

Probabilistic guarantees on the objective value for the scenario approach via sensitivity analysis

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Abstract—This paper is concerned with objective value performance of the scenario approach for robust convex optimization. A novel method is proposed to derive probabilistic bounds for the objective value from scenario programs with a finite number of samples. This method relies on a max-min reformulation and the concept of complexity of robust optimization problems. With additional continuity and regularity conditions, via sensitivity analysis, we also provide explicit bounds which outperform an existing result in the literature. To illustrate the improvements of our results, we also provide a numerical example.

I. INTRODUCTION

We consider the following robust convex optimization problem (RCP):

$$J^* := \inf_{x \in X} c^\top x \quad (1a)$$

$$\text{s.t. } f(x, \delta) \leq 0, \forall \delta \in \Delta \quad (1b)$$

where $x \in \mathbb{R}^d$ is the decision variable, δ is the uncertain parameter contained in the uncertain set $\Delta \subseteq \mathbb{R}^m$ with the uncertain constraint $f(x, \delta) \leq 0$, and $X \subseteq \mathbb{R}^d$ is some static constraint set. We would like to mention that, while there is only one uncertain constraint in (1), the results and discussions in the paper extend trivially to cases with multiple uncertain constraints. The following assumptions are made.

Assumption 1: $f : X \times \Delta \rightarrow \mathbb{R}$ is lower semi-continuous (l.s.c.) in $X \times \Delta$ and $X \subseteq \mathbb{R}^d$ is compact.

Assumption 2: There exists $x \in X$ such that $f(x, \delta) \leq 0$ for any $\delta \in \Delta$.

Assumption 3: $X \subseteq \mathbb{R}^d$ is convex and $f : X \times \Delta \rightarrow \mathbb{R}$ is convex in $x \in X$ for any $\delta \in \Delta$.

The problem above is a special class of uncertain optimization problems [1]. Classic approaches to tackle uncertainties can be found in [2]–[4]. This paper focuses on the scenario approach or scenario optimization which solves constrained uncertain optimization problems by sampling a finite set of constraints (also called scenarios), see, e.g., some early works [5], [6]. We refer to [7], [8] for a comprehensive view on this subject.

In the scenario approach, a significant amount of attention has been paid to probabilistic guarantees on the sample-based solution. Along this line, chance-constrained theorems

are often derived in terms of the measure of the violating subset of the uncertain constraints for both convex [5], [6], [9]–[12] and non-convex [13], [14] cases by using the concepts of support/essential constraints. In particular, the bounds in [9], [10] have been proved to be tight for a special class of uncertain convex programs called fully supported problems. While these theoretic results bring a lot of new perspectives into uncertain optimization problems, it takes further developments to apply the scenario approach to practical applications. For instance, many real-life physical properties of interest are directly related with the objective value of the underlying optimization problem, which means that the aforementioned chance-constrained theorems alone do not allow us to learn such properties. For this reason, formal analysis of the scenario approach is needed with respect to objective value performance. To the best of our knowledge, the first work providing such results is [15] under some continuity and regularity conditions on the basis of the chance-constrained theorems in [9], [10]. Though the results in [15] are built on an intermediate result that is tight in terms of the measure of the violating subset of the uncertain constraints (as proved in [9], [10]), they are not necessarily tight in terms of the objective value. In this paper, we show that they are indeed not tight by providing improved bounds under the same conditions. In addition, the chance-constrained theorems in [9], [10] require a non-degeneracy assumption. We show that such an assumption can be released when one is only concerned with the objective value. This paper is motivated by recent progress on data-driven analysis of complicated systems, see, e.g., [16]–[24].

The contributions of this paper are two-fold: First, we present a general method of deriving probabilistic guarantees on the objective value performance of the scenario approach for robust convex programs. Second, we show theoretic improvements of our results as compared to [15] under the same regularity conditions.

The rest of the paper is organized as follows. This section ends with the notation, followed by Section II on the review of preliminary results on the scenario approach. Section III presents a general method of deriving objective value performance guarantees for the scenario approach using a max-min reformulation. In Section IV, under additional regularity conditions, we show explicit objective value performance bounds via sensitivity analysis. Finally, an illustrative example is given in Section V. The proofs of all the lemmas, propositions and theorems can be found in [25].

Notation. The non-negative real number set and the non-negative integer set are denoted by \mathbb{R}^+ and \mathbb{Z}^+ respectively.

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Given any $r \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$, $\mathbb{B}_r(x)$ denotes the open ball in \mathbb{R}^n centered at x with radius r . For a square matrix P , $P \succ (\succeq) 0$ means that P is symmetric and positive definite (semi-definite), $\text{tr}(P)$ denotes the trace of P and $\det(P)$ denotes the determinant of P . For a symmetric matrix P , we denote by $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ the largest and smallest eigenvalues of P respectively. For any matrix $P \succ 0$, let $\kappa(P) := \lambda_{\max}(P)/\lambda_{\min}(P)$ and $\chi(P) := \sqrt{\det(P)/\lambda_{\min}(P)^n}$. For any $p \geq 1$, the ℓ_p norm of a vector $x \in \mathbb{R}^n$ is $\|x\|_p$ ($\|x\|$ is the ℓ_2 norm by default). For a set S , $\text{cl}(S)$ denote the closure of S .

II. PRELIMINARIES

This section reviews some known results on the scenario approach. Let $(\Delta, \mathcal{B}(\Delta), \mathbb{P})$ be a probability space where Δ is a metric space, $\mathcal{B}(\Delta)$ is the Borel σ -algebra of Δ and $\mathbb{P} : \mathcal{B}(\Delta) \rightarrow [0, 1]$ is the probability measure. In the framework of the scenario approach, the solution of the robust optimization problem in (1) is approximated with a finite number of samples. More precisely, we randomly generate $N \in \mathbb{Z}^+$ samples, denoted by $(\delta_1, \delta_2, \dots, \delta_N) \in \Delta^N$, from some probability measure with the support Δ , and formulate the following scenario convex program (SCP)

$$\begin{aligned} \mathcal{P}(\omega_N) : \quad J(\bar{\omega}_N) &:= \min_{x \in X} c^\top x & (2a) \\ \text{s.t. } f(x, \delta) &\leq 0, \forall \delta \in \bar{\omega}_N. & (2b) \end{aligned}$$

where $\omega_N := (\delta_1, \delta_2, \dots, \delta_N)$, and $\bar{\omega}_N$ denote the unordered sample set, i.e., $\bar{\omega}_N := \{\delta_1, \delta_2, \dots, \delta_N\}$, and $J : 2^\Delta \rightarrow \mathbb{R}$ with $J(\Delta) = J^*$.

A chance-constrained theorem is provided in [9], [10] for convex problems in terms of the violation probability, as shown below.

Theorem 1 ([9, Theorem 1], [10, Theorem 3.3]): For any $N \in \mathbb{Z}^+$ with $N > d$, suppose ω are N independent and identically distributed (i.i.d.) samples drawn according to the probability measure \mathbb{P} . Consider $\mathcal{P}(\omega)$ as defined in (2), let us assume that $\mathcal{P}(\omega)$ admits a unique optimal solution. Then, under Assumptions 1–3 and additional conditions (see Remark 1), for any $\epsilon \in (0, 1)$,

$$\mathbb{P}^N \{\omega \in \Delta^N : \mathcal{V}(\bar{\omega}) > \epsilon\} \leq \Phi_c(\epsilon; d, N) \quad (3)$$

where $\mathcal{V}(\bar{\omega}) := \mathbb{P}\{\delta : J(\bar{\omega} \cup \{\delta\}) > J(\bar{\omega})\}$, $\bar{\omega}$ is the set of the elements in ω , $J(\bar{\omega})$ is defined as in (2), \mathbb{P}^N is the probability measure in the N -Cartesian product of Δ , and

$$\Phi_c(\epsilon; k, N) := \sum_{i=0}^{k-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i}, \forall k \geq 1. \quad (4)$$

In fact, the bound in Theorem 1 is proved to be tight for a special class of convex programs called fully supported programs, see [9], [10] for details. For non-convex optimization problems, similar results are derived in [13], [14]. However, these results alone do not allow to bound the objective value of the original robust optimization.

Remark 1: The bound in (3) is derived by [9] and [10] based on different conditions: [9] requires that the feasible domain of $\mathcal{P}(\omega)$ has a nonempty interior for any ω ; [10]

assumes that $\mathcal{P}(\omega)$ is nondegenerate with probability one, see Definition 2.7 in [10] for the definition of degenerate problems.

In [15], theoretical links between $\mathcal{P}(\omega)$ and the original robust optimization problem RCP in (1) are further established in terms of the objective value under the assumptions in Theorem 1 and Slater's condition, i.e., there exists $x \in X$ such that $\sup_{\delta \in \Delta} f(x, \delta) < 0$. We briefly summarize the idea in [15] as follows. First, it is shown that the optimizer of SCP is a feasible solution to the following chance-constrained program (CCP) with some confidence level

$$\inf_{x \in X} c^\top x \quad (5a)$$

$$\text{s.t. } \mathbb{P}\{f(x, \delta) \leq 0\} \geq 1 - \epsilon \quad (5b)$$

where $\epsilon \in (0, 1)$. It is then shown that the feasible set of the program CCP is contained in the feasible set of a relaxed RCP in the form of

$$\inf_{x \in X} c^\top x \quad (6a)$$

$$\text{s.t. } f(x, \delta) \leq \xi, \forall \delta \in \Delta \quad (6b)$$

where $\xi \geq 0$. Finally, with the relations above, probabilistic objective bounds for the scenario approach can be derived from the relaxed RCP under Lipschitz continuity and other regularity assumptions. We call this procedure a two-step approach as it relies on the link from SCP to CCP to establish probabilistic objective value bounds. As we will show in the sequel, the step from SCP to CCP brings conservatism into the resulting bound.

In this paper, we revisit the problem of deriving probabilistic objective value bounds for the scenario approach from a sensitivity analysis perspective in the context of robust optimization. We develop a method that skips the step from SCP to CCP and enables us to directly establish probabilistic objective value bounds for SCP using a perturbation argument. Thus, our approach does not require the conditions in Remark 1. We believe that this latter fact constitutes an important improvement because these conditions can be violated in some control problems, see [22] for such an example.

III. PROBABILISTIC OBJECTIVE VALUE PERFORMANCE

This section presents a general method deriving probabilistic objective value bounds of the scenario approach for robust optimization problems. This method adopts a max-min reformulation of the original robust optimization problem (1).

A. Continuity properties

We first show some preliminary continuity results on parametric optimization, see, e.g., [26]–[28] for a comprehensive view. Given any $N \in \mathbb{Z}^+$, for any $\omega_N := (\delta_1, \delta_2, \dots, \delta_N) \in \Delta^N$, for ease of discussion, let us define

$$g_N(\omega_N) := J(\{\delta_1, \delta_2, \dots, \delta_N\}), \quad (7)$$

$$F_N(\omega_N) := \{x \in X : f(x, \delta_i) \leq 0, i = 1, 2, \dots, N\}, \quad (8)$$

$$F_N^o(\omega_N) := \{x \in X : f(x, \delta_i) < 0, i = 1, 2, \dots, N\}, \quad (9)$$

$$\phi_N(\omega_N) := \{x \in F_N(\omega_N) : c^\top x = g_N(\omega_N)\}, \quad (10)$$

Under Assumption 1, a semi-continuity property is stated below.

Lemma 1: Consider the RCP in (1) and the SCP in (2). Given any $N \in \mathbb{Z}^+$, for any $\omega \in \Delta^N$, let $g_N(\omega)$ be defined as in (7). Suppose Assumptions 1& 2 hold, then $g_N(\omega)$ is l.s.c. in Δ^N .

B. Max-min reformulation

With the continuity property above, we then define, for any $N \in \mathbb{Z}^+$,

$$g_N^* := \sup_{\omega \in \Delta^N} g_N(\omega) = \sup_{\omega \in \Delta^N} \inf_{x \in X} c^\top x + \chi_{S_N}(x, \omega), \quad (11)$$

where $S_N := \{(x, \delta_1, \delta_2, \dots, \delta_N) : f(x, \delta_i) \leq 0, i = 1, 2, \dots, N\}$ and $\chi_{S_N} : X \times \Delta^N \rightarrow \{0, +\infty\}$ is the characteristic function of S_N defined as follows: $\chi_{S_N}(x, \omega) = 0$ if $(x, \omega) \in S_N$ and $= +\infty$ everywhere else.

The next lemma states the equivalence between the RCP in (1) and the max-min formulation as defined in (11).

Lemma 2: Consider Problems (1) and (11) for any $N \in \mathbb{Z}^+$ with the optimal values J^* and g_N^* respectively. Suppose Assumptions 1 – 3 hold, then it holds that $J^* = g_N^*$ for any $N \geq d$;

In view of the result in Lemma 2, we define the concept of *complexity* for robust optimization problems with infinitely many constraints, following the definition of *complexity* for scenario programs with a finite number of constraints in [12].

Definition 1: Consider the RCP (1), the *complexity* of RCP is the smallest integer $k \in \mathbb{Z}^+$ such that $g_k^* = J^*$.

Similar to the *complexity* of SCP in [12], the *complexity* of RCP is intuitively the minimal number of samples needed to obtain the true objective value J^* with random sampling. With this formal definition, Lemma 2 basically means that the *complexity* of RCP is bounded by the number of decision variables under Assumptions 1 – 3.

Remark 2: When RCP is not feasible, i.e., Assumption 2 does not hold or $J^* = +\infty$, from Helly's theorem, it still holds that $J^* = g_{d+1}^*$ but not $J^* = g_d^*$ in Lemma 2. Thus, the *complexity* of RCP is bounded by $d + 1$ in any case.

C. Uniform level-set bounds

To establish probabilistic objective value performance guarantees, inspired from [15], [29], we define the following tail probability for any $N \in \mathbb{Z}^+$

$$p_N(\alpha) := \mathbb{P}^N \{\omega \in \Delta^N : g_N^* - \alpha \leq g_N(\omega)\}, \forall \alpha \geq 0. \quad (12)$$

From the semi-continuity property of g_N in Lemma 1, $p_N(\alpha)$ is well defined and monotonically increasing. In particular, when $N \geq d$, from Lemma 2, p_N defined as in (12) becomes

$$p_N(\alpha) = \mathbb{P}^N \{\omega \in \Delta^N : J^* - \alpha \leq g_N(\omega)\}, \forall \alpha \geq 0. \quad (13)$$

The continuity property of this tail probability function is stated in the following lemma.

Lemma 3: Consider the RCP (1) and the SCP (2). Suppose Assumptions 1 - 3 hold. For any $N \in \mathbb{Z}^+$, let the tail probability function $p_N : \mathbb{R}^+ \rightarrow [0, 1]$ be defined as in (12). Then, the following property holds: i) There exists a finite

$C \in \mathbb{R}^+$ such that $p_D(\alpha) = 1$ for any $\alpha \geq C$. ii) p_N is upper semi-continuous (u.s.c.) at any $\alpha \in \mathbb{R}^+$.

We then call $\tilde{\alpha} : (0, 1) \rightarrow \mathbb{R}$ a uniform level-set bound (ULB) if it satisfies

$$p_N(\tilde{\alpha}(\varepsilon)) \geq \varepsilon, \forall \varepsilon \in (0, 1). \quad (14)$$

The optimal ULB is defined as follows:

$$\alpha^*(\varepsilon) := \sup_{\alpha \geq 0} \alpha : p_N(\alpha) \leq \varepsilon, \forall \varepsilon \in (0, 1). \quad (15)$$

With the discussions above, we arrive at the main result of this section.

Proposition 1: Consider the RCP in (1) and the SCP in (2). Suppose Assumptions 1 – 3 hold. Given $N(N \geq d)$ i.i.d. samples, denoted by ω_N , drawn according to the probability measure \mathbb{P} , consider Problem (2) with the optimal value $g_N(\omega_N)$ as defined in (7). For any $\varepsilon \in (0, 1)$, let $\alpha^*(\varepsilon)$ be defined as in (15). Then, it holds that

$$\mathbb{P}^N \{\omega_N \in \Delta^N : J^* \leq g_N(\omega_N) + \alpha^*(\varepsilon)\} \geq \varepsilon \quad (16)$$

where J^* is given in (1).

While this is an elementary result, following from the bound on the *complexity* of RCP and the u.s.c. of p_N , it provides a general way of bounding the original objective value J^* . In practice, as computing the optimal ULB is not always possible, we turn to upper bounds of $\alpha^*(\varepsilon)$.

The definition of the tail probability function in (12) can be also extended to a functional space as follows: $\forall \alpha \in \mathcal{L}_N^+$,

$$p_N(\alpha) = \mathbb{P}^N \{\omega \in \Delta^N : J^* - \alpha(\omega) \leq g_N(\omega)\}. \quad (17)$$

where $\alpha : \Delta^N \rightarrow \mathbb{R}^+$ and \mathcal{L}_N^+ is the space of measurable positive real functions in Δ^N . Similarly, probabilistic upper bounds on J^* can be obtained by finding $\alpha \in \mathcal{L}_N^+$ such that $p_N(\alpha) \geq \varepsilon$ for some given $\varepsilon \in (0, 1)$.

IV. EXPLICIT BOUNDS VIA SENSITIVITY ANALYSIS

We now explicitly characterize ULBs as defined in (14) via sensitivity analysis.

A. Sensitivity analysis

With an assumption on the continuity of $g_d : \Delta^d \rightarrow \mathbb{R}$, a deterministic relation between the robust optimization problem RCP and the scenario program SCP is established in the following lemma. We will provide sufficient conditions to ensure this assumption later in this section.

Lemma 4: Suppose Assumptions 1 - 3 hold. Let $g_d(\omega)$ be defined as in (7) for any $\omega \in \text{cl}(\Delta)^d$. Assume that there exists a constant L_g such that, for any $\omega = (\delta_1, \delta_2, \dots, \delta_d) \in \text{cl}(\Delta)^d$ and $\omega' = (\delta'_1, \delta'_2, \dots, \delta'_d) \in \text{cl}(\Delta)^d$,

$$\|g_d(\omega) - g_d(\omega')\| \leq L_g \max_{i \in \{1, 2, \dots, d\}} \|\delta_i - \delta'_i\|. \quad (18)$$

Then, the following results holds: (i) There exists a set $\bar{\omega}_d^* := \{\delta_1^*, \delta_2^*, \dots, \delta_d^*\} \subset \text{cl}(\Delta)$ such that $J(\bar{\omega}_d^*) = J^*$. (ii) Given any finite subset $\bar{\omega} \subseteq \text{cl}(\Delta)$, $J(\bar{\omega}) \geq J^* - L_g \|\bar{\omega}_d^*\|_{\bar{\omega}}$.

When the uncertainty δ in RCP is a scalar, the condition (18) is simply the Lipschitz continuity with respect to the ℓ_∞ norm. In order to obtain an explicit ULB, we also need

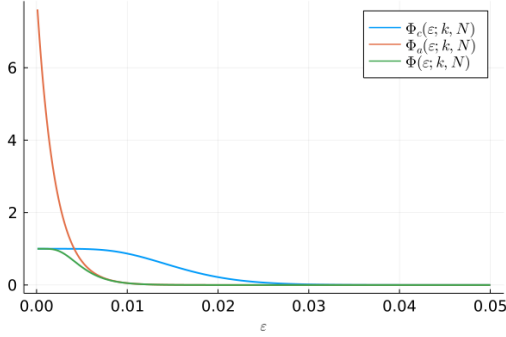


Fig. 1: Comparison of different probability bounds with $k = 8$ and $N = 500$.

a regularity condition on Δ with respect to the measure \mathbb{P} . We consider the same condition as in [15].

Assumption 4: For the probability space $(\Delta, \mathcal{B}(\Delta), \mathbb{P})$, there exists a strictly increasing function $\varphi: \mathbb{R}^+ \rightarrow [0, 1]$ such that $\forall \delta \in \text{cl}(\Delta), \forall r \in \mathbb{R}^+, \mathbb{P}\{\mathbb{B}_r(\delta) \cap \Delta\} \geq \varphi(r)$, where $\mathbb{B}_r(\delta)$ is an open ball centered at δ with radius r .

The following lemma shows an elementary probability result.

Lemma 5: Given the probability space $(\Delta, \mathcal{B}(\Delta), \mathbb{P})$, for any $k \in \mathbb{Z}^+$ and $\epsilon \in [0, 1]$, consider k subsets $\{B_1, B_2, \dots, B_k\}$ of Δ with $\mathbb{P}\{B_i\} = \epsilon$ for all $i = 1, 2, \dots, k$. For any $N \in \mathbb{Z}^+$, suppose ω is a vector of N independent and identically distributed (i.i.d.) samples drawn according to the probability measure \mathbb{P} . Then,

$$\mathbb{P}^N\{\omega \in \Delta^N : B_i \cap \bar{\omega} \neq \emptyset, \forall i\} \geq 1 - \Phi_a(\epsilon; k, N) \quad (19)$$

where $\bar{\omega}$ is the set of elements in ω and

$$\Phi_a(\epsilon; k, N) := k(1 - \epsilon)^N. \quad (20)$$

In fact, a tighter bound can be derived when ϵ in Lemma 5 is sufficiently small. However, the proof is too long to be included in the paper. We only comment on this in the following remark.

Remark 3: With the same conditions as Lemma 5, when $\epsilon \leq 1/k$, it can be verified that

$$\mathbb{P}^N\{\omega \in \Delta^N : B_i \cap \bar{\omega} \neq \emptyset, \forall i\} \geq 1 - \Phi(\epsilon; k, N) \quad (21)$$

where

$$\Phi(\epsilon; k, N) := \sum_{i=1}^k (-1)^{i-1} \binom{k}{i} (1 - i\epsilon)^N. \quad (22)$$

It can be also shown that $\Phi(\epsilon; k, N) \leq k(1 - \epsilon)^N$.

We compare the bounds in (4) (20), and (19) in Figure 1. It can be seen that $\Phi(\epsilon; k, N)$ is the best among these three bounds. As ϵ increases, $\Phi_a(\epsilon; k, N)$ becomes close to $\Phi(\epsilon; k, N)$ and outperforms $\Phi_c(\epsilon; k, N)$ eventually.

Based the lemmas above, we then present the main result of this section.

Theorem 2: Consider Problem (1) with the optimal objective value J^* , suppose Assumptions 1 – 4 hold. For any $N \in \mathbb{Z}^+$ with $N \geq d$, let $g_N(\omega)$ be defined as in (7) for any $\omega \in \text{cl}(\Delta)^N$. Assume that there exists a constant L_g such

that (18) is satisfied in $\text{cl}(\Delta)^d$. Let ω_N be a vector of N i.i.d. samples drawn according to the probability measure \mathbb{P} . Then, for any $\epsilon \in (0, 1)$, it holds that

$$\begin{aligned} \mathbb{P}^N\{\omega_N \in \Delta^N : J^* \leq g_N(\omega_N) + L_g \varphi^{-1}(\epsilon)\} \\ \geq 1 - \Phi_a(\epsilon; d, N) = 1 - d(1 - \epsilon)^N. \end{aligned} \quad (23)$$

From Theorem 2, we can derive an explicit ULB as follow.

Corollary 1: Suppose the conditions in Theorem 2 hold, a ULB satisfying (14) can be given as follow:

$$\tilde{\alpha}(\epsilon) = L_g \varphi^{-1}\left(1 - \sqrt[N]{\frac{1 - \epsilon}{d}}\right), \forall \epsilon \in (0, 1). \quad (24)$$

According to Remark 3, for any $\epsilon \in (0, 1/d]$, we can even replace the bound in (19) with the one in (21) in Theorem 2. Thus, a tighter ULB can be derived. We omit the details due to page limitation.

B. Lipschitz continuity

In the rest of this section, we discuss sufficient conditions for the continuity condition (18). First, we need Slater's condition of the RCP as given below.

Assumption 5: (Slater's condition) There exists $x_0 \in X$ such that $\sup_{\delta \in \Delta} f(x_0, \delta) < 0$.

Under this assumption, following [15], let us define the following constant:

$$L_{SP} := \frac{\min_{x \in X} c^\top x - c^\top x_0}{\sup_{\delta \in \Delta} f(x_0, \delta)} \quad (25)$$

We then assume Lipschitz continuity on $f(x, \delta)$ in δ inside the closure of Δ .

Assumption 6: For any $x \in X$, $f(x, \delta)$ is Lipschitz continuous in δ with constant L_δ in $\text{cl}(\Delta)$, i.e., $\forall \delta_1, \delta_2 \in \text{cl}(\Delta)$:

$$\|f(x, \delta_1) - f(x, \delta_2)\| \leq L_\delta \|\delta_1 - \delta_2\|, \quad (26)$$

where $\text{cl}(\Delta)$ denote the closure of Δ .

Based on these two additional assumptions, we claim that the continuity condition (18) is guaranteed, as shown in the next proposition.

Proposition 2: Given any $N \in \mathbb{Z}^+$, let $g_N(\omega)$ be defined as in (7) for any $\omega \in \text{cl}(\Delta)^N$. Suppose Assumptions 3 – 6 hold, then, for any $\omega = (\delta_1, \delta_2, \dots, \delta_N) \in \text{cl}(\Delta)^N$ and $\omega' = (\delta'_1, \delta'_2, \dots, \delta'_N) \in \text{cl}(\Delta)^N$, it holds that

$$\|g_N(\omega) - g_N(\omega')\| \leq L_{SP} L_\delta \max_{i \in \{1, 2, \dots, N\}} \|\delta_i - \delta'_i\|. \quad (27)$$

where L_{SP} and L_δ are given in (25) and (26) respectively.

The continuity property above then allows to derive explicit ULBs as mentioned above. The overall procedure is summarized in Figure 2. Putting Theorem 2 and Proposition 2 together, we immediately arrive at the following corollary.

Corollary 2: Consider the same conditions in Theorem 2 and suppose Assumptions 5 & 6 hold. For any $\beta \in (0, 1)$, it holds that

$$\begin{aligned} \mathbb{P}^N\{\omega_N \in \Delta^N : J^* - g_N(\omega_N) \\ \leq L_{SP} L_\delta \varphi^{-1}(\Phi_a^{-1}(\beta; d, N))\} \geq 1 - \beta. \end{aligned} \quad (28)$$

where $\Phi_a^{-1}(\cdot; d, N)$ denote the inverse function of $\Phi_a(\cdot; d, N)$ with fixed d and N .

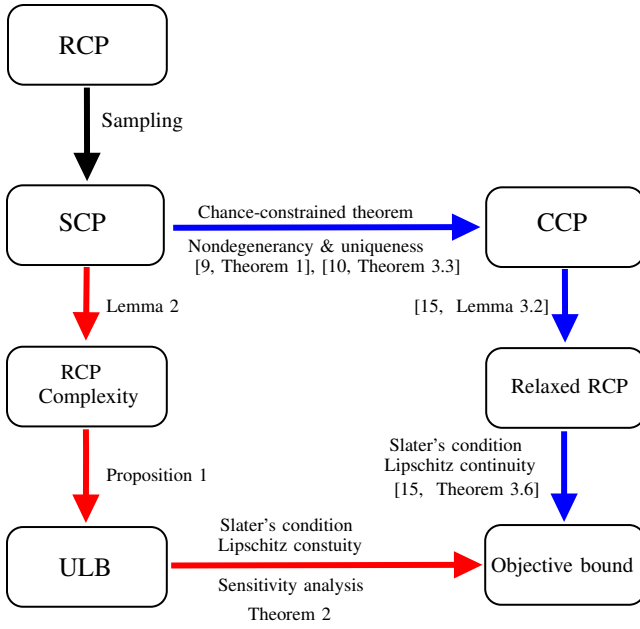


Fig. 2: Flowchart of the proposed method and its comparison with the two-step approach: The red arrows represent our method and the blue arrows represent the two-step approach.

Remark 4: When $\beta \geq \Phi(1/d; d, N)$, as mentioned in Remark 3, a better bound can be obtained

$$\mathbb{P}^N\{\omega_N \in \Delta^N : J^* - g_N(\omega_N) \leq L_{SP}L_\delta\varphi^{-1}(\Phi^{-1}(\beta; d, N))\} \geq 1 - \beta. \quad (29)$$

As a comparison, we also show the bound from Theorem 3.6 in [15] below

$$\mathbb{P}^N\{\omega_N \in \Delta^N : J^* - g_N(\omega_N) \leq L_{SP}L_\delta\varphi^{-1}(\Phi_c^{-1}(\beta; d, N))\} \geq 1 - \beta. \quad (30)$$

The difference of these bounds lies in the probability bounds $\Phi_a(\beta; d, N)$, $\Phi(\beta; d, N)$ and $\Phi_c(\beta; d, N)$, which are depicted in Figure 1.

V. AN ILLUSTRATIVE EXAMPLE

In this section, we present an example to illustrate the conservatism of the chance-constrained theorem in Theorem 1 for the derivation of the object value bound. Consider the following robust optimization problem

$$\begin{aligned} & \min_{x_1, x_2} x_2 \\ & \text{s.t. } |x_1 + b^\top \delta| \leq x_2, \forall \delta \in \Delta = \{\delta \in \mathbb{R}^2 : \|\delta\| = 1\}, \\ & |x_1| \leq 1, |x_2| \leq \sqrt{2}. \end{aligned}$$

where $b = [1; 1]$. As $b^\top \delta \in [-\sqrt{2}, \sqrt{2}]$ for all $\|\delta\| = 1$, the optimal $J^* = \sqrt{2}$ with the optimizer being $x_1^* = 0$ and $x_2^* = \sqrt{2}$. Given N i.i.d. samples $\omega_N := \{\delta_1, \delta_2, \dots, \delta_N\}$ for some $N \in \mathbb{Z}^+$, drawn from the uniform distribution \mathbb{P} on Δ , let

$$\underline{\eta}(\omega_N) := \min_{i=1,2,\dots,N} b^\top \delta_i, \quad \bar{\eta}(\omega_N) := \max_{i=1,2,\dots,N} b^\top \delta_i.$$

With these definitions, the optimizer of the sampled problem SCP is

$$x_1^*(\omega_N) = -\frac{\bar{\eta}(\omega_N) + \underline{\eta}(\omega_N)}{2}, \quad x_2^*(\omega_N) = \frac{\bar{\eta}(\omega_N) - \underline{\eta}(\omega_N)}{2}.$$

The tail probability $p_N(\alpha)$ defined as in (12) becomes

$$p_N(\alpha) = \mathbb{P}^N\{\omega_N \in \Delta^N : \sqrt{2} - \alpha \leq \frac{\bar{\eta}(\omega_N) - \underline{\eta}(\omega_N)}{2}\}$$

for $\alpha \in [0, \sqrt{2}]$. Given any $\alpha \in [0, \sqrt{2}]$, a relaxation of $p_N(\alpha)$ is given below

$$\tilde{p}_N(\alpha) = \mathbb{P}^N\{\omega_N : \bar{\eta}(\omega_N) \geq \sqrt{2} - \alpha, \underline{\eta}(\omega_N) \leq \alpha - \sqrt{2}\}.$$

We also define the following sets: $\bar{S}_\alpha := \{\delta \in \Delta : b^\top \delta \geq \sqrt{2} - \alpha\}$ and $\underline{S}_\alpha := \{\delta \in \Delta : b^\top \delta \leq \alpha - \sqrt{2}\}$, as illustrated in Figure 3. With simple manipulations, $\mathbb{P}\{\bar{S}_\alpha\} = \mathbb{P}\{\underline{S}_\alpha\} = \frac{\cos^{-1}(\frac{\sqrt{2}-\alpha}{\sqrt{2}})}{\pi}$. The relaxation $\tilde{p}_N(\alpha)$ can be considered as the probability that $\bar{\omega}_N \cap \bar{S}_\alpha \neq \emptyset$ and $\bar{\omega}_N \cap \underline{S}_\alpha \neq \emptyset$ with $\bar{\omega}_N := \{\delta_1, \delta_2, \dots, \delta_N\}$. Hence, from Lemma 5 and

Remark 3, $\tilde{p}_N(\alpha) = 1 - \Phi(\frac{\cos^{-1}(\frac{\sqrt{2}-\alpha}{\sqrt{2}})}{\pi}; 2, N) \leq 1 - \Phi_a(\frac{\cos^{-1}(\frac{\sqrt{2}-\alpha}{\sqrt{2}})}{\pi}; 2, N)$. Now, let us consider the optimal ULB of $\tilde{p}_N(\alpha)$: $\tilde{\alpha}^*(\varepsilon) := \sup_{\alpha \geq 0} \alpha : \tilde{p}_N(\alpha) \leq \varepsilon, \forall \varepsilon \in (0, 1)$. As $\tilde{p}_N(\alpha) \leq p_N(\alpha)$, $\alpha^*(\varepsilon) \leq \tilde{\alpha}^*(\varepsilon) = h(1 - \varepsilon) \leq h_a(1 - \varepsilon)$, where $h_a(\beta) := \sqrt{2}(1 - \cos(\pi\Phi_a^{-1}(\beta; 2, N)))$ and $h(\beta) := \sqrt{2}(1 - \cos(\pi\Phi^{-1}(\beta; 2, N)))$, $\beta \in (0, 1)$. Finally, we conclude that, given the confidence level $\beta \in (0, 1)$

$$\begin{aligned} & \mathbb{P}^N\{\omega_N \in \Delta^N : J^* - x_2^*(\omega_N) \leq h_a(\beta)\} \\ & \geq \mathbb{P}^N\{\omega_N \in \Delta^N : J^* - x_2^*(\omega_N) \leq h(\beta)\} \geq 1 - \beta. \end{aligned} \quad (31)$$

We now consider Theorem 1. For any $\varepsilon \in (0, 1)$,

$$\begin{aligned} & \mathbb{P}^N\{\omega_N \in \Delta^N : \mathbb{P}\{\delta : J(\bar{\omega}_N \cup \{\delta\}) > J(\bar{\omega}_N)\} \leq \varepsilon\} \\ & \geq 1 - \Phi_c(\varepsilon; d, N) \end{aligned}$$

In fact, the equality holds because this is a fully supported problem as defined in [10]. After some manipulations, the resulting objective value bound is the following:

$$\mathbb{P}^N\{\omega_N \in \Delta^N : J^* - x_2^*(\omega_N) \leq h_c(\beta)\} \geq 1 - \beta, \quad (32)$$

where $h_c(\beta) := \frac{\sqrt{2}(1 - \cos(\pi\Phi_c^{-1}(\beta; 2, N)))}{2}$ for any confidence level $\beta \in (0, 1)$. The comparison of these objective bounds is shown in Figure 4 with $N = 100$. From this example, we can see that Theorem 1 induces conservatism into the derivation of the objective value bound as the worst case has to be taken into account.

VI. CONCLUSIONS

In this paper, we have presented a novel method for deriving probabilistic guarantees on the objective value for the scenario approach for robust optimization problems via sensitivity analysis. This method is based in particular on the concept of *complexity* of a robust optimization problem, which is the minimal number of constraints that define the actual solution. For robust convex programs, we show that the *complexity* is bounded by the dimension of the decision variable by the use of a max-min reformulation and thereby

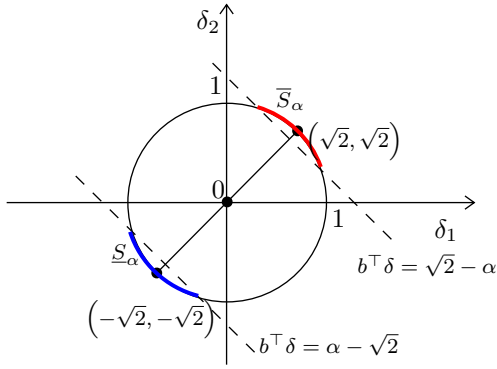


Fig. 3: Illustration of \bar{S}_α and \underline{S}_α .

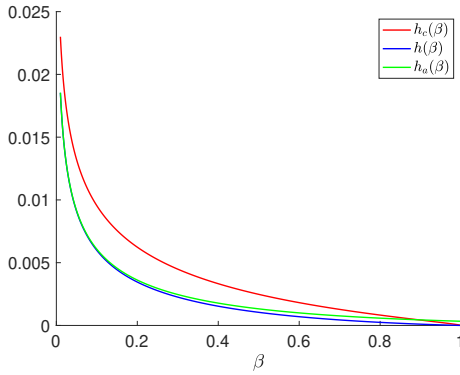


Fig. 4: Comparison of different objective bounds.

extend classical results, which are known for scenario programs but not the original robust optimization problem, which possibly contains an infinite number of constraints. With this observation and additional regularity conditions, we are able to build probabilistic objective value bounds for the scenario approach. Our results do not rely on the chance-constrained theorems in the literature as an intermediate step, which enables us to release some strong technical assumptions for ensuring these chance-constrained theorems. Under the same conditions, our bounds also outperform an existing result based on a tight chance-constrained theorem. We illustrate these improvements using a numerical example.

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