$, for each hyperplane game V.
<u>Axiom 7"</u> $\Sigma((\lambda V(S))_{S \in P(N)}\} = \{(\lambda Sh(S,v))_{S \in P(N)}\}$, for each $\lambda \in \Delta^N_{++}$, each game V, and each TU-game v such that V is equivalent to v.
<u>Axiom 7"'</u> $\Sigma(V) = \{(Sh(S,v))_{S \in P(N)}\}$, for each game V and each TU-game v such that V is equivalent to v.

Axioms 7', 7", and 7"', restrict the set of possible solutions, by imposing on them to take some paticular values on some particular class of games. More precisely, axiom 7' (resp. 7"') imposes on the solution to coincide with σ^* (resp. the Shapley value) on the class of hyperplane games (resp. games that are equivalent to some TU-game). On the other hand, axiom 7" is a combination of axiom 7"' and of the scale covariance property applied to the class of games that are equivalent to some TU-game. Clearly, axiom 7' implies axiom 7" which, in turn, implies axiom 7"'. As far as axiom 4' is concerned, let us consider a payoff configuration σ that belongs to the solution of a game V. If W is larger than V (which means that each coalition has at least as much cooperative opportunities in W than in V), and if, at the same time, σ_S is not Pareto-dominated in the game W (for each coalition S), then σ should belong to the solution of W as well. This axiom is a dual version of axiom 3 (IIA). It is directly inspired by axioms appearing in Thomson and Myerson (1980) and in Thomson (1981). Axiom 4' is implied by the conjunction of axioms 3, 4, and 6. We refer to lemma 1 in section 9 for the proof of this relation.

Proposition 1' The H-solution (restricted to $\mathcal{G}'_{PSm}$) is the only solution on $\mathcal{G}'_{PSm}$ that satisfies axioms 1, 3, 4', and 7".

Proposition 2' The KS-solution is the only solution on $\mathcal{G}$ that satisfies axioms 1, 3, 4', and 7"'.

Proposition 3' The MO-solution (restricted to $\mathcal{G}_{PSm}$) is the only solution on $\mathcal{G}_{PSm}$ that satisfies axioms 1, 3, 4', and 7'.

In propositions 1 and 2, axiom 4 plays a double role, as it is used in combination with axioms 1 and 5 (+2 for proposition 1) in order to deduce axioms 7" or 7"', and it is used as well in combination with axioms 3 and 6 in order to deduce the solution for games that are strictly NTU from the restriction on the solution imposed by axioms 7" or 7"'. In the propositions with a prime, we explicit the way axiom 4 is used in combination with axioms 3 and 6, by expressing axiom 4' which more clearly indicates how we relate the solution of NTU games to the solution of some hyperplane game, rescaled TU-game, or TU-game.

In this context, proposition 3' is probably more natural than proposition 3 from a mathematical point of view, as axiom 7 is not used in combination with axiom 4 in order to deduce the solution on the class of hyperplane games. On the other hand, axiom 7 is too strong in proposition 3, as axiom 7' is equivalent to axiom 7, only when applied to hyperplane games (cf. lemma 4 in section 9). Nevertheless, proposition 3 is probably a better axiomatization result than proposition 3', as the class of hyperplane games doesn't appear explicitely and the conditional superadditivity property is more natural and traditionally more used than the IUA property.

The bite of axioms 3 and 4' (or 3 and 4) depends on the class of games over which the solutions are assumed to be defined. This is why the validity of Hart's result is contingent on the domain of definition of the solutions one considers. The larger the class is, the strongest the two axioms are. The relation is strict when we move from $\mathcal{G}_{PSm}$, to $\mathcal{G}'_{PSm}$, and to $\mathcal{G}$. Indeed, it is easy to check that Σ_{MO} (resp. Σ_H) satisfies axioms 3 and 4' on $\mathcal{G}_{PSm}$ (resp. $\mathcal{G}'_{PSm}$), but that no solution that satisfies axioms 3 and 4' on $\mathcal{G}'_{PSm}$ (resp. $\mathcal{G}$) will coincide with Σ_{MO} (resp. Σ_H) on $\mathcal{G}_{PSm}$ (resp. $\mathcal{G}'_{PSm}$). On the other hand, axiom 7"' is implied by axiom 7" which, in turn, is implied by axiom 7'. The increasing restrictions implied by axioms 3 and 4' when we move from solutions defined on $\mathcal{G}_{PSm}$, to solutions defined on $\mathcal{G}'_{PSm}$, and then to solutions defined on $\mathcal{G}$, appears to be exactly balanced by the slack generated by the weakening of axiom 7', into axiom 7", and then into axiom 7"', as a unique solution is characterized in each of the three above propositions.

8 Localization

In order to better evaluate axioms 3 and 4', and in order to give some further support to our propositions 3 and 3', with respect to propositions 1, 1', 2, and 2', we note that the combination of axioms 3 and 4' is justified on $\mathcal{G}_{PSm}$, but not on $\mathcal{G}'_{PSm}$ and $\mathcal{G}$, by an adaptation[15] of the localization argument proposed by Nash (1953) in order to justify his IIA axiom (cf. also Thomson (1981)).

Let V and W be two games and let σ be a payoff configuration. Then, V is *locally equivalent to W around σ* if σ belongs to the interiors of V and of W, or σ belongs to the frontiers of V and of W and $\Lambda(V,\sigma) = \Lambda(W,\sigma)$, where $\Lambda(U,\tau)$ denotes the set of vectors $\lambda \in \times_{S\in P(N)}\Delta_+^S$ such that, for each $S \in P(N)$, λ_S is orthogonal to $U(S)$ at τ_S, for each game U and each payoff configuration $\tau \in \partial U$.

<u>Axiom 8</u> (Localization) $\sigma \in \Sigma(W)$, for each game $W \in G$ and each payoff configuration $\sigma \in W$ for which there exists a game V such that $\sigma \in \Sigma(V)$ and V is locally equivalent to W around σ.

We quote Professors Nash and Thomson in order to motivate axiom 8. Quoting Thomson (1981), axiom 8 "can be interpreted as formulating a requirement of informational simplicity of solutions since [it] amount[s] to stating that only local information is relevant to the determination of [a payoff configuration in] the solution". Quoting Nash (1953), axiom 8 means that "there is no "action at a distance" in the influence of the shape of [V] on the location of [a payoff configuration in the] solution. Thinking in terms of bargaining, it is as if a proposed deal [for each coalition] is to compete with small modifications of itself [given the credible threats of each of its strict subsets] and that ultimately the negotiation [in that

[15]Nash's original argument was mainly designed for positively smooth pure bargaining problems.

coalition] will be understood to be restricted to a narrow range of alternative deals and to be unconcerned with more remote alternatives".

It is easy to check that axiom 8 is equivalent to the combination of axioms 3 and 4' on $G = \mathcal{G}_{PSm}$ (for solutions that satisfy axiom 1), and that axiom 8 is implied by the conjunction of axioms 3 and 4' on $G = \mathcal{G}$ or $\mathcal{G}'_{PSm}$ (for solutions that satisy axiom 1), but that the converse is false. For instance, the MO-solution satisfies axiom 8 on $\mathcal{G}$ and $\mathcal{G}'_{PSm}$, but doesn't satisfy axioms 3 and 4' on these domains.

If we find that axioms 3 and 4' are too strong on $\mathcal{G}$ and that, instead, axiom 8 is more motivated, an interesting question is then to obtain an axiomatization of a unique meaningful solution defined on $\mathcal{G}$ that satisfies axioms 1, 7 , and 8. This solution could be specified *via* some topological closure of the graph of the MO-solution, and could be characterized by adding some axiom of continuity of the solution with respect to the games.

9 Proofs

We first state some logical relations between some axioms.

Lemma 1 *1) 7⇒7'⇒7"⇒7"'⇒5;*
2) 3+4+6⇒4';
3) 1+4+5⇒7"';
4) 2+7"'⇒7".

Proof: The implication 7⇒7' is an obvious consequence of lemma 4 (cf. subsection 9.3). The rest of part 1) as well as part 4) are easy to prove. Part 3) is proved in proposition 6.1 of Hart (1985). As far as part 2) is concerned, let V and W be two games such that $V \subseteq W$, let $\sigma \in \Sigma(V) \cap \partial W$, and let $\lambda \in \times_{S \in P(N)} \Delta_+^S$ be a (not necessarily unique) profile of normalized vectors such that λ_S is orthogonal to $W(S)$ at σ_S. Let W' be the game[16] defined by:

$$W'(S) := \{x \in \mathbb{R}^S | \lambda_S.x \leq 0\},$$

for each $S \in P(N)$. By axiom 6, $0 \in \Sigma(W')$. By axiom 4, $\sigma \in \Sigma(W^\lambda)$, where $W^\lambda := V + W'$. By axiom 3, $\sigma \in \Sigma(W)$, since $W \subseteq W^\lambda$.

□

[16] W' is not necessarily an hyperplane game, as λ doesn't necessarily belong to $\times_{S \in P(N)} \Delta_{++}^S$. Nevertheless, it is assumed that G includes W'. The sets $\mathcal{G}$, $\mathcal{G}'_{PSm}$, and $\mathcal{G}_{PSm}$ all satisfy this property.

9.1 Proof of propositions 1 and 1'

The proofs of propositions 1 and 1' are similar to the proof of the main theorem of Hart (1985). Nevertheless, we include them for the sake of completeness. In addition, the arguments are exposed a bit differently and, including them, will allow the reader to make comparisons with the proofs of propositions 2, 2', 3, and 3'.

We refer the reader to proposition 4.11 of Hart (1985) for the proof of the fact that the H-solution satisfies axioms 1, 2, 3, 4, 5, and 6 on $\mathcal{G}'_{PSm}$ (not difficult).

Lemma 2 *If a solution Σ defined on $\mathcal{G}'_{PSm}$ satisfies axioms 1, 3, 4', and 7", then $\Sigma = \Sigma_H$.*

Proof: Let $V \in \mathcal{G}'_{PSm}$, let $\sigma \in \partial V$, let $\lambda \in \Delta^N_{++}$ be the normalized vector that is orthogonal to $V(N)$ at σ_N, let V^λ_σ be the game defined by:

$$V^\lambda_\sigma(S) := \{x \in \mathbb{R}^S | \lambda.x \leq \lambda.\sigma_S\};$$

for each $S \in P(N)$, and let W be the game defined by $W(N) := V(N)$ and

$$W(S) := \{x \in \mathbb{R}^S | x \leq \sigma_S\},$$

for each $S \in P(N)\backslash\{N\}$. We note that $W \subseteq V^\lambda_\sigma$, that $W \subseteq V$, that $\sigma \in \partial V^\lambda_\sigma \cap \partial W$, and that both V^λ_σ and W are partially positively smooth. We have:

$$\begin{aligned} \sigma \in \Sigma_H(V) &\Longleftrightarrow \sigma \in \Sigma_H(V^\lambda_\sigma) && \text{(by definition of } \Sigma_H) \\ &\Longleftrightarrow \sigma \in \Sigma(V^\lambda_\sigma) && \text{(both } \Sigma \text{ and } \Sigma_H \text{ satisfy axiom 7")} \\ &\Longleftrightarrow \sigma \in \Sigma(W) && \text{(nc: ax. 3; sc: ax. 4')}^{17} \\ &\Longleftrightarrow \sigma \in \Sigma(V) && \text{(nc: ax. 4'; sc: ax. 3)} \end{aligned}$$

It is then easy to conclude the proof, thanks to the fact that both Σ and Σ_H satisfy axiom 1.

□

It is then easy to prove propositions 1 and 1', thanks to lemma 1.

9.2 Proof of propositions 2 and 2'

It is easy to check that the symmetric KS-solution satisfies axioms 1, 3, 4, 5 and 6 on $\mathcal{G}$.

[17] "nc" (resp. "sc") means "necessary condition" (resp. "sufficient condition").

Lemma 3 *If a solution Σ defined on $\mathcal{G}$ satisfies axioms 1, 3, 4', and 7"', then $\Sigma = \Sigma_{KS}$.*

Proof: Let $V \in \mathcal{G}$, let $\sigma \in \partial V$, let V_σ be the game defined by:

$$V_\sigma(S) := \{x \in \mathbb{R}^S | \sum_{i\in S} x_i \leq \sum_{i\in S} \sigma_{S,i}\};$$

for each $S \in P(N)$, and let $\hat{W}$ be the game defined by:

$$\hat{W}(S) := \{x \in \mathbb{R}^S | x \leq \sigma_S\},$$

for each $S \in P(N)$. We note that $\hat{W} \subseteq V_\sigma$, that $\hat{W} \subseteq V$, and that $\sigma \in \partial V_\sigma \cap \partial \hat{W}$. We have:

$$\begin{aligned}
\sigma \in \Sigma_{KS}(V) &\Longleftrightarrow \sigma \in \Sigma_{KS}(V_\sigma) && \text{(by definition of } \Sigma_{KS}\text{)}\\
&\Longleftrightarrow \sigma \in \Sigma(V_\sigma) && \text{(both } \Sigma \text{ and } \Sigma_{KS} \text{ satisfy axiom 7"')}\\
&\Longleftrightarrow \sigma \in \Sigma(\hat{W}) && \text{(nc: ax. 3; sc: ax. 4')}\\
&\Longleftrightarrow \sigma \in \Sigma(V) && \text{(nc: ax. 4'; sc: ax. 3)}
\end{aligned}$$

It is then easy to conclude the proof, thanks to the fact that both Σ and Σ_{KS} satisfy axiom 1.

□

It is then easy to prove propositions 2 and 2', thanks to lemma 1.

9.3 Proof of propositions 3 and 3'

Lemma 4 is an obvious consequence of lemma 1 of Maschler and Owen (1989). It states essentially that axiom 7 characterizes the MO-solution on the class of hyperplane games.

Lemma 4 *Let V be an hyperplane game. Then, the vector $\sigma^*(V)$ is the only vector $\sigma \in \times_{S\in P(N)}\mathbb{R}^S$ such that $\sigma_S = \frac{\sum_{i\in S} x^i(V(S),\sigma_{S-i})}{s}$, for each $S \in P(N)$.*

Let $V \in \mathcal{G}_{PSm}$ and let $\sigma \in \partial V$. Then $\lambda(\sigma, V) \in \times_{S\in P(N)}\Delta^S_{++}$ denotes the profile of normalized vectors whose S-component is orthogonal to $V(S)$ at σ_S, for each $S \in P(N)$. On the other hand, $V^{\lambda(\sigma,V)}$ denotes the hyperplane game that is characterized by the pair $(\lambda(\sigma, V), (\max_{y\in V(S)} \lambda_S(\sigma, V).y)_{S\in P(N)})$.

Lemma 5 *Σ_{MO}, as a solution defined on $\mathcal{G}_{PSm}$, satisfies axiom 7.*

Proof: Let $V \in \mathcal{G}_{PSm}$, let $\sigma \in \Sigma_{MO}(V)$, and let $S \in P(N)$ be such that $x^i(V(S), \sigma_{S-i})$ exists, for each $i \in S$, and $x := \frac{\sum_{i \in S} x^i(V(S), \sigma_{S-i})}{s} \in \partial_S V(S)$. We have: $\sigma_S = \sigma_S^*(V^{\lambda(\sigma,V)}) = \frac{\sum_{i \in S} x^i(V^{\lambda(\sigma,V)}(S), \sigma_{S-i})}{s}$ (cf. lemma 4). We prove that $x^i(V(S), \sigma_{S-i}) = x^i(V^{\lambda(\sigma,V)}(S), \sigma_{S-i})$, for each $i \in S$. This amounts to prove that $\max_{x \in V(S) \text{ s.t. } x_{S-i}=\sigma_{S-i}} x_i = \max_{x \in V^{\lambda(\sigma,V)}(S) \text{ s.t. } x_{S-i}=\sigma_{S-i}} x_i$. First, notice that $\max_{x \in V(S) \text{ s.t. } x_{S-i}=\sigma_{S-i}} x_i \leq \max_{x \in V^{\lambda(\sigma,V)}(S) \text{ s.t. } x_{S-i}=\sigma_{S-i}} x_i$, for each $i \in S$, since $V(S) \subseteq V^{\lambda(\sigma,V)}(S)$. Second, if there exists some player $i \in S$ such that $\max_{x \in V(S) \text{ s.t. } x_{S-i}=\sigma_{S-i}} x_i < \max_{x \in V^{\lambda(\sigma,V)}(S) \text{ s.t. } x_{S-i}=\sigma_{S-i}} x_i$, then $x < \sigma_S$, which contradicts the fact that $x \in \partial_S V(S)$ (weak and strong optimality are the same on $\mathcal{G}_{PSm}$).

□

Lemma 6 *(Conditional additivity of the MO-solution on the class of hyperplane games) Let $\lambda \in \times_{S \in P(N)} \Delta_{++}^S$ be a profile of normalized vectors, let $v : P(N) \to \mathbb{R}$ and $w : P(N) \to \mathbb{R}$ be two functions, and let V and W be the two hyperplane games that are characterized respectively by the pairs (λ, v) and (λ, w). Then, $\sigma^*(V + W) = \sigma^*(V) + \sigma^*(W)$.*

Proof: We prove that $\sigma_S^*(V + W) = \sigma_S^*(V) + \sigma_S^*(W)$, for each $S \in P(N)$, by induction on the cardinality of S. If S is a singleton, then the result is obvious. So, let us assume that $s \geq 2$ and that we already proved the statement for all the coalitions of cardinality $s - 1$. Notice that $V + W$ is the hyperplane game characterized by the pair $(\lambda, v + w)$. Let $i \in S$. We have:

$$\max_{x \in (V+W)(S) \text{ s.t. } x_{S-i}=\sigma_{S-i}^*(V+W)} x_i = \frac{(v + w)(S) - \sum_{j \in S-i} \lambda_{S,j} \sigma_{S-i,j}^*(V + W)}{\lambda_{S,i}}.$$

By the induction assumption, this last expression equals

$$\frac{v(S) - \sum_{j \in S-i} \lambda_{S,j} \sigma_{S-i,j}^*(V)}{\lambda_{S,i}} + \frac{w(S) - \sum_{j \in S-i} \lambda_{S,j} \sigma_{S-i,j}^*(W)}{\lambda_{S,i}},$$

which, in turn, equals

$$\max_{x \in V(S) \text{ s.t. } x_{S-i}=\sigma_{S-i}^*(V)} x_i + \max_{x \in W(S) \text{ s.t. } x_{S-i}=\sigma_{S-i}^*(W)} x_i.$$

So, $x^i((V + W)(S), \sigma_{S-i}^*(V + W)) = x^i(V(S), \sigma_{S-i}^*(V)) + x^i(W(S), \sigma_{S-i}^*(W))$, for each $i \in S$. It is then easy to conclude the proof, thanks to lemma 4.

□

Lemma 7 Σ_{MO}*, as a solution defined on* $\mathcal{G}_{PSm}$*, satisfies axiom* 4.

Proof: Let $V \in \mathcal{G}_{PSm}$ and $W \in \mathcal{G}_{PSm}$ be such that $U := V + W \in \mathcal{G}_{PSm}$. Let $\sigma \in \Sigma_{MO}(V)$ and $\sigma' \in \Sigma_{MO}(W)$ be such that $\sigma + \sigma' \in \partial U$. Then, $\lambda(\sigma, V) = \lambda(\sigma', W) = \lambda(\sigma + \sigma', U)$. For simplicity, let us call λ this profile of vectors. We note that $U^\lambda = V^\lambda + W^\lambda$. So, by lemma 6, we have: $\sigma^*(U^\lambda) = \sigma^*(V^\lambda) + \sigma^*(W^\lambda) = \sigma + \sigma'$. This concludes the proof.

□

Lemma 8 *If a solution* Σ *defined on* $\mathcal{G}_{PSm}$ *satisfies axioms 1, 3, 4', and 7', then* $\Sigma = \Sigma_{MO}$.

Proof: Let $V \in \mathcal{G}_{PSm}$ and let $\sigma \in \partial V$. We may thus consider the profile of normalized vectors $\lambda(\sigma, V)$. We have:

$$\begin{aligned} \sigma \in \Sigma_{MO}(V) &\Longleftrightarrow \sigma \in \Sigma_{MO}(V^{\lambda(\sigma,V)}) \quad \text{(by definition of } \Sigma_{MO}) \\ &\Longleftrightarrow \sigma \in \Sigma(V^{\lambda(\sigma,V)}) \quad \text{(both } \Sigma \text{ and } \Sigma_{MO} \text{ satisfy axiom 7')} \\ &\Longleftrightarrow \sigma \in \Sigma(V) \qquad\quad \text{(nc: ax. 3; sc: ax. 4').} \end{aligned}$$

It is then easy to conclude the proof, thanks to the fact that both Σ and Σ_{MO} satisfy axiom 1.

□

We proved in lemmas 5 and 7 that the MO-solution satisfies axioms 4 and 7 on $\mathcal{G}_{PSm}$. On the other hand, it is easy to check that it also satisfies axioms 1 and 3 on $\mathcal{G}_{PSm}$. Propositions 1 and 1' then follows from lemma 8, using lemma 1 (axiom 7 implies axiom 6 on $\mathcal{G}_{PSm}$).

Remark Looking at the proof of lemma 8, we observe that proposition 3' can be decomposed into two parts: Σ_{MO} is the minimal (resp. maximal) solution that satisfies axioms 1, 3, and 7' (resp. 1, 4', and 7'). In fact, if the MO-solution was single-valued (which is not true in general), either of the two parts would be sufficient in order to obtain the exact characterization. This point is well illustrated by the papers of Nash (1950), who adopts the approach with axiom 3, and of Thomson (1981), who adopts the approach with axiom 4', for their axiomatizations of the Nash bargaining solution (which is single-valued) on the class of pure bargaining games. The same kind of decomposition doesn't necessarily apply to propositions 1' and 2', as both axioms 3 and 4' are used in order to prove the two inclusions that are required in order to obtain the equality in the thesis of lemmas 2 and 3.

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