



Institute for Information
and Communication Technologies,
Electronics and Applied Mathematics

Generic Local Identifiability in Dynamical Networks: from Algebraic Characterization to Algorithms for Signal Selection

Antoine Legat

Thesis submitted in partial fulfillment
of the requirements for the degree of
Docteur en sciences de l'ingénieur

Dissertation committee:

Prof. Julien M. Hendrickx (UCLouvain, advisor)
Prof. Paul Van den Hof (Eindhoven University of Technology, Netherlands)
Prof. Pierre-Antoine Absil (UCLouvain)
Dr. Federica Garin (Inria Grenoble, France)
Prof. Bahman Ghahserifard (Queen's University, Canada)
Prof. Charles Pecheur (UCLouvain, chair)

Version of September 8, 2025.

Abstract

This thesis investigates the problem of identifiability in linear dynamic networks under partial excitation and measurement. In many practical scenarios, it is infeasible or costly to excite and measure all nodes in a network. Consequently, a central question arises: Which nodes should be excited and measured to ensure that the internal structure of the network can still be identified? We address this question through a three-pronged approach that combines algebraic analysis, graph-theoretic insight, and algorithmic design.

The first main contribution of the thesis is an algebraic characterization of generic local identifiability. We show that, generically, analyzing networks with scalar modules is equivalent to working with analytic transfer functions, which significantly simplifies the analysis. Using tools from manifold theory, we derive necessary and sufficient conditions for local identifiability and demonstrate that this property is pseudo-generic. This leads to a practical algorithm that tests identifiability with probability one by evaluating algebraic conditions on randomly sampled network instances.

We then explore how this algebraic characterization can be translated into graph-theoretic terms. We develop a combinatorial characterization of identifiability based on walk structures in the network, derive walk-based identifiability conditions and introduce the Meta-Walk Graph, a novel graph whose vertices represent bijective walk collections. This combinatorial framework provides a powerful lens for understanding identifiability and guides the design of algorithms.

The final part of the thesis focuses on the synthesis problem: selecting excitation and measurement sets that guarantee identifiability while minimizing the number of signals used. We prove that finding the optimal solution to this problem is NP-hard, motivating the development of heuristic algorithms. Leveraging the algebraic and combinatorial tools developed earlier, we propose several algorithms that balance computational efficiency with solution quality.

Remerciements

CES six années de thèse n'ont pas été une paisible course le long d'un canal monotone. Elles ont ressemblé à un parcours semé de côtes et de descentes, d'ornières et d'obstacles à franchir, avec des éclaircies, des draches et des orages. J'ai traversé des moments de découragement, mais aussi connu des élans d'adrénaline et de motivation. J'y ai laissé parfois mes genoux et mes tendons, mais surtout, même si la course à pied est un sport individuel, j'ai eu la chance de pouvoir compter sur un magnifique peloton, qui m'a aidé à franchir la ligne d'arrivée de ce marathon en un seul morceau. Dans ce peloton, j'ai trouvé tout le soutien dont j'avais besoin : des personnes inspirantes, qui m'ont transmis énormément, aussi bien sur le plan scientifique et mathématique que sur les plans organisationnel et communicationnel. Ce sont elles qui ont donné toute sa saveur à ces six années, et que je souhaite remercier ici, car sans elles, cette thèse n'aurait jamais pu voir le jour.

Tout d'abord, je souhaite remercier mon promoteur, Julien Hendrickx, pour sa disponibilité, ses conseils toujours précieux et avisés, ainsi que pour ses réponses rapides et pertinentes à mes nombreuses questions. Sa motivation inépuisable pour la recherche m'a profondément inspiré, et l'énergie qu'il y consacrait a été une source constante de motivation. J'ai trouvé en lui quelqu'un d'encore plus perfectionniste que moi, et il m'a donné l'envie de donner le meilleur de moi-même, de viser toujours plus haut, et d'apprendre le niveau académique de rigueur scientifique. Je le remercie également pour sa patience à l'égard de mon sens parfois relatif de la ponctualité, et pour avoir su, quand il le fallait, me pousser avec franchise et bienveillance à avancer.

Je souhaite également exprimer ma gratitude aux membres de mon comité d'encadrement. Je remercie tout particulièrement Paul Van den

Hof pour son accueil à Eindhoven, pour l'intérêt constant qu'il a manifesté envers mon travail et pour sa motivation. Je lui suis reconnaissant du temps qu'il a consacré à la lecture attentive de ma thèse, pour ses remarques pertinentes ainsi que pour les échanges stimulants et enrichissants que nous avons partagés. Je remercie également Pierre-Antoine Absil pour son aide précieuse en théorie des variétés, le temps qu'il m'a consacré, sa bienveillance et la qualité de ses conseils. Merci aussi aux autres membres de mon jury de thèse : Federica Garin, Bahman Gharesifard et Charles Pecheur. Ils et elle m'ont accordé un temps précieux pour enrichir mon travail via les discussions, conseils et remarques encourageantes.

Au-delà des aspects purement scientifiques, je souhaite exprimer ma gratitude à toutes celles et ceux qui ont contribué à faire de l'Euler un lieu de travail stimulant et convivial, rendant mon parcours doctoral plus serein et agréable. Pour commencer, je remercie Brieuc, mon collègue de bureau, pour sa présence douce, chaleureuse et rassurante, toujours à l'écoute et pleine de conseils avisés. Ce fut un réel plaisir de te retrouver au bureau, et je garde un excellent souvenir de nos conversations riches et stimulantes. Nous nous sommes beaucoup soutenus mutuellement tout au long de ces six années, et je t'en suis très reconnaissant : je n'aurais pas pu rêver meilleur partenaire de bureau.

Ensuite, j'aimerais remercier la team JUL, qui a été un soutien précieux, ainsi qu'une grande source de fun tout au long de ce périple, répondant toujours à l'appel pour une Ribouldingue, un Spikeball ou un jeu de société. Merci à Charles de m'avoir accueilli au sein de la team JUL et de m'avoir montré qu'il était parfaitement acceptable de dessiner des Pokémons pendant les réunions RevealFlight. Merci pour ta présence toujours joyeuse, ta culture geek infinie, tes innombrables blagues — même si je ne les comprenais pas toujours — et pour les délicieux cookies de chez MIAM. Merci à Seb d'avoir endossé la lourde responsabilité de fils spirituel de Julien, d'avoir répondu patiemment à toutes mes questions administratives, d'avoir organisé les retraites, group meetings, soirée jeux de société d'une main de maître, toujours accompagné des excellents produits de chez Pollen. Merci pour ton humour, ton esprit fédérateur, les spikeballs et les jeux de société à gogo. Merci à Anne, dont la détermination m'a inspiré, que ce soit en recherche, dans ton engagement écologique ou féministe, ou encore dans les missions cyclistes que tu relèves, toujours dans des conditions météorologiques extrêmes. Merci d'avoir été toujours présente pour m'écouter et me conseiller, et pour les conversations interminables avec Seb à refaire le monde. Anne,

c'est la collègue qui résout les problèmes de tout les membres de l'Euler, qu'ils soient de nature mathématiques ou plus personnelles. Merci à Nizar, toujours partant pour aller courir, même jusqu'à la corde les jours les plus audacieux, ou pour une séance de muscu. Un collègue rigoureux, fiable et solide, qui ne vacille que devant des jeux de société tels que *Sandwich* ou *Mito*. Merci à Yassine, qui a su mettre l'ambiance lors de nos group meetings, lorsqu'il était debout. Je ne sais pas ce qui m'impressionne le plus : tes talents de dessin à *Esquissé* ou ton art consommé de la triche au même jeu. Merci aussi pour tes *One man shows* réguliers, l'Euler aurait été boring sans toi ! Merci à Renato, joggeur invétéré et compagnon fidèle du royaume peu connu du -1. En tant que Président de l'IEEE, il peut légitimement se proclamer King of Networks, au nom de la Control Community et surtout le jour de la galette des Rois. Merci à Erwan pour ton écoute, ton rire, ton implication, ton engagement et ta motivation qui inspirent. J'ai adoré refaire le monde avec toi, et en apprendre plus sur la géopolitique (et sur la cuisine), tu es passionné et passionnant ! Merci à Martina, sur qui je sais pouvoir toujours compter pour partager une pizza à Eupen... même quand notre train part sans nous (après avoir déjà raté le précédent !). Ta motivation inépuisable pour la recherche force le respect, et tes glissades incontrôlées sur la glace à Monschau nous auront offert de beaux fous rires ! Merci à Moslem pour sa gentillesse, son sourire et sa culture scientifique sans limite : quel que soit ton problème, il connaît un article du XXe siècle, signé par un auteur russe méconnu, qui l'a résolu élégamment en trois lignes. Il peut en citer l'année de publication, la liste complète des auteurs, leur couleur préférée et même le numéro du théorème. Merci à Mingming, Isabella et Georgios d'avoir aussi fait partie de cette incroyable équipe.

Plus généralement, merci à toutes les autres belles personnes que j'ai eu la chance de côtoyer à l'Euler, et qui m'ont donné envie d'y venir plus souvent pour partager des pauses running, tartes, teambuildings, Benelux meetings, Photo Roulette, Aventure Parc, spikeball, ragots et autres : Philémon, Adrien, Clémence, Loïc, Florentin, Rémi, Julien, Thomas, Christophe, Guillaume, Guillaume, Guillaume, Guillaume, Léopold, Ange, Nathan, Matteo, Lucas, Astrid, Simon, Simon, Dâna, Bastien, et tous les autres.

Un grand merci à Marie-Christine, Pascale, Etienne, François, Astrid, John et Marine qui ont rendu ma vie de chercheur à l'Euler plus facile et bien plus belle !

Au-delà du travail, j'ai eu la chance d'être merveilleusement bien entouré par mes proches, que je remercie chaleureusement. Merci à la coloc du

Château, qui fut pour moi un véritable jardin d'apprentissages, foisonnant d'idées, de projets audacieux et parfois un peu fous. Au cours de ces quatre années, nous avons exploré des façons alternatives de vivre et de partager le quotidien, dans une ambiance à la fois chaleureuse, poétique, utopiste et un brin révolutionnaire.

Merci à mon oncle Vincent, pour sa présence chaleureuse, son humour légendaire, nos runnings et nos belles discussions sur le temps de midi. Merci à mon frère Benoît pour ses conseils précieux et pour l'exemple inspirant qu'il incarne à travers son excellence académique. Merci à mes parents pour leur soutien inconditionnel à tous les niveaux, leur disponibilité, et leurs bons conseils. Enfin, merci à Ysaline pour sa compréhension lors des moments de rush, et pour m'avoir aidé à instaurer une meilleure rigueur de travail. Merci surtout pour son soutien indéfectible, son écoute, sa patience, sa présence attentive, ses conseils avisés et son énergie solaire.

Contents

Contents	vi
List of symbols	x
List of acronyms	xvi
1 Introduction	1
1.1 State of the Art	4
1.1.1 Topology identification	4
1.1.2 Single transfer function identification	5
1.1.3 Full network identification	6
1.1.4 Network identifiability	7
1.1.5 Generic properties	7
1.1.6 Generic identifiability	9
1.1.7 Partial excitation and measurement	10
1.2 Research problem	13
1.3 Outline	15
2 Network identifiability	18
2.1 Model	18
2.2 Associated graph	22
2.2.1 Notions of graph theory	22
2.3 Global identifiability	23
2.4 Presence of loops	25
2.5 Genericity and pseudo-genericity	27
3 The full excitation (resp. measurement) case	30
3.1 Characterization with mutually vertex-disjoint walks	30
3.2 Algorithm for excitation allocation	33

3.2.1	Without known modules: pseudotrees	35
-------	--	----

I Algebraic characterization of local identifiability

4	Local identifiability and local injectivity	37
4.1	Local identifiability	37
4.2	Identifiability and injectivity	38
4.3	Reduction to the scalar case	41
4.4	Identifiability in the scalar case	49
5	Algebraic characterizations	53
5.1	Algebraic characterization of local injectivity	54
5.2	Algebraic characterization of local identifiability	60
5.3	Algorithm to test generic local identifiability	66
5.4	Summary	69
6	Coupled, decoupled and separable Excited and Measured networks	71
6.1	The coupled and decoupled EM networks	73
6.2	Separable networks	75
6.3	Identifiability of the coupled and decoupled networks	77
6.4	Decoupled identifiability and genericity	78
6.5	Decoupled identifiability and local identifiability	80
6.6	Contribution of this chapter	82

II Combinatorial and walk-based conditions for identifiability

7	Combinatorial characterization of network identifiability	84
7.1	Identifiability as matrix nonsingularity	85
7.2	Taylor expansion in the graph	86
7.3	Definitions on the graph	87
7.4	Determinant as sum of monomials	91
7.5	Combinatorial characterization	92
8	Walk-based conditions	94
8.1	Conditions with collections of walks	95
8.2	Decomposition of an assignment	95

8.3	Conditions with vertex-disjoint walks	97
8.4	Disproven conjecture	103
9	The Meta-Walk Graph	106
9.1	Meeting Nodes and MN-swaps	107
9.2	The Meta-Walk Graph	111
9.3	Quantitative properties	113
9.4	Sufficient conditions on identifiability	118
9.5	Characterization of identifiability for a specific case	120
9.6	Conclusion	121
 III Signal selection for identifiability		
10	Optimal signal selection is NP-hard	123
10.1	Classes P, NP, and NP-hard	124
10.1.1	<i>The class P</i>	124
10.1.2	<i>The class NP</i>	124
10.1.3	<i>NP-complete and NP-hard problems</i>	125
10.1.4	<i>Significance in practice</i>	126
10.2	A fundamental NP-complete problem: 3SAT	126
10.3	NP-hardness proof	127
11	Algorithms for signal selection	133
11.1	Tools for algorithm design	134
11.1.1	<i>Lower bound</i>	134
11.1.2	<i>Generic local identifiability test</i>	138
11.1.3	<i>Monotonicity of identifiability</i>	139
11.1.4	<i>SIMUG and pseudotree covering algorithms</i>	140
11.1.5	<i>Heuristics from walk-based conditions</i>	140
11.2	Algorithms description	142
11.2.1	<i>Exhaustive Search</i>	142
11.2.2	<i>Pruning Search</i>	143
11.2.3	<i>Pseudotree Search</i>	146
11.2.4	<i>SIMUG Search</i>	148
11.2.5	<i>Merging Search</i>	149
11.3	Dataset description	153
11.3.1	<i>Network topologies</i>	153
11.3.2	<i>Network metrics</i>	155
11.4	Results	155

11.4.1 Discussion 155

IV Conclusion

12 Conclusion 160

12.1 Research outcomes 160

12.2 Gap between generic global and generic local identifiability . . 162

12.3 Perspectives 164

12.3.1 Improvements on algorithms for signal selection 164

12.3.2 Identifiability of a subset of unknown modules 165

12.3.3 Extensions of algorithms for signal selection 166

Bibliography 167

List of symbols

This section lists the main notations introduced throughout this thesis with a reference to the page where they are defined or appear for the first time. They are listed in alphabetical order, starting with the typographical symbols, then the Latin symbols, and finally Greek symbols.

$*$ — A node of the network, can be different for each occurrence of $*$ (p. 87)

$\otimes$ — Kronecker product (p. 61)

$\mathcal{A} \uplus \mathcal{B}$ — Multiset union of $\mathcal{A}$ and $\mathcal{B}$ (p. 144)

$|\cdot|$ — Number of elements of a set (p. 19)

$A^\bullet$ — Matrix of known modules $\in \mathbb{C}^{L \times L}$, i.e. scalar version of $G^\bullet$ (p. 39)

A° — Matrix of unknown modules $\in \mathbb{C}^{L \times L}$, i.e. scalar version of G° (p. 39)

$A = A^\bullet + A^\circ \in \mathbb{C}^{L \times L}$, Matrix of all modules, i.e. scalar version of G (p. 39)

A_{ij} — Module $\in \mathbb{C}$ from node j to node i , i.e. scalar version of G_{ij} (p. 39)

$\check{A}^\bullet$ — Matrix of known modules $\in \mathbb{C}^{2L \times 2L}$ of the coupled EM network (p. 73)

$\check{A}^\circ$ — Matrix of unknown modules $\in \mathbb{C}^{2L \times 2L}$ of the coupled EM network (p. 73)

$\hat{A}^\bullet$ — Matrix of known modules $\in \mathbb{C}^{2L \times 2L}$ of the decoupled EM network (p. 74)

$\hat{A}^\circ$ — Matrix of unknown modules $\in \mathbb{C}^{2L \times 2L}$ of decoupled EM network (p. 74)

A_B — Submatrix of known modules $\in \mathbb{C}^{L_B \times L_B}$ of the excited subgraph of a separable EM network (p. 76)

A_C — Submatrix of known modules $\in \mathbb{C}^{L_C \times L_C}$ of the measured subgraph of a separable EM network (p. 76)

$A^\odot$ — Submatrix of unknown modules $\in \mathbb{C}^{L_C \times L_B}$ linking the two subgraphs of a separable network (p. 76)

$a^\bullet$ ————— = $\text{nonzeros}(A^\bullet) \in \mathbb{C}^{m^\bullet}$, Vector of known modules (p. 39)

a° ————— = $\text{nonzeros}(A^\circ) \in \mathbb{C}^{m^\circ}$, Vector of unknown modules (p. 39)

a ————— = $(a^\circ, a^\bullet) \in \mathbb{C}^{m^\circ + m^\bullet}$, Vector of all modules (p. 50)

$\bar{A}(a^\circ; a^\bullet)$ — = A , Rearranges vectors of modules in network structure (p. 39)

$\text{Ball}(x, \epsilon)$ — Set of points at a distant stricly smaller than ϵ from x . (p. 38)

$\mathcal{B}$ ————— Set of excited nodes (p. 22)

B ————— Excitations selection matrix (p. 19)

$\check{B}$ ————— Excitations selection matrix of the coupled EM network (p. 73)

$\hat{B}$ ————— Excitations selection matrix of the decoupled EM network (p. 74)

$\mathcal{B}^{\mathcal{P}}$ ————— Subset of $\mathcal{B}$ with excitations selected by $\mathcal{P}$ (p. 85)

$\mathcal{B}_c^{\mathcal{P}}$ ————— Set of excitations $b \in \mathcal{B}$ such that $(b, c) \in \mathcal{P}$ (p. 85)

$\mathcal{C}$ ————— Set of measured nodes (p. 22)

C ————— Measurements selection matrix (p. 19)

$\check{C}$ ————— Measurements selection matrix of the coupled EM network (p. 73)

$\hat{C}$ ————— Measurements selection matrix of the decoupled EM network (p. 74)

$\mathcal{C}^\infty$ ————— The class of infinitely differentiable function. (p. 42)

$\mathcal{C}^{\mathcal{P}}$ ————— Subset of $\mathcal{C}$ with measurements selected by $\mathcal{P}$ (p. 85)

$\mathcal{C}_b^{\mathcal{P}}$ ————— Set of excitations $c \in \mathcal{C}$ such that $(b, c) \in \mathcal{P}$ (p. 85)

$\mathcal{D}_h$ — = $\{(a^\circ; a^\bullet) \in \mathbb{C}^{m^\circ} \times \mathbb{C}^{m^\bullet} \text{ such that } \det(I - \bar{A}(a^\circ; a^\bullet)) = 0\}$ (p. 39)

$\mathcal{E}^\bullet$ ————— Set of edges corresponding to known transfer functions (p. 22)

$\mathcal{E}^\circ$ ————— Set of edges corresponding to unknown transfer functions (p. 22)

$\mathcal{E}$	_____	$= \{\mathcal{E}^\circ, \mathcal{E}^\bullet\}$, Set of all edges (p. 22)
$E(\mathcal{G})$	_____	Set of edges of graph $\mathcal{G}$ (p. 33)
$\mathbf{e}_\ell$	_____	The standard basis vector filled with zeros except 1 at ℓ -th entry (p. 42)
$\mathcal{G}_{\text{net}}$	_____	Associated graph (p. 22)
$G^\bullet$	_____	Matrix of known transfer functions (p. 19)
$\mathcal{G}_\mu$	_____	The Meta-Walk Graph associated to monomial μ (p. 111)
G°	_____	Matrix of unknown transfer functions (p. 19)
G	_____	$= G^\bullet + G^\circ$, Matrix of all transfer functions (p. 19)
G_{ij}	_____	Transfer function from node j to node i (p. 19)
h	_____	Function defined in (4.2) (p. 39)
I	_____	Identity matrix (p. 24)
I°	$\in \{0, 1\}^{L^2 \times m^\circ}$	selects the columns of the preceding matrix corresponding to the m° unknown modules (p. 61)
K	$= \nabla h$	matrix characterizing generic local identifiability, see (5.7) (p. 61)
$\hat{K}$	_____	Matrix characterizing decoupled identifiability, defined in (6.6) (p. 79)
$K_{\mathcal{P}}$	_____	Submatrix of K selecting the rows corresponding to $\mathcal{P}$ (p. 85)
λ	_____	Excited or measured node selected by $\mathcal{P}$, i.e. $\lambda \in \mathcal{B}^{\mathcal{P}} \cup \mathcal{C}^{\mathcal{P}}$ (p. 108)
L	_____	Number of nodes (p. 19)
L_B	_____	Number of nodes of excited subgraph of a separable EM network (p. 76)
L_C	_____	Number of nodes of measured subgraph of separable EM network (p. 76)
$m^\bullet$	_____	Number of known transfer functions (p. 22)
m°	_____	Number of unknown transfer functions (p. 22)
m	_____	$= m^\circ + m^\bullet$, Number of all transfer functions (p. 22)
$\mathbb{N}$	_____	Set of natural numbers (p. 21)
n_B	_____	Number of excited nodes (p. 19)

$n_{\mathcal{B}c}^{\mathcal{P}}$	Cardinality of set $\mathcal{B}_c^{\mathcal{P}}$ (p. 85)
n_C	Number of measured nodes (p. 19)
$n_{\mathcal{C}b}^{\mathcal{P}}$	Cardinality of set $\mathcal{C}_b^{\mathcal{P}}$ (p. 85)
$\mathcal{N}_\omega$	Set of all meeting nodes of ω (p. 108)
$\mathcal{P}$	Subset of cardinality m° of $\mathcal{B} \times \mathcal{C}$ (p. 85)
q	Delay operator (p. 19)
r	Excitation signals (p. 19)
$\mathbb{R}$	Set of real numbers (p. 21)
$r(\mu)$	The repetition of monomial μ , see (7.9) (p. 89)
$s(\alpha)$	The start node of unknown module α (p. 89)
$\text{sgn}(\sigma)$	Signature: assigns a binary number to assignation σ (p. 88)
$\text{sgn}(\omega)$	The signature of the assignation corresponding to ω (p. 88)
t	Time (p. 19)
$t(\alpha)$	The end node of unknown module α (p. 89)
$T(G(z;\theta))$	$= (I - G(z;\theta))^{-1}$ (p. 24)
$T(A)$	$= (I - A)^{-1}$ (p. 86)
$\text{vec}(\cdot)$	Vectorizes its $n_1 \times n_2$ matrix argument to a $n_1 \cdot n_2$ vector (p. 39)
$V(\mathcal{G})$	Set of nodes of graph $\mathcal{G}$ (p. 33)
w	Node signals (p. 19)
$\mathcal{W}$	Set of nodes in the associated graph (p. 22)
$\mathcal{W}_i^+$	The out-neighbors of node i (p. 32)
$\mathcal{W}_j^-$	The in-neighbors of node j (p. 33)
$\mathcal{W}_j^{\circ-}$	The in-neighbours of node j through unknown transfer functions (p. 33)
y	Measured signals (p. 19)

α _____	An unknown module of the network (p. 87)
$\mu(\omega)$ _____	The monomial cooresponding to ω , see (7.8) (p. 89)
$\mathcal{M}_{\mathcal{P}}$ _____	The set of all $\mu(\omega)$ corresponding to bijective ω with EM set $\mathcal{P}$ (p. 89)
ν _____	Node of the network, i.e. $\nu \in \mathcal{W}$ (p. 108)
ν^λ _____	Meeting node, with $\nu \in \mathcal{W}$ and $\lambda \in \mathcal{B}^{\mathcal{P}} \cup \mathcal{C}^{\mathcal{P}}$ (p. 108)
$\pi_{\nu\lambda}$ _____	Multiplicity of meeting node ν^λ (p. 108)
σ _____	Assigns a pair excitation-measurement to each unknown module (p. 88)
σ_B _____	Assigns an excitation to each unknown module (p. 97)
σ_{Bc} _____	The restriction of σ_B to unknown modules in $\sigma_C^{-1}(c)$ (p. 97)
$\sigma_B^{-1}(b)$ _____	The set of unknown modules assigned to excitation b by σ_B (p. 97)
σ_C _____	Assigns a measurement to each unknown module (p. 97)
σ_{Cb} _____	The restriction of σ_C to unknown modules in $\sigma_B^{-1}(b)$ (p. 97)
$\sigma_C^{-1}(c)$ _____	The set of unknown modules assigned to excitation c by σ_C (p. 97)
$\Sigma_B^{\mathcal{P}\sigma_C}$ _____	The set of all σ_B bijection-pairable and $\mathcal{P}$ -compatible with σ_C (p. 97)
$\Sigma_C^{\mathcal{P}\sigma_B}$ _____	The set of all σ_C bijection-pairable and $\mathcal{P}$ -compatible with σ_B (p. 97)
$\sigma_{\mathcal{P}\text{ref}}$ _____	A reference bijective assignation from which sgn is computed (p. 88)
$\Sigma_{\mathcal{P}}$ _____	Set of all bijective σ , with excitation-measurement set $\mathcal{P}$ (p. 88)
$\theta^\bullet$ _____	Parameters of known transfer matrix $G^\bullet$ (p. 21)
θ° _____	Parameters of unknown transfer matrix G° (p. 21)
θ _____	$= (\theta^\circ, \theta^\bullet)$, Parameters of transfer matrix G (p. 21)
$\Theta^\bullet$ _____	Set of all valid parameters $\theta^\bullet$ (p. 21)
Θ° _____	Set of all valid parameters θ° (p. 21)
Θ _____	Set of all valid parameters θ (p. 21)

$Y(\mathcal{T})$	_____	Set of roots of multi-rooted graph $\mathcal{T}$ (p. 33)
$\beta_{\mathcal{J} \rightarrow \mathcal{I}}$	_____	Max number of mutually vertex-disjoint walks from $\mathcal{J}$ to $\mathcal{I}$ (p. 31)
ω	_____	A collection of m° walks, see Definition 7.6 (p. 88)
$\omega_B(\alpha_i)$	_____	The start node of ω_i (p. 89)
$\omega_C(\alpha_i)$	_____	The end node of ω_i (p. 89)
ω_i	_____	A walk of ω (p. 88)
Ω_μ	_____	The set of all bijective ω corresponding to monomial μ (p. 89)
$\Omega_{\mathcal{P}}$	_____	The set of all bijective ω with excitation-measurement set $\mathcal{P}$ (p. 88)
Ω_σ	_____	The set of all ω corresponding to assignation σ (p. 88)
$\Omega_\mu^{\text{even deg}}$	_____	Set of all bijective ω with even degree in $\mathcal{G}_\mu$ (p. 119)

List of acronyms

DAG	Directed Acyclic Graph
EM network	Excited and Measured network
EM pair	Excitation-Measurement pair
EMP	Excitation and Measurement Pattern
MN	Meeting and Node
MN-swap	Meeting and Node-swap
SIMUG	Single-source Identifiable Multi-rooted Graph

1

Introduction

SYSTEM identification is a foundational discipline within the broader field of systems and control, concerned with building mathematical models of dynamic systems directly from observed data. In many cases, deriving models solely from physical principles is either impractical or too resource-intensive. System identification offers a powerful alternative by using input-output measurements to infer system behavior. Depending on the available prior knowledge, identification methods range from black-box approaches, which rely solely on data without assuming a physical structure, to grey-box models, which integrate partial physical insight with data-driven estimation of unknown parameters [Lju98].

With the rapid advancement of technology, real-world systems have grown increasingly interconnected and large-scale. Such systems are now common across a wide range of domains, including neuroscience [GGVC⁺20, Fri11, vESB⁺20, TVA⁺13], biology [DVNS08, DBJ⁺07], power systems [Kun07, SIR09], geology [BML06, JBVdH08], economics [BD94, SHR⁺18], medicine [CAO14], wind energy [KBS15], water networks [vGBY⁺24]. These systems are typically composed of many interdependent components that interact to perform sophisticated tasks. This interconnected nature leads to what are often referred to as dynamic networks or networked systems.

While classical system identification techniques were designed for relatively simple structures — such as open- or closed-loop multi-variable

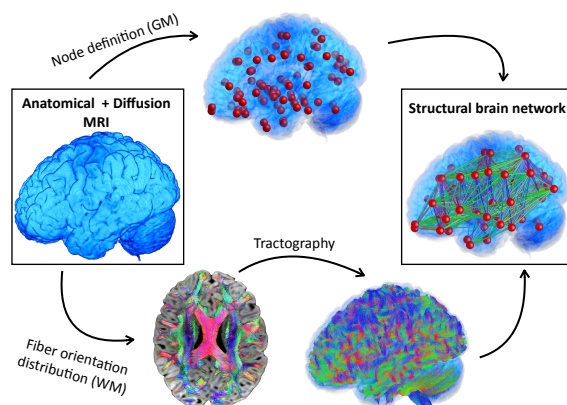


Fig. 1.1 Modeling the human brain as a dynamic network [Chi21].

systems — they fall short in addressing the structural and computational challenges posed by today’s complex networks. Modeling such interconnected systems is now essential not only for understanding how the individual components influence one another but also for enabling effective analysis, control, and optimization at the system level.

Besides, with ongoing progress in sensor technology, modern sensors have become not only more affordable but also significantly more accurate and versatile. Today’s sensors can capture a wide spectrum of physical quantities, including displacement, acceleration, humidity, chemical concentration, magnetic fields, light intensity, pH levels, and sound pressure — alongside more traditional measurements like temperature, pressure, voltage, and current. This growing capability enables detailed monitoring of complex systems across various domains. For example, in environmental monitoring, networks of sensors can track air quality, soil moisture, and pollutant levels. In structural health monitoring, embedded sensors detect vibrations, strain, and stress within bridges or buildings. The increasing diversity and quality of measurable variables have greatly enriched the amount of data available, opening new possibilities for identifying and modeling large-scale, interconnected systems using data-driven methods.

Neuroscience offers a compelling motivation for studying networked systems, as the brain itself is a highly complex, interconnected network, as illustrated in Fig. 1.1. Neural activity is not confined to isolated regions but emerges from interactions between numerous brain areas that communicate

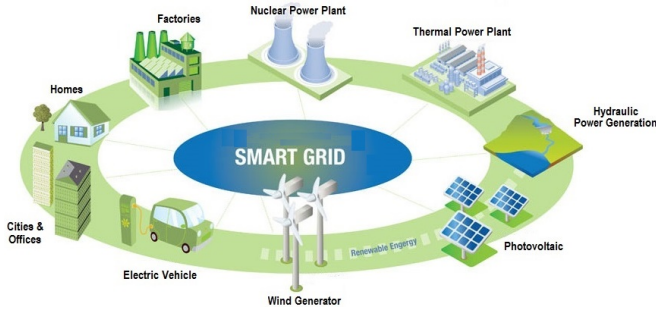


Fig. 1.2 Power grids are an example of networked systems [Con16].

dynamically over time. Understanding these interactions — often referred to as effective connectivity — is key to uncovering how cognitive functions arise and how disruptions in these networks may lead to neurological disorders. Modern brain imaging techniques, such as fMRI and EEG, provide rich datasets capturing the activity of multiple brain regions simultaneously. This abundance of multivariate, time-dependent data naturally calls for system-theoretic tools capable of modeling and analyzing dynamic networks. By treating the brain as a networked system, researchers can better capture its underlying structure, infer causal relationships, and ultimately gain deeper insights into its function and dysfunction.

Power grids are another example of complex networked systems, as illustrated in Fig. 1.2. The integration of renewable energy sources like solar and wind, alongside emerging components such as electric vehicles and decentralized storage systems (e.g., batteries or thermal storage), has transformed the traditional grid. While these additions promote a more sustainable and environmentally friendly energy infrastructure, they also introduce new challenges by increasing system complexity and causing voltage instabilities that can affect overall grid performance. As a result, there is a growing need for advanced methods in control, optimization, and monitoring to ensure the reliable and efficient operation of such complex networked systems.

This thesis focuses on the identification of such large-scale interconnected dynamical systems, leveraging data-driven approaches to address the challenges posed by their complexity and structure.

1.1 State of the Art

Over the past decade, the field of networked systems identification has evolved through a variety of research directions. In this section, we provide an overview of the main trends and methodologies that have emerged, with the aim of situating our work within the broader landscape of ongoing developments.

1.1.1 Topology identification

Topology identification is the problem of identifying the graph structure of a dynamical network using measured input/output data. In systems biology, it enables the reconstruction of gene regulatory networks to understand how genes influence one another during cellular processes [MCK⁺12]. In neuroscience, identifying the topology of brain networks from EEG or fMRI data provides insights into how different brain regions interact, supporting studies of cognition and neurological disorders [Fri11]. Similarly, in social network analysis, topology inference helps uncover patterns of influence or information diffusion, which is valuable for marketing strategies and behavioral studies [GRLK12]. Lastly, in economics and finance, topology identification techniques are applied to infer interdependencies among institutions or markets, helping assess systemic risks [DY14].

In communication networks, [CCN⁺02] introduced an MCMC-based method using end-to-end delay measurements to infer topology without internal cooperation. In contrast, power systems research often combines data with physical models: [ZWWZ20] proposed a two-step method combining regression with power flow equations, while [WLR16] used mutual information among voltage signals from distributed energy resources. Probabilistic inference based on real-time measurements is used in [SAMC12] to detect switching configurations under limited sensing.

More general approaches have leveraged statistical dependencies in time-series data. [GSK18] developed a kernel-based framework to capture nonlinear and time-varying dependencies, while [SP14] used spectral analysis and node-knockout experiments to reconstruct directed graphs in LTI systems. General conditions for solvability are formulated in [vWTC19b], where constrained Lyapunov methods are used to infer structure, while

[vWTC21] extends identifiability and reconstruction results to heterogeneous networks using subspace techniques.

1.1.2 Single transfer function identification

Identifying a single transfer function (also called module in the literature) in a dynamic network hinges on selecting a predictor model that ensures data-informativity, enabling consistent estimation. The direct method [VdHDHB13, VdHRDB19, VdHR20, RV20] formulates conditions on measured node signals as predictor inputs, using graph-based criteria. These conditions were later simplified and embedded in constructive node selection algorithms [VdHRF23], integrating structure and excitation requirements. This path-based framework aligns with the flexibility introduced in [RVdHD19], which combines direct and indirect approaches to broaden the admissible choices of measured and excited nodes. When all node signals are measurable, [SCDSVdH23] provides a graphical selection algorithm for single module identification.

To reduce complexity, [RCSVdH22] and [RBVdH21] avoid full parametric modeling of non-target modules by using non-parametric frequency response estimates and kernel-based regularization, respectively—both preserving consistency while minimizing computation and avoiding model order selection. These approaches share the goal of local focus seen in [GBdS18], which offers minimal excitation and measurement requirements based only on local topology, and in [DVdHBH15], which handles sensor noise via instrumental variables, extending prediction error methods to networks with flexible sensor placement.

When active excitation is not feasible, passive approaches using graphical identification conditions are used instead. The methods in [MS15, MS19] leverage d-separation to derive identification from partially observed networks, bridging control theory and probabilistic graphical models. This focus on structural constraints also underpins the abstraction framework in [WLEVdH20], which generalizes immersion techniques to identify modules invariant under signal removal.

In large-scale networks with limited accessibility, decentralized subspace methods [YV17, YCV19] reconstruct local models from local I/O data, treating unmeasurable interconnections as unknown inputs and enabling scalable identification through spatial-temporal decomposition. Finally, the

experiment design framework of [MBH⁺19] complements these methods by optimizing excitation signals to minimize identification cost while ensuring estimation quality, supporting efficient local identification strategies.

1.1.3 Full network identification

Full network identification aims to estimate all modules in a dynamic network with known topology. A common approach extends prediction error methods: [WVdHD18c] formulates a joint prediction problem with correlated, rank-deficient noise, leading to a constrained least-squares criterion achieving maximum likelihood estimates. [FFH20] adapts Weighted Null-Space Fitting (WNSF) to networks, avoiding local minima and matching the efficiency of PEM with ideal initialization. Both methods focus on statistical guarantees and performance with dense data. In the case of unknown disturbance topology, [FRVdH22] provides a scalable multi-step least squares method for network identification.

Sparse recovery methods address high-dimensional or data-scarce scenarios. [BVVN11] uses Group Lasso for AR networks, introducing the false connection score to ensure consistent recovery. [CP12] proposes a Bayesian method combining stable-spline kernels and sparsity-inducing priors, improving scalability for large ARMAX systems. Both treat network identification as variable selection under sparsity constraints, with complementary strategies.

Scalability and full network identification are further tackled in [HV14] and [HCGT16]. The former develops a decentralized subspace method using local I/O data for large-scale systems; the latter focuses on experiment design to ensure identifiability in sparse networks. Together, these works complement the statistical methods of [WVdHD18c, FFH20] and the sparse recovery approaches of [BVVN11, CP12], offering a rich toolbox for full network identification.

Diffusively coupled networks, characterized by symmetric interactions and modeled as undirected graphs, can be identified using adapted prediction error methods. Global and local identification strategies are proposed in [KVdH19, KVdH22a], with convex reformulations to handle nonconvexity. General identification conditions under relaxed sensing and actuation are established in [KVdH23], while [KVdH22b] enables consistent local identification from local signals through structured sensor-actuator placement.

1.1.4 Network identifiability

In contrast to identification methods focused on estimation, the concept of network identifiability is a theoretical concern. First introduced in [WD-VdH15] in the context of networked systems, it addresses whether the observed signals can be uniquely explained by a specific network model. If multiple models are consistent with the same data, then no identification method can guarantee a unique result. This becomes particularly critical when the network is expected to reflect a real-world system. Network identifiability is typically dependent on several structural properties of the networked system, such as the network topology, the placement of excited signals, and the set of measured signals.

Within this framework, [GBP17] provides a range of sufficient conditions for network identifiability, highlighting trade-offs between known external excitations and stochastic disturbances. The concept was further refined in [WVdHD18a], which extends identifiability to the level of single modules, introducing conditions under which a single transfer function can be uniquely identified. A more general and formal definition, accounting for the presence and structure of noise — including reduced-rank disturbance processes — was developed in [WVdHD18b], where identifiability conditions involve both excitation placement and disturbance correlations. Finally, [GBP18] adapts this framework to networks with singular noise spectra, shifting the focus from identifying a specific noise model to correctly recovering the noise spectrum, thereby accommodating scenarios with noise-free or highly correlated nodes.

1.1.5 Generic properties

As will be discussed in Section 1.1.6, the notion of genericity plays a central role in the study of network identifiability. To provide context, this section presents a concise review of the relevant literature on genericity. Generic properties — also referred to as structural in the literature — are properties that hold for almost all parameter values, that is, for all values except possibly those belonging to a lower-dimensional subset of the parameter space.

A classic example arises in the field of structural rigidity theory [Cra79, TD99]. The fundamental object of study is a framework: a graph whose vertices are embedded as points in Euclidean space (typically $\mathbb{R}^2$

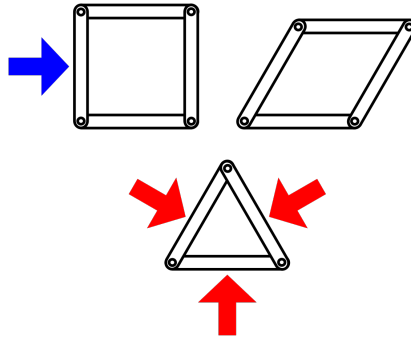


Fig. 1.3 Illustration of structural rigidity: the triangular framework is rigid, the square one is not [Wik25b].

or $\mathbb{R}^3$), and whose edges represent fixed-length bars connecting pairs of points (see Fig. 1.3). A framework is said to be rigid if it cannot be continuously deformed without changing the length of the bars — excluding trivial motions such as translations or rotations. If such deformations exist, the framework is flexible. At first glance, rigidity might appear to depend on both the combinatorial structure of the graph and the precise coordinates of the points. However, this dependency is highly structured and algebraic in nature. Rigidity is a generic property: it depends only on the graph for almost all point placements. More formally, rigidity holds for all configurations outside of a lower-dimensional algebraic variety in the configuration space [AR78, GSS93]. In practice, this means that a framework is either rigid for almost all placements of the points, or flexible for all placements. Therefore, randomly chosen configurations will almost surely lie in the generic case, while exceptions (e.g., collinear or symmetrically arranged points) form a degenerate set of measure zero.

Controllability is another classical example of generic property: for linear time-invariant systems of the form $\dot{x}(t) = Ax(t) + Bu(t)$, the set of parameter values (A, B) for which the system is controllable forms a dense open subset of the parameter space, while the uncontrollable cases lie in a proper algebraic variety [Bro15, Lin74]. It expresses the idea that controllability is not a fragile or exceptional condition, but rather a generic one.

The survey [DCVdW03] considers linear structured systems in state-

space form, where each matrix entry is either a fixed zero or a free parameter. The pattern of fixed zeros defines the system's structure, a modeling framework well-suited to many man-made physical systems. Many classical system-theoretic properties — such as controllability, finite and infinite zero structure, disturbance rejection, and input–output decoupling — can be studied in terms of genericity. Notably, these generic properties can often be verified using directed graphs naturally associated with the system's structure. This graph-based approach enables meaningful analysis even with limited numerical information.

The generic rank of a matrix with parameterized entries is defined as the maximum rank that the matrix can attain for any choice of the parameters. [Mur87] proves that such matrix achieves its generic rank for almost all parameters, i.e. for all except on a set of parameters of measure zero (typically a proper algebraic variety). The work in [dW91] considers structured systems, and establishes that the generic rank of the transfer matrices equals the maximum number of vertex disjoint paths from the input vertices to the output vertices in the graph that can be associated to the structured system.

1.1.6 Generic identifiability

The concept of generic network identifiability, first introduced in [BGHP17], concerns whether, for *almost all* parameterizations consistent with a given network topology, the dynamics of the network can be uniquely inferred from observed data.

A formal definition of generic identifiability is provided in [HGB18], in the case of full excitation (i.e. all nodes are excited) and partial measurements (i.e. not all nodes are measured). Necessary and sufficient conditions for generic identifiability are derived, based on the generic rank of transfer matrices, which can be equivalently formulated as the existence of *vertex-disjoint paths* between excited and measured nodes. These path-based conditions lay the foundation for a topological characterization of generic identifiability. Building on this, [WVdHD18a] derives path-based conditions for single module identifiability.

The work of [VWTC19a] extends the results of [HGB18], which focus on *generic* identifiability, to the stronger notion of identifiability that holds universally, i.e., for all parameterizations consistent with the network graph, and not only for almost all of them. To this end, the authors introduce the

graph simplification process: performing operations on the graph without affecting the rank of certain network matrices, leaving nongeneric identifiability unchanged. This yields necessary and sufficient topological conditions for nongeneric identifiability. These conditions not only generalize existing results but also allow for efficient verification via polynomial-time algorithms, highlighting the structural differences between generic and nongeneric settings.

In addition to analysis, recent research focuses on the synthesis of generic identifiable network configurations. [CSVdH19, CSVdH21] address the problem of determining optimal excitation locations to ensure generic identifiability when all node signals are measured. The authors introduce the notion of so-called *pseudotrees*, an extension of the classical tree structures from graph theory. The path-based conditions established in [HGB18] are reinterpreted and reformulated in terms of these pseudotrees in the graph. By covering the graph with disjoint pseudotrees, and placing excitation signals at their roots, generic identifiability is guaranteed. A dual problem — determining minimal measurement sets under full excitation — is tackled in a similar fashion using *antipseudotrees*. These results are extended in [DSC⁺22], where SIMUGs (single-source identifiable multi-rooted graphs) replace pseudotrees to account for fixed modules, which represent known transfer functions within the network.

The work in [SCVdH22] extends [CSVdH21, DSC⁺22] for the generic identifiability of single or multiple modules, instead of all unknown modules. The results account for known modules, and apply when all node signals are measured, and both measured excitations and unmeasured disturbances are present. The pseudotree-based conditions and algorithms of [CSVdH21, DSC⁺22] are formulated in terms of disconnecting sets in the graph, a graph-theoretical notion dual to vertex-disjoint paths.

1.1.7 Partial excitation and measurement

The common assumption in all the previously mentioned works is that either all nodes are excited, or they are all measured. [BGH19] is the first work to relax this assumption, and address the case of generic identifiability with partial excitation and measurement, i.e. when neither all nodes are excited, neither all nodes are measured. The authors analyze four prototypical edge set — edges entering a node, leaving a node, forming a tree, and forming a loop — and for each case, they characterize the excitation and measurement

patterns (EMPs) that are valid, i.e. that guarantee generic identifiability. A key insight is that, for a given set of edges to be generically identifiable, all nodes connected to these edges must be either excited or measured or both.

In the specific context of loop networks, [MGB22b] provides necessary and sufficient conditions for generic identifiability. These conditions enable a straightforward characterization of all valid EMPs for loops, demonstrating the power of topological insights in guiding excitation and measurement placement.

Directed acyclic graphs (DAGs) also receive particular attention due to their favorable structural properties and relevance in practical applications. The work in [MGB22a] addresses generic identifiability under partial excitation and measurement specifically for networks with DAG topology. The authors introduce a systematic procedure for designing valid EMPs (i.e. EMPs ensuring generic identifiability) in this setting. This constitutes the first synthesis result for partial excitation and measurement, albeit restricted to DAGs. Importantly, while the procedure guarantees identifiability for any network with a DAG topology, it does not provide optimality guarantees: the resulting excitation and measurement sets need not be minimal, and smaller sets may exist that would also ensure identifiability. Owing to the structural simplicity of DAGs, the procedure avoids the need for the more sophisticated graph-theoretical tools employed in [CSVdH21, DSC⁺22, SCVdH22].

Extending notions of [MGB22a] to non-DAGs, [MGB23] introduces the concepts of *dources* and *dinks*: nodes that present certain topological properties in the network, generalizing the notions *sources* and *sinks* respectively. The authors argue that, for an EMP to be valid, all *sources* and *dources* must be excited, and all *sinks* and *dinks* must be measured. They also show that *sources* and *dources* are the only node types that always require excitation, while *sinks* and *dinks* are the only ones that always require measurement.

While many results are derived under the assumption that the network to be identified is isolated, practical applications often require identifying subnetworks embedded within larger systems. The work in [MGB24] explores this issue, showing that an EMP valid for a subnetwork in isolation may not suffice once the subnetwork is embedded in a larger graph. The authors provide conditions under which such EMPs remain valid and propose enhancements to ensure identifiability in the embedded setting.

The work in [SCVdH23] accounts for fixed modules (i.e. known transfer

functions), which are not considered in [BGH19, MGB22b, MGB22a, MGB23, MGB24]. The authors introduce an equivalent network model structure that explicitly incorporates unmeasured noise signals as excitation signals. This reformulation allows any network with noise to be treated as a noise-free system with additional excitations, thereby simplifying the identifiability analysis and yielding less conservative conditions. Unlike previous works in partial excitation and measurement that focus on the generic identifiability of the whole network, this study targets the generic identifiability of a single module. Depending on whether the input and/or output of the target module is measured, four distinct graphical conditions are derived, covering all possible scenarios in single-module identification. These conditions support the development of synthesis procedures for selecting excitation and measurement nodes to ensure identifiability. The results are further generalized to the identifiability of multiple modules sharing the same starting or destination node.

In [CSLVdH23], a necessary condition for generic identifiability is developed based on the dependency structure of the transfer functions from excited to measured nodes. The analysis uses a bipartite graph representation of the network and introduces an edge-removal procedure to check identifiability. Moreover, the paper presents necessary and sufficient conditions for loop networks, offering a complementary view to [MGB22b].

Finally, [CSLVdH24] introduces the concept of transpose network, defined as the network obtained by reversing the direction of all edges and swapping excitations with measurements, and vice versa. The authors show that the identifiability of a network is equivalent to the identifiability of its transpose network, thereby offering a powerful dual perspective. Focusing on DAGs in which all modules are unknown, the authors propose two sufficient conditions for generic identifiability under partial excitation and measurement. The first condition relies on a vertex-wise analysis of identifiability, while the second is based on the graph-theoretic notions of tree and anti-tree coverings. This second approach leads to a synthesis algorithm for excitation and measurement placement that ensures generic identifiability of all modules in the network.

1.2 Research problem

The objective is as follows: given a network with some unknown transfer functions, determine a set of excitations and measurements such that all unknown transfer functions are generically identifiable. This task is referred to as the synthesis problem for generic identifiability under partial excitation and measurement. Although several algorithms have been proposed in the literature to address this problem, their applicability remains limited in scope:

- [MGB22a] proposes a systematic procedure tailored to networks with a Directed Acyclic Graph (DAG) topology. However, this method does not account for the presence of known modules.
- The algorithm in [SCVdH23] applies to any network topology, and can handle known modules, but it is restricted to the identifiability of a single module, or multiple modules that share the same start or end node. While one could apply this method to each module (or group of modules with a common start/end node) and combine the resulting Excitation and Measurement Patterns (EMPs), such a composition typically yields suboptimal solutions. Specifically, the resulting EMP would indeed guarantee identifiability, but it would ignore the coupling between different parts of the network, thereby missing opportunities to reduce the number of required excitations or measurements.
- [CSLVdH24] addresses the identifiability of all unknown modules, but is restricted to DAGs and does not consider known modules either.

What remains lacking is a synthesis algorithm for generic identifiability under partial excitation and measurement, that applies to arbitrary network topologies, beyond DAGs. The algorithm should account for known modules, and output a minimal EMP (i.e., with the smallest number of required excitations and measurements) that guarantees the identifiability of all unknown modules in the network.

Our objective is to design an algorithm similar to the one proposed in [DSC⁺22], but applicable to the partial excitation and measurement setting. Unfortunately, this setting is significantly more complex: the elegant

graph-theoretical algorithm developed in [DSC⁺22] — which assumes full excitation or full measurement — does not readily extend to the partial case.

To address this challenge, we adopt a hybrid approach, combining algebraic, graph-theoretical and combinatorial techniques.

1. **Efficient identifiability testing** — In Part I, we introduce so-called *local* identifiability, a notion amenable to linear analysis, and which proves to be a generic property. We develop a test for generic local identifiability based on an algebraic characterization. By exploiting its generic nature, the test efficiently evaluates whether a given Excitation and Measurement Pattern (EMP) guarantees generic local identifiability, using generic rank computations from a small number of random samples.
2. **Graph-theoretic analysis** — In Part II, we build on the algebraic characterization introduced in Part I to develop graph-theoretical insights. By introducing sophisticated auxiliary graph structures to support the analysis, we derive walk-based and combinatorial conditions that uncover how information propagates through the network. These conditions highlight key topological features that are critical for ensuring generic local identifiability.
3. **Algorithms for EMP synthesis** — Finally, in Part III, we combine the insights gained from Part II with the efficient testing tool from Part I to design algorithms for signal selection. These algorithms construct EMPs that ensure generic local identifiability of all unknown modules, while aiming to minimize the number of required excitations and measurements.

The graph-theoretic results developed in Part II serve not only as a foundation for deriving the algorithms presented in Part III, but also stand as meaningful theoretical contributions in their own right. Beyond their algorithmic utility, these results provide valuable insights into the problem of generic identifiability in networks, shedding light on its underlying structure and properties. Moreover, the methodological developments required to obtain these results are themselves of interest, as they illustrate a constructive and conceptually rich approach to addressing this class of problems.

1.3 Outline

This thesis investigates the identifiability of networked systems under partial excitation and measurement. Our approach is threefold: we first develop powerful algebraic tools based on manifold calculus; we then translate these results into graph-theoretic terms; and finally, we design algorithms that leverage these insights to solve the synthesis problem — selecting excitation and measurement sets that guarantee identifiability while minimizing the number of signals used.

Preliminaries

- **Chapter 2** introduces the linear dynamic network model along with its associated graph representation, which enables the use of graph-theoretic tools. It formalizes the notion of global identifiability and provides a detailed discussion of genericity. The related concept of pseudo-genericity is also introduced, as it will play a central role in the subsequent chapters.
- **Chapter 3** reviews foundational results from the literature on network identifiability under full excitation or measurement. It presents an algebraic condition based on the generic rank of a network matrix and its translation into graph-theoretic terms using walks. These insights inform algorithmic strategies for excitation allocation and inspire the approach taken in this thesis.

Part I: Algebraic characterization of local identifiability

In this part, we show that analyzing networks composed of scalar modules is equivalent to working with analytic transfer functions. This equivalence significantly simplifies the analysis throughout the thesis. We then derive an algebraic characterization of generic local identifiability, which in turn leads to a practical and efficient algorithm for testing identifiability with probability one.

- **Chapter 4** begins by introducing the notion of local identifiability — a weaker alternative to global identifiability that lends itself to linear analysis. Local identifiability is then characterized as the local injectivity of a specific function. Finally, we show that, generically, analyzing networks with scalar modules is equivalent to considering analytic transfer functions. This work is based on [LH24] and its preliminary version [LH20].

- **Chapter 5** starts by deriving an algebraic characterization of local scalar-coordinate injectivity using tools from manifold theory. As part of this analysis, we also show that scalar-coordinate injectivity is a pseudo-generic property. By applying this characterization to a specific function, we obtain an algebraic characterization of local identifiability and demonstrate that it too is a pseudo-generic property. This leads to necessary and sufficient conditions for generic local identifiability, both (i) for a single unknown module and (ii) for all unknown modules in the network. Leveraging the pseudo-generic nature of local identifiability, these algebraic conditions can be verified numerically with probability 1 by evaluating them on a randomly sampled instance of the network. We develop an algorithm based on this principle, which efficiently computes the generic local identifiability of any given network. This chapter is based on [LH24] and its preliminary version [LH20].
- **Chapter 6** shows how graphical insight can be obtained from the algebraic characterization of Chapter 5. This is achieved via the introduction of an associated network from the initial one, obtained by duplicating it. This approach leads to separable Excited and Measured networks (abbreviated separable EM networks), which possess the appealing property that generic global identifiability is equivalent to generic local identifiability. This property enables to address generic global identifiability with the algebraic tools of Chapter 5. Chapter based on [LH24] and its preliminary versions [LH21a, LH23].

Part II: Combinatorial and walk-based conditions for identifiability

In this part, we focus on the identifiability of *all* unknown modules simultaneously, i.e. *network identifiability*. This part is dedicated to trying to reformulate our algebraic characterization in graphical terms, that would provide insight and guide the design of the algorithms of Part III.

- **Chapter 7** develops a combinatorial characterization that translates the rank condition of Chapter 5 into network-level combinatorial terms. This reveals how network topology and walk-based patterns underpin the identifiability of its modules. This combinatorial perspective lays the foundation for the constructive results and algorithmic methods presented in the subsequent chapters. This chapter is based on [LH24] and its preliminary version [LH23].
- **Chapter 8** aims to translate the abstract characterization of identifi-

ability established in Chapter 7 into more tangible, graph-theoretic conditions. These conditions are intended to support the development of algorithmic methods for signal selection in partially measured and excited networks. Chapter is based on [LH21a, LH21b].

- **Chapter 9** constructs a new graph, called the Meta-Walk Graph, where each vertex corresponds to a bijective collection of walks, and vertices are connected by an edge if they can be obtained from each other by an MN-swap. The Meta-Walk Graph proves to be a powerful tool for analyzing network identifiability, when combined with the characterization of Chapter 7.

Part III: Signal selection for identifiability

This part is dedicated to the development of algorithms for selecting nodes to excite and measure in order to guarantee the generic local identifiability of the network. Since performing excitations and measurements is often costly in practice, our objective is to achieve identifiability while minimizing the number of such interventions.

- **Chapter 10** establishes that the problem of finding such *optimal* selection — i.e., excitation and measurement sets of minimal total cardinality that ensure identifiability — is *NP-hard*. Informally, NP-hard problems are widely regarded as computationally intractable for large instances.
- **Chapter 11** relinquishes the pursuit of guaranteed optimality, since the problem is NP-hard. Instead, our focus shifts to designing algorithms that aim to find high-quality solutions within reasonable computational time, thereby maintaining tractability even as network size increases. To address this challenge, our algorithms build upon a combination of algebraic conditions, walk-based criteria, and algorithmic frameworks developed in the previous chapters. By leveraging these diverse tools, we propose five distinct algorithms.

Part IV: Conclusion

We conclude the thesis by summarizing the research outcomes and discussing the open questions and perspectives in **Chapter 12**.

2

Network identifiability

WE begin by introducing the linear dynamic network model that forms the foundation of our study. We then construct the associated graph representation, which enables the application of graph-theoretical tools for analysis. The notion of global identifiability, a fundamental concept for this thesis, is subsequently defined. We also provide a detailed treatment of genericity, a key property in system theory, and introduce the related concept of pseudo-genericity, which will play a crucial role in the developments that follow.

2.1 Model

We consider L node signals stacked in the vector $w(t) = [w_1(t) \cdots w_L(t)]^\top$, where $t \in \mathbb{Z}$, that interact causally through a network encoded in the $L \times L$ transfer matrix $G(q)$, where q is the delay operator, i.e. $q^{-1}w_i(t) = w_i(t-1)$. The transfer matrix $G(q)$ characterizes how the different node signals influence one another through both present and past values, with the delay operator q accounting for temporal shifts in the signals. More precisely, each entry of $G(q)$ is a transfer function that describes how the output of a given node depends on the current and delayed inputs of another node. This representation is particularly useful because it captures the dynamic dependencies within the network in a compact and structured

form, making it a central tool for analysis and identifiability studies. An illustrative example is provided in Example 2.1 (continued) below. The network is activated by n_B known external excitation signals collected in vector $r(t)$, and we measure n_C node signals gathered in vector $y(t)$:

$$\begin{aligned} w(t) &= G(q) w(t) + B r(t) \\ y(t) &= C w(t) \end{aligned} \quad \forall t \in \mathbb{Z}, \quad (2.1)$$

where matrices $B \in \{0, 1\}^{L \times n_B}$ and $C \in \{0, 1\}^{n_C \times L}$ encode the n_B excited nodes and n_C measured nodes respectively. Matrix B is full column rank and each column contains one 1 and $L - 1$ zeros. Matrix C is full row rank and each row contains one 1 and $L - 1$ zeros, see Example 2.1. Some of those transfer functions are unknown and collected in matrix $G^\circ(q)$, and the others known, collected in matrix $G^\bullet(q)$, such that $G(q) = G^\circ(q) + G^\bullet(q)$, see Example 2.1. In our framework, the matrix $G(q)$ may contain nonzero diagonal elements, thereby allowing for loop transfer functions within the network. In much of the existing literature, such loops are typically excluded; see Section 2.4 for a discussion of why they are permitted in this thesis. A network model is entirely characterized by the quadruple $(G^\circ(q), G^\bullet(q), B, C)$, which we call an *EM network*, for Excited and Measured network.

Example 2.1. The EM network of Figure 2.1 is described by the matrices below.

$$\begin{aligned} G^\circ(q) &= \begin{bmatrix} 0 & 0 & G_{13}(q) \\ G_{21}(q) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad r(t) = r_1(t), \\ G^\bullet(q) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & G_{23}(q) \\ G_{31}(q) & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}. \end{aligned}$$

We consider rational transfer functions, which can be written as a finite sum of weighted and delayed terms; equivalently, each transfer function can be expressed as a ratio of finite-degree polynomials in the delay operator. This ensures that all summations involved are finite and well-defined. The transfer function from node signal i to j is denoted $G_{ji}(q)$, and is modeled by the following standard rational parameterization, standard in System

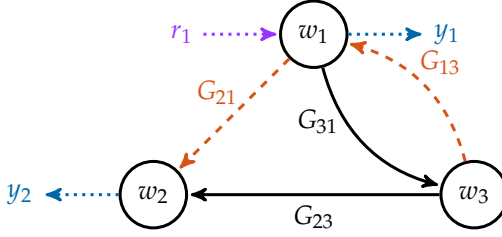


Fig. 2.1 Network of Example 2.1. Black plain arrows represent known transfer functions, orange dashed arrows represent unknown ones and purple (resp. blue) dotted ones represent excitations (resp. measurements). The arguments (t) and (q) are omitted to lighten notations.

Identification [Lju99]:

$$G_{ji}(q) = \frac{\beta_0 + \beta_1 q^{-1} + \dots + \beta_{n_\beta} q^{-n_\beta}}{1 + \alpha_1 q^{-1} + \dots + \alpha_{n_\alpha} q^{-n_\alpha}}, \quad (2.2)$$

where $n_\alpha, n_\beta \in \mathbb{N}_0$, and $n_\beta \leq n_\alpha$ to ensure that $G_{ji}(q)$ is proper. In other words, the degree of the numerator does not exceed that of the denominator, guaranteeing that the transfer function is physically meaningful and that its impulse response is causal and bounded. This is a standard and widely adopted assumption in System Identification, as it excludes non-proper or strictly improper transfer functions, which cannot typically be realized by causal, stable systems. The parameter β_0 represents the direct term or feedthrough component (also called the zero-delay term). If $\beta_0 \neq 0$, it means node i affects node j without delay. This does not create singularity; it just means there is a static gain from input to output in addition to the dynamic part. There are parameters $\alpha_1, \dots, \alpha_{n_\alpha}, \beta_0, \dots, \beta_{n_\beta} \in \mathbb{R}$ for each transfer function $G_{ji}(q)$.

In addition, the results presented are not inherently restricted to rational transfer functions. The statements and proofs remain valid for any analytic transfer function, as the arguments rely only on algebraic and analytic properties, and not on the rationality assumption itself. We have added a clarification to this effect in Note 4.12.

Example 2.1 (continued). Consider the following transfer functions:

$$G_{21}(q) = \frac{-2q^{-1}}{1 + 3q^{-1}}, \quad G_{23}(q) = \frac{4q^{-1}}{1 + 5q^{-2}}.$$

The node signal $w_2(t)$ is related to $w_1(t)$ and $w_3(t)$ as follows:

$$\begin{aligned} w_2(t) &= G_{21}(q)w_1(t) + G_{23}(q)w_3(t) \\ \Leftrightarrow w_2(t) &= \frac{-2q^{-1}}{1 + 3q^{-1}}w_1(t) + \frac{4q^{-1}}{1 + 5q^{-2}}w_3(t) \\ &\Leftrightarrow (1 + 3q^{-1})(1 + 5q^{-2})w_2(t) \\ &\quad = -2q^{-1}(1 + 5q^{-2})w_1(t) + 4q^{-1}(1 + 3q^{-1})w_3(t) \\ \Leftrightarrow (1 + 3q^{-1} + 5q^{-2} + 15q^{-3})w_2(t) \\ &\quad = (-2q^{-1} - 10q^{-3})w_1(t) + (4q^{-1} + 12q^{-2})w_3(t) \\ \Leftrightarrow w_2(t) + 3w_2(t-1) + 5w_2(t-2) + 15w_2(t-3) \\ &\quad = -2w_1(t-1) - 10w_1(t-3) + 4w_3(t-1) + 12w_3(t-2). \end{aligned}$$

The parameters of all unknown transfer functions of $G^\circ(q)$ are collected in vector θ° , and the parameters of all known transfer functions of $G^\bullet(q)$ are collected in vector $\theta^\bullet$. Vector $\theta \triangleq (\theta^\circ, \theta^\bullet)$ stores both, Θ° is the set of all valid parameters θ° , $\Theta^\bullet$ is the set of all valid parameters $\theta^\bullet$, and Θ is the set of all valid parameters θ [Lju99]. Notations $G^\circ(q; \theta^\circ)$, $G^\bullet(q; \theta^\bullet)$ and $G(q; \theta) = G^\circ(q; \theta^\circ) + G^\bullet(q; \theta^\bullet)$ denote the transfer matrices obtained from parameters $\theta = (\theta^\circ, \theta^\bullet)$.

We do not consider noise signal in our networked system, but this is actually not a loss of generality: under some mild assumptions, noise signals play a similar role as the excitation signals r . [SCVdH23] shows how an identifiability problem that includes noise signals can be reduced to a noiseless one. This paper introduces a concept of equivalence between network models, and develops an equivalent network model structure that incorporates noise signals as excitation signals r . Due to the contribution of the noise signals, a smaller number of measured excitation signals is needed for network identifiability.

Various alternative dynamic network models have been proposed in the literature. The primary distinction between these approaches and the one adopted in this work lies in the location of the sparsity constraints: in our framework, sparsity is imposed on the matrix of transfer functions, whereas

other works consider sparsity within the matrices of a particular state-space realization. The relationship between these two classes of models is highly non-trivial and far from straightforward. Establishing rigorous connections between them involves significant theoretical challenges and has been the subject of specialized studies in network realization theory (see, for example, [SSOJ23, VE15]).

2.2 Associated graph

To each EM network $(G^\circ(q), G^\bullet(q), B, C)$, we associate a directed graph. We begin by defining the following sets:

- $\mathcal{W}$ is a set of L nodes, representing the L node signals w_i . We refer to those nodes as $1, 2 \dots L$.
- Sets $\mathcal{B}$ and $\mathcal{C}$ respectively collect the excited nodes and measured ones.
- $\mathcal{E}^\circ \subseteq \mathcal{W} \times \mathcal{W}$ is the set of unknown edges between the nodes, defined by the nonzero entries of $G^\circ(q)$. We denote $m^\circ \triangleq |\mathcal{E}^\circ|$ the number of unknown transfer functions.
- $\mathcal{E}^\bullet \subseteq \mathcal{W} \times \mathcal{W}$ is the set of known edges between the nodes, defined by the nonzero entries of $G^\bullet(q)$. We denote $m^\bullet \triangleq |\mathcal{E}^\bullet|$ the number of known transfer functions.
- $\mathcal{E} = \{\mathcal{E}^\circ, \mathcal{E}^\bullet\}$ is the set of all edges. We denote $m \triangleq |\mathcal{E}| = m^\circ + m^\bullet$ the number of all transfer functions.

From those definitions, we construct directed graph $\mathcal{G}_{\text{net}} \triangleq (\mathcal{W}, \mathcal{E})$ associated to the network model (2.1). Besides, each instance $(G^\circ(q), G^\bullet(q), B, C)$ is uniquely characterized by its corresponding sets $(\mathcal{E}^\circ, \mathcal{E}^\bullet, \mathcal{B}, \mathcal{C})$.

2.2.1 Notions of graph theory

We introduce a few standard notions from graph theory that will be used throughout this thesis.

An edge $(i, j) \in \mathcal{E}$ — corresponding to a transfer function $G_{ji}(q)$ — is said to be an *incoming edge* to node j , and an *outgoing edge* from node i .

Accordingly, node i is called an *in-neighbor* of j , and j is an *out-neighbor* of i . In this edge, i is referred to as the *start node*, and j as the *end node*. A node with no incoming edges is called a *source*, while a node with no outgoing edges is called a *sink*.

A *walk* in the graph is a sequence of nodes $i_0, \dots, i_k$ such that $(i_{l-1}, i_l) \in \mathcal{E}$ for all $l = 1, \dots, k$. A *path* is a walk in which all nodes are distinct — in contrast to a walk, where repetition of nodes (and edges) is allowed. Two walks or paths are said to be *vertex-disjoint* if they do not share any common node, including the start and end nodes. A set of walks or paths are vertex-disjoint if all couples of the set are vertex-disjoint. This notion will play a key role in our work.

A *cycle* is a directed path that starts and ends at the same node, with all other nodes distinct. If such a cycle consists of a single edge from a node to itself, it is called a *loop*. A *Directed Acyclic Graph (DAG)* is a directed graph that contains no cycle.

A graph may have *multiple edges*, i.e., more than one edge connecting the same ordered pair of nodes. A graph without loops or multiple edges is called a *simple graph*. A graph is said to be *connected* if, for every pair of nodes $i, j \in \mathcal{W}$, there exists an undirected path (ignoring edge directions) connecting them.

Many works in the field of network identifiability focus on specific classes of network topologies — such as loops, trees, DAGs, or simple graphs — where the analysis is more manageable due to favorable structural properties. In contrast, this thesis considers arbitrary network topologies, including those with *multiple edges*, *loops*, and *cycles*, thereby addressing the full generality of the problem.

2.3 Global identifiability

In contrast to identification methods focused on estimation, the concept of network identifiability is a theoretical concern. First introduced in [WD-VdH15] in the context of networked systems, it addresses whether the observed signals can be uniquely explained by a specific network model. If multiple models are consistent with the same data, then no identification method can guarantee a unique result. This becomes particularly critical when the network is expected to reflect a real-world system.

In the subsequent analysis, we transition from using the shift operator q in transfer functions $G(\cdot; \theta)$ — with θ defined in Section 2.1 — to the complex frequency variable z , following standard system identification conventions [Lju99]. We define

$$T(G(z; \theta)) \triangleq (I - G(z; \theta))^{-1}, \quad (2.3)$$

the closed-loop transfer matrix that one would obtain if all nodes were excited and all nodes were measured. In our framework, however, only a subset of nodes are excited and only a subset of nodes are measured. Consequently, we have access only to the submatrix

$$C T(G(z; \theta)) B, \quad (2.4)$$

which contains all the data available from exciting nodes $\mathcal{B}$ and measuring nodes $\mathcal{C}$ at frequency z .

We assume that $C T(G(z; \theta)) B$ has been identified, that $G^\bullet$ is known, and the zero structure of G° is known. From this knowledge, the question is whether we can uniquely identify an entry of G° , a subset of them, or the whole matrix G° . This is precisely the notion captured by Definition 2.2, which reflects the standard definition of identifiability commonly used in the literature, and that we call here *global* identifiability, in line with [Lju99] and to avoid confusion with a notion of local identifiability that will be introduced in Section 4.1.

Definition 2.2. Consider a network $G(z; \theta) = G^\circ(z; \theta^\circ) + G^\bullet(z; \theta^\bullet)$ with excitations and measurements matrices (B, C) . The unknown transfer function G_{ij}° is *globally identifiable* at $\theta = (\theta^\circ, \theta^\bullet)$ if, for all $\tilde{\theta}^\circ \in \Theta^\circ$, there holds

$$\begin{aligned} C \left(I - G(z; (\theta^\circ, \theta^\bullet)) \right)^{-1} B &= C \left(I - G(z; (\tilde{\theta}^\circ, \theta^\bullet)) \right)^{-1} B \quad \forall z \\ \Rightarrow G_{ij}^\circ(z; (\theta^\circ, \theta^\bullet)) &= G_{ij}^\circ(z; (\tilde{\theta}^\circ, \theta^\bullet)) \quad \forall z. \end{aligned} \quad (2.5)$$

G_{ij}° is *generically* globally identifiable if it is globally identifiable at *almost all* $\theta \in \Theta$. This definition of identifiability of a single transfer function extends to a set of transfer functions by replacing G_{ij}° in the RHS of (2.5).

Definition 2.2 extends [HGB18] when some transfer functions are known ($G^\bullet$) and some are not (G°), as in [DSC⁺22, SCVdH22]. The notion of *almost all* will be clarified in the next section, dedicated to genericity.

Example 2.1 (continued). Equation (2.5) for both unknown transfer functions G_{21} and G_{13} gives

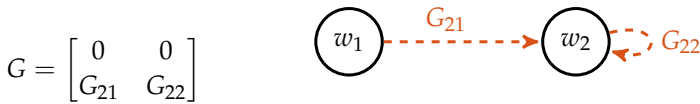
$$\begin{aligned} & \frac{1}{1 - G_{13}(z; \theta^\circ) G_{31}(z; \theta^\bullet)} \left[G_{21}(z; \theta^\circ) + G_{31}(z; \theta^\bullet) G_{23}(z; \theta^\bullet) \right] \\ &= \frac{1}{1 - G_{13}(z; \tilde{\theta}^\circ) G_{31}(z; \theta^\bullet)} \left[G_{21}(z; \tilde{\theta}^\circ) + G_{31}(z; \theta^\bullet) G_{23}(z; \theta^\bullet) \right] \quad \forall z \\ &\Rightarrow \begin{cases} G_{21}(z; \theta^\circ) = G_{21}(z; \tilde{\theta}^\circ) \\ G_{13}(z; \theta^\circ) = G_{13}(z; \tilde{\theta}^\circ) \end{cases} \quad \forall z. \end{aligned}$$

Since G_{31} and G_{23} are known, this implication is verified: both unknown transfer functions G_{21} and G_{13} are globally identifiable from exciting w_1 and measuring w_1 and w_2 .

2.4 Presence of loops

In much of the literature, it is commonly assumed that all diagonal entries of $G(z; \theta)$ must be zero, meaning that loops in the network are not permitted; equivalently, $G_{ii}(z; \theta) = 0$ for all i . In this section, we first present the reasoning underlying this assumption, and then explain why it does not apply in our framework.

The usual argument is that the presence of a loop prevents the network from being generically globally identifiable. To illustrate this reasoning, consider the following simple example with two nodes and two transfer functions (for clarity, we omit the arguments (t) and $(z; \theta)$ in the notation):



We have the following relation between node states 1 and 2:

$$w_2 = G_{21}w_1 + G_{22}w_2. \quad (2.6)$$

Multiplying both sides of the equation by 2 gives

$$2w_2 = 2G_{21}w_1 + 2G_{22}w_2, \quad (2.7)$$

and subtracting w_2 from both sides of the equation brings

$$w_2 = 2G_{21}w_1 + (2G_{22} - 1)w_2. \quad (2.8)$$

Assume that the network is generically globally identifiable, and the identified transfer functions are

$$G_{21} = f, \quad G_{22} = g,$$

Then, by (2.8), we automatically also have

$$w_2 = 2fw_1 + (2g - 1)w_2,$$

Hence $G_{21} = 2f, G_{22} = (2g - 1)$ is also a solution, which contradicts the assumption that the network is generically globally identifiable.

However, in our framework this argument does not apply, for the following reason. Consider that node 2 is excited. Equations (2.6), (2.7), (2.8) become, respectively:

$$w_2 = G_{21}w_1 + G_{22}w_2 + r_2 \quad (2.9)$$

$$\Leftrightarrow 2w_2 = 2G_{21}w_1 + 2G_{22}w_2 + 2r_2 \quad (2.10)$$

$$\Leftrightarrow w_2 = 2G_{21}w_1 + (2G_{22} - 1)w_2 + \textcolor{red}{2}r_2. \quad (2.11)$$

In this case, the argument developed above no longer holds: if the network is generically globally identifiable with $G_{21} = f$ and $G_{22} = g$, then the alternative choice $G_{21} = 2f$ and $G_{22} = (2g - 1)$ is *not* a valid solution. This is ruled out by the factor 2 in front of r_2 in (2.11), together with the fact that the entries of the matrices B and C are binary.

For clarity of exposition, the argument in this section has been illustrated using a simple two-node example, where an arbitrary factor of 2 was introduced to derive equation (2.7) from equation (2.6). Nevertheless, the argument extends to arbitrary networks containing loops, and the choice of the factor is not restricted to 2.

2.5 Genericity and pseudo-genericity

We will see that properties of local identifiability - which will be introduced in Section 4.1 - are generic (also denoted structural in the literature), or pseudo-generic. [DCVdW03, HGB18] define genericity on a parameterized model. We extend this definition to classes of functions, since we will see in Section 4.2 that local identifiability can be studied via the local injectivity of a function. Generic properties are defined similarly in other areas of mathematics, including topology [Bon06, EGK⁺24].

Definition 2.3. For two sets $\mathcal{M}$ and $\mathcal{N}$, let $\mathcal{F} \subseteq \{f : \mathcal{M} \rightarrow \mathcal{N}\}$ be a class of functions. A property of $f(x)$ is *generic* in $\mathcal{F}$ if, for each $f \in \mathcal{F}$, it either holds:

- (i) for *almost all* variables, i.e. for all x except possibly those lying on a set of dimension lower than $\dim \mathcal{M}$,
- (ii) for no x .

As an illustrative example, consider the class of polynomials, and take a polynomial p . The *nonzeroness* of $p(x)$ is a generic property in the polynomials: either (i) $p(x) \neq 0$ for all x except its roots, or (ii) p is the zero polynomial, which returns zero for all x . A handy consequence of Definition 2.3 is the following: showing that a generic property holds for one variable x implies that it holds for almost all x .

Some properties of local injectivity and local identifiability will be shown in Sections 5.1 and 5.2 to be only *pseudo-generic*, a new slightly more general notion that we introduce below.

Definition 2.4. For two sets $\mathcal{M}$ and $\mathcal{N}$, let $\mathcal{F} \subseteq \{f : \mathcal{M} \rightarrow \mathcal{N}\}$ be a class of functions. A property of $f(x)$ is *pseudo-generic* in $\mathcal{F}$ if, for each $f \in \mathcal{F}$, it either holds:

- (i) for *almost all* variables, i.e. for all x except possibly those lying on a set of dimension lower than $\dim \mathcal{M}$,
- (ii) for *almost no* variables, i.e. only for x lying on a set of dimension lower than $\dim \mathcal{M}$.

Typical examples of pseudo-generic properties are conditions which require that the rank of a matrix is greater than the rank of another matrix. A disadvantage of pseudo-genericity is that we lose the ‘handy consequence’ of genericity stated above: if the property holds for one variable x , it does not imply that it holds for almost all x : it could lie in the lower-dimensional set of case (ii). On the other hand, an advantage of pseudo-genericity is that it is preserved under any finite boolean function.

Lemma 2.5. *Any finite boolean function of pseudo-generic properties is pseudo-generic.*

Proof. First, any boolean function can be expressed in its disjunctive normal form, i.e. using only $\wedge, \vee, \neg$ (respectively logic AND, OR, NOT) [Whi12]. Recall De Morgan’s law: Let P, Q be boolean propositions, we have

$$P \wedge Q \Leftrightarrow \neg(\neg P \vee \neg Q).$$

Therefore, any boolean function can be expressed with only $\vee$ and $\neg$, and proving that $\vee$ and $\neg$ preserve pseudo-genericity will yield the result.

By definition, a pseudo-generic property either holds for almost all (AA) parameters, or for almost no (AN) parameters. If a property P holds for almost all parameters, then it does not hold for almost no parameters, i.e. $\neg P$ holds for almost no parameters. The reciprocal holds as well, and we can summarize this in the table below.

P	$\neg P$
AA	AN
AN	AA

Hence, $\neg$ preserves pseudo-genericity. We now consider $\vee$, and write the table below.

P	Q	$P \vee Q$
AA	AA	AA
AA	AN	AA
AN	AA	AA
AN	AN	AN

(Note that the last row would not hold for an infinite combination of $\vee$.) Thus, $\vee$ preserves pseudo-genericity. $\square$

Note 2.6. The property of Lemma 2.5 does not hold for genericity, since $\neg$ does not preserve genericity: P is generic $\nRightarrow \neg P$ is generic.

Most properties are neither generic, nor pseudo-generic. For instance, consider the class of linear functions $\mathcal{F} = \{y = ax + b\}$, and the property ‘increasing’ (i.e. $a > 0$). Half of the linear functions are increasing, and the other half are not. (Pseudo-)generic properties enable reasoning about the problem in a nearly universal manner. These properties are contingent on the problem’s structure rather than on specific instances of the problem, except for certain pathological cases.

3

The full excitation (resp. measurement) case

WHEN all nodes are excited (resp. measured), the matrix B (resp. C) reduces to the identity matrix, which simplifies the problem notably. In this chapter, we present key results from the literature that address this fully excited or fully measured setting. These works begin with an algebraic characterization of identifiability in terms of the generic rank of a network matrix, then translate this condition into graph-theoretic terms involving walks in the network. This reformulation enables the development of algorithms for excitation allocation. Inspired by these works, this thesis follows the same approach.

3.1 Characterization with mutually vertex-disjoint walks

This section presents results of [HGB18] that are key to this thesis. In this work, the authors revisited the notion of generic rank — originally introduced in [Dav77] — and applied it to the problem of generic global identifiability, in the case where all transfer functions are unknown, and all nodes are excited (respectively, measured).

The full-rankness of a parameterized matrix is a generic property. More precisely, the fact that a matrix attains its maximal rank is a generic property.

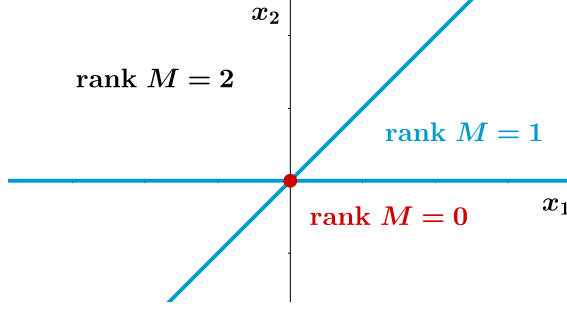


Fig. 3.1 Illustration of the notion of generic rank, Example 3.2.

This maximal rank is referred to as the *generic rank* of the matrix [Dav77].

Definition 3.1. For set $\mathcal{M}$ and naturals n_1, n_2 , let $M : \mathcal{M} \rightarrow \mathbb{C}^{n_1 \times n_2}$ be an analytical matrix function. The *generic rank* (also denoted structural rank in the literature) of M , denoted $\text{grank}(M)$, is the maximum rank that $M(x)$ attains for any value of $x \in \mathcal{M}$:

$$\text{grank}(M) = \max_{x \in \mathcal{M}} \text{rank}(M(x)).$$

As will be proved in Lemma 5.3, the generic rank of M is also the rank obtained for *almost all* x , i.e. for all x except possibly those lying on a set of dimension lower than $\dim \mathcal{M}$.

Example 3.2. The matrix $M = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 - x_2 \end{bmatrix}$ has generic rank 2. Its rank drops on $\{x_1 = 0\} \cup \{x_1 = x_2\}$, see Fig. 3.1.

The notion of generic rank plays a central role in our work and admits a meaningful graphical interpretation within our network model. In particular, it connects closely to the notion of mutually vertex-disjoint walks, introduced in Section 2.2.1. The proposition below is from [dW91, HGB18], and is illustrated in Example 3.4.

Proposition 3.3. Let $\mathcal{I}, \mathcal{J}$ be two sets of nodes of the associated graph of a network matrix $G(z, \theta)$. Let $T_{\mathcal{I}, \mathcal{J}}(z, \theta)$ be the restriction of $T(z, \theta) = (I - G(z, \theta))^{-1}$ to the rows corresponding to $\mathcal{J}$ and the columns corresponding to $\mathcal{I}$. Then, the generic rank of $T_{\mathcal{I}, \mathcal{J}}(z, \theta)$ over parameter θ is the maximum number of mutually vertex-disjoint walks from $\mathcal{J}$ to $\mathcal{I}$, denoted $\beta_{\mathcal{J} \rightarrow \mathcal{I}}$.

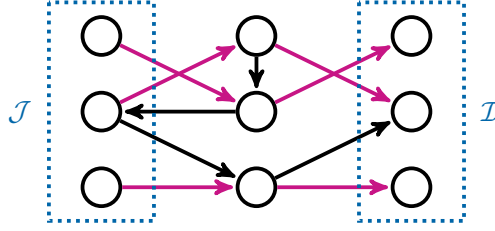


Fig. 3.2 One can route up to three vertex-disjoint walks from $\mathcal{J}$ to $\mathcal{I}$ (in pink), i.e. $\beta_{\mathcal{J} \rightarrow \mathcal{I}} = 3$.

Example 3.4. Consider the network depicted in Fig. 3.2. One can compute the generic rank of the matrix T corresponding to this network: it is 3. And we verify in Fig. 3.2 that one can route up to three vertex-disjoint walks from $\mathcal{J}$ to $\mathcal{I}$ (in pink), i.e. $\beta_{\mathcal{J} \rightarrow \mathcal{I}} = 3$.

Remark 3.5. In [HGB18] and related works, the notion of mutually vertex-disjoint *paths* is commonly used. In contrast, throughout this thesis we rely on the notion of *walks* rather than *paths*. Recall from Section 2.2.1 that a path is a walk in which all nodes are distinct — in contrast to a walk, where repetition of nodes (and edges) is allowed. This distinction is important in our context, since in Chapter 8 it is specifically the notion of walks that naturally arises in the characterization of generic local identifiability. Nevertheless, the difference is often negligible in practice, as a path can always be extracted from a walk.

The work in [HGB18] provides a necessary and sufficient condition for the generic global identifiability of all outgoing transfer functions of a node, when all transfer functions are unknown, and all nodes are excited. This condition is derived using the notion of generic rank, with the aid of Proposition 3.3, and is formulated in terms of mutually vertex-disjoint walks in the associated graph. This graph-theoretic formulation underscores the inherently topological nature of generic identifiability.

Theorem 3.6. *The transfer functions from node i to its out-neighbors $\mathcal{W}_i^+$ can generically all be globally identified when exciting all nodes and measuring nodes $\mathcal{C}$ if and only if $\beta_{\mathcal{W}_i^+ \rightarrow \mathcal{C}} = |\mathcal{W}_i^+|$.*

A more general version of Theorem 3.6, which addresses the generic global identifiability of any subset of $\mathcal{W}_i^+$, is presented in [HGB18]. The dual result, applicable under full excitation, is stated below.

Theorem 3.7. *The transfer functions to node j from its in-neighbors $\mathcal{W}_j^-$ can generically all be globally identified when exciting nodes $\mathcal{B}$ and measuring all nodes if and only if $\beta_{\mathcal{B} \rightarrow \mathcal{W}_j^-} = |\mathcal{W}_j^-|$.*

3.2 Algorithm for excitation allocation

This section presents results from [CSVdH21, DSC⁺22], which form the foundation of the algorithm developed in Sections 11.2.3 and 11.2.4. These works reformulate the vertex-disjoint walk condition of Theorem 3.7 in terms of coverings of subgraphs called *SIMUGs*. This reformulation enables the design of an excitation allocation algorithm that guarantees global identifiability under full measurement. Definitions 3.8 and 3.9 introduce the notion of SIMUG.

Notation: Consider a graph $\mathcal{G}$. We denote by $V(\mathcal{G})$ its set of nodes and by $E(\mathcal{G})$ its set of edges.

Definition 3.8. A connected directed graph $\mathcal{T}$, with $|V(\mathcal{T})| \geq 2$, is called a (directed) *multi-rooted graph* if there exists a path from each node in a nonempty set of *roots* $Y(\mathcal{T})$ to every node $i \in V(\mathcal{T})$.

We focus on a specific class of multi-rooted graphs in which each node has at most one incoming parametrized edge.

Definition 3.9. A multi-rooted graph $\mathcal{T}$ is called a *single-source identifiable multi-rooted graph* (SIMUG) if all nodes $j \in V(\mathcal{T})$ satisfy $|\mathcal{W}_j^{\circ-}| \leq 1$, where $\mathcal{W}_j^{\circ-} \triangleq \{i \mid (i, j) \in \mathcal{E}^{\circ}\}$ is the set of in-neighbours of j through unknown transfer functions.

Edge-disjoint SIMUGs are defined below. This notion will be seen to be closely related to the concept of vertex-disjoint walks.

Definition 3.10. Consider two multi-rooted graphs $\mathcal{T}_1$ and $\mathcal{T}_2$ as subgraphs of a directed graph $\mathcal{G}$. $\mathcal{T}_1$ and $\mathcal{T}_2$ are called *edge-disjoint* in $\mathcal{G}$ if the following two conditions hold:

1. $\mathcal{T}_1$ and $\mathcal{T}_2$ have no common edges;
2. For each node j in $\mathcal{T}_1 \cup \mathcal{T}_2$, all outgoing edges of j in $\mathcal{T}_1 \cup \mathcal{T}_2$ belong to either $\mathcal{T}_1$ or to $\mathcal{T}_2$. Hence $\mathcal{T}_1$ and $\mathcal{T}_2$ do not share the same root or any common internal node.

[CSVdH21] highlights how the notion of disjoint SIMUGs is closely related to that of vertex-disjoint walks. Consider $\mathcal{T}_1$ and $\mathcal{T}_2$ as two disjoint SIMUGs in $\mathcal{G}$. For any $j \in V(\mathcal{T}_1) \cap V(\mathcal{T}_2)$, if $|\mathcal{W}_j^-| \geq 2$, then there exist two in-neighbors of j located in $\mathcal{T}_1$ and $\mathcal{T}_2$ separately. Due to the fact that distinct SIMUGs cannot share any common root or internal vertex, we can find two vertex-disjoint walks in the union $\mathcal{T}_1 \cup \mathcal{T}_2$, starting from two roots in $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, and leading to two distinct in-neighbors of j .

Next, the concept of disjoint-edge covering for a directed graph is introduced.

Definition 3.11. Consider a directed graph $\mathcal{G}$, and let $\Pi \triangleq \{\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n\}$ be a collection of edge-disjoint SIMUGs of $\mathcal{G}$. The edges in a set $\mathcal{E} \subseteq E(\mathcal{G})$ are covered by Π , if $E(\mathcal{T}_1) \cup E(\mathcal{T}_2) \cup \dots \cup E(\mathcal{T}_n) = \mathcal{E}$, and Π is called an edge-disjoint SIMUG covering of $\mathcal{E}$.

By combining Theorem 3.7 for all nodes $\mathcal{W}$ with the notions introduced above, we obtain Theorem 3.12 — a sufficient condition for generic global identifiability based on disjoint SIMUGs. This result forms the theoretical foundation for the synthesis method presented in Algorithm 1.

Theorem 3.12. Consider a network $G(z, \theta) = G^\bullet(z, \theta^\bullet) + G^\circ(z, \theta^\circ)$ with excitations $\mathcal{B}$, measurements on all nodes and let $\mathcal{G}_{\text{net}} = (\mathcal{W}, \mathcal{E})$ be its associated graph. The network is generically globally identifiable if there exists an edge-disjoint SIMUG covering of $\mathcal{E}$, such that at least one of the roots of each SIMUG is excited.

Algorithm 1 provides a procedure to compute a set of excitations that guarantees generic global identifiability when all nodes are measured. It first constructs an edge-disjoint SIMUG covering, then assigns one excitation per SIMUG, at one of its roots, and finally prunes unnecessary excitations by testing whether global identifiability is preserved upon their removal. Note that there is no guarantee on the minimality of the number of excitations obtained by Algorithm 1. This algorithm serves as a basis for the method developed in Section 11.2.4.

Notations: $\mathcal{W}_j^{\circ-}$ is the set of in-neighbours of j through unknown transfer functions. $\beta_{\mathcal{I} \rightarrow \mathcal{J}}$ is the maximum number of mutually vertex-disjoint walks from $\mathcal{I}$ to $\mathcal{J}$.

Algorithm 1 Allocation of Excitations in Full Measurement

Require: The associated graph $\mathcal{G}_{\text{net}} = (\mathcal{W}, \{\mathcal{E}^\bullet, \mathcal{E}^\circ\})$.

Result: A set of excitations $\mathcal{B}$ such that the network is generically globally identifiable under full measurement.

- 1: Construct an edge-disjoint SIMUG covering Π with Algorithm 1 of [CSVdH21], adapted to the case with known transfer functions [DSC⁺22].
 - 2: Build excitation set $\mathcal{B}$ by selecting one root from each SIMUG in Π , except the SIMUGs that are composed of known edges only.
 - 3: For each $b_i \in \mathcal{B}$: remove b_i from $\mathcal{B}$ if $\beta_{\mathcal{B} \setminus \{b_i\} \rightarrow \mathcal{W}_j^{\circ-}} = |\mathcal{W}_j^{\circ-}|$ for all $j \in \mathcal{W}$.
 - 4: **return** $\mathcal{B}$.
-

3.2.1 Without known modules: pseudotrees

The notions and results presented in Section 3.2 are derived in the presence of *both known and unknown* modules, and originate from [DSC⁺22]. A preliminary version of this work, focusing on networks with only unknown modules, is presented in [CSVdH21]. The approaches in both works are closely related: the SIMUGs in [DSC⁺22] are replaced by *pseudotrees* in [CSVdH21], and the algorithms for merging them differ slightly due to the absence of known modules in the latter. Nonetheless, [CSVdH21] proposes an algorithm — conceptually very similar to Algorithm 1 — that constructs an excitation set ensuring identifiability in the absence of known modules, based on an edge-disjoint *pseudotree* covering. This algorithm forms the foundation for the method developed in Section 11.2.3.

Similarly to [HGB18], all results of this section can be derived analogously for the dual setting of full excitation, see [CSVdH21] for details.

PART I

**Algebraic characterization of
local identifiability**

4

Local identifiability and local injectivity

WE first introduce the notion of local identifiability, a weaker notion than global identifiability, but which is amenable to linear analysis. Local identifiability is then formulated as the local injectivity of a specific function. Finally, we establish that, generically, working with scalar modules on the edges is equivalent to working with analytic transfer functions. This chapter is based on [LH24] and its preliminary version [LH20].

4.1 Local identifiability

The graph-theoretical conditions for generic global identifiability obtained under full excitation or measurement (i.e. when B or $C = I$) in Chapter 3 from [HGB18, CSVdH21, DSC⁺22] were made possible by a linear algebraic reformulation of the generic global identifiability questions. No equivalent reformulation is available for partial excitations and measurements. Hence we will work on a weaker notion amenable to linear analysis: local identifiability, which corresponds to identifiability provided that $G(z; (\hat{\theta}^\circ, \theta^\bullet))$ is sufficiently close to $G(z; (\theta^\circ, \theta^\bullet))$. Definition 4.1 extends to networked systems the notion of local identifiability introduced in [Lju99].

Definition 4.1. Consider a network $G(z; \theta) = G^\circ(z; \theta^\circ) + G^\bullet(z; \theta^\bullet)$ with excitations and measurements matrices (B, C) . The unknown transfer function G_{ij}° is *locally identifiable* at $\theta = (\theta^\circ, \theta^\bullet)$ if there exists a continuous positive function $\epsilon : \mathbb{C} \rightarrow \mathbb{R}_0^+$ such that for all $\tilde{\theta}^\circ \in \Theta^\circ$, there holds:

$$\begin{aligned} \text{If } C \left(I - G(z; (\theta^\circ, \theta^\bullet)) \right)^{-1} B &= C \left(I - G(z; (\tilde{\theta}^\circ, \theta^\bullet)) \right)^{-1} B \text{ for all } z, \text{ then} \\ \text{either } G_{ij}^\circ(z; (\theta^\circ, \theta^\bullet)) &= G_{ij}^\circ(z; (\tilde{\theta}^\circ, \theta^\bullet)) \text{ for all } z \\ \text{or } G(z; (\tilde{\theta}^\circ, \theta^\bullet)) &\notin \text{Ball}(G(z; (\theta^\circ, \theta^\bullet)), \epsilon(z)) \text{ for all } z. \end{aligned} \quad (4.1)$$

G_{ij}° is *generically locally identifiable* if it is locally identifiable at *almost all* $\theta \in \Theta$. This definition of identifiability of a single transfer function extends to a set of transfer functions by replacing G_{ij}° in the second line of (4.1).

Local identifiability can be interpreted as no ambiguity with the $G(z; (\tilde{\theta}^\circ, \theta^\bullet))$ lying in a local neighborhood of $G(z; (\theta^\circ, \theta^\bullet))$, while global identifiability is no ambiguity with *all* $G(z; (\tilde{\theta}^\circ, \theta^\bullet))$. A similar concept in control theory is local stability, which refers to the stability of a system in a neighborhood around an equilibrium point.

Local identifiability is a necessary condition for global identifiability, as it represents the same notion restricted to a local neighborhood. It is *a priori* a weaker notion, yet no example of network generically locally identifiable but not generically globally identifiable is known to the authors, despite having checked all networks with up to five unknown transfer functions and no known transfer functions, see Section 12.2. In the case of full excitation or full measurement, the two notions are equivalent, see Proposition 5.13 in Section 5.2.

4.2 Identifiability and injectivity

The considered local identifiability questions can be reformulated in terms of local injectivity. In this Section 4.2, we consider the scalar case: we work with complex values $A_{ij} \in \mathbb{C}$ instead of transfer functions $G_{ij}(z; \theta)$. To highlight the distinction with transfer functions, we refer to those scalar complex values as *modules*. The scalar case will be proved to be equivalent to the standard “transfer function case” in Section 4.3. Matrices

$A, A^\circ, A^\bullet \in \mathbb{C}^{L \times L}$ (where L is the number of nodes) replace matrices of transfer functions $G(z; \theta), G^\circ(z; \theta^\circ), G^\bullet(z; \theta^\bullet)$ respectively, with same nonzero entries respectively. We collect in vector a° all the unknown modules, i.e. $a^\circ = \text{nonzeros}(A^\circ)$, where $\text{nonzeros}(\cdot)$ stacks in a vector the nonzero elements of its matrix argument. Vector $a^\bullet$ is defined analogously: $a^\bullet = \text{nonzeros}(A^\bullet)$. Conversely, we denote by $\bar{A}(a^\circ; a^\bullet) = A$ the network matrix rearranging vector of unknown modules a° and known modules $a^\bullet$ into the network structure of the problem.

Example 2.1 (continued). Recall that A_{21}, A_{13} are unknown, and A_{31}, A_{23} are known.

$$a^\circ = \begin{bmatrix} a_1^\circ \\ a_2^\circ \end{bmatrix} = \begin{bmatrix} A_{21} \\ A_{13} \end{bmatrix}, \quad a^\bullet = \begin{bmatrix} a_1^\bullet \\ a_2^\bullet \end{bmatrix} = \begin{bmatrix} A_{31} \\ A_{23} \end{bmatrix}, \quad \bar{A}(a^\circ; a^\bullet) = \begin{bmatrix} 0 & 0 & a_2^\circ \\ a_1^\circ & 0 & a_2^\bullet \\ a_1^\bullet & 0 & 0 \end{bmatrix} = A.$$

We define the function

$$h : \mathbb{C}^{m^\circ} \times \mathbb{C}^{m^\bullet} \setminus \mathcal{D}_h \rightarrow \mathbb{C}^{n_B \cdot n_C} : (a^\circ; a^\bullet) \rightarrow \text{vec} \left(C \left(I - \bar{A}(a^\circ; a^\bullet) \right)^{-1} B \right), \quad (4.2)$$

where $a^\circ \in \mathbb{C}^{m^\circ}$ are the variables of h , separated by a semicolon from its known parameters $a^\bullet \in \mathbb{C}^{m^\bullet}$. The parameters $a^\bullet$ of function $h(a^\circ; a^\bullet)$ represent the fixed modules in the scalar version of the problem and should not be confused with the parameters θ of function $G(z; \theta)$, which correspond to the rational parameterization (2.2) of the transfer functions. The set $\mathcal{D}_h \triangleq \{(a^\circ; a^\bullet) \in \mathbb{C}^{m^\circ} \times \mathbb{C}^{m^\bullet} \mid \det(I - \bar{A}(a^\circ; a^\bullet)) = 0\}$ collects the variables and parameters for which the inverse does not exist. n_B is the number of excitations, n_C the number of measurements and m° is the number of unknown modules. The vec function vectorizes the $n_B \times n_C$ matrix into a single $n_B \cdot n_C$ vector, which will ease the differential analysis below.

Example 4.2. $\text{vec} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = [M_{11} \quad M_{21} \quad M_{12} \quad M_{22}]^\top$.

Recovering a matrix A° from $C(I - A)^{-1}B$ can be seen as recovering a° from $h(a^\circ; a^\bullet)$, while recovering A_{ij}° corresponds to recovering a specific entry a_ℓ° of a° from $h(a^\circ; a^\bullet)$, where ℓ is the index of A_{ij}° in a° . *Local identifiability can then be seen as a question of local injectivity*: determining if $h(\tilde{a}^\circ; a^\bullet) = h(a^\circ; a^\bullet)$ implies $\tilde{a}^\circ = a^\circ$ for every $\tilde{a}^\circ$ in a neighborhood of a° . The local identifiability of a transfer function A_{ij}° becomes a question

of local coordinate-injectivity: determining if $h(\tilde{a}^\circ; a^\bullet) = h(a^\circ; a^\bullet)$ implies $\tilde{a}_\ell^\circ = a_\ell^\circ$ for every $\tilde{a}^\circ$ in a neighborhood of a° . The notion of local coordinate-injectivity is formally introduced below.

Definition 4.3. For sets $\mathcal{M}, \mathcal{N}$, and $\ell \in \{1, 2, \dots, \dim \mathcal{M}\}$, the function $f : \mathcal{M} \rightarrow \mathcal{N}$ is locally *scalar*-coordinate-injective for coordinate ℓ at $x \in \mathcal{M}$ if there exists $\epsilon > 0$ such that for all $\tilde{x} \in \text{Ball}(x, \epsilon)$, there holds

$$f(\tilde{x}) = f(x) \Rightarrow \tilde{x}_\ell = x_\ell. \quad (4.3)$$

This was for the scalar case, but when we work with transfer functions $G_{ij}(z; \theta)$, we need a slightly more elaborate notion.

Definition 4.4. For sets $\mathcal{M}, \mathcal{N}$ and $\ell \in \{1, 2, \dots, \dim \mathcal{M}\}$, the function $f : \mathcal{M} \rightarrow \mathcal{N}$ is locally *functional*-coordinate-injective for coordinate ℓ at function $g : \mathbb{C} \rightarrow \mathcal{M}$ if there exists a *continuous* positive function $\epsilon : \mathbb{C} \rightarrow \mathbb{R}_0^+$ such that for all analytic function $\tilde{g} : \mathbb{C} \rightarrow \mathcal{M}$, there holds:

$$\begin{aligned} & \text{If } f(g(z)) = f(\tilde{g}(z)) \text{ for all } z, \text{ then} \\ & \text{either } g_\ell(z) = \tilde{g}_\ell(z) \text{ for all } z \\ & \text{or } \tilde{g}(z) \notin \text{Ball}(g(z), \epsilon(z)) \text{ for all } z. \end{aligned} \quad (4.4)$$

For parameterized functions with the form of h , Definitions 4.3 and 4.4 become Definitions 4.5 and 4.6, respectively.

Definition 4.5. For sets $\mathcal{M}^\circ, \mathcal{M}^\bullet, \mathcal{D}_f, \mathcal{N}$, and $\ell \in \{1, 2, \dots, \dim \mathcal{M}^\circ\}$, the function $f : \mathcal{M}^\circ \times \mathcal{M}^\bullet \setminus \mathcal{D}_f \rightarrow \mathcal{N}$ is locally *scalar*-coordinate-injective for coordinate ℓ at $(x^\circ, x^\bullet) \in \mathcal{M}^\circ \times \mathcal{M}^\bullet \setminus \mathcal{D}_f$ if there exists $\epsilon > 0$ such that for all $\tilde{x}^\circ \in \text{Ball}(x^\circ, \epsilon)$, there holds

$$f(\tilde{x}^\circ; x^\bullet) = f(x^\circ; x^\bullet) \Rightarrow \tilde{x}_\ell^\circ = x_\ell^\circ. \quad (4.5)$$

Definition 4.6. For sets $\mathcal{M}^\circ, \mathcal{M}^\bullet, \mathcal{D}_f, \mathcal{N}, \Phi^\circ, \Phi^\bullet$ and $\ell \in \{1, \dots, \dim \mathcal{M}^\circ\}$, the function $f : \mathcal{M}^\circ \times \mathcal{M}^\bullet \setminus \mathcal{D}_f \rightarrow \mathcal{N}$ is locally *functional*-coordinate-injective for coordinate ℓ of function $g : \mathbb{C} \times \Phi^\circ \times \Phi^\bullet \rightarrow \mathcal{M}^\circ \times \mathcal{M}^\bullet$ at $(\phi^\circ, \phi^\bullet) \in \Phi^\circ \times \Phi^\bullet$ if there exists a *continuous* positive function $\epsilon : \mathbb{C} \rightarrow \mathbb{R}_0^+$ such that for all $\tilde{\phi}^\circ \in \Phi^\circ$, there holds:

$$\begin{aligned} & \text{If } f(g(z; (\phi^\circ, \phi^\bullet))) = f(g(z; (\tilde{\phi}^\circ, \phi^\bullet))) \text{ for all } z, \text{ then} \\ & \text{either } g_\ell(z; (\phi^\circ, \phi^\bullet)) = g_\ell(z; (\tilde{\phi}^\circ, \phi^\bullet)) \text{ for all } z \\ & \text{or } g(z; (\tilde{\phi}^\circ, \phi^\bullet)) \notin \text{Ball}(g(z; (\phi^\circ, \phi^\bullet)), \epsilon(z)) \text{ for all } z. \end{aligned} \quad (4.6)$$

Functional-coordinate-injectivity is a generalization of local identifiability for any function f , hence the resemblance between Definition 4.1 and Definition 4.6. This is formalized in Lemma 4.7.

Lemma 4.7. *Consider a network $G(z; \theta) = G^\circ(z; \theta^\circ) + G^\bullet(z; \theta^\bullet)$ with excitations and measurements matrices (B, C) . The unknown transfer function G_{ij}° is locally identifiable at $\theta = (\theta^\circ, \theta^\bullet)$ if and only if the corresponding function h defined in (4.2) is locally functional-coordinate-injective for coordinate ℓ of function*

$$g_{\text{net}} : \mathbb{C} \times \Theta^\circ \times \Theta^\bullet \rightarrow \mathbb{C}^{m^\circ} \times \mathbb{C}^{m^\bullet} : \\ (z; \theta^\circ, \theta^\bullet) \rightarrow \left(\text{nonzeros}(G^\circ(z; \theta^\circ)); \text{nonzeros}(G^\bullet(z; \theta^\bullet)) \right) \quad (4.7)$$

at $\theta = (\theta^\circ, \theta^\bullet)$, where ℓ is the index corresponding to G_{ij}° in the image of g_{net} .

Proof. Substitute the definitions of functions h (4.2) and g_{net} (4.7) into Definition 4.6: $f = h, g = g_{\text{net}}, \mathcal{M}^\circ = \mathbb{C}^{m^\circ}, \mathcal{M}^\bullet = \mathbb{C}^{m^\bullet}, \mathcal{D}_f = \mathcal{D}_h, \mathcal{N} = \mathbb{C}^{n_B \cdot n_C}, \phi^\circ = \theta^\circ, \Phi^\circ = \Theta^\circ, \phi^\bullet = \theta^\bullet, \Phi^\bullet = \Theta^\bullet$. We obtain exactly Definition 4.1. $\square$

4.3 Reduction to the scalar case

In this Section, we prove that when considered generically, the notions of Definition 4.3 and Definition 4.4 are equivalent for an analytic function. We first present three lemmas that support the proof of Theorem 4.11, which establishes the equivalence aforementioned.

Lemma 4.8 characterizes the intersection between a closed complex analytic variety and a complex analytic trajectory. It is illustrated in Fig. 4.1.

Lemma 4.8. *Let $\mathcal{M}$ be a manifold, $\mathcal{S} \subset \mathcal{M}$ be a closed complex analytic variety of lower dimension than $\mathcal{M}$, and $g : \mathbb{C} \rightarrow \mathcal{M}$ a complex analytic function. Either $\text{Im } g$ is entirely included in $\mathcal{S}$, or the intersection between $\text{Im } g$ and $\mathcal{S}$ is a discrete set of points.*

Proof. A complex analytic variety is the zero locus of a set of complex analytic functions. Denote those functions $s_1, \dots, s_s$ for $\mathcal{S}$, with $s_i : \mathcal{M} \rightarrow \mathbb{C}$ for each i . We have $\mathcal{S} = s_1^{-1}(0) \cap s_2^{-1}(0) \cap \dots \cap s_s^{-1}(0)$, where $s_i^{-1}(0)$ denotes the zero locus of function s_i . Since composition preserves the analytic character of functions, $s_i \circ g$ is also analytic. Hence:

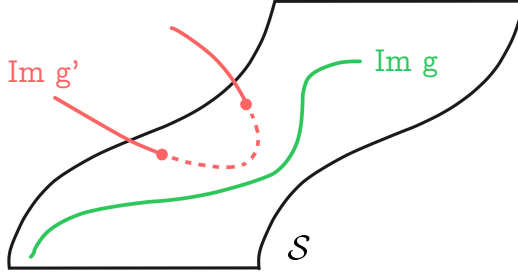


Fig. 4.1 Illustration of Lemma 4.8.

- either for all i , $s_i \circ g$ is the zero function, hence $\text{Im } g \subset \mathcal{S}$
- or there is i such that $s_i \circ g$ is zero only on a discrete set of points $z_j \in \mathbb{C}$. At those points z_j , $g(z_j)$ lies in the zero locus of s_i . Thus, $\text{Im } g$ intersects $s_i^{-1}(0)$ at a discrete set of points $\{g(z_j)\}$, and $\text{Im } g \cap \mathcal{S}$ is a discrete set of points.

□

Lemma 4.9 ensures that for each point u in a $\epsilon/2$ -neighborhood of $y \in \mathcal{M}$, scalar-coordinate-injectivity holds in a $\epsilon/2$ -neighborhood of u . This lemma, illustrated in Fig. 4.2, is exploited by the proof of Lemma 4.10. Notations y, w are just elements of manifold $\mathcal{M}$ in Lemma 4.9, they should not be confused with the node signals of our network model (2.1).

Lemma 4.9. *Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be a C^∞ function where $\mathcal{M}$ and $\mathcal{N}$ are smooth manifolds, $\mathcal{M}$ or $\mathcal{N}$ has finite dimension, $\mathcal{M}$ is open and for all $y \in \mathcal{M} : \nabla f(y)$ has the same rank and $\ker \nabla f(y) \perp \mathbf{e}_\ell$ (where $\mathbf{e}_\ell$ denotes the standard basis vector filled with zeros except 1 at the ℓ -th entry). We have :*

For all $y \in \mathcal{M}$, there exists $\epsilon > 0$ such that :

For each $u \in \text{Ball}(y, \epsilon/2)$, we have :

For each $w \in \text{Ball}(u, \epsilon/2)$, there holds :

$$f(w) = f(u) \Rightarrow w_\ell = u_\ell. \quad (4.8)$$

Proof. This proof relies on the rank theorem [AMR12], which states that if ∇f has constant rank k in a manifold of dimension m , then there are local coordinates where $m - k$ variables are redundant and can be eliminated.

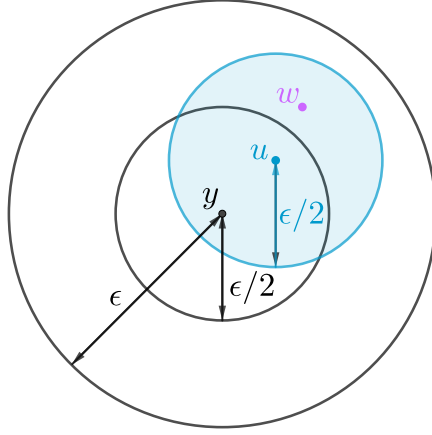


Fig. 4.2 Illustration of Lemma 4.9.

Consider $y \in \mathcal{M}$, and take $\epsilon > 0$ such that:

- $\text{Ball}(y, \epsilon) \subseteq \mathcal{M}$ (which is possible for any y since $\mathcal{M}$ is open).
- The local coordinates provided by the rank theorem [AMR12] applied on y are valid on the whole $\text{Ball}(y, \epsilon)$, i.e. each point $\chi \in \text{Ball}(y, \epsilon)$ can be expressed in the local coordinates provided by the rank theorem applied on y .

Consider a point $u \in \text{Ball}(y, \epsilon/2)$. The smoothness of f and constant rank of ∇f allow the application of the subimmersion theorem [AMR12]. In short, this theorem states that the level set of an image n_0 by such function is a submanifold of $\mathcal{M}$, and its tangent space coincides with the kernel of the gradient. Here, we take $n_0 = f(u)$ to make the connection with injectivity, and the theorem yields that for all u , $f^{-1}(f(u))$ is a submanifold of $\mathcal{M}$ with

$$T_w f^{-1}(f(u)) = \ker \nabla f(w) \quad \forall w \in f^{-1}(f(u)), \quad (4.9)$$

where $T_w A$ denotes the tangent space to manifold A at w .

Combining $\ker \nabla f(u) \perp \mathbf{e}_\ell$ with (5.2) yields $T_w f^{-1}(f(u)) \perp \mathbf{e}_\ell$ for all $w \in f^{-1}(f(u))$.

Take a point $w \in \text{Ball}(u, \epsilon/2) \cap f^{-1}(f(u))$, and assume that there is a path lying in $f^{-1}(f(u))$ between u and w . At every point of this path, the tangent space is orthogonal to direction ℓ , so the derivative along the path

is zero in the ℓ -coordinate. Integrating along the path gives a zero increase in ℓ -coordinate, hence $u_\ell = w_\ell$.

To complete the proof, we need to show that there is a path lying in $f^{-1}(f(u))$ between u and w . Such path can be found with the rank theorem [AMR12], which states that for all p in the manifold, there exist local coordinates centered at p and $f(p)$ in which $f(q) = (q_1, \dots, q_k, 0, \dots, 0)$, i.e. the last $m - k$ entries are zero, for any q in a local neighbourhood of p . In particular, two points have the same image by f if and only if the k first entries of their image by f are equal in those local coordinates.

For the rest of this proof, we work in the local coordinates provided by this theorem, centered at $p = y$. Recall that ϵ is such that each point $u \in \text{Ball}(y, \epsilon)$ can be expressed in those local coordinates. Since $u \in \text{Ball}(y, \epsilon/2)$ and $w \in \text{Ball}(u, \epsilon/2)$, the points $u, w \in \text{Ball}(y, \epsilon)$, and both u, w can be expressed in those local coordinates. The rank theorem gives $f(u) = (u_1, \dots, u_k, 0, \dots, 0)$ and $f(w) = (w_1, \dots, w_k, 0, \dots, 0)$ in the local coordinates. Since w lies in the manifold $f^{-1}(f(u))$, it must have the same image as u : $f(w) = f(u)$, and $w_1 = u_1, \dots, w_k = u_k$. An example of path between u and w is given by $(u_1, \dots, u_k, (1 - \lambda)u_{k+1} + \lambda w_{k+1}, \dots, (1 - \lambda)u_N + \lambda w_N)$, with $\lambda \in [0, 1]$. Any point of this path lies in $\text{Ball}(y, \epsilon)$, and applying the function f on any point of this path gives $f(u)$ in the local coordinates by the rank theorem, showing that the path is indeed included in the manifold $f^{-1}(f(u))$. $\square$

Lemma 4.10 ensures the existence of a *continuous* positive function ϵ inside which we have *scalar-coordinate-injectivity*. This is one of the keys for proving Theorem 4.11.

Lemma 4.10. *Let $\mathcal{M}$ be a smooth, open and finite-dimensional manifold, $\mathcal{N}$ be a smooth manifold, and $\mathcal{S} \subset \mathcal{M}$ be a closed complex analytic variety of lower dimension than $\mathcal{M}$. Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be a C^∞ function, such that for all $x \in \mathcal{M} \setminus \mathcal{S}$: $\nabla f(x)$ has the same rank and f is locally scalar-coordinate-injective for coordinate ℓ . Let $g : \mathbb{C} \rightarrow \mathcal{M}$ be a complex analytic function such that $\text{Im } g \not\subset \mathcal{S}$ (but $\text{Im } g \cap \mathcal{S}$ may be nonempty), and let g_ℓ be its ℓ -th entry.*

There exists a continuous positive function $\epsilon : \mathbb{C} \rightarrow \mathbb{R}_0^+$, such that for each $z \in \mathbb{C}$, we have scalar-coordinate-injectivity for coordinate ℓ at $g(z)$ on the whole $\text{Ball}(g(z), \epsilon(z))$, i.e. for all $\tilde{x} \in \text{Ball}(g(z), \epsilon(z))$, there holds

$$f(\tilde{x}) = f(g(z)) \Rightarrow \tilde{x}_\ell = g_\ell(z). \quad (4.10)$$

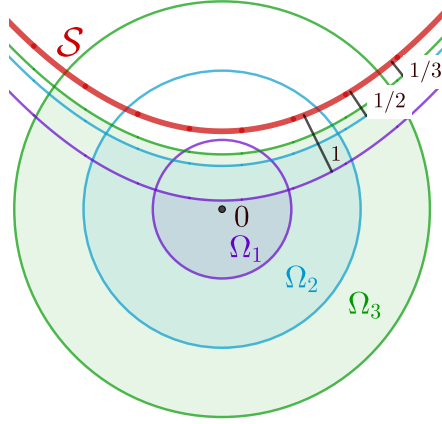


Fig. 4.3 Illustration of the sets Ω_n of Lemma 4.10.

Proof. We denote $\mathcal{M}_S \triangleq \mathcal{M} \setminus S$. The strategy of the proof is to apply Lemma 4.9 on $\mathcal{M}_S$, then define the sets Ω_n , illustrated in Fig. 4.3, extract a finite subcover of those sets and construct a continuous function ϵ from this subcover, as illustrated in Fig. 4.4.

The set $\mathcal{M}$ is open and S is closed, hence $\mathcal{M}_S$ is open. Since we assume that for all $y \in \mathcal{M}_S$: ∇f has the same rank and f is locally *scalar*-coordinate-injective for coordinate ℓ , we have $\ker \nabla f(y) \perp \mathbf{e}_\ell$ for all $y \in \mathcal{M}_S$ by Lemma 5.1. Therefore, Lemma 4.9 can be applied on $\mathcal{M}_S$: for all $y \in \mathcal{M}_S$, there exists $\rho_y > 0$ such that:

For each $u \in \text{Ball}(x, \rho_y/2)$, we have :

For each $w \in \text{Ball}(u, \rho_y/2)$, there holds :

$$f(w) = f(u) \Rightarrow w_\ell = u_\ell. \quad (4.11)$$

We define the sets Ω_n , which are illustrated in Fig. 4.3: $\Omega_n \triangleq \{y \in \mathcal{M} \text{ s.t. } \|y - \chi\| \geq 1/n \ \forall \chi \in S \text{ and } \|y\| \leq n\}$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, Ω_n is closed, bounded and finite-dimensional, hence compact. Define the sets $\Psi_n \triangleq \{\text{Ball}(y, \rho_y/2) \mid y \in \Omega_n\}$. Ψ_n is an infinite cover of Ω_n , i.e. $\Omega_n \subseteq \bigcup_{Y \in \Psi_n} Y$. Since Ω_n is compact, we can extract a finite subcover Φ_n from Ψ_n : we take $\Phi_n \subset \Psi_n$ such that Φ_n is finite and $\Omega_n \subseteq \bigcup_{Y \in \Phi_n} Y$.

Since Φ_n is finite, there is a minimum radius of all the balls that compose Φ_n , which we denote ϕ_n . Take a point $\chi \in \text{int } \Omega_n$. Since Φ_n is a cover of Ω_n ,

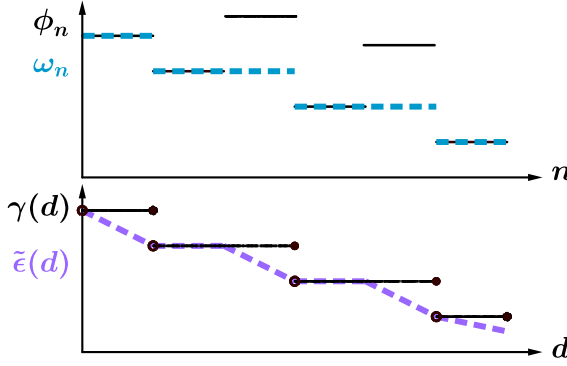


Fig. 4.4 Illustration of ϕ_n , ω_n , $\gamma(d)$ and $\tilde{\epsilon}(d)$ of Lemma 4.10.

there is $\text{Ball}(v, \rho_v/2) \in \Phi_n$ such that $\chi \in \text{Ball}(v, \rho_v/2)$, i.e. $\|\chi - v\| < \rho_v/2$. Also, by definition, $\phi_n \leq \rho_v/2$. Now take $\text{Ball}(\chi, \phi_n)$. This ball is entirely included in $\text{Ball}(v, \rho_v)$, i.e. $\text{Ball}(\chi, \phi_n) \subset \text{Ball}(v, \rho_v)$. Indeed, the distance between χ and the boundary of $\text{Ball}(v, \rho_v)$, which is given by $\rho_v - \|\chi - v\|$, can be bounded as: $\rho_v - \|\chi - v\| \geq \rho_v - \frac{\rho_v}{2} = \frac{\rho_v}{2} \geq \phi_n$, which is equivalent to $\text{Ball}(\chi, \phi_n) \subset \text{Ball}(v, \rho_v)$.

Also, from the definition of ϕ_n and Lemma 4.9, we have : for each $r \in \text{Ball}(\chi, \phi_n)$, there holds : $f(r) = f(\chi) \Rightarrow r_\ell = \chi_\ell$. In other words, for each $\chi \in \text{int } \Omega_n$, there is $\text{Ball}(v, \rho_v)$ such that $\text{Ball}(\chi, \phi_n) \subset \text{Ball}(v, \rho_v)$ and $\text{Ball}(v, \rho_v/2) \in \Phi_n$. Specifically, we have defined ϕ_n such that for each point $\chi \in \text{int } \Omega_n$ and for each radius $0 < r \leq \phi_n$, we have : for all points $\tilde{x} \in \text{Ball}(\chi, r)$, there holds $f(\tilde{x}) = f(\chi) \Rightarrow \tilde{x}_\ell = \chi_\ell$.

Define $\omega_n = \min(\phi_1, \phi_2, \dots, \phi_n)$. Define the function γ , illustrated in Fig. 4.4: $\gamma : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+ : \gamma(d) = \omega_{\lceil d \rceil}$. The function γ is discontinuous, but we can easily construct a *continuous* positive lower bound by linking linear interpolations between (d, ω_d) and $(d+1, \omega_{d+1})$ for each $d \in \mathbb{N}_0^+$. This gives the function $\tilde{\epsilon}$, a lower bound of γ which is *continuous*, positive and piecewise linear. It is illustrated in Fig. 4.4. $\tilde{\epsilon} : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ :$

$$\tilde{\epsilon}(d) = \begin{cases} \omega_1 - d \cdot (\omega_1 - \omega_2) & \text{if } 0 < d \leq 1 \\ \omega_2 - (d-1) \cdot (\omega_2 - \omega_3) & \text{if } 1 < d \leq 2 \\ \omega_3 - (d-2) \cdot (\omega_3 - \omega_4) & \text{if } 2 < d \leq 3 \\ \dots & \end{cases}$$

We define the function $\eta : \mathcal{M} \rightarrow \mathbb{R}_0^+ : y \rightarrow \max(\|y\|, 1/\text{dist}_{\mathcal{S}}(y))$, where $\text{dist}_{\mathcal{S}}(y)$ is the distance between y and $\mathcal{S}$, defined as $\text{dist}_{\mathcal{S}} : \mathcal{M} \rightarrow \mathbb{R}^+ : y \rightarrow \inf \{\|y - \chi\| : \chi \in \mathcal{S}\}$. The function $\tilde{\epsilon}(d)$ is continuous by construction. We know that the function distance to a set is continuous, hence so is the function $\text{dist}_{\mathcal{S}}$, and so is its inverse $1/\text{dist}_{\mathcal{S}}$. The function $g : \mathbb{C} \rightarrow \mathcal{M}$ is continuous since it is analytic, thus its norm $\|g\|$ is also continuous. Therefore, the function η is continuous, and so is the function $\epsilon : \mathbb{C} \rightarrow \mathbb{R}_0^+ : z \rightarrow \tilde{\epsilon}(\eta(g(z)))$.

This function ϵ is built such that for each $z \in \mathbb{C}$, $\epsilon(z) \leq \phi_n$ for the smallest n such that $g(z) \in \Omega_n$. Hence, we have *scalar-coordinate-injectivity* for coordinate ℓ at $g(z)$ on the whole $\text{Ball}(g(z), \epsilon(z))$, i.e. for all points $\tilde{x} \in \text{Ball}(g(z), \epsilon(z))$, there holds $f(\tilde{x}) = f(g(z)) \Rightarrow \tilde{x}_\ell = g_\ell(z)$. $\square$

Theorem 4.11 is one of the main results of this thesis. Building on Lemmas 4.8, 4.9 and 4.10, it establishes that, when considered generically, Definitions 4.3 and 4.4 are equivalent for analytic functions. Theorem 4.11 is stated here in its most general form, for any analytic function f . It will subsequently be formulated for parameterized analytic functions in Corollary 4.13.

Theorem 4.11. *Let $\mathcal{M}$ be a smooth, open and finite-dimensional manifold, $\mathcal{N}$ be a smooth manifold, $f : \mathcal{M} \rightarrow \mathcal{N}$ be an analytic function and $\ell \in \{1, \dots, \dim \mathcal{M}\}$. Let $\mathcal{S} \subset \mathcal{M}$ be a closed complex analytic variety of lower dimension than $\mathcal{M}$, such that $\nabla f(x)$ reaches its maximal rank (hence constant) for all $x \in \mathcal{M} \setminus \mathcal{S}$. The function f is locally scalar-coordinate-injective for coordinate ℓ at all $x \in \mathcal{M} \setminus \mathcal{S}$ if and only if f is locally functional-coordinate-injective for coordinate ℓ at all analytic function $g : \mathbb{C} \rightarrow \mathcal{M}$ such that $\text{Im } g \not\subset \mathcal{S}$ (but $\text{Im } g \cap \mathcal{S}$ may be nonempty).*

Proof. First, sufficiency is trivial. Assume that f is locally functional-coordinate-injective for coordinate ℓ at all function $g(z)$ except those with $\text{Im } g \subset \mathcal{S}$, where $\mathcal{S}$ is on a lower-dimensional complex analytic variety. Consider a point $x \notin \mathcal{S}$. Take the constant function $\bar{g}(z) = x \forall z$. There is a function $\epsilon(z)$ such that f is locally functional-coordinate-injective for coordinate ℓ at function $\bar{g}$. Take $\check{\epsilon} = \min_z \epsilon(z)$. With this $\check{\epsilon}$, f is also locally scalar-coordinate-injective for coordinate ℓ at x .

For necessity, the strategy of the proof is to consider an analytic function $g(z)$ that does not entirely lie in $\mathcal{S}$. Lemma 4.10 ensures the existence of

a *continuous* positive function ϵ ensuring scalar-coordinate-injectivity on the whole $\text{Ball}(g(z), \epsilon(z))$. Then we consider another analytic function $\tilde{g}$ such that for all z , we have $f(\tilde{g}(z)) = f(g(z))$. We prove that for almost all z_0 , either (i) $\tilde{g}(z_0)$ is close enough to $g(z_0)$ and local scalar-coordinate-injectivity applies, or (ii) it is too far and local scalar-coordinate-injectivity does not apply. Exploiting the fact that ϵ is continuous and g and $\tilde{g}$ are continuous analytic functions yields $\tilde{g}_\ell(z) = g_\ell(z)$ for all z .

Assume that f is generically locally scalar-coordinate-injective for coordinate ℓ . Take an analytic function $g(z)$ that does not entirely lie in $\mathcal{S}$, but may intersect it, i.e. $\text{Im } g \not\subset \mathcal{S}$, but $\text{Im } g \cap \mathcal{S}$ can be nonempty. Lemma 4.10 ensures the existence of a *continuous* positive function $\epsilon : \mathbb{C} \rightarrow \mathbb{R}_0^+$, such that for each $z \in \mathbb{C}$, $\text{Ball}(g(z), \epsilon(z)) \cap \mathcal{S} = \emptyset$, and such that f is scalar-coordinate-injective for coordinate ℓ at $g(z)$ on the whole $\text{Ball}(g(z), \epsilon(z))$, i.e. for all $\tilde{x} \in \text{Ball}(g(z), \epsilon(z))$, there holds

$$f(\tilde{x}) = f(g(z)) \Rightarrow \tilde{x}_\ell = g_\ell(z). \quad (4.12)$$

Consider an analytic function $\tilde{g} : \mathbb{C} \rightarrow \mathcal{M}$ such that for all z , we have $f(\tilde{g}(z)) = f(g(z))$. Consider $z_0 \in \mathbb{C}$ such that $g(z_0) \notin \mathcal{S}$. Two scenarios are possible:

- (i) either $\tilde{g}(z_0) \in \text{Ball}(g(z_0), \epsilon(z_0))$ and local scalar-coordinate-injectivity applies: we have $\tilde{x}_\ell(z_0) = x_\ell(z_0)$
- (ii) or $\tilde{g}(z_0)$ is too far from $g(z_0)$ and local scalar-coordinate-injectivity does not apply: we have $\tilde{g}(z_0) \notin \text{Ball}(g(z_0), \epsilon(z_0))$.

Since f is locally scalar-coordinate-injective for coordinate ℓ at all x except possibly on $\mathcal{S}$, this holds for all $g(z_0)$ except possibly on $\mathcal{S}$, and since we assumed $g(z)$ not entirely included in $\mathcal{S}$, applying Lemma 4.8 yields that it holds for all z_0 except possibly on a complex analytic variety of lower dimension than $\mathcal{S}$.

At this point of the proof we have, for almost all z , either (i) or (ii). In the last paragraph, we prove that we either have (i) for all z , or (ii) for all z .

Recall that ϵ, g and $\tilde{g}$ are continuous functions. Hence if there is z_0 such that $g(z_0) \notin \mathcal{S}$ and $\tilde{g}(z_0) \in \text{Ball}(g(z_0), \epsilon(z_0))$, then by continuity of ϵ, g and $\tilde{g}$, there is $Z > 0$ such that for all $\zeta \in \text{Ball}(z_0, Z)$, we have $g(\zeta) \notin \mathcal{S}$ and $\tilde{g}(\zeta) \in \text{Ball}(g(\zeta), \epsilon(\zeta))$.

Thus, we also have $\tilde{g}_\ell(\zeta) = g_\ell(\zeta)$ by local scalar-coordinate-injectivity.

Since $\tilde{g}_\ell(z)$ and $g_\ell(z)$ are analytic functions, their difference $\tilde{g}_\ell(z) - g_\ell(z)$ is also analytic and is either the zero function, or it is zero only on a complex analytic variety of lower dimension. Given the fact that $\tilde{g}_\ell(\zeta) = g_\ell(\zeta)$ on $\text{Ball}(z_0, Z)$, it must be the zero function. Thus $\tilde{g}_\ell(z) = g_\ell(z)$ for all z . $\square$

Note 4.12. Although we work with a rationally parameterized model (2.2) — a choice motivated by real-world applications and standard practice in System Identification [Lju99] — our results extend to arbitrary analytic functions. Indeed, Theorem 4.11 applies to *any* analytic functions, not just rational ones.

Applying Theorem 4.11 to parameterized analytic functions yields Corollary 4.13, which establishes the equivalence between the notions of Definitions 4.5 and 4.6, when considered generically. In Section 4.4, Corollary 4.13 will be applied on h defined in (4.2), with manifold $\mathcal{M} = \mathbb{C}^{m^\circ} \times \mathbb{C}^{m^\bullet} \setminus \mathcal{D}_h$, and manifold $\mathcal{N} = \mathbb{C}^{n_B \cdot n_C}$. Manifolds $\mathcal{M}$ and $\mathcal{N}$ could be defined more specifically in Corollary 4.13, but we keep the statement in its most general form.

Corollary 4.13. *Let $\mathcal{M}^\circ, \mathcal{M}^\bullet$ be smooth, open and finite-dimensional manifolds, $\mathcal{D}_f$ be a closed set, $\mathcal{N}$ be a smooth manifold, $\Phi^\circ, \Phi^\bullet$ be sets and denote $\mathcal{M} \triangleq \mathcal{M}^\circ \times \mathcal{M}^\bullet \setminus \mathcal{D}_f$. Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be an analytic function and $\ell \in \{1, 2, \dots, \dim \mathcal{M}^\circ\}$. Let $\mathcal{S} \subset \mathcal{M}$ be a closed complex analytic variety of lower dimension than $\mathcal{M}$, such that $\nabla f(x)$ reaches its maximal rank (hence constant) for all $x \in \mathcal{M} \setminus \mathcal{S}$. The function f is locally scalar-coordinate-injective for coordinate ℓ at all $x \in \mathcal{M} \setminus \mathcal{S}$ if and only if f is locally functional-coordinate-injective for coordinate ℓ of parameterized analytic function $g : \mathbb{C} \times \Phi^\circ \times \Phi^\bullet \rightarrow \mathcal{M}^\circ \times \mathcal{M}^\bullet$ at all $(\phi^\circ, \phi^\bullet) \in \Phi^\circ \times \Phi^\bullet$ such that $\text{Im } g(\cdot; (\phi^\circ, \phi^\bullet)) \not\subset \mathcal{S}$ (but $\text{Im } g(\cdot; (\phi^\circ, \phi^\bullet)) \cap \mathcal{S}$ may be nonempty).*

4.4 Identifiability in the scalar case

Corollary 4.13 establishes that, when considered generically, the results obtained for *scalar-coordinate* injectivity also apply to *functional-coordinate* injectivity for analytic parameterized functions. Consequently, in order to ease the analysis, we will work in the scalar setting in the remainder of this thesis:

- The transfer functions $G_{ij}(z; \theta)$ are now modeled by scalar values $A_{ij} \in \mathbb{C}$, and will be denoted by *modules*.

- Matrices $A, A^\circ, A^\bullet \in \mathbb{C}^{L \times L}$ replace matrices of transfer functions $G(z; \theta), G^\circ(z; \theta^\circ), G^\bullet(z; \theta^\bullet)$ respectively, with same nonzero entries respectively.
- Vectors of unknown and known modules $a^\circ = \text{nonzeros}(A^\circ)$, $a^\bullet = \text{nonzeros}(A^\bullet)$ are vectors of scalar values, i.e. $a^\circ \in \mathbb{C}^{m^\circ}$, $a^\bullet \in \mathbb{C}^{m^\bullet}$. Vector $a = (a^\circ, a^\bullet) \in \mathbb{C}^{m^\circ + m^\bullet}$ collects all unknown and known modules.
- The function $\bar{A}(a^\circ; a^\bullet)$ rearranges vector $a = (a^\circ, a^\bullet)$ in the network topology, i.e. $\bar{A}(a^\circ; a^\bullet) = A = A^\circ + A^\bullet$.
- Unless otherwise specified, mentions of local (coordinate)-injectivity should be understood as referring to the scalar version.

In the scalar case, the notions of global and local identifiability introduced in Definitions 2.2 and 4.1 take the following form.

Definition 4.14. Consider a network $A = A^\circ + A^\bullet$ with excitations and measurements matrices (B, C) . The unknown module A_{ij}° is *globally identifiable* at $(a^\circ, a^\bullet)$ if for all $\tilde{a}^\circ \in \mathbb{C}^{m^\circ}$, there holds

$$C \left(I - \bar{A}(a^\circ; a^\bullet) \right)^{-1} B = C \left(I - \bar{A}(\tilde{a}^\circ; a^\bullet) \right)^{-1} B \Rightarrow a_\ell^\circ = \tilde{a}_\ell^\circ, \quad (4.13)$$

where ℓ is the index corresponding to A_{ij}° in vector a° . Module A_{ij}° is *generically globally identifiable* if it is globally identifiable at *almost all* $(a^\circ, a^\bullet)$. This definition of global identifiability of a single module extends to a set of modules by replacing a_ℓ° in the RHS of (4.13).

Definition 4.15. Consider a network $A = A^\circ + A^\bullet$ with excitations and measurements matrices (B, C) . The unknown module A_{ij}° is *locally identifiable* at $(a^\circ, a^\bullet)$ if there exists $\epsilon > 0$ such that for all $\tilde{a}^\circ \in \text{Ball}(a^\circ, \epsilon)$, there holds

$$C \left(I - \bar{A}(a^\circ; a^\bullet) \right)^{-1} B = C \left(I - \bar{A}(\tilde{a}^\circ; a^\bullet) \right)^{-1} B \Rightarrow a_\ell^\circ = \tilde{a}_\ell^\circ, \quad (4.14)$$

where ℓ is the index corresponding to A_{ij}° in vector a° . Module A_{ij}° is *generically locally identifiable* if it is locally identifiable at *almost all* $(a^\circ, a^\bullet)$. This definition of local identifiability of a single module extends to a set of modules by replacing a_ℓ° in the RHS of (4.14).

The notion of local scalar-coordinate-injectivity that was introduced in Definition 4.5 is a generalization of the concept of local identifiability in the scalar case, presented in Definition 4.15. This is formalized in Lemma 4.16.

Lemma 4.16. *Consider a network $A = A^\circ + A^\bullet$ with excitations and measurements matrices (B, C) . The unknown module A_{ij}° is locally identifiable at $(a^\circ, a^\bullet)$ if and only if the corresponding function h defined in (4.2) is locally scalar-coordinate-injective for coordinate ℓ at $(a^\circ, a^\bullet)$, where ℓ is the index corresponding to A_{ij}° in vector a° .*

Proof. Substituting the expression of function h introduced in (4.2) in Definition 4.5 of local scalar-coordinate-injectivity yields precisely Definition 4.15. $\square$

Combining the results of Lemma 4.7, Corollary 4.13, and Lemma 4.16 gives Theorem 4.17, which establishes the equivalence between generic local identifiability in the transfer function setting and in the scalar case.

Theorem 4.17. *Consider a dynamic network $G(z; \theta) = G^\circ(z; \theta^\circ) + G^\bullet(z; \theta^\bullet)$ and its corresponding scalar network $A = A^\circ + A^\bullet$, with same excitations and measurements. The unknown transfer function G_{ij}° is generically locally identifiable (in the sense of Definition 4.1) if and only if the corresponding module A_{ij}° is generically locally identifiable (in the sense of Definition 4.15).*

Proof. First, Lemma 4.7 establishes that the unknown transfer function G_{ij}° is locally identifiable at $\theta = (\theta^\circ, \theta^\bullet)$ if and only if the corresponding function h defined in (4.2) is locally functional-coordinate-injective for coordinate ℓ of function g_{net} defined in (4.7) at $\theta = (\theta^\circ, \theta^\bullet)$, where ℓ is the index corresponding to G_{ij}° in the image of g_{net} . Second, Corollary 4.13 demonstrates that when considered generically, functional-coordinate-injectivity for parameterized functions of the form of g_{net} is equivalent to scalar-coordinate-injectivity. Finally, Lemma 4.16 shows that the notion of local scalar-coordinate-injectivity applied to function h gives precisely local identifiability in the scalar setting. $\square$

Theorem 4.17 provides a significant simplification: rather than analyzing transfer functions directly, we can equivalently work with complex-valued modules on the edges, which can be interpreted as edge weights. This reformulation reveals the inherently topological nature of generic local identifiability, thereby justifying our focus on this concept and motivating the methodology adopted. The following chapters will further reinforce

this initial intuition regarding the topological character of generic local identifiability.

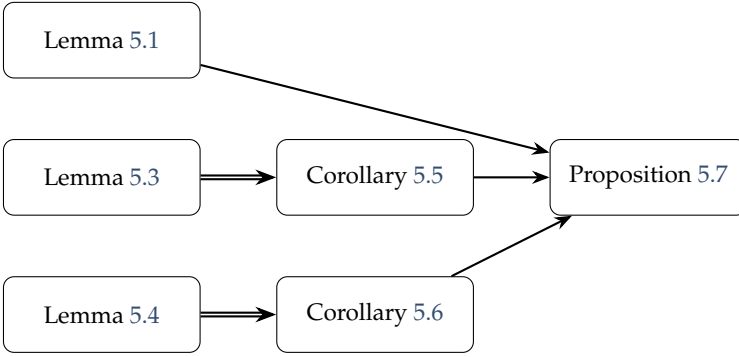
5

Algebraic characterizations

IN this chapter and throughout the remainder of the thesis, we restrict our attention to the scalar case, as Theorem 4.17 establishes the equivalence between generic local identifiability in the transfer function setting and in the scalar setting. We begin by deriving an algebraic characterization of local scalar-coordinate injectivity using tools from manifold theory. As part of this analysis, we also show that scalar-coordinate injectivity is a pseudo-generic property. By applying this characterization to the function h (defined in (4.2)), we obtain an algebraic characterization of local identifiability and demonstrate that it too is a pseudo-generic property. This leads to necessary and sufficient conditions for generic local identifiability, both (i) for a single unknown module and (ii) for all unknown modules in the network. Leveraging the pseudo-generic nature of local identifiability, together with the transition from the transfer function setting to the scalar setting, these algebraic conditions can be verified numerically with probability 1 by evaluating them on a randomly sampled instance of the network. We develop an algorithm based on this principle, which efficiently computes the generic local identifiability of any given network. This chapter is based on [LH24] and its preliminary version [LH20].

5.1 Algebraic characterization of local injectivity

In this section, we establish the equivalence of local scalar-coordinate-injectivity with an orthogonality relation of the gradient: see equation (5.1). We further demonstrate that local scalar-coordinate-injectivity is pseudo-generic by proving that (5.1) holds pseudo-generically. To this end, we present Lemmas 5.1, 5.3, and 5.4, which collectively support the proof of Proposition 5.7. In Section 5.2, we will show how local scalar-coordinate-injectivity connects to local identifiability, thereby establishing the pseudo-generic nature of the latter as well.



Intuitively, if the problem was one-dimensional, the derivative could be used to assess local injectivity. Indeed, if the function is constant — hence has a zero derivative — it cannot be locally injective. Conversely, h would be locally injective at $(a^\circ, a^\bullet)$ if $h'(a^\circ; a^\bullet) \neq 0$, where the derivative is taken with respect to the variable a° and the parameters $a^\bullet$ are fixed. In the multidimensional case, the condition on the derivative becomes: $\nabla h(a^\circ; a^\bullet) \cdot \delta \neq 0$ for any nonzero $\delta \in \mathbb{C}^{m^\circ}$, i.e. if $\nabla h(a^\circ; a^\bullet)$ has a trivial kernel. Similarly, h would be coordinate-injective at $(a^\circ, a^\bullet)$ for module ℓ if $\nabla h(a^\circ; a^\bullet) \cdot \delta \neq 0$ for any $\delta \in \mathbb{C}^{m^\circ}$ with $\delta_\ell \neq 0$, i.e. if the kernel of $\nabla h(a^\circ; a^\bullet)$ is orthogonal to $\mathbf{e}_\ell$, the standard basis vector filled with zeros except 1 at the ℓ -th entry.

This kernel condition is sufficient for local (coordinate-)injectivity, but another assumption is required to achieve necessity. For example, the real function $g(x) = x^3$ is injective, but $g'(0) = 0$ admits a non-trivial kernel, which contradicts the necessity of our previous statement. The

problem is that the derivative of g vanishes at 0, while being nonzero everywhere else. Adding an assumption on the conservation of information contained in the derivative (i.e. the rank of its gradient) resolves that issue. Lemma 5.1 shows that the intuitive idea behind the gradient is valid when ∇f has constant rank. We will then see in Proposition 5.7 how this rather constraining assumption can be removed provided one is only interested in local injectivity *at almost all* x , i.e. in a generic manner.

Lemma 5.1. *Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be a C^∞ function where $\mathcal{M}$ and $\mathcal{N}$ are smooth manifolds, $\mathcal{M}$ or $\mathcal{N}$ has finite dimension, $\nabla f(x)$ has the same rank for every $x \in \mathcal{M}$, and $\ell \in \{1, 2, \dots, \dim \mathcal{M}\}$. There holds*

(a) *For all $x \in \mathcal{M}$: if f is locally scalar-coordinate-injective for ℓ at x , then*

$$\ker \nabla f(x) \perp \mathbf{e}_\ell, \quad (5.1)$$

where $\mathbf{e}_\ell$ denotes the standard basis vector filled with zeros except 1 at the ℓ -th entry.

(b) *If (5.1) holds for all $x \in \mathcal{M}$, then f is locally scalar-coordinate-injective for ℓ at all $x \in \mathcal{M}$.*

Note 5.2. Observe the difference between (a) and (b) in Lemma 5.1. (a) can be summarized as: $\forall x : (4.3) \Rightarrow (5.1)$, while (b) would be summarized as: $(5.1) \forall x \Rightarrow (4.3) \forall x$. The formulation of (a) is stronger: the implication holds at each point of the manifold. On the other hand, for (b), (5.1) must hold at *all* points of the manifold.

To illustrate this difference, consider the following example. Suppose that (4.3) holds for all points in $\mathcal{M}$ except for one exceptional point x_0 . In that case, (a) still guarantees that (5.1) is satisfied for every point in $\mathcal{M} \setminus \{x_0\}$. On the other hand, for (b), the existence of a single point x_1 where (5.1) fails is sufficient to invalidate the assumption, so the statement does not apply and provides no information.

Proof. The smoothness of f and constant rank of ∇f allow the application of the subimmersion theorem [AMR12]. In short, this theorem states that the level set of an image n_0 by such function is a submanifold of $\mathcal{M}$, and its tangent space coincides with the kernel of the gradient. Here, we take $n_0 = f(x)$ to make the connection with injectivity, and the theorem yields that for all x , $f^{-1}(f(x))$ is a submanifold of $\mathcal{M}$ with

$$T_{\tilde{x}} f^{-1}(f(x)) = \ker \nabla f(\tilde{x}) \quad \forall \tilde{x} \in f^{-1}(f(x)), \quad (5.2)$$

where $T_{\tilde{x}}M$ denotes the tangent space to manifold M at $\tilde{x}$.

We first prove (a). Take $x \in \mathcal{M}$, and assume (4.3). Each $\tilde{x}$ belonging to a neighborhood of x in $f^{-1}(f(x))$ has the same ℓ -coordinate as x . Therefore, any tangent vector to $f^{-1}(f(x))$ at $\tilde{x}$ has a zero ℓ -component, and the tangent space to this manifold at $\tilde{x}$ is orthogonal to direction ℓ , i.e. $T_{\tilde{x}}f^{-1}(f(x)) \perp \mathbf{e}_\ell$. Equation (5.2) evaluated at $\tilde{x} = x$ then gives (5.1). Since this holds for any $x \in \mathcal{M}$, the proof of (a) is complete.

Now we prove (b). Assume (5.1) for all x , and consider a specific x . Combining (5.1) with (5.2) yields $T_{\tilde{x}}f^{-1}(f(x)) \perp \mathbf{e}_\ell$ for all $\tilde{x} \in f^{-1}(f(x))$. Take a specific $\tilde{x} \in f^{-1}(f(x))$, and assume that there is a path lying in $f^{-1}(f(x))$ between x and $\tilde{x}$. At every point of this path, the tangent space is orthogonal to direction ℓ , so the derivative along the path is zero in the ℓ -coordinate. Integrating along the path gives a zero increase in ℓ -coordinate, hence $\tilde{x}_\ell = x_\ell$. This holds for all $x \in \mathcal{M}$.

To complete the proof, we need to show that there exists $\epsilon > 0$ such that for all $\tilde{x} \in B(x, \epsilon) \cap f^{-1}(f(x))$, there is a path lying in $f^{-1}(f(x))$ between x and $\tilde{x}$. Such path can be found with the rank theorem [AMR12]. Roughly speaking, this theorem states that if ∇f has constant rank k in a manifold $\mathcal{M}$, then $\dim \mathcal{M} - k$ variables are redundant and can be eliminated. More precisely, for all p in the manifold, there exist local coordinates centered at p and $f(p)$ in which $f(q) = (q_1, \dots, q_k, 0, \dots, 0)$ for any q in the manifold.

For the rest of this proof, we work in the local coordinates provided by this theorem, centered at $p = x$. Since these are centered at x and $f(x)$, $x_1 = \dots = x_k = 0$. Let us now take $\epsilon > 0$, and $\tilde{x} \in B(x, \epsilon) \cap f^{-1}(f(x))$. The rank theorem yields that the last entries of $f(\tilde{x})$ are zero in those local coordinates, i.e. $f(\tilde{x}) = (\tilde{x}_1, \dots, \tilde{x}_k, 0, \dots, 0)$. Since $\tilde{x}$ lies in the manifold $f^{-1}(f(x))$, it must have the same image as x , hence $f(\tilde{x}) = f(x) = 0$, and $\tilde{x}_1 = \dots = \tilde{x}_k = 0$. An example of path between x and $\tilde{x}$ is given by $(0, \dots, 0, \lambda \tilde{x}_{k+1}, \dots, \lambda \tilde{x}_{\dim \mathcal{M}})$, with $\lambda \in [0, 1]$. Applying the function f on any point of this path gives 0 in the local coordinates by the rank theorem, showing that the path is indeed included in the manifold $f^{-1}(f(x))$. $\square$

Lemma 5.3 establishes the generic nature of the notion generic rank introduced in Definition 3.1. For a concrete illustration of this concept, refer to Example 3.2 and Fig. 3.1.

Lemma 5.3. *For set $\mathcal{M}$ and naturals n_1, n_2 , let $M : \mathcal{M} \rightarrow \mathbb{C}^{n_1 \times n_2}$ be a matrix analytic function, let $r = \text{grank}(M) \triangleq \max_{x \in \mathcal{M}} \text{rank } M(x)$ be its generic rank*

and $q \in \mathbb{N}$ such that $0 \leq q < r$. The set $\mathcal{M}$ is partitioned according to $\text{rank } M(x)$ as follows:

- (i) $\text{rank } M(x) = r$ for all $x \in \mathcal{M}$ except possibly those lying on a closed set of dimension $< \dim \mathcal{M}$.
- (ii) $\text{rank } M(x) = q$ for almost no x , i.e. only for x lying on a set of dimension $< \dim \mathcal{M}$.

Proof. Recall the following basic result of linear algebra [dW91]: the rank of a matrix is q if it has a nonzero q -th order minor, while every $(q + 1)$ -th order minor is zero. The minor of a matrix being the determinant of one of its submatrices, it is analytic in the entries of the matrix. Here, minors of $M(x)$ are also analytic in x since f is analytic and composition preserves analyticity. Since nonconstant analytic functions vanish only on a lower-dimensional set [D'A10, p. 162], the zero set of each minor of $M(x)$ is lower-dimensional.

First, we prove (i). As in [dW91], we define the set on which r is not reached:

$$\mathcal{W}_{<r} = \{x \in M \mid \text{rank } M(x) < r\}.$$

The above characterization of the rank yields

$$\mathcal{W}_{<r} = \{x \in M \mid \text{each } r\text{-th order minor of } M(x) \text{ is zero}\},$$

which is the finite intersection of lower-dimensional sets, hence is lower-dimensional. Besides, $\mathcal{W}_{<r}$ is also closed: the inverse image of a closed set by a continuous function is a closed set [Hau91], hence the zero sets of the minors are closed, and so is their intersection.

Next, we prove (ii). Let

$$\mathcal{W}_q = \{x \in M \mid \text{rank } M(x) = q\}.$$

Clearly, $\mathcal{W}_q \subseteq \mathcal{W}_{<r}$. Since $\mathcal{W}_{<r}$ is lower-dimensional, so is $\mathcal{W}_q$. □

Building on the generic nature of the notion of generic rank, Lemma 5.4 establishes the pseudo-generic character of the orthogonality relation (5.3), for any matrix analytic function M .

Lemma 5.4. *For set $\mathcal{M}$ and naturals n_1, n_2 , let $M : \mathcal{M} \rightarrow \mathbb{C}^{n_1 \times n_2}$ be a matrix*

analytic function, and $\ell \in \{1, \dots, \dim \mathcal{M}\}$. The orthogonality relation

$$\ker M(x) \perp \mathbf{e}_\ell \quad (5.3)$$

is pseudo-generic: it either holds

- (i) for almost all $x \in \mathcal{M}$, i.e. for all x except possibly those lying on a set of dimension $< \dim \mathcal{M}$.
- (ii) for almost no x , i.e. only for x lying on a set of dimension $< \dim \mathcal{M}$.

Proof. Equation (5.3) can be equivalently rewritten as $M(x) \cdot \delta = 0 \Rightarrow \delta_\ell = 0$ for all $\delta \in \mathbb{C}^{\dim \mathcal{M}}$, which is equivalent to the linear independence of the ℓ -th column of $M(x)$ with the other columns, i.e.

$$\text{rank } M(x) = \text{rank } [M(x)]_{\setminus \ell} + 1, \quad (5.4)$$

where $[M(x)]_{\setminus \ell}$ denotes the matrix $M(x)$ without its ℓ -th column. Equation (5.4) can be rewritten as

$$\bigvee_{i=0}^{i_{\max}} \left(\text{rank } M(x) = i + 1 \wedge \text{rank } [M(x)]_{\setminus \ell} = i \right), \quad (5.5)$$

where $i_{\max} \triangleq \min(\dim \mathcal{M}, n_1 n_2) - 1$. For each i , both rank equalities of (5.5) are pseudo-generic by Lemma 5.3. Then, equation (5.5) is pseudo-generic by Lemma 2.5. $\square$

Consider a parameterized function f , analytic in its variables and its parameters. Applying Lemma 5.3 and Lemma 5.4 for $M = \nabla f$ gives the two following corollaries.

Corollary 5.5. For sets $\mathcal{M}^\circ, \mathcal{M}^\bullet, \mathcal{D}_f, \mathcal{N}$ and $\mathcal{M} \triangleq \mathcal{M}^\circ \times \mathcal{M}^\bullet \setminus \mathcal{D}_f$, let $f : \mathcal{M} \rightarrow \mathcal{N}$ be a function analytic in its variables $x^\circ \in \mathcal{M}^\circ$ and in its parameters $x^\bullet \in \mathcal{M}^\bullet$, let $r = \max_{(x^\circ, x^\bullet) \in \mathcal{M}} \text{rank } \nabla f(x^\circ; x^\bullet)$ and $q \in \mathbb{N}$ such that $0 \leq q < r$. The set $\mathcal{M}$ is partitioned according to $\text{rank } \nabla f(x^\circ; x^\bullet)$ as follows:

- (i) $\text{rank } \nabla f(x^\circ; x^\bullet) = r$ for all $(x^\circ, x^\bullet) \in \mathcal{M}$ except possibly those lying on a closed set of dimension $< \dim \mathcal{M}$.
- (ii) $\text{rank } \nabla f(x^\circ; x^\bullet) = q$ for almost no $(x^\circ, x^\bullet)$, i.e. only for $(x^\circ, x^\bullet)$ lying on a set of dimension $< \dim \mathcal{M}$.

Corollary 5.6. For sets $\mathcal{M}^\circ, \mathcal{M}^\bullet, \mathcal{D}_f, \mathcal{N}$ and $\mathcal{M} \triangleq \mathcal{M}^\circ \times \mathcal{M}^\bullet \setminus \mathcal{D}_f$, let $f : \mathcal{M} \rightarrow \mathcal{N}$ be a function analytic in its variables $x^\circ \in \mathcal{M}^\circ$ and in its parameters $x^\bullet \in \mathcal{M}^\bullet$, and $\ell \in \{1, 2, \dots, \dim \mathcal{M}^\circ\}$. The orthogonality relation

$$\ker \nabla f(x^\circ; x^\bullet) \perp \mathbf{e}_\ell \quad (5.6)$$

is pseudo-generic: it either holds

- (i) for almost all $(x^\circ, x^\bullet) \in \mathcal{M}$, i.e. for all $(x^\circ, x^\bullet)$ except possibly those lying on a set of dimension $< \dim \mathcal{M}$.
- (ii) for almost no $(x^\circ, x^\bullet)$, i.e. only for $(x^\circ, x^\bullet)$ lying on a set of dimension $< \dim \mathcal{M}$.

Proposition 5.7 characterizes local scalar-coordinate-injectivity almost everywhere for analytic functions, building on Lemma 5.1, Corollary 5.5 and Corollary 5.6. Local scalar-coordinate-injectivity is closely related to the orthogonality relation (5.6), hence is pseudo-generic: it either holds almost everywhere, or almost nowhere. In Section 5.2, Proposition 5.7 will be applied on h , defined in (4.2), with manifolds $\mathcal{M}^\circ = \mathbb{C}^{m^\circ}$, $\mathcal{M}^\bullet = \mathbb{C}^{m^\bullet}$, $\mathcal{D}_f = \mathcal{D}_h$ and $\mathcal{N} = \mathbb{C}^{n_B \cdot n_C}$. Those manifolds could be defined more specifically in Proposition 5.7, but we keep the statement in its most general form.

Proposition 5.7. Let $\mathcal{M}^\circ, \mathcal{M}^\bullet$ be smooth, open manifolds, let $\mathcal{D}_f$ be a closed set, $\mathcal{M} \triangleq \mathcal{M}^\circ \times \mathcal{M}^\bullet \setminus \mathcal{D}_f$, $\mathcal{N}$ be a smooth manifold, and $\mathcal{M}$ or $\mathcal{N}$ has finite dimension. Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be a function analytic in its variables $x^\circ \in \mathcal{M}^\circ$ and in its parameters $x^\bullet \in \mathcal{M}^\bullet$, and $\ell \in \{1, 2, \dots, \dim \mathcal{M}^\circ\}$. Exactly one of the two following statements holds:

- (i) $\ker \nabla f(x^\circ; x^\bullet) \perp \mathbf{e}_\ell$ for almost all $(x^\circ, x^\bullet)$ and f is locally scalar-coordinate-injective for ℓ at almost all $(x^\circ, x^\bullet)$.
- (ii) For almost all $(x^\circ, x^\bullet) : \ker \nabla f(x^\circ; x^\bullet) \not\perp \mathbf{e}_\ell$ and f is locally non-scalar-coordinate-injective for ℓ at $(x^\circ, x^\bullet)$.

Proof. Let $\mathcal{S}_1 \subset \mathcal{M}$ be the set on which ∇f does not reach its maximal rank, and $\mathcal{S}_2 \subset \mathcal{M}$ the set on which (5.6) does not hold (case (i) of Corollary 5.6) or holds (case (ii) of Corollary 5.6). Corollary 5.5 ensures that $\mathcal{S}_1$ is closed and lower-dimensional, and Corollary 5.6 ensures that $\mathcal{S}_2$ is lower-dimensional. We write $\text{cl } \mathcal{S}_2$ for the closure of $\mathcal{S}_2$, i.e. the union of $\mathcal{S}_2$ and its boundary. Let $\mathcal{M}_\mathcal{S} \triangleq \mathcal{M} \setminus (\mathcal{S}_1 \cup \text{cl } \mathcal{S}_2)$. $\mathcal{M}_\mathcal{S}$ covers the whole domain $\mathcal{M}$ except a closed lower-dimensional set, and it is open since $\mathcal{M}$ is assumed open.

Thus, there is an ϵ for every $(p^\circ, p^\bullet) \in \mathcal{M}_S$ such that the rank of ∇f is constant over $\text{Ball}((p^\circ, p^\bullet), \epsilon)$, and (5.6) holds (case (i) of Corollary 5.6) or does not hold (case (ii) of Corollary 5.6) on the ball. Denote the restriction of f on $\text{Ball}((p^\circ, p^\bullet), \epsilon)$, with fixed parameters $p^\bullet$ by:

$$f_{(p^\circ, p^\bullet), \epsilon} : \text{Ball}((p^\circ, p^\bullet), \epsilon) \rightarrow \mathcal{N} : x^\circ \rightarrow f(x^\circ; p^\bullet).$$

The result (i) follows by applying Lemma 5.1 (b) on $f_{(p^\circ, p^\bullet), \epsilon}$ for all $(p^\circ, p^\bullet) \in \mathcal{M}_S$. Similarly, (ii) follows by applying the contraposition of Lemma 5.1 (a) on $f_{(p^\circ, p^\bullet), \epsilon}$ for all $(p^\circ, p^\bullet) \in \mathcal{M}_S$. $\square$

Note the difference between the formulation of (i) and (ii). For (i), the set on which the orthogonality relation holds does not need to coincide with the set on which f is locally coordinate-injective. On the other hand, for (ii), the set on which the orthogonality relation does not hold is the *same* as the one where we have local non-coordinate-injectivity. This comes from the difference between (a) and (b) in Lemma 5.1.

5.2 Algebraic characterization of local identifiability

In this section, we apply Proposition 5.7 to function h defined in (4.2). This leads to an algebraic characterization of local identifiability. We show that the local identifiability of a single unknown module is a pseudo-generic property, and that it becomes a generic property when the local identifiability of *all* unknown modules is considered.

Definition 5.8. We define the matrix function K which, as will be proved in Lemma 5.10, is the gradient of function h introduced in (4.2).

$$K : \mathbb{C}^{m^\circ + m^\bullet} \rightarrow \mathbb{C}^{n_B n_C \times m^\circ} : (a^\circ, a^\bullet) \rightarrow \left(B^\top \bar{T}^\top(a^\circ; a^\bullet) \otimes C \bar{T}(a^\circ; a^\bullet) \right) I^\circ, \quad (5.7)$$

where $\bar{T}(a^\circ; a^\bullet) \triangleq (I - \bar{A}(a^\circ; a^\bullet))^{-1}$, and $\bar{A}(a^\circ; a^\bullet)$ is the network matrix rearranging vector of unknowns a° and knowns $a^\bullet$ into the network structure of the problem, see Example 2.1 below. Symbol $\otimes$ denotes the Kronecker product and the matrix $I^\circ \in \{0, 1\}^{L^2 \times m^\circ}$ selects only the columns of the

preceding $n_B n_C \times L^2$ matrix corresponding to the m° unknown modules:

$$I^\circ \triangleq \begin{bmatrix} \text{vec}(\bar{A}(\mathbf{e}_1; 0)) & \cdots & \text{vec}(\bar{A}(\mathbf{e}_{m^\circ}; 0)) \end{bmatrix}, \quad (5.8)$$

where notation $\mathbf{e}_\ell$ stands for the standard basis vector filled with zeros except 1 at ℓ -th entry.

Note 5.9. For the functions $\bar{T}$ and $\bar{A}$, a° denotes the variables, which are separated by a semicolon from the parameters $a^\bullet$. In contrast, for the function K , both a° and $a^\bullet$ are treated as variables and jointly form the vector a . Accordingly, we will sometimes write $K(a)$ as a shorthand for $K(a^\circ, a^\bullet)$.

Example 2.1 (continued).

$$\begin{aligned} \bar{A}(a^\circ; a^\bullet) &= \begin{bmatrix} 0 & 0 & a_2^\circ \\ a_1^\circ & 0 & a_2^\bullet \\ a_1^\bullet & 0 & 0 \end{bmatrix}, & \bar{A}(\mathbf{e}_1, 0) &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \text{vec}(\bar{A}(\mathbf{e}_1, 0)) &= [0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^\top, \\ I^\circ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^\top, \\ K(a^\circ, a^\bullet) &= \begin{bmatrix} 0 & \frac{a_1^\bullet}{(a_2^\circ a_1^\bullet - 1)^2} \\ -1 & \frac{a_1^\bullet (a_1^\circ + a_2^\bullet a_1^\bullet)}{(a_2^\circ a_1^\bullet - 1)^2} \\ \frac{a_2^\circ a_1^\bullet - 1}{a_2^\circ a_1^\bullet - 1} & \frac{a_1^\bullet (a_1^\circ + a_2^\bullet a_1^\bullet)}{(a_2^\circ a_1^\bullet - 1)^2} \end{bmatrix}. \end{aligned}$$

Lemma 5.10 establishes that the gradient of function h is matrix K , introduced in Definition 5.8.

Lemma 5.10. *The gradient of function h defined in (4.2) is*

$$\nabla h(a^\circ; a^\bullet) = \left(B^\top \bar{T}^\top(a^\circ, a^\bullet) \otimes C \bar{T}(a^\circ; a^\bullet) \right) I^\circ = K(a^\circ, a^\bullet). \quad (5.9)$$

Besides, for all $\delta \in \mathbb{C}^{m^\circ}$, there holds

$$\text{mat}(\nabla h(a^\circ; a^\bullet) \delta) = C \bar{T}(a^\circ; a^\bullet) \bar{A}(\delta; 0) \bar{T}(a^\circ; a^\bullet) B, \quad (5.10)$$

where mat reorganizes its $n_B \cdot n_C$ vector argument in a $n_B \times n_C$ matrix.

Example 5.11. $\text{mat} \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \end{bmatrix}^\top = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix}.$

Proof. First note that $\nabla h(a^\circ; a^\bullet) \delta$ is the differential of h at a° evaluated on δ with parameters $a^\bullet$, which we denote $d_{a^\circ} h(\delta; a^\bullet)$. We start by deriving the expression of the differential (5.10), from which we compute the gradient (5.9). The differential of h follows from the differential of $\bar{T}$ by linearity: $\text{mat}(dh) = C d\bar{T}B$, which relies on its partial derivatives, developed as [PP12]:

$$\begin{aligned} \frac{\partial \bar{T}}{\partial a_\ell^\circ}(a^\circ; a^\bullet) &= -(I - \bar{A}(a^\circ; a^\bullet))^{-1} \frac{\partial(I - \bar{A})}{\partial a_\ell^\circ}(a^\circ; a^\bullet) (I - \bar{A}(a^\circ; a^\bullet))^{-1} \\ &= -\bar{T}(a^\circ; a^\bullet) \left(-\frac{\partial \bar{A}}{\partial a_\ell^\circ}(a^\circ; a^\bullet) \right) \bar{T}(a^\circ; a^\bullet) \\ &= \bar{T}(a^\circ; a^\bullet) \bar{A}(\mathbf{e}_\ell; 0) \bar{T}(a^\circ; a^\bullet). \end{aligned}$$

The partial derivatives then yield the differential:

$$\begin{aligned} d_{a^\circ} \bar{T}(\delta; a^\bullet) &= \sum_{\ell=1}^{m^\circ} \frac{\partial \bar{T}}{\partial a_\ell^\circ}(a^\circ; a^\bullet) \delta_\ell \\ &= \sum_{\ell=1}^{m^\circ} \bar{T}(a^\circ; a^\bullet) \bar{A}(\mathbf{e}_\ell; 0) \bar{T}(a^\circ; a^\bullet) \delta_\ell \\ &= \bar{T}(a^\circ; a^\bullet) \bar{A}(\delta; 0) \bar{T}(a^\circ; a^\bullet). \end{aligned}$$

For the differential of h , we obtain

$$\begin{aligned} \text{mat}(d_{a^\circ} h(\delta; a^\bullet)) &= C d_{a^\circ} \bar{T}(\delta; a^\bullet) B \\ &= C \bar{T}(a^\circ; a^\bullet) \bar{A}(\delta; 0) \bar{T}(a^\circ; a^\bullet) B. \end{aligned} \tag{5.11}$$

In order to derive the gradient of h , we vectorize (5.11):

$$\begin{aligned} \nabla h(a^\circ; a^\bullet) \delta &= \text{vec}(C \bar{T}(a^\circ; a^\bullet) \bar{A}(\delta; 0) \bar{T}(a^\circ; a^\bullet) B) \\ &= \left(B^\top \bar{T}^\top(a^\circ; a^\bullet) \otimes C \bar{T}(a^\circ; a^\bullet) \right) \text{vec}(\bar{A}(\delta; 0)), \end{aligned}$$

where vec stacks the columns of its $n_B \times n_C$ matrix argument into a $n_B \cdot n_C$ vector, see Example 4.2. Then, observing that $\text{vec}(\bar{A}(\delta; 0)) = I^\circ \delta$ gives $\nabla h(a^\circ; a^\bullet) \delta = K(a^\circ, a^\bullet) \delta \quad \forall \delta \in \mathbb{C}^{m^\circ}$, which yields (5.9). $\square$

Combining the expression of ∇h derived in Lemma 5.10 with Proposition 5.7 yields the following *algebraic necessary and sufficient characterization for local identifiability*: $\ker K(a) \perp \mathbf{e}_\ell$ for almost all a if and only if A_{ij} is

locally identifiable at almost all a , where ℓ is the index of module A_{ij} in vector a° . Theorem 5.12 is a stronger version of this statement: it provides characterization (ii) when (i) does not hold. Moreover, in (ii), the sets of a on which the orthogonality relation does not hold, and local identifiability does not hold, coincide exactly. The local identifiability of an unknown module is a pseudo-generic property: it either holds almost everywhere, or almost nowhere. It follows from the pseudo-genericity of local scalar-coordinate-injectivity, established in Proposition 5.7. Note that it is pseudo-generic in both the unknown modules a° , and the known modules $a^\bullet$.

Theorem 5.12. *Consider a network $A = A^\circ + A^\bullet$ with excitations matrix B , and measurements matrix C . Consider the unknown module A_{ij}° , let $K(a)$ be as defined in (5.7), and let ℓ be the index of module A_{ij}° in vector a° . Exactly one of the two following statements holds:*

- (i) $\ker K(a) \perp \mathbf{e}_\ell$ for almost all a and A_{ij}° is locally identifiable at almost all a .
- (ii) For almost all a : $\ker K(a) \not\perp \mathbf{e}_\ell$ and A_{ij}° is locally non-identifiable at a ,

where $a = (a^\circ, a^\bullet)$ contains the unknown and known modules. Besides, $\ker K(a) \perp \mathbf{e}_\ell$ is equivalent to the following implication holding $\forall \Delta^\circ \in \mathbb{C}^{L \times L}$ with same zero entries as A° :

$$C \bar{T}(a^\circ; a^\bullet) \Delta^\circ \bar{T}(a^\circ; a^\bullet) B = 0 \Rightarrow \Delta_{ij}^\circ = 0. \quad (5.12)$$

The proof of Theorem 5.12 relies on applying Proposition 5.7 on the gradient of h , characterized by Lemma 5.10.

Proof. Lemma 4.16 establishes that the local identifiability of module A_{ij} is equivalent to the local scalar-coordinate-injectivity of function h defined in (4.2) for coordinate ℓ . We verify the assumptions of Proposition 5.7 for h . First, h is analytic, and its domain and image sets have finite dimension. The openness of the domain $\mathbb{C}^{m^\circ} \times \mathbb{C}^{m^\bullet} \setminus \mathcal{D}_h$ follows from the continuity of $\det(I - \bar{A})$. Since the inverse image of a closed set by a continuous function is a closed set [Hau91], the zero set of $\det(I - \bar{A})$ is closed, and its complement is open. Its assumptions being fulfilled, Proposition 5.7 applies. The first part of the theorem then follows directly from the expression of ∇h given in (5.9). Besides, $\ker \nabla h(a^\circ; a^\bullet) \perp \mathbf{e}_\ell$ can be rewritten equivalently:

$$\nabla h(a^\circ; a^\bullet) \delta = 0 \Rightarrow \delta_\ell = 0.$$

Reorganizing this $n_B \cdot n_C$ vector into a $n_B \times n_C$ matrix gives

$$\text{mat}(\nabla h(a^\circ; a^\bullet) \delta) = 0 \Rightarrow \delta_\ell = 0,$$

and combining it with (5.10) then yields (5.12). $\square$

Building on Theorem 5.12, Proposition 5.13 establishes the equivalence between global and local identifiability in case of full excitation (resp. measurement).

Proposition 5.13. *Consider a network $A = A^\circ + A^\bullet$ with all nodes excited, and measurements matrix C (resp. all nodes measured, and excitations matrix B). For all $a = (a^\circ, a^\bullet) \in \mathbb{C}^{m^\circ + m^\bullet}$, each unknown module A_{ij}° is globally identifiable at a if and only if it is locally identifiable at a .*

Proof. Consider that all nodes are excited, i.e. $B = I$, and assume that A_{ij} is globally identifiable at $a = (a^\circ, a^\bullet)$. Definition 4.14 states that for all $\tilde{a}^\circ \in \mathbb{C}^{m^\circ}$, there holds

$$C \left(I - \bar{A}(\tilde{a}^\circ; a^\bullet) \right)^{-1} = C \left(I - \bar{A}(a^\circ; a^\bullet) \right)^{-1} \Rightarrow a_\ell^\circ = \tilde{a}_\ell^\circ, \quad (5.13)$$

where ℓ is the index corresponding to A_{ij} in vector a° . We pose $\Delta^\circ \triangleq \bar{A}(\tilde{a}^\circ; a^\bullet) - \bar{A}(a^\circ; a^\bullet)$, and drop the arguments $(a^\circ; a^\bullet)$ to lighten notations. Equation (5.13) becomes

$$\begin{aligned} C(I - \bar{A} - \Delta^\circ)^{-1} &= C(I - \bar{A})^{-1} && \Rightarrow \Delta_{ij}^\circ = 0 \\ \Leftrightarrow C &= C(I - \bar{A})^{-1}(I - \bar{A} - \Delta^\circ) && \Rightarrow \Delta_{ij}^\circ = 0 \\ \Leftrightarrow C &= C(I - \bar{A})^{-1}(I - \bar{A}) - C(I - \bar{A})^{-1}\Delta^\circ && \Rightarrow \Delta_{ij}^\circ = 0 \\ \Leftrightarrow C &= C \cdot I - C(I - \bar{A})^{-1}\Delta^\circ && \Rightarrow \Delta_{ij}^\circ = 0 \\ \Leftrightarrow C(I - \bar{A})^{-1}\Delta^\circ &= 0 && \Rightarrow \Delta_{ij}^\circ = 0 \\ \Leftrightarrow C(I - \bar{A})^{-1}\Delta^\circ(I - \bar{A})^{-1}B &= 0 && \Rightarrow \Delta_{ij}^\circ = 0, \end{aligned}$$

the last line is obtained because $(I - \bar{A})^{-1}$ is invertible and B is the identity. The last line matches equation (5.12), and is equivalent to generic local identifiability at a . If all nodes are measured, then $C = I$ and a similar reasoning yields the same result. $\square$

We saw in Theorem 5.12 that the local identifiability of an unknown module is a pseudo-generic property. If we consider the local identifiability

of a subset of the unknown modules, it is also pseudo-generic by Lemma 2.5: any boolean combination of pseudo-generic properties is pseudo-generic. But if we consider the local identifiability of *all* unknown modules, the property becomes *generic*, as shown in Corollary 5.14. Hence we have the following handy consequence of generic properties: if a network is locally identifiable for one a_0 , then it is locally identifiable for almost all a . We have the following *algebraic necessary and sufficient characterization for local identifiability of all unknown modules*: $\text{rank } K(a) = m^\circ$ for almost all a if and only if A° is locally identifiable at almost all a . Corollary 5.14 is a more precise version of this statement: it provides characterization (ii) when (i) does not hold.

Corollary 5.14. *Consider a network $A = A^\circ + A^\bullet$ with excitations matrix B , and measurements matrix C . Exactly one of the two following statements holds:*

- (i) $\text{rank } K(a) = m^\circ$ for almost all a and the network is locally identifiable at almost all a .
- (ii) $\text{rank } K(a) < m^\circ$ for all a and the network is locally non-identifiable at all a ,

where $a = (a^\circ, a^\bullet)$ contains the unknown and known modules. Besides, $\text{rank } K(a) = m^\circ$ is equivalent to the following implication holding $\forall \Delta^\circ \in \mathbb{C}^{L \times L}$ with same zero entries as A° :

$$C \bar{T}(a^\circ; a^\bullet) \Delta^\circ \bar{T}(a^\circ; a^\bullet) B = 0 \Rightarrow \Delta^\circ = 0. \quad (5.14)$$

The proof of Corollary 5.14 relies on combining Theorem 5.12 for all unknown modules.

Proof. First, $\ker K(a) \perp \mathbf{e}_\ell$ for each ℓ is equivalent to $\ker K(a) = \{0\}$, and therefore $\text{rank } K(a) = m^\circ$ by the rank-nullity theorem. From Corollary 5.5, the full rankness of $K(a) = \nabla h(a^\circ)$ is not only pseudo-generic, but actually generic: either (i) $\text{rank } K(a) = m^\circ$ for almost all a , or (ii) $\text{rank } K(a) < m^\circ$ for all a . In case (ii), for each a there is a direction ℓ for which $\ker K(a) \not\perp \mathbf{e}_\ell$, since the rank is non-full. Applying Theorem 5.12 (ii) for this a yields the non-identifiability of the module corresponding to ℓ at a . We apply this argument for each a , and obtain that in case (ii), the network is non-identifiable for *all* a . Then, (5.12) for each unknown module (i, j) yields (5.14). $\square$

Example 2.1 (continued). We have $\text{rank } K(a) = 2$ for all a except on $\{(a_1^\circ, a_2^\circ, a_1^\bullet, a_2^\bullet) \in \mathbb{C}^4 \mid a_1^\bullet = 0\}$, which is a set of complex dimension 3. From Corollary 5.14, the network is locally identifiable at almost all a .

We observe that condition (5.14) reduces to the necessary and sufficient condition for generic global identifiability in the full excitation (resp. measurement) case developed in [HGB18]. Indeed, since $\tilde{T} = (I - \tilde{A})^{-1}$ is by construction invertible, $C\tilde{T}\Delta^\circ\tilde{T}B = 0$ is equivalent to $C\tilde{T}\Delta^\circ = 0$ when B is the identity matrix (resp. to $\Delta^\circ\tilde{T}B$ when C is the identity matrix), which leads to Theorem 3.6 (resp. Theorem 3.7) in [HGB18].

5.3 Algorithm to test generic local identifiability

The necessary and sufficient conditions of Theorem 5.12 can be tested exactly by symbolic computation, but it becomes rapidly computationally intractable. Hence, we have designed a probability-1 algorithm that exploits the pseudo-generic character of local identifiability, as established in Theorem 5.12. Testing these conditions on a randomly selected matrix A with the network structure will indeed provide the correct result with a probability 1, and can be done very efficiently. The matrix A is generated randomly in the following manner: each entry corresponding to an edge is independently drawn from the uniform distribution on the interval $(0, 1)$. This extends the work of E. J. Davison in the controllability context [Dav77].

Although this idea allows to design an algorithm working with probability 1, an actual implementation could suffer from numerical issues for large networks, e.g. if the randomly selected values are close to the problematic lower-dimensional set. To increase the reliability of our results, the procedure is repeated `nsamples` times, where `nsamples` denotes the number of independent trials performed, as described in Algorithm 2. However, we did not observe any numerical issues in our sanity checks, as detailed below. For small networks, where generic local identifiability could be determined manually, the algorithm consistently produced results identical to our manual analysis. For larger networks of $\mathcal{O}(10^2)$ nodes, we verified robustness by performing the tests multiple times and confirming that each sample yielded identical outcomes. Across the 10^6 tests conducted, no numerical problems were detected. Although no numerical issues were observed during our sanity checks, we recommend using a value of `nsamples` = 10 for

networks of less than 500 nodes, to ensure robustness. For larger-scale networks, where computational cost becomes more significant, we recommend smaller values such as $n_{\text{samples}} = 1$ or 2.

We know from Corollary 5.14 that the local identifiability of the *whole* network is a generic property: if there is a sample for which matrix K has full column rank, then it has full column rank almost everywhere, and the network is generically locally identifiable (line 6). If the whole network is not generically locally identifiable, then the algorithm returns the generic local identifiability of each module. Theorem 5.12 ensures that the local identifiability of a *single* module (or a subset of the unknown modules) is *pseudo-generic*: it holds either almost everywhere, or almost nowhere. In order to compute the value taken by most samples, we average the values taken by all samples and round the result (lines 11-12). Our implementation is available at our GitHub repository [LH25], and includes a graphical representation of the results.

At this stage, the reader might understandably be confused by the simultaneous use of the terms *generic* and *pseudo-generic*. We therefore provide the following clarifications. Algorithm 2 is designed to determine the *generic* local identifiability of the unknown modules, as formally defined in Definition 4.15. In order to do so, the algorithm relies on its *pseudo-generic* character, as established in Theorem 5.12.

Regarding time complexity, the bottlenecks are the computation of rank K at line 5 and the computation of the basis of $\ker K$ at line 7: $\mathcal{O}(n_B \cdot n_C \cdot m^\circ \cdot \max(n_B n_C, m^\circ))$ for both, and experiments show that computing the basis of $\ker K$ is the most costly operation. Algorithm 2 typically takes 3 seconds per sample for an Erdős–Rényi network of 1000 nodes, 1500 modules among which 500 are unknown, 300 excitations and 300 measurements.

Fig. 5.1 shows examples of outputs of Algorithm 2. Fig. 5.1a (left) shows a nontrivial example of network where only one module is generically locally identifiable. Interestingly, switching excitation/measurement between nodes 4 and 5 allows to recover the identifiability of all unknown modules, see Fig. 5.1a (right). This result is not trivial by inspection, but is computed instantaneously by Algorithm 2.

Fig. 5.1b is the example of Section VII in [DSC⁺22], for which the authors provide an excitation set guaranteeing the generic global identifiability of all unknown modules, assuming that all nodes are measured. Algorithm 2

Algorithm 2 Generic local identifiability test

Require: The unknown and known edge topology, matrices B and C ,
 $nsamples$

Result: The generic local identifiability of each unknown module

- 1: Initialize modules to a vector of 0's of size m°
 (1 := generically locally identifiable, 0 := not generically locally identif.)
 - 2: **repeat** $nsamples$ **times:**
 - 3: Randomly generate A , a network matrix with nonzero structure dictated by the graph topology. Each nonzero entry of A is drawn independently from $\mathcal{U}(0,1)$.
 - 4: Build $K = (B^\top T^\top \otimes CT) I^\circ$, where $T = (I - A)^{-1}$
 - 5: **if** rank $K = m^\circ$ **then**
 - 6: **return** "Network generically locally identifiable"
 - 7: **else** Compute V , basis of $\ker K$
 - 8: Calculate v , a binary vector of size m° , with
 - 9: $v_i = 1$ if i -th entry of each vector of V is 0
 - 10: $= 0$ otherwise
 - 11: modules $\leftarrow$ modules + $v/nsamples$
 - 12: **return** round(modules)
-

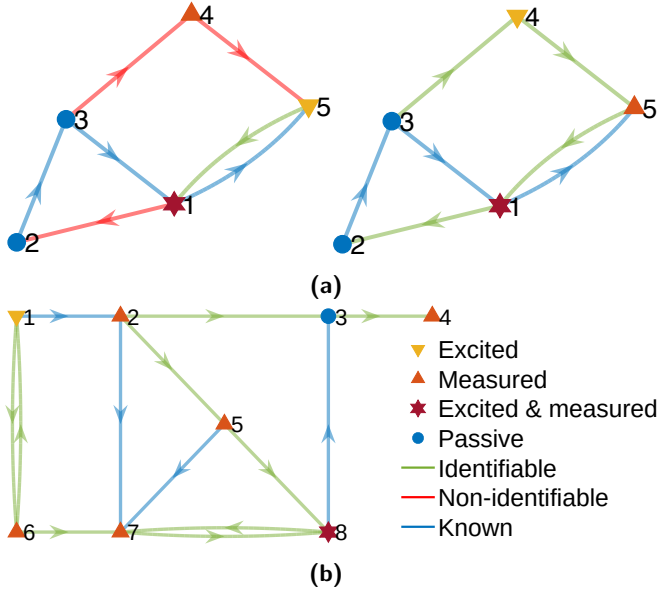


Fig. 5.1 Example of outputs from Algorithm 2: In (a), three modules cannot be identified (left subfigure), but switching excitation/measurement between nodes 4 and 5 allows to recover the identifiability of all unknown modules (right subfigure). (b) is an improvement on the example of Section VII in [DSC⁺22] showing that not all nodes need to be measured.

shows that all unknown modules can still be generically locally identified if nodes 1 and 3 are not measured, i.e. with less data than in [DSC⁺22].

5.4 Summary

To summarize the results of this chapter, we have derived algebraic necessary and sufficient conditions for generic local identifiability (i) for a single unknown module: Theorem 5.12, (ii) for all unknown modules: Corollary 5.14. We have established that local identifiability is a pseudo-generic property. Moreover, when all unknown modules are considered jointly, this property strengthens and becomes generic. These conditions can be easily checked numerically by Algorithm 2, implemented at [LH25].

These algebraic conditions allow rapidly testing generic local identi-

fiability for any given network, but do not provide any insight on the dynamics occurring in the network. We wish to find a combinatorial characterization for generic identifiability, that is expressed purely in terms of graph-theoretical properties, akin to what was done in the full excitation case e.g. in [HGB18]. Such characterization would in particular pave the way for optimizing the selection of nodes to be excited and measured, akin to the works in the full measurement case [CSVdH21, DSC⁺22].

6

Coupled, decoupled and separable Excited and Measured networks

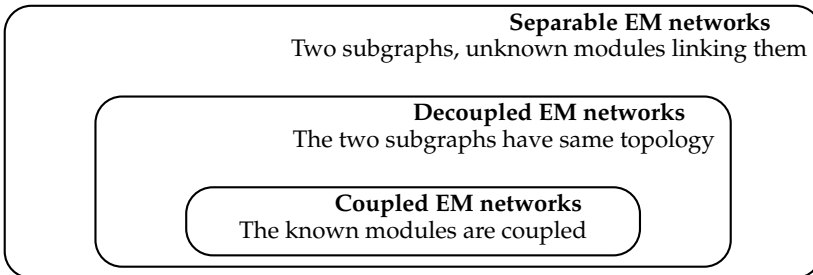
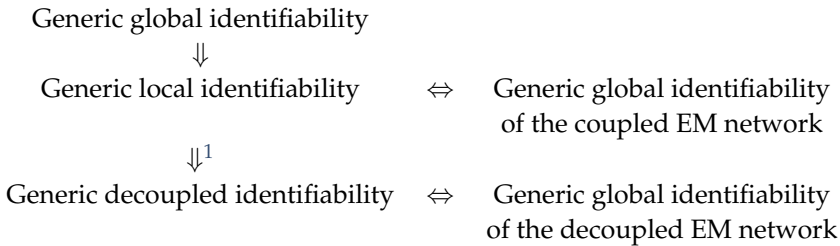
THEOREM 5.12 gives an algebraic characterization for generic local identifiability, that enables rapid testing with Algorithm 2. However, this characterization does not provide any understanding on the dynamics occurring in the network. In this chapter, we show how graphical insight can be obtained from equation (5.12). A quadruple $(A^\circ, A^\bullet, B, C)$ is called an EM network, for Excited and Measured network.

First, we show that equation (5.12) can be expressed as the global identifiability of an associated EM network, called the *coupled EM network*. The coupled EM network is obtained by duplicating the initial graph, applying the excitations on one copy, taking the measurements on the other copy and adding the unknown modules between the two copies, from the excited copy to the measured one. All the other modules, inside each copy, are known, and are *coupled*: each module of one copy must be equal to its corresponding module in the other copy. We will establish in Proposition 6.6 that generic local identifiability is equivalent to generic global identifiability of the associated coupled EM network.

The coupling constraint in the coupled EM network poses difficulties: it lacks a physical interpretation and is challenging to formalize. Relax-

ing these coupling relations gives birth to the *decoupled EM network*. The decoupled EM network is simply the coupled EM network with the coupling relations removed. The generic global identifiability of the decoupled network is denoted as generic decoupled identifiability. By construction, generic decoupled identifiability is necessary for generic local identifiability.

Allowing the two subgraphs of the decoupled EM network to have different topologies gives rise to the class of *separable EM networks*. Separable EM networks generalize decoupled networks and possess the appealing property that generic global identifiability is equivalent to generic local identifiability. This property enables to address generic global identifiability with the algebraic tools of Chapter 5.



Chapter based on [LH24] and its preliminary versions [LH21a, LH23].

¹When identifiability of all unknown modules is jointly considered.

6.1 The coupled and decoupled EM networks

Consider an EM network $(A^\circ, A^\bullet, B, C)$, with $A = A^\circ + A^\bullet$: its coupled network is obtained by duplicating A , applying the excitations on one copy, taking the measurements on the other copy and adding the unknown modules between the two copies, from the excited copy to the measured one. All the other modules, inside each copy, are known, and are *coupled*: each module of one copy must be equal to its corresponding module in the other copy. A formal definition is given below and it is illustrated in Fig. 6.1. The coupled network of Example 2.1 is in Fig. 6.2.

Definition 6.1. Consider an EM network $(A^\circ, A^\bullet, B, C)$ of L nodes, with $A = A^\circ + A^\bullet$. Its *coupled EM network*, $(\check{A}^\circ, \check{A}^\bullet, \check{B}, \check{C})$ is comprised of $2L$ nodes $\{1, \dots, L, 1', \dots, L'\}$, and its network matrices are as follows

$$\begin{aligned} \check{A}^\circ &= \begin{bmatrix} 0 & 0 \\ A^\circ & 0 \end{bmatrix}, & \check{A}^\bullet &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \\ \check{B} &= \begin{bmatrix} B \\ 0 \end{bmatrix}, & \check{C} &= \begin{bmatrix} 0 & C \end{bmatrix}. \end{aligned} \quad (6.1)$$

Note 6.2. Each unknown modules of the initial EM network A gives birth to three modules in its coupled EM network: a known module in the excited copy, a known module in the measured copy, which is equal to the one in the excited copy, and an unknown module between the two copies.

The decoupled EM network is a relaxed variant of the coupled EM network, in which the modules in the excited and measured subgraphs are allowed to differ, while the topology of the two subgraphs remains identical. A formal definition is given below and it is illustrated in Fig. 6.1.

Definition 6.3. Consider an EM network $(A^\circ, A^\bullet, B, C)$ of L nodes, with $A = A^\circ + A^\bullet$. Its *decoupled EM network*, $(\hat{A}^\circ, \hat{A}^\bullet, \hat{B}, \hat{C})$ is comprised of $2L$ nodes $\{1, \dots, L, 1', \dots, L'\}$, and its network matrices are as follows

$$\begin{aligned} \hat{A}^\circ &= \begin{bmatrix} 0 & 0 \\ A^\circ & 0 \end{bmatrix}, & \hat{A}^\bullet &= \begin{bmatrix} A_B & 0 \\ 0 & A_C \end{bmatrix}, \\ \hat{B} &= \begin{bmatrix} B \\ 0 \end{bmatrix}, & \hat{C} &= \begin{bmatrix} 0 & C \end{bmatrix}, \end{aligned} \quad (6.2)$$

where $A_B, A_C \in \mathbb{C}^{L \times L}$ have the same nonzero structure as A .

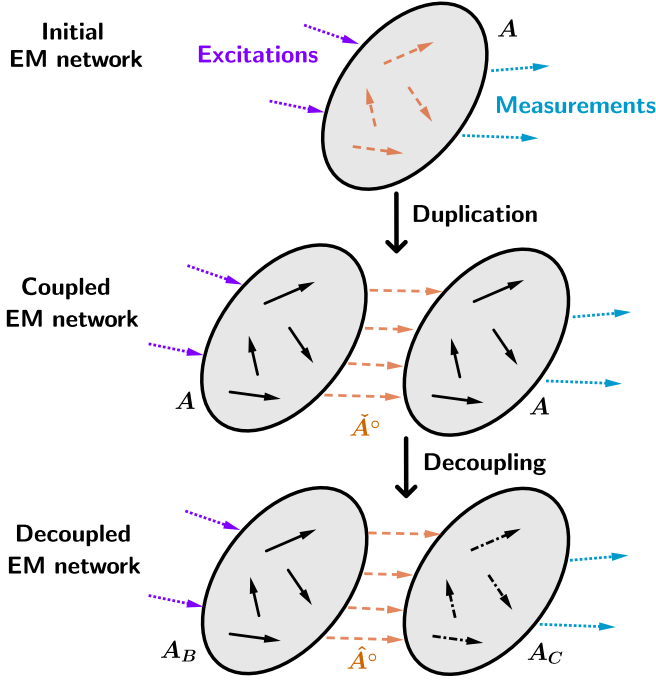


Fig. 6.1 Illustration of the coupled and decoupled EM networks. The initial network is duplicated, excitations are applied on the left copy, measurements taken on the right one. The unknown modules (dashed orange) link the excited and measured subgraphs. All other modules are known. Dotted-dashed arrows mean their modules differ from their plain copies.

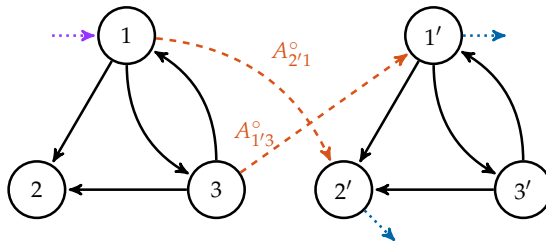


Fig. 6.2 The (de)coupled EM network of Example 2.1, represented in Fig. 2.1. Black plain arrows represent known modules, orange dashed arrows represent unknown ones and purple (resp. blue) dotted arrows are excitations (resp. measurements). Adding constraint $A_{ij} = A_{i'j'}$ gives the coupled EM network, otherwise it is the decoupled EM network.

6.2 Separable networks

We now introduce *separable EM networks*, which are a generalization of the decoupled EM network, where the excited and measured subgraphs are not required to have the same topology anymore. Our focus on separable EM networks is motivated by two considerations. First, every decoupled EM network is, by definition, separable. Second, despite attempts, we were not able to fully exploit the additional structure of decoupled EM networks in our analysis. For these reasons, it is natural to conduct the analysis in this section for the more general class of separable EM networks. Alternatively, separable EM networks can be defined as a class of EM networks for which excitations and measurements can be isolated in two distinct subgraphs, with no known module linking the two subgraphs. Between the two subgraphs lie all the unknown modules, going from the excited subgraph A_B to the measured one A_C . A formal definition in terms of network matrices is given in Definition 6.4, and an example is given in Fig. 6.3.

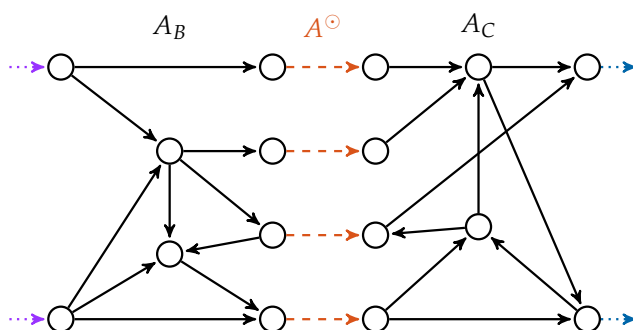


Fig. 6.3 An example of separable EM network: the excitations (dotted purple) are isolated in a subgraph where all modules are known, and the measurements (dotted blue) in another subgraph of known modules. The unknown modules (dashed orange) link the excited subgraph to the measured one. One can check that this EM network is generically globally identifiable.

Definition 6.4. A *separable EM network* is an EM network which nodes can be separated in two sets of nodes of size L_B and L_C , such that the network

matrices have the following block structure:

$$A^\circ = \begin{bmatrix} 0 & 0 \\ A^\odot & 0 \end{bmatrix}, \quad A^\bullet = \begin{bmatrix} A_B & 0 \\ 0 & A_C \end{bmatrix}, \quad B = \begin{bmatrix} B^\diamond \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & C^\diamond \end{bmatrix}, \quad (6.3)$$

where $A^\odot \in \mathbb{C}^{L_C \times L_B}$, $A_B \in \mathbb{C}^{L_B \times L_B}$, $A_C \in \mathbb{C}^{L_C \times L_C}$, $B^\diamond \in \mathbb{C}^{L_B \times n_B}$ and $C^\diamond \in \mathbb{C}^{n_C \times L_C}$.

We have the following important property: on separable EM networks, global and local identifiability are equivalent generically.

Proposition 6.5. *Consider a separable EM network $(A^\circ, A^\bullet, B, C)$. Each of its unknown modules A_{ij}° is generically locally identifiable if and only if it is generically globally identifiable.*

Proof. Consider a separable EM network: its matrices $(A^\circ, A^\bullet, B, C)$ have the block structure described in (6.3). Take $(a^\circ, a^\bullet)$ at which the EM network is globally identifiable. From Definition 4.14, we have: for all $\tilde{a}^\circ \in \mathbb{C}^{m^\circ}$, there holds

$$C \left(I - \bar{A}(a^\circ; a^\bullet) \right)^{-1} B = C \left(I - \bar{A}(\tilde{a}^\circ; a^\bullet) \right)^{-1} B \Rightarrow a_\ell^\circ = \tilde{a}_\ell^\circ, \quad (6.4)$$

where ℓ is the index corresponding to A_{ij} in vector a° , and $\bar{A}(a^\circ; a^\bullet)$ rearranges its arguments $(a^\circ; a^\bullet)$ in the network structure of $A = A^\circ + A^\bullet$. Denote $\bar{A}_B, \bar{A}_C$ and $\bar{A}^\odot$ such that

$$\bar{A}(a^\circ; a^\bullet) = \begin{bmatrix} \bar{A}_B(a^\bullet) & 0 \\ \bar{A}^\odot(a^\circ) & \bar{A}_C(a^\bullet) \end{bmatrix}.$$

Developing (6.4) from the block structure of (6.3) yields

$$\begin{aligned} & C^\diamond (I - \bar{A}_C(a^\bullet))^{-1} \bar{A}^\odot(a^\circ) (I - \bar{A}_B(a^\bullet))^{-1} B^\diamond \\ &= C^\diamond (I - \bar{A}_C(a^\bullet))^{-1} \bar{A}^\odot(\tilde{a}^\circ) (I - \bar{A}_B(a^\bullet))^{-1} B^\diamond \\ &\quad \Rightarrow a_\ell^\circ = \tilde{a}_\ell^\circ. \end{aligned}$$

Bringing out common terms gives

$$C^\circ (I - \bar{A}_C(a^\bullet))^{-1} \underbrace{(\bar{A}^\circ(a^\circ) - \bar{A}^\circ(\tilde{a}^\circ))}_{\triangleq \Delta} (I - \bar{A}_B(a^\bullet))^{-1} B^\circ = 0$$

$$\Rightarrow \underbrace{a_\ell^\circ - \tilde{a}_\ell^\circ}_{=\Delta_{ij}} = 0,$$

which is exactly what we obtain by developing (5.12) with the block structure of (6.3). To conclude, we know from Theorem 5.12 that (5.12) is a necessary and sufficient condition for generic local identifiability. $\square$

6.3 Identifiability of the coupled and decoupled networks

In Proposition 6.6, we establish that generically, the global identifiability of the coupled network encodes precisely the algebraic characterization of local identifiability of Theorem 5.12.

Proposition 6.6. *Consider an EM network $(A^\circ, A^\bullet, B, C)$. An unknown module A_{ij}° is generically locally identifiable if and only if in its associated coupled EM network $(\check{A}^\circ, \check{A}^\bullet, \check{B}, \check{C})$, the corresponding unknown module $\check{A}_{ij}^\circ$ is generically globally identifiable.*

Proof. Since the coupled network is a particular case of separable network, Proposition 6.5 assures that the generic global identifiability of its unknown modules is equivalent to their generic local identifiability. Then, formulating the algebraic necessary and sufficient condition of Theorem 5.12 for $\check{A}_{ij}^\circ$, and developing from the block structure of (6.1) in the same fashion as the proof of Proposition 6.5 yields precisely (5.12) for A_{ij} , which is equivalent to its generic local identifiability. $\square$

We introduce the notion of *decoupled identifiability*, which will be proved equivalent to the global identifiability of the decoupled EM network, when considered generically.

Definition 6.7. Consider an EM network $(A^\circ, A^\bullet, B, C)$. An unknown module A_{ij}° is *decoupled-identifiable* at $(a_B^\circ, a_B^\bullet, a_C^\circ, a_C^\bullet)$ if $\forall \Delta^\circ \in \mathbb{C}^{L \times L}$ with same zero entries as A° , there holds

$$C \bar{T}(a_C^\circ; a_C^\bullet) \Delta^\circ \bar{T}(a_B^\circ; a_B^\bullet) B = 0 \Rightarrow \Delta_{ij}^\circ = 0. \quad (6.5)$$

The module A_{ij}° is *generically* decoupled-identifiable if it is decoupled-identifiable at almost all $(a_B^\circ, a_B^\bullet, a_C^\circ, a_C^\bullet)$. This definition of decoupled identifiability of a single module extends to a set of modules by replacing Δ_{ij}° in the RHS of (6.5).

Note 6.8. Observe the similarity between equations (5.12) and (6.5): the only difference is that the two $\bar{T}$ matrix functions are not evaluated at the same $(a^\circ, a^\bullet)$ in (6.5). From this perspective, local identifiability can be seen as a special case of decoupled identifiability when $(a_B^\circ, a_B^\bullet) = (a_C^\circ, a_C^\bullet)$.

In Proposition 6.9, we establish that generically, the notion of decoupled identifiability we just introduced encodes the global identifiability of the decoupled EM network.

Proposition 6.9. *Consider an EM network $(A^\circ, A^\bullet, B, C)$. An unknown module A_{ij}° is generically decoupled-identifiable if and only if in its associated decoupled EM network $(\hat{A}^\circ, \hat{A}^\bullet, \hat{B}, \hat{C})$, the corresponding module $\hat{A}_{ij}^\circ$ is generically globally identifiable.*

The proof is very similar to the proof of Proposition 6.6.

Proof. Since the decoupled network is a particular case of separable network, Proposition 6.5 assures that the generic global identifiability of its unknown modules is equivalent to their generic local identifiability. Then, formulating the algebraic necessary and sufficient condition of Theorem 5.12 for $\hat{A}_{ij}^\circ$, and developing from the block structure of (6.2) in the same fashion as the proof of Proposition 6.5 yields precisely (6.5) for A_{ij} , which characterizes its generic decoupled identifiability by definition. $\square$

6.4 Decoupled identifiability and genericity

Similar to local identifiability, the decoupled identifiability of a single module — or of a subset of the unknown modules — is a pseudo-generic property (see Definition 2.4): it either holds for almost all (a_B, a_C) , or for almost none, where $a_B = (a_B^\circ, a_B^\bullet)$ and $a_C = (a_C^\circ, a_C^\bullet)$. This is formally established in Proposition 6.10. When all unknown modules are considered jointly, decoupled identifiability becomes a generic property, as shown in Corollary 6.11.

To support the proofs of this section, we define the matrix function $\hat{K}$ analogously to K in Definition 5.8, but with different $(a^\circ, a^\bullet)$ for the two $\bar{T}$:

$$\begin{aligned} \hat{K} : \quad \mathbb{C}^{2m^\circ + 2m^\bullet} &\rightarrow \mathbb{C}^{n_B n_C \times m^\circ} : \\ (a_B^\circ, a_B^\bullet, a_C^\circ, a_C^\bullet) &\rightarrow \left(B^\top \bar{T}^\top (a_B^\circ; a_B^\bullet) \otimes C \bar{T} (a_C^\circ; a_C^\bullet) \right) I^\circ, \end{aligned} \quad (6.6)$$

where $\bar{T}, I^\circ$ are defined in Definition 5.8.

Proposition 6.10. *Consider an EM network $(A^\circ, A^\bullet, B, C)$. An unknown module A_{ij}° is decoupled-identifiable at $(a_B^\circ, a_B^\bullet, a_C^\circ, a_C^\bullet)$ if and only if*

$$\ker \hat{K}(a_B, a_C) \perp \mathbf{e}_\ell, \quad (6.7)$$

where $a_B = (a_B^\circ, a_B^\bullet), a_C = (a_C^\circ, a_C^\bullet)$ and ℓ is the index of module A_{ij}° in vector a° . Besides, (6.7) is a pseudo-generic property: it either holds for almost all (a_B, a_C) , or for almost none.

Proof. Vectorizing (6.5) gives

$$\left(B^\top \bar{T} (a_B^\circ; a_B^\bullet)^\top \otimes C \bar{T} (a_C^\circ; a_C^\bullet) \right) \text{vec}(\Delta^\circ) = 0 \Rightarrow \text{vec}(\Delta_{ij}^\circ) = 0,$$

where vec stacks the columns of its $L \times L$ matrix argument into a L^2 vector, see Example 4.2. Then, observing that $\text{vec}(\Delta^\circ) = I^\circ \delta$ yields

$$\hat{K}(a_B, a_C) \delta = 0 \Rightarrow \delta_\ell = 0 \quad \forall \delta \in \mathbb{C}^{m^\circ},$$

which is equivalent to the orthogonality relation (6.7). Then, applying Lemma 5.4 to $\hat{K}$ establishes the pseudo-generic character of (6.7). $\square$

When all unknown modules are considered jointly, decoupled identifiability becomes a generic property, as shown in Corollary 6.11.

Corollary 6.11. *Consider an EM network $(A^\circ, A^\bullet, B, C)$. The network is decoupled-identifiable at $(a_B^\circ, a_B^\bullet, a_C^\circ, a_C^\bullet)$ if and only if*

$$\text{rank } \hat{K}(a_B, a_C) = m^\circ, \quad (6.8)$$

where $a_B = (a_B^\circ, a_B^\bullet), a_C = (a_C^\circ, a_C^\bullet)$. Besides, (6.8) is a generic property:

- (i) either $\text{rank } \hat{K}(a_B, a_C) = m^\circ$ for almost all (a_B, a_C) ,
- (ii) or $\text{rank } \hat{K}(a_B, a_C) < m^\circ$ for all (a_B, a_C) .

Proof. Combining the result of Proposition 6.10 for each unknown module as in the proof of Corollary 5.14 yields (6.8), which is generic by Lemma 5.3. $\square$

6.5 Decoupled identifiability and local identifiability

When the identifiability of all unknown modules is considered jointly, the notion of generic decoupled identifiability introduced in Definition 6.7 is necessary for generic local identifiability.

Proposition 6.12. *Consider an EM network $(A^\circ, A^\bullet, B, C)$. If the network is generically locally identifiable, then it is generically decoupled-identifiable.*

Proof. Take $(a^\circ, a^\bullet)$, at which the whole network A° is locally identifiable. Equation (5.14) holds, and it is decoupled-identifiable at $(a_B^\circ = a^\circ, a_B^\bullet = a^\bullet, a_C^\circ = a^\circ, a_C^\bullet = a^\bullet)$ by definition. We know from Corollary 6.11 that the decoupled identifiability of the entire network is a generic property. Since it holds for one $(a_B^\circ, a_B^\bullet, a_C^\circ, a_C^\bullet)$, then it holds for almost all $(a_B^\circ, a_B^\bullet, a_C^\circ, a_C^\bullet)$. $\square$

Note 6.13. Proposition 6.12 addresses the identifiability of the *whole network* A° , i.e. all unknown modules. The last step of the proof does not hold for a subset of the unknown modules, since their local identifiability is only a pseudo-generic property, not a generic one, see Proposition 6.10.

Example 2.1 (continued).

$$\hat{K}(a_B, a_C) = \begin{bmatrix} 0 & \frac{a_{C1}^\bullet}{(a_{B2}^\circ a_{B1}^\bullet - 1)(a_{C2}^\circ a_{C1}^\bullet - 1)} \\ -1 & \frac{a_{C1}^\bullet(a_{B1}^\circ + a_{B2}^\circ a_{B1}^\bullet)}{(a_{B2}^\circ a_{B1}^\bullet - 1)(a_{C2}^\circ a_{C1}^\bullet - 1)} \end{bmatrix}.$$

We have $\text{rank } \hat{K}(a_B, a_C) = 2$ for all (a_B, a_C) except on $\{(a_B, a_C) \in \mathbb{C}^8 \mid a_{C1}^\bullet = 0\}$, which has complex dimension 7. From Corollary 6.11, the network is generically decoupled-identifiable. This could also have been inferred from Proposition 6.12 and the fact that the network is generically locally identifiable. Besides, from Proposition 6.9, the associated decoupled network is generically globally identifiable.

Algorithm 2, presented in Section 5.3, can be adapted to compute the generic *decoupled* identifiability of an EM network. Instead of randomly

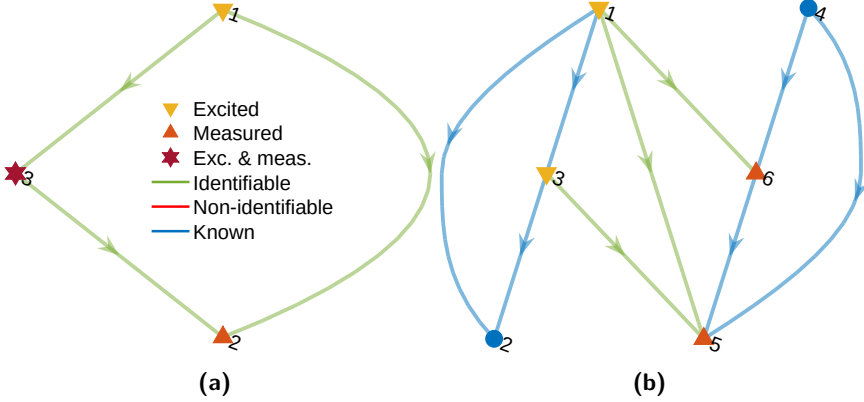


Fig. 6.4 Example of output from Algorithm 2 for generic decoupled identifiability: (a) displays the generic decoupled identifiability of the initial network, (b) displays the generic global identifiability of its decoupled network. Both are equivalent, as established in Proposition 6.9.

generating a single network matrix A and computing K , we randomly generate two matrices A_B and A_C with same nonzero structure, dictated by the graph topology, and compute $\hat{K} = (B^\top (I - A_B)^{-\top} \otimes C (I - A_C)^{-1}) I^\circ$. Calculating the rank of $\hat{K}$ then yields the generic decoupled identifiability of the network, and a basis of $\ker \hat{K}$ gives the generic decoupled identifiability of each unknown module.

There is a feature for the decoupled network in our implementation [LH25]. When the user opts for *decoupled* identifiability, the toolbox displays a graphical representation of both the initial network, *and* its decoupled network. And we know from Proposition 6.9 that a network is generically decoupled-identifiable if and only if its decoupled network is generically globally identifiable. An example of output is given in Fig. 6.4, and our toolbox is available at [LH25].

Proposition 6.12 establishes the necessity of generic decoupled identifiability of the entire network for generic local identifiability of the entire network. Regarding the potential sufficiency: no EM network that is generically decoupled-identifiable, but not generically locally identifiable is known to the authors, despite numerous automatic tests. In other words, experiments seem to show that when the identifiability of all unknown modules is considered jointly, generic decoupled identifiability is equivalent to generic local identifiability.

Conjecture 6.14. *Consider an EM network $(A^\circ, A^\bullet, B, C)$. The network is generically locally identifiable if and only if it is generically decoupled-identifiable.*

6.6 Contribution of this chapter

This chapter served primarily as an exploratory study. While it did not directly yield many actionable results for the main research problem, it provided valuable insights and a deeper understanding of the algebraic conditions developed in Chapter 5.

In particular, this chapter contributed to a better interpretation of Theorem 5.12 through the notion of the coupled EM network. We established, in Proposition 6.6, that generic local identifiability is equivalent to generic global identifiability of the associated coupled EM network.

Furthermore, by relaxing the coupling constraints, we introduced the concept of the decoupled EM network, which naturally led to the definition of separable EM networks. These separable networks have the appealing property that generic global identifiability coincides with generic local identifiability. This property opens the door to addressing generic global identifiability using the algebraic tools developed in Chapter 5.

PART II

**Combinatorial and
walk-based conditions for
identifiability**

7

Combinatorial characterization of network identifiability

THROUGHOUT the remainder of this thesis, starting from this chapter, we focus on the identifiability of *all* unknown modules simultaneously. This property is commonly referred to as the identifiability of the entire network, or simply *network identifiability*.

Corollary 5.14 provides an algebraic characterization of generic local network identifiability, formulated in terms of the rank of a matrix K , as defined in Definition 5.8. However, this algebraic condition offers limited insight into the structural or dynamical behavior of the network.

To bridge this gap, this chapter develops a combinatorial characterization that translates the rank condition of Corollary 5.14 into network-level combinatorial terms. This reveals how network topology and walk-based patterns underpin the identifiability of its modules. This combinatorial perspective lays the foundation for the constructive results and algorithmic methods presented in the subsequent chapters.

This chapter is based on [LH24] and its preliminary version [LH23].

7.1 Identifiability as matrix nonsingularity

A known result from linear algebra is that the rank of a matrix is the size of its largest square submatrix with nonzero determinant [HJ12]. Corollary 5.14 can then be rewritten in terms of determinant.

Corollary 7.1. *Consider an EM network $(A^\circ, A^\bullet, B, C)$. Exactly one of the two following statements holds:*

- (i) *The network is locally identifiable at almost all a , and there is a $m^\circ \times m^\circ$ submatrix of $K(a)$, denoted $K_{\mathcal{P}}(a)$, such that $\det K_{\mathcal{P}}(a) \neq 0$ for almost all a ;*
- (ii) *The network is locally non-identifiable at all a , and for each $m^\circ \times m^\circ$ submatrix of $K(a)$, denoted $K_{\mathcal{P}}(a)$, there holds $\det K_{\mathcal{P}}(a) = 0$ for all a .*

Matrix K , defined in Definition 5.8, is of dimensions $n_B n_C \times m^\circ$. Hence, extracting the submatrix $K_{\mathcal{P}}$ amounts to selecting m° rows of K . By construction of K , each of its rows corresponds to an excitation-measurement pair. We denote $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ the set of pairs selected by $K_{\mathcal{P}}$. This is formally introduced in Definition 7.2, and illustrated in Example 7.3.

Definition 7.2. Consider an EM network with m° unknown modules, excitation set $\mathcal{B}$ and measurement set $\mathcal{C}$. The notation $\mathcal{P}$ denotes a set of excitation-measurement pairs (denoted EM pairs) of cardinality m° :

$$\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}, \quad |\mathcal{P}| = m^\circ. \quad (7.1)$$

Notation $K_{\mathcal{P}}$ refers to the submatrix of K obtained by selecting the rows corresponding to $\mathcal{P}$. Since $|\mathcal{P}| = m^\circ$, $K_{\mathcal{P}}$ is square. We also write

$$\begin{aligned} \mathcal{B}^{\mathcal{P}} &\triangleq \{b \in \mathcal{B} \mid \exists c \in \mathcal{C} : (b, c) \in \mathcal{P}\}, & \mathcal{C}^{\mathcal{P}} &\triangleq \{c \in \mathcal{C} \mid \exists b \in \mathcal{B} : (b, c) \in \mathcal{P}\}, \\ \mathcal{B}_c^{\mathcal{P}} &\triangleq \{b \in \mathcal{B} \mid (b, c) \in \mathcal{P}\}, & \mathcal{C}_b^{\mathcal{P}} &\triangleq \{c \in \mathcal{C} \mid (b, c) \in \mathcal{P}\}, \\ n_{\mathcal{B}c}^{\mathcal{P}} &\triangleq |\mathcal{B}_c^{\mathcal{P}}|, & n_{\mathcal{C}b}^{\mathcal{P}} &\triangleq |\mathcal{C}_b^{\mathcal{P}}|. \end{aligned} \quad (7.2)$$

Example 7.3. Consider the EM network shown in Fig. 7.1. We have

$$\mathcal{P} = \{(1, 14), (2, 13), (2, 14)\} \subset \mathcal{B} \times \mathcal{C} = \{1, 2\} \times \{13, 14\},$$

which gives $\mathcal{B}^{\mathcal{P}} = \mathcal{B} = \{1, 2\}$, $\mathcal{C}^{\mathcal{P}} = \mathcal{C} = \{13, 14\}$, and

$$\begin{aligned} \mathcal{B}_{13}^{\mathcal{P}} &= \{2\}, & \mathcal{B}_{14}^{\mathcal{P}} &= \{1, 2\}, & \mathcal{C}_1^{\mathcal{P}} &= \{14\}, & \mathcal{C}_2^{\mathcal{P}} &= \{13, 14\}, \\ n_{\mathcal{B}13}^{\mathcal{P}} &= 1, & n_{\mathcal{B}14}^{\mathcal{P}} &= 2, & n_{\mathcal{C}1}^{\mathcal{P}} &= 1, & n_{\mathcal{C}2}^{\mathcal{P}} &= 2. \end{aligned}$$

If there are exactly $n_{\mathcal{B}}n_{\mathcal{C}}$ unknown modules, i.e. as many as the number of EM pairs, then $K_{\mathcal{P}} = K$ and $\mathcal{P} = \mathcal{B} \times \mathcal{C}$. If there are more unknown modules, then the network is not identifiable since there are more unknowns than equations available.

7.2 Taylor expansion in the graph

Our combinatorial characterization will be derived from the expansion of $\det K_{\mathcal{P}}$. By definition, the entries of the matrix $K_{\mathcal{P}}$ are functions of the matrix $T(A)$. In this section, we express the entries of $T(A)$ explicitly in terms of the modules A_{ij} . These expressions will serve as the foundation for the combinatorial characterization developed in the following sections.

The analytic matrix $T(A) \triangleq (I - A)^{-1}$ can be expanded in the Taylor series

$$T(A) = (I - A)^{-1} = I + \sum_{k=1}^{\infty} A^k, \quad (7.3)$$

which converges if $|\lambda_i| < 1$ for each eigenvalue λ_i of $A = \bar{A}(a^\circ, a^\bullet)$, with $a = (a^\circ, a^\bullet)$. For ϵ sufficiently small, this is true for $a \in \text{Ball}(0, \epsilon)$, for any network topology. Indeed, consider first the case $a = 0$. Then $A = 0$, and all eigenvalues of A are equal to zero. Since the eigenvalues of a matrix depend continuously on its entries [HJ12], there exists $\epsilon > 0$ such that for all $a \in \text{Ball}(0, \epsilon)$, the spectral radius $\rho(A)$ remains strictly less than one. We assume this holds for the rest of this chapter, and since Theorem 7.9 characterizes a generic property, the fact that it holds on $\text{Ball}(0, \epsilon)$ implies that it holds almost everywhere.

A classical result in graph theory is that $[A^k]_{ij}$ is the sum of all walks of

length k from j to i [Big93]:

$$\left[A^k\right]_{ij} = \sum_{\substack{\text{all } k\text{-walks} \\ j \rightarrow i}} \underbrace{A_{i*} \dots A_{*j}}_{k \text{ terms}} \quad (7.4)$$

where the notation $*$ denotes a node of the network, which can be different for each occurrence of $*$. Combining (7.3) and (7.4) gives the following lemma, which extends Lemma 2 of [CSLVdH24] for networks with cycles.

Lemma 7.4. *Consider the closed-loop transfer matrix $T(A) = (I - A)^{-1}$. Its entries are given as follows:*

1. *if there is no walk from j to i , $T_{ij}(A) = 0$*
2. *otherwise, $T_{ij}(A)$ is the (possibly infinite) sum of all the walks from j to i :*

$$T_{ij}(A) = \sum_{\substack{\text{all walks} \\ j \rightarrow i}} A_{i*} \dots A_{*j}. \quad (7.5)$$

Example 2.1 (continued). $T_{12}(A) = 0$,

$$\begin{aligned} T_{21}(A) &= A_{21} + A_{23}A_{31} + A_{21}A_{13}A_{31} + A_{23}A_{31}A_{13}A_{31} \\ &\quad + A_{21}(A_{13}A_{31})^2 + A_{23}A_{31}(A_{13}A_{31})^2 + \dots \\ &= (A_{21} + A_{23}A_{31}) (1 + A_{31}A_{13} + (A_{31}A_{13})^2 + \dots) \\ &= \frac{A_{21} + A_{23}A_{31}}{1 - A_{31}A_{13}}. \end{aligned}$$

7.3 Definitions on the graph

Definitions 7.5, 7.6 and 7.7 introduce notions for developing our combinatorial characterization, and Example 7.8 illustrates those notions. Those definitions assume a fixed set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° . In the sequel we denote unknown module A_{ij}° with letter α to lighten notations.

Definition 7.5. An *assignment* σ is a function that assigns a pair excitation-measurement to each unknown module:

$$\sigma : \mathcal{E}^\circ \rightarrow \mathcal{P} : \alpha \rightarrow (\sigma_B(\alpha), \sigma_C(\alpha)), \quad (7.6)$$

where $\sigma_B(\alpha)$ is the excitation assigned to α , $\sigma_C(\alpha)$ is its measurement. $\Sigma_{\mathcal{P}}$ is the set of all such *bijective* assignments, i.e. those for which each EM pair of $\mathcal{P}$ is assigned to one and only one unknown module.

Consider a bijective assignment $\sigma_1 \in \Sigma_{\mathcal{P}}$. Swapping the images of two elements, i.e., exchanging $\sigma_1(\alpha_i)$ and $\sigma_1(\alpha_j)$, yields another bijective assignment $\sigma_2 \in \Sigma_{\mathcal{P}}$. Such an operation is called a *transposition*. Fix an arbitrary bijective assignment $\sigma_{\mathcal{P}_{\text{ref}}} \in \Sigma_{\mathcal{P}}$ as a reference. Any assignment $\sigma \in \Sigma_{\mathcal{P}}$ can be obtained from $\sigma_{\mathcal{P}_{\text{ref}}}$ through a sequence of transpositions. Although the specific sequence is not unique, the number of transpositions used always has the same parity — either always even or always odd — regardless of the decomposition [DF⁺04]. The signature function

$$\text{sgn} : \Sigma_{\mathcal{P}} \rightarrow \{-1, 1\} \quad (7.7)$$

equals $+1$ if the number of transpositions used to obtain the considered assignment is even, and -1 otherwise.

Definition 7.6. A collection of walks ω is a set of m° walks ω_i : one ω_i for each unknown module α_i . Each walk ω_i is a sequence of connected edges that starts at an excited node b , goes through unknown module α_i , and ends at a measured node c , such that $(b, c) \in \mathcal{P}$. Its *B-part* is the subsequence between b and $s(\alpha_i)$, and its *C-part* is the subsequence between $t(\alpha_i)$ and c . On separable EM networks, each ω_i goes through one and only one unknown module α_i . There can be several different collections of walks corresponding to the same assignment. Ω_σ is the set of all ω corresponding to assignment σ , and $\text{sgn}(\omega)$ is the signature of the assignment corresponding to ω , as defined in (7.7).

We say that ω is *bijective* if no two walks of ω start at the same excitation and end at the same measurement — equivalently, the assignment σ corresponding to ω is bijective. $\Omega_{\mathcal{P}}$ is the set of all bijective collection of walks ω with set of EM pairs $\mathcal{P}$.

Definition 7.7. Consider a collection of walks ω . Its *monomial* $\mu(\omega)$ is the cumulative product for all unknown modules α_i , of the product of all modules taken by the corresponding walk ω_i , except α_i itself:

$$\mu(\omega) \triangleq \prod_{\alpha_i \in \mathcal{E}^\circ} A_{\omega_C(\alpha_i),*} \cdots A_{*,t(\alpha_i)} \cdot A_{s(\alpha_i),*} \cdots A_{*,\omega_B(\alpha_i)} \quad (7.8)$$

where $\omega_C(\alpha_i)$ and $\omega_B(\alpha_i)$ are respectively the end and start nodes of walk

ω_i , $t(\alpha_i)$ and $s(\alpha_i)$ are respectively the end and start nodes of module α_i , and each module A_{ij} may appear several times in $\mu(\omega)$, which would mean that it is taken several times by ω . On separable EM networks, due to their particular structure, each monomial is comprised of only known modules, and no unknown one.

$\mathcal{M}_{\mathcal{P}}$ is the set of all monomials μ corresponding to bijective collections of walks with set of EM pairs $\mathcal{P}$, and Ω_{μ} is the set of all bijective collections of walks ω corresponding to monomial μ .

We define the *repetition* of a monomial as the aggregate of the signatures of all bijective ω 's corresponding to that monomial.

$$r(\mu) \triangleq \sum_{\omega \in \Omega_{\mu}} \text{sgn}(\omega), \quad (7.9)$$

where $\text{sgn}(\omega)$ is the signature of the assignment corresponding to ω , as defined in (7.7).

Example 7.8. Consider the EM network shown in Fig. 7.1. This example represents a special case: a separable EM network. While this choice helps illustrate certain properties of ω and μ that are specific to separable EM networks, it is important to emphasize that Definitions 7.5, 7.6 and 7.7 apply to EM networks of arbitrary topology, not just the separable ones. Fig. 7.1 (a) represents the following collection of walks:

$$\omega = \begin{cases} \omega_1 = (A_{14,12}^{\bullet}, A_{12,9}^{\bullet}, A_{9,5}^{\circ}, A_{5,3}^{\bullet}, A_{3,1}^{\bullet}) \\ \omega_2 = (A_{14,10}^{\bullet}, A_{10,6}^{\circ}, A_{6,3}^{\bullet}, A_{3,2}^{\bullet}) \\ \omega_3 = (A_{13,12}^{\bullet}, A_{12,11}^{\bullet}, A_{11,7}^{\circ}, A_{7,2}^{\bullet}) \end{cases}$$

Observe that since the considered EM network is separable, each ω_i goes through one and only one unknown module. Besides, ω is bijective and corresponds to the following bijective assignment σ :

	α_1	α_2	α_3
$\sigma(\alpha_i)$	(1, 14)	(2, 14)	(2, 13)

The monomial of ω is given below

$$\mu(\omega) = A_{14,12}^{\bullet} A_{12,9}^{\bullet} A_{5,3}^{\bullet} A_{3,1}^{\bullet} A_{14,10}^{\bullet} A_{6,3}^{\bullet} A_{3,2}^{\bullet} A_{13,12}^{\bullet} A_{12,11}^{\bullet} A_{7,2}^{\bullet} \quad (7.10)$$

Observe that since the considered EM network is separable, $\mu(\omega)$ is comprised of only known modules, and no unknown ones.

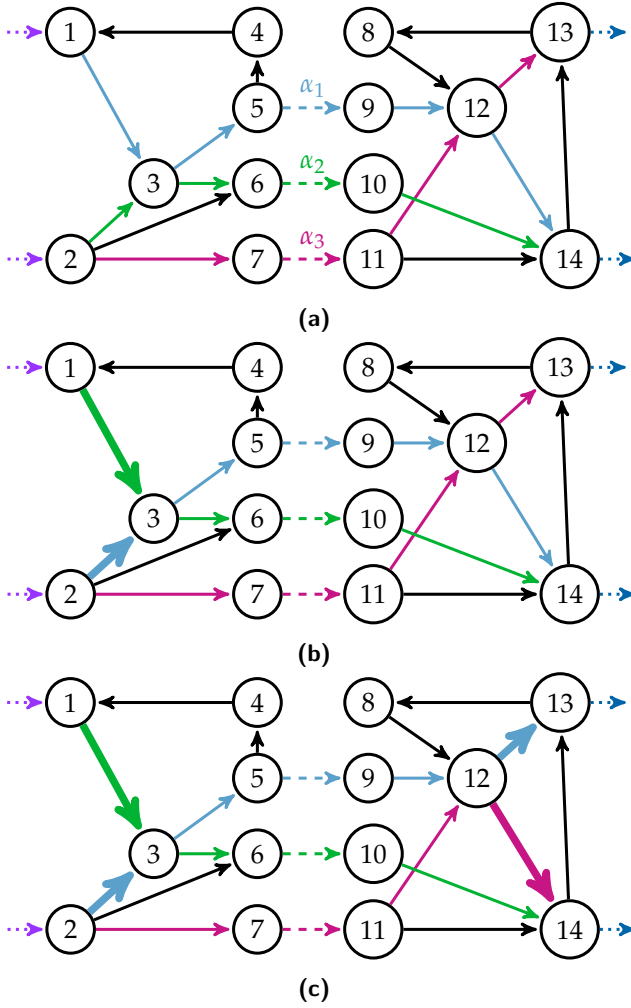


Fig. 7.1 The EM network of Example 7.8: dashed modules are unknown, purple dotted arrows are excitations and blue dotted arrows are measurements. The three subfigures represent the three different bijective collections of walks corresponding to monomial (7.10). Bold arrows highlight the changes between (a) and (b), (c). This example is a special case: a separable EM network. While this choice helps illustrate certain properties of ω and μ that are specific to separable EM networks, it is important to emphasize that Definitions 7.5, 7.6 and 7.7 apply to EM networks of arbitrary topology, not just the separable ones.

7.4 Determinant as sum of monomials

With Corollary 7.1, the expression of $T_{ij}(A)$ from Lemma 7.4, and the graph-theoretic definitions introduced in Section 7.3, we are now equipped to derive our combinatorial characterization from the expression of the matrix K as defined in Definition 5.8. We consider a fixed set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ with cardinality m° , such that the corresponding submatrix $K_{\mathcal{P}}$ is square.

We develop $\det K_{\mathcal{P}}$ as the sum over all possible row-column permutations by Leibniz formula:

$$\det K_{\mathcal{P}} = \sum_{\sigma \in \Sigma_{\mathcal{P}}} \text{sgn}(\sigma) \prod_{\alpha \in \mathcal{E}^\circ} T_{\sigma_C(\alpha), t(\alpha)} T_{s(\alpha), \sigma_B(\alpha)} \quad (7.11)$$

where $s(\alpha)$ and $t(\alpha)$ are respectively the start and end nodes of edge α , and each row-column permutation corresponds to a *bijective* assignment $\sigma \in \Sigma_{\mathcal{P}}$, introduced in Definition 7.5.

The Leibniz formula can be further developed by substituting (7.5) in the equation (7.11), thereby making the network modules A_{ij} appear explicitly:

$$\det K_{\mathcal{P}} = \sum_{\sigma \in \Sigma_{\mathcal{P}}} \text{sgn}(\sigma) \prod_{\alpha \in \mathcal{E}^\circ} \left[\left(\sum_{\substack{\text{all walks} \\ t(\alpha) \rightarrow \sigma_C(\alpha)}} A_{\sigma_C(\alpha), *}} \cdots A_{*, t(\alpha)} \right) \cdot \left(\sum_{\substack{\text{all walks} \\ \sigma_B(\alpha) \rightarrow s(\alpha)}} A_{s(\alpha), *} \cdots A_{*, \sigma_B(\alpha)} \right) \right]$$

Distributing the product over all unknown modules α gives the monomials of all bijective collections of walks corresponding to the bijective assignment σ , i.e. all $\omega \in \Omega_\sigma$ (see Definition 7.6 for more details about those notions):

$$\det K_{\mathcal{P}} = \sum_{\sigma \in \Sigma_{\mathcal{P}}} \text{sgn}(\sigma) \sum_{\omega \in \Omega_\sigma} \mu(\omega), \quad (7.12)$$

where $\mu(\omega)$ is the *monomial* corresponding to ω , introduced in Definition 7.7. From (7.12), we group by monomials of same ω : instead of summing

over σ and then over ω , we sum directly over ω :

$$\det K_{\mathcal{P}} = \sum_{\omega \in \Omega_{\mathcal{P}}} \text{sgn}(\omega) \mu(\omega), \quad (7.13)$$

where $\Omega_{\mathcal{P}}$ is the set of all bijective collections of walks ω , and $\text{sgn}(\omega)$ is the signature of the assignment corresponding to ω .

Observe that from a same group of modules, it may be possible to build different bijective ω (see the three ω 's of Fig. 7.1), such that these different ω 's have the same monomial μ . Regrouping the same monomials together yields

$$\det K_{\mathcal{P}} = \sum_{\mu \in \mathcal{M}_{\mathcal{P}}} \left(\sum_{\omega \in \Omega_{\mu}} \text{sgn}(\omega) \right) \mu, \quad (7.14)$$

where $\mathcal{M}_{\mathcal{P}}$ and Ω_{μ} are introduced in Definition 7.7. Equation (7.14) can be rewritten concisely using the *repetition* $r(\mu)$ defined in (7.9):

$$\det K_{\mathcal{P}} = \sum_{\mu \in \mathcal{M}_{\mathcal{P}}} r(\mu) \mu. \quad (7.15)$$

Equation (7.15) plays a central role in our analysis. As established in Corollary 7.1, generic local network identifiability is characterized by the determinants of square submatrices of K , denoted $K_{\mathcal{P}}$. Equation (7.13) expresses $\det K_{\mathcal{P}}$ as a sum over all bijective collections of walks ω , each weighted by its signature. Grouping together all ω corresponding to the same monomial μ leads to the more structured form in Equation (7.15).

7.5 Combinatorial characterization

The repetition $r(\mu)$ accounts for all ω corresponding to monomial μ , and plays a crucial role in the combinatorial characterization below. By combining Equation (7.15) with Corollary 7.1, we obtain a necessary and sufficient combinatorial condition for generic local identifiability of the entire network, formulated in terms of $r(\mu)$.

Theorem 7.9. *Consider an EM network $(A^{\circ}, A^{\bullet}, B, C)$. The network is generically locally identifiable if and only if there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of*

cardinality m° such that the associated set of monomials $\mathcal{M}_{\mathcal{P}}$ contains at least one monomial μ with nonzero repetition, i.e., $r(\mu) \neq 0$.

Proof. From Corollary 7.1, we know that A° is generically locally identifiable if and only if there is a $m^\circ \times m^\circ$ submatrix of $K(a)$, denoted $K_{\mathcal{P}}(a)$, such that $\det K_{\mathcal{P}}(a) \neq 0$ for almost all a . Selecting one such submatrix $K_{\mathcal{P}}$ amounts to selecting a subset of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° . Equation (7.15) expresses $\det K_{\mathcal{P}}$ as a sum over all monomials μ of the set $\mathcal{M}_{\mathcal{P}}$, weighted by their repetition $r(\mu)$. A sum of distinct monomials is generically nonzero if and only if at least one of the monomials has a nonzero coefficient. Therefore, for this sum to be nonzero for almost all a , at least one μ must have a nonzero repetition $r(\mu)$. $\square$

Example 7.8 (continued). Consider the EM network depicted in Fig. 7.1, and the monomial μ expressed in (7.10). Its set Ω_μ is comprised of three bijective collections of walks, represented in Fig. 7.1, we denote them $\omega^a, \omega^b, \omega^c$ respectively. Fix $\sigma_{\mathcal{P}_{\text{ref}}}$ as the assignation corresponding to ω^a . We have

$$r(\mu) = \sum_{\omega \in \Omega_\mu} \text{sgn}(\omega) = \text{sgn}(\omega^a) + \text{sgn}(\omega^b) + \text{sgn}(\omega^c) = 1 - 1 + 1 = 1,$$

and Theorem 7.9 ensures that this network is generically locally identifiable.

Theorem 7.9 provides a necessary and sufficient combinatorial characterization of generic local network identifiability, expressed in terms of monomial repetitions in the graph. These monomials represent collections of walks, and their repetition aggregates their associated signatures. While this characterization is exact, it is not directly exploitable for algorithmic purposes due to its combinatorial complexity. The subsequent chapters aim to extract more practical, tractable conditions from this framework, ultimately leading to algorithmic methods developed in Part III.

8

Walk-based conditions

BUILDING on the combinatorial framework established in Chapter 7, this chapter aims to translate the abstract characterization of identifiability into more tangible, graph-theoretic conditions.

We begin by deriving direct consequences of Theorem 7.9, which lay the groundwork for the more refined results that follow. To deepen those results, we then introduce a decomposition of the assignment function σ , enabling a finer analysis of the algebraic characterization of Chapter 5.

This refinement leads to the central result of the chapter: Theorem 8.7, which extends earlier results from the full excitation (respectively, measurement) case [HGB18]. The theorem provides a necessary condition for identifiability based on the existence of vertex-disjoint walks, along with a sufficient condition of similar flavor. The proof of Theorem 8.7 relies on two key lemmas. By dissecting the structure of σ at a more granular level, these lemmas reveal the presence of vertex-disjoint walk patterns that arise naturally from the algebraic framework.

The structural insight offered by Theorem 8.7 gives rise to a conjecture suggesting a possible strengthening of the result. Although this conjecture appears plausible, it is disproven by a counterexample. This refutation not only clarifies the limitations of the current approach but also underscores the need for deeper combinatorial insight to fully resolve the network identifiability question in the general case of partial excitation and measurement. This chapter is based on [LH21a, LH21b].

8.1 Conditions with collections of walks

A first set of conditions can be directly extracted from Theorem 7.9. These include a necessary condition and a sufficient one, formulated in terms of the existence and uniqueness of bijective collections of walks. Together, they form the conceptual basis for the analysis developed in this chapter and will be refined in the sections that follow.

Proposition 8.1. *Consider an EM network $(A^\circ, A^\bullet, B, C)$. We have the two following conditions for generic local network identifiability:*

- (i) *If the network is generically locally identifiable, then there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° such that there is at least one bijective collection of walks $\omega \in \Omega_{\mathcal{P}}$, i.e. $|\Omega_{\mathcal{P}}| \geq 1$.*
- (ii) *If there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° such that there is only one bijective collection of walks $\omega \in \Omega_{\mathcal{P}}$, i.e. $|\Omega_{\mathcal{P}}| = 1$, then the network is generically locally identifiable.*

Proof. (i) Assume that A° is generically locally identifiable. Then, from Theorem 7.9, there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° such that the associated set of monomials $\mathcal{M}_{\mathcal{P}}$ contains at least one monomial μ with nonzero repetition, i.e., $r(\mu) \neq 0$. Hence $\Omega_\mu \neq \emptyset$, i.e. there is at least one bijective collection of walks ω corresponding to μ .

(ii) Assume there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° such that there is *only one* bijective collection of walks $\omega \in \Omega_{\mathcal{P}}$. Consider its corresponding monomial μ . We have $r(\mu) = \pm 1$, and the network is generically locally identifiable by Theorem 7.9. □

8.2 Decomposition of an assignment

This section introduces and illustrates key notions that will be essential for the developments in Section 8.3. We begin by showing how the assignment function σ can be decomposed into two components: σ_B , which handles the assignment of excitations, and σ_C , which handles that of measurements.

We then introduce terminology to describe whether a pair (σ_B, σ_C) constitutes a bijective assignment and whether it is compatible with a given set

$\mathcal{P}$. Furthermore, we show how σ_C can be decomposed into sub-assignments σ_{Cb} , one for each excitation, and how their respective signatures are related.

We acknowledge that the definitions introduced in this section are somewhat heavy and technical. To facilitate their understanding, we provide a concrete illustration in Example 8.3 and Fig. 8.1.

Definition 8.2. Consider a fixed set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° . The assignment function σ introduced in Definition 7.5 can be decomposed into a B -assignment and a C -assignment:

$$\sigma_B : \mathcal{E}^\circ \rightarrow \mathcal{B}^\mathcal{P}, \quad \sigma_C : \mathcal{E}^\circ \rightarrow \mathcal{C}^\mathcal{P}. \quad (8.1)$$

- σ_B (resp. σ_C) is *connected* if for each unknown module α , there is a walk from its assigned excitation $\sigma_B(\alpha)$ to its start node $s(\alpha)$ (resp. from its end node $t(\alpha)$ to its assigned measurement $\sigma_C(\alpha)$).
- σ_B and σ_C are said *bijection-pairable* if the resulting assignment $\sigma = (\sigma_B, \sigma_C)$ is bijective.
- σ_B and σ_C are said *$\mathcal{P}$ -compatible* if the resulting assignment $\sigma = (\sigma_B, \sigma_C)$ only assigns EM pairs that belong to $\mathcal{P}$, i.e. $\text{Im } \sigma = \mathcal{P}$.
- $\Sigma_C^{\mathcal{P}\sigma_B}$ is the set of all σ_C bijection-pairable and $\mathcal{P}$ -compatible with σ_B .
- $\sigma_B^{-1}(b) \subseteq \mathcal{E}^\circ$ is the set of unknown modules assigned to excitation b by σ_B .

Consider an excitation b , and its corresponding set $\sigma_B^{-1}(b)$. The restriction of σ_C to this subset of $\mathcal{E}^\circ$ gives the following sub-assignment:

$$\sigma_{Cb} : \sigma_B^{-1}(b) \rightarrow \mathcal{C}_b^\mathcal{P} : \alpha \rightarrow \sigma_C(\alpha), \quad (8.2)$$

which is bijective if σ_B and σ_C are bijection-pairable, with $\mathcal{C}_b^\mathcal{P}$ introduced in Definition 7.2. $\Sigma_B^{\mathcal{P}\sigma_C}, \sigma_C^{-1}(c)$ and σ_{Bc} are defined analogously.

Let $\sigma_{\mathcal{P}_{\text{ref}}}$ be as defined in Definition 7.5, consider an excitation b , and let $\sigma_{\mathcal{P}_{\text{ref}},Cb}$ be its sub- C -assignment corresponding to excitation b . Consider a bijective assignment $\sigma \in \Sigma_{\mathcal{P}}$, and its sub- C -assignment σ_{Cb} . It can be obtained via a sequence of transpositions from $\sigma_{\mathcal{P}_{\text{ref}},Cb}$. The signature $\text{sgn}(\sigma_{Cb})$ is defined as in (7.7): it equals $+1$ if the number of such transpositions is even, and -1 otherwise. The signature $\text{sgn}(\sigma_{Bc})$ is defined analogously. We

define

$$\text{sgn}(\sigma_B) \triangleq \prod_{c \in \mathcal{C}} \text{sgn}(\sigma_{Bc}), \quad \text{sgn}(\sigma_C) \triangleq \prod_{b \in \mathcal{B}} \text{sgn}(\sigma_{Cb}).$$

Example 8.3. Consider the network depicted in Fig. 8.1. We have

$$\mathcal{P} = \{(1, 4), (1, 5), (2, 5), (3, 4), (3, 5)\} \subset \mathcal{B} \times \mathcal{C} = \{1, 2, 3\} \times \{4, 5\}.$$

The colors in Fig. 8.1 indicate the bijective assignment σ given below:

	α_1	α_2	α_3	α_4	α_5
$\sigma(\alpha_i)$	(1, 4)	(2, 5)	(1, 5)	(3, 4)	(3, 5)

Assignment σ can be decomposed into σ_B and σ_C , which are connected, and bijection-pairable and $\mathcal{P}$ -compatible with each other:

	α_1	α_2	α_3	α_4	α_5
$\sigma_B(\alpha_i)$	1	2	1	3	3
$\sigma_C(\alpha_i)$	4	5	5	4	5

From σ_B and σ_C , the following sets can be derived:

$$\begin{aligned} \sigma_B^{-1}(1) &= \{\alpha_1, \alpha_3\}, & \sigma_B^{-1}(2) &= \{\alpha_2\}, & \sigma_B^{-1}(3) &= \{\alpha_4, \alpha_5\}, \\ \sigma_C^{-1}(4) &= \{\alpha_1, \alpha_4\}, & \sigma_C^{-1}(5) &= \{\alpha_2, \alpha_3, \alpha_5\}, \end{aligned}$$

and σ_B and σ_C can be decomposed into the following sub-assignments:

$\sigma_{C,1}(\alpha_i)$	α_1	α_3	$\sigma_{C,2}(\alpha_i)$	α_2	$\sigma_{C,3}(\alpha_i)$	α_4	α_5
	4	5		5		4	5
$\sigma_{B,4}(\alpha_i)$	α_1	α_4	$\sigma_{B,5}(\alpha_i)$	α_2	α_3	α_5	
	1	3		2	1	3	

8.3 Conditions with vertex-disjoint walks

In this section, we present the main result of the chapter: Theorem 8.7. To establish this result, we first introduce two technical lemmas that serve as key components in the proof. We first present Lemma 8.4, which addresses the signature decomposition of the assignment function.

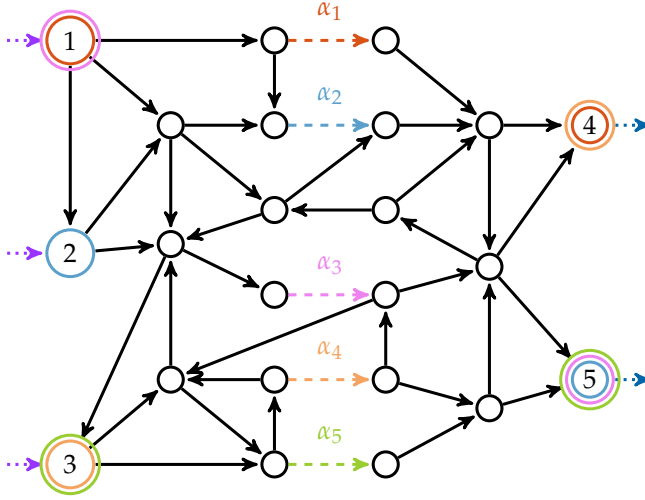


Fig. 8.1 The EM network of Example 8.3: dashed edges represent unknown modules, purple dotted arrows are excitations and blue dotted arrows are measurements. Colors denote the bijective assignment σ of Example 8.3

Lemma 8.4. Let $\sigma \in \Sigma_{\mathcal{P}}$ be a bijective assignment comprised of σ_B and σ_C . Its signature decomposes into

$$\text{sgn}(\sigma) = \text{sgn}(\sigma_B) \text{sgn}(\sigma_C). \quad (8.3)$$

Proof. If $|\text{Im } \sigma_B| = 1$ (resp. $|\text{Im } \sigma_C| = 1$), then $\text{sgn}(\sigma_B) = 1$ (resp. $\text{sgn}(\sigma_C) = 1$) and (8.3) follows. Otherwise, we have $|\text{Im } \sigma_B| \geq 2$ and $|\text{Im } \sigma_C| \geq 2$. Assignment σ can be obtained from $\sigma_{\mathcal{P}_{\text{ref}}}$ by a sequence of transpositions that swap two unknown modules, denote them $\alpha_{1,1}$ and $\alpha_{2,2}$, initially assigned to (b_1, c_1) and (b_2, c_2) respectively. Since the assignment is bijective, either (i) $b_1 = b_2, c_1 \neq c_2$, (ii) $b_1 \neq b_2, c_1 = c_2$, or (iii) $b_1 \neq b_2, c_1 \neq c_2$. Besides, every transposition of case (iii) can be decomposed into transpositions of type (i) and (ii), as shown in Table 8.1.

(b_1, c_1)	$\boxed{\alpha_{1,1}}$	$\xrightarrow{(i)}$	$\alpha_{1,2}$	$\xrightarrow{(ii)}$	$\boxed{\alpha_{1,2}}$	$\xrightarrow{(i)}$	$\alpha_{2,2}$
	$\updownarrow$				$\updownarrow$		
(b_1, c_2)	$\boxed{\alpha_{1,2}}$		$\boxed{\alpha_{1,1}}$		$\boxed{\alpha_{2,2}}$		$\alpha_{1,2}$
			$\updownarrow$				
(b_2, c_1)	$\alpha_{2,1}$		$\alpha_{2,1}$		$\alpha_{2,1}$		$\alpha_{2,1}$
(b_2, c_2)	$\alpha_{2,2}$		$\boxed{\alpha_{2,2}}$		$\alpha_{1,1}$		$\alpha_{1,1}$

Table 8.1 Decomposition of a class-(iii) transposition into class-(i) and -(ii) transpositions. The boxes and arrows indicate the swap operated to obtain the next column.

Hence, any assignment σ can be obtained from $\sigma_{\mathcal{P}\text{ref}}$ by a sequence of class-(i) and -(ii) transpositions only. Take one such sequence, and denote its number of transpositions $\eta(\sigma)$. Within this sequence, consider the transpositions of class (i) with same fixed excitation $b \in \mathcal{B}^{\mathcal{P}}$. Combining these transpositions gives the sub-assignment σ_{Cb} , defined in equation (8.2). The number of such transpositions, denoted $\eta(\sigma_{Cb})$, gives the parity of σ_{Cb} . The same holds for all $b \in \mathcal{B}^{\mathcal{P}}$, and for the class-(ii) transpositions, for each $c \in \mathcal{C}^{\mathcal{P}}$, $\eta(\sigma_{Bc})$ is defined analogously. Enumerating all $\eta(\sigma)$ transpositions of the considered sequence allows to rewrite it as a sum:

$$\eta(\sigma) = \sum_{c \in \mathcal{C}^{\mathcal{P}}} \eta(\sigma_{Bc}) + \sum_{b \in \mathcal{B}^{\mathcal{P}}} \eta(\sigma_{Cb}).$$

Since $\text{sgn}(\sigma) = (-1)^{\eta(\sigma)}$ by definition, this yields

$$\text{sgn}(\sigma) = \prod_{c \in \mathcal{C}^{\mathcal{P}}} (-1)^{\eta(\sigma_{Bc})} \prod_{b \in \mathcal{B}^{\mathcal{P}}} (-1)^{\eta(\sigma_{Cb})}.$$

□

We now introduce Lemma 8.5, which forms the cornerstone of Theorem 8.7. This is where the core of the analysis takes place and the key ideas emerge. By dissecting the expression of $\det K_{\mathcal{P}}$, and leveraging the framework developed in Section 8.2 as well as Lemma 8.4, this result reveals how vertex-disjoint walks naturally arise from the algebraic characterization of Chapter 5.

Lemma 8.5. *Consider an EM network $(A^{\circ}, A^{\bullet}, B, C)$. If the network is generically locally identifiable, then there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° such that there is at least one B-assignment σ_B (resp. C-assignment σ_C) as*

defined in (8.1), such that¹:

- (a) σ_B (resp. σ_C) is connected
- (b) Each excitation $b \in \mathcal{B}^{\mathcal{P}}$ is the image of $n_{\mathcal{C}_b}^{\mathcal{P}}$ unknown modules by σ_B .
- (c) (resp. each measurement $c \in \mathcal{C}^{\mathcal{P}}$ is the image of $n_{\mathcal{B}_c}^{\mathcal{P}}$ unknown modules by σ_C .)
- (d) For each excitation $b \in \mathcal{B}^{\mathcal{P}}$, there are $n_{\mathcal{C}_b}^{\mathcal{P}}$ vertex-disjoint walks starting at the end nodes of the unknown modules assigned to b by σ_B , and ending at the measurements $\mathcal{C}_b^{\mathcal{P}}$.
- (e) (resp. for each measurement $c \in \mathcal{C}^{\mathcal{P}}$, there are $n_{\mathcal{B}_c}^{\mathcal{P}}$ vertex-disjoint walks starting at the excitations $\mathcal{B}_c^{\mathcal{P}}$, and ending at the start nodes of the unknown modules assigned to c by σ_C .)

If there is only one such B -assignment σ_B (resp. C -assignment σ_C), then this condition is also sufficient for generic local network identifiability.

Note 8.6. Conditions (b), (c) are direct consequences of conditions (d), (e) respectively, but we include them explicitly to improve clarity and aid the reader's understanding.

Proof. This proof is derived for σ_B , but an analogous argument can be made for σ_C . Consider an EM network which is generically locally identifiable. Corollary 7.1 yields that there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° such that $\det K_{\mathcal{P}} \neq 0$ generically. As in (7.11), the Leibniz formula for the determinant gives:

$$\begin{aligned} \det K_{\mathcal{P}} &= \sum_{\sigma \in \Sigma_{\mathcal{P}}} \text{sgn}(\sigma) \prod_{\alpha \in \mathcal{E}^\circ} T_{\sigma_C(\alpha), t(\alpha)} T_{s(\alpha), \sigma_B(\alpha)} \\ &\downarrow \text{grouping } \sigma \text{ with same } \sigma_B, \text{ Lemma 8.4 for sgn} \\ &= \sum_{\sigma_B} \text{sgn}(\sigma_B) \sum_{\sigma_C \in \Sigma_C^{\mathcal{P}\sigma_B}} \text{sgn}(\sigma_C) \prod_{\alpha \in \mathcal{E}^\circ} T_{\sigma_C(\alpha), t(\alpha)} T_{s(\alpha), \sigma_B(\alpha)} \end{aligned}$$

where $\Sigma_C^{\mathcal{P}\sigma_B}$ is the set of all σ_C bijection-pairable and $\mathcal{P}$ -compatible with σ_B , defined in Definition 8.2. Bringing out common T terms yields

$$\det K_{\mathcal{P}} = \sum_{\sigma_B} \text{sgn}(\sigma_B) \prod_{\alpha \in \mathcal{E}^\circ} T_{s(\alpha), \sigma_B(\alpha)} \underbrace{\sum_{\sigma_C \in \Sigma_C^{\mathcal{P}\sigma_B}} \text{sgn}(\sigma_C) \prod_{\alpha \in \mathcal{E}^\circ} T_{\sigma_C(\alpha), t(\alpha)}}_{\triangleq \gamma_{\sigma_B}}$$

¹Notations $n_{\mathcal{C}_b}^{\mathcal{P}}, n_{\mathcal{B}_c}^{\mathcal{P}}, \mathcal{C}_b^{\mathcal{P}}, \mathcal{B}_c^{\mathcal{P}}$ are introduced in Definition 7.2

We develop further γ_{σ_B} , the factor accounting for the assignments σ_C bijection-pairable and $\mathcal{P}$ -compatible with σ_B . We decompose the product on all unknown modules α into subgroups of α assigned to the same excitation b in σ_B (subgroups denoted by $\sigma_B^{-1}(b)$, defined in Definition 8.2):

$$\gamma_{\sigma_B} = \sum_{\sigma_C \in \Sigma_C^{\mathcal{P}\sigma_B}} \text{sgn}(\sigma_C) \prod_{b \in \mathcal{B}^{\mathcal{P}}} \prod_{\alpha \in \sigma_B^{-1}(b)} T_{\sigma_C(\alpha), t(\alpha)} \quad (8.4)$$

The C -assignment σ_C can be decomposed sub-assignments σ_{Cb} for the unknown modules α assigned to the same excitation b in σ_B (i.e. the α belonging to $\sigma_B^{-1}(b)$), as illustrated in Example 8.3.

Below is the key step of the proof: those sub-assignments σ_{Cb} do not depend on each other, thus (8.4) can be rewritten by regrouping in the same factor the modules α assigned to the same excitation b in the B -assignment σ_B :

$$\gamma_{\sigma_B} = \prod_{b \in \mathcal{B}^{\mathcal{P}}} \sum_{\sigma_{Cb} \in \Sigma_{Cb}^{\mathcal{P}\sigma_B}} \text{sgn}(\sigma_{Cb}) \prod_{\alpha \in \sigma_B^{-1}(b)} T_{\sigma_{Cb}(\alpha), t(\alpha)}, \quad (8.5)$$

where $\Sigma_{Cb}^{\mathcal{P}\sigma_B}$ denotes the set of sub-assignments σ_{Cb} of assignments $\sigma_C \in \Sigma_C^{\mathcal{P}\sigma_B}$, i.e. bijection-pairable and $\mathcal{P}$ -compatible with σ_B . Furthermore, each sub-assignment $\sigma_{Cb} \in \Sigma_{Cb}^{\mathcal{P}\sigma_B}$ is bijective: since each pair $(b, c) \in \mathcal{P}$ must be assigned to only one unknown module α for bijectivity, each α of $\sigma_B^{-1}(b)$ must be assigned to a different measurement, i.e. σ_{Cb} must be injective. Besides, $\sigma_B^{-1}(b)$ is composed of $n_{Cb}^{\mathcal{P}}$ modules, so σ_{Cb} is bijective.

By definition, $\Sigma_{Cb}^{\mathcal{P}\sigma_B}$ contains all sub-assignments σ_{Cb} of assignments $\sigma_C \in \Sigma_C^{\mathcal{P}\sigma_B}$, i.e. bijection-pairable and $\mathcal{P}$ -compatible with σ_B . Thus $\Sigma_{Cb}^{\mathcal{P}\sigma_B}$ contains all possible bijections σ_{Cb} . Hence, each factor of (8.5) is a determinant:

$$\gamma_{\sigma_B} = \prod_{b \in \mathcal{B}^{\mathcal{P}}} \det T_{C_b^{\mathcal{P}}, \sigma_B^{-1}(b)}.$$

Putting it all together, we have

$$\det K_{\mathcal{P}} = \sum_{\sigma_B} \text{sgn}(\sigma_B) \underbrace{\prod_{\alpha \in \mathcal{E}^{\circ}} T_{s(\alpha), \sigma_B(\alpha)} \prod_{b \in \mathcal{B}^{\mathcal{P}}} \det T_{C_b^{\mathcal{P}}, \sigma_B^{-1}(b)}}_{\triangleq \tau_B(\sigma_B)} \quad (8.6)$$

Since $\det K_{\mathcal{P}} \neq 0$ generically, there is at least one B -assignment σ_B with a

nonzero term $\tau_B(\sigma_B)$. For this assignation, each factor of the product must be nonzero generically, i.e. (i) $T_{s(\alpha), \sigma_B(\alpha)} \neq 0$ for all unknown module α and (ii) $\det T_{C_b^{\mathcal{P}}, \sigma_B^{-1}(b)} \neq 0$ for each excitation $b \in \mathcal{B}^{\mathcal{P}}$.

Lemma 7.4 ensures that T_{ij} is generically nonzero if and only if there is a walk from j to i , so (i) gives condition (a). Besides, (ii) is equivalent to $\text{rank } T_{C_b^{\mathcal{P}}, \sigma_B^{-1}(b)} = n_{C_b}^{\mathcal{P}}$ for all excitation b , and Proposition 3.3 yields condition (d). Finally, condition (b) is a trivial consequence of condition (d). If more than one assignation σ_B have a nonzero term $\tau_B(\sigma_B)$, the condition is not sufficient since terms could cancel each other. But if there is only one such assignation, the condition is also sufficient. $\square$

Theorem 8.7 is the central result of this chapter. It extends Proposition 8.1 by adding conditions in terms of vertex-disjoint walks, thereby offering a stronger statement. The proof builds directly on Lemma 8.5, and essentially consists of constructing a bijective collection of walks ω using the σ_B (respectively, σ_C) obtained from that lemma. This theorem provides a necessary condition for the generic local identifiability of networks, and a sufficient one.

Theorem 8.7. *Consider an EM network $(A^\circ, A^\bullet, B, C)$. If the network is generically locally identifiable, then there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° with at least one bijective collection of walks $\omega \in \Omega_{\mathcal{P}}$ such that one of the two following conditions holds:*

- (A) *Either: for each excitation $b \in \mathcal{B}^{\mathcal{P}}$, the C-parts² of the walks starting at b are vertex-disjoint.*
- (B) *Or: for each measurement $c \in \mathcal{C}^{\mathcal{P}}$, the B-parts of the walks ending at c are vertex-disjoint.*

If there is only one such collection of walks $\omega \in \Omega_{\mathcal{P}}$, then this statement is also sufficient for the generic local identifiability of network A° .

Proof. This proof is derived for an ω satisfying condition (A), from a σ_B obtained from Lemma 8.5. An analogous argument can be made for condition (B), from a σ_C obtained from Lemma 8.5.

Applying Lemma 8.5 to the considered EM network gives a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° such that there is at least one σ_B such that the conditions (a), (b) and (d) of Lemma 8.5 hold. The idea of this proof is to

²C-parts and B-parts are defined in Definition 7.6.

construct a collection of paths ω from this σ_B , and show that it is bijective, it belongs to $\Omega_{\mathcal{P}}$ and it satisfies condition (A) of Theorem 8.7.

Condition (a) ensures that σ_B is connected. Thus, from Definition 8.2, for each unknown module $\alpha_i \in \mathcal{E}^\circ$, there is a walk from its assigned excitation $\sigma_B(\alpha_i)$ to its start node $s(\alpha_i)$. Besides, condition (d) ensures that for each excitation $b \in \mathcal{B}^{\mathcal{P}}$, there are $n_{\mathcal{C}_b}^{\mathcal{P}}$ vertex-disjoint walks starting at the end nodes of the unknown modules assigned to b by σ_B , and ending at the measurements $\mathcal{C}_b^{\mathcal{P}}$. We construct a collection of walks ω from the connectedness of σ_B and its vertex-disjoint walks: for each unknown module $\alpha_i \in \mathcal{E}^\circ$, the first part of the walk ω_i starts at $\sigma_B(\alpha_i)$ and follows the sequence assured by connectedness. Then the walk goes through α_i , follows one of the vertex-disjoint walks, and ends at a node of $\mathcal{C}_{\sigma_B(\alpha_i)}^{\mathcal{P}}$. Collecting the ω_i constructed in this manner for each $\alpha_i \in \mathcal{E}^\circ$ yields the collection of walks ω .

By construction, for each unknown module $\alpha_i \in \mathcal{E}^\circ$, the couple made by the start node and end node of ω_i is an element of $\mathcal{P}$, i.e. $(\omega_B(\alpha_i), \omega_C(\alpha_i)) \in \mathcal{P}$. Besides, if there are i, j such that $\omega_C(\alpha_i) = \omega_C(\alpha_j)$, then by construction $\omega_B(\alpha_i)$ must be different to $\omega_B(\alpha_j)$. Hence each element of $\mathcal{P}$ is not assigned more than once by the assignation σ corresponding to ω , i.e. σ is injective. Also $|\mathcal{E}^\circ| = |\mathcal{P}|$, thus σ is bijective, and so is ω . Finally, by construction, condition (A) follows directly from condition (d).

If there is only one such ω , then there is only one σ_B satisfying (a), (b) and (d), and Lemma 8.5 yields the generic local identifiability of the network. $\square$

8.4 Disproven conjecture

Theorem 8.7 generalizes Theorem 3.6, originally established in the full excitation setting [HGB18], to scenarios with partial excitation and measurement. Theorem 8.7 introduces two mutually exclusive conditions, (A) and (B), separated by an exclusive *or*. This naturally raises the question of whether a stronger result could be obtained by requiring both conditions to hold simultaneously. This motivates the following conjecture, which is refuted by the counterexample presented in Example 8.9 and illustrated in Fig. 8.2.

Conjecture 8.8. *Consider an EM network $(A^\circ, A^\bullet, B, C)$. If the network is generically locally identifiable, then there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of*

cardinality m° with at least one bijective collection of walks $\omega \in \Omega_{\mathcal{P}}$ such that both following conditions hold simultaneously:

- (A) For each excitation $b \in \mathcal{B}^{\mathcal{P}}$, the C-parts of the walks starting at b are vertex-disjoint.
- (B) For each measurement $c \in \mathcal{C}^{\mathcal{P}}$, the B-parts of the walks ending at c are vertex-disjoint.

Example 8.9. Consider the EM network depicted in Fig 8.2 (a). Running Algorithm 2 on it outputs that the whole network is generically locally identifiable. In this example, since $n_{\mathcal{B}}n_{\mathcal{C}} = m^\circ$, there is only one option for a subset $\mathcal{P}$ of $\mathcal{B} \times \mathcal{C}$: the whole set itself: $\mathcal{P} = \mathcal{B} \times \mathcal{C} = \{1, 2\} \times \{3, 4\}$. There is only one monomial μ corresponding to bijective collections of walks: it takes each known module exactly once. There are only three bijective collections of walks corresponding to that monomial μ , denoted ω^b, ω^c and ω^d , depicted in Fig 8.2 (b), (c) and (d) respectively. However, none of them satisfies both (A) and (B) simultaneously:

- ω_1^b and ω_2^b , both starting from the same excitation, meet at node 4, which is highlighted with a red star to emphasize this phenomenon. Such node will be denoted a *Meeting Node* (MN), a notion introduced in Chapter 9.
- ω^c satisfies neither (A), nor (B). ω_1^c and ω_2^c , both starting from the same excitation, meet at node 4. ω_2^c and ω_3^c , both ending at the same measurement, meet at node 3. In Chapter 9, a walk such as ω_2^c is referred to as *MN-MN*, indicating that it passes through meeting nodes in both its B-part and C-part. The MN-MN nature of ω^c is what makes this EM network identifiable, a phenomenon we examine in more detail in Chapter 9.
- ω_2^d and ω_3^d , both ending at the same measurement, meet at node 3.

The disproven conjecture and its counterexample reveal the deeper complexity of our problem. This motivates the introduction of a new, more detailed and powerful framework, one capable of capturing the full range of subtleties that influence network identifiability in partial excitation and measurement. Unlike simpler approaches such as Conjecture 8.8, which overlook crucial hidden mechanisms, this new tool is designed to leave no detail unaccounted for. While it may appear cumbersome at first glance, its necessity becomes clear in light of the limitations of previous attempts. The development and exploration of this tool form the core of Chapter 9.

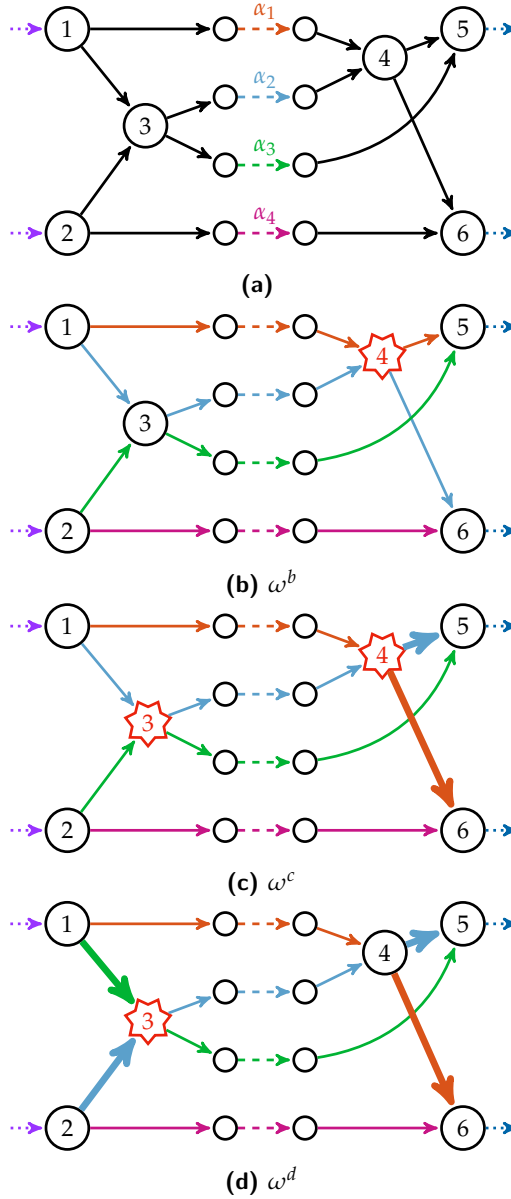


Fig. 8.2 Subfigure (a) represents the EM network of Example 8.9: dashed modules are unknown, purple dotted arrows are excitations and blue dotted arrows are measurements. Subfigures (b), (c), (d) represent the three only bijective collections of walks of this EM network. Bold arrows highlight the changes between (b) and (c), (d). Nodes highlighted with a red star are *Meeting Nodes*, and will be introduced in Chapter 9.

9

The Meta-Walk Graph

WE possess a combinatorial characterization, derived in Chapter 7. However, its combinatorial nature makes it difficult to distill practical or easily verifiable conditions from it. Walk-based conditions have been extracted from it in Chapter 8, resulting in Theorem 8.7, which presents vertex-disjoint walk conditions that extend prior results obtained in the full excitation or measurement setting [HGB18].

Still, the situation remains unsatisfactory. The condition in Theorem 8.7 is expressed as a logical OR: it requires either a condition on the excitation side or a condition on the measurement side. What we seek is a unified condition : one that simultaneously captures both sides and encodes the full combinatorial richness of the identifiability problem. This goal led us to formulate Conjecture 8.8. However, a counterexample refuted it, exposing a deeper complexity in the problem of network identifiability under partial excitation and measurement — one that is absent in the full excitation or measurement cases. The failure of the conjecture spurs the need for a more sophisticated framework, capable of capturing and exposing the subtleties that were hidden in Conjecture 8.8.

Previous results, both from the literature and earlier chapters of this thesis, highlight the central role of walks — particularly vertex-disjoint walks — in network identifiability. A key concept introduced in this chapter is that of *Meeting Nodes* (MN): nodes at which two non-vertex-disjoint walks intersect. These nodes are critical, as they represent points where signals

from different walks interfere, leading to potential information loss. To analyze such interactions, we define the notion of an *MN-swap*: an operation that exchanges the suffixes (or prefixes) of walks meeting at a given MN, thereby producing a new collection of walks.

The core idea of this chapter is to construct a new graph, called the Meta-Walk Graph, where each vertex corresponds to a bijective collection of walks, and vertices are connected by an edge if they can be obtained from each other by an MN-swap. The Meta-Walk Graph proves to be a powerful tool for analyzing network identifiability. In certain cases, its structure fully characterizes whether the network is identifiable or not. We show that the Meta-Walk Graph is bipartite, and even multipartite. Moreover, we derive quantitative properties, such as the degree of its vertices and its total number of vertices, under some assumptions.

By combining this new framework with the combinatorial characterization from Chapter 7, and leveraging tools from graph theory, we obtain new sufficient conditions for identifiability.

9.1 Meeting Nodes and MN-swaps

Meeting nodes generalize the notion of vertex-disjoint walks by providing a more precise way to characterize where and how walks fail to be vertex-disjoint.

Definition 9.1. Consider an EM network $(A^\circ, A^\bullet, B, C)$, a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , and a bijective collection of walks $\omega \in \Omega_{\mathcal{P}}$, composed of m° walks $\omega_1 \dots \omega_{m^\circ}$. Let $v \in \mathcal{W}$ be a node of the network, and let λ be an excited or measured node selected by $\mathcal{P}$, i.e. $\lambda \in \mathcal{B}^{\mathcal{P}} \cup \mathcal{C}^{\mathcal{P}}$.

We say that v^λ is a *Meeting Node* (MN) of ω when :

- (a) if λ is an excited node : the C -parts of several walks $\omega_i \dots \omega_j$, starting from the *same* excitation λ , meet at v ,
- (b) if λ is a measured node : the B -parts of several walks $\omega_i \dots \omega_j$, ending at the *same* measurement λ , meet at v .

The number of such different walks $\omega_i \dots \omega_j$ meeting at v^λ is the *multiplicity* of the meeting node, denoted π_{v^λ} . We denote by $\mathcal{N}_\omega$ the set of all distinct meeting nodes of ω . Some special cases require additional clarification:

- A single node v may serve as a meeting node for multiple different excitations or measurements $\lambda_1, \dots, \lambda_k$, this gives rise to multiple distinct meeting nodes $v^{\lambda_1}, \dots, v^{\lambda_k}$.
- If λ is both excited and measured, it is treated as two distinct nodes: the excitation λ_B , and the measurement λ_C . If (a) holds for λ_B and (b) holds for λ_C , then we have two distinct meeting nodes, denoted v^{λ_B} and v^{λ_C} .
- When the C-part (resp. B-part) of a walk traverses κ times one of its meeting nodes v^λ — i.e. it contains $\kappa - 1$ distinct cycles passing through node v — then this gives rise to κ distinct meeting nodes, one for each passage. If ℓ walks make multiple passages, this gives rise to $\kappa_1 \kappa_2 \dots \kappa_\ell$ distinct meeting nodes, where κ_l is the number of passages of the l -th walk.

Note 9.2. We emphasize that the notion of meeting node is defined relative to a specific collection of walks ω , and does not depend solely on the network topology. In the same graph, a node may be a meeting node for one collection of walks ω^b , yet not for another collection ω^d . This context-dependent nature is illustrated in Figure 9.1.

Example 9.3. Consider the EM network depicted in Fig. 9.1, with $\mathcal{P} = \{(1, 6), (2, 5), (2, 6)\}$. Let monomial $\mu \in \mathcal{M}_{\mathcal{P}}$ be as represented in Fig. 9.1(a). The set of bijective collections of walks corresponding to μ is composed of three elements: $\Omega_\mu = \{\omega^b, \omega^c, \omega^d\}$, represented in subfigures (b), (c), (d) respectively. Meeting nodes are highlighted by red stars in the figure. Each meeting node has multiplicity 2 in this example. We have:

$$\mathcal{N}_{\omega^b} = \{3^6\}, \quad \mathcal{N}_{\omega^c} = \{3^6, 4^2\}, \quad \mathcal{N}_{\omega^d} = \{4^2\}.$$

We introduce the operation of MN-swap, which consists in swapping the suffixes (or prefixes) of two walks that intersect at a given meeting node. This operation serves as the adjacency relation in the construction of the Meta-Walk Graph, as will be detailed in the following sections.

Definition 9.4. Consider an EM network $(A^\circ, A^\bullet, B, C)$, a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , and a bijective collection of walks $\omega \in \Omega_{\mathcal{P}}$, composed of m° walks $\omega_1 \dots \omega_{m^\circ}$. Let v^λ be a meeting node (MN) of ω , where walks ω_i and ω_j meet.

An *MN-swap* is an operation applied to ω to produce a new collection of walks $\tilde{\omega}$, identical to ω except for a modification at the meeting node v^λ

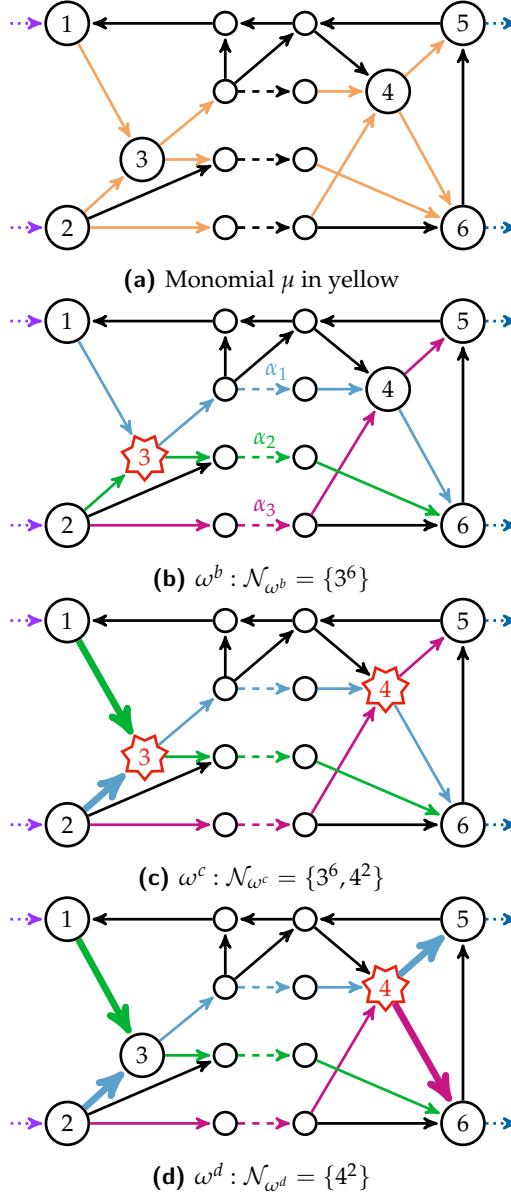


Fig. 9.1 The EM network of Example 9.3: dashed modules are unknown, purple dotted arrows are excitations and blue dotted arrows are measurements. Subfigure (a) represents monomial μ , and subfigures (b), (c), (d) represent the three bijective ω corresponding to μ . Bold arrows highlight the changes between ω^b and ω^c, ω^d . Red stars are meeting nodes.

on the two walks ω_i and ω_j . The modification depends on whether λ is an excitation or a measurement:

- If λ is an excitation, the suffixes of walks ω_i and ω_j *after* the meeting node v are exchanged. That is, $\tilde{\omega}_i$ follows the initial segment of ω_i up to v , then continues along the suffix of ω_j ; similarly for $\tilde{\omega}_j$.
- If λ is a measurement, the prefixes of walks ω_i and ω_j *before* the meeting node v are exchanged. That is, $\tilde{\omega}_i$ follows the prefix of ω_j up to v , then continues along the suffix of ω_i ; and analogously for $\tilde{\omega}_j$.

The collections of walks ω and $\tilde{\omega}$ correspond to the same monomial μ and are said *neighbors*. An MN-swap is a bidirectional operation: if $\tilde{\omega}$ can be obtained from ω by an MN-swap at meeting node v^λ , then ω can likewise be obtained from $\tilde{\omega}$ by an MN-swap at the same node v^λ .

Note 9.5. Recall the type (i) and type (ii) transpositions introduced in the proof of Lemma 8.4. An MN-swap applied to a bijective collection of walks corresponds to performing one of these transpositions on its associated assignation, thereby flipping its signature. This observation will be formalized in Proposition 9.9.

Example 9.3 (continued). Consider the EM network depicted in Fig. 9.1. ω^b and ω^c are neighbors: they can be obtained from each other by an MN-swap at meeting node 3^6 . Similarly, ω^c and ω^d are neighbors: they can be obtained from each other by an MN-swap at meeting node 4^2 .

Proposition 9.6 establishes that the operation of an MN-swap is well-defined: it preserves the bijective character of the collection of walks.

Proposition 9.6. *Consider an EM network $(A^\circ, A^\bullet, B, C)$, a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , and a bijective collection of walks $\omega \in \Omega_{\mathcal{P}}$, composed of m° walks $\omega_1 \dots \omega_{m^\circ}$. Let v^λ be a meeting node (MN) of ω , where walks ω_i and ω_j meet. The collection of walks $\tilde{\omega}$, obtained by an MN-swap between walks ω_i and ω_j at v^λ , is bijective.*

Proof. We assume that λ is an excitation, an analogous reasoning can be done if it is a measurement. Say that ω_i and ω_j are the walks corresponding to unknown edges α_i and α_j respectively, and they end at measurements c_i and c_j respectively. In other words, the assignation corresponding to ω assigns (λ, c_i) to α_i and (λ, c_j) to α_j . Since ω is bijective, no other unknown edge is assigned to neither (λ, c_i) nor (λ, c_j) . Now consider $\tilde{\omega}$. Since it is obtained by an MN-swap between walks ω_i and ω_j at v^λ , its corresponding

assignment assigns (λ, c_j) to α_i and (λ, c_i) to α_j . The rest of $\tilde{\omega}$ is the same as ω , hence no other unknown edge is assigned to neither (λ, c_i) nor (λ, c_j) , and $\tilde{\omega}$ is bijective. $\square$

9.2 The Meta-Walk Graph

Armed with the notions introduced in the previous section, we are now ready to present the central concept of this chapter. As shown in Chapter 7, bijective collection of walks play a key role in the determinant of $K_{\mathcal{P}}$, especially their signature. Moreover, performing an MN-swap transforms one bijective collection of walks into another, with the important property of flipping the signature.

This observation motivates the construction of a new graph, where each vertex represents a bijective collection of walks, and edges correspond to MN-swaps between them. We refer to this graph as the Meta-Walk Graph. By studying the structure and properties of the Meta-Walk Graph, we aim to uncover deeper insights into the identifiability of the underlying network.

Definition 9.7. Consider an EM network $(A^\circ, A^\bullet, B, C)$, a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , and a monomial $\mu \in \mathcal{M}_{\mathcal{P}}$. The *Meta-Walk Graph* $\mathcal{G}_\mu$ of this monomial is defined as follows:

- The vertices of the graph are the bijective collections of walks corresponding to μ , i.e. the set of vertices of $\mathcal{G}_\mu$ is Ω_μ .
- Two collections of walks are adjacent in $\mathcal{G}_\mu$ (i.e. connected by an edge) if they are neighbors in the sense of Definition 9.4, i.e. if they can be obtained from each other by an MN-swap.

The Meta-Walk Graph is *undirected* since MN-swaps are a bidirectional operation, see Definition 9.4.

Example 9.3 (continued). The Meta-Walk Graph of the monomial depicted in Fig. 9.1(a) is given in Fig. 9.2. Observe that the EM network of Example 8.9 has the exact same Meta-Walk Graph for its only monomial corresponding to bijective collections of walks, composed of all known modules.

We begin by recalling the fundamental definitions of multipartite and bipartite graphs.

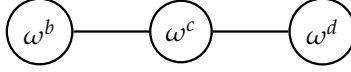


Fig. 9.2 The Meta-Walk Graph of the monomial μ of Example 9.3, depicted in Fig. 9.1(a). The three bijective collections of walks corresponding to μ are represented in Figure 9.1 (b), (c), (d).

Definition 9.8. [W⁺01] A *multipartite graph* is a type of graph in which the vertices can be divided into distinct independent sets such that no two vertices within the same set are adjacent. If a graph can be divided into two sets, it is called a bipartite graph. In general, for k sets, the graph is called a k -partite graph.

Building on Note 9.5, Proposition 9.9 establishes a key qualitative property of the Meta-Walk Graph: it is bipartite, and even multipartite.

Proposition 9.9. Consider an EM network $(A^\circ, A^\bullet, B, C)$, a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , and a monomial $\mu \in \mathcal{M}_{\mathcal{P}}$. Its Meta-Walk Graph $\mathcal{G}_\mu$ is

- k -partite, where each partition set consists of all the collections of walks corresponding to the same bijective assignment,
- bipartite: the partition set denoted $+$ (resp. $-$) contains all bijective collections of walks whose associated bijective assignments have positive (resp. negative) signature.

Proof. If there is no edge in $\mathcal{G}_\mu$, then $k = |\Omega_\mu|$ and the result is trivial. Otherwise, consider one edge $\{\omega, \tilde{\omega}\}$ and the corresponding bijective assignments σ and $\tilde{\sigma}$ respectively. ω and $\tilde{\omega}$ are just one MN-swap away from each other. For their corresponding assignments, this results in σ and $\tilde{\sigma}$ being just one transposition away from each other. Hence $\sigma \neq \tilde{\sigma}$ and $\text{sgn}(\sigma) = -\text{sgn}(\tilde{\sigma})$. Since the argument holds for every edge of $\mathcal{G}_\mu$, the proof follows. $\square$

Example 9.3 (continued). Consider again the EM network depicted in Fig. 9.1, and its Meta-Walk Graph represented in Fig. 9.2. Each bijective collection of walks of $\Omega_\mu = \{\omega^b, \omega^c, \omega^d\}$ corresponds to a different bijective assignment: the Meta-Walk Graph $\mathcal{G}_\mu$ is 3-partite. $\mathcal{G}_\mu$ is also bipartite: ω^b and ω^d have same signature, opposite to the one of ω^c .

9.3 Quantitative properties

The degree (i.e. number of neighbors) of a bijective collection of walks in the Meta-Walk Graph can be easily computed from the multiplicities of its meeting nodes.

Proposition 9.10. Consider an EM network $(A^\circ, A^\bullet, B, C)$, a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , a monomial $\mu \in \mathcal{M}_{\mathcal{P}}$ and its Meta-Walk Graph $\mathcal{G}_\mu$. Let ω be a bijective collection of walks of Ω_μ , i.e. a vertex of $\mathcal{G}_\mu$. The degree (i.e. number of neighbors) of ω in $\mathcal{G}_\mu$ is given by

$$\deg(\omega) = \sum_{v^\lambda \in \mathcal{N}_\omega} \binom{\pi_{v^\lambda}}{2} \quad \text{where} \quad \binom{\pi_{v^\lambda}}{2} = \frac{\pi_{v^\lambda}!}{2! (\pi_{v^\lambda} - 2)!}, \quad (9.1)$$

where $\mathcal{N}_\omega$ is the set collecting all meeting nodes of ω and π_{v^λ} is the multiplicity of v^λ , see Definition 9.1.

Proof. From Definition 9.7, we know that each edge in $\mathcal{G}_\mu$ corresponds to an MN-swap. Consider a meeting node v^λ of ω . One can perform $\binom{\pi_{v^\lambda}}{2}$ different MN-swaps at v^λ from ω : it is the number of different pairs that can be extracted from π_{v^λ} elements. Summing this for all meeting nodes of ω yields (9.1). $\square$

Example 9.3 (continued). Consider again the EM network depicted in Fig. 9.1, and its Meta-Walk Graph represented in Fig. 9.2. We have

$$\deg(\omega^b) = \binom{2}{2} = 1, \quad \deg(\omega^c) = \binom{2}{2} + \binom{2}{2} = 2, \quad \deg(\omega^d) = \binom{2}{2} = 1.$$

A bijective collection of walks is said *MN-MN* if one of its walks has meeting nodes in both its *B*-part and *C*-part. This notion is closely related to the concept of *independent* meeting nodes, as defined below.

Definition 9.11. Consider an EM network $(A^\circ, A^\bullet, B, C)$, a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , and a bijective collection of walks $\omega \in \Omega_{\mathcal{P}}$, composed of m° walks $\omega_1 \dots \omega_{m^\circ}$.

We say that ω is *MN-MN* if at least one of the walks ω_i of ω has a meeting node in its *B*-part *and* a meeting node in its *C*-part¹.

¹The *B*-part and *C*-part of a walk ω_i are introduced in Definition 7.6.

Besides, consider a collection of walks ω with at least two meeting nodes $v_1^{\lambda_1}$ and $v_2^{\lambda_2}$. These meeting nodes are said *independent* if either:

- (i) λ_1 and λ_2 are both excitations (resp. measurements);
- (ii) λ_1 is an excitation (resp. measurement), λ_2 is a measurement (resp. excitation), and there is no walk ω_i for which both $v_1^{\lambda_1}$ and $v_2^{\lambda_2}$ are meeting nodes.

Those two notions are closely related: a collections of walks is MN-MN if it has at least two non-independent meeting nodes.

Example 9.3 (continued). In the example of Fig. 9.1, ω^c is MN-MN: 3^6 and 4^2 are not independent. ω^b and ω^d are not MN-MN.

Lemma 9.12 establishes that MN-swaps preserve the independence of meeting nodes.

Lemma 9.12. *Consider an EM network $(A^\circ, A^\bullet, B, C)$, and a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° . Let ω be a bijective collection of walks of $\Omega_{\mathcal{P}}$, with two independent meeting nodes $v_1^{\lambda_1}$ and $v_2^{\lambda_2}$. Consider $\tilde{\omega}$, a collection of walks obtained by MN-swap between walks ω_i and ω_j meeting at $v_1^{\lambda_1}$. In $\tilde{\omega}$, $v_2^{\lambda_2}$ is still a meeting node, and is still independent of $v_1^{\lambda_1}$.*

Proof. The independence of $v_1^{\lambda_1}$ and $v_2^{\lambda_2}$ arises from two possible cases: (i) and (ii) of Definition 9.11. This proof addresses each case separately, following the same order.

- (i) λ_1 and λ_2 are both excitations (resp. measurements). Consider they are both excitations, an analogous proof can be made if they are both measurements. If $v_2^{\lambda_2}$ is meeting node of neither ω_i nor ω_j , then the result is trivial. Otherwise, there are three cases : $v_2^{\lambda_2}$ is meeting node of : (a) ω_i , (b) ω_j , (c) ω_i and ω_j . Consider case (a), and denote ω_k one of the walks meeting ω_i at $v_2^{\lambda_2}$. By Definition 9.1, ω_j and ω_i come from the same excitation λ_1 , and ω_k and ω_i come from the same excitation λ_2 . Hence $\lambda_1 = \lambda_2$. Denote $\tilde{\omega}_l$ the walk in $\tilde{\omega}$ corresponding to ω_l in ω , for any l . There are two cases:

- A. $v_1^{\lambda_1}$ comes before $v_2^{\lambda_2}$ in ω_i . In $\tilde{\omega}$, $v_2^{\lambda_2}$ is now a meeting node for $\tilde{\omega}_j$ and $\tilde{\omega}_k$. For $\tilde{\omega}_j$, both $v_1^{\lambda_1}$ and $v_2^{\lambda_2}$ are meeting nodes, and they are independent.

- B. $v_1^{\lambda_1}$ comes after $v_2^{\lambda_2}$ in ω_i . In $\tilde{\omega}$, $v_2^{\lambda_2}$ is still a meeting node for $\tilde{\omega}_i$ and $\tilde{\omega}_k$. For $\tilde{\omega}_i$, both $v_1^{\lambda_1}$ and $v_2^{\lambda_2}$ are meeting nodes, and they are independent.

The argument for case (b) follows symmetrically, with the roles of indices i and j reversed. In case (c), the same argument as in (a) (or equivalently (b)) applies directly, without any modification.

- (ii) λ_1 is an excitation (resp. measurement), λ_2 is a measurement (resp. excitation), and there is no walk for which both $v_1^{\lambda_1}$ and $v_2^{\lambda_2}$ are meeting nodes. Consequently, neither ω_i nor ω_j meets any other walk at $v_2^{\lambda_2}$. By Definition 9.4, performing an MN-swap between walks ω_i and ω_j at $v_1^{\lambda_1}$ does not impact other walks than ω_i and ω_j , i.e. $\tilde{\omega}_l = \omega_l$ for all $l \neq i, j$. Therefore, $v_2^{\lambda_2}$ remains a meeting node in $\tilde{\omega}$, and there is still no walk for which both $v_1^{\lambda_1}$ and $v_2^{\lambda_2}$ are meeting nodes in $\tilde{\omega}$: $v_1^{\lambda_1}$ and $v_2^{\lambda_2}$ are still independent.

□

Lemma 9.12 is illustrated in Example 9.13.

Example 9.13. Consider the EM network depicted in Fig. 9.3, with $\mathcal{P} = \{(1, 4), (1, 5), (1, 6), (2, 6), (3, 6)\}$. Let monomial $\mu \in \mathcal{M}_{\mathcal{P}}$ be as represented in Fig. 9.3, where each yellow module is taken exactly once. Its associated Meta-Walk Graph $\mathcal{G}_{\mu}$ is represented in Figure 9.4, with meeting nodes highlighted by red stars. One verifies that the two meeting nodes remain independent for each $\omega \in \Omega_{\mu}$, as ensured by Lemma 9.12.

If there is no MN-MN bijective collection of walks in a connected component of the Meta-Walk Graph, then its number of vertices can be explicitly computed, as shown by Proposition 9.14.

Proposition 9.14. Consider an EM network $(A^{\circ}, A^{\bullet}, B, C)$, a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , a monomial $\mu \in \mathcal{M}_{\mathcal{P}}$ and its Meta-Walk Graph $\mathcal{G}_{\mu}$. Let ω be a bijective collection of walks of Ω_{μ} with at least one meeting node, and such that there is no MN-MN bijective collection of walks in the connected component of ω in $\mathcal{G}_{\mu}$, denoted $\mathcal{G}_{\mu, \omega}$. The vertex set of $\mathcal{G}_{\mu, \omega}$ is denoted by $\Omega_{\mu, \omega}$. Denote the meeting nodes of ω by $v_1^{\lambda_1}, \dots, v_{\eta_{\omega}}^{\lambda_{\eta_{\omega}}}$, where η_{ω} is the number of meeting nodes of ω . The number of collections of walks in $\Omega_{\mu, \omega}$ is even, and is given by the multiplicities of the meeting nodes of ω :

$$|\Omega_{\mu, \omega}| = \pi_{v_1^{\lambda_1}}! \cdots \pi_{v_{\eta_{\omega}}^{\lambda_{\eta_{\omega}}}}! \quad (9.2)$$

Proof. We proceed by construction. We start with $\tilde{\Omega} = \emptyset$, and fill this set until $\tilde{\Omega} = \Omega_{\mu, \omega}$. Consider a meeting node $v_1^{\lambda_1}$ of ω , of multiplicity $\pi_{v_1 \lambda_1}$. There are $\pi_{v_1 \lambda_1}!$ permutations of the $\pi_{v_1 \lambda_1}$ walk prefixes/suffixes at $v_1^{\lambda_1}$ (including ω), which all give bijective collections of walks of same monomial μ . They can all be obtained from ω by a sequence of MN-swaps, hence belong to $\Omega_{\mu, \omega}$, the vertex set of the connected component of ω in $\mathcal{G}_\mu$. We append those $\pi_{v_1 \lambda_1}!$ collections of walks in $\tilde{\Omega}$.

Now consider another meeting node $v_2^{\lambda_2}$ of ω . Since there is no MN-MN collection of walks in $\Omega_{\mu, \omega}$, $v_1^{\lambda_1}$ and $v_2^{\lambda_2}$ are independent in the sense of Definition 9.11. Hence, by Lemma 9.12, for each $\tilde{\omega} \in \tilde{\Omega}$, $v_2^{\lambda_2}$ is also a meeting node, still independent of $v_1^{\lambda_1}$. Thus, for each $\tilde{\omega} \in \tilde{\Omega}$, there are $\pi_{v_2 \lambda_2}!$ permutations of the $\pi_{v_2 \lambda_2}$ walk prefixes/suffixes at $v_2^{\lambda_2}$ (including $\tilde{\omega}$), which all give bijective collections of walks of same monomial μ . They can be obtained by a sequence of MN-swaps from $\tilde{\omega}$, hence belong to $\Omega_{\mu, \omega}$ as well. Appending those new collections of walks to $\tilde{\Omega}$ for all $\tilde{\omega} \in \tilde{\Omega}$ gives $|\tilde{\Omega}| = \pi_{v_1 \lambda_1}! \pi_{v_2 \lambda_2}!$. Repeating this reasoning for all meeting nodes yields the result. Since all meeting nodes are independent, each collection of walks of $\Omega_{\mu, \omega}$ is counted once and only once. The multiplicity $\pi_{v_1 \lambda_1} \geq 2$ by definition, hence the result is even. $\square$

Example 9.13 (continued). Consider again the EM network depicted in Fig. 9.3, with monomial μ composed of each known module taken exactly once. Its associated Meta-Walk Graph $\mathcal{G}_\mu$ is represented in Fig. 9.4. First, we illustrate the result of Proposition 9.10 on this example: One verifies in Fig. 9.4 that each collection of walks of Ω_μ has degree $\binom{2}{2} + \binom{3}{2} = 4$, as predicted by Proposition 9.10.

Second, we illustrate the result of Proposition 9.14. Let ω be as represented in the top left corner² of Fig. 9.4. There is no MN-MN walk in $\Omega_{\mu, \omega}$, and there are one meeting node of multiplicity 2 in the B -parts, and one meeting node of multiplicity 3 in the C -parts. The assumptions of Proposition 9.14 are satisfied, and one verifies that $|\Omega_{\mu, \omega}| = 2! 3! = 12$, in accordance with Proposition 9.14. Besides, it is the only connected component of $\mathcal{G}_\mu$, hence $|\Omega_\mu| = |\Omega_{\mu, \omega}|$.

²It is an arbitrary choice, and any other choice yields the same result.

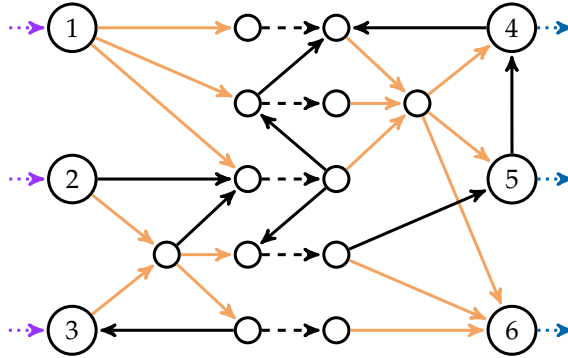


Fig. 9.3 The EM network of Example 9.13. Dashed modules are unknown, plain modules are known, purple dotted arrows are excitations, and blue dotted ones measurements. The monomial μ is highlighted in yellow.

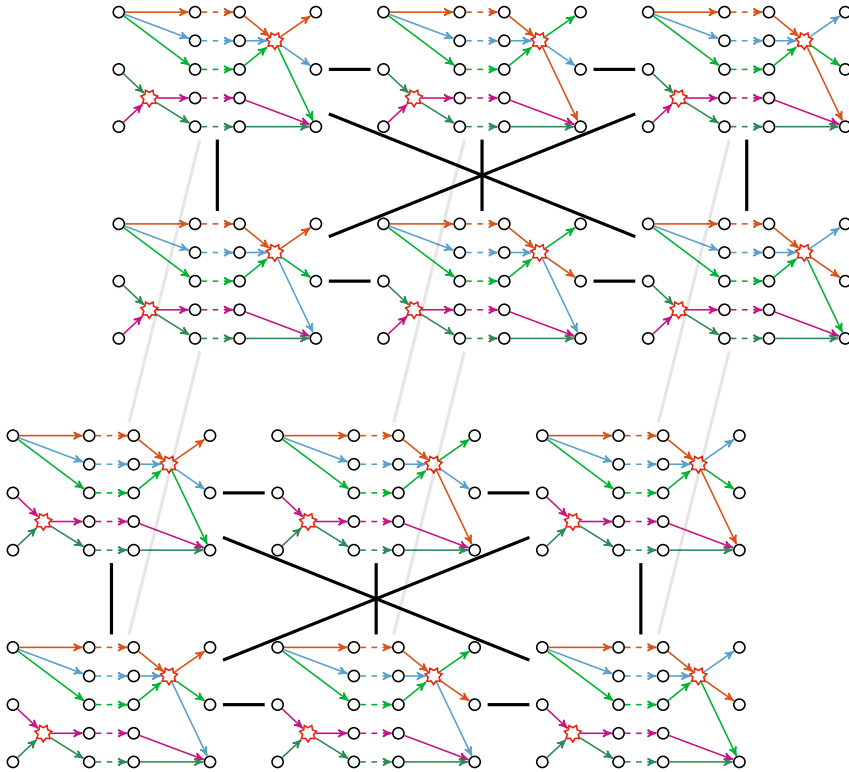


Fig. 9.4 The Meta-Walk Graph $\mathcal{G}_\mu$ of Example 9.13, Fig. 9.3. Excitations, measurements and known modules not taken by μ are omitted. Gray edges represent MN-swaps in the B -parts, black ones MN-swaps in the C -parts.

9.4 Sufficient conditions on identifiability

Proposition 9.15 (a) is a straightforward extension of Proposition 8.1 (ii), and can be seen as an immediate corollary of the combinatorial characterization of Theorem 7.9.

Proposition 9.15 (b) is more refined. It leverages the structure of the Meta-Walk Graph to formulate a more specific and insightful condition. This result is derived by combining Proposition 9.14, which characterizes the size of connected components in the absence of MN-MN configurations, with Proposition 9.15 (a).

Proposition 9.15. *Consider an EM network $(A^\circ, A^\bullet, B, C)$. We have the two following sufficient conditions for generic local network identifiability:*

- (a) *If there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , and a monomial $\mu \in \mathcal{M}_{\mathcal{P}}$ for which the number of corresponding bijective collections of walks $|\Omega_\mu|$ is odd, then the network is generically locally identifiable.*
- (b) *If there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , and a monomial $\mu \in \mathcal{M}_{\mathcal{P}}$ such that in Ω_μ there is an odd number of bijective collection of walks with no meeting node, and no MN-MN bijective collection of walks, then the network is generically locally identifiable.*

Proof. (a) Assume there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , and a monomial $\mu \in \mathcal{M}_{\mathcal{P}}$ for which the number of corresponding bijective collections of walks $|\Omega_\mu|$ is odd. Then the repetition $r(\mu)$ (defined in (7.9)) cannot be zero, and the network is generically locally identifiable by Theorem 7.9.

(b) Assume there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , and a monomial $\mu \in \mathcal{M}_{\mathcal{P}}$ such that in Ω_μ there is an odd number of bijective collection of walks with no meeting node, and no MN-MN bijective collection of walks. Since there is no MN-MN bijective collection of walks, Proposition 9.14 applies to each connected component of $\mathcal{G}_\mu$, except for the isolated vertices. It follows that the total number of bijective collections of walks — excluding isolated vertices — is even. These isolated vertices correspond precisely to the bijective collections of walks with no meeting node, and their number determines the parity of $r(\mu)$. If the number of such walk collections is odd, then $|\Omega_\mu|$ is odd, and the argument of (a) concludes the proof. $\square$

Example 9.13 (continued). Applying Proposition 9.15 (a) to the EM network of Example 9.13 ensures its generic local network identifiability. The same reasoning applies to Example 8.9.

The following fundamental result from graph theory is a direct consequence of the handshaking lemma [W⁺01].

Lemma 9.16. *Consider an undirected graph. If the number of vertices with an even degree is odd, then the total number of vertices is odd.*

Proof. We know from the handshaking lemma that for any undirected graph, the sum of the degrees of all vertices equals twice the number of edges [W⁺01]. Hence the number of vertices with an odd degree is even, which yields the result. $\square$

By combining Theorem 7.9 with Lemma 9.16, we obtain our most general sufficient condition, stated in Theorem 9.18. While this condition may initially seem difficult to apply, it is important to note that the degree of a bijective collection of walks in $\mathcal{G}_\mu$ can be explicitly computed using the expression derived in Proposition 9.10. Before presenting Theorem 9.18, we introduce the following notation to clarify its statement.

Definition 9.17. Consider an EM network $(A^\circ, A^\bullet, B, C)$, a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , a monomial $\mu \in \mathcal{M}_{\mathcal{P}}$ and its Meta-Walk Graph $\mathcal{G}_\mu$. We denote the set of bijective collections of walks with an even degree in $\mathcal{G}_\mu$ as follows:

$$\Omega_\mu^{\text{even deg}} \triangleq \{\omega \in \Omega_\mu \mid 0 \equiv \deg(\omega) \pmod{2}\}, \quad (9.3)$$

where $\deg(\omega)$ is the degree of ω (i.e. neighbors number) as vertex of $\mathcal{G}_\mu$.

Theorem 9.18. *Consider an EM network $(A^\circ, A^\bullet, B, C)$. If there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , and a monomial $\mu \in \mathcal{M}_{\mathcal{P}}$ such that $|\Omega_\mu^{\text{even deg}}|$ is odd, then the network is generically locally identifiable.*

Proof. Assume there exists a set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , and a monomial $\mu \in \mathcal{M}_{\mathcal{P}}$ such that $|\Omega_\mu^{\text{even deg}}|$ is odd. Applying Lemma 9.16 on the Meta-Walk Graph $\mathcal{G}_\mu$ yields that $|\Omega_\mu|$ is odd. Then, Proposition 9.15 (a) directly implies the result and completes the proof. $\square$

Example 9.13 (continued). Applying Theorem 9.18 to the EM network of Example 9.13 ensures its generic local network identifiability. The same reasoning applies to Example 8.9.

9.5 Characterization of identifiability for a specific case

Consider an EM network $(A^\circ, A^\bullet, B, C)$, and assume we *only have access to one* set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° . If the two following conditions are satisfied:

- (a) There is *only one* monomial μ corresponding to bijective collections of walks, i.e. $|\mathcal{M}_{\mathcal{P}}| = 1$;
- (b) The Meta-Walk Graph $\mathcal{G}_\mu$ of this monomial is connected;

then the Meta-Walk Graph $\mathcal{G}_\mu$ entirely characterizes the generic local identifiability of network A° . In other words, all the information needed to decide whether the network is generically locally identifiable is encoded in the structure of $\mathcal{G}_\mu$.

The procedure to verify identifiability under these conditions proceeds as follows. Two cases must be considered:

- (i) If $|\Omega_\mu|$, the number of vertices of $\mathcal{G}_\mu$, is *odd*, then Proposition 9.15 (a) implies the generic local identifiability of A° .
- (ii) If $|\Omega_\mu|$ is *even*, we leverage the bipartite structure of $\mathcal{G}_\mu$. Specifically, we count how many ω belong to the $+$ and $-$ partition sets. If these counts differ, then the repetition satisfies $r(\mu) \neq 0$, and Theorem 7.9 guarantees that A° is generically locally identifiable. Conversely, if the counts are equal, then $r(\mu) = 0$, and the network A° is *not* generically locally identifiable.

We provide a brief explanation regarding the need for the two conditions (a) and (b), as well as the assumption on $\mathcal{P}$:

- (a) If $|\mathcal{M}_{\mathcal{P}}| > 1$, then it suffices to find a single monomial $\mu \in \mathcal{M}_{\mathcal{P}}$ for which the network is generically locally identifiable to conclude that A° is generically locally identifiable. However, to conclude that A° is not generically locally identifiable, all monomials in $\mathcal{M}_{\mathcal{P}}$ must be examined and found to yield zero repetition.
- (b) If $\mathcal{G}_\mu$ is not connected, we lack information about the relative signatures of bijective collections of walks belonging to different connected components. That is, $\mathcal{G}_\mu$ does not encode whether two such ω have the same or opposite signatures. As a result, the approach described in (ii) cannot be applied.

- (c) If we have access to more than one set of EM pairs $\mathcal{P} \subseteq \mathcal{B} \times \mathcal{C}$ of cardinality m° , the same problem as in (a) arises.

9.6 Conclusion

This chapter introduced a novel approach to studying network identifiability. By leveraging the notions of meeting nodes and MN-swaps, the Meta-Walk Graph reveals the structural properties of the network that directly influence identifiability. When combined with the combinatorial characterization developed in Chapter 7, this framework yields new sufficient conditions on identifiability.

Constructing the Meta-Walk Graph of an EM network allows one to distill and visualize the key features that govern identifiability. In the specific setting of Section 9.5, we demonstrated how the Meta-Walk Graph fully encapsulates all the relevant information for identifiability, and enables a straightforward identifiability check.

However, in the general case — where multiple choices for $\mathcal{P}$ and μ exist — several Meta-Walk Graphs need to be considered, rendering exhaustive analysis impractical. Moreover, when the Meta-Walk Graph is not connected, essential information about the relative signatures of walks across connected components is missing, making identifiability undeterminable from the sole knowledge of the Meta-Walk Graph.

These limitations underscore the intrinsic complexity of the identifiability problem. Even with the Meta-Walk Graph, it remains a fundamentally hard and deeply combinatorial challenge in the general case.

PART III
Signal selection for
identifiability

10

Optimal signal selection is NP-hard

PART III is dedicated to the development of algorithms for selecting nodes to excite and measure in order to guarantee the generic local identifiability of the network. Since performing excitations and measurements is often costly in practice, our objective is to achieve identifiability while minimizing the number of such interventions.

In this chapter, we establish that the problem of finding such *optimal* selection — i.e., excitation and measurement sets of minimal total cardinality that ensure identifiability — is *NP-hard*. NP-hard problems are at least as difficult as the hardest problems in NP, the class of problems whose solutions can be verified in polynomial time. Informally, NP-hard problems are widely regarded as computationally intractable for large instances, meaning that no efficient (i.e. polynomial-time) algorithm is known for solving them in general. As a result, one typically relies on heuristic, approximate, or problem-specific methods for practical purposes.

We begin by reviewing foundational notions in computational complexity, including the classes P, NP, and the notion of NP-hardness. We then introduce the classical NP-complete problem 3SAT, which plays a central role in complexity theory and serves as a basis for our NP-hardness proof. Finally, we show that our signal selection problem is NP-hard by constructing a polynomial-time reduction from 3SAT. That is, we demonstrate how

any instance of 3SAT can be transformed into an instance of our problem. Consequently, if our problem could be solved in polynomial time, then 3SAT could also be solved in polynomial time, which would imply $P = NP$, a conclusion that is widely believed to be false.

10.1 Classes P, NP, and NP-hard

Understanding the computational difficulty of a problem is essential, particularly when the design or analysis task involves combinatorial structures or large-scale systems. In this section, we provide a brief overview of the fundamental complexity classes P, NP, and the concept of NP-hardness. Additional background can be found in the references [Sip96, Pap03, AB09].

10.1.1 The class P

The class P (for *Polynomial time*) consists of all decision problems that can be solved by a deterministic Turing machine in time that is upper bounded by a polynomial function of the size of the input. That is, a problem is in P if there exists an algorithm whose worst-case running time is $\mathcal{O}(n^k)$ for some constant k , where n is the size of the input.

Problems in P are typically regarded as *tractable*, meaning they can be efficiently solved even for large instances, in contrast to problems outside of this class. Examples of problems in P include shortest path computation, maximum flow, matrix multiplication, and solving systems of linear equations.

10.1.2 The class NP

The class NP (for *Nondeterministic Polynomial time*) contains all decision problems for which a proposed solution can be verified in polynomial time by a deterministic Turing machine. In other words, even if we do not know how to efficiently find a solution, we can efficiently check whether a given solution is correct. For instance, while factoring a large integer into primes is not known to be solvable in polynomial time, checking whether a given factorization is correct is straightforward and fast.

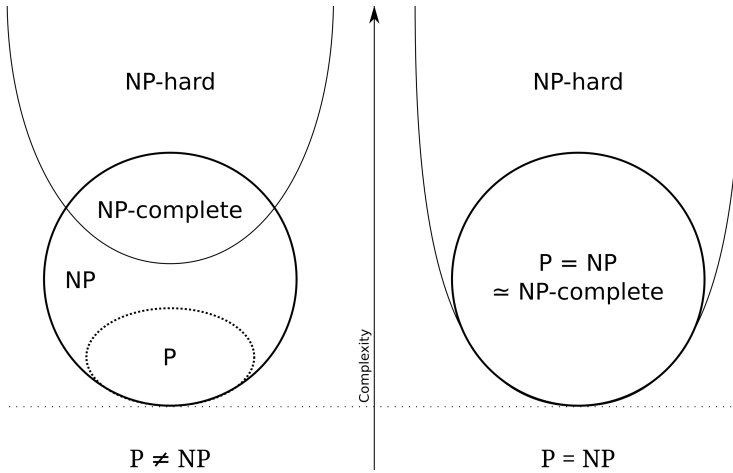


Fig. 10.1 Euler diagram for classes P, NP, NP-complete and NP-hard [Wik25a].

The acronym NP reflects the historical perspective of nondeterministic computation. A nondeterministic Turing machine is a hypothetical model of computation that can “guess” a solution nondeterministically and then verify it in polynomial time. Though such machines do not exist physically, this abstraction gives rise to the class NP. Importantly, $P \subseteq NP$, since any problem whose solution can be computed in polynomial time can also have its solution verified in polynomial time.

A classic example of a problem in NP is the *Boolean satisfiability problem* (SAT), where the goal is to determine whether there exists a truth assignment that satisfies a given Boolean formula.

10.1.3 NP-complete and NP-hard problems

A decision problem is said to be *NP-complete* if it is both in NP and as hard as any other problem in NP, in the sense that every problem in NP can be reduced to it in polynomial time. The notion of polynomial-time reduction allows to formally compare the difficulty of computational problems.

The class of NP-complete problems represents the most challenging problems in NP: if any one of them can be solved in polynomial time, then

every problem in NP can also be solved in polynomial time, implying $P = NP$, a famous open question in theoretical computer science.

A problem is called NP-*hard* if it is at least as hard as the hardest problems in NP, meaning that every problem in NP can be reduced to it in polynomial time. Unlike NP-complete problems, an NP-hard problem is not required to be in NP, and may not even be a decision problem. For instance, optimization problems like the Traveling Salesman Problem (TSP) in its optimization form are NP-hard but not in NP.

10.1.4 Significance in practice

In control theory and system identification, many design or verification problems turn out to be NP-hard. This means that, unless $P = NP$, there does not exist a polynomial-time algorithm that solves all instances of the problem efficiently. Consequently, for large-scale instances, one must resort to heuristic, approximate, or problem-specific methods.

In the next section, we will present one of the most fundamental NP-complete problems, known as 3SAT, which will serve as a basis for proving the NP-hardness of our problem.

10.2 A fundamental NP-complete problem: 3SAT

The Boolean satisfiability problem (SAT) is a central problem in computer science and mathematical logic. Given a Boolean formula, the goal is to determine whether there exists a truth assignment to its variables that makes the entire formula evaluate to true. The SAT problem was the first problem shown to be NP-complete, as demonstrated by Stephen Cook in his seminal work on the complexity of theorem-proving procedures [Coo23].

A particularly important restriction of SAT is the 3SAT problem, where each clause in the Boolean formula contains exactly three literals. Despite this apparent simplification, 3SAT remains NP-complete, and it is frequently used as a base problem for NP-hardness reductions due to its simple and constrained structure. Formally, an instance of 3SAT is defined as follows:

- Let $\{x_1, x_2, \dots, x_n\}$ be a set of Boolean variables, where each variable $x_i \in \{0, 1\}$ for $i = 1, 2, \dots, n$, with 1 representing true and 0 representing false.

- Let Φ be a Boolean formula expressed in *3-conjunctive normal form* (3CNF), meaning $\Phi = c_1 \wedge c_2 \wedge \cdots \wedge c_m$, where $\wedge$ stands for logical AND and each clause c_j is a disjunction (logical OR) of exactly three literals.
- A *literal* is either a variable x_i or its negation $\neg x_i$.
- The question is: Does there exist a truth assignment (i.e., a mapping of each variable to true or false) that makes Φ evaluate to true?

As an example, consider the following Boolean formula in 3SAT form:

$$\Phi = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3).$$

This formula has seven clauses, each containing exactly three literals. To satisfy Φ , one must assign (x_1, x_2, x_3) to (false, true, false) so that every clause has at least one literal that evaluates to true. This example illustrates that, while verifying a solution to a 3SAT instance is straightforward, finding such a solution can be computationally challenging.

The 3SAT problem is NP-complete, and this result is foundational in computational complexity theory. It provides a starting point for proving that other problems are NP-hard via polynomial-time reductions. Due to its generality and simplicity, it is a canonical problem in the study of computational intractability.

We refer the reader to the standard texts such as [GJ02, Pap03] for a more detailed presentation of SAT, 3SAT, and their central role in the theory of NP-completeness.

10.3 NP-hardness proof

Now that we have introduced the complexity classes P, NP, and NP-hard, and presented the 3SAT problem, we proceed to establish the NP-hardness of our problem by constructing a reduction from 3SAT.

Theorem 10.1. *Consider a network with unknown modules and known modules. The following problems are NP-hard.*

- (A) *Let no node be a priori excited, and all nodes be a priori measured. The problem of selecting a minimal excitation set that ensures generic global identifiability of all unknown modules.*
- (B) *Let all nodes be a priori excited, and no node be a priori measured. The problem of selecting a minimal measurement set that ensures generic global identifiability of all unknown modules.*
- (C) *Let no node be a priori excited, and some nodes be a priori measured. The problem of selecting a minimal excitation set that ensures generic global identifiability of all unknown modules.*
- (D) *Let some nodes be a priori excited, and no node be a priori measured. The problem of selecting a minimal measurement set that ensures generic global identifiability of all unknown modules.*
- (E) *Let some nodes be a priori excited, and some nodes be a priori measured. The problem of selecting an excitation set and a measurement set of minimal total cardinality, that ensures generic global identifiability of all unknown modules.*
- (F) *Let no node be a priori excited, and no node be a priori measured. The problem of selecting an excitation set and a measurement set of minimal total cardinality, that ensures generic global identifiability of all unknown modules.*

Proof. We first prove that (A) is NP-hard using a reduction from 3SAT. Given an arbitrary 3CNF formula Φ , we construct a network $\mathcal{G}_{3\text{SAT}}(\Phi)$ as follows. Let (β, γ) denote respectively the number of literals and the number of clauses in Φ .

For each variable x_i of Φ , we construct a block composed of four nodes, two known modules and one unknown module as represented in Fig. 10.2 (left). For each clause c_j of Φ , we add an unknown module and its start and target nodes: $\alpha_j^c := \{w_{c_j}^s, w_{c_j}^t\}$, see Fig. 10.2 (right). Then we connect the start node $w_{c_j}^s$ to three in-neighbours chosen among nodes $\{w_{i\Box}, w_{i\blacksquare} \mid 1 \leq i \leq \beta\}$, according to the clause itself: Let x_{j_1} (resp. $\neg x_{j_1}$) be one of the three literals of clause c_j : we add a known module $\{w_{j_1\Box}, w_{c_j}^s\}$ (resp. $\{w_{j_1\blacksquare}, w_{c_j}^s\}$) to the network constructed. This procedure is repeated for the two other literals of clause c_j . The resulting network constructed, denoted $\mathcal{G}_{3\text{SAT}}(\Phi)$,

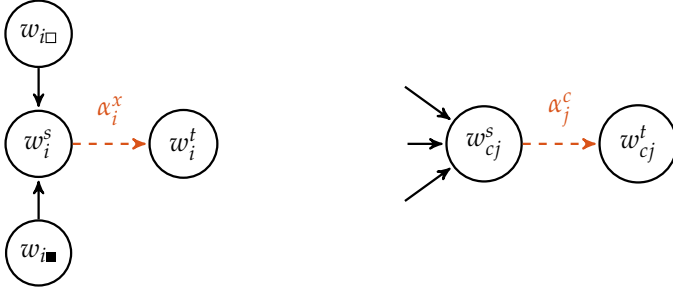


Fig. 10.2 Left: The block corresponding to x_i in the network construction. Right: The construction added for clause j .

contains exactly $4\beta + 2\gamma$ nodes, $\beta + \gamma$ unknown modules and $2\beta + 3\gamma$ known modules. Fig. 10.3 shows an example of this network construction.

We consider the generic global identifiability of all unknown modules of $\mathcal{G}_{3SAT}$, under the assumption that all nodes are a priori measured, and none is a priori excited. We first present two claims that support the proof.

Claim 10.2. Consider a 3CNF formula Φ , its corresponding network $\mathcal{G}_{3SAT}(\Phi)$ with all nodes a priori measured, and $i \in \{1, \dots, \beta\}$. The unknown module α_i^x is generically globally identifiable if and only if at least one node among $\{w_{i\Box}, w_i^s, w_{i\blacksquare}\}$ is excited.

Proof of Claim 10.2. Applying Theorem 3 of [SCVdH22] yields that α_i^x is generically globally identifiable if and only if there is a walk starting at an excitation and ending at w_i^s . Given the topology of $\mathcal{G}_{3SAT}(\Phi)$, this excitation can only be among the three nodes $\{w_{i\Box}, w_i^s, w_{i\blacksquare}\}$. $\square$

Claim 10.3. Consider a 3CNF formula Φ with β variables, and its corresponding network $\mathcal{G}_{3SAT}(\Phi)$ with all nodes a priori measured. If all unknown modules are generically globally identifiable with exactly β excitations, then for each $i \in \{1, \dots, \beta\}$, there is exactly one excitation among the three nodes $\{w_{i\Box}, w_i^s, w_{i\blacksquare}\}$.

Proof of Claim 10.3. From Claim 10.2, for each $i \in \{1, \dots, \beta\}$, there must be at least one excitation among $\{w_{i\Box}, w_i^s, w_{i\blacksquare}\}$ for the generic global identifiability of α_i^x . Since there are exactly β excitations, it must be exactly one excitation among those three nodes. $\square$

Claim 10.2 implies that β is a lower bound on the number of excitations needed for the generic global identifiability of all unknown modules of

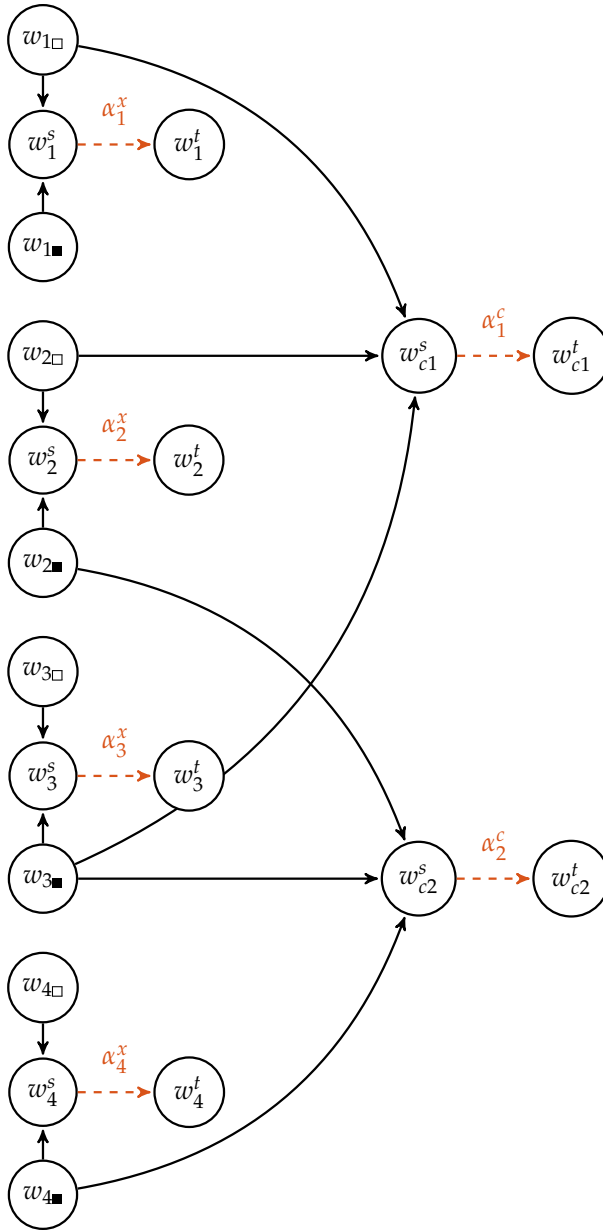


Fig. 10.3 Example of the network construction for the following 3CNF formula: $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_3 \vee \neg x_4)$. The black plain arrows are known modules, the orange dashed ones are unknown modules

$\mathcal{G}_{3\text{SAT}}(\Phi)$. Therefore, solving problem (A) starts with the search for an excitation set of size β such that all unknown modules of $\mathcal{G}_{3\text{SAT}}(\Phi)$ are generically globally identifiable, which we prove to be NP-hard below.

We claim that there is an excitation set of size β such that all unknown modules of the network $\mathcal{G}_{3\text{SAT}}(\Phi)$ are generically globally identifiable if and only if the original formula Φ is satisfiable. As usual for “if and only if” statements, the proof consists of two parts.

$\Rightarrow$ Suppose there is an excitation set $\mathcal{B}^*$ of size β such that the network is generically globally identifiable. By Claim 10.3, for each $i \in \{1, \dots, \beta\}$ there is exactly one excitation among the three nodes $\{w_{i\Box}, w_i^s, w_{i\blacksquare}\}$.

For each $j \in \{1, \dots, \gamma\}$, since α_j^c is generically globally identifiable, there must be a walk starting at an excitation and ending at w_{cj}^s , by Theorem 3 of [SCVdH22]. Let $\{\ell_{j1}, \ell_{j2}, \ell_{j3}\}$ be the three literals of clause j in Φ , where $\ell_{jk} \in \{x_{jk}, \neg x_{jk}\}$. Let $\clubsuit_k := \Box$ if $\ell_{jk} = x_{jk}$, and $\clubsuit_k := \blacksquare$ if $\ell_{jk} = \neg x_{jk}$. All β excitations have already been allocated: exactly one among the three nodes $\{w_{i\Box}, w_i^s, w_{i\blacksquare}\}$ for $i = 1, \dots, \beta$. For all (i, j) , there is no walk from w_i^s to w_{cj}^s . Hence, given the topology of the network, there must be at least one excitation among the three nodes $\{w_{j1\clubsuit_1}, w_{j2\clubsuit_2}, w_{j3\clubsuit_3}\}$.

Construct the assignment $\tau^* : \{x_1 \dots x_\beta\} \rightarrow \{\text{true}, \text{false}\}$ from $\mathcal{B}^*$ as follows. For each $i \in \{1, \dots, \beta\}$: if $w_{i\Box}$ is excited, then $\tau^*(x_i) := \text{true}$. Else, if $w_{i\blacksquare}$ is excited, then $\tau^*(x_i) := \text{false}$. Otherwise (when w_i^s is excited), assign arbitrarily $\tau^*(x_i) := \text{true}$ (but it could be assigned to false and it would not matter).

For each clause $j \in \{1, \dots, \gamma\}$, there is at least one excitation among the three nodes $\{w_{j1\clubsuit_1}, w_{j2\clubsuit_2}, w_{j3\clubsuit_3}\}$. Hence, the construction of assignment τ^* ensures that at least one literal of clause j in Φ is satisfied, hence clause j is satisfied. By Claim 10.3, for each $i \in \{1, \dots, \beta\}$ there cannot be an excitation on both $w_{i\Box}$ and $w_{i\blacksquare}$, which ensures that $\tau^*(x_i)$ is well-defined: it is not assigned to both true and false. Hence τ^* is a satisfying assignment to Φ , and Φ is satisfiable.

$\Leftarrow$ Suppose Φ is satisfiable. Fix a satisfying assignment $\tau^\dagger$, and construct an excitation set $\mathcal{B}^\dagger$ from $\tau^\dagger$ as follows. Start with $\mathcal{B}^\dagger := \emptyset$. For $i = 1, \dots, \beta$: if $\tau^\dagger(x_i) = \text{true}$, then add $w_{i\Box}$ to $\mathcal{B}^\dagger$. Else (when $\tau^\dagger(x_i) = \text{false}$), add $w_{i\blacksquare}$ to $\mathcal{B}^\dagger$. This construction gives a set $\mathcal{B}^\dagger$ of size β . For each $i \in \{1, \dots, \beta\}$, the unknown module α_i^x is generically globally

identifiable by Claim 10.2, because either $w_{i\Box}$ or $w_{i\blacksquare}$ is excited. For each $j \in \{1, \dots, \gamma\}$, there is at least one excitation among the three nodes $\{w_{j_1\clubsuit_1}, w_{j_2\clubsuit_2}, w_{j_3\clubsuit_3}\}$, where $\{\ell_{j_1}, \ell_{j_2}, \ell_{j_3}\}$ are the three literals of clause j in Φ , with $\ell_{j_k} \in \{x_{j_k}, \neg x_{j_k}\}$, and $\clubsuit_k := \Box$ if $\ell_{j_k} = x_{j_k}$, and $\clubsuit_k := \blacksquare$ if $\ell_{j_k} = \neg x_{j_k}$. Given the topology of the network, there is a walk from this excitation to w_{cj}^s , hence the unknown module α_j^c is generically globally identifiable, by Theorem 3 of [SCVdH22]. Therefore, the network $\mathcal{G}_{3SAT}(\Phi)$ is generically globally identifiable with excitation set $\mathcal{B}^\dagger$ of size β .

Transforming any 3CNF formula Φ into the graph $\mathcal{G}_{3SAT}(\Phi)$ takes polynomial time, even using a brute-force construction. Thus, if we could solve (A) in polynomial time, then we could also solve 3SAT in polynomial time, by following the procedure below, where each step takes polynomial time:

1. Construct network $\mathcal{G}_{3SAT}(\Phi)$ from the given formula Φ ,
2. Find a minimal excitation set $\mathcal{B}^*$ such that all unknown modules are generically globally identifiable,
3. Construct τ^* from $\mathcal{B}^*$.

But that would imply $P = NP$, which is widely believed to be false. We conclude that (A) is NP-hard.

We now show that (C) is NP-hard. If it could be solved in polynomial time, then so could (A), since the latter is a special case of the former where *all* nodes are assumed to be measured. But we have already established that (A) is NP-hard. It follows that (C) must also be NP-hard.

We now show that (E) is NP-hard using a similar argument. If this problem could be solved in polynomial time, then so could (A), since the latter is a special case of (E) in which no node is a priori excited, and all nodes are already measured, hence no additional measurements need to be selected. As we have already shown that (A) is NP-hard, it follows that (E) is NP-hard as well.

The NP-hardness of (B) and (D) can be established through an analogous argument, using a dual construction of the network.

Finally, the NP-hardness of (F) can be established using the same reduction as for (A). We construct the same graph $\mathcal{G}_{3SAT}(\Phi)$ and invoke Lemma 1 from [SCVdH23], which requires measuring the nodes w_i^t and w_{cj}^t for all $i \in 1, \dots, \beta$ and $j \in 1, \dots, \gamma$.

□

11

Algorithms for signal selection

WE have established that the problem of optimal signal selection is computationally intractable in the general case, as no polynomial-time algorithm exists, unless $P = NP$. Consequently, attempting to compute exact optimal solutions becomes impractical for large-scale networks due to the exponential growth in computational complexity.

Given the intractability of the problem, we relinquish the pursuit of guaranteed optimality. With the exception of the Exhaustive Search — which is primarily employed to benchmark the optimality gap of our proposed methods — none of the four other algorithms presented can ensure optimal solutions. Instead, our focus shifts to designing algorithms that aim to find high-quality solutions within reasonable computational time, thereby maintaining tractability even as network size increases.

To address this challenge, our algorithms build upon a combination of algebraic conditions, walk-based criteria, and algorithmic frameworks developed earlier in this thesis. By leveraging these diverse tools, we propose five distinct algorithms, each embodying a different methodological approach. To evaluate their performance, we constructed a dataset comprising networks of varying topologies and sizes. The chapter concludes with an analysis and discussion of the algorithms' performance on this dataset. Implementations available in our GitHub repository [LH25].

11.1 Tools for algorithm design

This section compiles the key tools that underpin the design of our algorithms. Some of these tools are novel contributions — such as the lower bound on the number of required signals and the monotonicity property of identifiability — while others are concise, black-box recaps of results introduced earlier, including Algorithms 2 and 1. Finally, we revisit the results from Chapter 8 through the lens of algorithm design. We examine how each condition could be leveraged in practice, and this analysis naturally leads to heuristics that will guide the development of the method presented in Section 11.2.5.

11.1.1 Lower bound

In this section, we derive a lower bound on the number of signals required to ensure the identifiability of all unknown modules. This bound will serve as guidance for selecting an initial iterate or as a stopping criterion for the algorithms developed in this chapter.

To guarantee identifiability, the number of excitation–measurement pairs must be greater or equal to the number of unknowns, i.e. $n_B \cdot n_C \geq m^\circ$. To determine the lower bound, we minimize the total number of signals, i.e. the sum of excitations and measurements. Denoting $x \triangleq n_B \in \mathbb{N}$ and $y \triangleq n_C \in \mathbb{N}$, the problem reduces to:

$$\begin{aligned} \min_{x,y} \quad & f(x,y) = x + y \\ \text{s.t.} \quad & x \cdot y \geq m^\circ \\ & x, y \in \mathbb{N} \end{aligned} \tag{P1}$$

A lower bound of our signal selection problem is given by the optimal condition of (P1), which is given by the proposition below.

Proposition 11.1. *The optimal solution of (P1) is*

$$(x_1^*, y_1^*) = \left(\left\lceil \sqrt{m^\circ} \right\rceil, \left\lceil \frac{m^\circ}{\left\lceil \sqrt{m^\circ} \right\rceil} \right\rceil \right). \tag{S1}$$

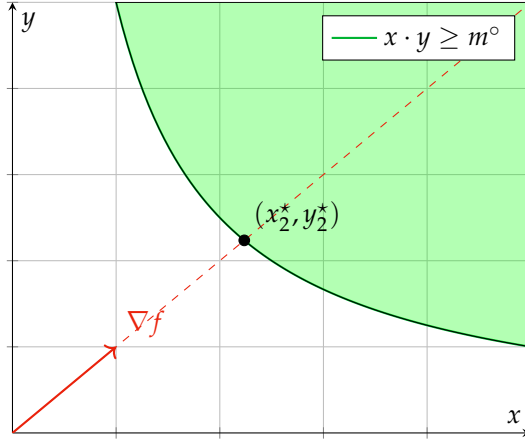


Fig. 11.1 Illustration of problem (P2) and its optimal solution.

Proof. Consider the relaxation of problem (P1), where variables x, y can take real values:

$$\begin{aligned} \min_{x,y} \quad & f(x, y) = x + y \\ \text{s.t.} \quad & x \cdot y \geq m^\circ \\ & x, y \in \mathbb{R}^+ \end{aligned} \tag{P2}$$

Problem (P2) is illustrated in Figure 11.1 and its solution is $(x_2^*, y_2^*) = (\sqrt{m^\circ}, \sqrt{m^\circ})$, with optimal objective value $f_2^* = f(x_2^*, y_2^*) = 2\sqrt{m^\circ}$. Since (P2) is a relaxation of (P1), its optimal objective value is a lower bound to the one of (P1), i.e. we have $2\sqrt{m^\circ} \leq x_1^* + y_1^*$, where (x_1^*, y_1^*) is the optimal solution of problem (P1).

Consider now the natural-valued problem (P1). The problem parameter m° can take any natural value, and four distinct cases can be detailed, illustrated in Fig. 11.2. Observe that those four cases form a partition of $\mathbb{N}$.

- (a) m° is a square number, i.e. there exists $r \in \mathbb{N}$ such that $m^\circ = r^2$. Then the optimal solution of (P2) is admissible, and is also the optimal solution of (P1):

$$(x_1^*, y_1^*) = (x_2^*, y_2^*) = (\sqrt{m^\circ}, \sqrt{m^\circ}) = (r, r) \in \mathbb{N}^2.$$

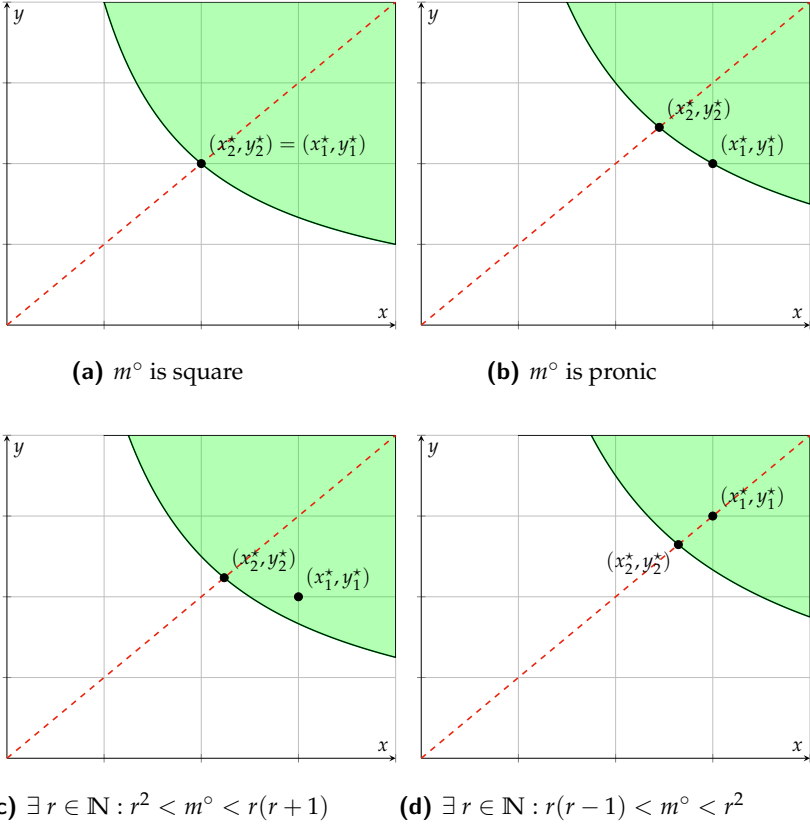


Fig. 11.2 The possible values of m° can be partitioned in four distinct cases.

Since $\sqrt{m^\circ} \in \mathbb{N}$, we have $\left(\lceil \sqrt{m^\circ} \rceil, \left\lceil \frac{m^\circ}{\lceil \sqrt{m^\circ} \rceil} \right\rceil\right) = (\sqrt{m^\circ}, \sqrt{m^\circ}) = (r, r)$.

- (b) m° is a pronic number, i.e. there exists $r \in \mathbb{N}$ such that $m^\circ = r \cdot (r+1)$. Since m° is not square, the optimal solution of (P2) $(\sqrt{m^\circ}, \sqrt{m^\circ}) \notin \mathbb{N}^2$, and its optimal objective $f_2^* = 2\sqrt{m^\circ}$ is unreachable. Since m° is pronic, we have $2\sqrt{m^\circ} = 2\sqrt{r(r+1)} > 2r$. Thus $2r$ is a strict lower bound to the objective, and if we find (x, y) such that $f(x, y) = 2r+1$, then it is optimal. This objective value is obtained by the following

solution, which is feasible since m° is a pronic number:

$$(x_1^*, y_1^*) = (r + 1, r) \in \mathbb{N}^2.$$

We have $\lceil \sqrt{m^\circ} \rceil = \lceil \sqrt{r(r+1)} \rceil = r + 1$ since $r < \sqrt{r(r+1)} < r + 1$. Hence we verify that the expression (S1) equals this solution: $\left(\lceil \sqrt{m^\circ} \rceil, \left\lceil \frac{m^\circ}{\lceil \sqrt{m^\circ} \rceil} \right\rceil \right) = (r + 1, r)$.

(c) There exists $r \in \mathbb{N}$ such that:

$$r^2 < m^\circ < r(r + 1). \quad (11.1)$$

Since m° is not square, the optimal solution of (P2) $(\sqrt{m^\circ}, \sqrt{m^\circ}) \notin \mathbb{N}^2$, and its optimal objective $f_2^* = 2\sqrt{m^\circ}$ is unreachable. From (11.1), we have $2\sqrt{m^\circ} > 2r$. Then the same reasoning as case (b) applies: $2r$ is a strict lower bound to the objective, and if we find (x, y) such that $f(x, y) = 2r + 1$, then it is optimal. This objective value is obtained by the following solution, which is ensured feasible by (11.1):

$$(x_1^*, y_1^*) = (r + 1, r) \in \mathbb{N}^2.$$

From (11.1), we have $r < \sqrt{m^\circ} < \sqrt{r(r+1)} < r + 1$, hence $\lceil \sqrt{m^\circ} \rceil = r + 1$. Equation (11.1) also yields $\frac{r^2}{r+1} < \frac{m^\circ}{r+1} < r$, and $\frac{r^2}{r+1} = \frac{r^2-1}{r+1-1} = \frac{(r+1)(r-1)}{r} > r - 1$. Combining those inequalities gives $r - 1 < \frac{m^\circ}{r+1} < r$, thus $\lceil \frac{m^\circ}{r+1} \rceil = r$, which confirms that this solution equals expression (S1).

(d) There exists $r \in \mathbb{N}$ such that:

$$r(r - 1) < m^\circ < r^2. \quad (11.2)$$

Since m° is not square, the optimal solution of (P2) $(\sqrt{m^\circ}, \sqrt{m^\circ}) \notin \mathbb{N}^2$, and its optimal objective $f_2^* = 2\sqrt{m^\circ}$ is unreachable. From (11.2), we have $2\sqrt{m^\circ} > 2\sqrt{r(r-1)} > 2(r-1) = 2r - 2$. Thus $2r - 2$ is a strict lower bound to the objective, and if we find (x, y) such that $f(x, y) = 2r - 1$, then it is optimal. We prove that there is no such feasible (x, y) , i.e. that solving the following problem for (x, y) gives

no solution:

$$\begin{aligned} x + y &= 2r + 1 \\ r(r-1) &< m^\circ \leq xy \\ x, y &\in \mathbb{N} \end{aligned} \tag{11.3}$$

We have $y = 2r - 1 - x$ and after plugging this expression for y , the inequality gives $x^2 + (1 - 2r)x + r(r + 1) = (x - r) \cdot (x - (r - 1)) < 0$, thus x must lie in the open interval $]r - 1, r[$, which is impossible since $x \in \mathbb{N}$. Since $2r - 1$ is not reachable by any feasible solution, it is a strict lower bound to the objective, and if we find (x, y) such that $f(x, y) = 2r$, then it is optimal. Such objective is obtained by the following solution, which is ensured feasible by (11.2):

$$(x_1^*, y_1^*) = (r, r) \in \mathbb{N}^2.$$

From (11.2), we have $r - 1 < \sqrt{r(r-1)} < \sqrt{m^\circ} < r$, hence $\lceil \sqrt{m^\circ} \rceil = r$. Equation (11.2) also gives $\frac{r(r-1)}{r} < \frac{m^\circ}{r} < \frac{r^2}{r}$, thus $r - 1 < \frac{m^\circ}{r} < r$, which yields $\lceil \frac{m^\circ}{r} \rceil = r$. This confirms that this solution equals expression (S1).

□

11.1.2 Generic local identifiability test

In Chapter 5, we developed Algorithm 2, a procedure that exploits the algebraic condition and the pseudo-generic nature of local identifiability, as established in Theorem 5.12. The algorithm efficiently tests these conditions on a randomly chosen matrix A consistent with the network structure, and returns the correct result with probability 1.

Algorithm 2:

input The network graph $\mathcal{G}_{\text{net}} = (\mathcal{W}, \{\mathcal{E}^\bullet, \mathcal{E}^\circ\})$, excitation set $\mathcal{B}$ and measurement set $\mathcal{C}$

output The generic local identifiability of *each distinct* unknown module

Algorithm 2 serves as a central and efficient tool in the development of our signal selection algorithms. It enables the verification of generic local identifiability for any given EM network, i.e. for any network topology, and

any excitations and measurements sets. As will be seen in the following sections, our signal selection procedures rely heavily on this algorithm.

Algorithm 2 is based on the matrix $K = (B^\top T^\top \otimes CT) I^\circ$, as defined in Definition 5.8, where $T = (I - A)^{-1}$. This matrix has dimensions $n_B n_C \times m^\circ$, and on large-scale instances, repeatedly computing K can become computationally expensive.

Consider a procedure that repeatedly invokes Algorithm 2 to search for minimal excitation and measurement sets that ensure generic local identifiability of a given network. Although the excitation and measurement sets vary across calls, the underlying network topology remains fixed throughout the procedure.

This observation enables a significant computational optimization: instead of recomputing the matrix $K = (B^\top T^\top \otimes CT) I^\circ$ at each iteration, one can precompute and store the full matrix $K_{\text{full}} = (T^\top \otimes T) I^\circ$ just once at the beginning. Then, for each specific excitation and measurement configuration, the relevant submatrix of K can be efficiently extracted from K_{full} , via careful indexing to respect the Kronecker product structure of K_{full} .

However, implementing $K_{\text{full}} = (T^\top \otimes T) I^\circ$ naively involves first constructing the full $L^2 \times L^2$ Kronecker product matrix $T^\top \otimes T$, and then selecting a subset of its columns via multiplication with I° . This approach is computationally inefficient and memory-intensive, especially for large networks, as the vast majority of the entries in $T^\top \otimes T$ are ultimately discarded by the multiplication with I° .

To address this, we implemented a dedicated function, `kron_submatrix`, which computes K_{full} in a more efficient manner. Instead of forming the full Kronecker product, it directly computes only the relevant entries selected by the post-product I° , thereby reducing both computation time and memory usage. The implementation is available at [LH25]: `identifiable / algebraic_identifiability_test / kron_submatrix`.

11.1.3 Monotonicity of identifiability

Identifiability — whether generic or not, global or local — exhibits a form of monotonicity with respect to excitations and measurements, in the following sense:

- Adding excitations or measurements to an identifiable network preserves identifiability.

- Conversely, removing excitations or measurement signals from a non-identifiable network cannot render it identifiable.

In our setting, we are interested in minimal configurations that ensure identifiability : cases where the network is identifiable, but the removal of any single excitation or measurement breaks this property.

This type of monotonicity is reminiscent of similar phenomena in other domains. For example, in graph connectivity, once a graph is connected, adding edges cannot make it disconnected; and conversely, a disconnected graph cannot be made connected by removing edges. The minimal number of edges that ensures connectivity corresponds to a spanning tree, just as we seek minimal sets of signals ensuring identifiability.

11.1.4 SIMUG and pseudotree covering algorithms

In section 3.2, we presented Algorithm 1 from [DSC⁺22] : a procedure to compute a set of excitations that guarantees generic global identifiability of the input network, when all nodes are measured, some modules are unknown and some modules are known. It first constructs an edge-disjoint SIMUG covering, then assigns one excitation per SIMUG root, and finally prunes unnecessary excitations by testing whether global identifiability is preserved upon their removal. Note that there is no guarantee on the optimality of the number of excitations obtained by Algorithm 1.

Algorithm 1 :

input The network graph $\mathcal{G}_{\text{net}} = (\mathcal{W}, \{\mathcal{E}^\bullet, \mathcal{E}^\circ\})$

output A set of excitations $\mathcal{B}$ such that the network is generically globally identifiable under full measurement

As discussed in Section 3.2.1, [CSVdH21] develops a closely related algorithm for the case without known modules, where SIMUGs are replaced by *pseudotrees*.

11.1.5 Heuristics from walk-based conditions

In this section, we revisit the conditions established in Chapter 8 and explore how they can be leveraged in the design of our algorithms. For the reader's convenience, we provide a brief summarized statement of each condition

below, using “identifiability” as a shorthand for “generic local network identifiability”.

- Proposition 8.1 (i), summarized as :

$$\text{Identifiability} \Rightarrow \exists \mathcal{P} : \exists \text{ bijective } \omega \in \Omega_{\mathcal{P}}$$

This necessary condition is too permissive: it is satisfied by a wide range of network configurations that are not generically locally identifiable, as evidenced by empirical analysis.

- Proposition 8.1 (ii), summarized as :

$$\exists \mathcal{P} : \exists \text{ only 1 bijective } \omega \in \Omega_{\mathcal{P}} \Rightarrow \text{Identifiability}$$

This sufficient condition is overly restrictive, particularly for highly connected networks. Empirical observations show that most generically locally identifiable networks fail to meet this criterion. Moreover, verifying the uniqueness of a bijective collection of walks requires checking all possible combinations — a task that quickly becomes computationally intractable when designing algorithms intended to scale with arbitrary network sizes and topologies.

- Theorem 8.7 — the necessary condition, summarized as :

$$\text{Identifiability} \Rightarrow \exists \mathcal{P} : \exists \text{ bijective } \omega \in \Omega_{\mathcal{P}} \text{ without MN in its C-parts} \\ \text{(resp. B-parts)}$$

This necessary condition refines Proposition 8.1 (i) by being more restrictive, while being practical to implement:

- To enforce the bijectivity constraint, an assignment σ can be constructed incrementally while iterating over $\mathcal{B}$ and $\mathcal{C}$, ensuring that it remains both bijective and connected at each step. Such a bijective and connected assignment σ guarantees the existence of a corresponding bijective collection of walks ω .
- The condition requiring the absence of meeting nodes is equivalent to the existence of vertex-disjoint walks, which can be efficiently verified by checking the rank of the corresponding T matrices, as established in Proposition 3.3.

- Theorem 8.7 — the sufficient condition, summarized as :

$$\exists \mathcal{P} : \exists \text{ only 1 bijective } \omega \in \Omega_{\mathcal{P}} \text{ without MN in its C-parts} \Rightarrow \text{Identif.} \\ \text{(resp. B-parts)}$$

Similar to Proposition 8.1 (ii), this sufficient condition is difficult to handle in practice: verifying the uniqueness of such a collection of

walks ω would require enumerating all possible candidates, which becomes computationally intractable for large and highly connected networks.

As discussed in Section 8.4, it is tempting to strengthen the necessary condition Theorem 8.7 by enforcing the vertex-disjoint walk constraints (A) and (B) on both the excitation and measurement sides. This leads to Conjecture 8.8, a necessary condition which was disproven in Section 8.4 through a counterexample.

In our approach, we use this (disproven) necessary condition as a filtering step for candidate EMP sets. Validated candidates are then tested using Algorithm 2 to check whether they ensure generic local network identifiability.

Relying on this disproven condition as a necessary test will inevitably exclude some valid solutions and result in optimality loss. Nevertheless, we will show that, when combined with other algorithmic tools, its practical impact can be mitigated.

11.2 Algorithms description

This section presents our signal selection algorithms, built upon the tools introduced in Section 11.1. All implementations are available in the GitHub repository [LH25].

11.2.1 Exhaustive Search

Algorithm 3 presents our Exhaustive Search: starting from the lower bound calculated in Proposition 11.1, the procedure tests the generic local identifiability of all possible combinations of excitations and measurements, using the algebraic criterion of Corollary 5.14. As soon as the criterion is met, the algorithm returns the considered excitation and measurement sets, with the guarantee that their total cardinality is optimal.

The notation $K_{\text{full}, \mathcal{I}^*}$ denotes the submatrix of K_{full} consisting of the rows indexed by $\mathcal{I}$ and all columns. It is important to highlight that the set Ξ constructed at line 6 is combinatorially large. Moreover, the lower bound provided in Proposition 11.1 is only achieved in ideal scenarios, where

Algorithm 3 Exhaustive Search**Require:** The network graph $\mathcal{G}_{\text{net}} = (\mathcal{W}, \{\mathcal{E}^\circ, \mathcal{E}^\bullet\})$ **Result:** Excitation set $\mathcal{B}^*$, measurement set $\mathcal{C}^*$ such that the EM network $(\mathcal{G}_{\text{net}}, \mathcal{B}^*, \mathcal{C}^*)$ is generically locally identifiable, and $|\mathcal{B}^*| + |\mathcal{C}^*|$ *minimal*

- 1: Randomly sample matrices $A^\circ, A^\bullet$ consistent with $\mathcal{E}^\circ, \mathcal{E}^\bullet$
- 2: $A := A^\circ + A^\bullet$ and $T := (I - A)^{-1}$
- 3: Compute $K_{\text{full}} = (T^\top \otimes T) I^\circ$ with function `kron_submatrix`
- 4: $n_{\mathbb{R}} := \lceil \sqrt{m^\circ} \rceil + \lceil \frac{m^\circ}{\sqrt{m^\circ}} \rceil$
- 5: **for** $n = n_{\mathbb{R}} \rightarrow 2L$ **do**
- 6: Construct $\Xi := \{(\mathcal{B}, \mathcal{C}) : \mathcal{B} \subseteq \mathcal{W}, \mathcal{C} \subseteq \mathcal{W}, |\mathcal{B}| + |\mathcal{C}| = n\}$, set of all combinations of excitations and measurements with total cardinality n
- 7: **for** $(\mathcal{B}, \mathcal{C}) \in \Xi$ **do**
- 8: Compute $\mathcal{I}$, the row indices corresponding to $(\mathcal{B}, \mathcal{C})$ in K_{full}
- 9: **if** $\text{rank } K_{\text{full}, \mathcal{I}^*} = m^\circ$ **then**
- 10: **return** $(\mathcal{B}^*, \mathcal{C}^*) := (\mathcal{B}, \mathcal{C})$

each excitation-measurement pair uniquely identifies a distinct unknown module — a situation that is rarely encountered in practice.

As a result, the Exhaustive Search is expected to be highly inefficient, as it must evaluate a vast number of combinations before finding a valid solution. In practice, the method becomes intractable for networks of moderate size. For instance, our experiments show that even on networks with as few as 20 nodes, the algorithm becomes prohibitively slow.

This outcome is expected, given the inherently naive and exhaustive nature of the algorithm. Nevertheless, despite its simplicity, it is the only method among those we propose that guarantees optimality. As such, its solutions serve as a valuable benchmark for evaluating the performance of our other, more scalable algorithms on small-scale networks.

11.2.2 Pruning Search

Algorithm 4 outlines our Pruning Search strategy. The procedure begins with all nodes excited and all nodes measured, ensuring that the initial configuration guarantees identifiability. It then performs n_{pass} independent pruning passes, where each pass consists of arbitrarily removing one excita-

tion or measurement and subsequently verifying whether identifiability is preserved. The algorithm finally returns the best solution across these n_{pass} trials, defined as the configuration with the lowest total number of signals.

Algorithm 4 Pruning Search

Require: The network graph $\mathcal{G}_{\text{net}} = (\mathcal{W}, \{\mathcal{E}^\circ, \mathcal{E}^\bullet\})$, parameter n_{pass}

Result: Excitation set $\mathcal{B}^*$, measurement set $\mathcal{C}^*$ such that the EM network $(\mathcal{G}_{\text{net}}, \mathcal{B}^*, \mathcal{C}^*)$ is generically locally identifiable

- 1: Randomly sample matrices $A^\circ, A^\bullet$ consistent with $\mathcal{E}^\circ, \mathcal{E}^\bullet$
 - 2: $A := A^\circ + A^\bullet$ and $T := (I - A)^{-1}$
 - 3: Compute $K_{\text{full}} = (T^\top \otimes T) I^\circ$ with function `kron_submatrix`
 - 4: Initialize $\mathcal{B}_0 := \mathcal{W}$ and $\mathcal{C}_0 := \mathcal{W}$
 - 5: **return** $(\mathcal{B}^*, \mathcal{C}^*) := \text{pruning_multipass}(\mathcal{G}_{\text{net}}, K_{\text{full}}, \mathcal{B}_0, \mathcal{C}_0, n_{\text{pass}})$
-

Each pruning pass follows the same principle: starting from the fully excited and measured configuration, it iteratively attempts to remove excitations and measurements while preserving generic local identifiability. The order in which signals are considered for removal is randomized in each pass, which impacts the quality of the resulting solution. Repeating this randomized pruning procedure n_{pass} times increases the likelihood of finding a solution with minimal cardinality.

To improve readability and facilitate reuse across algorithms, we define a companion function, `pruning_multipass`, which manages the execution of the n_{pass} pruning passes. Notation $\mathcal{B}_0 \uplus \mathcal{C}_0$ at line 6 of Algorithm 5 denotes the multiset union, where common elements of $\mathcal{B}_0$ and $\mathcal{C}_0$ appear twice. Line 16 returns the current solution if it is as good as the lower bound n_{R} derived in Section 11.1.1. Line 18 updates the best solution found so far with the current one, if its number of signals is lower.

Compared to the Exhaustive Search, the Pruning Search leverages randomization to significantly improve computational efficiency. While this comes at the cost of losing the guarantee of optimality, it yields a tractable algorithm. However, the Pruning Search remains agnostic to the structural properties of the problem — in particular, it does not exploit the network topology or the walk-based conditions developed earlier. The subsequent algorithms address this limitation by incorporating structural insights and tailored heuristics into their design, aiming to enhance both efficiency and solution quality.

Algorithm 5 pruning_multipass

Require: $\mathcal{G}_{\text{net}}, K_{\text{full}}, \mathcal{B}_0, \mathcal{C}_0, n_{\text{pass}}$ **Result:** $\mathcal{B}^*, \mathcal{C}^*$ such that $(\mathcal{G}_{\text{net}}, \mathcal{B}^*, \mathcal{C}^*)$ is generically locally identifiable

```

1:  $n_{\mathbb{R}} := \lceil \sqrt{m^\circ} \rceil + \left\lceil \frac{m^\circ}{\lceil \sqrt{m^\circ} \rceil} \right\rceil$ 
2:  $n_0 := |\mathcal{B}_0| + |\mathcal{C}_0|$ 
3: Initialize  $\mathcal{B}^* := \mathcal{B}_0$ ,  $\mathcal{C}^* := \mathcal{C}_0$  and  $n^* := n_0$ 
4: if  $n_0 > n_{\mathbb{R}}$  then
5:   for  $p = 1 \rightarrow n_{\text{pass}}$  do
6:     for  $\lambda$  randomly picked in  $\mathcal{B}_0 \uplus \mathcal{C}_0$  do
7:       if  $\lambda$  was picked from  $\mathcal{B}_0$  then
8:          $\mathcal{B}_{\text{new}} := \mathcal{B} \setminus \lambda$  and  $\mathcal{C}_{\text{new}} := \mathcal{C}$ 
9:       else
10:         $\mathcal{B}_{\text{new}} := \mathcal{B}$  and  $\mathcal{C}_{\text{new}} := \mathcal{C} \setminus \lambda$ 
11:        Compute  $\mathcal{I}$ , the indices of  $(\mathcal{B}_{\text{new}}, \mathcal{C}_{\text{new}})$  in  $K_{\text{full}}$ 
12:        if rank  $K_{\text{full}, \mathcal{I}^*} = m^\circ$  then
13:           $\mathcal{B} := \mathcal{B}_{\text{new}}$  and  $\mathcal{C} := \mathcal{C}_{\text{new}}$ 
14:         $n := |\mathcal{B}| + |\mathcal{C}|$ 
15:        if  $n \leq n_{\mathbb{R}}$  then
16:          return  $(\mathcal{B}^*, \mathcal{C}^*) := (\mathcal{B}, \mathcal{C})$ 
17:        else if  $n < n^*$  then
18:           $\mathcal{B}^* := \mathcal{B}, \mathcal{C}^* := \mathcal{C}$  and  $n^* := n$ 

```

11.2.3 Pseudotree Search

Algorithm 6 presents our Pseudotree Search strategy, which closely follows the general procedure of the Pruning Search. The key difference lies in the initialization step. Instead of starting from full excitation and full measurement, the algorithm begins by invoking the pseudotree allocation algorithm from [CSVdH21]. As discussed in Section 3.2.1, this algorithm provides an excitation set that guarantees generic global identifiability when *all* modules are unknown and *all* nodes are measured.

In our context, since some modules are known, this initialization likely leads to an overestimated excitation set, though this is not formally guaranteed. To obtain an initial measurement set, we apply the same pseudotree allocation procedure to the transposed network, in the sense of [CSLVdH24]: the adjacency matrix is transposed, and the roles of excitations and measurements are swapped. The transposed network of a given network $\mathcal{G}_{\text{net}}$ is denoted by $\mathcal{G}_{\text{net}}^\top$.

Algorithm 6 Pseudotree Search

Require: The network graph $\mathcal{G}_{\text{net}} = (\mathcal{W}, \{\mathcal{E}^\circ, \mathcal{E}^\bullet\})$, parameter n_{pass}

Result: Excitation set $\mathcal{B}^*$, measurement set $\mathcal{C}^*$ such that the EM network $(\mathcal{G}_{\text{net}}, \mathcal{B}^*, \mathcal{C}^*)$ is generically locally identifiable

- 1: Randomly sample matrices $A^\circ, A^\bullet$ consistent with $\mathcal{E}^\circ, \mathcal{E}^\bullet$
 - 2: $A := A^\circ + A^\bullet$ and $T := (I - A)^{-1}$
 - 3: Compute $K_{\text{full}} = (T^\top \otimes T) I^\circ$ with function `kron_submatrix`
 - 4: $\mathcal{B}_0 :=$ solution of the pseudotree algorithm of [CSVdH21] with input $\mathcal{G}_{\text{net}}$ where all modules are assumed unknown, and all nodes measured
 - 5: $\mathcal{C}_0 :=$ solution of the pseudotree algorithm of [CSVdH21] with input $\mathcal{G}_{\text{net}}^\top$ where all modules are assumed unknown, and all nodes measured
 - 6: **return** $(\mathcal{B}^*, \mathcal{C}^*) = \text{adding_pruning_multipass}(\mathcal{G}_{\text{net}}, K_{\text{full}}, \mathcal{B}_0, \mathcal{C}_0, n_{\text{pass}})$
-

Although empirical evidence suggests that the resulting excitation and measurement sets often ensure generic local identifiability, no guarantee is provided. Therefore, we begin with an addition phase, where excitations and measurements are incrementally added at random until the network becomes generically locally identifiable. This is followed by a pruning phase, analogous to Algorithm 5. The entire procedure is repeated n_{pass}

times, and the best (i.e., lowest cardinality) solution among these runs is returned, as described by Algorithm 7.

Algorithm 7 adding_pruning_multipass

Require: $\mathcal{G}_{\text{net}}, K_{\text{full}}, \mathcal{B}_0, \mathcal{C}_0, n_{\text{pass}}$

Result: $\mathcal{B}^*, \mathcal{C}^*$ such that $(\mathcal{G}_{\text{net}}, \mathcal{B}^*, \mathcal{C}^*)$ is generically locally identifiable

```

1:  $n_{\mathbb{R}} := \lceil \sqrt{m^\circ} \rceil + \lceil \frac{m^\circ}{\lceil \sqrt{m^\circ} \rceil} \rceil$ 
2:  $n_0 := |\mathcal{B}_0| + |\mathcal{C}_0|$ 
3: Initialize  $\mathcal{B}^* := \mathcal{B}_0, \mathcal{C}^* := \mathcal{C}_0$  and  $n^* := n_0$ 
4: if  $n_0 > n_{\mathbb{R}}$  then
5:   for  $p = 1 \rightarrow n_{\text{pass}}$  do
6:     Compute  $\mathcal{I}$ , the indices of  $(\mathcal{B}_0, \mathcal{C}_0)$  in  $K_{\text{full}}$ 
7:     while  $\text{rank } K_{\text{full}, \mathcal{I}^*} < m^\circ$  do
8:       Randomly add an excitation or measurement to  $(\mathcal{B}_0, \mathcal{C}_0)$ 
9:       Update  $\mathcal{I}$  accordingly
10:    end while
11:    for  $\lambda$  randomly picked in  $\mathcal{B}_0 \uplus \mathcal{C}_0$  do
12:      if  $\lambda$  was picked from  $\mathcal{B}_0$  then
13:         $\mathcal{B}_{\text{new}} := \mathcal{B} \setminus \lambda$  and  $\mathcal{C}_{\text{new}} := \mathcal{C}$ 
14:      else
15:         $\mathcal{B}_{\text{new}} := \mathcal{B}$  and  $\mathcal{C}_{\text{new}} := \mathcal{C} \setminus \lambda$ 
16:      Compute  $\mathcal{I}$ , the indices of  $(\mathcal{B}_{\text{new}}, \mathcal{C}_{\text{new}})$  in  $K_{\text{full}}$ 
17:      if  $\text{rank } K_{\text{full}, \mathcal{I}^*} = m^\circ$  then
18:         $\mathcal{B} := \mathcal{B}_{\text{new}}$  and  $\mathcal{C} := \mathcal{C}_{\text{new}}$ 
19:       $n := |\mathcal{B}| + |\mathcal{C}|$ 
20:      if  $n \leq n_{\mathbb{R}}$  then
21:        return  $(\mathcal{B}^*, \mathcal{C}^*) := (\mathcal{B}, \mathcal{C})$ 
22:      else if  $n < n^*$  then
23:         $\mathcal{B}^* := \mathcal{B}, \mathcal{C}^* := \mathcal{C}$  and  $n^* := n$ 

```

11.2.4 SIMUG Search

Algorithm 8 presents our SIMUG Search strategy, which differs from the Pseudotree Search only in its initialization step. Instead of using the pseudotree allocation algorithm from [CSVdH21], it initializes with the SIMUG allocation algorithm introduced in [DSC⁺22], discussed in Section 11.1.4, detailed in Section 3.2, and denoted as Algorithm 1.

Algorithm 8 SIMUG Search

Require: The network graph $\mathcal{G}_{\text{net}} = (\mathcal{W}, \{\mathcal{E}^\circ, \mathcal{E}^\bullet\})$, parameter n_{pass}

Result: Excitation set $\mathcal{B}^*$, measurement set $\mathcal{C}^*$ such that the EM network $(\mathcal{G}_{\text{net}}, \mathcal{B}^*, \mathcal{C}^*)$ is generically locally identifiable

- 1: Randomly sample matrices $A^\circ, A^\bullet$ consistent with $\mathcal{E}^\circ, \mathcal{E}^\bullet$
 - 2: $A := A^\circ + A^\bullet$ and $T := (I - A)^{-1}$
 - 3: Compute $K_{\text{full}} = (T^\top \otimes T) I^\circ$ with function `kron_submatrix`
 - 4: $\mathcal{B}_0 := \text{Algorithm 1}(\mathcal{G}_{\text{net}})$
 - 5: $\mathcal{C}_0 := \text{Algorithm 1}(\mathcal{G}_{\text{net}}^\top)$
 - 6: **return** $(\mathcal{B}^*, \mathcal{C}^*) = \text{adding_pruning_multipass}(\mathcal{G}_{\text{net}}, K_{\text{full}}, \mathcal{B}_0, \mathcal{C}_0, n_{\text{pass}})$
-

Empirical evidence also indicates that, unlike in Algorithm 6, the initialized excitation and measurement sets frequently fail to guarantee identifiability. As a result, the additional augmentation step (adding pass) is often required in this case to achieve identifiability. The lack of identifiability of the initialized excitation and measurement sets can be explained as follows. The initial excitation set, $\mathcal{B}_0$, is generated by Algorithm 1, which, under the assumption that all nodes are measured, produces a set of excitations that guarantees generic global identifiability. Since the algorithm is designed to minimize the size of $\mathcal{B}_0$ (although without guaranteeing optimality), the resulting set tends to be relatively restrictive and small. The initial measurement set, $\mathcal{C}_0$, is obtained in an analogous manner by applying Algorithm 1 to the transposed network. Consequently, both $\mathcal{B}_0$ and $\mathcal{C}_0$ are typically restrictive and small, and the corresponding EM network obtained by combining them is often non-identifiable, hence the frequent use of the adding pass. This issue was less pronounced for Algorithm 6, since known modules were treated as unknown during the initialization. As a result, the initial sets $\mathcal{B}_0$ and $\mathcal{C}_0$ were less restrictive and often ensured identifiability.

11.2.5 Merging Search

Algorithm 9 presents our Merging Search strategy, which leverages the heuristics introduced in Section 11.1.5. The algorithm initializes the excitation and measurement sets as the start and end nodes of the unknown modules respectively, an initialization that guarantees identifiability. It then iteratively alternates between B -merges (excitation merges) and C -merges (measurement merges), aiming to reduce the total number of signals while preserving identifiability.

In a B -merge, the algorithm considers replacing a group of excitations by one of their common ancestors — i.e., a node w from which there exists a path to each excitation in the group. Guided by the heuristics from Section 11.1.5, it checks that such a merge does not introduce any meeting node in either the B -parts or the C -parts. This verification is based on the current assignments σ_B and σ_C , which are updated throughout the merging process. Additionally, it ensures that the pair (σ_B, σ_C) remains bijective. If these walk-based conditions are satisfied, the algorithm then tests for generic local identifiability using the algebraic criterion, and performs the merge if the condition holds. C -merges are carried out analogously, on the measurement nodes.

After the merging phase, the algorithm performs n_{pass} pruning passes to eliminate potentially redundant signals and reduce the overall cardinality of the solution. This entire procedure is repeated n_{run} times, and the solution with the lowest cardinality among all runs is returned.

There are two variants of the merging subroutines: the `ranking` version and the `uniform` version. The distinction between the two lies in the order in which candidate pairs $(w, \text{candidates})$ are selected and evaluated for merging. The `uniform` version samples these pairs uniformly at random. In contrast, the `ranking` version leverages the network structure to guide the search more efficiently. For instance, it is pointless to consider potential B -merges toward a sink node. More promising candidates are nodes from which paths reach many excitation points, an intuition captured by the ranking vector `ranking`, which determines the processing order in the `ranking` version.

Empirically, with a single run, the `ranking` version tends to outperform the `uniform` version, both in speed and in producing solutions with fewer signals. However, when multiple runs are allowed, the `uniform` version

explores a broader space of solutions and can eventually yield better results, whereas the deterministic ranking version does not benefit from repeated runs.

The number of potential merges grows combinatorially with the network size, making exhaustive exploration increasingly impractical. To mitigate this, a stopping criterion is introduced to terminate the search early, with its threshold adapted to the size of the network.

To keep the document concise, we include only the pseudocode for the `merging_ranking_B` subroutine, presented in Algorithm 10. Its measurement version `merging_ranking_C` is analogous, while `merging_uniform_B` and `merging_uniform_C` are simpler variants using random ordering. All implementations are available in the GitHub repository [LH25].

Algorithm 9 Merging Search

Require: The network graph $\mathcal{G}_{\text{net}} = (\mathcal{W}, \{\mathcal{E}^\circ, \mathcal{E}^\bullet\})$, parameters $n_{\text{run}}, n_{\text{pass}}$ and parameter $\text{stop} \in [0, 1]$

Result: Excitation set $\mathcal{B}^*$, measurement set $\mathcal{C}^*$ such that the EM network $(\mathcal{G}_{\text{net}}, \mathcal{B}^*, \mathcal{C}^*)$ is generically locally identifiable

- 1: $n_{\mathbb{R}} := \lceil \sqrt{m^\circ} \rceil + \lceil \frac{m^\circ}{\lceil \sqrt{m^\circ} \rceil} \rceil$
- 2: $T^\dagger := \text{ancestors_weighted}(\mathcal{G}_{\text{net}})$
- 3: Initialize $\mathcal{B}_0 := \{s(\alpha) \mid \alpha \in \mathcal{E}^\circ\}$ and $\mathcal{C}_0 := \{t(\alpha) \mid \alpha \in \mathcal{E}^\circ\}$
- 4: Initialize $\sigma_B := \mathcal{E}^\circ \rightarrow \mathcal{W} : \alpha \rightarrow s(\alpha)$ and $\sigma_C := \mathcal{E}^\circ \rightarrow \mathcal{W} : \alpha \rightarrow t(\alpha)$
- 5: Initialize $\mathcal{B}^* := \mathcal{B}_0$, $\mathcal{C}^* := \mathcal{C}_0$ and $n^* := 2m^\circ$
- 6: **for** $r = 1 \rightarrow n_{\text{run}}$ **do**
- 7: Randomly sample matrices $A^\circ, A^\bullet$ consistent with $\mathcal{E}^\circ, \mathcal{E}^\bullet$
- 8: $A := A^\circ + A^\bullet$ and $T := (I - A)^{-1}$ and $T^\dagger := T$ with zero diagonal
- 9: Compute $K_{\text{full}} = (T^\top \otimes T) I^\circ$ with function `kron_submatrix`
- 10: **if** $r = 1$ **then**
- 11: $(\mathcal{B}, \mathcal{C}) = \text{merging_ranking_B}(\mathcal{G}_{\text{net}}, T^\dagger, T, K_{\text{full}}, \mathcal{B}_0, \mathcal{C}_0, \sigma_B, \sigma_C, \text{stop})$
- 12: **else if** $r = 2$ **then**
- 13: $(\mathcal{B}, \mathcal{C}) = \text{merging_ranking_C}(\mathcal{G}_{\text{net}}, T^\dagger, T, K_{\text{full}}, \mathcal{B}_0, \mathcal{C}_0, \sigma_B, \sigma_C, \text{stop})$
- 14: **else**
- 15: **if** $\text{rand}(0, 1) < 1/2$ **then**
- 16: $(\mathcal{B}, \mathcal{C}) = \text{merging_uniform_B}(\mathcal{G}_{\text{net}}, T^\dagger, T, K_{\text{full}}, \mathcal{B}_0, \mathcal{C}_0, \sigma_B, \sigma_C, \text{stop})$
- 17: **else**
- 18: $(\mathcal{B}, \mathcal{C}) = \text{merging_uniform_C}(\mathcal{G}_{\text{net}}, T^\dagger, T, K_{\text{full}}, \mathcal{B}_0, \mathcal{C}_0, \sigma_B, \sigma_C, \text{stop})$
- 19: $(\mathcal{B}, \mathcal{C}) := \text{pruning_multipass}(\mathcal{G}_{\text{net}}, K_{\text{full}}, \mathcal{B}, \mathcal{C}, n_{\text{pass}})$
- 20: $n := |\mathcal{B}| + |\mathcal{C}|$
- 21: **if** $n \leq n_{\mathbb{R}}$ **then**
- 22: **return** $(\mathcal{B}^*, \mathcal{C}^*) := (\mathcal{B}, \mathcal{C})$
- 23: **else if** $n < n^*$ **then**
- 24: $\mathcal{B}^* := \mathcal{B}, \mathcal{C}^* := \mathcal{C}$ and $n^* := n$

Algorithm 10 merging_ranking_B**Require:** $\mathcal{G}_{\text{net}}, T^\dagger, T, K_{\text{full}}, \mathcal{B}, \mathcal{C}, \sigma_B, \sigma_C, \text{stop}$ **Result:** $(\mathcal{B}_{\text{out}}, \mathcal{C}_{\text{out}})$ s.t. $(\mathcal{G}_{\text{net}}, \mathcal{B}_{\text{out}}, \mathcal{C}_{\text{out}})$ is generically locally identifiable

```

1: if stop criterion is reached then
2:   return  $(\mathcal{B}_{\text{out}}, \mathcal{C}_{\text{out}}) := (\mathcal{B}, \mathcal{C})$ 
3: Compute vector ranking from  $T^\dagger$ 
4: for  $w \in \mathcal{W}$ , ordered by ranking do
5:   Construct groups = the set of combinations of excitations reachable
   by a walk from  $w$ 
6:   for candidates  $\in$  groups, in decreasing order of group size do
7:     if Replacing candidates by  $w$  in  $\mathcal{B}$  does not create MNs in the
       C-parts, according to  $\sigma_B$  then
8:        $\sigma_{B_{\text{new}}} :=$  update of  $\sigma_B$  when replacing candidates by  $w$  in  $\mathcal{B}$ 
9:       if  $(\sigma_{B_{\text{new}}}, \sigma_C)$  is bijective then
10:        if Replacing candidates by  $w$  in  $\mathcal{B}$  does not create MNs
          in the B-parts, according to  $\sigma_C$  then
11:           $\mathcal{B}_{\text{new}} := \mathcal{B} \cup w \setminus \text{candidates}$ 
12:          Compute  $\mathcal{I}$ , the indices of  $(\mathcal{B}_{\text{new}}, \mathcal{C})$  in  $K_{\text{full}}$ 
13:          if rank  $K_{\text{full}, \mathcal{I}^*} = m^\circ$  then
14:            if Not all C-merges have been tried then
15:               $(\mathcal{B}, \mathcal{C}) = \text{merging\_ranking\_C}(\mathcal{G}_{\text{net}}, T^\dagger, T, K_{\text{full}},$ 
                 $\mathcal{B}_{\text{new}}, \mathcal{C}, \sigma_{B_{\text{new}}}, \sigma_C, \text{stop})$ 
16:              return
17:            else
18:               $(\mathcal{B}, \mathcal{C}) = \text{merging\_ranking\_B}(\mathcal{G}_{\text{net}}, T^\dagger, T, K_{\text{full}},$ 
                 $\mathcal{B}_{\text{new}}, \mathcal{C}, \sigma_{B_{\text{new}}}, \sigma_C, \text{stop})$ 
19:              return
20:            if Not all C-merges have been tried then
21:               $(\mathcal{B}, \mathcal{C}) = \text{merging\_ranking\_C}(\mathcal{G}_{\text{net}}, T^\dagger, T, K_{\text{full}}, \mathcal{B}, \mathcal{C}, \sigma_B, \sigma_C, \text{stop})$ 
22:            else
23:               $(\mathcal{B}_{\text{out}}, \mathcal{C}_{\text{out}}) := (\mathcal{B}, \mathcal{C})$ 

```

11.3 Dataset description

To evaluate and compare the performance of our algorithms, we generated a dataset comprising networks of varying sizes and topologies, available in the GitHub repository [LH25]. This section provides a detailed description of the dataset.

11.3.1 Network topologies

We consider five distinct graph types, each exhibiting different structural properties, presented in Fig. 11.3. This selection is intended to cover a broad spectrum of network topologies, thereby aiming for generality in our evaluation. The chosen graph types include both synthetic structures and topologies inspired by practical applications or real-world networks.

Erdős–Rényi (ER) networks are a class of random graphs introduced in the late 1950s by Paul Erdős and Alfréd Rényi. In the ER model, a graph is constructed by connecting each pair of nodes independently with fixed probability. This simple generative rule leads to graphs with homogeneous degree distributions and no inherent community structure. Despite their simplicity, ER networks serve as a fundamental baseline in network science and are widely used for benchmarking algorithms due to their well-understood theoretical properties.

Lattice graphs are regular graphs in which each node is connected to its nearest neighbors in a structured, grid-like pattern. In two dimensions, they are often represented as nodes placed on the points of a square grid, where each node is connected to its four immediate neighbors. These graphs are highly regular and exhibit strong locality, making them useful for modeling spatial systems or physical networks such as electrical grids, crystal lattices, or image processing tasks. Their regularity and simplicity make them a common benchmark for evaluating the behavior of algorithms under structured topologies.

The *IEEE standard power networks* are a collection of benchmark test cases widely used in the power systems community for the evaluation of algorithms in areas such as state estimation, optimization, and control. These networks, maintained and distributed by the IEEE, provide realistic topologies, line parameters, and operational data that reflect practical constraints

and characteristics of real-world power grids. They serve as a standard reference to ensure reproducibility and comparability of results across studies, and are available at [Ins25].

Random geometric graphs (RGGs) are constructed by placing L nodes uniformly at random in a metric space (typically the unit square in $\mathbb{R}^2$), and connecting pairs of nodes whose Euclidean distance is below a given threshold. These graphs capture spatial locality and are widely used to model wireless communication networks, sensor networks, and other systems where connections depend on physical proximity. Unlike Erdős–Rényi graphs, RGGs naturally incorporate geometric constraints and clustering, making them useful for studying network algorithms in spatial settings.

Watts–Strogatz graphs are generated through a rewiring process that interpolates between regular lattices and random graphs, aiming to replicate the small-world phenomenon observed in many real-world networks. The construction starts from a ring lattice where each node is connected to its k nearest neighbors. Then, with a rewiring probability p , each edge is randomly rewired to a new node. This model yields graphs that simultaneously exhibit high clustering (like regular lattices) and short average path lengths (like random graphs), thereby capturing the small-world property. These graphs are often used to model social, biological, and technological networks where local cohesion and global connectivity coexist.

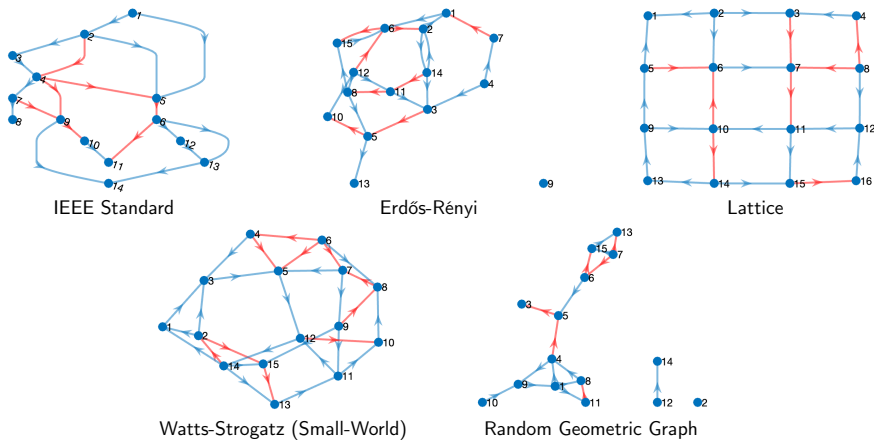


Fig. 11.3 The five network topologies of our dataset. Blue edges represent known modules, red edges unknown ones.

11.3.2 Network metrics

For each of those five network topologies, we generated networks with number of nodes $L \in \{10, 15, 20, 30, 40, 50, 100\}$ and with:

- Average degree $2m/L \in [2, 4]$, where m is the number of edges (both unknown and known)
- Ratio of unknown modules $m^\circ / m = 1/3$

For each network instance, we sampled 10 different configurations of unknown modules that adhere to these criteria.

11.4 Results

We applied the five algorithms described in Section 11.2 to the network dataset introduced in Section 11.3. All algorithms were run with $n_{\text{pass}} = 10$, and the Merging Search was run with $n_{\text{run}} = 10$. Each network instance includes 10 different configurations of unknown modules. For each configuration, we executed every algorithm 10 times (except for Exhaustive Search, which is deterministic). We then computed the average number of selected signals and the average computation time across these 100 executions. The results are summarized in Table 11.1 and Table 11.2, respectively. The Exhaustive Search was executed only on networks with 16 nodes or fewer, as it becomes computationally prohibitive for larger instances. Implementation available in our GitHub repository [LH25].

11.4.1 Discussion

An initial observation from Table 11.1 is that, although the Pruning, Pseudotree, SIMUG, and Merging strategies do not offer formal guarantees of optimality, their empirical performance on small networks (where the optimal solution is available via Exhaustive Search) remains consistently close to optimal. The most significant deviation is observed for Pseudotree Search on the ER15 instance, which yields a score of 6.22, corresponding to an optimality gap of 3.67%. While these results do not provide theoretical assurances regarding performance on larger networks, they offer encouraging empirical evidence of the algorithms' practical effectiveness in such settings.

To facilitate comparison across algorithms, for each network in the dataset, we highlight in yellow the lowest score, referred to as a “win” for the corresponding algorithm. The final row of the table summarizes the total number of such wins achieved by each method. From this overview, it is evident that Pseudotree Search and SIMUG Search frequently yield the best results. Nevertheless, both Pruning Search and Merging Search also secure a non-negligible number of wins, indicating their competitiveness in certain scenarios. Importantly, no clear pattern emerges from the table that would suggest a single algorithm consistently outperforms the others across specific network topologies or sizes.

It can be observed that the Pseudotree Search generally outperforms the SIMUG Search. At first glance, this result may seem counterintuitive, as the Pseudotree Search treats known modules as unknown, thereby using less information and offering a less precise initialization. The explanation is likely as follows: as discussed earlier, the initialization phase of the SIMUG Search often fails to guarantee identifiability, necessitating an additional augmentation step (the adding pass). By contrast, in the Pseudotree Search, the adding pass is almost never required, as supported by empirical evidence. The adding pass in SIMUG Search operates by randomly introducing additional excitations or measurements until identifiability is achieved. This additional layer of randomness allows the SIMUG Search to explore a broader range of possible solutions, whereas the Pseudotree Search, with its deterministic initialization, does not benefit from this extra exploratory step.

An examination of Table 11.2 reveals that, with the exception of Exhaustive Search, all proposed algorithms complete execution in under one second for networks containing up to 50 nodes. For networks with 100 nodes, execution times remain efficient, with all algorithms completing in less than 13 seconds.

In conclusion, all four algorithms demonstrate practical relevance. For an real-world application, a reasonable strategy would be to execute all four methods and select the best-performing solution among them. This approach leverages the complementary strengths of the algorithms and maximizes the likelihood of obtaining a high-quality result.

	Exhaust.	Pruning	Pseudotree	SIMUG	Merging
ER10	5.6	5.61	5.6	5.6	5.61
ER15	6	6	6.22	6.04	6.01
ER20		7.5	7.6	7.5	7.56
ER30		9.13	9.11	9.1	9.1
ER40		10.37	10.1	10.23	10.22
ER50		14.17	14.17	14.08	14.9
ER100		23.85	23.67	23.71	23.51
Lattice3x3	4	4.01	4.1	4	4.02
Lattice4x4	6.4	6.45	6.42	6.45	6.45
Lattice4x5		7.65	7.68	7.67	7.67
Lattice5x6		11.61	11.58	11.52	11.53
Lattice6x7		15.07	14.98	14.97	15.49
Lattice7x7		14.94	14.85	14.83	15.22
Lattice10x10		23.44	22.99	23.29	23.06
IEEE14	6.2	6.24	6.3	6.21	6.22
IEEE24		9.11	9.1	9.11	9.33
IEEE30		11.12	11.06	11.08	11.34
IEEE39		13.82	13.81	13.8	13.97
IEEE57		19.22	19.16	19.27	19.9
IEEE118		37.88	37.5	37.72	38.02
RGG10	5.2	5.31	5.23	5.32	5.28
RGG15	6.9	6.91	7	6.9	6.91
RGG20		7.47	7.71	7.51	7.71
RGG30		14.29	14.2	14.29	14.2
RGG40		17.81	17.62	17.88	17.95
RGG50		22.31	22.19	22.29	22
RGG100		40.57	40.27	40.48	39.97
Watts10	6	6	6.1	6.03	6.01
Watts15	7	7.02	7.02	7	7.05
Watts20		8.77	8.71	8.71	8.69
Watts30		11.9	11.7	11.84	11.82
Watts40		12.94	12.48	12.7	13.09
Watts50		15.81	15.25	15.69	15.8
Watts100		31.93	29.89	31.78	31.99
Wins		5	15	12	6

Table 11.1 Average number of selected signals for each algorithm across all networks in the dataset. Averages are computed over 10 configurations of unknown modules, each run 10 times. Yellow highlights indicate the lowest score in each row, excluding Exhaustive Search. The final row reports the number of such “wins” per algorithm.

	Exhaust.	Pruning	Pseudotree	SIMUG	Merging
ER10	0.094	0.008	0.004	0.005	0.021
ER15	0.485	0.016	0.003	0.002	0.020
ER20		0.056	0.009	0.013	0.041
ER30		0.108	0.061	0.064	0.072
ER40		0.087	0.037	0.043	0.105
ER50		0.217	0.090	0.100	0.328
ER100		3.130	1.830	1.856	5.208
Lattice3x3	0.002	0.001	0.002	0.002	0.003
Lattice4x4	3.653	0.039	0.008	0.007	0.064
Lattice4x5		0.054	0.016	0.015	0.032
Lattice5x6		0.057	0.034	0.036	0.081
Lattice6x7		0.140	0.070	0.085	0.223
Lattice7x7		0.228	0.093	0.121	0.343
Lattice10x10		4.003	2.460	2.682	8.044
IEEE14	0.798	0.018	0.003	0.004	0.023
IEEE24		0.101	0.047	0.059	0.110
IEEE30		0.123	0.095	0.105	0.103
IEEE39		0.156	0.126	0.160	0.078
IEEE57		0.347	0.220	0.221	0.507
IEEE118		6.073	5.932	6.727	12.157
RGG10	0.025	0.006	0.003	0.002	0.011
RGG15	8.922	0.030	0.008	0.005	0.046
RGG20		0.058	0.005	0.009	0.029
RGG30		0.134	0.022	0.067	0.113
RGG40		0.106	0.041	0.069	0.159
RGG50		0.238	0.092	0.130	0.385
RGG100		3.453	1.652	2.358	4.473
Watts10	0.017	0.003	0.002	0.001	0.004
Watts15	1.094	0.016	0.003	0.003	0.032
Watts20		0.091	0.006	0.016	0.136
Watts30		0.056	0.019	0.035	0.542
Watts40		0.132	0.032	0.073	0.608
Watts50		0.330	0.092	0.188	0.918
Watts100		4.498	2.181	4.263	12.037

Table 11.2 Average execution time for each algorithm across all networks in the dataset. Averages are computed over 10 configurations of unknown modules, each run 10 times.

PART IV

Conclusion

12

Conclusion

TO CONCLUDE this thesis, we first summarize the main research contributions developed throughout the chapters. We then reflect on the relationship between generic local and generic global identifiability. While the former is known to be a necessary condition for the latter, whether it is also sufficient remains an open question. To date, no counterexample has been found.

Finally, we outline several perspectives for future research, including potential extensions of the theoretical framework, algorithmic improvements, and applications to different problem settings.

12.1 Research outcomes

In Chapter 4, we introduced the concept of local identifiability and established its connection to the local injectivity of a carefully constructed function h . A key result of this chapter is that, generically, analyzing networks with scalar modules is equivalent to working with analytic transfer functions. This equivalence justifies restricting the analysis to the scalar case throughout the remainder of the thesis, with the assurance that the results extend to the broader class of analytic systems. This simplification not only streamlines the mathematical treatment but also highlights the underlying combinatorial and structural nature of the identifiability problem.

Chapter 5 developed an algebraic characterization of local scalar-coordinate injectivity using tools from manifold theory. As part of this analysis, we established that scalar-coordinate injectivity is a pseudo-generic property. By applying this characterization to the function h , we obtained an algebraic criterion for local identifiability and demonstrated that it too is pseudo-generic. This led to necessary and sufficient conditions for generic local identifiability (i) for a single unknown module, (ii) for a subset of unknown modules, and (iii) for all unknown modules in the network. Crucially, the pseudo-generic nature of these conditions allows them to be verified numerically with probability one by evaluating them on a randomly sampled instance of the network. Based on this principle, we developed an efficient algorithm that determines the generic local identifiability of any given network.

However, this algebraic condition offers limited insight into the structural or dynamical behavior of the network. Chapter 7 reformulated the rank condition established in Chapter 5 into a combinatorial framework at the network level. This translation reveals how the identifiability of network modules is fundamentally governed by the topology of the network and the structure of walk-based patterns within it.

Building on the algebraic and combinatorial characterizations developed in Chapters 5 and 7, Chapter 8 derives walk-based conditions for identifiability. Specifically, it introduces a necessary condition based on the existence of vertex-disjoint walks, along with a sufficient condition of similar nature. These results generalize earlier findings from the full excitation (or full measurement) setting [HGB18], extending them to the more general case of partial excitation and measurement.

Chapter 9 introduced a novel perspective through the construction of the Meta-Walk Graph, a new graph derived from the original network in which each node represents a collection of walks. This framework provides deeper insight into the combinatorial complexity underlying the identifiability problem. By analyzing the structure and connectivity of the Meta-Walk Graph, the chapter established new sufficient conditions for identifiability and offered a powerful tool for understanding the interplay between walk-based structures and network identifiability.

In Part III, we began by establishing the NP-hardness of the problem of selecting optimal sets of excitations and measurements that ensure identifiability. Given this computational intractability, our focus shifted to the

design of efficient algorithms capable of producing high-quality solutions within reasonable computational time, even for large-scale networks.

To tackle this challenge, we developed five algorithms that build upon the algebraic conditions, walk-based criteria, and structural insights introduced in the earlier parts of the thesis. These algorithms are designed to exploit the combinatorial structure of the network while maintaining computational tractability. We conducted a comparative evaluation of the algorithms' performance on a dataset of networks with diverse topologies and sizes.

12.2 Gap between generic global and generic local identifiability

Generic local identifiability is a necessary condition for generic global identifiability, as it captures the same concept but restricted to a local neighborhood of the network. While it is a priori a weaker notion, the extent of this weakness is not immediately clear. This section investigates the relationship between the two concepts and explores whether generic local identifiability may, in practice, be nearly as strong as generic global identifiability.

In cases of full excitation or full measurement, the notions of generic local and generic global identifiability are equivalent, as established by Proposition 5.13 in Section 5.2. Similarly, for separable EM networks, the two notions coincide, as demonstrated in Proposition 6.5.

In the general case of partial excitation, partial measurement, and arbitrary network topology, the relationship between generic global and generic local identifiability remains an open question. A natural intuition might suggest that if a function is locally injective almost everywhere, it should also be globally injective almost everywhere. However, this reasoning is flawed, as illustrated by the classical counterexample $f(x) = x^2$, which is locally injective on $\mathbb{R}_0$, but not globally injective on $\mathbb{R}$.

If a counterexample to the equivalence between generic local and global identifiability exists, we believe that it is likely to be found in a small network. This view is based on intuition rather than on any formal result. To investigate this, and with the assistance of Anne Rubbens, we manually examined all structurally interesting networks consisting of 3 nodes, 4 unknown edges, 2 excitations, and 2 measurements. Despite this exhaustive inspection, no counterexample was found.

To extend this investigation, we developed a script aimed at finding a network that is generically locally identifiable but not generically globally identifiable. Such a network would serve as a counterexample, demonstrating that local identifiability is not sufficient to guarantee global identifiability. The script is available at [LH25]: `identifiable / gap_global_local`.

The script focuses exclusively on the identifiability of the entire network, assessing generic local identifiability via the rank condition on the matrix K . Several classes of networks are excluded from the search space to streamline the analysis:

- Networks with full excitation or full measurement are omitted, as the equivalence between local and global identifiability in these cases has already been established.
- Disconnected graphs are excluded, since their identifiability properties can be inferred from those of their connected components.
- Graphs with self-loops are excluded from our analysis, as we observed that they tend to produce misleading results : what appear to be counterexamples but are not. This behavior is likely due to numerical instabilities or pathological cases in the underlying system equations, which cause MATLAB's `solve` function to perform unreliably on such structures.
- Known edges are not included, because we believe that if counterexamples exist, then there is one counterexample without known edges. This view is based on intuition rather than on any formal result.
- We restrict our analysis to EM networks satisfying $m^\circ = n_B n_C$. This choice is motivated by the following considerations:
 - If $m^\circ > n_B \cdot n_C$, the system is underdetermined, with more unknowns than equations, and thus not generically identifiable.
 - If $m^\circ < n_B \cdot n_C$, the system contains some redundant or non-informative equations. In such cases, the EM network can often be reduced to an equivalent one satisfying $m^\circ = n_B n_C$ (a hypothesis we consider plausible). Moreover, we expect that any potential counterexample to the sufficiency of generic local identifiability is most likely to arise in this critical case where the number of unknowns matches the number of equations.

Excluding the special cases discussed above, we have exhaustively examined all EM networks with up to five unknown edges and no known edges. Across this exhaustive search — covering more than two million distinct EM networks — no counterexample to the sufficiency of generic local identifiability for generic global identifiability was found. Script available at our GitHub repository [LH25].

12.3 Perspectives

12.3.1 Improvements on algorithms for signal selection

Our signal selection algorithms rely on Corollary 5.14, but do not currently exploit the more detailed structural insights provided by Theorem 5.12. In particular, there is potential to enhance these algorithms by incorporating information about which specific modules are not identifiable.

When testing whether a given set of excitations and measurements ensures generic local identifiability — via the rank condition on the matrix K — a failure to satisfy this condition indicates that the system is not fully identifiable. In such cases, the structure of $\ker K$ through its orthogonality relations, can reveal which modules are unidentifiable and which may still be identifiable. This finer-grained information could be leveraged to guide the search process more intelligently, potentially improving the efficiency of our algorithms.

Besides, there remains potential to extract additional identifiability conditions from the Meta-Walk Graph approach. While a few useful properties have already been derived, it is possible that not all relevant insights have been uncovered. In particular, the degree of a bijective collection of walks in $\mathcal{G}_\mu$, which can be explicitly computed using the expression in Proposition 9.10, may offer further opportunities. We believe that combining this expression with Theorem 9.18 could lead to actionable identifiability conditions, especially in specific network topologies or structured cases.

Currently, our algorithms do not directly leverage the conditions derived from the Meta-Walk Graph framework. However, this presents an opportunity: it may be possible to design a dedicated algorithm that explicitly incorporates the structural properties and identifiability conditions revealed by the Meta-Walk Graph. Such an approach could enhance both

the theoretical understanding and practical performance of signal selection strategies.

12.3.2 Identifiability of a subset of unknown modules

Part I gives an algebraic characterization for the identifiability of any set of unknown modules: a single unknown module, a subset of them or all of them. However, Parts II and III focus exclusively on the identifiability of *all* unknown modules simultaneously — a property commonly referred to as *network identifiability*. This distinction highlights promising perspectives for further work.

Indeed, when the conditions presented in Part II indicate that the network is not identifiable, they do not provide any information about the identifiability of individual modules. In particular, they do not reveal whether some unknown modules may still be identifiable, nor do they specify which ones.

If one seeks to extend the conditions presented in Part II to address the identifiability of only a subset of the unknown modules, composed of more than one element, the situation becomes more complex. In particular, one cannot simply treat the remaining, non-targeted unknown modules as if they were known.

One challenge is that the identifiability of a subset of unknown modules is not a truly generic property, but rather a pseudo-generic one. Unlike generic properties, pseudo-genericity lacks the convenient implication that validity at a single point ensures validity almost everywhere.

The identifiability condition is formulated in terms of the orthogonality of $\ker K$, as stated in Theorem 5.12. This orthogonality can be reformulated as the linear independence condition in (5.4), which in turn can be expressed as (5.5). For the case of full network identifiability, combining these conditions across all unknown modules leads to a single rank condition. This can be further reduced to the non-vanishing of a determinant, ultimately yielding a combinatorial characterization.

However, in the general case — when only a subset of modules is considered — we do not yet know how to simplify the condition into a more tractable form. As it stands, the condition is difficult to apply in practice. Developing a more manageable formulation remains an open problem for future research.

Part III presents algorithms for designing excitation and measurement

sets that guarantee generic local identifiability of the *entire* network. However, when the objective shifts to identifying only a subset of the unknown modules — potentially with fewer signals than those required for full network identifiability — the current algorithms do not directly address this more targeted scenario.

Nevertheless, the Pruning Search procedure can be readily adapted to this setting, leveraging the algebraic characterization provided by Theorem 5.12, which explicitly determines which modules are identifiable and which are not.

12.3.3 Extensions of algorithms for signal selection

Several potential directions for extending the algorithms presented in Part III can be envisioned.

- Incorporating the constraint that certain nodes cannot be excited or measured requires only minor modifications to our algorithms and is conceptually straightforward.
- An economic setting where excitation and measurement at each node incur different costs. For example, all excitations may have a uniform cost of p_B , and all measurements a cost of p_C , with $p_B < p_C$.
- If a set of excitations and measurements is already fixed, either provided for free or previously deployed. The Pruning Search can readily accommodate such constraints.

From a conceptual standpoint, these extensions appear relatively straightforward and would likely require only minor adjustments to the existing algorithms. While some adaptation is needed, they do not seem to demand fundamentally new ideas or theoretical developments.

Bibliography

- [AB09] Sanjeev Arora and Boaz Barak. *Computational complexity: a modern approach*. Cambridge University Press, 2009.
- [AMR12] Ralph Abraham, Jerrold E Marsden, and Tudor Ratiu. *Manifolds, tensor analysis, and applications*, volume 75. Springer Science & Business Media, 2012.
- [AR78] Leonard Asimow and Ben Roth. The rigidity of graphs. *Transactions of the American Mathematical Society*, 245:279–289, 1978.
- [BD94] Patrick Bolton and Mathias Dewatripont. The firm as a communication network. *The Quarterly Journal of Economics*, 109(4):809–839, 1994.
- [BGH19] Alexandre S Bazanella, Michel Gevers, and Julien M Hendrickx. Network identification with partial excitation and measurement. In *58th Conference on Decision and Control (CDC)*, pages 5500–5506. IEEE, 2019.
- [BGHP17] Alexandre S Bazanella, Michel Gevers, Julien M Hendrickx, and Adriane Parraga. Identifiability of dynamical networks: which nodes need be measured? In *IEEE 56th Annual Conference on Decision and Control (CDC)*, 2017.
- [Big93] Norman Biggs. *Algebraic graph theory*. Number 67. Cambridge university press, 1993.
- [BML06] Jean-Stéphane Bailly, Pascal Monestiez, and Philippe Lagacherie. Modelling spatial variability along drainage networks with geostatistics. *Mathematical geology*, 38(5):515–539, 2006.

- [Bon06] Christian Bonatti. Generic properties of dynamical systems. *Encyclopedia of Mathematical Physics*, pages 494–502, 2006.
- [Bro15] Roger W Brockett. *Finite dimensional linear systems*. SIAM, 2015.
- [BVVN11] Andrew Bolstad, Barry D Van Veen, and Robert Nowak. Causal network inference via group sparse regularization. *IEEE transactions on signal processing*, 59(6):2628–2641, 2011.
- [CAO14] Christopher R Christie, Luke E K Achenie, and Babatunde A Ogunnaike. A control engineering model of calcium regulation. *The Journal of Clinical Endocrinology & Metabolism*, 99(8):2844–2853, 2014.
- [CCN⁺02] Mark Coates, Rui Castro, Robert Nowak, Manik Gadhiok, Ryan King, and Yolanda Tsang. Maximum likelihood network topology identification from edge-based unicast measurements. *ACM SIGMETRICS Performance Evaluation Review*, 30(1):11–20, 2002.
- [Chi21] Benjamin Chiêm. *Interweaving structure and function in human brain networks*. PhD thesis, Catholic University of Louvain, Louvain-la-Neuve, Belgium, 2021.
- [Con16] Electrical Concepts. Smart grid – an overview, 2016. URL: <https://electricalbaba.com/smart-grid-an-overview/>.
- [Coo23] Stephen A Cook. The complexity of theorem-proving procedures. In *Logic, automata, and computational complexity: The works of Stephen A. Cook*, pages 143–152. 2023.
- [CP12] Alessandro Chiuso and Gianluigi Pillonetto. A bayesian approach to sparse dynamic network identification. *Automatica*, 48(8):1553–1565, 2012.
- [Cra79] Henry Crapo. Structural rigidity. *Structural topology*, 1979, núm. 1, 1979.
- [CSLVdH23] Xiaodong Cheng, Shengling Shi, Ioannis Lestas, and Paul MJ Van den Hof. A necessary condition for network identifiability with partial excitation and measurement. *IEEE Transactions on Automatic Control*, 68(11):6820–6827, 2023.

- [CSLVdH24] Xiaodong Cheng, Shengling Shi, Ioannis Lestas, and Paul MJ Van den Hof. Identifiability in dynamic acyclic networks with partial excitation and measurement. *IEEE Transactions on Automatic Control*, 2024.
- [CSVdH19] Xiaodong Cheng, Shengling Shi, and Paul MJ Van den Hof. Allocation of excitation signals for generic identifiability of dynamic networks. In *Proceedings of the IEEE Conference on Decision and Control*, 2019.
- [CSVdH21] Xiaodong Cheng, Shengling Shi, and Paul MJ Van den Hof. Allocation of excitation signals for generic identifiability of linear dynamic networks. *IEEE Transactions on Automatic Control*, 67(2):692–705, 2021.
- [D’A10] John P D’Angelo. *An introduction to complex analysis and geometry*, volume 12. American Mathematical Soc., 2010.
- [Dav77] Edward J Davison. Connectability and structural controllability of composite systems. *Automatica*, 13(2):109–123, 1977.
- [DBJ⁺07] Natalie C Duarte, Scott A Becker, Neema Jamshidi, Ines Thiele, Monica L Mo, Thuy D Vo, Rohith Srivas, and Bernhard Ø Palsson. Global reconstruction of the human metabolic network based on genomic and bibliomic data. *Proceedings of the National Academy of Sciences*, 104(6):1777–1782, 2007.
- [DCVdW03] Jean-Michel Dion, Christian Commault, and Jacob Van der Woude. Generic properties and control of linear structured systems: a survey. *Automatica*, 2003.
- [DF⁺04] David Steven Dummit, Richard M Foote, et al. *Abstract algebra*, volume 3. Wiley Hoboken, 2004.
- [DSC⁺22] HJ Dreef, Shengling Shi, Xiaodong Cheng, MCF Donkers, and Paul MJ Van den Hof. Excitation allocation for generic identifiability of linear dynamic networks with fixed modules. *IEEE Control Systems Letters*, 6:2587–2592, 2022.
- [DVdHBH15] Arne Dankers, Paul MJ Van den Hof, Xavier Bombois, and Peter SC Heuberger. Errors-in-variables identification in

- dynamic networks—consistency results for an instrumental variable approach. *Automatica*, 62:39–50, 2015.
- [DVNS08] Domitilla Del Vecchio, Alexander J Ninfa, and Eduardo D Sontag. Modular cell biology: retroactivity and insulation. *Molecular systems biology*, 4(1):161, 2008.
- [dW91] Jacob Willem Van der Woude. A graph-theoretic characterization for the rank of the transfer matrix of a structured system. *Mathematics of Control, Signals and Systems*, 4(1):33–40, 1991.
- [DY14] Francis X Diebold and Kamil Yilmaz. On the network topology of variance decompositions: Measuring the connectedness of financial firms. *Journal of econometrics*, 182(1):119–134, 2014.
- [EGK⁺24] Márton Elekes, Boglárka Gehér, Tamás Kátay, Tamás Keleti, Anett Kocsis, and Máté Pálffy. Generic properties of topological groups. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, pages 1–31, 2024.
- [FFH20] Stefanie Fonken, Mina Ferizbegovic, and Håkan Hjalmarsson. Consistent identification of dynamic networks subject to white noise using weighted null-space fitting. *IFAC-PapersOnLine*, 53(2):46–51, 2020.
- [Fri11] Karl J Friston. Functional and effective connectivity: a review. *Brain connectivity*, 1(1):13–36, 2011.
- [FRVdH22] Stefanie JM Fonken, Karthik Raghavan Ramaswamy, and Paul MJ Van den Hof. A scalable multi-step least squares method for network identification with unknown disturbance topology. *Automatica*, 141:110295, 2022.
- [GBdS18] Michel Gevers, Alexandre Sanfelice Bazanella, and Gian Vianna da Silva. A practical method for the consistent identification of a module in a dynamical network. *IFAC-PapersOnLine*, 51(15):862–867, 2018.
- [GBP17] Michel Gevers, Alexandre S Bazanella, and Adriane Paraga. On the identifiability of dynamical networks. *IFAC-PapersOnLine*, 50(1):10580–10585, 2017.

- [GBP18] Michel Gevers, Alexandre Sanfelice Bazanella, and Guilherme Araujo Pimentel. Identifiability of dynamical networks with singular noise spectra. *IEEE Transactions on Automatic Control*, 64(6):2473–2479, 2018.
- [GGVC⁺20] Leonardo Gutiérrez-Gómez, Jakub Vohryzek, Benjamin Chiêm, Philipp S Baumann, Philippe Conus, Kim Do Cuenod, Patric Hagmann, and Jean-Charles Delvenne. Stable biomarker identification for predicting schizophrenia in the human connectome. *NeuroImage: Clinical*, 27:102316, 2020.
- [GJ02] Michael R Garey and David S Johnson. *Computers and intractability*, volume 29. wh freeman New York, 2002.
- [GRLK12] Manuel Gomez-Rodriguez, Jure Leskovec, and Andreas Krause. Inferring networks of diffusion and influence. *ACM Transactions on Knowledge Discovery from Data (TKDD)*, 5(4):1–37, 2012.
- [GSK18] Georgios B Giannakis, Yanning Shen, and Georgios Vasileios Karanikolas. Topology identification and learning over graphs: Accounting for nonlinearities and dynamics. *Proceedings of the IEEE*, 106(5):787–807, 2018.
- [GSS93] Jack E Graver, Brigitte Servatius, and Herman Servatius. *Combinatorial rigidity*. Number 2. American Mathematical Soc., 1993.
- [Hau91] Felix Hausdorff. *Set theory*. American Mathematical Society (RI), 1991.
- [HCGT16] David Hayden, Young Hwan Chang, Jorge Goncalves, and Claire J Tomlin. Sparse network identifiability via compressed sensing. *Automatica*, 68:9–17, 2016.
- [HGB18] Julien M Hendrickx, Michel Gevers, and Alexandre S Bazanella. Identifiability of dynamical networks with partial node measurements. *IEEE Transactions on Automatic Control*, 64(6):2240–2253, 2018.
- [HJ12] Roger A Horn and Charles R Johnson. *Matrix analysis*. Cambridge university press, 2012.

- [HV14] Aleksandar Haber and Michel Verhaegen. Subspace identification of large-scale interconnected systems. *IEEE Transactions on Automatic Control*, 59(10):2754–2759, 2014.
- [Ins25] Information Trust Institute. Literature-based power flow test cases, 2025. URL: <https://icseg.iti.illinois.edu/power-cases/>.
- [JBVdH08] Jan-Dirk Jansen, Okko H Bosgra, and Paul MJ Van den Hof. Model-based control of multiphase flow in subsurface oil reservoirs. *Journal of Process Control*, 18(9):846–855, 2008.
- [KBS15] Torben Knudsen, Thomas Bak, and Mikael Svenstrup. Survey of wind farm control—power and fatigue optimization. *Wind Energy*, 18(8):1333–1351, 2015.
- [Kun07] Prabha Kundur. Power system stability. *Power system stability and control*, 10(1):7–1, 2007.
- [KVdH19] EMM Lizan Kivits and Paul MJ Van den Hof. A dynamic network approach to identification of physical systems. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 4533–4538. IEEE, 2019.
- [KVdH22a] EMM Kivits and Paul MJ Van den Hof. Identification of diffusively coupled linear networks through structured polynomial models. *IEEE Transactions on Automatic Control*, 68(6):3513–3528, 2022.
- [KVdH22b] EMM Lizan Kivits and Paul MJ Van den Hof. Local identification in diffusively coupled linear networks. In *2022 IEEE 61st Conference on Decision and Control (CDC)*, pages 874–879. IEEE, 2022.
- [KVdH23] EMM Lizan Kivits and Paul MJ Van den Hof. Identifiability of diffusively coupled linear networks with partial instrumentation. *IFAC-PapersOnLine*, 56(2):2395–2400, 2023.
- [LH20] Antoine Legat and Julien M Hendrickx. Local network identifiability with partial excitation and measurement. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 4342–4347. IEEE, 2020.

- [LH21a] Antoine Legat and Julien M Hendrickx. Path-based conditions for local network identifiability. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 3024–3029. IEEE, 2021.
- [LH21b] Antoine Legat and Julien M Hendrickx. Path-based conditions for local network identifiability—full version. *arXiv preprint arXiv:2110.00126*, 2021.
- [LH23] Antoine Legat and Julien M Hendrickx. Combinatorial characterization for global identifiability of separable networks with partial excitation and measurement. In *2023 62nd IEEE Conference on Decision and Control (CDC)*, pages 2471–2476. IEEE, 2023.
- [LH24] Antoine Legat and Julien M Hendrickx. Identifiability in networked systems: Algebraic and combinatorial conditions. *in review process for IEEE Transactions on Automatic Control*, 2024.
- [LH25] Antoine Legat and Julien M Hendrickx. Generic local identifiability toolbox, 2025. URL: <https://github.com/alegat/identifiable>.
- [Lin74] Ching-Tai Lin. Structural controllability. *IEEE Transactions on Automatic Control*, 19(3):201–208, 1974.
- [Lju98] Lennart Ljung. System identification. In *Signal analysis and prediction*, pages 163–173. Springer, 1998.
- [Lju99] Lennart Ljung. *System identification: theory for the user*, volume 28. 1999.
- [MBH⁺19] Federico Morelli, Xavier Bombois, Håkan Hjalmarsson, Laurent Bako, and Kévin Colin. Optimal experiment design for the identification of one module in the interconnection of locally controlled systems. In *2019 18th European Control Conference (ECC)*, pages 363–368. IEEE, 2019.
- [MCK⁺12] Daniel Marbach, James C Costello, Robert Küffner, Nicole M Vega, Robert J Prill, Diogo M Camacho, Kyle R Allison, Manolis Kellis, James J Collins, et al. Wisdom of crowds for

- robust gene network inference. *Nature methods*, 9(8):796–804, 2012.
- [MGB22a] Eduardo Mapurunga, Michel Gevers, and Alexandre S Bazanella. Excitation and measurement patterns for the identifiability of directed acyclic graphs. In *2022 IEEE 61st Conference on Decision and Control (CDC)*, pages 1616–1621. IEEE, 2022.
- [MGB22b] Eduardo Mapurunga, Michel Gevers, and Alexandre S Bazanella. Necessary and sufficient conditions for the identifiability of isolated loops. *IEEE Control Systems Letters*, 6:2276–2280, 2022.
- [MGB23] Eduardo Mapurunga, Michel Gevers, and Alexandre S Bazanella. Identifiability of dynamic networks: The essential rôle of dources and dinks. *IEEE Transactions on Automatic Control*, 69(2):1303–1308, 2023.
- [MGB24] Eduardo Mapurunga, Michel Gevers, and Alexandre S Bazanella. Extending identifiability results from isolated networks to embedded networks. *arXiv preprint arXiv:2402.14144*, 2024.
- [MS15] Donatello Materassi and Murti V Salapaka. Identification of network components in presence of unobserved nodes. In *2015 54th IEEE Conference on Decision and Control (CDC)*, pages 1563–1568. IEEE, 2015.
- [MS19] Donatello Materassi and Murti V Salapaka. Signal selection for estimation and identification in networks of dynamic systems: A graphical model approach. *IEEE Transactions on Automatic Control*, 65(10):4138–4153, 2019.
- [Mur87] Kazuo Murota. Systems analysis by graphs and matroids, volume 3 of algorithms and combinatorics, 1987.
- [Pap03] Christos H Papadimitriou. Computational complexity. In *Encyclopedia of computer science*, pages 260–265. 2003.
- [PP12] KB Petersen and MS Pedersen. The matrix cookbook, version 20121115. *Technical Univ. Denmark, Kongens Lyngby, Denmark, Tech. Rep.*, 3274, 2012.

- [RBVdH21] Karthik Raghavan Ramaswamy, Giulio Bottegal, and Paul MJ Van den Hof. Learning linear modules in a dynamic network using regularized kernel-based methods. *Automatica*, 129:109591, 2021.
- [RCSVdH22] Karthik Raghavan Ramaswamy, Péter Zoltán Csurscia, Johan Schoukens, and Paul MJ Van den Hof. A frequency domain approach for local module identification in dynamic networks. *Automatica*, 142:110370, 2022.
- [RV20] Karthik R Ramaswamy and Paul MJ Vandenhof. A local direct method for module identification in dynamic networks with correlated noise. *IEEE Transactions on Automatic Control*, 2020.
- [RVdHD19] Karthik R Ramaswamy, Paul MJ Van den Hof, and Arne G Dankers. Generalized sensing and actuation schemes for local module identification in dynamic networks. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 5519–5524. IEEE, 2019.
- [SAMC12] Yoav Sharon, Anuradha M Annaswamy, Alexis L Motto, and Amit Chakraborty. Topology identification in distribution network with limited measurements. In *2012 IEEE PES Innovative Smart Grid Technologies (ISGT)*, pages 1–6. IEEE, 2012.
- [SCDSVdH23] Shengling Shi, Xiaodong Cheng, Bart De Schutter, and Paul MJ Van den Hof. Signal selection for local module identification in linear dynamic networks: A graphical approach. *IFAC-PapersOnLine*, 56(2):2407–2412, 2023.
- [SCVdH22] Shengling Shi, Xiaodong Cheng, and Paul MJ Van den Hof. Generic identifiability of subnetworks in a linear dynamic network: the full measurement case. *Automatica*, 137:110093, 2022.
- [SCVdH23] Shengling Shi, Xiaodong Cheng, and Paul MJ Van den Hof. Single module identifiability in linear dynamic networks with partial excitation and measurement. *IEEE Transactions on Automatic Control*, 2023.

- [SHR⁺18] Syed Jawad Hussain Shahzad, Jose Areola Hernandez, Mobeen Ur Rehman, Khamis Hamed Al-Yahyaee, and Muhammad Zakaria. A global network topology of stock markets: Transmitters and receivers of spillover effects. *Physica A: Statistical Mechanics and its Applications*, 492:2136–2153, 2018.
- [Sip96] Michael Sipser. Introduction to the theory of computation. *ACM Sigact News*, 27(1):27–29, 1996.
- [SIR09] Murari Mohan Saha, Jan Jozef Izykowski, and Eugeniusz Rosolowski. *Fault location on power networks*. Springer Science & Business Media, 2009.
- [SP14] Shahin Shahrampour and Victor M Preciado. Topology identification of directed dynamical networks via power spectral analysis. *IEEE Transactions on Automatic Control*, 60(8):2260–2265, 2014.
- [SSO]23] Șerban Sabău, Andrei Sperilă, Cristian Oară, and Ali Jad-babaie. Network realization functions for optimal distributed control. *IEEE Transactions on Automatic Control*, 68(12):8059–8066, 2023.
- [TD99] Michael F Thorpe and Phillip M Duxbury. *Rigidity theory and applications*. Springer Science & Business Media, 1999.
- [TVA⁺13] Emma K Towlson, Petra E Vértés, Sebastian E Ahnert, William R Schafer, and Edward T Bullmore. The rich club of the c. elegans neuronal connectome. *Journal of Neuroscience*, 33(15):6380–6387, 2013.
- [VdHDB13] Paul MJ Van den Hof, Arne Dankers, Peter SC Heuberger, and Xavier Bombois. Identification of dynamic models in complex networks with prediction error methods—basic methods for consistent module estimates. *Automatica*, 49(10):2994–3006, 2013.
- [VdHR20] Paul MJ Van den Hof and Karthik R Ramaswamy. Path-based data-informativity conditions for single module identification in dynamic networks. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 4354–4359. IEEE, 2020.

- [VdHRDB19] Paul MJ Van den Hof, Karthik R Ramaswamy, Arne G Dankers, and Giulio Bottegal. Local module identification in dynamic networks with correlated noise: the full input case. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 5494–5499. IEEE, 2019.
- [VdHRF23] Paul MJ Van den Hof, Karthik R Ramaswamy, and Stefanie JM Fonken. Integrating data-informativity conditions in predictor models for single module identification in dynamic networks. *IFAC-PapersOnLine*, 56(2):2377–2382, 2023.
- [VE15] Andalam Satya Mohan Vamsi and Nicola Elia. Optimal distributed controllers realizable over arbitrary networks. *IEEE Transactions on Automatic Control*, 61(1):129–144, 2015.
- [vESB⁺20] Rik JC van Esch, Shengling Shi, Antoine Bernas, Svitlana Zinger, Albert P Aldenkamp, and Paul MJ Van den Hof. A bayesian method for inference of effective connectivity in brain networks for detecting the mozart effect. *Computers in Biology and Medicine*, 127:104055, 2020.
- [vGBY⁺24] JJH van Gemert, V Breschi, DR Yntema, KJ Keesman, and M Lazar. Exploiting structural observability and graph colorability for optimal sensor placement in water distribution networks. *arXiv preprint arXiv:2412.09965*, 2024.
- [VWTC19a] Henk J Van Waarde, Pietro Tesi, and M Kanat Camlibel. Necessary and sufficient topological conditions for identifiability of dynamical networks. *IEEE Transactions on Automatic Control*, 65(11):4525–4537, 2019.
- [vWTC19b] Henk J van Waarde, Pietro Tesi, and M Kanat Camlibel. Topology reconstruction of dynamical networks via constrained lyapunov equations. *IEEE Transactions on Automatic Control*, 64:4300–4306, 2019.
- [vWTC21] Henk J van Waarde, Pietro Tesi, and M Kanat Camlibel. Topology identification of heterogeneous networks: Identifiability and reconstruction. *Automatica*, 123:109331, 2021.
- [W⁺01] Douglas Brent West et al. *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River, 2001.

- [WDVdH15] Harm HM Weerts, Arne G Dankers, and Paul MJ Van den Hof. Identifiability in dynamic network identification. *IFAC-PapersOnLine*, 2015.
- [Whi12] John Eldon Whitesitt. *Boolean algebra and its applications*. Courier Corporation, 2012.
- [Wik25a] Wikipedia. Np (complexity), 2025. URL: [https://en.wikipedia.org/wiki/NP_\(complexity\)](https://en.wikipedia.org/wiki/NP_(complexity)).
- [Wik25b] Wikipedia. Structural rigidity, 2025. URL: https://en.wikipedia.org/wiki/Structural_rigidity.
- [WLEVdH20] Harm HM Weerts, Jonas Linder, Martin Enqvist, and Paul MJ Van den Hof. Abstractions of linear dynamic networks for input selection in local module identification. *Automatica*, 117:108975, 2020.
- [WLR16] Yang Weng, Yizheng Liao, and Ram Rajagopal. Distributed energy resources topology identification via graphical modeling. *IEEE Transactions on Power Systems*, 32(4):2682–2694, 2016.
- [WVdHD18a] Harm Weerts, Paul MJ Van den Hof, and Arne Dankers. Single module identifiability in linear dynamic networks. In *2018 IEEE Conference on Decision and Control (CDC)*, pages 4725–4730. IEEE, 2018.
- [WVdHD18b] Harm HM Weerts, Paul MJ Van den Hof, and Arne G Dankers. Identifiability of linear dynamic networks. *Automatica*, 89:247–258, 2018.
- [WVdHD18c] Harm HM Weerts, Paul MJ Van den Hof, and Arne G Dankers. Prediction error identification of linear dynamic networks with rank-reduced noise. *Automatica*, 98:256–268, 2018.
- [YCV19] Chengpu Yu, Jie Chen, and Michel Verhaegen. Subspace identification of individual systems in a large-scale heterogeneous network. *Automatica*, 109:108517, 2019.
- [YV17] Chengpu Yu and Michel Verhaegen. Subspace identification of individual systems operating in a network (si2 on). *IEEE Transactions on Automatic Control*, 63(4):1120–1125, 2017.

- [ZWWZ20] Jiawei Zhang, Yi Wang, Yang Weng, and Ning Zhang. Topology identification and line parameter estimation for non-pmu distribution network: A numerical method. *IEEE Transactions on Smart Grid*, 11(5):4440–4453, 2020.