Alex Grossmann, from Nested Hilbert spaces to Partial Inner Product Spaces and Wavelets

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Abstract This chapter is devoted to the long collaboration between Alex Grossmann and I, the focal point being the notion of PIP-spaces, introduced around 1970. Next I will review the basic facts of the theory. Then I will explore all the consequences and applications of that notion, to operator theory, spectral analysis of self-adjoint operators, metric operators that generate lattices of Hilbert spaces, signal processing, etc. As a matter of fact, new developments are still arising today. In addition, we'll describe the interaction with Alex that lead to 2D continuous wavelets.

1 Introduction: Some history

In these lines, I will describe my collaboration with Alex and its rich legacy, starting in Spring 1968. Some years before, upon the suggestion of V. Bargmann, I had studied the notion of Rigged Hilbert Space (RHS) developed by I.M.Gel'fand et al. Next, in 1966, I had written a PhD thesis on a formulation of quantum mechanics in the new formalism) [1, 2]. Then, in the Spring 1968, a visitor, called Alex Grossmann, came from Marseille and gave a talk in our department, on a topic unknown to most of us, namely Nested Hilbert Spaces (NHS). Actually both RHS and NHS (a detailed description will be given in the sequel) have the same goal, to describe in a rigorous manner those elements that do not belong to the fundamental Hilbert space, yet play a crucial role, for instance in the Dirac bra-ket formalism (plane waves, δ-function, distributions in general). In other words, both Alex and I were doing implicitly the same thing. After discussion, it became clear to us that there ought to be a more general formalism englobing both NHS and RHS. Thus Alex invited me to spend some time with him in Marseille.

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The trip with the family was planned for May 1968. But there was a bug : no gas station was operational along the roads of France ! Thus we had to wait until June, and finally we settled in Cassis (where Alex lived) while working in Marseille, at the CNRS center (not yet transferred to Luminy). The first task that Alex suggested was to clean up his closet, it contained hundreds of pages on various topics, ranging from analyticity in quantum field theory to basics of Nested Hilbert Spaces. Alex was a man with such a wide scientific culture that he had original ideas about almost every topic...but rarely went to their conclusion, so that there were many loose ends. We sorted and classified everything, ending up with three fat folders, quickly christened "the skeleton". We made two copies, one for each, and I still have mine (I used it recently). It also generated funny situations, for instance when I sent a postcard to Alex (no internet at the time!), with the sentence "Do you agree if I use the skeleton?" (in French "Es-tu d'accord que j'utilise le squelette ?", to be well understood by the postal service employees...)

Having all this material at our disposal, we concentrated on our goal, to find a comprehensive formalism that would generalize both NHS and RHS. The net result was the concept of Partial inner product space that emerged in1973 [3]. The basic papers were published in 1976 [4, 5, 6]. [Actually we had written *three* papers and submitted all three together. A first journal was willing to take one.The next one would accept two. So we gave up and the result is the two articles [4, 5] just cited. [I don't remember what happened to the third one . . .]. I continued along the same lines [7, 8], but Alex quickly moved to an emerging new concept, namely, wavelets...but this is another story. Now we will switch to mathematics, without too much technical details.

2 Rigged Hilbert Spaces

This is by now a standard tool in analysis, thanks to the work of Sobolev, Gel'fand, Maurin, Schwartz, etc. [9]. A rigged Hilbert space (RHS) consists of a triplet of spaces:

$$\Phi \subset \mathcal{H} \subset \Phi^{\times}, \tag{1}$$

where $\mathcal{H}$ is a Hilbert space, Φ is a dense subspace of $\mathcal{H}$, endowed with a locally convex topology τ_Φ finer than the norm topology inherited from $\mathcal{H}$ (i.e., a stronger notion of convergence), and $\Phi^{\times}$ is the space of continuous conjugate linear functionals $F(\phi)$ on Φ. By duality, each space in (1) is dense in the next one and all embeddings are linear and continuous. Standard examples of rigged Hilbert spaces are the Schwartz distribution spaces over $\mathbb{R}$ or $\mathbb{R}^3$, namely $\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}^{\times}$ or $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^{\times}$. In general, one requires that the space Φ be complete, reflexive and nuclear [we skip the technical details]. The third property implies that one can use the Gel'fand-Maurin spectral theorem. According to the latter, a self-adjoint operator A in $\mathcal{H}$ that maps Φ into itself continuously (for τ_Φ) possesses a complete orthonormal set of generalized eigenvectors $\xi_\lambda \in \Phi^{\times}$. The vector ξ_λ is called a generalized eigenvector of A if it satisfies the equation $A^\times \xi_\lambda = \lambda^* \xi_\lambda$, where $A^\times$ is the extension (by duality) to $\Phi^\times$ of the adjoint A^*. Then the statement of the theorem is that, for any $\phi, \psi \in \Phi$, one has

$$\langle \phi | \psi \rangle = \int_{\mathbb{R}} \xi_\lambda(\phi)\, \overline{\xi_\lambda(\psi)}\, \mathrm{d}\mu(\lambda) := \int_{\mathbb{R}} \langle \phi | \xi_\lambda \rangle \langle \xi_\lambda | \psi \rangle\, \mathrm{d}\mu(\lambda), \tag{2}$$

where $\langle \cdot | \cdot \rangle$ is the sesquilinear form putting Φ and $\Phi^\times$ in duality and μ is some measure on $\mathbb{R}$.

In quantum mechanics, if one splits the measure μ into a discrete part (essentially a sum of δ-functions) and a continuous part, one recovers the distinction between normalizable $\xi_\lambda \in \mathcal{H}$ and non-normalizable eigenvectors ($\xi_\lambda \in \Phi^\times \setminus \mathcal{H}$), that is, the discrete and the continuous part of the spectrum of the observable A. Thus (2) justifies Dirac's formalism.

This is very good, but this approach has a serious drawback: it requires mastering nontrivial mathematical notions from functional analysis. Another problem with the RHS (1) is that, besides the Hilbert space vectors, it contains only two types of elements, "very good" functions in Φ and "very bad" ones in $\Phi^\times$. If one wants a fine control on the behavior of individual elements, one has to interpolate somehow between the two extreme spaces. In the case of the Schwartz triplet, $\mathcal{S} \subset L^2 \subset \mathcal{S}^\times$, a well-known solution is given by a chain of Hilbert spaces, the so-called Hermite representation of tempered distributions [10]. Actually this was anticipated by Alex already in 1965 [11], as we shall see in Section 6.

These are two reasons that urged us, and Alex in the first place, to invent a powerful new language, namely, Nested Hilbert Spaces and Partial inner product spaces, to which we turn now.

3 Nested Hilbert Spaces

The original definition of Grossmann runs as follows [12, 13]. Consider the triple $V_I = [V_r; E_{sr}; I]$, where

(1) I is an index set, directed to the right, with an order-reversing involution $r \leftrightarrow \overline{r}$ and a self-dual element $o = \overline{o}$.
(2) For each $r \in I$, V_r is a Hilbert space.
(3) For every $p \in I$ and every $q \geq p$, E_{qp} is an injective, bounded linear mapping from V_p into V_q, with dense range, such that E_{pp} is the identity on V_p for every $p \in I$ and $E_{qp} = E_{qr} E_{rp}$ whenever $q \geq r \geq p$ (E_{qp} is called a *nesting*).
(4) V_I is the algebraic inductive limit of the family $\{V_r, r \in I\}$ with respect to the nestings E_{qp}; this means the following: in the disjoint union $\bigcup_{r \in I} V_r$, define an equivalence relation by writing $f_p \sim f_r$, ($f_p \in V_p, f_r \in V_r$) if there is an $s \geq p, r$ such that $E_{sp} f_p = E_{sr} f_r$; then the set of equivalence classes is a vector space, noted V_I. One notes E_{Ir} the natural embedding of V_r into V_I.

Then the triple $V_I = [V_r; E_{sr}; I]$ is called a *nested Hilbert space (NHS)* if the following two conditions are satisfied:

(nh$_1$) V_I is stable under intersection: for any two $r, q \in I$, there is a $p \leq r, q$ such that $E_{Ip}V_p = E_{Ir}V_r \cap E_{Iq}V_q$;

(nh$_2$) For every $r \in I$, there exists a (unique) unitary map $u_{r\overline{r}}$ from V_r onto $V_{\overline{r}}$ such that $u_{oo} = 1$ and $(E_{sr})^*_{rs} = u_{r\overline{r}} E_{\overline{r}\overline{s}} u_{\overline{s}s}$.

Condition (nh$_1$) guarantees that the family $\{V_r, r \in I\}$ is a lattice, whereas Condition (nh$_2$) states that $V_{\overline{r}}$ is the conjugate dual of V_r, with the dual norm, $u_{r\overline{r}}$ being the unitary (Riesz) operator establishing the duality between the two. Notice that the embeddings E_{qp} need *not* reduce to the identity, they have only to satisfy Condition (3). Examples are given in [14, Sec.2.4.1].

The next step is to define a partial inner product on the NHS V_I. Given a vector $f \in V_I$, we consider the set $J(f) = \{r \in I : f \in V_r\}$, that characterizes the behavior of f. Write $\overline{J}(h) = \{r \in I : \overline{r} \in J(h)\}$. Then, given $f, g \in V_I$ such that $\overline{J}(f) \cap J(g)$ is not empty, the number $\langle f|g\rangle := (u_{r\overline{r}} f_{\overline{r}}, g_r) = (f_{\overline{r}}, u_{\overline{r}r} g_r)$ is independent of r and called the inner product of f and g. This is obviously a sesquilinear form defined on any pair $f, g \in V_I$ such that $\overline{J}(f) \cap J(g) \neq \emptyset$. Such a pair will be called *compatible* and $\langle\cdot|\cdot\rangle$ a *partial inner product* on V_I.

Next we introduce operators on a NHS. The idea is to replace unbounded, or even very singular, operators by a (coherent) collection of bounded operators $\{A_{sr} : V_r \to V_s\}$, called *representatives*. Actually the original definition of [12] considers operators as algebraic inductive limit of sets of representatives, but a much simpler definition, yet equivalent, will given in the context of PIP-spaces. For simplicity, we will give that version only, in Sec. 4.

Suffice it to say that every operator A has an adjoint $A^\times$ verifying the natural relation $\langle y|Av\rangle = \langle A^\times y|v\rangle$, when all the expressions are well defined. Also two operators A, B can be multiplied only when matching conditions are satisfied, and then $(AB)^\times = B^\times A^\times$. It follows that operators on a NHS form a partial *-algebra [14, 15].

Actually the final description of NHS given in [12, 13] was preceded by the paper [16], which clearly shows the original motivation, namely, to find a elegant and rigorous formulation of quantum mechanics. The same applies to several papers on application of NHS, for instance [17, 18]. As explained in Sec.1, the search for a more general formulation gradually turned to our next topic, namely, Partial inner product spaces (PIP-spaces). Before that, Alex had already tried to generalize the NHS, under various names, such as "Espaces hilbertiens généralisés" or Dirac spaces [19]. the latter, in particular, is interesting, because it is defined as a collection of Banach spaces or Hilbert spaces: PIP-spaces are not far!

4 Towards Partial inner product spaces

In a RHS, the form $\langle\cdot|\cdot\rangle$ defines a partial inner product on $\Phi^\times$, in the sense that the product $\langle\xi|\eta\rangle$ is defined *only* when one of the two elements ξ, η belongs to Φ. For a NHS V_I, we have seen above that the inner product $\langle f|g\rangle$ is defined only if f, g are compatible in the sense that $\overline{J}(f) \cap J(g) \neq \emptyset$. Thus, in both cases, the crucial notion is that of compatibility (although that of a RHS is poor, as pointed out in Sec.2). This prompted Alex and I to base our formalism on an abstract notion of *linear compatibility* on a vector space V. That is, a symmetric binary relation $f\,\#\,g$ which preserves linearity:

$$f\,\#\,g \iff g\,\#\,f,\ \forall\, f,g \in V,$$
$$f\,\#\,g, f\,\#\,h \implies f\,\#\,(\alpha g + \beta h),\ \forall\, f,g,h \in V,\ \forall\, \alpha,\beta \in \mathbb{C}.$$

As a consequence, for every subset $S \subset V$, one has

$$S^{\#\#} = (S^{\#})^{\#} \supseteq S,\ S^{\#\#\#} = S^{\#}.$$

We will call *assaying subspace* of V a subspace S such that $S^{\#\#} = S$ and denote by $\mathcal{F}(V,\#)$ the family of all assaying subsets of V, ordered by inclusion. Let F be the isomorphy class of $\mathcal{F}$, that is, $\mathcal{F}$ considered as an abstract partially ordered set. Elements of F will be denoted by $r,q,\ldots$, and the corresponding assaying subsets $V_r, V_q, \ldots$. By definition, $q \leq r$ if and only if $V_q \subseteqq V_r$. We also write $V_{\overline{r}} = V_r^{\#}$, $r \in F$. In other words, vectors should not be considered individually, but only in terms of assaying subspaces, which are the building blocks of the whole structure. For instance, $f\,\#\,g$ if and only if there is a $r \in F$ such that $f \in V_r$ and $g \in V_{\overline{r}}$.

Then one has the following standard result from universal algebra:

Theorem 1. *The family $\mathcal{F}(V,\#) \equiv \{V_r, r \in F\}$, ordered by inclusion, is a complete involutive lattice, i.e., it is stable under the following operations, arbitrarily iterated:*

$$\begin{array}{ll} .\ \textit{involution:} & V_r \leftrightarrow V_{\overline{r}} = (V_r)^{\#}, \\ .\ \textit{infimum:} & V_{p\wedge q} := V_p \wedge V_q = V_p \cap V_q, \qquad (p,q,r \in F) \\ .\ \textit{supremum:} & V_{p\vee q} := V_p \vee V_q = (V_p + V_q)^{\#\#}. \end{array}$$

The smallest element of $\mathcal{F}(V,\#)$ is $V^{\#} = \bigcap_r V_r$ and the greatest element is $V = \bigcup_r V_r$. By definition, the index set F is also a complete involutive lattice; for instance,

$$(V_{p\wedge q})^{\#} = V_{\overline{p\wedge q}} = V_{\overline{p}\vee\overline{q}} = V_{\overline{p}} \vee V_{\overline{q}}.$$

A *partial inner product* on $(V,\#)$ is a Hermitian form $\langle\cdot|\cdot\rangle$ defined exactly on compatible pairs of vectors. A *partial inner product space* (PIP-space) is a vector space V equipped with a linear compatibility and a partial inner product. The latter is usually assumed to be positive definite, but this is not compulsory.

The partial inner product clearly defines a notion of *orthogonality:* $f \perp g$ if and only if $f\,\#\,g$ and $\langle f|g\rangle = 0$. The PIP-space $(V,\#,\langle\cdot|\cdot\rangle)$ is *nondegenerate* if $(V^{\#})^{\perp} = \{0\}$, that is, if $\langle f|g\rangle = 0$ for all $f \in V^{\#}$ implies $g = 0$. From now on, we will assume that our PIP-space $(V,\#,\langle\cdot|\cdot\rangle)$ is nondegenerate. As a consequence, $(V^{\#}, V)$ and every couple $(V_r, V_{\overline{r}})$, $r \in F$, are a dual pair in the sense of topological vector spaces [20].

Now, one wants the topological structure to match the algebraic structure, in particular, the topology τ_r on V_r should be such that its conjugate dual be $V_{\overline{r}}$: $(V_r[\tau_r])^{\times} = V_{\overline{r}}$, $\forall r \in F$. This implies that the topology τ_r must be finer than the weak topology $\sigma(V_r, V_{\bar{r}})$ and coarser than the Mackey topology $\tau(V_r, V_{\bar{r}})$:

$$\sigma(V_r, V_{\bar{r}}) \;\leq\; \tau_r \;\leq\; \tau(V_r, V_{\bar{r}}).$$

Throughout the theory, we will assume that every V_r carries its Mackey topology $\tau(V_r, V_{\bar{r}})$. This choice has two interesting consequences. First, if $V_r[\tau_r]$ is a Hilbert space or a reflexive Banach space, then $\tau(V_r, V_{\bar{r}})$ coincides with the norm topology. Next, $r < s$ implies $V_r \subset V_s$, and the embedding operator $E_{sr} : V_r \to V_s$ is continuous and has dense range. In particular, $V^{\#}$ is dense in every V_r.

Standard examples include sequence spaces, spaces of locally integrable functions, Rigged Hilbert spaces, and Nested Hilbert spaces.

As we have seen, $\mathcal{F}(V, \#)$ is a huge lattice (it is complete!) and assaying subspaces may be complicated, such as Fréchet spaces, nonmetrizable spaces, etc. The theory may be developed in full generality, as shown in [7, 8]. In practice, however, a more restrictive class is largely sufficient, namely, lattices of Banach spaces (LBS) or Hilbert spaces (LHS). In order to situate these within the general theory, one chooses an involutive sublattice $\mathcal{I} \subset \mathcal{F}$, indexed by I, such that

(i) $\mathcal{I}$ is generating: $f \# g \;\Leftrightarrow\; \exists\, r \in I$ such that $f \in V_r, g \in V_{\bar{r}}$;
(ii) every $V_r, r \in I$, is a Hilbert space or a reflexive Banach space;
(iii) there is a unique self-dual, Hilbert, assaying subspace $V_o = V_{\bar{o}}$.

In the Hilbert case, one has, of course, recovered the notion of NHS (with unit nestings). In both cases, one equips intersections and unions with the so-called projective, resp. inductive, norms, inherited from interpolation theory (see [14]).

Familiar examples of LBS/LHS are

1. Chains of Banach or Hilbert spaces, such as the chain of Lebesgue spaces on a finite interval $\mathcal{I} = \{L^p([0,1], dx),\ 1 < p < \infty\}$ or the scale of Hilbert spaces $\{\mathcal{H}_n, n \in \mathbb{Z}\}$ built on powers of an unbounded self-adjoint operator $A = A^* \geq 1$.
2. Various lattices of weighted Hilbert spaces of sequences or of locally integrable functions.
3. The lattice generated by the spaces $L^p(\mathbb{R}, dx)$, $1 < p < \infty$, which is not a chain.
4. Many more spaces used in analysis or signal processing, such as Mixed norm Lebesgue spaces $L^{p,q}_m(\mathbb{R}^2)$, Amalgam spaces, Modulation spaces $M^{p,q}_m(\mathbb{R}^d)$ (see [14]).
5. Several PIP-spaces of entire functions with Bargmann's space as central Hilbert space [21]. Given the Gaussian measure on $\mathbb{C}$, $\mathrm{d}\mu(z) = \pi{-}1 \exp(-|z|^2\, \mathrm{d}z$, and a measurable real-valued function ρ on $\mathbb{C}$, define the Hilbert space $\mathcal{F}(\rho)$ of all entire functions f such that

$$\|f\|^2_\rho := \int_{\mathbb{C}} |f(z)|^2 e^{-\rho(z)}\, \mathrm{d}\mu(z) < \infty.$$

Then one can build several PIP-spaces from families of (assaying) spaces $\mathcal{F}(\rho)$, which were in fact introduced by Alex in a talk at UCL.

5 Operators on PIP-spaces

5.1 General definitions

As already mentioned, the basic idea of PIP-spaces is that vectors should not be considered individually, but only in terms of the subspaces V_r ($r \in F$ or $r \in I$), the building blocks of the structure. Correspondingly, an operator on a PIP-space should be defined in terms of assaying subspaces only, with the proviso that only bounded operators between Hilbert or Banach spaces are allowed. Thus an operator is a *coherent collection* of bounded operators. For simplicity, we restrict the definition to the case of a LBS/LHS, the general case is essentially identical. More precisely, Given a LHS or LBS $V_I = \{V_r,\, r \in I\}$, an *operator* on V_I is a map $A : \mathcal{D}(A) \to V$, where

(i) $\mathcal{D}(A) = \bigcup_{q \in \mathsf{d}(A)} V_q$, with $\mathsf{d}(A)$ a nonempty subset of I;
(ii) for every $q \in \mathsf{d}(A)$, there exists a $p \in I$ such that the restriction of A to V_q is linear and continuous into V_p (we denote this restriction by A_{pq});
(iii) A has no proper extension satisfying (i) and (ii).

The set of all operators on V_I is denoted $\mathrm{Op}(V_I)$. Operators between two LBS/LHS V_I, Y_K are defined in the same way and their set is denoted by $\mathrm{Op}(V_I, Y_K)$.

The bounded operator $A_{pq} : V_q \to V_p$ id called a *representative*. The operator A may be characterized by the set $\mathsf{j}(A) = \{(q,p) \in I \times I : A_{pq} \text{ exists}\}$. Thus the operator A may be identified with the collection of its representatives and every one of them uniquely defines A. The collection $\mathsf{j}(A)$ is *coherent*, in the sense that, whenever $p < p', q > q'$, one has $A_{p'q'} = E_{p'p} A_{pq} E_{qq'}$.

The crucial fact in the definition is the maximality condition (iii), which implies that an operator A cannot have any extension, contrary to unbounded operators in a Hilbert space, for instance.

The idea behind the notion of operator is to keep also the *algebraic operations* on operators, namely:

(i) *Adjoint :* every $A \in \mathrm{Op}(V_I)$ has a unique adjoint $A^\times \in \mathrm{Op}(V_I)$, defined by the relation

$$\langle A^\times x | y \rangle = \langle x | Ay \rangle, \text{ for } y \in V_r,\ r \in \mathsf{d}(A), \text{ and } x \in V_{\overline{s}},\ s \in \mathsf{i}(A), \tag{3}$$

that is, $(A^\times)_{\overline{rs}} = (A_{sr})^*$ (usual Hilbert/Banach space adjoint). Here one has defined

$$\mathsf{i}(A) = \{p \in I : \text{there is a } q \text{ such that } A_{pq} \text{ exists}\},$$

a set that plays the role of the range of A. It follows from the definition (3) that $A^{\times\times} = A$, for every $A \in \mathrm{Op}(V_I)$: no extension is allowed, by the maximality condition (iii).

(ii) *Partial multiplication: AB* is defined if and only if there is a $q \in \mathsf{i}(B) \cap \mathsf{d}(A)$, that is, if and only if there is a continuous factorization through some V_q :

$$V_r \xrightarrow{B} V_q \xrightarrow{A} V_s, \quad \text{i.e.,} \quad (AB)_{sr} = A_{sq} B_{qr} \,.$$

It is worth noting that, for a LHS/LBS, the domain $\mathcal{D}(A)$ is always a vector subspace of V (this is not true for a general PIP-space). Therefore, $\mathrm{Op}(V_I)$ is a vector space and a *partial *-algebra* [15].

5.2 Homomorphisms and orthogonal projections

There are many classes of operators on a PIP-space. The one we need most is that of *homomorphisms*

Let V_I, Y_K be two LHSs or LBSs. An operator $A \in \mathrm{Op}(V_I, Y_K)$ is called a *homomorphism* if

(i) for every $r \in I$ there exists $u \in K$ such that both A_{ur} and $A_{\overline{ur}}$ exist;

(ii) for every $u \in K$ there exists $r \in I$ such that both A_{ur} and $A_{\overline{ur}}$ exist.

Equivalently, for every $r \in I$, there exists $u \in K$ such that $(r,u) \in \mathrm{j}(A)$ and $(\overline{r},\overline{u}) \in \mathrm{j}(A)$, and for every $u \in K$, there exists $r \in I$ with the same property.[1]

Such homomorphisms are needed in two instances. The first one is a formulation of the theory in the language of categories [22]. This is interesting, since operators on a PIP-space constitute sheaves and cosheaves, the latter being rather uncommon.

The second case where homomorphisms play a crucial role is the definition of *orthogonal projections* [6] or [14, Sec.3.4]. Since PIP-spaces tend to mimic the properties of Hilbert spaces, one may wonder whether this applies also to geometry, in particular the bijection between (closed) subspaces and orthogonal projections. To that effect we need a good definition of the latter. We adopted the following one : an orthogonal projection in a nondegenerate PIP-space is a homomorphism P satisfying the conditions $P^2 = P = P^\times$. Then the main result is that a subspace W of a PIP-space V is orthocomplemented if and only if it is the range of an orthogonal projection P :

$$W = PV \text{ and } V = W \oplus Z.$$

There are equivalent conditions, for instance, the vector subspace W is orthocomplemented if it satisfies the following two conditions:

(i) for every assaying subset $V_r \subseteq V$, the intersections $W_r = W \cap V_r$ and $W_{\overline{r}} = W \cap V_{\overline{r}}$ are a dual pair in V;

(ii) the intrinsic Mackey topology $\tau(W_r, W_{\overline{r}})$ coincides with the Mackey topology $\tau(V_r, V_{\overline{r}})|_{W_r}$ induced by V_r.

The upshot is that we have in this way the natural notion of PIP-subspace, that generalizes that of Hilbert subspace. And indeed, the set $\mathrm{Proj}(V_I$ of all orthogonal projections enjoys lattice properties similar to their Hilbert space analogues.

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[1] Contrary to what is stated in [14, Def. 3.3.4], the conditions (i) and (ii) do *not* imply $\mathrm{j}(A) = I \times K$ and $\mathrm{j}(A^\times) = K \times I$.

As a final remark, one may note that, in a nondegenerate PIP-space V, a finite-dimensional subspace W of V is orthocomplemented if and only if it is contained in $V^{\#}$ and nondegenerate, i.e., $W \cap W^{\perp} = \{0\}$. This may have significance in problems of approximation.

5.3 Symmetric operators and self-adjointness

A standard problem is to determine whether a symmetric operator in a Hilbert space $\mathcal{H}$ is in fact self-adjoint. This is the case, in particular, for quantum Hamiltonians. The usual technique is to start from the restriction to a smaller subspace and exploit the theory of self-adjoint extensions of von Neumann. But there is an alternative, namely, to start from a space larger than $\mathcal{H}$ and search for self-adjoint *restrictions* to $\mathcal{H}$. This approach, leading to the KLMN theorem, [KLMN stands for Kato, Lax, Lions, Milgram, Nelson], a technique pioneered by Nelson [23] and translated by us in PIP-space language [5]. The theorem runs as follows.

Theorem 2. . *Let V_I be a nondegenerate, positive definite PIP-space with a central Hilbert space $V_o = V_{\overline{o}}$ and let $A = A^{\times} \in \mathrm{Op}(V_I)$ be a symmetric operator. Assume there exists a $\lambda \in \mathbb{R}$ such that $A - \lambda$ has an invertible representative $A_{sr} - \lambda E_{sr}$ from a "small" V_r onto a "big" V_s (i.e., $V_r \subseteq V_o \subseteq V_s$). Then there exists a unique restriction of A_{sr} to a self-adjoint operator A in the Hilbert space V_o. The number λ does not belong to the spectrum of A. The domain of A is obtained by eliminating from V_r exactly the vectors f that are mapped by A_{sr} beyond V_o (i.e., satisfy $A_{sr} f \notin V_o$). The resolvent $(\mathsf{A} - \lambda)^{-1}$ is compact (trace class, etc.) if and only if the natural embedding $E_{sr} : V_r \to V_s$ is compact (trace class, etc.).*

Several variants of the theorem are available, for instance in terms of quadratic forms, see or instance [14, Sec. 3.3.5]. Applications include the analysis (in momentum space) of Schrödinger (or Dirac) Hamiltonians in the presence of singular interactions [24] or solid state Hamiltonians [25] (see Sec. 6.3 below).

Another application of KLMN-type approaches is a generalization of the Gel'fand-Maurin spectral theorem in a PIP-space [29]. Actually this discussion needed a full analysis of spectral properties of a self-adjoint operator, including the resolvent and its analyticity properties, generalized eigenvectors, and so on, going beyond the standard RHS result of Sec. 2. On the way, we had to define the inverse of an PIP-space operator, a missing piece of the puzzle.

6 Applications in quantum mechanics

6.1 General formulation

As it is well known, the natural framework of nonrelativistic quantum mechanics is the Schwartz triplet, $\mathcal{S} \subset L^2 \subset \mathcal{S}^\times$, suggested by the operators of position q and momentum p. A familiar formulation is the so-called Hermite representation of tempered distributions introduced by B. Simon [10], which consists in a chain of Hilbert spaces. Actually this was anticipated and generalized by Alex already in 1965 [11]. He defines recursively the operators

$$\begin{aligned} M_0 &= 1 \\ M_1 &= x^2 + p^2 \\ &\vdots \\ M_n &= xM_{n-1}x + pM_{n-1}p. \end{aligned}$$

Then he considers the numbers $(u, M_n u), u \in \mathcal{S}$ and the Hilbert space H_n as the set of functions $u \in \mathcal{S}$ for which

$$\sum_n [\Gamma(\alpha n)^{-2}(u, M_n u) < \infty,\ \alpha > 1/2.$$

Since the numbers $(u, M_n u)$ can be reexpressed in terms of Hermite functions, we obviously recover the usual formulation, but in fact one gets much more. Given a sequence $\{\beta\} = \{\beta_0, \beta_1, \ldots, \beta_n\}$ of positive numbers, denote by $\mathcal{S}(\beta)$ the set of all $u \in \mathcal{S}$ such that

$$\sum_n \beta_n(u, M_n u) < \infty,$$

Then $\mathcal{S}(\beta)$ is a Hilbert space of test functions, specified by the sequence $\{\beta\}$, and its dual is a Hilbert space of distributions, possibly much larger than $\mathcal{S}^\times$. This construction manifestly anticipates PIP-spaces !

An interesting characteristic of $\mathcal{S}(\beta)$ is that, for β_n decreasing sufficiently fast as $n \to \infty$, elements $u \in \mathcal{S}(\beta)$ may be analytically continued to an entire function $u(x + iy) = u(z)$. This leads us to another domain of interest of Alex.

6.2 Resonances, analyticity properties

In the skeleton, we had found many pieces of work concerning analytic functions in quantum mechanics. This is not surprising, since Alex had published, together with T.T.Wu [26], a pioneering paper on analyticity properties of Schrödinger scattering amplitudes. In the same vein, resonances may be defined as poles of the analytically continued amplitude. Actually, there are other possible definitions and it is not clear how they compare. In view of this cacophony, Ludwig Streit convened in 1984 a meeting in Bielefeld in order to clean the subject. Alex was of course invited. When his time came, he went to the overhead, put an empty transparency on it, and started to think . . . to think . . . to think (he had visibly not prepared a talk !). This lasted for several minutes, the audience was getting nervous, until finally Alex started to write something, of course original and interesting! You need some patience to work with him . . . .

As mentioned in Section 4, PIP-spaces of entire functions with Bargmann's space as central Hilbert space have been defined in [21], following suggestions from Alex. Actually these are in fact closely related to some of the spaces $\mathcal{S}(\beta)$ described above. Interestingly, Bargmann's space, which is at the core of the Bargmann-Segal (or Fock-Bargmann) representation of quantum mechanics, is also the source of many examples and counterexamples about PIP-spaces. In addition, since it consists of functions $u(x + iy) = u(z)$, it is a *phase space representation,* another subject omnipresent in Alex's work. It provides also a link to frame theory, a key topic in Alex's collaboration with Ingrid Daubechies [27]. Finally, it is well known that resonances are beautifully described in the theory of dilation-analytic operators, and here too Alex produced nice results. [28]. A comprehensive study of PIP-spaces of entire functions may be found in [14, Sec.4.6]. More recently, analyticity properties surfaced in PIP-spaces [29], as mentioned already in Sec. 5.3, when studying the spectrum of a self-adjoint operator.

6.3 Condensed matter physics

A beautiful application of NHS or PIP-spaces is the analysis of Schrödinger Hamiltonians in the presence of singular interactions, exemplified by the Kronig-Penney model [24], whose "potential" consists of a periodic array of δ-functions in one dimension.

Consider, in momentum space and arbitrary dimension ν, the kinetic energy operator T, which is a multiplication operator $(T\phi)(p) = t(p)\phi(p)$ by a positive unbounded function $t(p)$. Then define the following family of Hilbert spaces

$$\mathcal{H}_r(\mathbb{R}^\nu) = \{\phi : \int (t(p)+1)^r\, |\phi(p)|^2 \, \mathrm{d}^\nu(p) < \infty\}.$$

This is again a scale of Hilbert spaces, thus a PIP-space, and one has

$$\mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_0 = L^2 \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2}.$$

Next, consider exponentials, which correspond to δ-functions in the x-representation. Then the key point is that each exponential $e^x_\nu \sim e^{ix\cdot p}, x, p \in \mathbb{R}^\nu$, in dimension ν belongs to some minimal space $\mathcal{H}_r$, where r varies with ν. For instance,

$$\begin{aligned}
&e^x_1 \in \mathcal{H}_{-1}(\mathbb{R})\\
&e^x_2 \notin \mathcal{H}_{-1}(\mathbb{R}^2),\ e^x_2 \in \mathcal{H}_{-2}(\mathbb{R}^2)\\
&e^x_3 \notin \mathcal{H}_{-1}(\mathbb{R}^3),\ e^x_3 \in \mathcal{H}_{-2}(\mathbb{R}^3)\\
&e^x_\nu \notin \mathcal{H}_{-2}(\mathbb{R}^\nu),\ \text{if } \nu \geq 4.
\end{aligned}$$

Next one considers the resolvent $R(E) = (T - E)^{-1}$ of T and one observes that $R(E)$ is bounded with bounded inverse from $\mathcal{H}_r$ to $\mathcal{H}_{r+2}$, and similarly for $R^{1/2}(E) : \mathcal{H}_r \to \mathcal{H}_{r+1}$. Then it only remains to exploit the KLMN theorem of Sec. 5.3 to obtain by restriction self-adjoint operators in $\mathcal{H}_0$.

Now, in order to obtain a decent Hamiltonian, the formal analogue of $T + |\Psi\rangle\langle\Psi|$, $\Psi \in \mathcal{H}_{-1}$ (mildly singular perturbation) or $\Psi \in \mathcal{H}_{-2}$ (strongly singular perturbation), one defines its resolvent and develops the familiar resolvent formalism, exploiting the properties of every operator mapping $\mathcal{H}_r$ to $\mathcal{H}_s$. In particular, one obtains a full description of the spectra.The strength of the PIP-space method is that one can treat both mildly and strongly singular perturbations by the *same* formalism. A similar analysis can be applied to many different situations, such as rows or lattices of δ-functions in 1 or 3 dimensions, derivatives of δ-functions or model molecules.

A similar analysis allows to study (reduced) Bloch Hamiltonians of the type $H = T + V$, where T is a fairly general kinetic energy and V is a periodic crystal field [25]. "Reduced" means that one considers the Hamiltonian H_k for fixed quasi-momentum k , which acts in the triplet of Hilbert spaces built on the powers of the kinetic energy $t(p)$, that is,

$$\mathcal{H}_1 \subset \mathcal{H}_0 \subset \mathcal{H}_{-1},$$

where $\mathcal{H}_0$ is the usual ℓ_2 sequence space. The distinguished feature of this triplet is that all embeddings (nestings) are compact. This implies that the reduced Hamiltonian has a purely discrete spectrum, which proves the existence of bands in the full spectrum.

7 The legacy: Operator partial algebras

As I said before, my active collaboration with Alex on PIP-spaces lasted until a new wind started to blow, soon to become a storm that engulfed the Centre de Physique Théorique of Marseille, namely wavelets. The story started around 1980 with a collaboration between Alex and Jean Morlet [30], soon followed by many others. Alex got excited, started to play on a computer and, at least for a while, quit PIP-spaces. However, before jumping on the wavelet train (see Section. 7 below), I continued, having dragged into the subject two PhD students, Françoise Mathot and Juma Shabani., but Alex did not follow. As he said himself, his only contribution to the subject was the title "POp*-algebras" (pop stars) !!

Françoise was actually considering the partial algebra of operators on a NHS V_I [31], with the idea of finding genuine algebras inside, including a von Neumann algebra. Her result is that $\mathrm{Op}(V_I)$ contains three *-algebras $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$. Among these, $\mathcal{A}$ is a *-algebra, $\mathcal{B}$ is a Banach algebra and a C* algebra under certain conditions, and $\mathcal{C}$ is a von Neumann algebra, thus generated by its orthogonal projections — which are therefore sufficiently numerous. Next she studied the spectral properties of operators contained in each of the three algebras [32]. In both papers, she actually interacted significantly with Alex.

As for Shabani, he studied the varions types of commutants and bicommutants of a subset $\mathcal{R}$ of a PIP-space V_I [33, 34]. This then paved the way for the analysis of a general partial *-algebra of operators, the subject matter of our later textbook [15]. Actually this approach allowed him to obtain a nice definition of an unsmeared quantum field (field at a point) [35], parallel to that given by Alex [17].

8 Towards 2D wavelets

Another decisive interaction with Alex took place in Spring 1987. I had invited him in UCL, Louvain-la-Neuve, where he gave a course on phase space quantum mechanics. He also gave a talk on 1D wavelets, in particular for advertising them to our chemists colleagues who struggled with signal processing. We had already started to collaborate on application of wavelets in NMR spectroscopy — and indeed I had become a sort of permanent visitor at the CPT, where the action was. Later this activity was greatly expanded with the help of EU Marie Curie Fellowships (see for instance [36, 37].

The story starts in the coffee room of the Institut de Physique Théorique of UCL. Thus the two of us were discussing a possible PhD topic for a young African student, called Romain Murenzi. The latter had just concluded a master's thesis on five-dimensional quantum field theory, a subject hardly practical for a developing country! So the idea came up, why not try to do in two dimensions what had been so successful in 1D, namely, wavelet analysis? The topic seemed tractable, involving moderate amounts of mathematics and some simple computing technology, and if it worked out, there could be very interesting practical applications. The problem was that nobody knew how to do it! The next summer, Romain went down to Marseille and started to work with Alex and Ingrid Daubechies who happened to be there too.

When Romain came back three months later, the solution was clear. The key is to start from the operations that one wants to apply to an image, namely, translations in the image plane, rotations for choosing a direction of sight, and global magnification (zooming in and out). The problem is to combine these three elements in such a way that the wavelet machine could start rolling (there are mathematical conditions to satisfy here). The result of Romain was that the so-called similitude group yields a solution (actually, the only one). There remained to put it all together, to turn the mathematical crank and to apply the resulting formalism to a real problem, namely, 2D fractals, and the PhD thesis was within reach [38]. In addition, Romain visited Alain Arnéodo and his group in Bordeaux, where 2D fractals were the daily staple, and taught them how to use 2D wavelets. Many papers followed, more MSc or

PhD students got involved over the years, thus continuing the interaction between Louvain-la-Neuve and Marseille.

The key point is that this solution is universal: in order to design a continuous wavelet transform (CWT) on some manifold, one has to identify the operations one wants to apply to a signal that lives on it. Then, if these operations constitute a group with the required properties, the construction is straightforward. This method has been applied successfully to the CWT in 3D, on the 2-sphere, on a hyperboloid, on space-time. See [39] for a detailed analysis.

9 Epilogue

As we have seen, the influence of Alex's ideas has spread in many directions. Here I am limiting myself to PIP-spaces and their descendants, on one hand, and to wavelets, on the other hand. After the original papers [4, 5, 6], the structure surfaced again around 2007, in a serendipitous way. I had heard several times Hans Feichtinger (Vienna) claim that a single L^p space (on [0,1] or on $\mathbb{R}$) has no intrinsic meaning, one has to consider the whole family $L^p, 1 \leq p \leq \infty$ and operators acting globally on it, like translations or Fourier transform. Then I realized that PIP-spaces provided precisely such a formalism, so that it might be time to write a comprehensive account of it. Camillo Trapani (Palermo) agreed and was interested in such an enterprise since he had worked extensively on RHS, distributions and related matters. So we started and the result was our book [14].

I already mentioned the analysis of spectral properties of self-adjoint operators, with incidence on the formulation of quantum mechanics. Another development pertains to signal processing, in connection with frames, semi-frames and reproducing pairs. In all cases, the problem is to represent a signal in terms of some measurable functions on a manifold. And there too, PIP-spaces provide a valuable insight and far reaching generalizations [40, 41].

Another yet application has to do with the metric operators, a favorite tool in the so-called $\mathcal{PT}$-symmetric quantum mechanics, based on non-self-adjoint Hamiltonians. Indeed, a metric operator generates a whole lattice of Hilbert spaces, so that PIP-spaces are the obvious formalism to use. See the papers [42, 43, 44] for a review of this development.

. . . And the adventure continues. . . . Thank you, Alex !

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