FUNDAMENTAL GROUPS IN $\mathbb{E}$-SEMI-ABELIAN CATEGORIES

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Fundamental groups in $\mathbb{E}$-semi-abelian categories

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Dissertation présentée en vue de l'obtention du grade de Docteur en Sciences, le 9 septembre 2013, à l'Institut de Recherche en Mathématique et Physique de l'Université catholique de Louvain, à Louvain-la-Neuve.

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AMS Subject Classification (2010) : 18G50 (Non abelian homological algebra)

Remerciements

Je remercie Marino. Il aura été mon guide dans l'incroyable univers des catégories. A tous points de vue, il a été un promoteur formidable, et surtout, une personne d'une grande humanité. Par son intermédiaire, j'ai rencontré des catégoriciens remarquables: Manuela Sobral, Maria Manuel Clementino, Diana Rodelo, George Janelidze, Dominique Bourn, Tim, Tomas, Julia, Andrea, Alan et bien d'autres encore que j'apprécie énormément. Je remercie aussi tout les membres de mon jury qui, par leur exemple et leurs remarques, m'ont beaucoup appris et permis d'améliorer cette thèse. Je voudrais remercier en particulier Enrico qui, avec Monsieur Félix, a été le premier à m'accueillir au cyclotron, et qui m'a permis de m'y sentir chez moi. Je remercie tous les camarades - actuels ou anciens -, chacun pour ses qualités uniques: Hector, Alex, Valérian, Bao, Olivette, Corentin, Mathieu D.,... . Chacun le sait, pas de recherche sans longues discussions. Merci Alexandre, mon ami, pour ces années passées et nos échanges sur l'épistémologie, les sciences, le monde en général. Je remercie ma famille, qui, malgré un intérêt général à peu près nul pour les mathématiques, m'a encouragé dans cette voie. Enfin, je remercie Cécile dont l'amour et le soutien avant, pendant, après et autour du processus de rédaction de cette thèse me sont vitaux autant que l'air.

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Introduction

The main aim of my thesis is to begin the study of the notion of *fundamental group*, coming from the *categorical Galois theory*, in the framework of $\mathbb{E}$-semi-abelian categories which is sufficiently wide to include several categories of "algebraic" nature such as the semi-abelian categories **[52]**, the integral almost abelian categories **[72]** and the categories of topological semi-abelian algebras **[8]**. The fundamental groups corresponding to many adjunctions in this context are described by generalisations of the Brown-Ellis-Hopf formulae for the integral homology of groups. I shall start from these formulae to explain the genesis of the main ideas in this thesis and how they relate to other ones in our field of study.

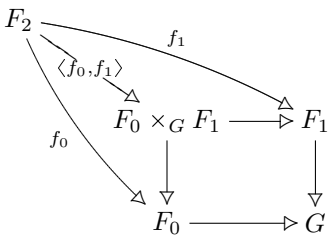
The Brown-Ellis-Hopf formulae. In 1942, H. Hopf **[45]** showed (among other things) that if

$$F \xrightarrow{\ f\ } B$$

is any free presentation of a group B, i.e. f is a surjective homomorphism (an *extension*) and F a free group, then the quotient

$$\frac{[F,F] \cap \mathsf{Ker}\,(f)}{[F, \mathsf{Ker}\,(f)]}. \tag{\textbf{A}}$$

is an invariant of the group B. This led to the introduction of homology (and cohomology) of groups (see **[64]**) and this quotient (**A**) is by now known as the second integral homology group $H_2(B,\mathbb{Z})$ of B. In 1988, R. Brown and G. J. Ellis gave, using topological methods, formulae similar to (**A**) describing the higher (integral) homology objects of a group **[20]**. For instance, given a commutative diagram

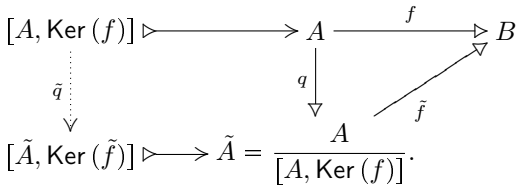


(**B**)

in which all morphisms are surjective homomorphisms (a *double extension*) and F_0, F_1, F_2 are free, they showed that

$$H_3(G,\mathbb{Z}) \cong \frac{[F_2,F_2] \cap \mathsf{Ker}\,(f_0) \cap \mathsf{Ker}\,(f_1)}{[\mathsf{Ker}\,(f_0), \mathsf{Ker}\,(f_1)] \cdot [\mathsf{Ker}\,(f_0) \cap \mathsf{Ker}\,(f_1), F_2]}. \tag{\textbf{C}}$$

A categorical analysis of the Brown-Ellis-Hopf formulae. In 1995, at a conference at Halifax [**49**], G. Janelidze explained that the quotients in the formulae (**A**) and (**C**) were related to *central extensions* of groups and suitably defined *double central extensions* [**48**] of group. Let us explain this. A surjective group homomorphism $f\colon A \to B$ (an extension of B) is classically called a central extension (of B) if its kernel $\mathsf{Ker}\,(f)$ is in the center of its domain A or, equivalently, if the *commutator* $[\mathsf{Ker}\,(f), A]$ vanishes. Every surjective homomorphism $f\colon A \to B$ can be turned into a central extensions in a universal way: its factorisation $\tilde{f}$ through the canonical quotient q of A by $[A, \mathsf{Ker}\,(f)]$ is a central extension



In fact one can prove that the restriction $\tilde{q}$ of q to the commutators is a surjective homomorphism. So the quotient in the Hopf formula (**A**) is precisely what is needed to turn the free presentation f into a central extension, in fact, a *weakly universal central extension* of B (a weak initial object in the category of central extensions of B). Then, the basic reflection $\mathsf{Gp} \to \mathsf{Ab}$ induces another reflection $\mathsf{Ext}(\mathsf{Gp}) \to \mathsf{CExt}(\mathsf{Gp})$ where $\mathsf{Ext}(\mathsf{Gp})$ is the category of extensions and $\mathsf{CExt}(\mathsf{Gp})$ the category of central extensions.

The categorical Galois theory. The central extensions of groups can be studied from the point of view of the categorical Galois Theory of G. Janelidze (see [**47, 9**] and the references therein). A core result of this theory (generalizing the classical one concerning field extensions) is a category equivalence

$$\mathsf{Spl}_\Gamma(E,p) \cong \mathcal{B}^{\mathsf{Gal}(E,p)}$$

stated in the context of a *Galois structure* Γ, that is an adjunction

$$\mathcal{B} \underset{H}{\overset{I}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{A}$$

and two classes of morphisms $\mathcal{E}$ and $\mathcal{Z}$ (in $\mathcal{A}$ and $\mathcal{B}$ respectively) with some suitable stability properties, and for $p\colon E \to B$ a morphism of relative Galois descent with respect to Γ. I shall not enter into the details here but I shall only give the interpretation of this result for the case of the Galois structure

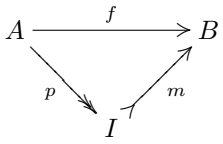
$$\mathsf{Ab} \underset{H}{\overset{\mathsf{ab}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Gp} \qquad (\mathbf{D})$$

with $\mathcal{E}$ (respectively $\mathcal{Z}$) the class of surjective homomorphisms in Gp (resp. Ab) and (E,p) a weakly universal central extension. In this case the category $\mathsf{Spl}_\Gamma(E,p)$ (the extensions "split" by (E,p)) coincides with the category of central extensions of B, and is equivalent to $\mathsf{Ab}^{\mathsf{Gal}(E,p)}$ the category of internal

covariant presheaves in Ab on the Galois groupoid $\mathsf{Gal}(E,p)$ of (E,p). This latter is the image of the kernel pair $\mathsf{Eq}(p)$ of p under the reflector $\mathsf{ab}\colon \mathsf{Gp} \to \mathsf{Ab}$. The next observation is that the reflection $\mathsf{Ext}(\mathsf{Gp}) \to \mathsf{CExt}(\mathsf{Gp})$ induces an associated notion of double central extension. Again, a double extension like (**B**) is central with respect to this new induced adjunction if and only if $[\mathsf{Ker}\,(f_0), \mathsf{Ker}\,(f_1)]\cdot[\mathsf{Ker}\,(f_0)\cap\mathsf{Ker}\,(f_1), F_2]=0$. These principles lead inductively to a chain of Galois structures and to a connection between the Brown-Ellis-Hopf formulae and higher central extensions of groups.

$\mathbb{E}$-semi-abelian categories. Nearly as old as the category theory is the quest for a suitable categorical framework for the study of groups and similar non-abelian algebraic structures. Various authors proposed different lists of axioms, each one allowing a categorical treatment of some aspect of group theory. In 2002, G. Janelidze, L. Marki and W. Tholen published [**52**] in which they defined the particularly simple and elegant notion of semi-abelian category: a category $\mathcal{A}$ is semi-abelian if it is pointed, Barr-exact [**2**], Bourn-protomodular [**11**] and has all binary coproducts. They showed in particular the equivalence of these "new style" axioms with some "old style" axioms, and they provided evidence that their setting was good enough to capture many important features of the category of groups. The main examples of semi-abelian categories are, beside the category of groups Gp, the categories of Lie algebras, commutative algebras, (not necessarily unitary) rings, compact Hausdorff groups, C^*-algebras, Heyting semilattices [**63**], the dual of the category of pointed sets and, of course, any abelian category. For reason which will be clear soon, I have chosen to work in a slightly weaker context than the one of semi-abelian categories where the Barr-exactness condition is relaxed in some way. Let us recall here that a finitely complete category $\mathcal{A}$ is exact if it is regular [**2**], i.e. it has coequalisers of kernel pairs and regular epimorphisms are pullback stable, and if every internal equivalence relation is effective, i.e. the kernel pair of some morphism. These conditions can be rephrased in the following way:

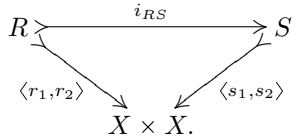
(1) ($\mathcal{A}$ is regular) every morphism $f\colon A \to B$ can be factorised as a regular epimorphism p followed by a monomorphism m



and the regular epimorphisms are pullback-stable. Equivalently, the pair $(\mathsf{RegEpi}(\mathcal{A}), \mathsf{Mono}(\mathcal{A}))$ is a (proper) stable *factorisation system* on $\mathcal{A}$.

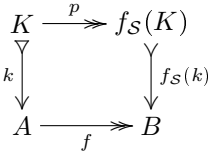
(2) for every internal equivalence relation R, there exists an effective equivalence relation S such that R factorises through S with a morphism

i_{RS} which is in $\mathsf{RegEpi}(\mathcal{A})$ (hence an isomorphism)



(**E**)

If one replaces in the conditions (1)-(2) above the pair $(\mathsf{RegEpi}(\mathcal{A}), \mathsf{Mono}(\mathcal{A}))$ by a pair $(\mathbb{E}, \mathbb{M})$ of classes of morphisms in $\mathcal{A}$, one obtains what I call an $\mathbb{E}$-*exact category* (note that the class $\mathbb{E}$ determines the pair $(\mathbb{E}, \mathbb{M})$). An example of an $\mathbb{E}$-exact category which is not Barr-exact is the category of topological groups $\mathsf{Gp}(\mathsf{Top})$. Nevertheless, it is Epi-exact and also regular. The condition (2) can be interpreted in this case, saying that every internal equivalence in $\mathsf{Gp}(\mathsf{Top})$ is "almost" effective: only the topology on it might prevent it from being effective. In this thesis, I shall work in the context of an $\mathbb{E}$-semi-abelian category: a homological (i.e. a pointed, regular, protomodular) category which is $\mathbb{E}$-exact (for some $\mathbb{E}$) and has all binary coproducts. This allows one to work, for instance, with the category of topological semi-abelian algebras [**8**] and also any integral almost abelian category [**72**]. I must say here that the idea of considering this kind of category has been inspired by [**25, 16**]. In [**25**], M.M. Clementino, D. Dikranjan and W. Tholen work with categories equipped with a proper stable factorisation system $\mathcal{S} = (\mathbb{E}, \mathbb{M})$ and satisfying some axioms relative to this system. One axiom is particularly relevant for the present work. Given a commutative square



where f is in $\mathbb{E}$, k is a normal monomorphism and $f_{\mathcal{S}}(k) \circ p$ is the $\mathcal{S}$-factorisation of $f \circ k$, then $f_{\mathcal{S}}(k)$ is necessarily a normal monomorphism. This axiom, which will be called (INN) in the following, is satisfied by the homological category $\mathsf{Gp}(\mathsf{Top})$ of topological groups, for instance, with $\mathcal{S} = (\mathsf{Epi}, \mathsf{Mono})$. This axiom in the case $(\mathsf{RegEpi}, \mathsf{Mono})$ was known to be equivalent to Barr-exactness in the context of a homological category. More generally, for any $\mathcal{S} = (\mathbb{E}, \mathbb{M})$ the above axiom is equivalent to $\mathbb{E}$-exactness (propositions 1.5.12 and 1.5.9). Another source of inspiration has been the notion of efficiently regular homological category introduced by D. Bourn in [**16**]. The main condition is the following: whenever one has a diagram like (**E**) in which R is an internal equivalence relation, S an effective equivalence relation and i_{RS} an equalizer, then necessarily R is effective. In particular any $\mathbb{E}$-semi-abelian is efficiently regular. Let us just note here that the notion of $\mathbb{E}$-semi-abelian category is not directly related to the notion of (incomplete) relative semi-abelian category [**60, 61**]. In the series of papers [**60, 61, 62, 58**] and in her thesis [**59**], T. Janelidze has developed another notion of relativisation with respect to a class $\mathcal{E}$ of regular epimorphisms. In particular, she shows that the relative versions of the so-called "old"

and "new" axioms for semi-abelian categories are equivalent and that some relative version of homological lemmas and various results on $\mathcal{E}$-relations hold in her context. This approach (which is related to the work of N. Yoneda on quasi-abelian categories [**76**]) is based on the idea that the class of short exact sequences

$$0 \rightarrowtriangle K \xrightarrow{k} A \xrightarrow{f} B \longrightarrow 0$$

with f in $\mathcal{E}$ should behave like the usual short exact sequences in a semi-abelian (homological) category.

A categorical proof of the Brown-Ellis-Hopf formulae. The results presented at the conference [**49**] have not been published immediately. A different approach was introduced by T. Everaert, M. Gran and T. Van der Linden in [**36**], where some generalised Brown-Ellis-Hopf formulae were shown to describe the Barr-Beck cotriple homology when the coefficient functor is a reflector $\mathcal{A} \to \mathcal{B}$ with $\mathcal{A}$ a monadic semi-abelian category and $\mathcal{B}$ a *Birkhoff subcategory* of $\mathcal{A}$ (it was known before that the integral homology of a group corresponds to the cotriple homology with the abelianisation functor as coefficient).

Fundamental group. The original proof by G. Janelidze was published in [**50**], where he proved in the more general context of a pointed Galois structure with abstract commutators that the *Galois group* $\mathsf{Gal}(E, p, 0)$, the group of automorphisms of 0 of the Galois groupoid $\mathsf{Gal}(E, p)$ of any weakly universal normal extensions (E, p) of B, is an invariant of this object and can be described by the formula (**A**) (note that in the case of groups, normal extensions and central extensions coincide). This invariant is called the fundamental group of B by analogy with another Galois structure for which the terminology agrees with the usual one. Indeed, if one considers the Galois structure

$$\mathsf{Set} \underset{H}{\overset{\pi_0}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{LoCo}\,,$$

where LoCo is the category of locally connected topological spaces, π_0 the connected component functor, $\mathcal{E}$ the class of local homeomorphisms and $\mathcal{Z}$ the class of all morphisms in the category Set of sets, then for a "good" topological space admitting a universal covering, the Galois group of this covering turns out to be precisely the fundamental group of the space [**9**]. So, in the semi-abelian context it was rather natural to compare the two independent approaches (both based on a study of higher central extensions) and to see if the definition of invariants through the Galois groups could bring further insights into the subject. One of the goals was to further extend the context in order to obtain new description of the fundamental group in the category of topological groups. Another hope was to be able to describe the fundamental groups corresponding to new reflections where the reflective subcategory was no longer required to be stable under quotients, as in the Birkhoff case.

A description of the fundamental group in terms of commutators and closure operators. A first result in these directions was obtained in [**29**] where we considered composite Galois structures

$$\mathcal{F} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{B} \underset{H}{\overset{I}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{A} \qquad (\mathbf{F})$$

where $\mathcal{A}$ is a semi-abelian category with enough projective objects, I a *Birkhoff reflector* and $\mathcal{F}$ a (regular epi)-reflective subcategory of $\mathcal{B}$ with the property that the reflector $F\colon \mathcal{B} \to \mathcal{F}$ is *protoadditive* [**34**] (i.e. preserves split short exact sequences). For a Galois structure of this kind, we proved that the fundamental group of any object B of $\mathcal{A}$ can be described by

$$\pi_1(B) \cong \frac{\overline{([F,F]_{\mathcal{B}})_F}^{\mathcal{F}} \cap \mathsf{Ker}\,(f)}{\overline{([F,\mathsf{Ker}\,(f)]_{\mathcal{B}})_{\mathsf{Ker}\,(f)}}^{\mathcal{F}}}$$

where (F, f) is a projective presentation of B, $[-,-]_{\mathcal{B}}$ the relative commutator associated with $\mathcal{B}$ [**38, 51, 37**], and $\overline{\cdot}^{\mathcal{F}}$ the homological closure operator associated with $\mathcal{F}$ [**18**]. In fact, my co-authors had already studied some Galois structures as above where the functor F is also a Birkhoff reflector (so that the composite is a Birkhoff reflector and the situation is the same as the one treated in [**36**]). In that cases, the protoadditivity of F allowed them to better explicit the Hopf-formulae [**35**]. In some specific example, topological closures appeared in these formulae (see section 4.6). Subsequently, I made explicit the connection with the homological closure operators.

$\mathcal{E}$-protoadditive reflectors. I extend the results proved in [**29**] to the context of $\mathbb{E}$-semi-abelian categories and define higher fundamental groups in the same way as G. Janelidze did in the special case of groups: $\pi_n(B)$ is the first fundamental group of any "good" $(n-1)$-projective presentation of B. I also give some descriptions of them. The main results are stated for reflectors which preserves pullbacks of the form

$$\begin{array}{ccc} P & \xrightarrow{f'} & E \\ {\scriptstyle p'}\downarrow & \lrcorner & \uparrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array} \qquad s\colon B \to E$$

where f and p are in $\mathcal{E}$, and s a section of p. I call them *$\mathcal{E}$-protoadditive reflectors*. Both the Birkhoff and the protoadditive reflectors have this property. When the coefficient functor factors through a protoadditive functor as in (**F**), it is possible to give better description of the higher fundamental groups in which closures occur (Theorem 3.2.5).

Idempotent $\mathcal{E}$-closure operators. Although the only *closure operator* which appears in our formulae is the one associated with $\mathcal{F}$, some induced closure operators (at the level of categories of higher extensions) are needed in the proof of the Theorem 3.2.5 and lead me to define a suitable notion of closure

operator allowing one to work in categories without all finite limits. At some point, I was glad to realize that a good notion for this purpose had just been introduced by W. Tholen in [**73**]. This allowed me to prove Theorem 1.6.10 which extends the bijection between (regular-epi)-reflective subcategories of a homological category and homological closure operators to a bijection between $\mathcal{E}$-reflective subcategories of a category with a "good" class of extensions $\mathcal{E}$ and idempotent $\mathcal{E}$-closure operators.

The structure of the text.

Chapter 1: The Context. In this chapter, we introduce (almost) all the basic notions and results we shall need in the subsequent chapters. In particular, we recall some basic facts about internal structures, factorisation systems, homological categories, classes of extensions and exact cubes (higher extensions). In the fifth section, we introduce the notion of $\mathbb{E}$-semi-abelian category and give some examples: the semi-abelian categories, the categories of topological semi-abelian algebras, and the integral almost abelian categories are all $\mathbb{E}$-semi-abelian. In the last section, we study idempotent $\mathcal{E}$-closure operators and show that they are in bijection with full replete $\mathcal{E}$-reflective subcategories (in the appropriate framework).

Chapter 2: Derived Galois structures. This chapter is devoted to the study of different types of Galois structures inducing a chain of similar higher order Galois structures. We first recall the basic notion of Galois structure and introduce the notion of quasi-normal extension. After this, we introduce the notion of an $\mathcal{E}$-(pre)radical in a homological category which satisfies (INN) and show that it induces an $\mathcal{E}^1$-preradical (at the level of the category of extensions) which is a radical if and only if the category of quasi-normal extensions (associated with the $\mathcal{E}$-radical) is reflective in the category of extensions. We then study some classes of $\mathcal{E}$-preradicals which ensure that the induced preradicals are also radicals. In the third section, we pay particular attention to a special kind of $\mathcal{E}$-preradical. It is for this particular kind of $\mathcal{E}$-preradical that we are able to give better descriptions of the fundamental groups in chapter 3. All results of this chapter are valid in the context of $\mathbb{E}$-semi-abelian categories.

Chapter 3: Fundamental groups. We begin this chapter by recalling the notion of fundamental group in the context of a Galois structure satisfying some suitable properties. We then prove that if one works in a more specific situation (essentially the one of chapter 2 which gives a tower of induced Galois structures), the fundamental group of any object exists and can be described by the Hopf formula. Using this, we can define the higher fundamental groups, and also provide some descriptions for them. We finally show that for a Galois structure of the kind (**F**), these descriptions can be refined using the $\mathcal{E}$-closure operator introduced in the first chapter.

Chapter 4: Examples. We start this chapter with a recollection of some results on projective objects which are important in order to obtain the existence of weakly universal quasi-normal extensions. We then recall some classical notions, such as the one of variety of Ω-groups and the one of Huq-commutator. Finally, we give some particular instances of our theory, and provide the corresponding descriptions of the fundamental groups obtained in these cases.

CHAPTER 1

The context

In this chapter, we introduce (almost) all the basic notions and results we shall need in the subsequent chapters. In particular, we recall some basic facts about internal structures, factorisation systems, homological categories, classes of extensions and exact cubes (higher extensions). In the fifth section, we introduce the notion of $\mathbb{E}$-semi-abelian category and give some examples: the semi-abelian categories, the categories of topological semi-abelian algebras, and the integral almost abelian categories are all $\mathbb{E}$-semi-abelian. In the last section, we study idempotent $\mathcal{E}$-closure operators and show that they are in bijection with full replete $\mathcal{E}$-reflective subcategories (in the appropriate framework).

1.1. Internal structures

The reader may consult [**4**] for more information on internal structures.

Definition 1.1.1. An *internal relation* from A to B in a category $\mathcal{A}$ is a triple $(R, r_1\colon R \to A, r_2\colon R \to B)$ such that r_1 and r_2 are jointly monic. In the presence of binary products in $\mathcal{A}$, this amounts to saying that the induced morphism $\langle r_1, r_2\rangle\colon R \to A \times B$ is a monomorphism.

A relation $(R, r_1\colon R \to X, r_2\colon R \to X)$ (on X) is *symmetric* if there exists a morphism $\sigma\colon R \to R$ such that $r_1 \circ \sigma = r_2$ and $r_2 \circ \sigma = r_1$. Let $(R \times_X R, p_1, p_2)$ denote the pullback of r_1 along r_2:

$$\begin{array}{ccc} R \times_X R & \xrightarrow{p_2} & R \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle r_1} \\ R & \xrightarrow[r_2]{} & X. \end{array}$$

The relation $(R, r_1\colon R \to X, r_2\colon R \to X)$ is *transitive* if there is a morphism $\tau\colon R \times_X R \to R$ such that $r_1 \circ \tau = r_1 \circ p_1$ and $r_2 \circ \tau = r_2 \circ p_2$. The relation $(R, r_1\colon R \to X, r_2\colon R \to X)$ is *reflexive* if there is a morphism $\delta\colon X \to R$ such that $r_1 \circ \delta = 1_X$ and $r_2 \circ \delta = 1_X$. When $\mathcal{A}$ has binary products, this is equivalent to saying that the diagonal $\langle 1_X, 1_X\rangle\colon X \to X \times X$ factors through $\langle r_1, r_2\rangle\colon R \to X \times X$. An *internal equivalence relation* is a reflexive, symmetric and transitive internal relation. An internal equivalence relation is called *effective* if it is the kernel pair of some morphism. In this case, it is necessarily an internal equivalence relation. For f a morphism in a category, $(\mathsf{Eq}(f), \pi_1^f, \pi_2^f)$, or $(\mathsf{Eq}(f), \pi_1, \pi_2)$ for short, will denote its kernel pair when it exists. For an object A in a category $\mathcal{A}$, $\mathsf{Eq}(A)$ will denote the poset of effective internal equivalence relations on A.

Definition 1.1.2. An *internal category* $\mathbf{X}$ in a category $\mathcal{A}$ with pullbacks is a diagram of the form

$$X_1 \times_{X_0} X_1 \underset{p_2}{\overset{p_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\overset{m}{\longrightarrow}}}}} X_1 \underset{d_2}{\overset{d_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\overset{s}{\longleftarrow}}}}} X_0$$

where $(X_1 \times_{X_0} X_1, p_1, p_2)$ is the pullback of d_1 along d_2

$$\begin{array}{ccc} X_1 \times_{X_0} X_1 & \xrightarrow{p_2} & X_1 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ X_1 & \xrightarrow[d_2]{} & X_0 \end{array}$$

such that the following equalities hold

(1) $d_1 \circ s = 1_{X_0} = d_2 \circ s$;
(2) $m \circ \langle 1_{X_1}, s \circ d_2 \rangle = 1_{X_1} = m \circ \langle s \circ d_1, 1_{X_1} \rangle$;
(3) $d_1 \circ p_1 = d_1 \circ m$;
(4) $d_2 \circ p_2 = d_2 \circ m$;
(5) $m \circ (1_{X_1} \times_{X_0} m) = m \circ (m \times_{X_0} 1_{X_1})$.

Definition 1.1.3. An *internal groupoid* in $\mathcal{A}$ is an internal category in $\mathcal{A}$ equipped with a morphism $\sigma \colon X_1 \to X_1$ such that the following equalities hold

(1) $d_1 \circ \sigma = d_2$;
(2) $d_2 \circ \sigma = d_1$;
(3) $m \circ \langle 1_{X_1}, \sigma \rangle = s \circ d_1$;
(4) $m \circ \langle \sigma, 1_{X_1} \rangle = s \circ d_2$.

Remark 1.1.4. An internal equivalence relation is an internal groupoid.

Definition 1.1.5. An *internal functor* $f \colon \mathbf{X} \to \mathbf{Y}$ is a pair (f_0, f_1) of arrows in $\mathcal{A}$ as in the diagram

$$\begin{array}{ccccc} X_1 \times_{X_0} X_1 & \xrightarrow{m_X} & X_1 & \underset{d_2}{\overset{d_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\overset{s}{\longleftarrow}}}}} & X_0 \\ {\scriptstyle f_2 = f_1 \times_{f_0} f_1}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ Y_1 \times_{Y_0} Y_1 & \xrightarrow{m_Y} & Y_1 & \underset{d_2}{\overset{d_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\overset{s}{\longleftarrow}}}}} & Y_0 \end{array}$$

such that

(1) $d_1 \circ f_1 = f_0 \circ d_1$;
(2) $d_2 \circ f_1 = f_0 \circ d_2$;
(3) $f_1 \circ s = s \circ f_0$;
(4) $f_1 \circ m_X = m_Y \circ f_2$.

Definition 1.1.6. For two given internal functors $f, g \colon \mathbf{X} \to \mathbf{Y}$, an *internal natural transformation* from f to g is a morphism $\alpha \colon X_0 \to Y_1$ in $\mathcal{A}$ such that

(1) $d_1 \circ \alpha = f_0$;
(2) $d_2 \circ \alpha = g_0$;

(3) $m_Y \circ \langle f_1, \alpha \circ d_2 \rangle = m_Y \circ \langle \alpha \circ d_1, g_1 \rangle$

$$\begin{array}{ccc} X_1 & \xrightarrow{\langle f_1, \alpha \circ d_2 \rangle} & Y_1 \times_{Y_0} Y_1 \\ {\scriptstyle \langle \alpha \circ d_1, g_1 \rangle}\downarrow & & \downarrow{\scriptstyle m_Y} \\ Y_1 \times_{Y_0} Y_1 & \xrightarrow[m_Y]{} & Y_1. \end{array}$$

Definition 1.1.7. An internal natural transformation $\alpha \colon f \Rightarrow g$ is called a *homotopy* if it is invertible, i.e. if there exists an internal natural transformation $\beta \colon g \Rightarrow f$ such that

(1) $m_Y \circ \langle \alpha, \beta \rangle = s \circ f_0$;
(2) $m_Y \circ \langle \beta, \alpha \rangle = s \circ g_0$.

Remark 1.1.8. If $\mathbf{Y}$ is an internal relation, so that d_1 and d_2 are jointly monic, then (3) in Definition 1.1.6 is satisfied as soon as (1) and (2) are.

Remark 1.1.9. An internal natural transformation between groupoids is automatically a homotopy.

Proof. $\sigma \circ \alpha$ is the inverse of α. To see this, it suffices to precompose with α in the axioms (3) and (4) for σ. □

Definition 1.1.10. A *strong equivalence* of internal categories $\mathbf{X}$ and $\mathbf{Y}$ is an internal functor $f \colon \mathbf{X} \to \mathbf{Y}$ such that there exists an internal functor $\tilde{f} \colon \mathbf{Y} \to \mathbf{X}$ and homotopies $f \circ \tilde{f} \Rightarrow id_{\mathbf{Y}}$ and $\tilde{f} \circ f \Rightarrow id_{\mathbf{X}}$.

Remark 1.1.11. A strong equivalence $f \colon \mathbf{X} \to \mathbf{Y}$ is always *cartesian*, i.e. the square

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle \langle d_1, d_2 \rangle}\downarrow & & \downarrow{\scriptstyle \langle d_1, d_2 \rangle} \\ X_0 \times X_0 & \xrightarrow[f_0 \times f_0]{} & Y_0 \times Y_0 \end{array}$$

is a pullback. This is the internalisation of the notion of fully faithful functor.

Definition 1.1.12. A morphism $f \colon \mathbf{X} \to \mathbf{Y}$ of internal categories is called a *discrete fibration* if the square

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle d_2} \\ X_0 & \xrightarrow[f_0]{} & Y_0 \end{array}$$

is a pullback.

Remark 1.1.13. If $\mathbf{X}$ and $\mathbf{Y}$ are internal groupoids, $f\colon \mathbf{X} \to \mathbf{Y}$ is a discrete fibration if and only if the square

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ X_0 & \xrightarrow[f_0]{} & Y_0 \end{array}$$

is a pullback.

1.2. Factorisation systems

Let us recall some classical notions and results which can be found for instance in [**22**]. Let $\mathcal{A}$ be a category and p, i two morphisms in $\mathcal{A}$. We shall write $p \downarrow i$ if for every pair of morphisms u, v with $v \circ p = i \circ u$, there exists a unique morphism w such that the following diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{p} & \cdot \\ {\scriptstyle u}\downarrow & \overset{w}{\swarrow} & \downarrow{\scriptstyle v} \\ \cdot & \xrightarrow[i]{} & \cdot \end{array}$$

is commutative. In this case, one says that p and i are *orthogonal*. If $\mathcal{H}$ is a class of morphisms, we also write $\mathcal{H}^{\uparrow} = \{p \mid p \downarrow h \; \forall h \in \mathcal{H}\}$ and $\mathcal{H}^{\downarrow} = \{i \mid h \downarrow i \; \forall h \in \mathcal{H}\}$.

Definition 1.2.1. A *prefactorisation system* $(\mathbb{E}, \mathbb{M})$ on $\mathcal{A}$ is a pair of classes of maps of $\mathcal{A}$ such that $\mathbb{E} = \mathbb{M}^{\uparrow}$ and $\mathbb{M} = \mathbb{E}^{\downarrow}$.

Proposition 1.2.2. *Let $(\mathbb{E}, \mathbb{M})$ be a prefactorisation system. Then*

1. *$\mathbb{M}$ contains the isomorphisms and is closed under composition;*
2. *every pullback of a morphism in $\mathbb{M}$ is in $\mathbb{M}$;*
3. *if $f \circ g$ is in $\mathbb{M}$ so is g, provided that f is either in $\mathbb{M}$ or a monomorphism.*

Proof. (1): Let $i\colon C \to D$ be an isomorphism and let us consider a commutative diagram of the form

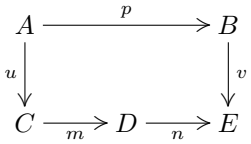
$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow[i]{} & D \end{array}$$

with $p \in \mathbb{E}$. Then $w = i^{-1} \circ v\colon B \to C$ is the unique morphism such that

$$w \circ p = u \quad \text{and} \quad i \circ w = v$$

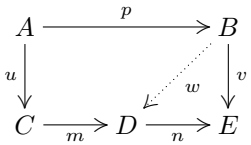
since i is in particular a monomorphism. This shows that $i \in \mathbb{M}$. Now, let $m\colon C \to D$ and $n\colon D \to E$ be two morphisms in $\mathbb{M}$ and let us consider a

commutative diagram of the form



with $p \in \mathbb{E}$. Since $n \in \mathbb{M}$, there exists a unique morphism $w\colon B \to D$ such that

$$w \circ p = m \circ u \quad \text{and} \quad n \circ w = v$$

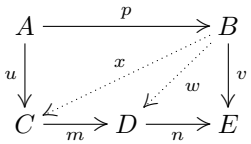


and for a morphism $x\colon B \to C$, one has

$$\left\{ \begin{array}{l} x \circ p = u \\ m \circ x = w \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x \circ p = u \\ n \circ m \circ x = v \end{array} \right. .$$

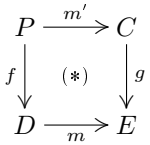
Consequently and because m is also in $\mathbb{M}$, there exists a unique morphism $x\colon B \to C$ such that

$$x \circ p = u \quad \text{and} \quad (n \circ m) \circ x = v.$$

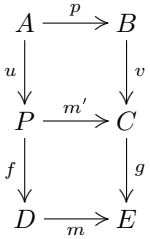


Thus, we have that $n \circ m \in \mathbb{M}$.

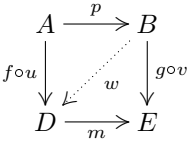
(2): Let us consider a pullback diagram in $\mathcal{A}$



with $m \in \mathbb{M}$ and let us suppose that we have morphisms $p \in \mathbb{E}$, u and v such that the diagram



is commutative. Then there exists a unique morphism $w\colon B \to D$ such that the square

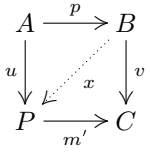


commutes and, since $(*)$ is a pullback, one has for a morphism $x\colon B \to P$

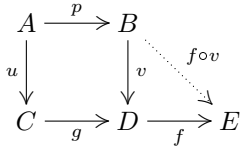
$$\left\{\begin{array}{l} f\circ x = w \\ m'\circ x = v \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} (f\circ x)\circ p = f\circ u \\ m\circ(f\circ x) = g\circ v \\ m'\circ x = v \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} f\circ(x\circ p) = f\circ u \\ m'\circ(x\circ p) = v\circ p \\ m'\circ x = v \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x\circ p = u \\ m'\circ x = v \end{array}\right. .$$

So $\langle w, v\rangle$ is the unique morphism $x\colon B \to P$ such that



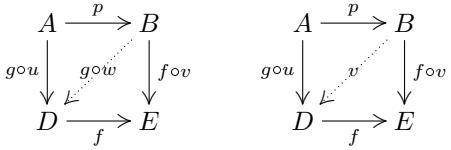
commutes and $m' \in \mathbb{M}$.
(3): Let us assume that we have a commutative diagram



in which $f\circ g \in \mathbb{M}$ and $p \in \mathbb{E}$. Then there exists a unique morphism $w\colon B \to C$ such that

$$w\circ p = u \quad \text{and} \quad (f\circ g)\circ w = f\circ v.$$

If f is a monomorphism, one can easily deduce that g is in $\mathbb{M}$. If f is in $\mathbb{M}$, then $g\circ w = v$ since both diagrams



commute. □

Corollary 1.2.3. *Let $(\mathbb{E}, \mathbb{M})$ be a prefactorisation system. Then*

(1) *$\mathbb{E}$ contains the isomorphisms and is closed under composition;*
(2) *every pushout of a morphism in $\mathbb{E}$ is in $\mathbb{E}$;*
(3) *if $f\circ g$ is in $\mathbb{E}$ so is f, provided that g is either in $\mathbb{E}$ or an epimorphism;*
(4) *$\mathbb{E} \cap \mathbb{M} = \mathsf{Iso}$, the class of isomorphisms.*

PROOF. (1)-(3): follow by duality. (4): Corollary 1.2.3 (1) and Proposition 1.2.2 (1) already give us $\mathsf{Iso} \subseteq \mathbb{E} \cap \mathbb{M}$, so that it only remains to show that $\mathbb{E} \cap \mathbb{M} \subseteq \mathsf{Iso}$. Let $f\colon A \to B$ be in $\mathbb{E} \cap \mathbb{M}$. Then, since one has a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1_A}\| & & \|{\scriptstyle 1_B} \\ A & \xrightarrow[f]{} & B, \end{array}$$

it follows that there exists $w\colon B \to A$ such that

$$w \circ f = 1_A \quad \text{and} \quad f \circ w = 1_B.$$

and that f is an isomorphism. □

Proposition 1.2.4. *When $\mathcal{A}$ is either finitely complete or finitely cocomplete, the following properties of a prefactorisation system $(\mathbb{E}, \mathbb{M})$ are equivalent:*

(1) *every morphism in $\mathbb{E}$ is an epimorphism;*
(2) *if $f \circ g$ is in $\mathbb{M}$ so is g;*
(3) *every equalizer is in $\mathbb{M}$;*
(4) *every split monomorphism is in $\mathbb{M}$.*

PROOF. (1) ⇒ (2): Let us suppose we have a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{p} & B & & \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & \searrow^{f\circ v} & \\ C & \xrightarrow[g]{} & D & \xrightarrow[f]{} & E \end{array}$$

in which $f \circ g$ is in $\mathbb{M}$ and p is an epimorphism in $\mathbb{E}$. Then there exists a unique morphism $w\colon B \to C$ such that

$$w \circ p = u \quad \text{and} \quad (f \circ g) \circ w = f \circ v.$$

This morphism w is in fact such that

$$w \circ p = u \quad \text{and} \quad g \circ w = v$$

since $g \circ w \circ p = g \circ u = v \circ p$ and p is epi.

(1) ⇒ (3): Let $e\colon E \to X$ be the equalizer of a pair of morphisms $f, g\colon X \to Y$ and let us suppose we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ E & \xrightarrow[e]{} & X \end{array}$$

in which p is in $\mathbb{E}$ and, by assumption, an epimorphism. Then

$$f \circ v \circ p = f \circ e \circ u = g \circ e \circ u = g \circ v \circ p$$

imply $f \circ v = g \circ v$. Thus there exists a morphism $w\colon B \to E$ such that $v = e \circ w$ and, since e is in particular a monomorphism, $u = w \circ p$.

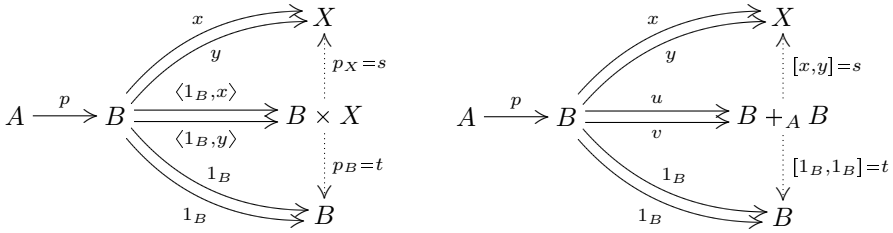
(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4): hold trivially.

(4) $\Rightarrow$ (1): Let $p\colon A \to B$ be in $\mathbb{E}$ and $x, y\colon B \to X$ such that $x \circ p = y \circ p$. If $\mathcal{A}$ has binary products, we set

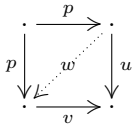
$$u = \langle 1_B, x\rangle, v = \langle 1_B, y\rangle\colon B \to B \times X.$$

If not, $\mathcal{A}$ has pushouts and we choose u, v to be the cokernel-pair of p. In both cases, there exist some t and s such that

$$u \circ p = v \circ p, \quad t \circ u = t \circ v = 1_B, \quad x = s \circ u, \quad y = s \circ v :$$



Since v is a split monomorphism, it is in $\mathbb{M}$ by (4) and we have a commutative diagram



Then $w = t \circ v \circ w = t \circ u = 1_B$ gives that $u = v$, which finally gives that $x = y$. □

By duality, one obtains

Corollary 1.2.5. *When $\mathcal{A}$ is either finitely complete or finitely cocomplete, the following properties of a prefactorisation system $(\mathbb{E}, \mathbb{M})$ are equivalent:*

(1) *every morphism in $\mathbb{M}$ is a monomorphism;*
(2) *if $f \circ g$ is in $\mathbb{E}$ so is f;*
(3) *every coequalizer is in $\mathbb{E}$;*
(4) *every split epimorphism is in $\mathbb{E}$.* □

Definition 1.2.6. A prefactorisation system $(\mathbb{E}, \mathbb{M})$ on a category $\mathcal{A}$ is a *factorisation system* if every map of $\mathcal{A}$ can be factorised as $m \circ e$ with $m \in \mathbb{M}$ and $e \in \mathbb{E}$.

Proposition 1.2.7. *A pair $(\mathbb{E}, \mathbb{M})$ of classes of morphisms in $\mathcal{A}$ is a factorisation system on $\mathcal{A}$ if and only if*

- *$\mathbb{E}$ and $\mathbb{M}$ contain the class of isomorphisms and are closed under composition with isomorphisms;*
- *every map of $\mathcal{A}$ can be factorised as $m \circ e$ with $m \in \mathbb{M}$ and $e \in \mathbb{E}$;*
- *$e \downarrow m$ for all $e \in \mathbb{E}$ and $m \in \mathbb{M}$.* □

Definition 1.2.8. A factorisation system $(\mathbb{E}, \mathbb{M})$ is *proper* if every morphism in $\mathbb{E}$ is an epimorphism and every morphism in $\mathbb{M}$ is a monomorphism.

Definition 1.2.9. A factorisation system $(\mathbb{E}, \mathbb{M})$ is *stable* if $\mathbb{E}$ is stable under pullback, i.e. if whenever one has a pullback square

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ C & \longrightarrow & D \end{array}$$

in which f is in $\mathbb{E}$, then necessarily f' is also in $\mathbb{E}$.

Now, let us suppose that $\mathcal{A}$ is a category with pullbacks and equipped with a factorisation system $\mathcal{S} = (\mathbb{E}, \mathbb{M})$. Then for every morphism $f\colon A \to B$ there is an adjunction

$$\mathbb{M}/A \underset{f_{\mathcal{S}}^{-1}(-)}{\overset{f_{\mathcal{S}}(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbb{M}/B,$$

where $f_{\mathcal{S}}(-)$ is the functor which assigns to a morphism $m\colon M \to A$ of $\mathbb{M}$ its image under f given by the $(\mathbb{E}, \mathbb{M})$-factorisation of $f \circ m$ and $f_{\mathcal{S}}^{-1}(-)$ is the pullback functor which assigns to a morphism $m'\colon M' \to B$ of $\mathbb{M}$ its inverse image under f. $(\mathsf{Im}_{\mathcal{S}}(f), \mathsf{im}_{\mathcal{S}}(f))$ will denote $f_{\mathcal{S}}(1_A)$.

1.3. Homological categories

Let $\mathcal{A}$ be a category equipped with a factorisation system $(\mathbb{E}, \mathbb{M})$.

Definition 1.3.1. $\mathcal{A}$ is $\mathbb{E}$-*regular* if it has finite limits and $(\mathbb{E}, \mathbb{M})$ is proper and stable.

Example 1.3.2. A category $\mathcal{A}$ is $\mathsf{RegEpi}(\mathcal{A})$-regular if and only if it is regular [**1**]. We shall write $\mathcal{S}_r = (\mathsf{RegEpi}(\mathcal{A}), \mathsf{Mono}(\mathcal{A}))$.

Let $\mathsf{Pt}(\mathcal{A})$ be the category of split epimorphisms in a category $\mathcal{A}$ with pullbacks of split epimorphisms along any morphism. $\mathsf{Pt}(\mathcal{A})$ is given by

Objects: pairs $(p\colon A \to C, s\colon C \to A)$ such that $p \circ s = 1_C$;
Arrows: $f\colon (p, s\colon A \leftrightarrows C) \to (p', s'\colon A' \leftrightarrows C')$, pairs $f = (f_1\colon A \to A', f_0\colon C \to C')$ such that

$$p' \circ f_1 = p \circ f_0 \quad \text{and} \quad f_1 \circ s = s' \circ f_0.$$

(with obvious composition and identities). The forgetful functor

$$\pi\colon \mathsf{Pt}(\mathcal{A}) \to \mathcal{A}\colon (p, s\colon A \leftrightarrows C) \to C \quad f = (f_1, f_0) \to f_0$$

is a fibration which is called the *fibration of points*. It has some strong classifying properties [**6**].

Definition 1.3.3. The fibre over an object C, denoted by $\mathsf{Pt}_{\mathcal{A}}(C)$ and called the *category of pointed objects over* C, has

Objects: triples $(A, p\colon A \to C, s\colon C \to A)$ such that $p \circ s = 1_C$;

Arrows: $f\colon (p, s\colon A \leftrightarrows C) \to (p', s'\colon A' \leftrightarrows C)$ morphisms $f\colon A \to A'$ in $\mathcal{A}$ such that

$$p' \circ f = p \quad \text{and} \quad f \circ s = s'.$$

Remark 1.3.4. $\mathsf{Pt}_{\mathcal{A}}(C)$ is in fact the category $((C, 1_C)/(\mathcal{A}/C))$. This explains the chosen name.

Definition 1.3.5. **[11]** A category $\mathcal{A}$ is *(Bourn-)protomodular* if

(1) $\mathcal{A}$ has pullbacks along split epimorphisms;
(2) for every morphism $v\colon B \to C$, the pullback functor

$$v^*\colon \mathsf{Pt}_{\mathcal{A}}(C) \to \mathsf{Pt}_{\mathcal{A}}(B)$$

is conservative, i.e. reflects isomorphisms.

Protomodular categories are such that

Proposition 1.3.6. **[13]** *In any pullback*

$$\begin{array}{ccc} E \times_B A & \xrightarrow{f} & A \\ \downarrow & & \underset{s}{\overset{p}{\downarrow\uparrow}} \\ E & \longrightarrow & B \end{array}$$

where p is split by s, the pair (f, s) is jointly extremally epimorphic. This means that whenever f and s factor through the same monomorphism, this monomorphism is an isomorphism. □

Finitely complete protomodular categories are also Mal'tsev categories.

Definition 1.3.7. **[23]** A category with finite limits is *Mal'tsev* if every reflexive internal relation is an equivalence.

Concerning Mal'tsev categories, one has the following result

Proposition 1.3.8. **[15]** *A regular category $\mathcal{A}$ is Mal'tsev if and only if morphisms in $\mathsf{SplitEpi}(\mathcal{A})$ with f_1, f_0 regular epimorphisms*

$$\begin{array}{ccc} A & \xrightarrow{f_1} & A' \\ \underset{p}{\overset{s}{\downarrow\uparrow}} & & \underset{p'}{\overset{s'}{\uparrow\downarrow}} \\ C & \xrightarrow[f_0]{} & C' \end{array}$$

are in fact regular pushouts, i.e. the comparison morphism $\langle p, f_1\rangle\colon A \to C \times_{C'} A'$ is a regular epimorphism. □

Let us recall that a category is *pointed* if it has a terminal object which is also initial. In this case, we denote this object by 0 and call it the zero object (it is determined up to isomorphism). In this kind of category, the unique morphism $A \to B$ which factors through 0 is called a zero morphism and denoted by 0_{AB} or just by 0. The *kernel* of a morphism f is defined as the equalizer of the pair $(f, 0)$ when it exists. The notion of *cokernel* is defined dually. The kernel of a morphism $f\colon A \to B$ is denoted by $\mathsf{ker}\,(f)\colon \mathsf{Ker}\,(f) \to A$, or just by $(\mathsf{Ker}\,(f), \mathsf{ker}\,(f))$, and its cokernel by $(\mathrm{Coker}(f), \mathsf{coker}\,(f))$ whenever

they exist. A morphism which is the kernel of some morphism is called a *normal monomorphism*. One defines dually the notion of *normal epimorphism*. We shall use the following convention: arrows depicted in diagrams (or somewhere else) by

$$\cdot \rightarrowtail \cdot \qquad \cdot \triangleright\!\!\rightarrow \cdot \qquad \cdot \twoheadrightarrow \cdot \qquad \cdot \longrightarrow\!\!\triangleright \cdot$$

will represent respectively monomorphism, normal monomorphism, epimorphism and normal epimorphism. From Proposition 1.3.6, one can infer the following

Proposition 1.3.9. **[12]** *A pointed protomodular category is* unital*: for every pair of objects A and B, the pair of morphisms*

$$A \xrightarrow{\langle 1,0\rangle} A \times B \xleftarrow{\langle 0,1\rangle} B$$

is jointly extremally epimorphic. □

Definition 1.3.10. **[6]** A category $\mathcal{A}$ is *homological* when it is regular, pointed and protomodular.

It is well known that any regular epimorphism in a homological category $\mathcal{A}$ is a normal epimorphism, thus the cokernel of its kernel **[11]**. We recall here the notion of exact sequence. Let $\mathcal{A}$ be a homological category.

Definition 1.3.11. The category $\mathsf{Ch}(\mathcal{A})$ of *chain complexes* in $\mathcal{A}$ is given by

Objects: a chain complex is a family $C_\bullet = (d_n \colon C_n \to C_{n-1})_{n\in\mathbb{Z}}$

$$\cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow \cdots$$

of morphisms in $\mathcal{A}$ such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$.

Arrows: a morphism of chains complexes $f \colon C_\bullet \to C'_\bullet$ is a family of morphisms $f = (f_n \colon C_n \to C'_n)_{n\in\mathbb{Z}}$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_2 & \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \\
\cdots & \longrightarrow & C'_2 & \xrightarrow{d'_2} & C'_1 & \xrightarrow{d'_1} & C'_0 & \longrightarrow & \cdots
\end{array}$$

such that $f_{n-1} \circ d_n = d'_n \circ f_n$ for all $n \in \mathbb{Z}$.

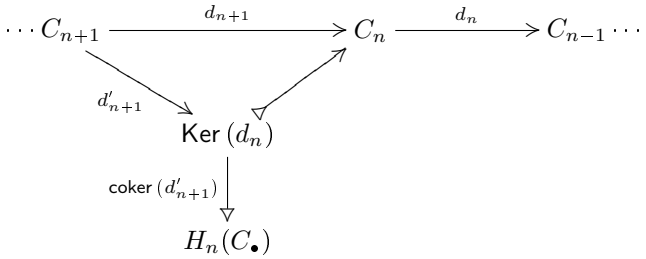
Let $\mathcal{A}$ be a homological category equipped with a proper stable factorisation system $\mathcal{S} = (\mathbb{E}, \mathbb{M})$ (possibly different from $\mathcal{S}_r = (\mathsf{RegEpi}, \mathsf{Mono})$).

Definition 1.3.12. A chain complex $C_\bullet$ is called $\mathcal{S}$-*proper* or $\mathbb{E}$-*proper* when $\mathsf{im}_{\mathcal{S}}(d_n)$ is a normal monomorphism for all $n \in \mathbb{Z}$.

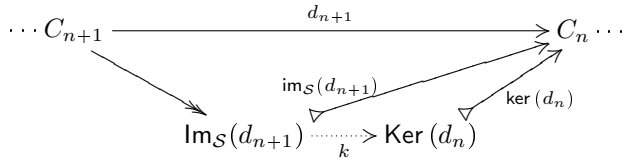
Definition 1.3.13. A chain complex $C_\bullet$ is $\mathcal{S}$-*exac*t or $\mathbb{E}$-*exact* at n when $\mathsf{im}_{\mathcal{S}}(d_{n+1}) = \mathsf{ker}\,(d_n)$.

The nth homology object $H_n(C_\bullet)$ of an $\mathbb{E}$-proper chain complex $C_\bullet$ can be defined as usual. First, one constructs the kernel $\mathsf{ker}\,(d_n)$ of the map d_n and

then the factorisation d'_{n+1} of d_{n+1} through this kernel. The (object part of the) cokernel of d'_{n+1} is $H_n(C_\bullet)$.



Observe that the cokernel of d'_{n+1} exists: this is easily seen using the commutative diagram



in which k is of course a normal monomorphism whose cokernel exists and is precisely $\mathsf{coker}\,(d'_{n+1})$. This also allows us to see that $C_\bullet$ is $\mathbb{E}$-exact at n if and only if $H_n(C_\bullet) = 0$. We shall say that $C_\bullet$ is exact at n if it is $\mathcal{S}_r$-exact.

A sequence depicted by

$$0 \longrightarrow K \overset{k}{\rhd\!\!\longrightarrow} A \overset{f}{\longrightarrow\!\!\rhd} B \longrightarrow 0$$

is called a *short exact sequence*. This means that $k = \mathsf{ker}\,(f)$ and $f = \mathsf{coker}\,(k)$. Then, in a homological category, a short exact sequence is the same thing as a regular epimorphism equipped with its kernel.

Lemma 1.3.14. **[13]** *(Short Five Lemma) For every morphism of short exact sequences in a homological category*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \rhd\!\!\longrightarrow & A & \longrightarrow\!\!\rhd & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & K' & \rhd\!\!\longrightarrow & A' & \longrightarrow\!\!\rhd & B & \longrightarrow & 0 \end{array}$$

with u and w isomorphisms, v is also an isomorphism. □

We also have the following useful result:

Lemma 1.3.15. **[13]** *Let $\mathcal{A}$ be a homological category and consider the following commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \rhd\!\!\longrightarrow & A & \longrightarrow\!\!\rhd & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & (1) & \downarrow{\scriptstyle v} & (2) & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & K' & \rhd\!\!\longrightarrow & A' & \longrightarrow\!\!\rhd & B' & \longrightarrow & 0 \end{array}$$

where both rows are exact.

- *if* (1) *is a pullback then* w *is a monomorphism.*
- *if* u *is an isomorphism then* (2) *is a pullback.* □

Note, that in the previous lemma, the converse implications hold in fact in any pointed category. One also has the

Lemma 1.3.16. *(*3×3*-lemma) Let* $\mathcal{A}$ *be a homological category and*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K'' & \rightarrowtail & X'' & \twoheadrightarrow & Y'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \rightarrowtail & X' & \twoheadrightarrow & Y' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \rightarrowtail & X & \twoheadrightarrow & Y & \longrightarrow & 0
\end{array}$$

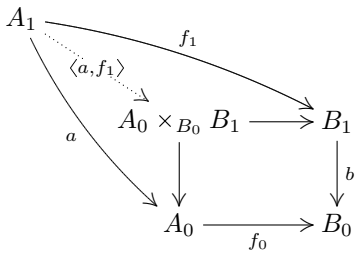
be a commutative diagram in $\mathcal{A}$ *where every row is short exact and the middle column is zero. If two columns are short exact, then the third one is also short exact.* □

1.4. Categories with a class of extensions

Let $\mathcal{E}$ denote a class of arrows in a category $\mathcal{A}$ which has pullbacks along morphisms in $\mathcal{E}$. We shall write $\mathcal{E}^1$ for the class of arrows in $\mathsf{Arr}(\mathcal{A})$ such that an arrow $f = (f_1, f_0)\colon a \to b$ in $\mathsf{Arr}(\mathcal{A})$

$$\begin{array}{ccc}
A_1 & \xrightarrow{f_1} & B_1 \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
A_0 & \xrightarrow[f_0]{} & B_0
\end{array}$$

is in $\mathcal{E}^1$ if and only if all the arrows in the diagram (lying in $\mathcal{A}$)



are in $\mathcal{E}$.

If $\mathcal{E}$ is a class of arrows in a category $\mathcal{A}$, we denote by $\mathcal{A}_\mathcal{E}$ the full subcategory of $\mathcal{A}$ whose objects are the objects of $\mathcal{A}$ which are domains or codomains of elements of $\mathcal{E}$. The class of objects of $\mathcal{A}_\mathcal{E}$ will be denoted $\mathcal{E}^-$. Let $\mathcal{A}$ be a homological category and $\mathcal{E}$ a class of morphisms in $\mathcal{A}$ such that $0 \in \mathcal{E}^-$.

Definition 1.4.1. We shall call $\mathcal{E}$ a *class of extensions* (following [**31**]) if six conditions are fullfilled:

(A1) $\mathcal{E}$ contains all isomorphisms $f\colon A \to B$ such that $A, B \in \mathcal{E}^-$;

(A2) the pullback of any morphism $f\colon A \to B$ in $\mathcal{E}$ along any morphism $g\colon C \to B$ with $C \in \mathcal{E}^-$ exists in $\mathcal{A}$ and is in $\mathcal{E}$, i.e. the pullback

$$\begin{array}{ccc} C \times_B A & \longrightarrow & A \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & B \end{array}$$

exists and f' is in $\mathcal{E}$ whenever $f \in \mathcal{E}$ and $C \in \mathcal{E}^-$;

(A3) $\mathcal{E}$ is closed under composition;

(A4) if the composite $A \to B \xrightarrow{g} C \in \mathcal{E}$ and $B \in \mathcal{E}^-$ then $g \in \mathcal{E}$;

(A5) For every morphism of short exact sequences in $\mathcal{A}$ of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \rightarrowtail & A & \overset{f}{\twoheadrightarrow} & B & \longrightarrow & 0 \\ & & {\scriptstyle k}\downarrow & & {\scriptstyle l}\downarrow & & \| & & \\ 0 & \longrightarrow & K' & \rightarrowtail & A' & \underset{f'}{\twoheadrightarrow} & B & \longrightarrow & 0 \end{array}$$

and such that $f, f', k \in \mathcal{E}$, one has $l \in \mathcal{E}$.

(A6) If $0 \to K \to A \xrightarrow{f} B \to 0$ is a short exact sequence in $\mathcal{A}$ with K and A in $\mathcal{E}^-$ then $f \in \mathcal{E}$.

Remark 1.4.2. Let us notice that, unlike in [**31**], we do not require that $\mathcal{E} \subseteq \mathsf{RegEpi}(\mathcal{A})$. Using (A1) and (A4), one sees that any split epimorphism $f\colon A \to B$ is in $\mathcal{E}$ provided that $A, B \in \mathcal{E}^-$. If $\mathcal{E}^-$ is the class of all objects of $\mathcal{A}$, then it follows from (A6) that $\mathsf{RegEpi}(\mathcal{A}) \subseteq \mathcal{E}$. By (A2), the (object part of the) kernel of any morphism in $\mathcal{E}$ is in $\mathcal{E}^-$ since $0 \in \mathcal{E}^-$.

Lemma 1.4.3. *For every morphism of short exact sequences in a homological $\mathbb{E}$-regular category $\mathcal{A}$ of the form*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \rightarrowtail & A & \overset{f}{\twoheadrightarrow} & B & \longrightarrow & 0 \\ & & {\scriptstyle k}\downarrow & & {\scriptstyle a}\downarrow & & \| & & \\ 0 & \longrightarrow & K' & \rightarrowtail & A' & \underset{f'}{\twoheadrightarrow} & B & \longrightarrow & 0, \end{array}$$

where $k \in \mathbb{E}$, one has $a \in \mathbb{E}$.

Proof. Let $A \xrightarrow{p} I \xrightarrow{m} B$ be the $\mathcal{S}$-factorisation of a. Then the morphism $g = f' \circ m$ is of course a normal epimorphism (one uses Corollary 1.2.5 (2) applied to the factorisation system $\mathcal{S}_r$). This implies that one has a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{Ker}\,(g) & \overset{\ker(g)}{\rightarrowtail} & I & \overset{g}{\twoheadrightarrow} & B & \longrightarrow & 0 \\ & & {\scriptstyle \hat{m}}\downarrow & & {\scriptstyle m}\downarrow & & \| & & \\ 0 & \longrightarrow & K' & \rightarrowtail & A' & \underset{f'}{\twoheadrightarrow} & B & \longrightarrow & 0, \end{array}$$

where the left square is a pullback diagram. Therefore $\hat{m}$ is in $\mathbb{M}$. But k ($\in \mathbb{E}$) factors as $\hat{m} \circ \hat{k}$ with $\hat{k} \in \mathbb{E}$, therefore $\hat{m}$ is in $\mathbb{E} \cap \mathbb{M} = \mathsf{Iso}$. By the Short Five Lemma, m is an isomorphism and then a belongs to $\mathbb{E}$. □

Lemma 1.4.4. *If $\mathcal{A}$ is a homological $\mathbb{E}$-regular category, then $\mathbb{E}$ is a class of extensions.*

Proof. First, let us notice that $\mathcal{E}^-$ is the whole class of objects of $\mathcal{A}$. For (A1)-(A6), one uses the fact that $(\mathbb{E}, \mathbb{M})$ is a proper stable factorisation system, corollaries 1.2.3 and 1.2.5, and Lemma 1.4.3. For instance, (A1) and (A3) simply follow from Corollary 1.2.3 (1). □

Example 1.4.5. If $\mathcal{E}$ and $\mathcal{E}'$ are both classes of extensions in a homological category, then so is their intersection. For $\mathcal{E}$ a class of arrows in a pointed category, we shall write $\underline{\mathcal{E}}$ for $\mathcal{E} \cap \mathsf{NormEpi}(\mathcal{A})$. Hence, if $\mathcal{E}$ is a class of extensions in a homological category, $\underline{\mathcal{E}}$ is also a class of extensions.

Let $\mathcal{E}$ be a class of extensions in a homological category $\mathcal{A}$.

Definition 1.4.6. **[48]** A morphism in $\mathsf{Arr}(\mathcal{A})$ is a *double extension* if it is in $\mathcal{E}^1$.

Lemma 1.4.7. **[31]** *Consider in a homological category $\mathcal{A}$ a commutative diagram with short exact rows, such that f, f', l and m are extensions:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \rightarrowtail & A & \overset{f}{\twoheadrightarrow} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle m} & & \\
0 & \longrightarrow & K' & \rightarrowtail & A' & \underset{f'}{\twoheadrightarrow} & B & \longrightarrow & 0.
\end{array}$$

The right hand square is a double extension if and only if k is an extension.

Proof. We can decompose the above diagram as follows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \rightarrowtail & A & \overset{f}{\twoheadrightarrow} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle k} & \lrcorner & \downarrow{\scriptstyle \langle l,f\rangle} & & \| & & \\
0 & \longrightarrow & K' & \rightarrowtail & A' \times_{B'} B & \overset{p_2}{\twoheadrightarrow} & B & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle p_1} & \lrcorner & \downarrow{\scriptstyle m} & & \\
0 & \longrightarrow & K' & \rightarrowtail & A' & \underset{f'}{\twoheadrightarrow} & B' & \longrightarrow & 0.
\end{array}$$

Now, let us notice that in the above diagram K' is in $\mathcal{E}^-$ by Remark 1.4.2 and that p_2 is an extension by (A2). So, if the morphism $\langle l, f\rangle$ is in $\mathcal{E}$, then so is k by (A2). Conversely, if k is an extension, then $\langle l, f\rangle$ is also an extension by (A5). □

In the sequel, we will use the notation for the finite ordinals: $0 = \varnothing$ and $n = \{0, \dots, n-1\}$ for $n \geqslant 1$. Following **[36]**, we will write $\mathscr{P}(n)$ for the poset of subsets of n viewed as a category: for $S, T \subseteq n$,

$$\mathsf{Hom}_{\mathscr{P}(n)}(S,T) = \begin{cases} \{(S,T)\} \text{ if } S \subseteq T; \\ \varnothing \text{ else.} \end{cases}$$

Definition 1.4.8. The category $\mathsf{Cub}^n\mathcal{A} = \mathcal{A}^{(\mathscr{P}(n)^{\mathsf{op}})}$ is given by

Objects: *n-cubes* in $\mathcal{A}$: functors of type $\mathscr{P}(n)^{\mathsf{op}} \to \mathcal{A}$;
Arrows: natural transformations between them.

We shall use the notation $A_S = A(S)$, $a^T_S = A(S,T)$ (for $S \subseteq T \subseteq n$) and $a_i = a^n_{n\backslash\{i\}} = A(n\backslash\{i\}, n)$. Let us point out that we assume here that

$$\mathsf{Hom}_{\mathscr{P}(n)^{\mathsf{op}}}(T,S) = \mathsf{Hom}_{\mathscr{P}(n)}(S,T)$$

but of course $A(S,T)$ is a morphism from $A(T)$ to $A(S)$. For $\varnothing \neq I \subseteq n$, we shall write $\downarrow I$ for the full subcategory of $\mathscr{P}(n)^{\mathsf{op}}$ whose class of objects is $\{J \subsetneq I\}$ (there is no ambiguity with this notation). For a n-cube A, its restriction to $\downarrow I$ shall be denote by $A_{\downarrow I}$. Let us define for $i \in \mathbb{N}$, $s_i \colon \mathbb{N} \to \mathbb{N}$ by

$$s_i(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geqslant i \end{cases}$$

and $-^i \colon \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ by $S^i = \{s_i(k) \,|\, k \in S\}$. This gives functors for each $n \geqslant 1$ and $0 \leqslant i \leqslant n-1$,

$$-^i \colon \mathscr{P}(n-1)^{\mathsf{op}} \to \mathscr{P}(n)^{\mathsf{op}}$$

and

$$-^i \cup \{i\} \colon \mathscr{P}(n-1)^{\mathsf{op}} \to \mathscr{P}(n)^{\mathsf{op}}$$

connected by a (unique) natural transformation $\alpha_i \colon -^i \cup\{i\} \to -^i$. The functor $-^i$ restricts for every $I \subseteq n-1$ to an isomorphism

$$-^i \colon \downarrow I \to \downarrow I^i.$$

For $n \geqslant 1$ there are different isomorphisms between the categories $\mathsf{Cub}^n\mathcal{A}$ and $\mathsf{ArrCub}^{n-1}\mathcal{A}$ which we shall denote by δ_i $(0 \leqslant i \leqslant n-1)$:

$$\delta_i \colon \mathsf{Cub}^n\mathcal{A} \to \mathsf{ArrCub}^{n-1}\mathcal{A}$$

maps a n-cube A to the arrow $A * \alpha_i = 1_A * \alpha_i$:

$$\mathscr{P}(n-1)^{\mathsf{op}} \underset{-^i}{\overset{-^i \cup \{i\}}{\underset{\Downarrow \alpha_i}{\rightrightarrows}}} \mathscr{P}(n)^{\mathsf{op}} \underset{A}{\overset{A}{\underset{\Downarrow 1_A}{\rightrightarrows}}} \mathcal{A},$$

i.e. the natural transformation

$$(a^{S^i \cup \{i\}}_S)_{S \subseteq n-1} \colon (A_{S^i \cup \{i\}})_{S \subseteq n-1} \to (A_{S^i})_{S \subseteq n-1}.$$

An arrow $f \colon A \to B$ between n-cubes is sent to $(f * -^i, f * -^i \cup \{i\})$:

$$\begin{array}{ccc} (A_{S^i \cup \{i\}})_{S \subseteq n-1} & \xrightarrow{(f_{S^i \cup \{i\}})_{S \subseteq n-1}} & (B_{S^i \cup \{i\}})_{S \subseteq n-1} \\ \downarrow & & \downarrow \\ (A_{S^i})_{S \subseteq n-1} & \xrightarrow[(f_{S^i})_{S \subseteq n-1}]{} & (B_{S^i})_{S \subseteq n-1} \end{array}$$

We shall often use the following notation: $a\colon A_1 \to A_0$ for $\delta_{n-1}A$, $(f_0, f_1)\colon a \to b$ for $\delta_{n-1}(f\colon A \to B)$:

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ {\scriptstyle a}\downarrow & \Rightarrow & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow[f_0]{} & B_0, \end{array}$$

δ_i^+ for $\mathsf{Cod} \circ \delta_i = - \circ -^i$ and δ_i^- for $\mathsf{Dom} \circ \delta_i = - \circ (-^i \cup \{i\})$. Here the functors Cod and Dom are the codomain and domain functors (they send an arrow to its codomain and domain, respectively).

Lemma 1.4.9. **[31]** *If $j < i$, then $s_i \circ s_j = s_j \circ s_{i-1}$.* □

Lemma 1.4.10. *For $n \geqslant 2$, A in $\mathsf{Cub}^n(\mathcal{A})$ and $0 \leqslant j < i \leqslant n-1$, the diagram (in $\mathsf{Cub}^{n-2}\mathcal{A}$)*

$$\begin{array}{ccc} \cdot & \xrightarrow{\delta_{i-1}(\delta_j^-(A))} & \cdot \\ {\scriptstyle \delta_j(\delta_i^-(A))}\downarrow & & \downarrow{\scriptstyle \delta_j(\delta_i^+(A))} \\ \cdot & \xrightarrow[\delta_{i-1}(\delta_j^+(A))]{} & \cdot \end{array}$$

is commutative.

Proof. Using the previous lemma several times, one finds for $S \subseteq n-2$

$$\begin{aligned} &\big(\delta_j(\delta_i^+(A)) \circ \delta_{i-1}(\delta_j^-(A))\big)_S \\ =&A((S^j)^i, (S^j \cup \{j\})^i) \circ A((S^{i-1})^j \cup \{j\}, (S^{i-1} \cup \{i-1\})^j \cup \{j\}) \\ =&A((S^j)^i, (S^{i-1} \cup \{i-1\})^j \cup \{j\}) \\ =&A((S^j)^i, (S^{i-1})^j \cup \{i\} \cup \{j\}) \\ =&A((S^{i-1})^j, (S^j)^i \cup \{j\} \cup \{i\}) \\ =&A((S^{i-1})^j, (S^j \cup \{j\})^i \cup \{i\}) \\ =&A((S^{i-1})^j, (S^{i-1} \cup \{i-1\})^j) \circ A((S^j)^i \cup \{i\}, (S^j \cup \{j\})^i \cup \{i\}) \\ =&\big(\delta_{i-1}(\delta_j^+(A)) \circ \delta_j(\delta_i^-(A))\big)_S. \end{aligned}$$

□

Lemma 1.4.11. **[33]** *Let $\mathcal{E}$ be a class of extensions in a homological category $\mathcal{A}$. Then, every split epimorphism $f = (f_0, f_1)\colon a \to b$ of split epimorphisms in $\mathcal{E}$, i.e. every diagram in $\mathcal{A}_\mathcal{E}$*

$$\begin{array}{ccc} A_1 & \underset{\overline{f_1}}{\overset{f_1}{\rightleftarrows}} & B_1 \\ {\scriptstyle a}\downarrow\uparrow{\scriptstyle \overline{a}} & & {\scriptstyle \overline{b}}\downarrow\uparrow{\scriptstyle b} \\ A_0 & \underset{f_0}{\overset{\overline{f_0}}{\rightleftarrows}} & B_0 \end{array}$$

such that $f_0 \circ a = b \circ f_1$, $\overline{f_0} \circ b = a \circ \overline{f_1}$, $\overline{b} \circ f_0 = f_1 \circ \overline{a}$, $\overline{a} \circ \overline{f_0} = \overline{f_1} \circ \overline{b}$ and $f_0 \circ \overline{f_0} = 1_{B_0}$, $f_1 \circ \overline{f_1} = 1_{B_1}$, $a \circ \overline{a} = 1_{A_0}$, $b \circ \overline{b} = 1_{B_0}$, is a double extension.

Proof. One uses Lemma 1.4.7 and Remark 1.4.2. □

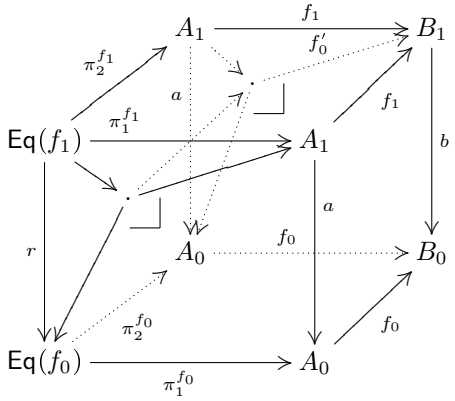
The following lemma is an adaptation of Lemma 3.2 in [**33**]

Lemma 1.4.12. *Let $\mathcal{E}$ be a class of extensions in a homological category $\mathcal{A}$. Consider a diagram*

$$\begin{array}{ccccc}
\mathsf{Eq}(f_1) & \underset{\pi_2^{f_1}}{\overset{\pi_1^{f_1}}{\rightrightarrows}} & A_1 & \xrightarrow{f_1} & B_1 \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\
\mathsf{Eq}(f_0) & \underset{\pi_2^{f_0}}{\overset{\pi_1^{f_0}}{\rightrightarrows}} & A_0 & \xrightarrow[f_0]{} & B_0
\end{array}$$

in $\mathcal{A}$ with a, b, f_0 and f_1 in $\mathcal{E}$. If either of the left commutative squares is in $\mathcal{E}^1$, then the right one is in $\mathcal{E}^1$.

Proof. Let us suppose that the square $a \circ \pi_1^{f_1} = \pi_1^{f_0} \circ r$ is in $\mathcal{E}^1$. Then, in the commutative cube



the commutative square

$$\begin{array}{ccc}
\mathsf{Eq}(f_1) & \xrightarrow{\pi_2^{f_1}} & A_1 \\
{\scriptstyle \langle r, \pi_1^{f_1} \rangle}\downarrow & & \downarrow{\scriptstyle \langle a, f_1 \rangle} \\
\mathsf{Eq}(f_0) \times_{A_0} A_1 & \xrightarrow[\pi_2^{f_0} \times_{f_0} f_1]{} & A_0 \times_{B_0} B_1
\end{array}$$

is such that

- $\langle r, \pi_1^{f_1} \rangle$ is in $\mathcal{E}$ by assumption;
- $\pi_2^{f_1}$ is in $\mathcal{E}$ since it is the pullback of f_1 in $\mathcal{E}$ along f_1 and (A2) holds;
- $\pi_2^{f_0} \times_{f_0} f_1$ is in $\mathcal{E}$ since it is the pullback of f_1 in $\mathcal{E}$ along f_0'.

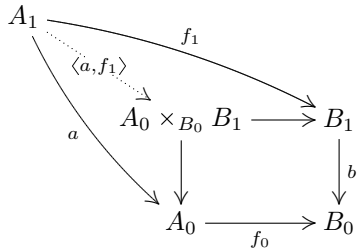
Then by axioms (A3) and (A4), the morphism $\langle a, f_1 \rangle$ is in $\mathcal{E}$ and $f = (f_0, f_1)$ is a double extension. □

It has been proved in [**31**] that for $\mathcal{E}$ a class of regular epimorphisms which is a class of extensions in a homological category $\mathcal{A}$, $\mathcal{E}^1$ is also a class of extensions in $\mathsf{Arr}\mathcal{A}$. The same thing can be proved more generally for any class of extensions $\mathcal{E}$:

Proposition 1.4.13. *If $\mathcal{E}$ is a class of extensions in a homological category $\mathcal{A}$, then $\mathcal{E}^1$ is a class of extensions in the homological category $\mathsf{Arr}\mathcal{A}$.*

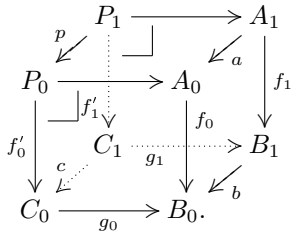
Proof. First, let us notice that $0_{\mathsf{Arr}\mathcal{A}} = 1_0 \in \mathcal{E} = (\mathcal{E}^1)^-$.

(A1): An isomorphism $(f_0, f_1)\colon a \to b$ in $\mathsf{Arr}\mathcal{A}$ between arrows a and b in $\mathcal{E}$

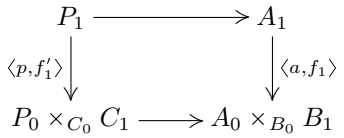


is of course in $\mathcal{E}^1$. (A1) for $\mathcal{E}$ gives f_1 and f_0 in $\mathcal{E}$ and then, (A2) and (A3) gives $\langle a, f_1\rangle$ in $\mathcal{E}$ since it is the composite of isomorphisms in $\mathcal{E}$.

(A2): Let us take the pullback of an $\mathcal{E}^1$-extension $(f_0, f_1)\colon a \to b$ along an arrow $(g_0, g_1)\colon c \to b$ (with c in $(\mathcal{E}^1)^- = \mathcal{E}$) in $\mathsf{Arr}\mathcal{A}$

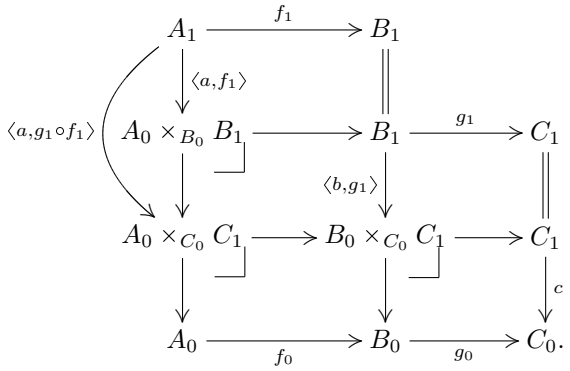


We have to prove that $(f_0', f_1')\colon p \to c$ is in $\mathcal{E}^1$. So, let us consider the following commutative diagram



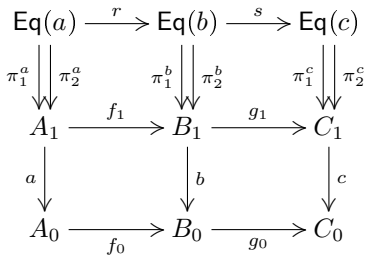
which turns out to be a pullback. Since $f_0' \in \mathcal{E}$ and $C_1 \in \mathcal{E}^-$, using (A2) one finds that $P_0 \times_{C_0} C_1 \in \mathcal{E}^-$. Now, by definition, $\langle a, f_1\rangle$ is in $\mathcal{E}$, so that by axiom (A2) for $\mathcal{E}$, $\langle p, f_1'\rangle$ is in $\mathcal{E}$. It follows from axiom (A3) and (A2) for $\mathcal{E}$ that (f_0', f_1') is in $\mathcal{E}^1$.

(A3): Let us suppose that $(f_0, f_1)\colon a \to b$ and $(g_0, g_1)\colon b \to c$ are in $\mathcal{E}^1$ and consider the following commutative diagram



Then, by axiom (A2) for $\mathcal{E}$, all "straight" arrows in this diagram are in $\mathcal{E}$. One concludes by axiom (A3) for $\mathcal{E}$ that $(g_0 \circ f_0, g_1 \circ f_1)$ is in $\mathcal{E}^1$.

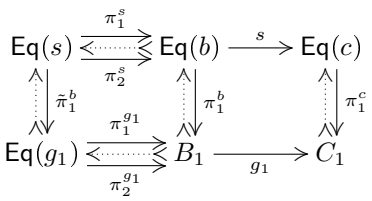
(A4): Let $f = (f_0, f_1)\colon a \to b$ and $g = (g_0, g_1)\colon b \to c$ be two morphisms in $\mathsf{Arr}(\mathcal{A})$ such that $g \circ f$ is in $\mathcal{E}^1$ and b in $\mathcal{E}$. We can produce a diagram



in which

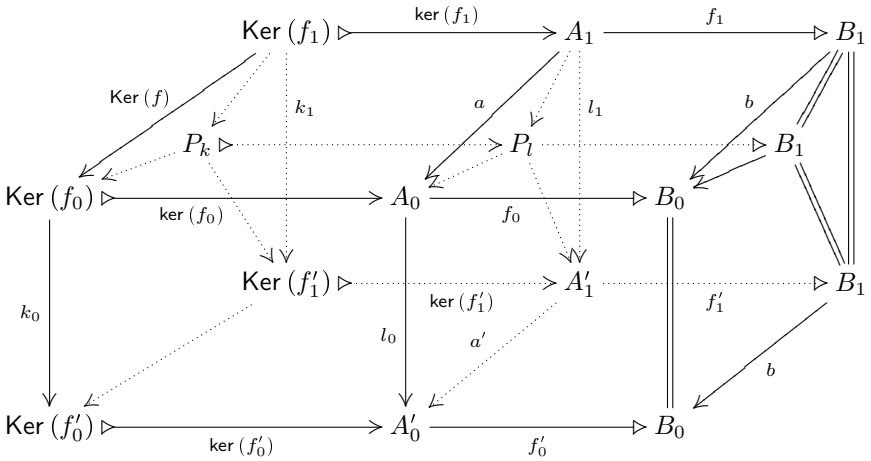
- b is in $\mathcal{E}$ by assumption;
- a, c are in $(\mathcal{E}^1)^- = \mathcal{E}$;
- g_1, g_0 are in $\mathcal{E}$ since $g_1 \circ f_1$ and $g_0 \circ f_0$ are in $\mathcal{E}$ and $(A4)$ holds for $\mathcal{E}$;
- $s \circ r$ and thus s are in $\mathcal{E}$ by (A2) for $\mathcal{E}^1$ (proved above).

So if the morphism (g_1, s) is in $\mathcal{E}^1$, g is in $\mathcal{E}^1$ by Lemma 1.4.12. But this follows, once again, from Lemma 1.4.12 and from the fact that in the diagram



the morphism $(\pi_1^s, \pi_1^{g_1})$ is a double extension by Lemma 1.4.11.

(A5): One starts from a commutative diagram



where $P_k = \mathsf{Ker}\,(f_0) \times_{\mathsf{Ker}\,(f'_0)} \mathsf{Ker}\,(f'_1)$, $P_l = A_0 \times_{A'_0} A'_1$ and $k = (k_0, k_1), f = (f_0, f_1), f' = (f'_0, f'_1)$ are all in $\mathcal{E}^1$. Then using (A2), (A4) and (A5) for $\mathcal{E}$, one finds that $l = (l_0, l_1) \in \mathcal{E}^1$.

(A6): Let $f = (f_0, f_1)\colon a \to b$ be a regular epimorphism in $\mathsf{Arr}\mathcal{A}$ with $a \in \mathcal{E}$ equipped with its kernel

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{Ker}\,(f_1) & \overset{\ker(f_1)}{\rhd\!\!\longrightarrow} & A_1 & \overset{f_1}{\longrightarrow\!\!\rhd} & B_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\ 0 & \longrightarrow & \mathsf{Ker}\,(f_0) & \underset{\ker(f_0)}{\rhd\!\!\longrightarrow} & A_0 & \underset{f_0}{\longrightarrow\!\!\rhd} & B_0 & \longrightarrow & 0. \end{array}$$

If $k \in \mathcal{E}$, then $\mathsf{Ker}\,(f_1)$ and $\mathsf{Ker}\,(f_0)$ are in $\mathcal{E}^-$ and by (A6) for $\mathcal{E}$, f_1 and f_0 are in $\underline{\mathcal{E}}$ and by (A3) and (A4), $b \in \mathcal{E}$. One concludes using Lemma 1.4.7. □

The previous Proposition 1.4.13 allows us to define a class $\mathcal{E}^n$ of extensions in $\mathsf{Cub}^n(\mathcal{A})$ by induction as follows: $\mathcal{E}^0 = \mathcal{E}$ and for $n \geqslant 1$, $f\colon A \to B \in \mathcal{E}^n$ if and only if $\delta_{n-1}f\colon \delta_{n-1}(A) \to \delta_{n-1}(B)$ is in $(\mathcal{E}^{n-1})^1$ (see Corollary 1.13 in [**31**]).

Remark 1.4.14. If $\mathcal{F} \subseteq \mathcal{E}$ are classes of extensions, then $\mathcal{F}^n \subseteq \mathcal{E}^n$.

Now, let $\mathcal{E}$ be a class of morphisms in a category $\mathcal{A}$.

Definition 1.4.15. **[28]** Let $n \geqslant 1$ and A be a n-cube in $\mathcal{A}$. Then A is *$\mathcal{E}$-exact* if for all $\varnothing \neq I \subseteq n$ the limit $\lim A_{\downarrow I}$ exists and the comparison morphism $A_I \to \lim A_{\downarrow I}$ is in $\mathcal{E}$.

Of course, an $\mathcal{E}$-exact 1-cube is just an extension and, for $n \geqslant 2$, a n-cube A is $\mathcal{E}$-exact if and only if for all $0 \leqslant k \leqslant n-1$, $\delta_k^+(A)$ is $\mathcal{E}$-exact and $A_n \to \lim A_{\downarrow n}$ is in $\mathcal{E}$ (for $0 \leqslant k \leqslant n-1$ and $\varnothing \neq I \leqslant n-1$, $A_{I^k} \to \lim A_{\downarrow I^k} = \delta_k^+(A)_I \to \lim(\delta_k^+(A))_{\downarrow I}$). By convention, we shall denote by $\mathsf{ExCub}^0_{\mathcal{E}}(\mathcal{A})$ the category $\mathcal{A}_{\mathcal{E}}$. With this convention, one has $(\mathsf{Cub}^0(\mathcal{A}))_{\mathcal{E}^0} = \mathcal{A}_{\mathcal{E}} = \mathsf{ExCub}^0_{\mathcal{E}}(\mathcal{A})$.

Definition 1.4.16. The full subcategory $\mathsf{ExCub}^n_{\mathcal{E}}(\mathcal{A})$ of $\mathsf{Cub}^n(\mathcal{A})$ is given by

Objects: $\mathcal{E}$-exact n-cubes in $\mathcal{A}$;
Arrows: natural transformations between them.

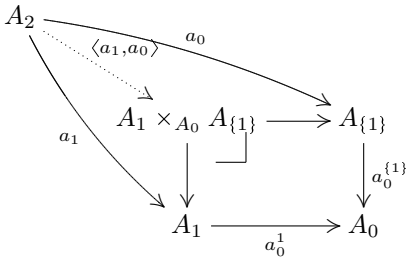
One has $(\mathsf{Cub}^1(\mathcal{A}))_{\mathcal{E}^1} = \mathsf{ExCub}^1_{\mathcal{E}}(\mathcal{A})$.

Theorem 1.4.17. *Let $n \geqslant 2$ and A be a n-cube in $\mathcal{A}$. For any $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant n-2$, the following are equivalent*

(1) *A is $\mathcal{E}$-exact;*
(2) *$\delta_i(A) \in \mathcal{E}^{n-1}$;*
(3) *$\delta_j\delta_i(A) \in (\mathcal{E}^{n-2})^1$.*

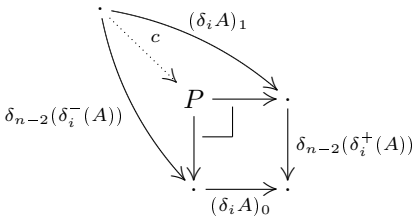
Thus, one has $(\mathsf{Cub}^n(\mathcal{A}))_{\mathcal{E}^n} = \mathsf{ExCub}^n_{\mathcal{E}}(\mathcal{A})$ for all $n \in \mathbb{N}$.

Proof. This is an adaptation of a proof in [**33**]. The result is proved by induction. Let us notice that if (1)-(3) are equivalent for some n, this implies that $(\mathsf{Cub}^n(\mathcal{A}))_{\mathcal{E}^n} = \mathsf{ExCub}^n_{\mathcal{E}}(\mathcal{A})$. In fact, when (1)-(3) are equivalent, it is easy to show that a n-cube A is $\mathcal{E}$-exact if and only if 1_A is in $\mathcal{E}^n$. Now for $n = 2$, (1)-(3) are equivalent since a 2-cube A is $\mathcal{E}$-exact if and only if in the commutative diagram



the arrows a_0^1, $a_0^{\{1\}}$ and $\langle a_1, a_0\rangle$ are in $\mathcal{E}$ if and only if every arrow in this diagram is in $\mathcal{E}$. Let $n \geqslant 3$ be such that (1)-(3) are equivalent for $2 \leqslant l \leqslant n-1$.

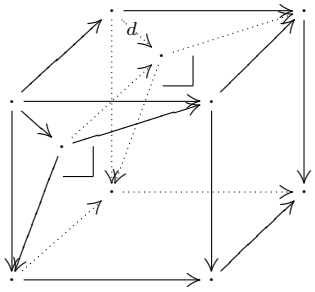
(1) $\Rightarrow$ (2): By definition, $\delta_i(A) \in \mathcal{E}^{n-1}$ if and only if $\delta_{n-2}(\delta_i(A)) \in (\mathcal{E}^{n-2})^1$, i.e. that every arrow in the following commutative diagram



(1.1)

is in $\mathcal{E}^{n-2}$. Since A is $\mathcal{E}$-exact, $\delta_k^+(A)$ is $\mathcal{E}$-exact for every $0 \leqslant k \leqslant n-1$. So by assumption, the bottom and right arrows in this diagram are in $\mathcal{E}^{n-2}$ and $P = \mathsf{Cod}(c)$, in $(\mathcal{E}^{n-2})^-$ by (A2), is $\mathcal{E}$-exact. It remains to show that c is in $\mathcal{E}^{n-2}$. For this, we can check that $C = \delta_{n-2}^{-1}(c)$ is exact. We already know that $\delta_{n-2}^+(C)$ is $\mathcal{E}$-exact. For $0 \leqslant k \leqslant n-3$, let us apply the isomorphism δ_k to the

above diagram, so that we have a cube



in which $d = \delta_{n-3}(\delta_k^+(C))$ is in $\mathcal{E}^{n-2}$ since the back square is a double $\mathcal{E}^{n-2}$-extension. Then, by assumption, $\delta_k^+(C)$ is $\mathcal{E}$-exact. It remains to show that the comparison morphism $C_{n-1} \to \lim C_{\downarrow n-1}$ is in $\mathcal{E}$. For this, one shows that it coincides with the comparison morphism $A_n \to \lim A_{\downarrow n}$.

(2) $\Rightarrow$ (1): If $\delta_i(A) \in \mathcal{E}$, one has a diagram like (1.1) in which all arrows are in $\mathcal{E}^{n-2}$. In particular $\delta_{n-2}(\delta_i^+(A))$ and c are in $\mathcal{E}^{n-2}$, so that $\delta_i^+(A)$ and C (as above) are $\mathcal{E}$-exact. If one applies, once again the functor δ_k to the diagram (1.1), one finds a cube in which the back face has all its arrows in $\mathcal{E}^{n-3}$ so that $\delta_k^+(A)$ is $\mathcal{E}$-exact by assumption. The comparison morphism $A_n \to \lim A_{\downarrow n}$, which corresponds to $C_{n-1} \to \lim C_{\downarrow n-1}$, is in $\mathcal{E}$ since C is $\mathcal{E}$-exact.

(2) $\Leftrightarrow$ (3) can be proved using (1) $\Leftrightarrow$ (2) and the same kind of principle as above. □

Remark 1.4.18. We shall sometimes speak of *exact cubes* if it is clear from the context which is the relative class of extensions.

Let $n \geqslant 1$ and A be a n-cube in a category $\mathcal{A}$. For every $0 \leqslant i \leqslant n-1$, one can define a $(n-1)$-cube $\rho_i A$ in $\mathsf{Arr}\mathcal{A}$ in the following way

$$(a_{S^i}^{S^i \cup \{i\}} \colon A_{S^i \cup \{i\}} \to A_{S^i})_{S \subseteq n-1}.$$

These associations extend to functors $\rho_i \colon \mathsf{Cub}^n\mathcal{A} \to \mathsf{Cub}^{n-1}(\mathsf{Arr}\mathcal{A})$ which are isomorphisms. Let us denote by (ρ_i, ρ_i), the functor of type $\mathsf{ArrCub}^n\mathcal{A} \to \mathsf{ArrCub}^{n-1}\mathsf{Arr}\mathcal{A}$ which sends an object $f \colon A \to B$ in $\mathsf{ArrCub}^n\mathcal{A}$ to $\rho_i f \colon \rho_i A \to \rho_i B$. One has the following useful results

Lemma 1.4.19. **[31]** *Let $n \geqslant 1$ and $0 \leqslant i, j \leqslant n-1$. If $i < j$, then $\delta_{j-1} \circ \rho_i = (\rho_i, \rho_i) \circ \delta_j$. If $i > j$, then $\delta_j \circ \rho_i = (\rho_{i-1}, \rho_{i-1}) \circ \delta_j$.*

Proof. This result is rather long to prove. We shall only prove here that $\delta_{j-1} \circ \rho_i$ and $(\rho_i, \rho_i) \circ \delta_j$ are the same on objects when $i < j$. Let A be a n-cube in $\mathcal{A}$. Then $\delta_{j-1}(\rho_i A)$ is the natural transformation whose component at $S \subseteq n-2$ is

$$\left(A((S^{j-1})^i \cup \{i\}, (S^{j-1} \cup \{j\})^i \cup \{i\})),\, A((S^{j-1})^i), (S^{j-1} \cup \{j-1\})^i\right)$$

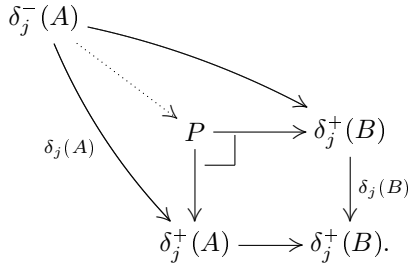
and $(\rho_i, \rho_i)(\delta_j A)$ has component

$$\left(A((S^i \cup \{i\})^j, (S^i \cup \{i\})^j \cup \{j\})\,,\, A((S^i)^j), (S^i)^j \cup \{j\}\right).$$

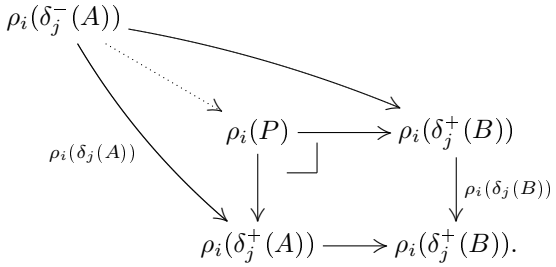
One concludes using Lemma 1.4.9. □

Lemma 1.4.20. **[31]** *Let $n \geqslant 1$. For $0 \leqslant i \leqslant n-1$, $\rho_i(\mathcal{E}^n) = (\mathcal{E}^1)^{n-1}$.*

Proof. By induction. For $n = 1$, the result holds since $\rho_0 = 1_{\mathsf{Cub}^n(\mathcal{A})}$ and $\mathcal{E}^n = (\mathcal{E}^1)^0 = (\mathcal{E}^1)^{n-1}$. Let $n > 1$ be such that the result holds for $0 \leqslant k < n$, $0 \leqslant i < n-1$ (the remaining case $i = n-1$ can be proved similarly) and an arrow $f\colon A \to B$ in $\mathsf{Cub}^n(\mathcal{A})$. If f is in $\mathcal{E}^n$, then using Theorem 1.4.17, one sees that $\delta_j(f)$ is in $(\mathcal{E}^{n-1})^1$ and all morphisms in the commutative diagram



are in $\mathcal{E}^{n-1}$. Applying the functor (ρ_i, ρ_i) to this diagram one finds a commutative diagram



in which, by assumption, all morphisms are in $(\mathcal{E}^1)^{n-2}$. But this square (in $((\mathcal{E}^1)^{n-2})^1$) , is the image by δ_{j-1} (by Lemma 1.4.19) of the morphism $\rho_i(f)\colon \rho_i(A) \to \rho_i(B)$ which is then, once again by Theorem 1.4.17, in $(\mathcal{E}^1)^{n-1}$. We have just prove that $\rho_i(\mathcal{E}^n) \subseteq (\mathcal{E}^1)^{n-1}$. The converse inclusion is proved in a similar way. □

Theorem 1.4.21. **[31]** *Let $n \geqslant 1$. For any $0 \leqslant i \leqslant n-1$ and A a n-cube in $\mathcal{A}$, the following are equivalent*

1. *A is an $\mathcal{E}$-exact n-cube;*
2. *$\rho_i A$ is an $\mathcal{E}^1$-exact $(n-1)$-cube.* □

Definition 1.4.22. **[69]** Let $n \geqslant 1$. The *direction* of a n-cube A is

$$\partial A = \bigcap_{i=0}^{n-1} \mathsf{Ker}\,(a_i).$$

Lemma 1.4.23. *Let $n \geqslant 1$. For A a n-cube in a homological category $\mathcal{A}$, $\mathcal{E}$ a class of extensions in $\mathcal{A}$ and $0 \leqslant i \leqslant n-1$, one has $\partial A \cong \mathsf{Ker}\,(\partial(\rho_i A))$. Thus, if A is $\mathcal{E}$-exact, $\partial A \in \mathcal{E}^-$.*

PROOF. The first assertion is proved in the following way

$$\begin{array}{rcl}
\mathsf{Ker}\,(\partial\rho_i A) & = & \mathsf{Ker}\bigcap_{j=0}^{n-2}\mathsf{Ker}\,((\rho_i A)^{n-1}_{n-1\setminus\{j\}}) \\
& = & \bigcap_{j=0}^{n-2}\mathsf{Ker}\,\mathsf{Ker}\,((\rho_i A)^{n-1}_{n-1\setminus\{j\}}) \\
& = & \bigcap_{j=0,j\neq i}^{n-1}\mathsf{Ker}\,(a_i)\cap\mathsf{Ker}\,(a_j) \\
& = & \mathsf{Ker}\,(a_i)\cap\bigcap_{j=0,j\neq i}^{n-1}\mathsf{Ker}\,(a_j) \\
& = & \partial A.
\end{array}$$

The second one is proved by induction. Remark 1.4.2 shows that for $n = 1$, the assertion holds. Now, let $n \geqslant 2$ and let us suppose that the assertion holds for any $1 \leqslant k < n$. For P an $\mathcal{E}$-exact n-cube, $\rho_0(P)$ is an $\mathcal{E}^1$-exact $(n-1)$-cube and then by hypothesis $\partial(\rho_0 P) \in (\mathcal{E}^1)^- = \mathcal{E}$. Then the first assertion and, once again, Remark 1.4.2 give us $\partial P = \mathsf{Ker}\,(\partial(\rho_0 P)) \in \mathcal{E}^-$. □

Sometimes, like in **[33]**, we shall only consider couples of the form $(\mathcal{A}, \mathcal{E})$ where $\mathcal{A}$ is a category and $\mathcal{E}$ a class of morphisms of $\mathcal{A}$ satisfying some conditions:

(E1) $\mathcal{E}$ contains all isomorphisms in $\mathcal{A}$;
(E2) pullbacks of morphisms in $\mathcal{E}$ exist in $\mathcal{A}$ and are in $\mathcal{E}$;
(E3) $\mathcal{E}$ is closed under composition;
(E4) if $g \circ f \in \mathcal{E}$ then $g \in \mathcal{E}$;
(E5) the $\mathcal{E}$-Mal'tsev axiom: any split epimorphism in $\mathsf{Arr}\mathcal{A}$

$$\begin{array}{ccc}
A_1 & \underset{}{\overset{f_1}{\rightleftarrows}} & B_1 \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
A_0 & \underset{f_0}{\rightleftarrows} & B_0
\end{array}$$

where a and b are in $\mathcal{E}$, is in $\mathcal{E}^1$, i.e. is a *double $\mathcal{E}$-extension.*

(E6) (when $\mathcal{A}$ is pointed) kernels of morphisms in $\mathcal{E}$ have cokernels in $\mathcal{E}$.

Remark 1.4.24. In presence of (E1)-(E4), every split epimorphism is in $\mathcal{E}$. The Proposition 1.3.8 explains the choice of the name for the axiom (E5).

Let $\mathsf{Ext}_{\mathcal{E}}\mathcal{A}$ denote the full subcategory of $\mathsf{Arr}\mathcal{A}$ whose objects are elements of $\mathcal{E}$, then $(\mathsf{Ext}_{\mathcal{E}}\mathcal{A}, \mathcal{E}^1)$ satisfies (E1)-(E5) (see **[33]**) and by induction $(\mathsf{ExCub}^n_{\mathcal{E}}\mathcal{A}, \mathcal{E}^n)$. The proof for (E1)-(E3) has been given above (see 1.4.13). For the proof of (E4)-(E5), one uses the following useful lemma

Lemma 1.4.25. **[33]** *Let $(\mathcal{A}, \mathcal{E})$ be such that* (E1)-(E4) *hold. The following are equivalent*

(1) $(\mathsf{Ext}_{\mathcal{E}}\mathcal{A}, \mathcal{E}^1)$ *satisfies* (E4)*;*
(2) $(\mathcal{A}, \mathcal{E})$ *satisfies* (E5)*;*
(3) *every split epimorphism of split epimorphisms, i.e. every diagram*

$$\begin{array}{ccc}
A_1 & \underset{\overline{f_1}}{\overset{f_1}{\rightleftarrows}} & B_1 \\
{\scriptstyle a}\downarrow\uparrow{\scriptstyle \overline{a}} & & {\scriptstyle \overline{b}}\downarrow\uparrow{\scriptstyle b} \\
A_0 & \underset{f_0}{\overset{\overline{f_0}}{\rightleftarrows}} & B_0
\end{array} \tag{1.2}$$

such that $f_0 \circ a = b \circ f_1$, $\overline{f_0} \circ b = a \circ \overline{f_1}$, $\overline{b} \circ f_0 = f_1 \circ \overline{a}$, $\overline{a} \circ \overline{f_0} = \overline{f_1} \circ \overline{b}$ *and* $f_0 \circ \overline{f_0} = 1_{B_0}$, $f_1 \circ \overline{f_1} = 1_{B_1}$, $a \circ \overline{a} = 1_{A_0}$, $b \circ \overline{b} = 1_{B_0}$, *is a double extension;*

(4) *In a commutative diagram*

$$\begin{array}{ccccc} \mathsf{Eq}(f_1) & \underset{\pi_2^{f_1}}{\overset{\pi_1^{f_1}}{\rightrightarrows}} & A_1 & \xrightarrow{f_1} & B_1 \\ {\scriptstyle r}\downarrow & & {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ \mathsf{Eq}(f_0) & \underset{\pi_2^{f_0}}{\overset{\pi_1^{f_0}}{\rightrightarrows}} & A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

where a, b, f_1 *and* f_0 *are in* $\mathcal{E}$, *the arrow* r *is in* $\mathcal{E}$ *if and only if the right hand side square is in* $\mathcal{E}^1$. □

Proposition 1.4.26. *If* $\mathcal{E}$ *is a class of extensions in a homological category* $\mathcal{A}$, *then the pair* $(\mathcal{A}_\mathcal{E}, \mathcal{E})$ *satisfies the axioms* (E1)-(E6).

Proof. Axioms (E1)-(E4) hold for obvious reasons. For (E6), one begins with an exact sequence (in $\mathcal{A}$):

$$0 \longrightarrow K \overset{k}{\rhd\!\!\longrightarrow} A \overset{f\in\mathbb{E}}{\twoheadrightarrow} B$$

and takes the S_r-factorisation of f. The regular epi part of this factorisation is in $\mathbb{E}$ by (A6) and it is a cokernel of its kernel k. For (E5), one uses Lemma 1.4.11 and Lemma 1.4.25. □

Proposition 1.4.27. *If* $\mathcal{E}$ *is a class of extensions in a homological category* $\mathcal{A}$, *then* $\mathcal{E} \cap \mathsf{NormEpi}(\mathcal{A}) = \mathcal{E} \cap \mathsf{NormEpi}(\mathcal{A}_\mathcal{E})$, *so that* $\underline{\mathcal{E}}$ *will denote either of these classes with no ambiguity.*

Proof. Let us consider a short exact sequence

$$0 \longrightarrow K \xrightarrow{k} A \xrightarrow{f} B \longrightarrow 0$$

in $\mathcal{A}$ with $f \in \mathcal{E}$. Then K, A and B are objects of $\mathcal{E}^-$ and the sequence is also exact in $\mathcal{A}_\mathcal{E}$. Conversely, if we start with $f\colon A \to B$ a morphism in $\mathcal{E} \cap \mathrm{Coker}(\mathcal{A}_\mathcal{E})$, then it is in fact the cokernel (in $\mathcal{A}_\mathcal{E}$) of its kernel (K, k) in $\mathcal{A}$. Then taking the cokernel (C, c) of k in $\mathcal{A}$, one finds, using (A6), that C is in $\mathcal{E}^-$ and, finally, that $B \cong C$. Thus f is of course a cokernel in $\mathcal{A}$. □

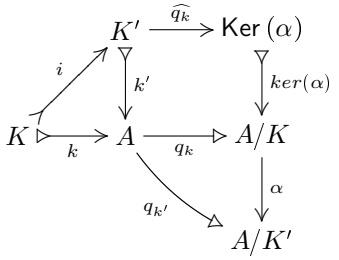
When $(\mathcal{A}, \mathcal{E})$ is such that $\mathcal{A}$ is pointed and $\mathcal{E}$ satisfies (E2), one defines $\mathcal{M}(\mathcal{E})$ as the class of kernels of morphisms in $\mathcal{E}$. This class has some stability properties if $\mathcal{E}$ satisfies (E1)-(E4)+(E6):

Lemma 1.4.28. $\mathcal{M}(\mathcal{E})$ *contains all isomorphisms in* $\mathcal{A}$, *is closed under composition with isomorphisms and satisfies the following implication*

$$k = k' \circ i \in \mathcal{M}(\mathcal{E}), k' \in \mathcal{M}(\mathcal{E}) \Rightarrow i \in \mathcal{M}(\mathcal{E}).$$

Proof. An isomorphism $i\colon A \to A'$ is the kernel of $0\colon A' \to 0$ which is in $\mathcal{E}$. If $k\colon K \to A$ is the kernel of $f\colon A \to B \in \mathcal{E}$, and $i : A \to A'$ and $j\colon K' \to K$ are isomorphisms, then $i \circ k$ is the kernel of $f \circ i^{-1}$ which is an extension and $k \circ j$ is the kernel of f. If $k = k' \circ i\colon K \to K' \to A$ and $k, k' \in \mathcal{M}(\mathcal{E})$, then one

can see that i is the kernel of the morphism $\widehat{q_k}$ in the following commutative diagram



where α, the factorisation of the cokernel $q_{k'}$ of k' through the cokernel q_k of k, is an extension and then $\widehat{q_k}$ is an extension since the square is a pullback. □

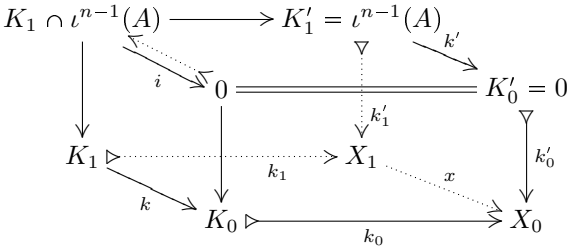
If $(\mathcal{A}, \mathcal{E})$ is such that $\mathcal{A}$ is pointed and $\mathcal{E}$ satisfies (E1)-(E5) then the pair $(\mathsf{ExCub}^n_{\mathcal{E}}(\mathcal{A}), \mathcal{E}^n)$ satisfies (E1)-(E5) and is also pointed. Let $\iota^n \colon \mathcal{A} \to \mathsf{ExCub}^n_{\mathcal{E}}(\mathcal{A})$ be the functor which sends an object A in $\mathcal{A}$ to the n-cube $\iota^n A$ defined by $(\iota^n A)_n = A$ and $(\iota^n A)_S = 0$ for $S \subsetneq n$. This functor has a left adjoint $\mathsf{Dom}^n \colon \mathsf{ExCub}^n_{\mathcal{E}}(\mathcal{A}) \to \mathcal{A}$. Let $\mathcal{O}$ be a subclass of the class of objects of $\mathcal{A}$. We shall sometimes need more limits than the ones already available in $(\mathcal{A}, \mathcal{E})$ and shall require that the triple $(\mathcal{A}, \mathcal{E}, \mathcal{O})$ satisfies

(O1) for all $K \to X$, $K' \to X$ in $\mathcal{M}(\mathcal{E})$ such that $K' \in \mathcal{O}$, one can construct the intersection $K \cap K'$, and $K \cap K'$ is in $\mathcal{O}$.

Remark 1.4.29. (O1) implies that $\mathcal{O}$ is closed under kernels of extensions: $K \to A$ in $\mathcal{M}(\mathcal{E})$ and $A \in \mathcal{O}$ give $K \in \mathcal{O}$. If we write $\mathcal{O}^n = \iota^n \mathcal{O}$, one has $\delta_i \mathcal{O}^n = (\mathcal{O}^{n-1})^1$ for any $0 \leqslant i \leqslant n-1$.

Lemma 1.4.30. *If* $(\mathcal{A}, \mathcal{E}, \mathcal{O})$ *satisfies* (O1)*, then* $(\mathsf{ExCub}^n_{\mathcal{E}}\mathcal{A}, \mathcal{E}^n, \mathcal{O}^n)$ *satisfies also* (O1)*.*

Proof. By induction. Let $n \geqslant 1$, $K \to X$, $K' \to X$ be in $\mathcal{M}(\mathcal{E})$ with K' of the form $\iota^n A$ for some A in $\mathcal{A}$. We can see that the result holds using the following diagram in $\mathsf{ExCub}^{n-1}_{\mathcal{E}}\mathcal{A}$



i is of course in $\mathcal{E}^{n-1}(\mathcal{A})$ (as a split epimorphism in $\mathsf{ExCub}^{n-1}_{\mathcal{E}}\mathcal{A}$) and $\delta^{-1}_{n-1} i$ is the intersection $K \cap K'$ in $\mathcal{O}^n$. □

Remark 1.4.31. An object of the form $\iota^n(A)$ is in fact in $\mathsf{ExCub}^n_{\mathcal{F}}\mathcal{A}$ for every class $\mathcal{F}$ of extensions in $\mathcal{A}$ and $A \in \mathcal{F}^-$.

For some reasons which shall appear later, we have to extend the concept of a class of extensions in a homological category. For $\mathcal{E}$ a class of morphisms in

a homological category $\mathcal{A}$ and $\mathcal{O}$ a subclass of the class of objects of $\mathcal{A}$, such that $\mathcal{O} \subseteq \mathcal{E}^-$

Definition 1.4.32. $\mathcal{E}$ is a class of extensions in $(\mathcal{A}, \mathcal{O})$ if it satisfies axioms (A1)-(A5) (see Definition 1.4.1) and the axiom

(A6') If $0 \to K \to A \xrightarrow{f} B \to 0$ is a short exact sequence in $\mathcal{A}$, $A \in \mathcal{E}^-$ and $K \in \mathcal{O}$, then B is in $\mathcal{E}^-$.

A class $\mathcal{E}$ of extensions in a homological category $\mathcal{A}$ in the sense of Definition 1.4.1 is a class of extensions in $(\mathcal{A}, \mathcal{E}^-)$.

Lemma 1.4.33. *If $\mathcal{E}$ is a class of extensions in $(\mathcal{A}, \mathcal{O})$, then $\mathcal{E}^1$ is also a class of extensions in $(\mathsf{Arr}(\mathcal{A}), \mathcal{O}^1)$.*

Proof. Reading carefully the proof of Proposition 1.4.13, one sees that we have just to prove axiom (A6') for $\mathcal{E}^1$. So let us consider a short exact sequence

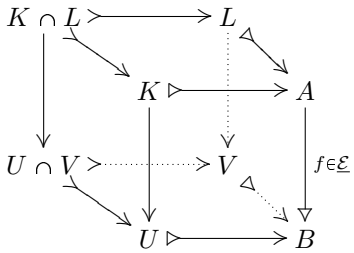
$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \overset{k}{\rhd\!\!\longrightarrow} & A_1 & \overset{f}{\longrightarrow\!\!\rhd} & B_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\ 0 & \longrightarrow & 0 & \rhd\!\!\longrightarrow & A_0 & \xrightarrow{\cong} & B_0 & \longrightarrow & 0. \end{array}$$

in $\mathsf{Arr}(\mathcal{A})$ with $a \in (\mathcal{E}^1)^- = \mathcal{E}$ and K in $\mathcal{O}$. Since K is in $\mathcal{O}$, B_1 is in $\mathcal{E}^-$ by (A6') for $\mathcal{E}$. Thus the axiom (A4) for $\mathcal{E}$ gives $b \in \mathcal{E} = (\mathcal{E}^1)^-$. □

Corollary 1.4.34. *For $\mathcal{E}$ a class of extensions in $(\mathcal{A}, \mathcal{O})$ and $n \geqslant 1$, $\mathcal{E}^n$ is a class of extensions in $(\mathsf{Cub}^n(\mathcal{A}), \mathcal{O}^n)$.* □

The following lemma is a slightly modified version of Lemma 5.2 in [**50**].

Lemma 1.4.35. *Let $(\mathcal{A}, \mathcal{E})$ be such that $\mathcal{A}$ is pointed and $\underline{\mathcal{E}}$ satisfies* (E2)*. Let us suppose that we have a commutative cube*

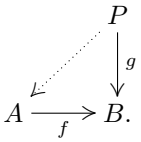


where $f \in \underline{\mathcal{E}}$, $K = f^{-1}(U)$ and $L = f^{-1}(V)$. Then

$$U \cap V \cong \frac{K \cap L}{\mathsf{Ker}\,(f)}.$$

□

Projective objects. An object P of a category $\mathcal{A}$ with a class $\mathcal{E}$ of morphisms is projective with respect to $\mathcal{E}$ if any arrow $g\colon P \to B$ admits a lifting along a morphism $f\colon A \to B$ in $\mathcal{E}$:



This can be rephrased, saying that the functor

$$\mathsf{Hom}_{\mathcal{A}}(P, f)\colon \mathsf{Hom}_{\mathcal{A}}(P, A) \to \mathsf{Hom}_{\mathcal{A}}(P, B)$$

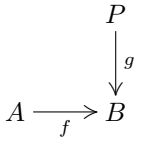
is surjective whenever $f\colon A \to B$ is in $\mathcal{E}$. A category has enough projectives with respect to $\mathcal{E}$ if every object A of this category can be "covered" with a morphism in $\mathcal{E}$ with a domain projective w.r.t. $\mathcal{E}$:

$$P \xrightarrow{f \in \mathcal{E}} A$$

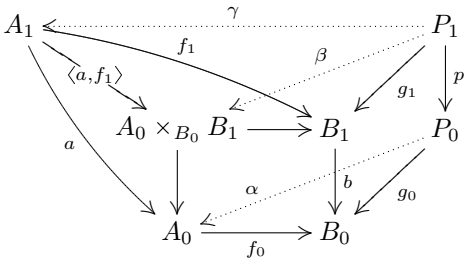
Let $\mathcal{A}_{\mathcal{E}}$ be a category with enough $\mathcal{E}$-projective objects and $A \in \mathcal{E}^{-}$. For $n \geqslant 0$, one calls an $\mathcal{E}$-exact n-cube P in A, an *$\mathcal{E}$-projective n-fold presentation* of A if $P_0 = A$ and P_S is projective, w.r.t. $\mathcal{E}$, for every $0 \neq S \subseteq n$. Let $\mathcal{A}$ be a homological category and $\mathcal{E}$ a class of extensions in $(\mathcal{A}, \mathcal{O})$ for some $\mathcal{O} \subseteq \mathcal{E}^{-}$.

Lemma 1.4.36. **[36]** *Let $n \in \mathbb{N}$. The objects of $\mathsf{Cub}^n\mathcal{A}$ that are projective with respect to the class of extensions $\mathcal{E}^n$ are precisely those n-cubes P such that P_S is $\mathcal{E}$-projective for every $S \subseteq n$.*

Proof. By induction. It is true for $n = 0$. Let $n \geqslant 1$ be such that the result is true for $n - 1$. Let us start with the situation



where P is such that P_S is projective for every $S \subseteq n$ and f is in $\mathcal{E}^n$. Then applying the isomorphism δ_{n-1}, one constructs the following commutative diagram



where α exists because P_0 is projective with respect to $\mathcal{E}^{n-1}$ and f_0 is in $\mathcal{E}^{n-1}$, β is induced by the universal property of the pullback $A_0 \times_{B_0} B_1$, and γ exists because P_1 is projective (w.r.t. $\mathcal{E}^{n-1}$) and $\langle a, f_1\rangle$ is in $\mathcal{E}^{n-1}$. The morphism $\tilde{g} = \delta_{n-1}^{-1}(\gamma, \alpha)\colon P \to A$ is such that $g = f \circ \tilde{g}$.

Conversely, let P be a n-cube which is projective with respect to $\mathcal{E}^n$ and $P_1 \xrightarrow{p} P_0$ its image by δ_{n-1}. Then one can show that P_1 and P_0 are projective with respect to $\mathcal{E}^{n-1}$ and infer from this that P_S is projective w.r.t. $\mathcal{E}$ for all $S \subseteq n$. If one starts with a diagram

$$\begin{array}{ccc} & & P_0 \\ & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f\in\mathcal{E}^{n-1}]{} & B, \end{array}$$

one can construct a commutative diagram

$$\begin{array}{ccccc} & & & & P_1 \\ & & & \swarrow^{g\circ p} & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B & & P_0 \\ \| & & \| & \swarrow^{g} & \\ A & \xrightarrow[f]{} & B & & \end{array}$$

since p is, of course, projective w.r.t $(\mathcal{E}^{n-1})^1$. Similarly, one can prove that P_1 is projective w.r.t. $\mathcal{E}^{n-1}$. A diagram like

$$\begin{array}{ccc} & & P_1 \\ & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f\in\mathcal{E}^{n-1}]{} & B, \end{array}$$

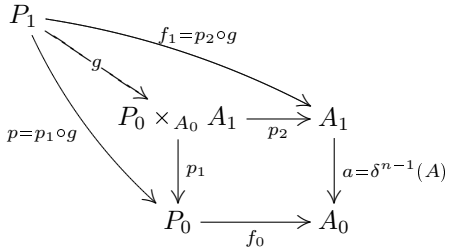
is completed in the following way

$$\begin{array}{ccccc} & & & & P_1 \\ & & & \swarrow^{g} & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B & & P_0. \\ \downarrow & & \downarrow & \swarrow^{0} & \\ 0 & = & 0 & & \end{array}$$

□

Lemma 1.4.37. **[31]** *If $\mathcal{A}_{\mathcal{E}}$ has enough $\mathcal{E}$-projective objects, $\mathsf{ExCub}^n_{\mathcal{E}}\mathcal{A}$ has enough $\mathcal{E}^n$-projective objects.*

PROOF. By induction. Let $n \geqslant 1$ be such that the property holds for $n-1$ and let A be in $\mathsf{ExCub}^n_{\mathcal{E}}\mathcal{A}$. Let us consider the following commutative diagram



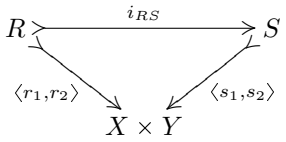
where $f_0 \colon P_0 \to A_0$ is any $\mathcal{E}$-projective $(n-1)$ presentation of A_0 and $g \colon P_1 \to P_0 \times_{A_0} A_1$ any $\mathcal{E}$-projective $(n-1)$ presentation of $P_0 \times_{A_0} A_1$. Then $f = \delta_{n-1}(f_1, f_0) \colon P \to A$ is an $\mathcal{E}$-projective n-fold presentation of A. □

Lemma 1.4.38. **[31]** *Every object $\mathcal{A} \in \mathcal{E}^-$ has at least one n-fold presentation provided that $\mathcal{A}_{\mathcal{E}}$ has enough projective objects w.r.t. $\mathcal{E}$.*

PROOF. By induction. Let $n \geqslant 1$ such that for $n-1$ the property holds and $A \in \mathcal{E}^-$. Then one has a $(n-1)$-fold presentation of A, let us say $P_0 \in \mathsf{ExCub}^{n-1}_{\mathcal{E}}\mathcal{A}$. Since $\mathsf{ExCub}^{n-1}_{\mathcal{E}}\mathcal{A}$ has enough $\mathcal{E}^{n-1}$-projective objects, one can cover P_0 by $p \colon P_1 \to P_0$ with P_1 projective and p in $\mathcal{E}^{n-1}$. $P = \delta_{n-1}^{-1}(p)$ is of course an $\mathcal{E}$-projective n-fold presentation of A. □

1.5. 𝔼-semi-abelian categories

In this section, we introduce a context suitable to develop the whole theory, namely the one of 𝔼-semi-abelian categories. We also give the main examples of this kind of categories. Let (R, r_1, r_2) and (S, s_1, s_2) be two internal relations from X to Y in $\mathcal{A}$ with binary products and $\mathcal{E}$ a class of morphisms in $\mathcal{A}$. We write $R \leqslant_{\mathcal{E}} S$ if $R \leqslant S$ and the (unique) factorisation of R through S is in $\mathcal{E}$, i.e. when we have a commutative diagram



where i_{RS} is in $\mathcal{E}$. Let $\mathcal{A}$ be a category equipped with a factorisation system $(\mathbb{E}, \mathbb{M})$.

Definition 1.5.1. $\mathcal{A}$ is $\mathbb{E}$-*exact* if it is $\mathbb{E}$-regular and for every internal equivalence relation R there exists an effective equivalence relation S such that $R \leqslant_{\mathbb{E}} S$.

Example 1.5.2. A category $\mathcal{A}$ is RegEpi-exact if and only if it is Barr-exact [**1**]. Let us recall here that a category is Barr-exact if it is regular (RegEpi-regular) and every internal equivalence relation in this category is effective. Since an equivalence relation R is effective precisely when there exists an effective equivalence relation S for which the morphism i_{RS} above is an isomorphism , the

result follows (a monomorphism which is a regular epimorphism is an isomorphism).

Let us recall the important

Definition 1.5.3. **[52]** A category is *semi-abelian* if it is

- pointed;
- (Barr-)exact;
- protomodular;
- with binary coproducts.

This is the notion we are going to extend. Let $\mathcal{A}$ be a category equipped with a factorisation system $(\mathbb{E}, \mathbb{M})$.

Definition 1.5.4. $\mathcal{A}$ is $\mathbb{E}$-*semi-abelian* if it is homological (pointed, regular and protomodular), $\mathbb{E}$-exact and has binary coproducts.

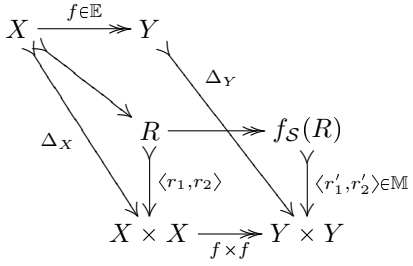
Of course, any semi-abelian category is a RegEpi-semi-abelian category. Before giving further examples, we shall prove some results concerning $\mathbb{E}$-semi-abelian categories.

Lemma 1.5.5. *Let $\mathcal{A}$ be a category which is $\mathbb{E}$-semi-abelian for some factorisation system $\mathcal{S} = (\mathbb{E}, \mathbb{M})$ on it and (R, r_1, r_2) be a reflexive relation on X in $\mathcal{A}$. Then $\langle r_1, r_2 \rangle \colon R \to X \times X \in \mathbb{M}$ if and only if R is an effective equivalence relation.*

Proof. $\mathcal{A}$ being a Mal'tsev category, R is an equivalence relation. Therefore, there exists an effective equivalence relation S such that $R \leqslant_{\mathbb{E}} S$. If $\langle r_1, r_2 \rangle$ is in $\mathbb{M}$, $i_{RS} \in \mathbb{E} \cap \mathbb{M} = \mathsf{Iso}$ (by Proposition 1.2.2 (3)) and R is effective. Conversely, if R is an effective equivalence relation, then $\langle r_1, r_2 \rangle$ factors as $\langle r'_1, r'_2 \rangle \circ p$ where $\langle r'_1, r'_2 \rangle$ is in $\mathbb{M}$ and is an equivalence relation which is effective by the above and has the same coequalizer as R. Then p is an isomorphism and $\langle r_1, r_2 \rangle$ is in $\mathbb{M}$. □

Lemma 1.5.6. *Let $\mathcal{A}$ be a category which is $\mathbb{E}$-semi-abelian for some factorisation system $\mathcal{S} = (\mathbb{E}, \mathbb{M})$ on it. The $\mathcal{S}$-images of reflexive relations along morphisms in $\mathbb{E}$ are effective equivalence relations.*

Proof. Let (R, r_1, r_2) be a reflexive relation on X and $f \colon X \to Y$ be in $\mathbb{E}$. Using the commutative diagram



one easily sees that there exists a morphism $\alpha \colon Y \to f_{\mathcal{S}}(R)$ such that

$$\langle r'_1, r'_2 \rangle \circ \alpha = \Delta_Y$$

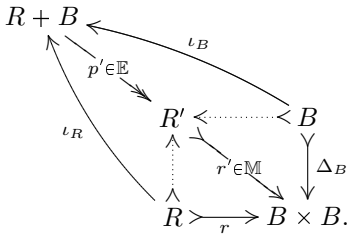
since f and $\langle r'_1, r'_2 \rangle$ are orthogonal. Then $f_{\mathcal{S}}(R)$ is reflexive and, by Lemma 1.5.5, an effective equivalence relation. □

Proposition 1.5.7. *An $\mathbb{E}$-semi-abelian category has all finite colimits.*

Proof. This is a straightforward adaptation of the proof of Proposition 3.10 in [**5**] . One has to prove that all coequalizers exist. Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

be a pair of parallel morphisms and $A \xrightarrow{p} R \xrightarrow{r} B \times B$ be the $(\mathbb{E}, \mathbb{M})$-factorisation of the induced morphism $\langle f, g \rangle \colon A \to B \times B$. Let us consider the commutative diagram



in which $r' \circ p'$ is the $\mathcal{S}$-factorisation of $[r, \Delta_B] \colon R + B \to B \times B$. Then (R', r') is of course an effective equivalence relation (by Lemma 1.5.5) whose coequalizer is the coequalizer of (f, g). □

Let $\mathcal{A}$ be a pointed category. One says that $\mathcal{A}$ satisfies the property (INN) (for the image of a normal monomorphism is a normal monomorphism) with respect to some proper factorisation system $\mathcal{S} = (\mathbb{E}, \mathbb{M})$ on $\mathcal{A}$ if

Property 1.5.8. (INN) *Given a commutative square in $\mathcal{A}$*

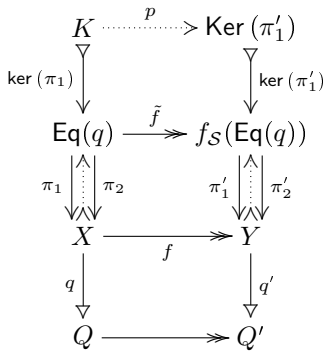
$$\begin{array}{ccc} K & \overset{p}{\twoheadrightarrow} & f_{\mathcal{S}}(K) \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f_{\mathcal{S}}(k)} \\ A & \underset{f}{\twoheadrightarrow} & B \end{array} \tag{1.3}$$

where f is in $\mathbb{E}$, k is a normal monomorphism and $f_{\mathcal{S}}(k) \circ p$ is the $\mathcal{S}$-factorisation of $f \circ k$, then $f_{\mathcal{S}}(k)$ is necessarily a normal monomorphism.

Let $\mathcal{A}$ be a category equipped with a factorisation system $\mathcal{S} = (\mathbb{E}, \mathbb{M})$.

Proposition 1.5.9. *If $\mathcal{A}$ is $\mathbb{E}$-semi-abelian then it satisfies* (INN) *with respect to $\mathcal{S}$.*

PROOF. A diagram of the form (1.3) can be presented and completed as in



where q is the cokernel of k and q' is the coequalizer of $f_{\mathcal{S}}(\mathsf{Eq}(q))$ and $k = \pi_2 \circ \mathsf{ker}\,(\pi_1)$. Since $f_{\mathcal{S}}(\mathsf{Eq}(q))$ is an effective equivalence relation, one has that $k' = \pi'_2 \circ \mathsf{ker}\,(\pi'_1)$ is the kernel of q'. Since $\tilde{f}$ is in $\mathbb{E}$, the bottom square is in $\mathbb{E}^1$ and p is in $\mathbb{E}$ (see Lemma 1.4.25). Therefore the kernel k' is the $\mathcal{S}$-factorisation of k. □

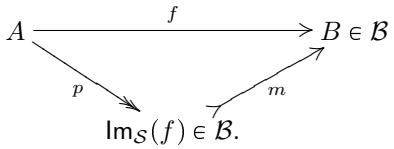
Proposition 1.5.10. *If $\mathcal{A}$ is $\mathbb{E}$-semi-abelian for some class $\mathbb{E}$ then $\mathsf{Cub}^n(\mathcal{A})$ is $\mathbb{E}_n$-semi-abelian where $\mathbb{E}_n$ denotes the class of natural transformations such that all their components are in $\mathbb{E}$. If $\mathcal{A}$ is homological and satisfies* (INN) *for some proper (stable) factorisation system $(\mathbb{E}, \mathbb{M})$, then $\mathsf{Cub}^n(\mathcal{A})$ is also homological and satisfies the Property* (INN) *for the proper (stable) factorisation system $(\mathbb{E}_n, \mathbb{M}_n)$ (where $\mathbb{M}_n$ is completely determined by $\mathbb{E}_n$ and is defined similarly to $\mathbb{E}_n$).* □

Let us recall that a *Birkhoff subcategory* $\mathcal{B}$ of a homological category $\mathcal{A}$ is a full reflective subcategory $\mathcal{B}$ closed under subobjects and quotients in $\mathcal{A}$. A Birkhoff subcategory of a homological category $\mathcal{A}$ is also homological.

Proposition 1.5.11. *Let $\mathcal{B}$ be a Birkhoff subcategory of a homological category $\mathcal{A}$ with a proper (stable) factorisation system $\mathcal{S} = (\mathbb{E}, \mathbb{M})$, $\mathbb{E}_{\mathcal{B}} = \mathbb{E} \cap \mathcal{B}$ and $\mathbb{M}_{\mathcal{B}} = \mathbb{M} \cap \mathcal{B}$. Then $\mathcal{S}_{\mathcal{B}} = (\mathbb{E}_{\mathcal{B}}, \mathbb{M}_{\mathcal{B}})$ is a proper (stable) factorisation system on $\mathcal{B}$ and*

1. *If $\mathcal{A}$ satisfies the property* (INN)*, $\mathcal{B}$ also satisfies it;*
2. *If $\mathcal{A}$ is $\mathbb{E}$-semi-abelian, $\mathcal{B}$ is $\mathbb{E}_{\mathcal{B}}$-semi-abelian.*

PROOF. One uses Proposition 1.2.7 to show that $\mathcal{S}_{\mathcal{B}}$ is a factorization system in $\mathcal{B}$. Let us just point out here that the $\mathcal{S}_{\mathcal{B}}$-factorisation of a morphism $f\colon A \to B$ in $\mathcal{B}$ coincides with its $\mathcal{S}$-factorisation in $\mathcal{A}$ ($\mathcal{B}$ is closed under subobjects and $\mathcal{S}$ proper):



The fact that $\mathcal{S}$ is proper (stable and $\mathcal{B}$ closed under pullbacks) implies that $\mathcal{S}_\mathcal{B}$ is a proper (stable) factorisation system.

(1): Let us consider the following diagram in $\mathcal{B}$

$$\begin{array}{ccc} K & \xrightarrow{p} & f_{\mathcal{S}_\mathcal{B}}(K) \\ {\scriptstyle k\in\mathbb{M}_\mathcal{B}}\downarrow & & \downarrow{\scriptstyle f_{\mathcal{S}_\mathcal{B}}(k)} \\ A & \xrightarrow[f\in\mathbb{E}_\mathcal{B}]{} & B \end{array}$$

with k a normal monomorphism in $\mathcal{B}$. Then k is also a normal monomorphism in $\mathcal{A}$ ($\mathcal{B}$ is a Birkhoff subcategory of $\mathcal{A}$) and, since $f_{\mathcal{S}_\mathcal{B}}(k) = f_\mathcal{S}(k)$ and $\mathcal{A}$ satisfies (INN), $f_{\mathcal{S}_\mathcal{B}}(k)$ is a normal monomorphism in $\mathcal{A}$ and consequently in $\mathcal{B}$.

(2): Let us consider an internal equivalence relation $(R, r_1 \colon R \to B, r_2 \colon R \to X)$ in $\mathcal{B}$. Since R is of course an internal equivalence relation in $\mathcal{A}$ and $\mathcal{A}$ is $\mathbb{E}$-exact, there exists an effective internal equivalence relation S in $\mathcal{A}$ such that $R \leqslant_\mathcal{E} S$. Since $\mathcal{B}$ is a Birkhoff subcategory of $\mathcal{A}$, S and its coequalizer turn out to be both in $\mathcal{B}$: $\mathcal{B}$ is $\mathbb{E}_\mathcal{B}$-exact. □

Proposition 1.5.12. *A pointed protomodular category $\mathcal{A}$ which satisfies the Property* (INN) *with respect to a proper stable factorisation system $(\mathbb{E}, \mathbb{M})$ is $\mathbb{E}$-exact.*

Proof. This is a straightforward adaptation of a proof in [**52**] (see Proposition 3.5 on page 381). Let (R, r_1, r_2) be an equivalence relation on A, $k = \mathsf{ker}\,(r_1)$ and let $k' \circ p$ be the $\mathcal{S}$-factorisation of $r_2 \circ k$

$$\begin{array}{ccc} \mathsf{Ker}\,(r_1) & \xrightarrow{p\in\mathbb{E}} & K' \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k'\in\mathbb{M}} \\ R & \xrightarrow[r_2]{} & A. \end{array}$$

Then k' is the kernel of a normal epimorphism $q \colon A \to Q$. Let $(\mathsf{Eq}(q), \pi_1, \pi_2)$ be the kernel pair of q. Since k and δ (the common section of r_1 and r_2) are jointly epimorphic by Proposition 1.3.6 and

$$q \circ r_1 \circ k = 0 \circ p = q \circ k' \circ p = q \circ r_2 \circ k$$

and

$$q \circ r_1 \circ \delta = q = q \circ r_2 \circ \delta,$$

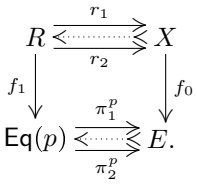
there is an inclusion $i \colon R \to \mathsf{Eq}(q)$ such that $r_1 = \pi_1 \circ i$ and $r_2 = \pi_2 \circ i$. It remains to show that i is in $\mathbb{E}$. But we have a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \xrightarrow{k'} & A & \xrightarrow{r_1} & B & \longrightarrow & 0 \\ & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle i} & & \| & & \\ 0 & \longrightarrow & K' & \xrightarrow[h]{} & \mathsf{Eq}(q) & \xrightarrow[\pi_1]{} & B & \longrightarrow & 0, \end{array}$$

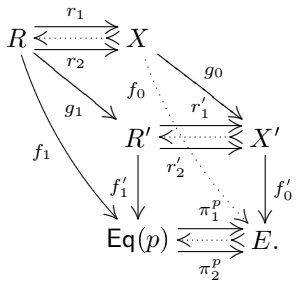
where h is the factorisation of k' through π_1 and π_2, therefore, by Lemma 1.4.3, the result follows. □

Descent theory. Let $\mathcal{A}$ be a category with pullbacks and $p\colon E \to B$ a morphism in $\mathcal{A}$. $\mathsf{Des}(p)$, the *category of descent data* for p, has

Objects: discrete fibrations $(R, f\colon R \to \mathsf{Eq}(p))$ where R is an internal equivalence relation in $\mathcal{A}$



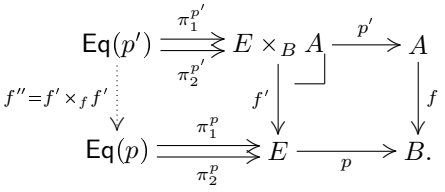
Arrows: $g\colon (R, f) \to (R', f')$ internal functors $g\colon R \to R'$ such that $f = f' \circ g$



For $p\colon E \to B$, one defines a functor

$$\mathsf{K}^p\colon \mathcal{A}/B \to \mathsf{Des}(p)$$

on objects in the following way. Take a morphism $f\colon A \to B$ and consider the following diagram



The pair $\mathsf{K}^p(f) = (f'', f')$ is in fact a discrete fibration $\mathsf{Eq}(p') \to \mathsf{Eq}(p)$. The functor is defined on arrows in a natural way (using the universal property of pullbacks).

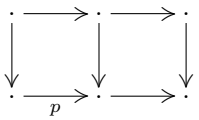
Definition 1.5.13. $p\colon E \to B$ is an *effective descent morphism* if the functor K^p determines a category equivalence.

If $\mathcal{A}$ is a regular category, one has the following useful characterization:

Theorem 1.5.14. **[55]** *p is an effective descent morphism if and only if p is a regular epimorphism and for every discrete fibration (R, f) over $\mathsf{Eq}(p)$, the equivalence relation R is effective.* □

The regular epimorphisms in a regular category have a useful property

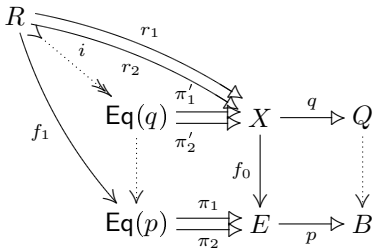
Proposition 1.5.15. [**51**] *Given a commutative diagram in a regular category*



in which p is a regular epimorphism and the left square and the outer rectangle are pullbacks. Then the right square is also a pullback. □

Proposition 1.5.16. *Every regular epimorphism in an 𝔼-semi-abelian category 𝒜 is an effective descent morphism.*

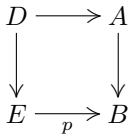
Proof. This is done essentially in the same way as in [**16**]. By Theorem 1.5.14, one just has to prove that a discrete fibration (R, f) over the kernel pair $\mathsf{Eq}(p)$ of a regular epimorphism $p\colon E \to B$ is such that R is also effective. Let us consider the following diagram



where $q\colon X \to Q$ is the coequalizer of (r_1, r_2) and the inclusion $i\colon R \to \mathsf{Eq}(q)$ is in $\mathbb{E}$ by $\mathbb{E}$-exactness. By the universal property of the pullback square $f_0 \circ r_1 = \pi_1 \circ f_1$, i has a retraction and so is in $\mathbb{M} \cap \mathbb{E} = \mathsf{Iso}$. □

Proposition 1.5.17. [**56**] *Let 𝒟 be a category with pullbacks, and 𝒞 a full subcategory in 𝒟 closed under pullbacks. Then for a morphism $p\colon E \to B$ in 𝒞, which is an effective descent morphism in 𝒟, the following conditions are equivalent:*

(1) *$p\colon E \to B$ is an effective descent morphism in 𝒞;*
(2) *for every pullback diagram of the form*



in 𝒟, if D is in 𝒞, then so is A. □

We now recall some classical definitions.

Definition 1.5.18. A category $\mathcal{A}$ is *additive* if it has a terminal object, binary products and is enriched in Ab (the category of abelian groups), i.e. that each hom-set $\mathsf{Hom}_{\mathcal{A}}(A, B)$ is an abelian group and the composition (both on the

left and on the right) is a group homomorphism. This implies the existence of biproducts, i.e. for any A and B, one has a diagram

$$A \underset{\iota_A}{\overset{p_A}{\leftrightarrows}} A \oplus B \underset{\iota_B}{\overset{p_B}{\leftrightarrows}} B$$

such that $(A \oplus B, p_A, p_B)$ is a product of A and B, $(A \oplus B, \iota_A, \iota_B)$ a coproduct of A and B, $p_B \circ \iota_A = 0$, $p_A \circ \iota_B = 0$, $p_A \circ \iota_A = 1_A$, $p_B \circ \iota_B = 1_B$ and $\iota_A \circ p_A + \iota_B \circ p_B = 1_{A \oplus B}$.

Definition 1.5.19. A functor $F \colon \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are additive is called *additive* if it preserves biproducts.

Here are some results relative to additive categories.

Proposition 1.5.20. *Any split epimorphism $f \colon X \to Y$ in an additive category produces an isomorphism $X \cong \mathsf{Ker}\,(f) \oplus Y$.* □

Theorem 1.5.21. **[11]** *An additive finitely complete category $\mathcal{A}$ is pointed and protomodular.* □

Definition 1.5.22. An additive category $\mathcal{A}$ is *almost abelian* **[72]** if

(1) every morphism in $\mathcal{A}$ has a kernel and a cokernel. This implies that a morphism $f \colon A \to B$ can be factorised as

$$A \xrightarrow{\mathsf{coker}\,(\mathsf{ker}\,(f))} \mathsf{Coim}(f) \xrightarrow{\quad \dot{f} \quad} \mathsf{Im}(f) \xrightarrow{\mathsf{ker}\,(\mathsf{coker}\,(f))} B; \tag{1.4}$$

(2) normal epimorphisms are pullback-stable and normal monomorphisms are pushout-stable.

Proposition 1.5.23. **[72]** *For every $f \colon A \to B$ in an almost abelian category $\mathcal{A}$, the morphism $\dot{f}$ in (1.4) is a bimorphism, i.e. both monic and epic.*

Proof. (see Proposition 1 in **[72]**) Let $f \colon A \to B$ be a morphism in $\mathcal{A}$ and let us write $c = \mathsf{coker}\,(\mathsf{ker}\,(f))$ and $d = \mathsf{ker}\,(\mathsf{coker}\,(f))$. We prove that $\dot{f}$ is monic using the fact that normal epimorphisms are pullback stable. Then, the result follows by duality. Let us consider the pullback

$$\begin{array}{ccc} A \times_{\mathsf{Coim}(f)} \mathsf{Ker}\,(\dot{f}) & \xrightarrow{\ g\ } & \mathsf{Ker}\,(\dot{f}) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathsf{ker}\,(\dot{f})} \\ A & \xrightarrow[\ c\ ]{} & \mathsf{Coim}(f) \end{array}$$

where g is in particular an epimorphism. Thus,

$$f \circ h = d \circ \dot{f} \circ c \circ h = d \circ \dot{f} \circ \mathsf{ker}\,(\dot{f}) \circ g = 0$$

implies that h factors as $\mathsf{ker}\,(f) \circ \tilde{h} = \mathsf{ker}\,(c) \circ \tilde{h}$. So, one has $\mathsf{ker}\,(\dot{f}) \circ g = c \circ h = 0 = 0 \circ g$ and, finally, $\mathsf{ker}\,(\dot{f}) = 0$. □

Remark 1.5.24. The previous proposition shows that an almost abelian category $\mathcal{A}$ admits $(\mathsf{NormEpi}(\mathcal{A}), \mathsf{Mono}(\mathcal{A}))$ and $(\mathsf{Epi}(\mathcal{A}), \mathsf{NormMono}(\mathcal{A}))$ as proper factorisation systems. Theorem 1.5.21 shows that an almost abelian category and its dual are protomodular. In fact, almost abelian categories turn out to

be precisely the Raïkov semi-abelian categories [**67**] which are known to be homological and cohomological categories [**71**] (see remark (1) on page 311). Of course, these categories satisfy (INN) with respect to the factorisation system (Epi, NormMono). Recently, it was proved in [**42**] that these categories are in fact *almost exact* in the sense of [**54**], i.e. regular epimorphisms are effective for descent in these categories.

Examples 1.5.25. Main examples of almost abelian categories are the category of Banach spaces (with continuous linear operators as morphisms), the category of abelian locally compact groups, of normed linear spaces, of Fréchet spaces and a lot of categories of interest in functional analysis [**67, 72**]. Also, all abelian categories are almost abelian.

Definition 1.5.26. An almost abelian category is called *left-integral* [**72**] if epimorphisms are pullback-stable.

Remark 1.5.27. If an almost abelian category is left-integral, it is also *right-integral*, i.e. monomorphisms are pushout-stable (see [**72**] Corollary on page 173). An almost abelian category which is left and right integral is called *integral* in the terminology of [**72**]. We shall give a large class of examples of integral almost abelian categories. Of course, not all almost abelian categories are integral. A rather simple example is the category of Banach spaces. In this category, a morphism $L\colon X \to Y$ is an epimorphism if and only if $\overline{\mathsf{Im}(L)} = Y$. Now, let us consider the following situation. Let $L\colon X \to Y$ be a bimorphism with $\mathsf{Im}(L) \neq Y$ and $Z \neq 0$ a closed subspace of Y such that $Z \cap \mathsf{Im}(L) = 0$. One has thus a pullback diagram

$$\begin{array}{ccc} 0 = Z \times_Y X & \rightarrowtail & X \\ {\scriptstyle \tilde{L}}\downarrow & & \downarrow{\scriptstyle L} \\ Z & \rightarrowtail & Y \end{array}$$

where $\overline{\tilde{L}(0)} = \overline{0} = 0 \neq Z$ and $\tilde{L}$ is not an epimorphism. Such a situation exists: simply take $Y = L^2[-1, 1]$, X the subspace generated by the polynomial functions and Z the subspace generated by the exponential function.

We have here some new examples of 𝔼-semi-abelian categories

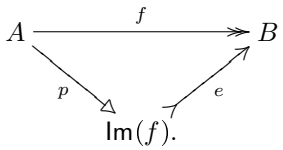
Proposition 1.5.28. *A left-integral almost abelian category* [**72**] *is* Epi*-semi-abelian.*

Proof. In presence of biproducts, the existence of cokernels and kernels provides us with all equalizers and coequalizers. This gives finite limits and colimits. Protomodularity comes from Theorem 1.5.21. The regularity and the existence of a second proper stable factorisation system (Epi, NormMono) come from the axioms and Proposition 1.5.23 as explained above. Epi-exactness follows from the fact that one has a system (Epi, NormMono) (see Proposition 1.5.12). □

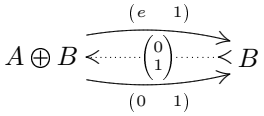
We are grateful to G. Janelidze for pointing us the following

Proposition 1.5.29. *If an additive category $\mathcal{A}$ is $\mathbb{E}$-semi-abelian then $\mathbb{E} = \mathsf{Epi}$. This implies that for an additive category, being $\mathbb{E}$-semi-abelian (for some $\mathbb{E}$) becomes a property.*

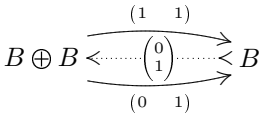
Proof. We have to show that bimorphisms are in $\mathbb{E}$. This is in fact sufficient as we are going to explain. Let $f\colon A \to B$ be an epimorphism an let us take its $(\mathsf{RegEpi}(\mathcal{A}), \mathsf{Mono}(\mathcal{A}))$-factorisation $e \circ p$



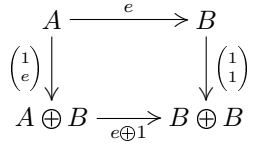
Then e is of course a bimorphism and p is in $\mathsf{RegEpi}(\mathcal{A}) \subseteq \mathbb{E}$ (one uses Corollary 1.2.5 (3)). Then if e is in $\mathbb{E}$, f is also in $\mathbb{E}$ since $\mathbb{E}$ is closed under composition. Then one has $\mathsf{Epi}(\mathcal{A}) \subseteq \mathsf{RegEpi}(\mathcal{A})$ and the result follows ($(\mathbb{E}, \mathbb{M})$ is proper). So, let $e\colon A \to B$ be a bimorphism. The reflexive graph



is in fact an equivalence relation (since e is a monomorphism and $\mathcal{A}$ is a Mal'tsev category) and its coequalizer, the cokernel c of e is zero (since e is an epimorphism). Then the comparison morphism $e \oplus 1\colon A \oplus B \to B \oplus B$ to the kernel pair of c



is in $\mathbb{E}$ since $\mathcal{A}$ is $\mathbb{E}$-semi-abelian. Now, one can check that the following commutative square



is a pullback (for $\begin{pmatrix} a \\ b \end{pmatrix}$ and b' such that $e \oplus 1 \circ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \circ b'$, a is the required induced morphism). Thus, by pullback stability of morphisms in $\mathbb{E}$, e is in $\mathbb{E}$. □

Remark 1.5.30. So far, I have not managed to determine whether the uniqueness of $\mathbb{E}$ holds for general categories.

We are going to give a large class of examples of integral almost abelian categories, thus of $\mathbb{E}$-semi-abelian categories. First, let us recall some classical notions. Let $\mathcal{A}$ be a pointed category with kernels and cokernels.

Definition 1.5.31. [27, 57] A *torsion theory* in $\mathcal{A}$ is a couple $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathcal{A}$ such that:

(1) every morphism $f\colon T \to F$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$ is zero;
(2) for any object A in $\mathcal{A}$ there exists a short exact sequence:

$$0 \longrightarrow T \overset{k}{\rhd\!\!\longrightarrow} A \overset{f}{\longrightarrow\!\!\rhd} F \longrightarrow 0$$

with $T \in \mathcal{T}$, $F \in \mathcal{F}$.

For a torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, the subcategories $\mathcal{T}$ and $\mathcal{F}$ are called, respectively, the *torsion part* and the *torsion-free part* of the torsion theory. As one can easily check from the definition, $\mathcal{T}$ is in fact a full replete normal mono-coreflective subcategory of $\mathcal{A}$ and $\mathcal{F}$ a full replete normal epi-reflective subcategory of $\mathcal{A}$. We shall denote the (uniquely determined up to isomorphism) short exact sequence associated with an object A of $\mathcal{A}$ by

$$0 \longrightarrow T(A) \overset{\mu_A}{\rhd\!\!\longrightarrow} A \overset{\eta_A}{\longrightarrow\!\!\rhd} F(A) \longrightarrow 0.$$

Definition 1.5.32. A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ is *hereditary* if $\mathcal{T}$ is closed under subobjects in $\mathcal{A}$.

Example 1.5.33. The typical example of hereditary torsion theory is given by the pair $(\mathsf{Ab}_{t.}, \mathsf{Ab}_{t.f.})$ in Ab where $\mathsf{Ab}_{t.}$ is the *category of torsion groups* and $\mathsf{Ab}_{t.f.}$ the *category of torsion-free groups*. Let us recall that a group G is a *torsion group* if all its elements have finite order and a *torsion-free group* if it satisfies the axiom

$$\forall g \in G\, \forall n \in \mathbb{N}_0 \,:\, g^n = 1 \Rightarrow g = 1,$$

i.e. 1 is the only element whose order is finite in G.

We collect in the following lemma some useful properties of the torsion-free part of a torsion theory.

Lemma 1.5.34. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in an abelian category $\mathcal{A}$. Then*

(1) *$\mathcal{F}$ is additive and closed under limits in $\mathcal{A}$;*
(2) *$\mathcal{F}$ has cokernels;*
(3) *a morphism in $\mathcal{F}$ is a normal epimorphism in $\mathcal{F}$ if and only if it is a normal epimorphism in $\mathcal{A}$;*
(4) *a morphism is an epi in $\mathcal{F}$ if and only if its cokernel in $\mathcal{A}$ is in $\mathcal{T}$.*
(5) *$\mathcal{F}$ is closed under extensions: given a short exact sequence*

$$0 \longrightarrow A \overset{l}{\rhd\!\!\longrightarrow} B \overset{q}{\longrightarrow\!\!\rhd} C \longrightarrow 0 \tag{1.5}$$

in $\mathcal{A}$ with A and C in $\mathcal{F}$, then B is also in $\mathcal{F}$.

Proof. (1): The torsion-free subcategory $\mathcal{F}$ contains the zero object and is enriched in Ab as a full subcategory of $\mathcal{A}$. $\mathcal{F}$ is also closed under limits in $\mathcal{A}$ since it is reflective.

(2): Let $f\colon A \to B$ be a morphism in $\mathcal{F}$ and $c\colon B \to C$ its cokernel in $\mathcal{A}$. Let us consider the short exact sequence associated with C

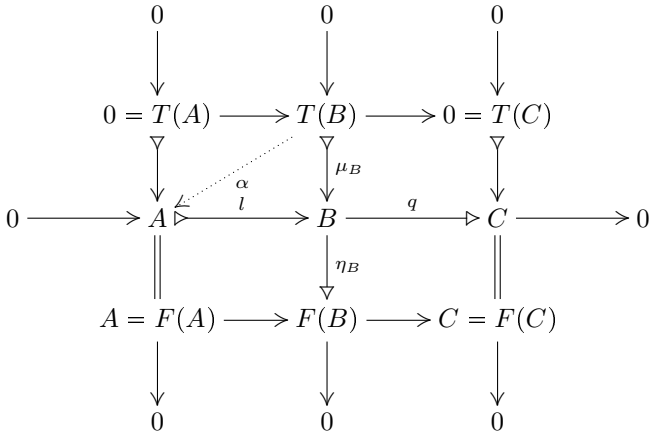
$$0 \longrightarrow T(C) \overset{\mu_C}{\rhd\!\!\longrightarrow} C \overset{\eta_C}{\longrightarrow\!\!\rhd} F(C) \longrightarrow 0.$$

Then, as one can easily check, $\eta_C \circ c\colon B \to F(C)$ is in fact the cokernel of f in $\mathcal{F}$. This shows that $\mathcal{F}$ has cokernels.

(3): If $f\colon A \to B$ is a cokernel in $\mathcal{F}$, it is also one in $\mathcal{A}$ since it is an epimorphism in $\mathcal{A}$ which is abelian.

(4): If f is epi in $\mathcal{F}$, then its cokernel in $\mathcal{F}$ is zero and consequently its cokernel is in $\mathcal{T}$. Conversely, if its cokernel in $\mathcal{A}$ is in $\mathcal{T}$ then $F(C)$ in the above short exact sequence is zero, which implies that f is epi.

(5): If one starts with a sequence like in (1.5), then one can construct the following commutative diagram

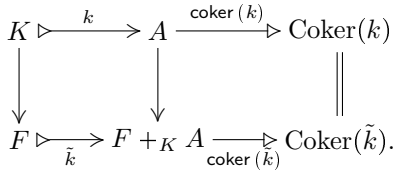


Of course μ_B must factor as $l \circ \alpha$ (with α a monomorphism) since $q \circ \mu_B = 0$. But since $T(B)$ is in $\mathcal{T}$ and A in $\mathcal{F}$, α is zero and $T(B) = 0$, i.e. B is in $\mathcal{F}$. □

The following is part of Theorem 2 on page 191 in **[72]**

Proposition 1.5.35. *The torsion-free part of a hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{A}$ is almost-abelian and left-integral, hence* Epi*-semi-abelian.*

Proof. We know by Lemma 1.5.34 that $\mathcal{F}$ is additive and has kernels and cokernels and that limits exist in $\mathcal{F}$ and are the same as in $\mathcal{A}$. Then, still by Lemma 1.5.34, normal epimorphisms are pullback-stable. Now take a normal monomorphism $k\colon K \to A$ in $\mathcal{F}$ and $\tilde{k}$ its pushout in $\mathcal{A}$ along a morphism $K \to F$ in $\mathcal{F}$ and their cokernels in $\mathcal{A}$



From Lemma 1.5.34 (3), one can deduce that $\mathrm{Coker}(k)$ is in fact in $\mathcal{F}$. But $\mathcal{F}$ is closed under extensions, hence $F +_K A$ is in $\mathcal{F}$ (and the pushout in $\mathcal{A}$ is in

fact a pushout in $\mathcal{F}$) and $\tilde{k}$ a kernel in $\mathcal{F}$. $\mathcal{F}$ is almost abelian. Now, let us consider a pullback diagram in $\mathcal{F}$

$$\begin{array}{ccc} F \times_B A & \xrightarrow{\tilde{g}} & A \\ \downarrow & & \downarrow \\ F & \xrightarrow[g]{} & B \end{array}$$

with g epi in $\mathcal{F}$. Taking cokernels of g and $\tilde{g}$ in $\mathcal{A}$, one sees that the morphism $\mathrm{Coker}(\tilde{g}) \to \mathrm{Coker}(g)$ induced by the square is in fact a monomorphism. But $\mathrm{Coker}(g)$ is in $\mathcal{T}$ and $\mathcal{T}$ closed under subobjects, then $\mathrm{Coker}(\tilde{g})$ is in $\mathcal{T}$ and $\tilde{g}$ epic in $\mathcal{F}$. $\mathcal{F}$ is left-integral (and by Remark 1.5.26, integral). □

Remark 1.5.36. In fact, it is proved in [**72**] (see the corollary on page 193) that for a category $\mathcal{A}$, the following conditions are equivalent

- $\mathcal{A}$ is almost abelian;
- $\mathcal{A}$ is equivalent to a torsion-free subcategory of an abelian category;
- $\mathcal{A}$ is equivalent to a torsion subcategory of an abelian category.

Let $\mathcal{A}$ be an almost abelian category.

Definition 1.5.37. The category $\mathsf{mod}_1(\mathcal{A})$ is given by

Objects: monomorphisms $d_1 \colon C_1 \to C_0$ in $\mathcal{A}$;
Arrows: commutative squares in $\mathcal{A}$

$$\begin{array}{ccc} C_1 & \xrightarrow{d_1} & C_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ C_1' & \xrightarrow[d_1']{} & C_0' \end{array}$$

modulo "homotopy": $f = (f_1, f_0) \cong g = (g_1, g_0)$ if and only if there exists $s_0 \colon C_0 \to C_1'$ such that $f_0 - g_0 = d_1' \circ s_0$.

Remark 1.5.38. The above relation of "homotopy" is the usual one defined for morphisms of chain complexes with $C_n = 0$ for $n \neq 0$, 1 and d_1 a monomorphism.

The following proposition is a corollary of Theorem 1 on page 184 in [**72**]

Proposition 1.5.39. *The category $\mathsf{mod}_1(\mathcal{A})$ of an almost abelian category $\mathcal{A}$ is almost abelian and integral, hence* Epi*-semi-abelian.* □

Let us also recall the following useful result from [**19**]:

Theorem 1.5.40. *A finitary algebraic theory $\mathbb{T}$ is semi-abelian, i.e. has a semi-abelian category $\mathbb{T}(\mathsf{Set})$ of models, precisely when, for some natural number n, the theory $\mathbb{T}$ contains*

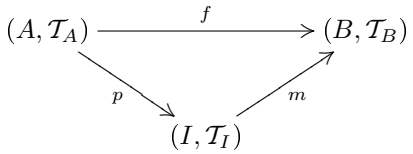
- *a unique constant* 0*;*
- *n binary operations $\alpha_1(X, Y)$, …, $\alpha_n(X, Y)$ satisfying $\alpha_i(X, X) = 0$;*
- *an $(n+1)$-ary operation $\theta(X_1, \ldots, X_{n+1})$ satisfying*

$$\theta(\alpha_1(X, Y), \ldots, \alpha_n(X, Y), Y) = X.$$

Let us recall that a *topological semi-abelian variety* [**8**] is the category of models in Top of some semi-abelian theory. An example is the category of topological groups $\mathsf{Grp}(\mathsf{Top})$ which is known to be not exact and not additive. Therefore $\mathsf{Gp}(\mathsf{Top})$ is neither a semi-abelian category nor an almost abelian category. However it is a Epi-semi-abelian category. More generally:

Proposition 1.5.41. *Topological semi-abelian varieties are* Epi*-semi-abelian.*

Proof. By Theorem 50 on page 446 in [**8**], these categories are cocomplete and homological. The comparison morphism for an internal equivalence relation R is an epimorphism since the forgetful functor U to $\mathbb{T}(\mathsf{Set})$ is topological thus faithful and limits and colimits preserving [**4**]. The existence of the proper stable factorisation system $(\mathsf{Epi}, \mathsf{NormMono})$ is clear, one uses the factorisation which comes from $\mathbb{T}(\mathsf{Set})$ and one chooses the induced topology on the image. Let $f\colon (A, \mathcal{T}_A) \to (B, \mathcal{T}_B)$ be a continuous homomorphism of topological $\mathbb{T}$-algebras and let $m \circ p$ be the $\mathcal{S}_r$-factorisation of $U(f)$ in $\mathbb{T}(\mathsf{Set})$. Then, putting the topology $\mathcal{T}_I$ on $I = \mathsf{Im}_{\mathcal{S}_r}(U(f))$ induced by the one on B, one finds a commutative diagram in $\mathbb{T}(\mathsf{Top})$



where p is an epimorphism and m a normal monomorphism. □

1.6. Closure operators

In [**18**] the notion of *homological closure operator* was introduced and a bijective correspondence was established between such closure operators and regular epi-reflective subcategories of a given homological category. Some recent developments (see [**36, 31, 29**]) suggests to extend this correspondence and various results which were first proved in [**18**]. The main point is that we have to work without finite limits since one would like to define closure operators in categories like $\mathsf{ExCub}^n_{\mathcal{E}}(\mathcal{A})$, for instance. A good definition, for our purpose, has been introduced recently in [**73**]. We adopt here the notation used in [**18**].

Let $\mathcal{M}$ be a class of monomorphisms in a category $\mathcal{A}$ such that:

(a) $\mathcal{M}$ contains all isomorphisms;
(b) $\mathcal{M}$ is closed under composition with isomorphisms;
(c) $\mathcal{M}$ satisfies the left-cancellation properties:

$$n \circ m \in \mathcal{M}, n \in \mathcal{M} \Rightarrow m \in \mathcal{M}.$$

Such a class $\mathcal{M}$ can be viewed as a full replete subcategory of $\mathsf{Arr}\mathcal{A}$. We shall write Cod for the functor of type $\mathcal{M} \to \mathcal{A}$ which sends an object of $\mathcal{M}$ to its codomain. For two monomorphisms $k\colon K \to X$ and $l\colon L \to X$ such that k factors through l, we shall write $K \leqslant L$.

Definition 1.6.1. [**73**] A *closure operator* of $\mathcal{M}$ in $\mathcal{A}$ is an endofunctor

$$\overline{\cdot}\colon \mathcal{M} \to \mathcal{M}$$

such that:

(1) $\mathsf{Cod} = \mathsf{Cod} \circ \overline{\cdot}$;
(2) $\forall K \in \mathcal{M} : K \leqslant \overline{K}$;
(3) $\forall K, L \in \mathcal{M} : K \leqslant L \Rightarrow \overline{K} \leqslant \overline{L}$.

Remark 1.6.2. Condition (3) follows in fact from (1) and the functoriality of $\overline{\cdot}$. We shall speak of a closure operator in $(\mathcal{A}, \mathcal{M})$, and denote i_k the factorisation of k through $\overline{k}$.

Definition 1.6.3. **[73]** A closure operator in $(\mathcal{A}, \mathcal{M})$ is *idempotent* if

(4) $\forall K \in \mathcal{M} : \overline{\overline{K}} = \overline{K}$.

Let $\mathcal{E}$ be a class of arrows such that elements of $\mathcal{M}$ are pullback-stable along morphisms of $\mathcal{E}$.

Definition 1.6.4. A closure operator $\overline{\cdot}$ in $(\mathcal{A}, \mathcal{M})$ is called an *$\mathcal{E}$-closure operator* if morphisms in $\mathcal{E}$ are $\overline{\cdot}$-open:

(5) $\overline{f^{-1}(K)} = f^{-1}(\overline{K})$ whenever $f\colon B \to A$ is in $\mathcal{E}$ and $k\colon K \to A$ is in $\mathcal{M}$.

A morphism $k\colon K \to A$ in $\mathcal{M}$ will be called *closed* (*dense*) with respect to a closure operator $\overline{\cdot}$ in $(\mathcal{A}, \mathcal{M})$ if $\overline{K} = K$ (resp. $\overline{K} = A$).

Definition 1.6.5. **[73]** A closure operator in $(\mathcal{A}, \mathcal{M})$ is *weakly hereditary* if $\overline{i_k}$ is dense for all $k \in \mathcal{M}$.

Example 1.6.6. The typical example of closure operator is of course the Kuratowski closure operator which can be defined via the classical notion of closure of a subset in a topological space.

Let $(\mathcal{A}, \mathcal{E})$ be such that (E1)-(E4)+(E6) hold, and let $\mathcal{M}(\mathcal{E})$ be the class of the kernels of elements in $\mathcal{E}$. Then $\mathcal{M}(\mathcal{E})$ satisfies the conditions (a)-(c) (see Lemma 1.4.28).

Example 1.6.7. If $\mathcal{A}$ is a category with pullbacks, a closure operator is simply a closure operator in the sense of **[26]**.

Example 1.6.8. If $\mathcal{A}$ is a homological category and $\mathcal{E}$ is the class of regular epimorphisms in $\mathcal{A}$, then $\mathcal{M}(\mathcal{E})$ is the category of normal monomorphisms in $\mathcal{A}$. An idempotent closure operator $\overline{\cdot}\colon \mathcal{M}(\mathcal{E}) \to \mathcal{M}(\mathcal{E})$ is a closure operator in the sense of **[18]** and an idempotent $\mathcal{E}$-closure operator is a homological closure operator (also in the sense of **[18]**).

We generalise here some results from **[18]**.

Lemma 1.6.9. *Let $\overline{\cdot}$ be an $\mathcal{E}$-closure operator in $(\mathcal{A}, \mathcal{M}(\mathcal{E}))$ such that* (E1)-(E4)+(E6) *hold, and $\mathcal{E} \subseteq \mathsf{Epi}(\mathcal{A})$. If $k\colon K \to A \in \mathcal{M}(\mathcal{E})$ and $f\colon B \to A \in \mathcal{E}$, then k is closed (respectively dense) if and only if $f^{-1}(k)$ is closed (dense).*

Proof. If k is closed, one has the following equalities

$$\overline{f^{-1}(K)} \overset{(5)}{=} f^{-1}(\overline{K}) = f^{-1}(K)$$

i.e. $f^{-1}(k)$ is closed; if k is dense, then

$$\overline{f^{-1}(K)} \overset{(5)}{=} f^{-1}(\overline{K}) = f^{-1}(A) = B$$

i.e. $f^{-1}(k)$ is dense. Conversely, let us consider the following commutative diagram

$$\begin{array}{ccccc} f^{-1}(K) & \overset{i_{f^{-1}(k)}}{\rightarrowtail} & \overline{f^{-1}(K)} & \overset{\overline{f^{-1}(k)}}{\triangleright\!\!\rightarrow} & B \\ {\scriptstyle g}\downarrow & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle f} \\ K & \underset{i_k}{\triangleright\!\!\rightarrow} & \overline{K} & \underset{\overline{k}}{\triangleright\!\!\rightarrow} & A. \end{array}$$

Since $\overline{\cdot}$ is an $\mathcal{E}$-closure operator and $\mathcal{E}$ satisfies (E2), the right square above is a pullback and g' is in $\mathcal{E}$. Then if $f^{-1}(K)$ is closed, i.e. $i_{f^{-1}(k)}$ is an isomorphism, one has that i_k is also an isomorphism since $\mathcal{E} \subseteq \mathsf{Epi}(\mathcal{A})$. If $f^{-1}(k)$ is dense, one has that $\overline{f^{-1}(k)}$ is an iso and then $\overline{k}$ is an iso as well. □

For a pair $(\mathcal{A}, \mathcal{E})$ where $\mathcal{E}$ is any class of morphisms in a category $\mathcal{A}$, we shall say that a reflective subcategory of $\mathcal{A}$ is *$\mathcal{E$-reflective*} if the unit of the reflection has all its components in $\mathcal{E}$.

Theorem 1.6.10. *Let $(\mathcal{A}, \mathcal{E})$ be such that $\mathcal{A}$ is pointed and $\mathcal{E} \subseteq \mathsf{Coker}(\mathcal{A})$ satisfies* (E1)-(E4)*. There is a bijection between the full replete $\mathcal{E}$-reflective subcategories of $\mathcal{A}$ and the idempotent $\mathcal{E}$-closure operators in $(\mathcal{A}, \mathcal{M}(\mathcal{E}))$.*

Proof. If $\mathcal{B}$ is a full replete $\mathcal{E}$-reflective subcategory of $\mathcal{A}$ with reflector I and unit η, one defines an endofunctor $\overline{\cdot}^{\mathcal{B}} \colon \mathcal{M}(\mathcal{E}) \to \mathcal{M}(\mathcal{E})$ in the following way:

$$\overline{(k \colon K \to A)}^{\mathcal{B}} = \mathsf{ker}(\eta_{A/K} \circ q_k) \cong q_k^{-1}(\mathsf{Ker}\,(\eta_{A/K}))$$

where $q_k \colon A \to A/K$ (in $\mathcal{E}$) is the cokernel of k. This endofunctor is clearly an idempotent closure operator. For $k \colon K \to A$ in $\mathcal{M}(\mathcal{E})$ and $f \colon B \to A$ in $\mathcal{E}$, $f^{-1}(K)$ has cokernel $q_k \circ f$ (in $\mathcal{E}$) and the following morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^{-1}(K) & \rightarrowtail & B & \overset{q_k \circ f}{\triangleright\!\!\rightarrow} & A/K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \| & & \\ 0 & \longrightarrow & K & \underset{k}{\triangleright\!\!\rightarrow} & A & \underset{q_k}{\triangleright\!\!\rightarrow} & A/K & \longrightarrow & 0 \end{array}$$

induces the following one

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{f^{-1}(K)}^{\mathcal{B}} & \rightarrowtail & A & \overset{\eta_{A/K} \circ q_k \circ f}{\triangleright\!\!\rightarrow} & I(A/K) & \longrightarrow & 0 \\ & & \downarrow & (1) & \downarrow{\scriptstyle f} & & \| & & \\ 0 & \longrightarrow & \overline{K}^{\mathcal{B}} & \underset{\overline{k}^{\mathcal{B}}}{\rightarrowtail} & A & \underset{\eta_{A/K} \circ q_k}{\triangleright\!\!\rightarrow} & I(A/K) & \longrightarrow & 0. \end{array}$$

It follows that the square (1) is a pullback and that $\overline{f^{-1}(K)}^{\mathcal{B}} = f^{-1}(\overline{K}^{\mathcal{B}})$: $\overline{\cdot}^{\mathcal{B}}$ is an idempotent $\mathcal{E}$-closure operator. Given an idempotent $\mathcal{E}$-closure operator $\overline{\cdot} \colon \mathcal{M}(\mathcal{E}) \to \mathcal{M}(\mathcal{E})$, the full replete subcategory $\{B \in |\mathcal{A}| \mid 0 \text{ is closed in } B\}$ of

$\mathcal{A}$ is $\mathcal{E}$-reflective. One defines the reflection $I(A)$ of an object A of $\mathcal{A}$ as $A/\overline{0}_A$ and the component η_A of the unit η at A as $\mathsf{coker}\,(\overline{0}_A)$ (which is in $\mathcal{E}$). Let B be an object such that $0 = \overline{0}_B$ and $f\colon A \to B$ an arrow in $\mathcal{A}$. The following commutative square

$$\begin{array}{ccc} 0 & \rightarrowtriangle & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ 0 & \rightarrowtriangle & B \end{array}$$

gives us, by functoriality of $\overline{\cdot}$, the following commutative square

$$\begin{array}{ccc} \overline{0}_A & \rightarrowtriangle & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ 0 = \overline{0}_B & \rightarrowtriangle & B \end{array}$$

which can be completed in the expected way

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{0}_A & \rightarrowtriangle & A & \xrightarrow{\eta_A} & A/\overline{0}_A & \longrightarrow & 0. \\ & & & \searrow^{0} & \downarrow{\scriptstyle f} & & \swarrow & & \\ & & & & B & & & & \end{array}$$

Let us prove that the described processes are mutually inverse. First, let us take $C = \overline{\cdot}$ an $\mathcal{E}$-closure operator in $(\mathcal{A}, \mathcal{E})$ and $\mathcal{B}_C$ the associated full replete $\mathcal{E}$-reflective subcategory of $\mathcal{A}$. We have to prove that $C = \overline{\cdot}^{\mathcal{B}_C}$. So let $K\colon k \to A$ be in $\mathcal{M}(\mathcal{E})$ and q_k its cokernel (in $\mathcal{E}$). One has

$$\overline{K}^{\mathcal{B}_C} = q_k^{-1}(\overline{0}_{A/K}) \overset{(5)}{=} \overline{q_k^{-1}(0)} = \overline{K}.$$

Conversely, let $\mathcal{B}$ be a full replete $\mathcal{E}$-reflective subcategory of $\mathcal{A}$ with reflector I and $C = \overline{\cdot}^{\mathcal{B}}$. We have to prove that $\mathcal{B} = \mathcal{B}_C$. This holds simply because a cokernel is the cokernel of its kernel so that the reflector I_C to $\mathcal{B}_C$ is given

$$I_C(A) = A/\overline{0}_A^{\mathcal{B}} = A/\mathsf{Ker}\,(\eta_A) = I(A). \qquad \square$$

Remark 1.6.11. There are natural orders on the class of closure operators in $(\mathcal{A}, \mathcal{E})$ and on the class of full subcategories of $\mathcal{A}$: iff $\overline{\cdot}$ and $\overline{\cdot}'$ are closure operators in $(\mathcal{A}, \mathcal{E})$, one has, by definition, that $\overline{\cdot} \leqslant \overline{\cdot}'$ if $\forall K \in \mathcal{M}(\mathcal{E}) : \overline{K} \leqslant \overline{K}'$ and for $\mathcal{F}$ and $\mathcal{F}'$ two full subcategories of $\mathcal{A}$, $\mathcal{F} \leqslant \mathcal{F}'$ iff $\forall B \in \mathcal{F} : B \in \mathcal{F}'$. The above bijection is in fact a decreasing morphism of preordered classes.

Lemma 1.6.12. *Let $\mathcal{B}$ be a full replete $\mathcal{E}$-reflective subcategory of $(\mathcal{A}, \mathcal{E})$, such that $\mathcal{A}$ is pointed and $\mathcal{E} \subseteq \mathsf{Coker}(\mathcal{A})$ satisfies* (E1)-(E4)*, with reflector $I\colon \mathcal{A} \to \mathcal{B}$. Then for all $k\colon K \to A \in \mathcal{M}(\mathcal{E})$.*

(1) The following sequence is exact

$$0 \longrightarrow K \overset{i_k}{\rightarrowtriangle} \overline{K} \xrightarrow{\widehat{q_k}} \overline{0}^{\mathcal{B}}_{A/K} \longrightarrow 0;$$

(2) K is closed if and only if $I(A/K) = A/K$;
(3) K is dense if and only if $I(A/K) = 0$.

PROOF. (1) This can be seen in the proof of Lemma 1.4.28. (2) We have that K is closed if and only if (iff) i_k is an isomorphism iff $\overline{0}^{\mathcal{B}}_{A/K} = 0$ iff $A/K \in \mathcal{B}$ iff $I(A/K) = A/K$.
(3) We have that K is dense iff $\overline{k}$ is an isomorphism iff $\mathsf{ker}\,(\eta_{A/K})$ is an isomorphism (since $\mathcal{E}$ satisfies (E3)-(E4)) iff $I(A/K) = 0$. □

Let $(\mathcal{A}, \mathcal{E})$ be such that $\mathcal{A}$ is pointed and $\mathcal{E} \subseteq \mathsf{Coker}(\mathcal{A})$ satisfies (E2). We extend the definition of torsion theory to this context.

Definition 1.6.13. A *torsion theory* in $(\mathcal{A}, \mathcal{E})$ is a couple $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathcal{A}$ such that:

(1) every morphism $f\colon T \to F$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$ is null;
(2) for all object A in $\mathcal{A}$ there exists a short exact sequence:

$$0 \longrightarrow T \overset{k}{\rhd\!\!\longrightarrow} A \overset{f}{\longrightarrow\!\!\rhd} F \longrightarrow 0$$

with $T \in \mathcal{T}$, $F \in \mathcal{F}$ and $f \in \mathcal{E}$.

The torsion-free part of a torsion theory in $(\mathcal{A}, \mathcal{E})$ is of course a full replete $\mathcal{E}$-reflective subcategory of $\mathcal{A}$. The A-component of the unit is given by the morphism f in the exact sequence above.

Theorem 1.6.14. *If $\mathcal{E} \subseteq \mathsf{Coker}(\mathcal{A})$ satisfies* (E1)-(E4)*, there is a bijection between torsion theories in $(\mathcal{A}, \mathcal{E})$ and weakly-hereditary $\mathcal{E}$-idempotent closure operators in $(\mathcal{A}, \mathcal{E})$.*

PROOF. By Theorem 1.6.10, the only thing to prove is that a full replete $\mathcal{E}$-reflective subcategory is torsion-free if and only if its associated closure operator is weakly hereditary. First, let us suppose that the closure operator $\overline{\cdot}^{\mathcal{B}}$, associated with a full replete $\mathcal{E}$-reflective subcategory $\mathcal{B}$ of $\mathcal{A}$, is weakly-hereditary. Then the kernel $i_0\colon 0 \to \overline{0}^{\mathcal{B}}_A$, the factorisation of 0 through its closure $\overline{0}^{\mathcal{B}}_A = \mathsf{Ker}\,(\eta_A)$ (in A an object of $\mathcal{A}$), is dense. This is equivalent, by Lemma 1.6.12, to $I(\overline{0}^{\mathcal{B}}_A) = 0$. Then the couple $(\mathcal{T}, \mathcal{B})$, where $\mathcal{T}$ is the full replete subcategory of $\mathcal{A}$ given by

$$\mathcal{T} = \{\, A \,|\, I(A) = 0 \,\},$$

is a torsion theory in $(\mathcal{A}, \mathcal{E})$. Conversely, given $k\colon K \to A$ in $\mathcal{M}(\mathcal{E})$, one sees that $i_0\colon 0 \to \overline{0}^{\mathcal{B}}_{A/K}$ is dense (since $\overline{0}^{\mathcal{B}}_{A/K} = \mathsf{Ker}\,(\eta_{A/K})$ is in the torsion subcategory) and that the left square in the following commutative diagram

$$\begin{array}{ccccc} K & \overset{i_k}{\rhd\!\!\longrightarrow} & \overline{K}^{\mathcal{B}}_A & \rhd\!\!\longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \widehat{q_k}} & & \downarrow{\scriptstyle q_k} \\ 0 & \underset{i_0}{\rhd\!\!\longrightarrow} & \overline{0}^{\mathcal{B}}_{A/K} & \rhd\!\!\longrightarrow & A/K \end{array}$$

is a pullback. By Lemma 1.6.9, i_k is dense. □

The following lemma was already proved in [**29**].

Lemma 1.6.15. *Let $\mathcal{B}$ be a regular epi-reflective subcategory of a homological category $\mathcal{A}$ such that the supremum of two normal subobjects always exists and $\overline{\cdot}^{\mathcal{B}}$ the induced homological closure operator (see Example 1.6.8). The following properties hold:*

(1) *$\forall A \in \mathcal{A}$, $\forall K$ a normal subobject of A ($K \lhd A$) one has*

$$\overline{K}^{\mathcal{B}}_A = \overline{(K \vee \overline{0}^{\mathcal{B}}_A)}^{\mathcal{B}}_A;$$

(2) *$\mathcal{B}$ is a Birkhoff subcategory if and only if $\forall A \in \mathcal{A}$, $\forall K \lhd A$ one has*

$$K \vee \overline{0}^{\mathcal{B}}_A = \overline{K}^{\mathcal{B}}_A;$$

(3) *if $\mathcal{B}$ is a Birkhoff subcategory, then $\forall A \in \mathcal{A}$, $\forall K, L \lhd A$ one has*

$$\overline{(K \vee L)}^{\mathcal{B}}_A = \overline{K}^{\mathcal{B}}_A \vee \overline{L}^{\mathcal{B}}_A.$$

Proof. (1) On the one hand we have $K \leqslant K \vee \overline{0}^{\mathcal{B}}_A$, which implies $\overline{K}^{\mathcal{B}}_A \leqslant \overline{(K \vee \overline{0}^{\mathcal{B}}_A)}^{\mathcal{B}}_A$. On the other hand, since $K \vee \overline{0}^{\mathcal{B}}_A \leqslant \overline{K}^{\mathcal{B}}_A$, we find that

$$\overline{(K \vee \overline{0}^{\mathcal{B}}_A)}^{\mathcal{B}}_A \leqslant \overline{(\overline{K}^{\mathcal{B}}_A)}^{\mathcal{B}}_A = \overline{K}^{\mathcal{B}}_A.$$

(2) By the previous property it suffices to prove that $\mathcal{B}$ is a Birkhoff subcategory of $\mathcal{A}$ if and only if $K \vee \overline{0}^{\mathcal{B}}_A$ is closed in A. Let us first of all remark that for any short exact sequence

$$0 \longrightarrow K \longrightarrow A \longrightarrow B \longrightarrow 0$$

in $\mathcal{A}$ we have that K is closed in A if and only B lies in $\mathcal{B}$. Now, if $\mathcal{B}$ is a Birkhoff subcategory of $\mathcal{A}$ with reflector $I \colon \mathcal{A} \to \mathcal{B}$, then $K \vee \overline{0}^{\mathcal{B}}_A$ is closed in A since the quotient $A/K \vee \overline{0}^{\mathcal{B}}_A$ lies in $\mathcal{B}$ as a regular quotient of $A/\overline{0}^{\mathcal{B}}_A = I(A) \in \mathcal{B}$. Conversely, if we assume that $K \vee \overline{0}^{\mathcal{B}}_A$ is closed in A, for any short exact sequence as above, then if A lies in $\mathcal{B}$, so that $\overline{0}^{\mathcal{B}}_A = 0$, we have that K is closed in A, hence $B \in \mathcal{B}$.

(3) If $\mathcal{B}$ is a Birkhoff subcategory of $\mathcal{A}$ one has the following equalities:

$$\begin{aligned}\overline{(K \vee L)}^{\mathcal{B}}_A &\overset{(2)}{=} (K \vee L) \vee \overline{0}^{\mathcal{B}}_A \\ &= (K \vee \overline{0}^{\mathcal{B}}_A) \vee (L \vee \overline{0}^{\mathcal{B}}_A) \\ &\overset{(2)}{=} \overline{K}^{\mathcal{B}}_A \vee \overline{L}^{\mathcal{B}}_A.\end{aligned}$$

□

CHAPTER 2

Derived Galois structures

This chapter is devoted to the study of different types of Galois structures inducing a chain of similar higher order Galois structures. We first recall the basic notion of Galois structure and introduce the notion of quasi-normal extension. After this, we introduce the notion of an $\mathcal{E}$-(pre)radical in a homological category which satisfies (INN) and show that it induces an $\mathcal{E}^1$-preradical (at the level of the category of extensions) which is a radical if and only if the category of quasi-normal extensions (associated with the $\mathcal{E}$-radical) is reflective in the category of extensions. We then study some classes of $\mathcal{E}$-preradicals which ensure that the induced preradicals are also radicals. In the third section, we pay particular attention to a special kind of $\mathcal{E}$-preradical. It is for this particular kind of $\mathcal{E}$-preradical that we are able to give better descriptions of the fundamental groups in chapter 3. All results of this chapter are valid in the context of $\mathbb{E}$-semi-abelian categories.

2.1. Galois structures and normal extensions

We now recall the definitions of a Galois structure and of a normal extension. We shall restrict ourselves to the case of reflective subcategories [**51**].

Definition 2.1.1. [**47**] A *Galois structure* Γ is a system $(\mathcal{A}, \mathcal{B}, I, H, \eta, \epsilon, \mathcal{E})$ where:

(1) $\mathcal{B}$ is a full replete reflective subcategory of $\mathcal{A}$ with inclusion functor H and left adjoint I, unit η and counit ϵ (which is an isomorphism, see Theorem 1 on page 90 in [**65**] for instance)

$$\mathcal{B} \underset{H}{\overset{I}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{A},$$

(2) $\mathcal{E}$ is a class of morphisms in $\mathcal{A}$, such that:
   (a) $\mathcal{E}$ satisfies (E1)-(E3);
   (b) $HI(\mathcal{E}) \subseteq \mathcal{E}$.

Remark 2.1.2. A full replete $\mathcal{E}$-reflective subcategory $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{E}$ satisfies (E1)-(E4) determines a Galois structure. If $\mathcal{A}$ is pointed, $\mathcal{E}$ satisfies moreover (E6), $\underline{\mathcal{E}}$ satisfies (E1)-(E4), and $\mathcal{B}$ is $\underline{\mathcal{E}}$-reflective, such a Galois structure is associated with an idempotent $\underline{\mathcal{E}}$-closure operator on $\mathcal{M}(\mathcal{E}) = \mathcal{M}(\underline{\mathcal{E}})$ (see Theorem 1.6.10). We shall sometimes use the notation $\Gamma = (\mathcal{A}, \mathcal{B}, I, \eta, \mathcal{E})$ since H and ϵ are completely determined by the above condition (1).

A Galois structure induces an adjunction

$$(I^B, H^B, \eta^B, \epsilon^B) : \mathcal{A} \downarrow B \rightharpoonup \mathcal{B} \downarrow I(B)$$

for any B in $\mathcal{A}$, where

(1) $\mathcal{A} \downarrow B$ is the full subcategory of the slice category $\mathcal{A}/B$ whose objects, the *extensions of* B, are the arrows in $\mathcal{E}$ with codomain B, and $\mathcal{B} \downarrow I(B)$ is the full subcategory of $\mathcal{B}/I(B)$ whose objects are in $\mathcal{E}$; we write (A, f) or $(A, f\colon A \to B)$ to denote an extension of B;
(2) $I^B(A, f : A \to B) = (I(A), I(f) : I(A) \to I(B))$;
(3) $H^B(X, \phi : X \to I(B))$ is defined as the first projection π_1 of the following pullback:

$$\begin{array}{ccc} B \times_{HI(B)} H(X) & \xrightarrow{\pi_2} & H(X) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle H(\phi)} \\ B & \xrightarrow[\eta_B]{} & HI(B); \end{array}$$

(4) $(\eta^B)_{(A,f)} = \langle f, \eta_A \rangle : A \to B \times_{HI(B)} HI(A)$;
(5) $(\epsilon^B)_{(X,\phi)} = \epsilon_X \circ I(\pi_2) : I(B \times_{HI(B)} H(X)) \to IH(X) \to X$.

Definition 2.1.3. A Galois structure is *admissible* when, for any object B in $\mathcal{A}$, ϵ^B is an isomorphism.

The admissibility of the Galois structure can be expressed in several equivalent ways:

- H^B is fully faithful for every $B \in \mathcal{A}$.
- I preserves pullback of the form

$$\begin{array}{ccc} P & \longrightarrow & H(X) \\ \downarrow & & \downarrow{\scriptstyle H(\phi)} \\ B & \xrightarrow[\eta_B]{} & HI(B) \end{array}$$

  where $X \in \mathcal{B}$ and $\phi\colon X \to I(B) \in \mathcal{E}$.

Definition 2.1.4. A morphism $p : E \to B$ in $\mathcal{E}$ is a *monadic extension* (with respect to $\mathcal{E}$) if the pullback functor $p^* : \mathcal{A} \downarrow B \to \mathcal{A} \downarrow E$ is monadic.

Remark 2.1.5. Let $\mathsf{Des}_{\mathcal{E}}(p)$ denote the full subcategory of $\mathsf{Des}(p)$ whose objects are descent data (R, f) such that f_0 and f_1 are in $\mathcal{E}$

$$\begin{array}{ccc} R & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & X \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ \mathsf{Eq}(p) & \overset{\pi_1^p}{\underset{\pi_2^p}{\rightrightarrows}} & E. \end{array}$$

Then K^p restricts to a functor $\mathsf{K}^p_{\mathcal{E}}\colon \mathcal{A} \downarrow B \to \mathsf{Des}_{\mathcal{E}}(p)$. A morphism $p : E \to B$ in $\mathcal{E}$ is a monadic extension if and only if $\mathsf{K}^p_{\mathcal{E}}$ is an equivalence of categories [**3, 55**].

Lemma 2.1.6. **[35]** *Let $\mathcal{E}$ be a class of extensions in a homological category $\mathcal{A}$ and let $p \in \mathcal{E}$ be an effective descent morphism in $\mathcal{A}$. Then p is a monadic extension w.r.t. $\mathcal{E}$ in $\mathcal{A}_{\mathcal{E}}$.*

Proof. In fact, it can be shown, using a modified version of Proposition 1.5.17, that p is an effective descent morphism in $\mathcal{A}_{\mathcal{E}}$. Indeed, if $f\colon A \to B$ is a morphism in $\mathcal{A}$ and

$$\begin{array}{ccc} D & \xrightarrow{p'} & A \\ \downarrow & & \downarrow f \\ E & \xrightarrow[p]{} & B \end{array}$$

is a pullback in $\mathcal{A}$ with $D \in \mathcal{E}^-$, then $\mathsf{Ker}\,(p') = \mathsf{Ker}\,(p) \in \mathcal{E}^-$ by (A2) and p' is in $\mathcal{E}$ by (A6) which incidentally shows that $A \in \mathcal{E}^-$. Now, since p is an effective descent morphism in $\mathcal{A}_{\mathcal{E}}$, the functor

$$\mathsf{K}^p\colon (\mathcal{A}_{\mathcal{E}})/B \to \mathsf{Des}(p)$$

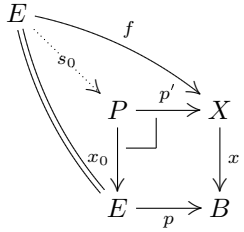
is an equivalence of categories which restricts to another equivalence

$$\mathsf{K}^p_{\mathcal{E}}\colon \mathcal{A} \downarrow B \to \mathsf{Des}_{\mathcal{E}}(p)$$

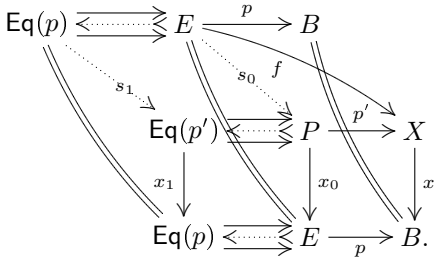
by axioms (A2) and (A4) for $\mathcal{E}$. Thus, by the previous remark, p is monadic w.r.t. $\mathcal{E}$. □

Proposition 2.1.7. *Let $\mathcal{E}$ be a class of extensions in a homological category $\mathcal{A}$ and let us suppose that regular epimorphisms are effective for descent in $\mathcal{A}$. Then the monadic extensions are precisely the morphisms in $\underline{\mathcal{E}}$.*

Proof. Let $p\colon E \to B$ be a morphism in $\mathcal{E}$. If p is a regular epimorphism, it is by assumption a morphism of effective descent and by the previous lemma a monadic extension. Now, as explained for instance in **[32]**, if p is monadic and is viewed as a morphism $p\colon (E,p) \to (B,1_B)$ in $\mathcal{A} \downarrow B$, it is a regular epimorphism in that category. Let us prove this. If $f\colon (E,p) \to (X,x)$ is a morphism in $\mathcal{A} \downarrow B$ such that $f \circ \pi_1^p = f \circ \pi_2^p\colon (\mathsf{Eq}(p), p \circ \pi_1^p) \to (X,x)$, then one has a commutative diagram



and then a diagram



Then since $\mathsf{K}^p_{\mathcal{E}}$ is an equivalence, there exists a morphism $s\colon B \to X$ such that

$$x \circ s = 1_B \quad \text{and} \quad s \circ p = p' \circ s_0 = f.$$

$s\colon (B, 1_B) \to (X, x)$ is the unique morphism such that $f = s \circ p$. Now, since $0\colon B \to 0$ is in $\mathcal{E}$ (as a split epimorphism in $\mathcal{A}_{\mathcal{E}}$), the functor $0^*\colon \mathcal{A}_{\mathcal{E}} \cong \mathcal{A} \downarrow 0 \to \mathcal{A} \downarrow B$ is well defined and has a left adjoint Σ_0 which sends the regular epimorphism $p\colon (E, p) \to (B, 1_B)$ to $p\colon E \to B$ which is thus a regular epimorphism in $\mathcal{A}_{\mathcal{E}}$ and belongs to $\underline{\mathcal{E}}$ (see Proposition 1.4.27). □

Definition 2.1.8. A morphism $f : E \to B$ in $\mathcal{E}$ is said to be a Γ-*trivial extension* if the following commutative square is a pullback:

$$\begin{array}{ccc} E & \xrightarrow{\eta_E} & HI(E) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle HI(f)} \\ B & \xrightarrow[\eta_B]{} & HI(B) \end{array}$$

Notice that, if the Galois structure is admissible, a morphism $f\colon E \to B$ in $\mathcal{E}$ is a trivial extension precisely when it lies in the replete image of the fully faithful functor H^B.

Definition 2.1.9. An extension f is a Γ-*quasi-normal extension* if $f^*(f)$ is a trivial extension.

Definition 2.1.10. A Γ-*normal extension* is a Γ-quasi-normal extension which is monadic w.r.t. $\mathcal{E}$.

Sometimes, we shall speak of ($\mathcal{B}$-)trivial, ($\mathcal{B}$-)quasi-normal, ($\mathcal{B}$-)normal extensions if the Galois structure Γ is understood.

2.2. Derived Galois structures

Until the end of this chapter, $\mathcal{A}$ will always denote a homological category which satisfies (INN) with respect to a (proper) factorization system $\mathcal{S} = (\mathbb{E}, \mathbb{M})$ and $\mathcal{E}$ a class of extensions in $\mathcal{A}$ (possibly different from $\mathbb{E}$).

Definition 2.2.1. An $\mathcal{E}$-*preradical* of the category $\mathcal{A}$ is a normal subfunctor $\mu\colon \mathbf{r} \to 1_{\mathcal{A}_{\mathcal{E}}}$ such that every component of μ is in $\mathcal{M}(\mathcal{E})$.

One can associate with such an $\mathcal{E}$-preradical a short exact sequence of functors

$$0 \longrightarrow \mathbf{r} \xrightarrow{\mu} 1_{\mathcal{A}_\mathcal{E}} \xrightarrow{\eta} I \longrightarrow 0$$

with every component of η in $\mathcal{E}$, and the full replete subcategories of $\mathcal{A}_\mathcal{E}$

$$\mathsf{F}_\mathbf{r} = \{\, X \mid \mathbf{r}(X) = 0\}$$

and

$$\mathsf{T}_\mathbf{r} = \{\, X \mid \mathbf{r}(X) = X\}.$$

Definition 2.2.2. An $\mathcal{E}$-preradical is an *$\mathcal{E}$-radical* if $\mathbf{r}(X/\mathbf{r}(X)) = 0$ for all X in $\mathcal{A}_\mathcal{E}$.

The $\mathcal{E}$-radicals correspond bijectively with the full replete $\underline{\mathcal{E}}$-reflective subcategories of $\mathcal{A}_\mathcal{E}$ via the above correspondence.

Proposition 2.2.3. *From every $\mathcal{E}$-preradical in $\mathcal{A}$ one can obtain a short exact sequence of functors of type* $\mathsf{Ext}_\mathcal{E}\mathcal{A} \to \mathsf{Ext}_\mathcal{E}\mathcal{A}$

$$0 \longrightarrow \mathbf{r}_1 \xrightarrow{\mu^1} 1_{\mathsf{Ext}_\mathcal{E}\mathcal{A}} \xrightarrow{\eta^1} I_1 \longrightarrow 0$$

with every component of η^1 in $\mathcal{E}^1$.

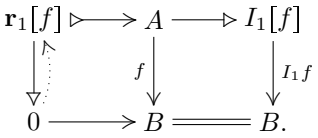
Proof. Let $(\mathsf{Eq}(f), \pi_1, \pi_2)$ be the kernel pair of a morphism $f\colon A \to B$ in $\mathcal{E}$ and define $\mathbf{r}_1[f] = \mathsf{Ker}\,(\mathbf{r}\pi_1)$ and $\mathbf{r}_1(f) = \iota\mathbf{r}_1[f]\colon \mathbf{r}_1[f] \to 0$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{r}_1[f] & \xrightarrow{\ker(\mathbf{r}\pi_1)} & \mathbf{r}\mathsf{Eq}(f) & \underset{\mathbf{r}\pi_2}{\overset{\mathbf{r}\pi_1}{\rightrightarrows}} & \mathbf{r}A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle k} & (4) & \downarrow{\scriptstyle \mu_{\mathsf{Eq}(f)}} & & \downarrow{\scriptstyle \mu_A} & & \\
0 & \longrightarrow & \mathsf{Ker}\, f & \xrightarrow[\ker(\pi_1)]{} & \mathsf{Eq}(f) & \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} & A & \longrightarrow & 0.
\end{array}$$

Let us now define $\mu^1\colon \mathbf{r}_1 \Rightarrow 1_{\mathsf{Arr}\mathcal{A}}$ and $\eta^1\colon 1_{\mathsf{Arr}\mathcal{A}} \Rightarrow I_1$. Simply take as components in an arrow $f\colon A \to B$, $\mu^1_f = (\mu_A \circ \mathbf{r}\pi_2 \circ \mathsf{ker}(\mathbf{r}\pi_1), \alpha_B)$ (where α_B is the unique arrow $0 \to B$) and $\eta^1_f = (\rho^1_f, 1_A) = \mathsf{coker}(\mu^1_f)$. Indeed, μ^1_f is a normal monomorphism. First of all, $\mu_A \circ \mathbf{r}\pi_2 \circ \mathsf{ker}(\mathbf{r}\pi_1)$ is in $\mathbb{M}$. Indeed, $\mu_A \circ \mathbf{r}\pi_2 \circ \mathsf{ker}(\mathbf{r}\pi_1) = \pi_2 \circ \mathsf{ker}(\pi_1) \circ k$, and $\pi_2 \circ \mathsf{ker}(\pi_1)$ and k are in $\mathbb{M}$ since they are kernels of f and $\eta_{\mathsf{Eq}(f)} \circ \mathsf{ker}(\pi_1)$ respectively because (4) is a pullback. For this last reason, $\mathsf{ker}(\pi_1) \circ k$ is also a normal monomorphism and then $\mu_A \circ \mathbf{r}\pi_2 \circ \mathsf{ker}(\mathbf{r}\pi_1) = (\pi_2)_\mathcal{S}(\mathsf{ker}(\pi_1) \circ k)$

$$\begin{array}{ccc}
\mathbf{r}_1[f] & = & \mathbf{r}_1[f] \\
{\scriptstyle \mathsf{ker}(\pi_1)\circ k}\big\downarrow & & \big\downarrow{\scriptstyle \mu_A\circ\mathbf{r}\pi_2\circ\mathsf{ker}(\mathbf{r}\pi_1)\in\mathbb{M}} \\
\mathsf{Eq}(f) & \xrightarrow[\pi_2]{} & A,
\end{array}$$

is a normal monomorphism (it is the $\mathcal{S}$-image of the normal monomorphism $\mathsf{ker}(\pi_1) \circ k$ by π_2). Since $\mathbf{r}_1[f]$ is in $\mathcal{E}^-$, $\mathbf{r}_1(f)$ is in $\mathcal{E}$ (Remark 1.4.2) and $\eta^1_f\colon f \to I_1 f$ is in $\mathcal{E}^1$:



□

From now on, we shall denote the normal monomorphism $\mu_A \circ \mathbf{r}\pi_2 \circ \mathsf{ker}(\mathbf{r}\pi_1)$ by ν_f^1.

Remark 2.2.4. An $\mathcal{E}$-radical $\mathbf{r}$ of $\mathcal{A}$ induces a Galois structure

$$\Gamma_{\mathbf{r}} = (\mathcal{A}_{\mathcal{E}}, \mathsf{F}_{\mathbf{r}}, I, \eta, \mathcal{E}).$$

Using Lemma 1.3.15, one sees that

$$\mathsf{F}_{\mathbf{r}_1} = \mathsf{QNExt}_{\Gamma_{\mathbf{r}}}\mathcal{A}$$

where $\mathsf{QNExt}_{\Gamma_{\mathbf{r}}}\mathcal{A}$ is the full replete subcategory of the category of extensions $\mathsf{Ext}_{\mathcal{E}}\mathcal{A}$ whose objects are the $\Gamma_{\mathbf{r}}$-quasi-normal extensions.

In the rest of this chapter, we shall be interested in different types of $\mathcal{E}$-radicals which give rise to Galois structures

$$\Gamma_{\mathbf{r}_1} = (\mathsf{Ext}_{\mathcal{E}}\mathcal{A}, \mathsf{QNExt}_{\Gamma_{\mathbf{r}}}, I_1, \eta^1, \mathcal{E}^1) :$$

this essentially means that the preradical $\mathbf{r}_1$ is also a radical.

Let $\mathcal{C}, \mathcal{D}$ be pointed protomodular categories.

Definition 2.2.5. **[34]** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *protoadditive* if it preserves *split short exact sequences*: if one has a split short exact sequence in $\mathcal{C}$

$$0 \longrightarrow \mathsf{Ker}\,(p) \overset{\mathsf{ker}\,(p)}{\rightarrowtail} A \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

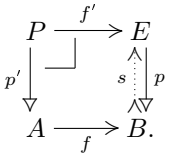
(s is a section of p), then its image by F is also a split short exact sequence in $\mathcal{D}$:

$$0 \longrightarrow F(\mathsf{Ker}\,(p)) \overset{F(\mathsf{ker}\,(p))}{\rightarrowtail} F(A) \underset{F(s)}{\overset{F(p)}{\rightleftarrows}} F(B) \longrightarrow 0$$

This notion generalises the one of additive functor (see Proposition 1.5.20). For $\mathcal{E}$ a class of extensions in $\mathcal{A}$, $\mathcal{A}_{\mathcal{E}}$ is pointed protomodular, so that it makes sense to speak of *protoadditive $\mathcal{E}$-preradical*. One has the following

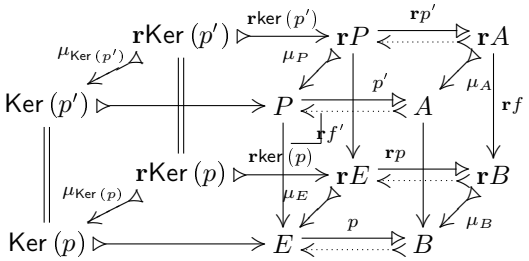
Lemma 2.2.6. **[35]** *Let $\mathbf{r}$ be an $\mathcal{E}$-preradical. Then it preserves 0 and the following are equivalent:*

(1) $\mathbf{r}$ *is protoadditive;*
(2) $\mathbf{r}$ *preserves pullbacks in $\mathcal{A}_{\mathcal{E}}$ of the form*



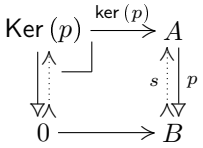
where s is a section of p.

PROOF. (1) ⇒ (2): Let us take such a pullback. Then, one has a commutative diagram

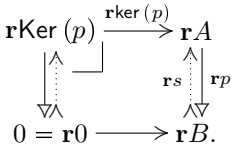


where $\mathbf{r}\mathsf{ker}\,(p') = \mathsf{ker}\,(\mathbf{r}p')$ and $\mathbf{r}\mathsf{ker}\,(p) = \mathsf{ker}\,(\mathbf{r}p)$, so that the right back face is of course a pullback (by Lemma 1.3.15).

(2) ⇒ (1): A short exact sequence can be presented as a pullback



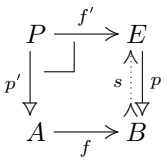
which is sent by $\mathbf{r}$ to a pullback



Then $\mathbf{r}\mathsf{ker}\,(p) = \mathsf{ker}\,(\mathbf{r}p)$ and, since $\mathbf{r}p$ is a split epimorphism, it is the cokernel of its kernel: $\mathbf{r}p = \mathsf{coker}\,(\mathbf{r}\mathsf{ker}\,(p))$ ($\mathcal{A}$ is protomodular). □

Let $\mathbf{r}$ be an $\mathcal{E}$-preradical of $\mathcal{A}$.

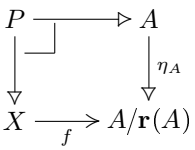
Definition 2.2.7. $\mathbf{r}$ is *$\mathcal{E}$-protoadditive* if it preserves the pullbacks of the form



(2.1)

where f is in $\mathcal{E}$, $E \in \mathcal{E}^-$ and s a section of p.

Definition 2.2.8. $\mathbf{r}$ is *admissible* if it preserves the pullbacks of the form



(2.2)

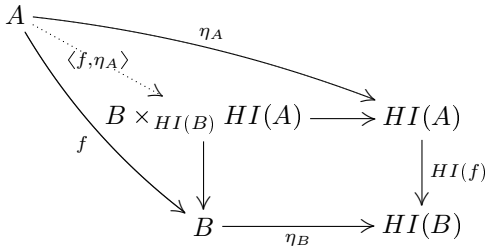
where $X \in \mathsf{F}_{\mathbf{r}}$ and $f \in \mathcal{E}$.

Let us introduce a definition inspired by **[31, 36]**. We suppose here that $\mathcal{E} \subseteq \mathsf{Epi}(\mathcal{A})$.

Definition 2.2.9. A full replete reflective subcategory $\mathcal{B}$ of $\mathcal{A}_{\mathcal{E}}$

$$\mathcal{B} \underset{H}{\overset{I}{\underset{\longleftarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{A}_{\mathcal{E}}$$

is a *strongly $\mathcal{E}$-Birkhoff subcategory* of $\mathcal{A}$ if, for every morphism $f\colon A \to B$ in $\mathcal{E}$, all arrows in the diagram



are in $\mathcal{E}$ and the components of η in $\underline{\mathcal{E}}$. One says in that case that I is an *$\mathcal{E}$-Birkhoff reflector*.

One gives now a definition also inspired by **[18, 25]**.

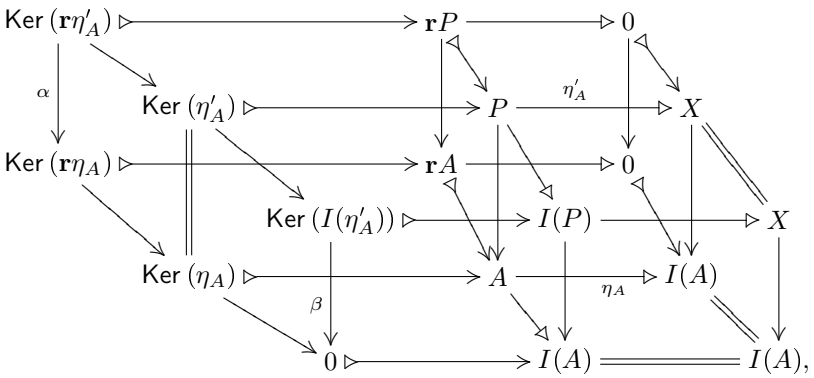
Definition 2.2.10. If $\mathcal{E} \subseteq \mathsf{Epi}(\mathcal{A})$, one says that an $\mathcal{E}$-preradical $\mathbf{r}$ is *$\mathcal{E}$-Birkhoff* if it preserves morphisms in $\mathcal{E}$: if $f\colon A \to B$ is in $\mathcal{E}$, so is $\mathbf{r}f\colon \mathbf{r}A \to \mathbf{r}B$.

One can now prove the following

Proposition 2.2.11. *For $\mathbf{r}$ a given $\mathcal{E}$-radical, the following properties hold*

(1) $\mathbf{r}$ *preserves pullbacks of the form (2.1) if and only if I preserves them;*
(2) $\mathbf{r}$ *is admissible if and only if I is admissible;*
(3) $\mathbf{r}$ *is protoadditive if and only if I is protoadditive;*
(4) $\mathbf{r}$ *is $\mathcal{E}$-Birkhoff if and only if I is $\mathcal{E}$-Birkhoff.*

Proof. (2): Let us take a pullback of the form (2.2). By the 3×3-lemma, the two left downward columns in the commutative diagram



are short exact sequences, so that we have, using Lemma 1.3.15, that $\mathbf{r}$ is admissible if and only if α is an isomorphism if and only if β is an isomorphism if and only if I is admissible. (1) can be proved in a very similar way.

(3) is proved as in **[35]**. If one starts from a split short exact sequence

$$0 \longrightarrow \mathsf{Ker}\,(p) \overset{\ker(p)}{\rightarrowtail} A \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

in $\mathcal{A}_{\mathcal{E}}$, one has a commutative diagram of the form

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathbf{r}\mathsf{Ker}\,(p) & \rightarrowtail & \mathbf{r}A & \underset{\mathbf{r}s}{\overset{}{\rightleftarrows}} & \mathbf{r}B & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{Ker}\,(p) & \overset{\ker(p)}{\rightarrowtail} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & I(\mathsf{Ker}\,(p)) & \rightarrowtail & I(A) & \rightleftarrows & I(B) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in $\mathcal{A}$. Then by the 3×3-lemma, $\mathbf{r}$ is protoadditive if and only if I is protoadditive.

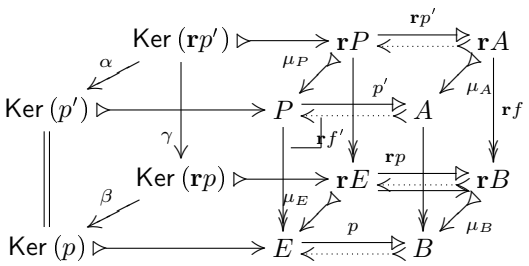
(4): Let us consider a morphism $f\colon A \to B$ in $\mathcal{E} \subseteq \mathsf{Epi}(\mathcal{A})$ and the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{r}(A) & \rightarrowtail & A & \overset{\eta_A}{\twoheadrightarrow} & I(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathbf{r}f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle I(f)} & & \\
0 & \longrightarrow & \mathbf{r}(B) & \rightarrowtail & B & \overset{\eta_B}{\twoheadrightarrow} & I(B) & \longrightarrow & 0.
\end{array}$$

Then by Lemma 1.4.7, $\mathbf{r}f$ is in $\mathcal{E}$ if and only if the right square is a double extension. □

Proposition 2.2.12. *If $\mathbf{r}$ is an $\mathcal{E}$-Birkhoff preradical then it preserves the pullback of the form (2.1).*

PROOF. Let us consider a pullback of the form (2.1). Then one can construct a commutative diagram



where α, β are normal monomorphisms and then γ is a normal monomorphism. Since $\mathbf{r}$ is $\mathcal{E}$-Birkhoff, $\mathbf{r}f'$ and $\mathbf{r}f$ are in $\mathcal{E}$ and the right back face is in $\mathcal{E}^1$, so that γ is in $\mathcal{E} \subseteq \mathsf{Epi}(\mathcal{A})$. Then γ is an isomorphism and the back right face is in fact a pullback (by Lemma 1.3.15). □

The previous result shows that the notion of $\mathcal{E}$-protoadditive $\mathcal{E}$-radical can be seen as a common generalization of both the notions of protoadditive radical and of $\mathcal{E}$-Birkhoff radical.

Proposition 2.2.13. *Let $\mathcal{E} \subseteq \mathsf{Epi}(\mathcal{A})$. Then if I is an $\mathcal{E}$-Birkhoff reflector, I is also admissible.*

PROOF. The admissibility can be proved following the same lines as in [**36**] Proposition 2.6. Let B be an object in $\mathcal{A}$, one has, using Proposition 1.5.15, that η^*_B and then H^B reflect isomorphisms. For $\phi \in \mathcal{B} \downarrow I(B)$, it follows, by the triangular identity $(H^B(\epsilon^B_\phi)) \circ \eta^B_{H^B(\phi)} = 1_{H^B(\phi)}$ and because $\eta^B_f = \langle f, \eta_A \rangle \colon f \to H^B I^B(f)$ is an epimorphism for all $f \in \mathcal{A} \downarrow B$, that $\eta^B_{H^B(\phi)}$ is an isomorphism and so is $H^B(\epsilon^B_\phi)$. The functor H^B being conservative, the result follows. □

Remark 2.2.14. A protoadditive reflector is not necessarily admissible. The following example has been given in [**35**]: The reflector $I \colon \mathsf{Ab} \to \mathcal{F}$, where $\mathcal{F}$ is the quasi-variety of abelian groups which satisfy the axiom $4x = 0 \Rightarrow 2x = 0$, is additive and thus protoadditive. But it is not admissible with respect to surjective homomorphism. To see this, it suffices to note that the pullback diagram

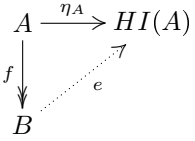
$$\begin{array}{ccc} C_4 \times_{C_2} \mathbb{Z} & \longrightarrow & C_4 \\ \downarrow & & \downarrow{\scriptstyle \eta_{C_4}} \\ \mathbb{Z} & \longrightarrow & C_2, \end{array}$$

where C_n is the cyclic group of order n, is not preserved by I.

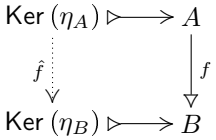
For the next main theorem, we shall need the following lemmas

Lemma 2.2.15. [**29**] (1) *Let $\mathcal{C}$ be any category and $\mathcal{D}$ be a reflective subcategory of $\mathcal{C}$ with reflector $I \colon \mathcal{C} \to \mathcal{D}$ and inclusion functor $H \colon \mathcal{D} \to \mathcal{C}$. If $f \colon A \to B$ is*

an epimorphism such that the unit η_A factors through f

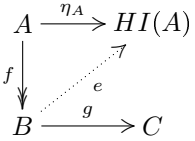


then the factorisation e is necessarily the unit $\eta_B\colon B \to HI(A) \cong HI(B)$.
(2) *In particular, in the case of a pointed category $\mathcal{C}$ with kernels, if $f\colon A \to B$ is a normal epimorphism such that $\mathsf{Ker}\,(f) \leqslant \mathsf{Ker}\,(g)$ for some $g \in I^{-1}(\mathsf{Iso})$ (i.e. $I(g)$ is an isomorphism), it follows that the induced commutative square*



is a pullback.

Proof. (1): Let us consider a commutative diagram

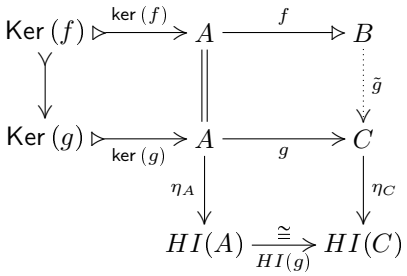


in $\mathcal{C}$ with f an epimorphism and $C \in \mathcal{D}$. Then for any morphism $x\colon HI(A) \to C$

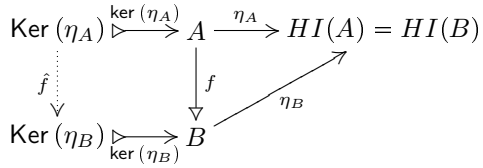
$$g = x \circ e \Leftrightarrow g \circ f = x \circ \eta_A = x \circ e \circ f$$

and the universality of $(I(A), \eta_A)$ provides us a unique morphism $\alpha\colon HI(A) \to C$ such that $g = \alpha \circ e$.

(2): Let us start with a normal epimorphism $f\colon A \to B$ such that there exists an inverted morphism $g\colon A \to C$ such that $\mathsf{Ker}\,(f) \leqslant \mathsf{Ker}\,(g)$. Then one has a commutative diagram

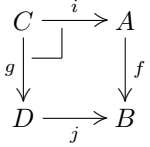


and η_A factors as $e \circ f$ for $e = HI(g)^{-1} \circ \eta_C \circ \tilde{g}$. By (1), e is the unit η_B and one has a commutative diagram



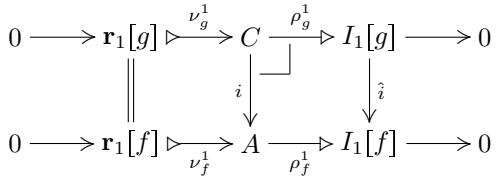
in which the left square is a pullback. □

Lemma 2.2.16. *If* $\mathbf{r}$ *is an* $\mathcal{E}$*-protoadditive* $\mathcal{E}$*-radical and we have a pullback diagram of the form*

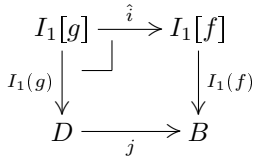


(2.3)

where all morphisms are in $\mathcal{E}$*, then one has a commutative morphism of short exact sequences*

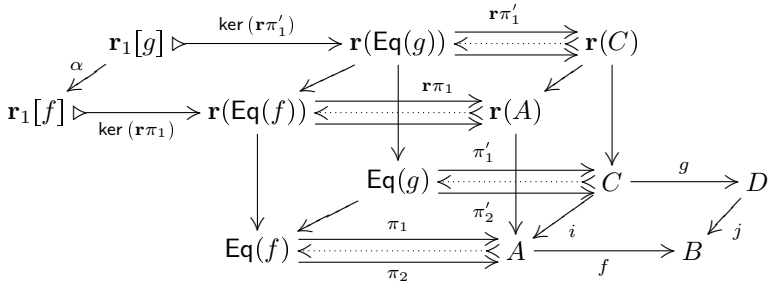


which also gives a pullback diagram

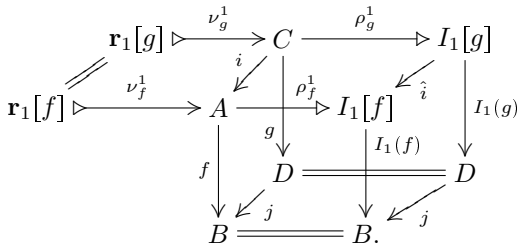


of the type (2.3) *(all morphisms in it are in* $\mathcal{E}$*). Then one has* $\mathbf{r}_1[f] = 0$ *if and only if* $\mathbf{r}_1[g] = 0$*, and* $\mathbf{r}_1[I_1(f)] = 0$ *if and only if* $\mathbf{r}_1[I_1(g)] = 0$*.*

Proof. The morphism α in the following diagram



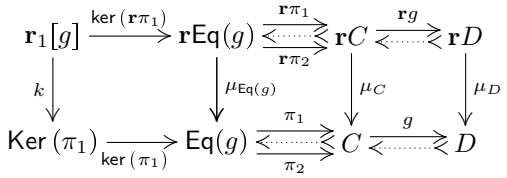
is an isomorphism since the square linking π_1 and π_1' is a pullback of the form (2.1). This gives a morphism of short exact sequence as claimed, so that we find a commutative diagram



Since ρ_f^1 is a regular epimorphism, using Proposition 1.5.15, one concludes the proof. □

Lemma 2.2.17. *If* $\mathbf{r}$ *is an* $\mathcal{E}$*-protoadditive* $\mathcal{E}$*-radical and* $g\colon C \to D$ *is a split epimorphism in* $\mathcal{E}$*, then* $\mathbf{r}_1[g] = \ker(\mathbf{r}g)$ *and* $\nu_g^1 = \mu_C \circ \ker(\mathbf{r}g)$.

Proof. Since $\mathbf{r}$ preserves pullbacks of the form (2.1), $(\mathbf{r}\mathsf{Eq}(g), \mathbf{r}\pi_1, \mathbf{r}\pi_2)$ is the kernel pair of $\mathbf{r}g$:

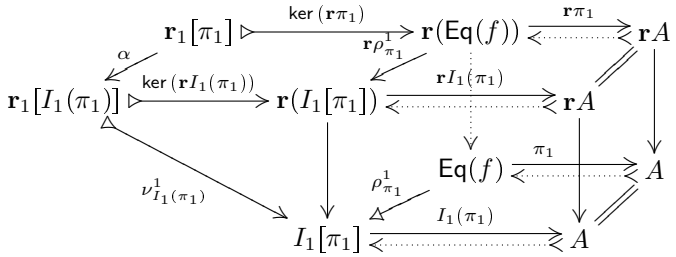


and then $\mathbf{r}\pi_2 \circ \ker(\mathbf{r}\pi_1) = \ker(\mathbf{r}g)$. □

Theorem 2.2.18. *Let* $\mathbf{r}$ *be an* $\mathcal{E}$*-preradical.*

(1) *If* $\mathbf{r}$ *is an* $\mathcal{E}$*-protoadditive* $\mathcal{E}$*-radical, then* $\mathbf{r}_1$ *is an* $\mathcal{E}^1$*-protoadditive* $\mathcal{E}^1$*-radical;*
(2) *if* $\mathbf{r}$ *is* $\mathcal{E}$*-Birkhoff then it is an* $\mathcal{E}$*-radical, and* $\mathbf{r}_1$ *is an* $\mathcal{E}^1$*-Birkhoff radical.*

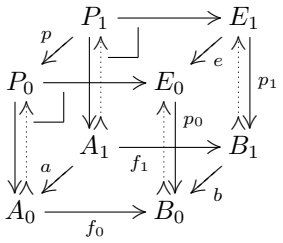
Proof. (1): First, let us prove that $\mathbf{r}_1$ is a radical. We have to show that $\mathbf{r}_1[I_1(f)] = 0$ for any $f \in \mathcal{E}$. For this, it suffices to show that $\mathbf{r}_1[I_1(\pi_1)] = 0$ by Lemma 2.2.16. So let us consider the following commutative diagram



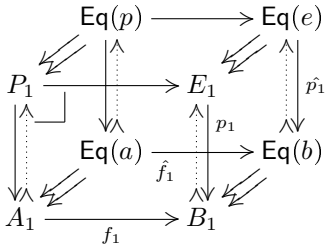
Since $\rho_{\pi_1}^1$ is inverted by I (see Lemma 2.2.15), one has that $\mathbf{r}\rho_{\pi_1}^1$ is a regular epimorphism by pullback stability of regular epimorphism and then also α is a

regular epimorphism, again by pullback stability. Thus one has $\mathbf{r}_1[I_1(\pi_1)] = 0$ since $\nu^1_{I_1(\pi_1)} \circ \alpha = \rho^1_{\pi_1} \circ \mu_{\mathsf{Eq}(f)} \circ \ker(\mathbf{r}\pi_1) = 0$ by definition of $\rho^1_{\pi_1}$.

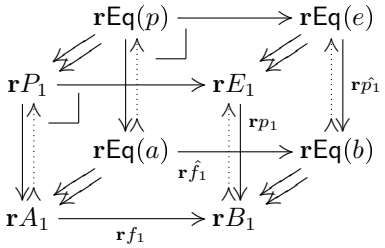
Now, let us consider a pullback



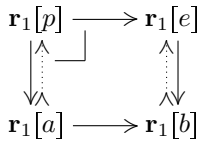
with $f = (f_1, f_0) \in \mathcal{E}^1$, $e \in \mathcal{E}$ and $p = (p_0, p_1)$ a split epimorphism. If one takes kernel pairs, then by commutation of limits and by Lemma 1.4.25, one finds a diagram



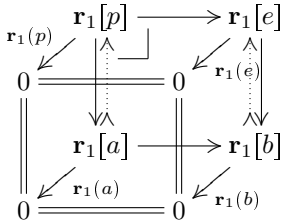
where the front and back faces are pullbacks of the form (2.1). Then applying $\mathbf{r}$ to this diagram one obtains a cube



Consequently, one finds a pullback



and then a cube



which is a pullback.

(2): Let us show that the $\mathcal{E}$-preradical $\mathbf{r}$ is an $\mathcal{E}$-radical. Let A be an object in $\mathcal{A}_\mathcal{E}$. If we consider the following commutative diagram:

$$\begin{array}{ccc} \mathbf{r}A & \xrightarrow{\mathbf{r}\eta_A} & \mathbf{r}I(A) \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_{I(A)}} \\ A & \xrightarrow[\eta_A]{} & I(A) \end{array}$$

we see that, since $\eta \in \mathcal{E}$ and $\mathbf{r}$ is $\mathcal{E}$-Birkhoff, $\mathbf{r}\eta_A$ is in $\mathcal{E}(\subseteq \mathsf{Epi}(\mathcal{A}))$ and $\mathbf{r}I(A) = 0$. So, to prove (2), it remains to show now that $\mathbf{r}$ is $\mathcal{E}^1$-Birkhoff. Let $f = (f_1, f_0)\colon a \to b$ be a morphism in $\mathcal{E}^1$, $(\mathsf{Eq}(a), \pi_1^a, \pi_2^a)$ the kernel pair of a and $(\mathsf{Eq}(b), \pi_1^b, \pi_2^b)$ the kernel pair of b. Since $\mathbf{r}$ preserves the morphisms in $\mathcal{E}$, both $\mathbf{r}f_1$ and $\mathbf{r}\hat{f}_1$ ($\hat{f}_1$ is the restriction of f_1 to the kernels pairs of a and b) are in $\mathcal{E}$. It follows that the right-hand (solid) square in the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{r}_1[b] & \rhd\!\!\longrightarrow & \mathbf{r}\mathsf{Eq}(b) & \underset{\mathbf{r}\pi_1^b}{\rightleftarrows} & \mathbf{r}B_1 & \longrightarrow & 0 \\ & & {\scriptstyle \tilde{f}}\downarrow & & {\scriptstyle \mathbf{r}\hat{f}_1}\downarrow & & \downarrow{\scriptstyle \mathbf{r}f_1} & & \\ 0 & \longrightarrow & \mathbf{r}_1[a] & \rhd\!\!\longrightarrow & \mathbf{r}\mathsf{Eq}(a) & \underset{\mathbf{r}\pi_1^a}{\rightleftarrows} & \mathbf{r}A_1 & \longrightarrow & 0 \end{array}$$

is in $\mathcal{E}^1$, and then that the left-hand downward pointing arrows is in $\mathcal{E}$ (Lemma 1.4.7). Hence $\mathbf{r}_1(f)$:

$$\begin{array}{ccc} \mathbf{r}_1[b] & \rightleftarrows & 0 \\ {\scriptstyle \tilde{f}}\downarrow & & \| \\ \mathbf{r}_1[a] & \rightleftarrows & 0 \end{array}$$

is in $\mathcal{E}^1$. □

Corollary 2.2.19. *If an $\mathcal{E}$-radical $\mathbf{r}$ is $\mathcal{E}$-protoadditive, the full replete subcategory $\mathsf{F}_{\mathbf{r}_1} = \mathsf{QNExt}_\mathcal{E}\mathcal{A}$ of $\mathsf{Ext}_\mathcal{E}\mathcal{A}$ is reflective and pointed protomodular.* □

Let $\mathbf{r}$ be an $\mathcal{E}$-protoadditive $\mathcal{E}$-radical. We shall write $\mathbf{r}_0 = \mathbf{r}$ and $I_0 = I$. We have induced short exact sequences

$$0 \longrightarrow \mathbf{r}_n \xrightarrow{\mu^n} 1_{\mathsf{ExCub}^n_\mathcal{E}\mathcal{A}} \xrightarrow{\eta^n} I_n \longrightarrow 0$$

obtained using the isomorphism δ_{n-1}:

$$\begin{array}{ccc} \mathsf{ExCub}^n_\mathcal{E}\mathcal{A} & \xrightarrow{\mathbf{r}_n} & \mathsf{ExCub}^n_\mathcal{E}\mathcal{A} \\ {\scriptstyle \delta_{n-1}}\downarrow & & \downarrow{\scriptstyle \delta_{n-1}} \\ \mathsf{Ext}_{\mathcal{E}^{n-1}}\mathsf{Cub}^{n-1}\mathcal{A} & \xrightarrow[(\mathbf{r}_{n-1})_1]{} & \mathsf{Ext}_{\mathcal{E}^{n-1}}\mathsf{Cub}^{n-1}\mathcal{A}. \end{array}$$

Let $\mathcal{O}^0 = \mathcal{O}$ be such that $(\mathcal{A}, \mathcal{E}, \mathcal{O}, \mathbf{r})$ satisfies (O1) and

(O2) $\mathbf{r}A \in \mathcal{O}$ for all $A \in \mathcal{A}_\mathcal{E}$.

By induction one can show that $\mathbf{r}_n A \in \mathcal{O}^n$ for all A in $\mathsf{ExCub}^n_{\mathcal{E}}\mathcal{A}$. The functor $[-]_{n,\mathbf{r}} \colon \mathsf{ExCub}^n_{\mathcal{E}}\mathcal{A} \to \mathcal{A}_{\mathcal{E}}$ will denote the factorisation of $\mathbf{r}_n$ through ι^n. We shall sometimes just write $[A]_n$ when it is clear where A lies, and $[A]$ if A is in $\mathcal{A}_{\mathcal{E}}$.

Definition 2.2.20. For $n \geqslant 1$, an $\mathcal{E}$-exact n-cube A in $\mathsf{ExCub}^n_{\mathcal{E}}(\mathcal{A})$ is called a *quasi-normal $\mathcal{E}$-exact n-cube* if $\delta_{n-1}A$ is $\Gamma_{\mathbf{r}_{n-1}}$-quasi-normal.

If in the above diagrams, one writes δ_i instead of δ_{n-1}, the diagram still commutes. This will ensure symmetry of quasi-normal extensions in all degrees. This fact was first proved in Theorem 3.3 in [**31**] in a stricter context, but a careful inspection of the proof shows that it only relies on symmetry of higher extensions (see Theorem 1.4.17) and commutation of limits.

Theorem 2.2.21. *Let $n \geqslant 1$. For any $0 \leqslant i \leqslant n-1$ and A an $\mathcal{E}$-exact n-cube in $\mathcal{A}$, we have*

$$\delta_i \circ \mathbf{r}_n = (\mathbf{r}_{n-1})_1 \circ \delta_i;$$

and

$$\iota^{n-1}[A]_n = (\mathbf{r}_{n-1})_1[\delta_i A]$$

as objects. Consequently, the following conditions are equivalent

(1) *A is a quasi-normal n-cube.*
(2) *$\delta_i A$ is a $\Gamma_{\mathbf{r}_{n-1}}$-quasi-normal extension.*

Proof. We only have to check the first claim. We shall prove by induction that $(\mathbf{r}_{n-1})_1[\delta_i A] = (\mathbf{r}_{n-1})_1[\delta_j A]$ for every $n \geqslant 0$ and $0 \leqslant i < j \leqslant n-1$. For $n = 1$, there is nothing to prove. So, let $n \geqslant 2$, $0 \leqslant i < j \leqslant n-1$ and let us suppose that the property holds for $n-1$. We have to compare the top left objects in the constructions

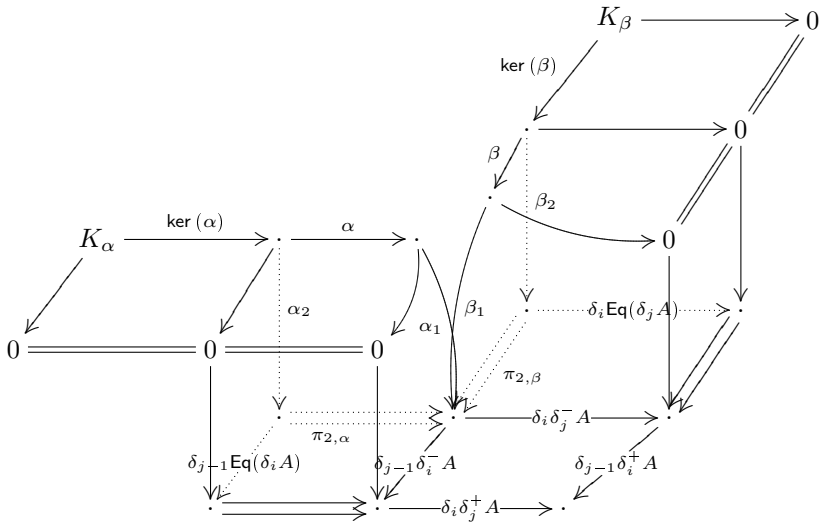
$$\begin{array}{ccccc}
(\mathbf{r}_{n-1})_1[\delta_i A] & \xrightarrow{\ker(\mathbf{r}_{n-1}\pi_1^i)} & \mathbf{r}_{n-1}\mathsf{Eq}(\delta_i A) & \overset{\mathbf{r}_{n-1}\pi_1^i}{\underset{\mathbf{r}_{n-1}\pi_2^i}{\rightrightarrows}} & \mathbf{r}_{n-1}\delta_i^- A \\
 & & \downarrow{\scriptstyle \mu^{n-1}_{\mathsf{Eq}(\delta_i A)}} & & \downarrow{\scriptstyle \mu^{n-1}_{\delta_i^- A}} \\
 & & \mathsf{Eq}(\delta_i A) & \overset{\pi_1^i}{\underset{\pi_2^i}{\rightrightarrows}} & \delta_i^- A \xrightarrow{\delta_i A} \delta_i^+ A
\end{array}$$

and

$$\begin{array}{ccccc}
(\mathbf{r}_{n-1})_1[\delta_j A] & \xrightarrow{\ker(\mathbf{r}_{n-1}\pi_1^j)} & \mathbf{r}_{n-1}\mathsf{Eq}(\delta_j A) & \overset{\mathbf{r}_{n-1}\pi_1^j}{\underset{\mathbf{r}_{n-1}\pi_2^j}{\rightrightarrows}} & \mathbf{r}_{n-1}\delta_j^- A \\
 & & \downarrow{\scriptstyle \mu^{n-1}_{\mathsf{Eq}(\delta_j A)}} & & \downarrow{\scriptstyle \mu^{n-1}_{\delta_j^- A}} \\
 & & \mathsf{Eq}(\delta_j A) & \overset{\pi_1^j}{\underset{\pi_2^j}{\rightrightarrows}} & \delta_j^- A \xrightarrow{\delta_j A} \delta_j^+ A,
\end{array}$$

so that we have to know what $\ker(\mathbf{r}_{n-1}\pi_1^i)$ and $\ker(\mathbf{r}_{n-1}\pi_1^j)$ are. By induction, $\mathbf{r}_{n-1}$ is defined via $\mathbf{r}_{n-2}$ choosing any one of the functors δ_k. So let us apply δ_{j-1} to the first diagram and δ_i to the second one. We arrive at the following

diagram

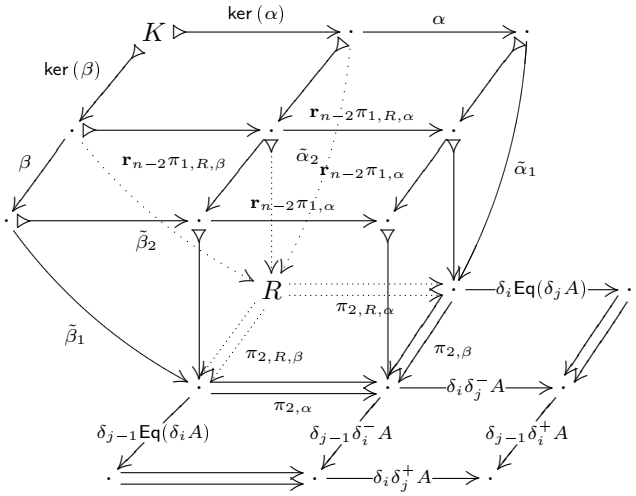


where

- $\alpha_1 = \nu^1_{\delta_{j-1}\delta_i^- A}$
- $\alpha_2 = \nu^1_{\delta_{j-1}\mathsf{Eq}(\delta_i A)}$
- $(\alpha, 0) = \delta_{j-1}\mathbf{r}_{n-1}\pi_1^i$
- $\beta_1 = \nu^1_{\delta_i\delta_j^- A}$
- $\beta_2 = \nu^1_{\delta_i\mathsf{Eq}(\delta_j A)}$
- $(\beta, 0) = \delta_i\mathbf{r}_{n-1}\pi_1^j$
- $(K_\alpha, 0) = \delta_{j-1}(\mathbf{r}_{n-1})_1[\delta_i A]$
- $(K_\beta, 0) = \delta_i(\mathbf{r}_{n-1})_1[\delta_j A]$

and $(\pi_{1,\beta}, \pi_{2,\beta})$, $(\pi_{1,\alpha}, \pi_{2,\alpha})$ are the kernel pairs of $\delta_{j-1}\delta_i^- A$ and $\delta_i\delta_j^- A$ respectively. One can prove that $K_\alpha = K_\beta$. Of course, by commutation of limits, $\delta_{j-1}\mathsf{Eq}(\delta_i A)$ and $\delta_i\mathsf{Eq}(\delta_j A)$ have kernel pairs $(R_\beta, \pi_{1,R,\beta}, \pi_{2,R,\beta})$ and $(R_\alpha, \pi_{1,R,\alpha}, \pi_{2,R,\beta})$ with the same object part $R_\beta = R_\alpha$. Let us denote it by R

as in the following diagram



So, again by commutation of limits, α and β have the same kernel K. Finally, one also has that

$$\begin{aligned}\pi_{2,\alpha} \circ \alpha_2 \circ \mathsf{ker}\,(\alpha) &= \pi_{2,\alpha} \circ \pi_{2,R\beta} \circ \tilde{\alpha}_2 \circ \mathsf{ker}\,(\alpha) \\ &= \pi_{2,\beta} \circ \pi_{2,R\alpha} \circ \tilde{\beta}_2 \circ \mathsf{ker}\,(\beta) \\ &= \pi_{2,\beta} \circ \beta_2 \circ \mathsf{ker}\,(\beta). \qquad \square\end{aligned}$$

Using Theorem 2.2.21, one can prove the following

Theorem 2.2.22. *For any $n \geqslant 1$ and $0 \leqslant i \leqslant n-1$, the square*

$$\begin{array}{ccc} \mathsf{ExCub}^n_{\mathcal{E}}\mathcal{A} & \xrightarrow{\mathbf{r}_n} & \mathsf{ExCub}^n_{\mathcal{E}}\mathcal{A} \\ \downarrow{\scriptstyle \rho_i} & & \downarrow{\scriptstyle \rho_i} \\ \mathsf{ExCub}^{n-1}_{\mathcal{E}^1}\mathsf{Arr}\mathcal{A} & \xrightarrow[(\mathbf{r}_1)_{n-1}]{} & \mathsf{ExCub}^{n-1}_{\mathcal{E}^1}\mathsf{Arr}\mathcal{A} \end{array}$$

commutes and $\iota^1[P]_n = [\rho_i P]_{n-1}$ for any P in $\mathsf{ExCub}^n_{\mathcal{E}}\mathcal{A}$. $\square$

2.3. Special derived Galois structures

We are going to study a special kind of adjunction:

$$\mathcal{F} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{B} \underset{H}{\overset{I}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{A}_{\mathcal{E}} \tag{2.4}$$

where $\mathcal{A}$ is a homological category which satisfies the property (INN), $\mathcal{E}$ a class of extensions in $\mathcal{A}$, $\mathcal{B}$ a full replete $\underline{\mathcal{E}}$-reflective subcategory of $\mathcal{A}_{\mathcal{E}}$ with I preserving pullbacks of the form (2.1), and $\mathcal{F}$ an $\underline{\mathcal{E}}$-reflective subcategory of $\mathcal{B}$ with the property that the reflector $F\colon \mathcal{B} \to \mathcal{F}$ is *protoadditive*.
The functor $F \circ I$ is left adjoint to $H \circ U$. We shall write η and ϵ for the unit and the counit of this composite adjunction. The unit and the counit of the adjunction $I \dashv H$ will be denoted by η^1 and ϵ^1, whereas the unit and the counit of the adjunction $F \dashv U$ will be denoted η^2 and ϵ^2, respectively. This kind

of adjunction gives a Galois structure $\Gamma = (\mathcal{A}_{\mathcal{E}}, \mathcal{F}, F \circ I, \eta, \mathcal{E})$. The following theorem is an adaptation of Theorem 6.2 in [**29**]

Theorem 2.3.1. *Let $f : A \to B$ be in $\mathcal{E}$. The following conditions are equivalent:*

(1) *f is an $\mathcal{F}$-quasi-normal extension;*
(2) *f is $\mathcal{B}$-quasi-normal and $\mathsf{Ker}\,(f) \in \mathcal{F}$.*

Proof. (1) $\Rightarrow$ (2): Let us assume that f is a $\mathcal{F}$-quasi-normal extension. Then in the following commutative diagram the composite of the left pointing squares is a pullback:

$$\begin{array}{ccccccc}
FI(A \times_B A) & \longleftarrow & I(A \times_B A) & \longleftarrow & E \times_B A & \longrightarrow & A \\
{\scriptstyle FI(\pi_1)}\downarrow & & {\scriptstyle I(\pi_1)}\downarrow & & {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\
FI(A) & \longleftarrow & I(A) & \longleftarrow & A & \xrightarrow[f]{} & B
\end{array}$$

Since the middle square is in $\underline{\mathcal{E}}^1$, this implies that this square is, in fact, a pullback (see Lemma 1.1 in [**39**]), and we find that f is a $\mathcal{B}$-quasi-normal extension. The fact that $\mathsf{Ker}\,(f)$ lies in $\mathcal{F}$ for any $\mathcal{F}$-quasi-normal extension f is a very general and well-known fact. It suffices to note that the above pullbacks induce the isomorphisms

$$\mathsf{Ker}\,(FI(\pi_1)) \cong \mathsf{Ker}\,(\pi_1) \cong \mathsf{Ker}\,(f),$$

which imply that $\mathsf{Ker}\,(f) \in \mathcal{F}$, as (the domain of) the kernel of a morphism in $\mathcal{F}$.

(2) $\Rightarrow$ (1): Suppose now that f is $\mathcal{B}$-quasi-normal and $\mathsf{Ker}\,(f) \in \mathcal{F}$. Then one sees that in the commutative diagram

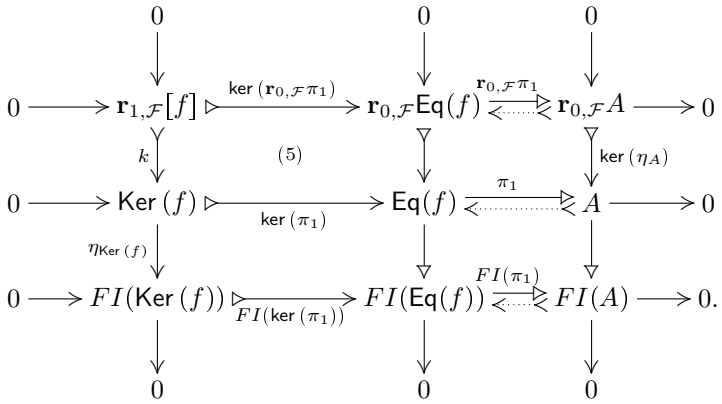
$$\begin{array}{ccccc}
\mathsf{Ker}\,(f) & \overset{\ker(\pi_1)}{\rhd\!\!\longrightarrow} & \mathsf{Eq}(f) & \underset{\lessdot}{\overset{\pi_1}{\rightrightarrows}} & A \\
{\scriptstyle \eta^1_{\mathsf{Ker}\,(f)}}\downarrow & & {\scriptstyle \eta^1_{\mathsf{Eq}(f)}}\downarrow & (3) & \downarrow{\scriptstyle \eta^1_A} \\
I(\mathsf{Ker}\,(f)) & \xrightarrow{I(\ker(\pi_1))} & I(\mathsf{Eq}(f)) & \underset{\lessdot}{\overset{I(\pi_1)}{\rightrightarrows}} & I(A) \\
{\scriptstyle \eta^2_{I(\mathsf{Ker}\,(f))}}\downarrow & & \downarrow & (4) & \downarrow \\
FI(\mathsf{Ker}\,(f)) & \underset{FI(\ker(\pi_1))}{\longrightarrow} & FI(\mathsf{Eq}(f)) & \underset{\lessdot}{\overset{FI(\pi_1)}{\rightrightarrows}} & FI(A)
\end{array} \tag{2.5}$$

the square (3) is a pullback (since f is $\mathcal{B}$-quasi-normal) and $\eta^1_{\mathsf{Ker}\,(f)}$ an isomorphism. It follows that the second row is a split short exact sequence. By protoadditivity of F, the third row is also a split short exact sequence, and we obtain from Lemma 1.3.15 that (4) is a pullback because $\eta^2_{I(\mathsf{Ker}\,(f))}$ is an isomorphism, by assumption. Thus (3)+(4) is a pullback and f is a $\mathcal{F}$-quasi-normal extension. □

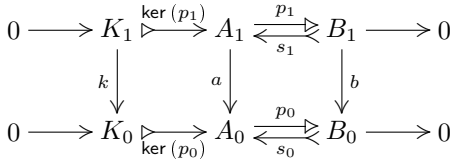
Let us write $\mathsf{QNExt}^1_{\mathcal{E},\mathcal{B}}\mathcal{A}$ for the full subcategory of $\mathsf{Ext}^1_{\mathcal{E}}\mathcal{A}$ whose objects are $\mathcal{B}$-quasi-normal extensions, $\mathbf{r}_{0,\mathcal{F}}$ the radical associated with $F \circ I \colon \mathcal{A}_{\mathcal{E}} \to \mathcal{F}$, and $\mathbf{r}_{1,\mathcal{F}}$ the radical associated with $\mathbf{r}_{0,\mathcal{F}}$. We shall also write $I_{\mathcal{B}}$ for I, and $I_{\mathcal{F}}$ for $F \circ I$.

Lemma 2.3.2. *Let $f \colon A \to B$ be in $\mathsf{QNExt}_{\mathcal{E},\mathcal{B}}\mathcal{A}$. Then $r_{1,\mathcal{F}}[f] = \overline{0}^{\mathcal{F}}_{\mathsf{Ker}\,(f)} = \mathsf{Ker}\,(\eta_{\mathsf{Ker}\,(f)}) = \mathbf{r}_{0,\mathcal{F}}(\mathsf{Ker}\,(f))$. This implies that $\mathbf{r}_{1,\mathcal{F}}$ restricted to $\mathsf{QNExt}_{\mathcal{E},\mathcal{B}}\mathcal{A}$ is protoadditive and therefore, that the restriction $F_{1,\mathcal{F}}$ of $I_{1,\mathcal{F}}$ to $\mathsf{QNExt}_{\mathcal{E},\mathcal{B}}\mathcal{A}$ is also protoadditive.*

Proof. First note that $\mathsf{Ker}\,(f)$ lies in $\mathcal{B}$ since f is a $\mathcal{B}$-central extension (see the proof of (1) ⇒ (2) in Theorem 2.3.1). Then, as in the proof of (2) ⇒ (1) in Theorem 2.3.1, we see that the last row in the following diagram is exact:

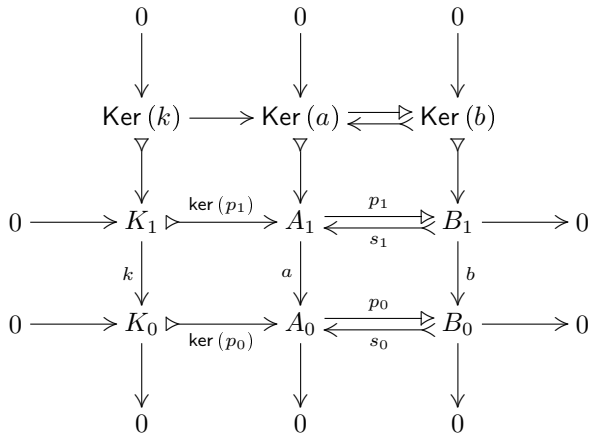


It follows that the left column in this diagram is also exact and the first part of the result follows. Now, let us start with a split short exact sequence



(2.6)

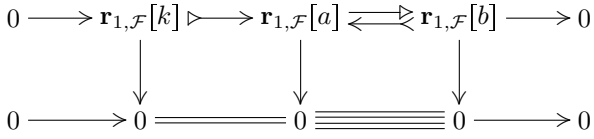
in $\mathsf{QNExt}_{\mathcal{E},\mathcal{B}}\mathcal{A}$ (this is also a split short exact sequence in $\mathsf{Arr}\mathcal{A}$). Then, by commutation of limits, one finds that the first row in the commutative diagram



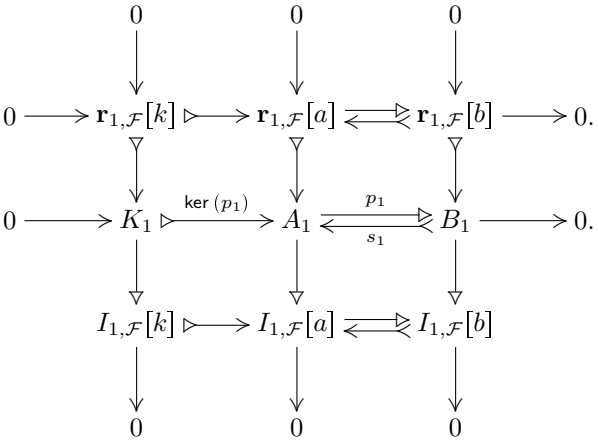
is a split short exact sequence and, since $\mathbf{r}_{0,\mathcal{F}}$ restricted to $\mathcal{B}$ is protoadditive, that one has a split short exact sequence

$$0 \longrightarrow \mathbf{r}_1[k] \rhd\!\!\longrightarrow \mathbf{r}_1[a] \rightleftarrows \mathbf{r}_1[b] \longrightarrow 0.$$

Consequently, the image of (2.6) by $\mathbf{r}_{1,\mathcal{F}}$ is



and $\mathbf{r}_{1,\mathcal{F}}$ is protoadditive. Finally, applying the 3×3-lemma, one finds that the last row in the commutative diagram



is also a split short exact sequence and the image of the split short exact sequence (2.6) by $F_{1,\mathcal{F}}$ is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{1,\mathcal{F}}[k] & \rhd\!\!\longrightarrow & I_{1,\mathcal{F}}[a] & \rightleftarrows & I_{1,\mathcal{F}}[b] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_0 & \underset{\ker(p_0)}{\rhd\!\!\longrightarrow} & A_0 & \underset{s_0}{\overset{p_0}{\rightleftarrows}} & B_0 & \longrightarrow & 0 \end{array}$$

which is a split short exact sequence: the functor $F_{1,\mathcal{F}}$ is protoadditive. □

The results of this chapter show that for an adjunction of the type (2.4) we have an induced Galois structure

$$\Gamma' = (\mathsf{Ext}^1_{\mathcal{E}}\mathcal{A}, \mathsf{QNExt}^1_{\mathcal{E},\mathcal{F}}\mathcal{A}, F_1 \circ I_1, \eta^1, \mathcal{E}^1)$$

of the same type and then, by induction, the

Theorem 2.3.3. *One has a chain of Galois structures of the type (2.4)*

$$\Gamma_n = (\mathsf{ExCub}^n_{\mathcal{E}}\mathcal{A}, \mathsf{QNExCub}^n_{\mathcal{E},\mathcal{F}}\mathcal{A}, F_{n,\mathcal{F}} \circ I_{n,\mathcal{B}}, \eta^n, \mathcal{E}^n).$$

□

The following proposition is similar to Theorem 6.3 in [**35**].

Proposition 2.3.4. *Let Γ be an admissible Galois structure of the type (2.4) with I an $\mathcal{E}$-Birkhoff reflector. Then*

$$\Gamma' = (\mathsf{Ext}^1_{\mathcal{E}}\mathcal{A}, \mathsf{QNExt}^1_{\mathcal{E},\mathcal{F}}\mathcal{A}, F_1 \circ I_1, \eta^1, \mathcal{E}^1)$$

is admissible and I_1 is an $\mathcal{E}^1$-Birkhoff reflector, so that for all $n \in \mathbb{N}$, the Galois structure

$$\Gamma_n = (\mathsf{ExCub}^n_{\mathcal{E}}\mathcal{A}, \mathsf{QNExCub}^n_{\mathcal{E},\mathcal{F}}\mathcal{A}, F_{n,\mathcal{F}} \circ I_{n,\mathcal{B}}, \eta^n, \mathcal{E}^n).$$

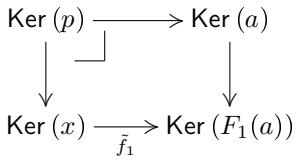
is admissible and $I_{n,\mathcal{B}}$ is an $\mathcal{E}^n$-Birkhoff reflector.

Proof. By Theorem 2.2.18 (2), Proposition 2.2.11 and Proposition 2.2.13, I_1 is an $\mathcal{E}^1$-Birkhoff reflector and is admissible. Then, it remains to show that F_1 is admissible. So, let us consider a pullback in $\mathsf{QNExt}^1_{\mathcal{E},\mathcal{B}}\mathcal{A}$

$$\begin{array}{ccccc} & & P_1 & \longrightarrow & A_1 \\ & \overset{p}{\swarrow} & \downarrow & \overset{a}{\swarrow} & \downarrow{\scriptstyle \rho^1_a} \\ X_0 & \longrightarrow & A_0 & & \\ \| & & \downarrow{\scriptstyle h} & & \| \\ & & X_1 & \longrightarrow & I_{1,\mathcal{F}}[a] \\ & \overset{x}{\swarrow} & \| & \overset{F_1(a)}{\swarrow} & \| \\ X_0 & \underset{f_0}{\longrightarrow} & A_0 & & \end{array}$$

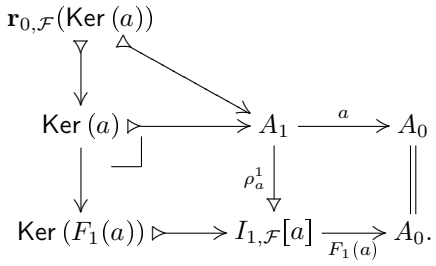
with $f = (f_0, f_1)$ a double extension and x a $\mathcal{F}$-quasi-normal extension. We are going to prove that $(h, 1_{X_0})\colon p \to x$ (in $\mathcal{E}^1$) is sent to an isomorphism by

F_1. Taking kernels, we find a pullback diagram in $\mathcal{B}$

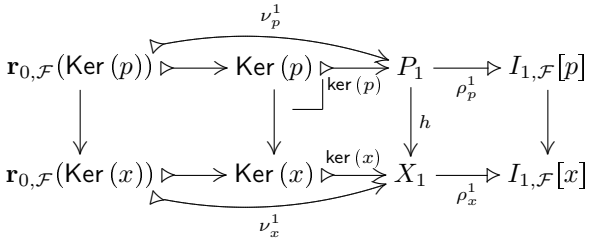


where

- $\mathsf{Ker}\,(x)$ is in $\mathcal{F}$ since it is the kernel of an $\mathcal{F}$-quasi-normal extension;
- $\tilde{f}_1$ is in $\mathcal{E}$ by Lemma 1.4.7;
- $\mathsf{Ker}\,(F_1(a)) \cong \mathsf{Ker}\,(a)/\mathbf{r}_{0,\mathcal{F}}(\mathsf{Ker}\,(a))$:



It follows that this pullback is preserved by F and $F(\mathsf{Ker}\,(p)) \cong F(\mathsf{Ker}\,(x))$. This implies in particular that in the commutative diagram



the left hand square is a pullback. Consequently, the right downward arrow is an isomorphism and $(h, 1_{X_0})$ is sent to an isomorphism. □

CHAPTER 3

Fundamental groups

We begin this chapter by recalling the notion of fundamental group in the context of a Galois structure satisfying some suitable properties. We then prove that if one works in a more specific situation (essentially the one of chapter 2 which gives a tower of induced Galois structures), the fundamental group of any object exists and can be described by the Hopf formula. Using this, we can define higher fundamental groups, and also provide some descriptions for them. We finally show that for a Galois structure of the kind (2.4), these descriptions can be refined using the $\mathcal{E}$-closure operator introduced in the first chapter.

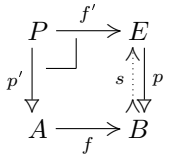
3.1. The fundamental group

Let $\Gamma = (\mathcal{A}, \mathcal{B}, I, H, \eta, \epsilon, \mathcal{E})$ be a Galois structure such that

- (E4) holds;
- η_A is in $\mathcal{E}$ for every A in $\mathcal{A}$;
- $\mathcal{A}$ is a pointed Bourn-protomodular category with intersections

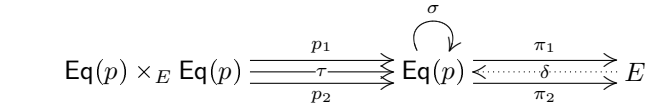
$$(K, k\colon K \to A) \cap (\mathsf{Ker}\,(\eta_A), \mathsf{ker}\,(\eta_A)\colon \mathsf{Ker}\,(\eta_A) \to A).$$

  for every K in $\mathcal{M}(\mathcal{E})$;
- the reflector I preserves pullbacks of the form



  where f is in $\mathcal{E}$ and s is a section of p.

With any quasi-normal extension $(E, p : E \to B)$ in $\mathcal{E}$ one can associate an internal groupoid $\mathsf{Gal}(E, p)$, called its *Galois groupoid*, and an internal group $\mathsf{Gal}(E, p, 0)$, its *Galois group*. If the diagram



represents the kernel pair relation of p (viewed as an internal groupoid in $\mathcal{A}$), then:

Definition 3.1.1. **[50]** The *Galois groupoid* $\mathsf{Gal}(E,p)$ is the image under HI of the kernel pair of p:

$$HI(\mathsf{Eq}(p)\times_E \mathsf{Eq}(p)) \underset{HI(p_2)}{\overset{HI(p_1)}{\underset{HI(\tau)}{\rightrightarrows}}} HI(\mathsf{Eq}(p)) \overset{HI(\sigma)}{\circlearrowright} \underset{HI(\pi_2)}{\overset{HI(\pi_1)}{\underset{HI(\delta)}{\rightleftarrows}}} HI(E).$$

Definition 3.1.2. **[50]** The *Galois group* of p is defined as the object $\mathsf{Gal}(E,p,0)$ in the following pullback:

$$\begin{array}{ccc} \mathsf{Gal}(E,p,0) & \rightarrowtriangle & HI(\mathsf{Eq}(p)) \\ \downarrow & & \downarrow{\scriptstyle \langle HI(\pi_1),HI(\pi_2)\rangle} \\ 0 & \longrightarrow & HI(E)\times HI(E). \end{array}$$

One has

Theorem 3.1.3. **[50]** *The Galois group of any quasi-normal extension $p\colon E \to B$ can be constructed and*

$$\mathsf{Gal}(E,p,0) \cong \mathsf{Ker}\,(p) \cap \mathsf{Ker}\,(\eta_E).$$

Proof. One first sees that $\mathsf{ker}\,(HI(\pi_1))$ exists, since $HI(\pi_1)$ is in $\mathcal{E}$, and that $HI(\pi_2)\circ \mathsf{ker}\,(HI(\pi_1)) = \eta_E \circ \mathsf{ker}\,(p)$ since p is a quasi-normal extension:

$$\begin{array}{ccccccc} \mathsf{Ker}\,(p) = \mathsf{Ker}\,(\pi_1) & \rightarrowtriangle^{\mathsf{ker}\,(\pi_1)} & \mathsf{Eq}(p) & \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} & E & \xrightarrow{p} & B \\ \| & & \downarrow{\scriptstyle \eta_{\mathsf{Eq}(p)}} & & \downarrow{\scriptstyle \eta_E} & & \\ \mathsf{Ker}\,(HI(\pi_1)) & \underset{\mathsf{ker}\,(HI(\pi_1))}{\rightarrowtriangle} & HI(\mathsf{Eq}(p)) & \underset{HI(\pi_2)}{\overset{HI(\pi_1)}{\rightrightarrows}} & HI(E). & & \end{array}$$

Then, one can construct $i = \mathsf{ker}\,(HI(\pi_2)\circ \mathsf{ker}\,(HI(\pi_1)))$ as follows:

$$\begin{array}{ccccc} \mathsf{Ker}\,(p)\cap \mathsf{Ker}\,(\eta_E) & \xrightarrow{i} & \mathsf{Ker}\,(p) & & \\ \downarrow & & \downarrow{\scriptstyle \mathsf{ker}\,(p)} & & \\ \mathsf{Ker}\,(\eta_E) & \underset{\mathsf{ker}\,(\eta_E)}{\rightarrowtriangle} & E & \xrightarrow{HI(\pi_2)\circ \mathsf{ker}\,(HI(\pi_1))} & \\ \downarrow & & \downarrow{\scriptstyle \eta_E} & & \\ 0 & \longrightarrow & I(E). & & \end{array}$$

One can show quite easily that $\mathsf{ker}\,(\langle HI(\pi_1), HI(\pi_2)\rangle) = \mathsf{ker}\,(HI(\pi_1))\circ i$ using the following diagram:

$$\begin{array}{ccccccc} \mathsf{Ker}\,(p)\cap \mathsf{Ker}\,(\eta_E) & \xrightarrow{i} & \mathsf{Ker}\,(p) & \xrightarrow{\mathsf{ker}\,(HI(\pi_1))} & HI(\mathsf{Eq}(p)) & \underset{HI(\pi_2)}{\overset{HI(\pi_1)}{\rightrightarrows}} & HI(E) \\ & & & & \downarrow{\scriptstyle \langle HI(\pi_1),HI(\pi_2)\rangle} & & \| \\ & & & & HI(E)\times HI(E) & \underset{p^2_{HI(E)}}{\overset{p^1_{HI(E)}}{\rightrightarrows}} & HI(E). \end{array}$$

If one has a morphism $x\colon X \to HI(\mathsf{Eq}(p))$ such that $\langle HI(\pi_1), HI(\pi_2)\rangle \circ x = 0$, then

$$HI(\pi_1) \circ x = p^1_{HI(E)} \circ \langle HI(\pi_1), HI(\pi_2)\rangle \circ x = p^1_{HI(E)} \circ 0$$

and there exists $\tilde{x}\colon X \to \mathsf{Ker}\,(p)$ such that $\mathsf{ker}\,(HI(\pi_1)) \circ \tilde{x} = x$. Finally,

$$HI(\pi_2) \circ \mathsf{ker}\,(HI(\pi_1)) \circ \tilde{x} = p^2_{HI(E)} \circ \langle HI(\pi_1), HI(\pi_2)\rangle \circ x = 0$$

gives us a (unique) morphism $\check{x}\colon X \to \mathsf{Ker}\,(p) \cap \mathsf{Ker}\,(\eta_E)$ such that $i \circ \check{x} = \tilde{x}$ and

$$\mathsf{ker}\,(HI(\pi_1)) \circ i \circ \check{x} = x. \qquad \square$$

The Galois group $\mathsf{Gal}(E, p, 0)$ inherits an internal group structure from the composition of the groupoid $\mathsf{Gal}(E, p)$ and can be viewed, internally, as the group of automorphisms of 0. Since $\mathcal{A}$ is a protomodular category, any of its objects underlies at most one internal group structure [**7**] so that we do not have to keep track of the group structure: only the object itself will be of interest. In fact, Proposition 1.3.9 shows that if a group multiplication $m\colon A \times A \to A$ is defined on a objet A of $\mathcal{A}$, then it must make the following diagram commute

$$\begin{array}{ccc} A \times A & \xleftarrow{\langle 0,1\rangle} & A \\ {\scriptstyle\langle 1,0\rangle}\uparrow & \overset{m}{\searrow} & \| \\ A & = & A, \end{array}$$

and must be unique. Let $\mathcal{P} \subseteq \mathcal{E}$ be a class of morphisms closed under composition with isomorphisms.

Definition 3.1.4. A quasi-normal extension $f\colon E \to B$ in $\mathcal{P}$ is *$\mathcal{P}$-weakly universal* if, given any other quasi-normal extension $f'\colon E' \to B$ in $\mathcal{P}$, there is a morphism $u\colon E \to E'$ such that $f = f' \circ u$.

It can be proved, using the same kind of argument as in [**50**], that the Galois group $\mathsf{Gal}(E, p, 0)$ of $p\colon E \to B$, a $\mathcal{P}$-weakly universal quasi-normal extension of B, is an invariant of B:

Definition 3.1.5. The *fundamental group* $\pi_1(B)$ of B is the Galois group of any of its $\mathcal{P}$-weakly universal quasi-normal extension $p\colon E \to B$ (when this latter exists).

Let us recall here why the fundamental group is an invariant of B. Let $p\colon E \to B$ and $p'\colon E' \to B$ be two $\mathcal{P}$-weakly universal quasi-normal extensions of B. Then, there exist two morphisms $u\colon E \to E'$ and $u'\colon E' \to E$ such that

$$p = p' \circ u \quad \text{and} \quad p' = p \circ u'$$

which induce two internal functors

$$(\tilde{u}, u)\colon \mathsf{Eq}(p) \to \mathsf{Eq}(p') \quad \text{and} \quad (\tilde{u'}, u')\colon \mathsf{Eq}(p') \to \mathsf{Eq}(p)$$

which are mutually inverse up to homotopy (see remarks 1.1.8 and 1.1.9). So, $(\tilde{u}, u)$ is sent to an internal functor $(HI(\tilde{u}), HI(u))\colon \mathsf{Gal}(\mathsf{E}, \mathsf{p}) \to \mathsf{Gal}(\mathsf{E}', \mathsf{p}')$

which is still a strong equivalence and, by Remark 1.1.11, cartesian. Then the result is easily seen to hold, looking at the following pullback diagram

$$\begin{array}{ccccc}
\mathsf{Gal}(E,p,0) & \rhd\!\!\longrightarrow & HI(E\times_B E) & \xrightarrow{HI(\tilde{u})} & HI(E'\times_{B'} E') \\
\downarrow & & \downarrow{\scriptstyle \langle HI(\pi_1),HI(\pi_2)\rangle} & & \downarrow{\scriptstyle \langle HI(\pi'_1),HI(\pi'_2)\rangle} \\
0 & \longrightarrow & HI(E)\times HI(E) & \xrightarrow[HI(u)]{} & HI(E')\times HI(E').
\end{array}$$

Furthermore, the Theorem 3.1.3 shows that $\pi_1(B)$ is isomorphic to the following intersection

$$\pi_1(B) \cong \mathsf{Ker}\,(p) \cap \mathsf{Ker}\,(\eta_E). \tag{3.1}$$

where $(E, p\colon E \to B)$ is any $\mathcal{P}$-weakly universal quasi-normal extension of B, and $\mathsf{Ker}\,(p) \cap \mathsf{Ker}\,(\eta_E)$ denotes the domain of the intersection of the kernels $\mathsf{ker}\,(p)$ and $\mathsf{ker}\,(\eta_E)$ of p and η_E, respectively.

3.2. Higher fundamental groups and Hopf formulae

Let $\mathcal{A}$ be a homological category which satisfies (INN) with respect to some factorisation system $(\mathbb{E}, \mathbb{M})$ on $\mathcal{A}$, $\mathcal{E}$ a class of extensions in $\mathcal{A}$, $\mathbf{r}$ an $\mathcal{E}$-radical of $\mathcal{A}$ preserving pullbacks of the forms (2.1), and $\mathcal{O}$ such that (O1)-(O2) hold. In this case, the Galois structure $\Gamma_{\mathbf{r}} = (\mathcal{A}_{\mathcal{E}}, \mathcal{B}, I, \eta, \mathcal{E})$ is such that $\mathcal{A}_{\mathcal{E}}$ has intersections

$$(K, k\colon K \to A) \cap (\mathsf{Ker}\,(\eta_A), \mathsf{ker}\,(\eta_A)\colon \mathsf{Ker}\,(\eta_A) \to A),$$

and we are in the situation of the previous section. Let us suppose also that $\mathcal{A}_{\mathcal{P}}$ has enough projectives with respect to $\mathcal{P}$ a class of extensions in $(\mathcal{A},\mathcal{O})$ such that $\mathcal{O} \subseteq \mathcal{P}^-$ and $\mathcal{P} \subseteq \mathcal{E}$. By Corollary 2.2.19, the full replete subcategory $\mathsf{QNExt}_{\mathcal{E},\mathcal{B}}\mathcal{A}$ is reflective in $\mathsf{Ext}_{\mathcal{E}}\mathcal{A}$ and one has the following

Lemma 3.2.1. *One can construct, for any B in $\mathcal{A}_{\mathcal{P}}$, a $\mathcal{P}$-weakly universal quasi-normal extension of B.*

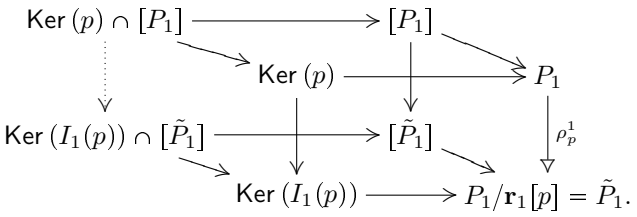
Proof. Let B be an object in $\mathcal{P}^-$ and $p : P_1 \to B$ a $\mathcal{P}$-projective presentation of B. Then $I_1(p)$ is also in $\mathcal{P}$ (see Definition 1.4.32 (A6') and Proposition 2.2.3) and is a quasi-normal extension of B. It is also $\mathcal{P}$-weakly universal. If (E', p') is in $\mathcal{P}$ and is a quasi-normal extension of B, since P_1 is projective with respect to $\mathcal{P}$, there exists an arrow $\alpha : P \to E$ such that $p = p' \circ \alpha$. By the universal property of $I_1(p)$, one gets a factorisation of $I_1(p)$ through p'. □

In that case, one has the following

Theorem 3.2.2. *Let B be an object of $\mathcal{P}^-$ and P a $\mathcal{P}$-projective 1-fold presentation of B. Then*

$$\pi_1(B) \cong \frac{[P_1] \cap \partial P}{[P]_1}$$

PROOF. Let us first recall the convention $\delta_0 P = p\colon P_1 \to P_0 = B$. Then, one can remark that all the faces in the following cube are pullbacks



Indeed, $\mathsf{Ker}\,(\rho_p^1) = [P]_1 \leqslant_{P_1} [P_1]$ entails that the right hand face is a pullback, by the second part of Lemma 2.2.15. The fact that its front face is a pullback follows from Lemma 1.3.15, so that all the other faces are pullbacks as well. One concludes by Lemma 1.4.35. □

One can obtain, still with the same assumptions, some derived Galois structure of the same type. One defines for $n \geqslant 1$, the n^{th} *fundamental group* of an object B in $\mathcal{A}_{\mathcal{P}}$ as the top object of the first fundamental group of a $\mathcal{P}$-projective $(n-1)$-fold presentation Q of B:

$$\pi_n(B) \cong (\pi_1(Q))_{n-1}.$$

Let $\tilde{P}$ be a $\mathcal{P}^{n-1}$-projective 1-fold presentation of Q so that in particular $\tilde{P}$ is in $\mathsf{Ext}_{\mathcal{P}^{n-1}}\mathsf{Cub}^{n-1}\mathcal{A}$. Then by Theorem 3.2.2,

$$\pi_1(Q) \cong \frac{[\tilde{P}_1] \cap \partial\tilde{P}}{[\tilde{P}]_1}.$$

$\tilde{P}$ can be viewed as a $\mathcal{P}$-projective presentation of B, let us say P. $\pi_n(B)$ is also an invariant of B which can be described by the formula

$$\pi_n(B) \cong \frac{[P_n]_0 \cap \partial P}{[P]_n} \tag{3.2}$$

where P is a $\mathcal{P}$-projective n-fold presentation of B. We shall use, following [**31**], the notation

$$\Delta_n(P) = \frac{[P_n]_0 \cap \partial P}{[P]_n}.$$

Again, the proof given in [**31**] still works in our wider context.

Lemma 3.2.3. [**31**] *$\iota^1\Delta_n P = \Delta_{n-1}(\rho_i P)$ for any $n \geqslant 2$, $0 \leqslant i \leqslant n-1$, and any $\mathcal{P}$-projective n-fold presentation of an object B in $\mathcal{P}^-$.*

PROOF. We want to prove that

$$\iota^1\frac{[P_n] \cap \partial P}{[P]_n} = \frac{\iota^1([P_n] \cap \partial P)}{\iota^1[P]_n} = \frac{[(\rho_i P)_{n-1}] \cap \partial(\rho_i P)}{[\rho_i P]_{n-1}}.$$

One has by Theorem 2.2.22, that $\iota^1[P]_n = [\rho_i P]_{n-1}$. One can also prove that $[P_n] \cap \mathsf{Ker}\,(p_i) = \mathbf{r}_1[p_i]$. Of course, $p_i\colon P_n \to P_{n\setminus\{i\}}$ and $1_{P_{n\setminus\{i\}}}\colon P_{n\setminus\{i\}} \to P_{n\setminus\{i\}}$ are $\mathcal{P}$-projective presentations of $P_{n\setminus\{i\}}$, so that

$$\frac{[P_n] \cap \mathsf{Ker}\,(p_i)}{\mathbf{r}_1[p_i]} = \Delta p_i \cong \Delta 1_{P_{n\setminus\{i\}}} = \frac{[P_{n\setminus\{i\}}] \cap \mathsf{Ker}\,(1_{P_{n\setminus i}})}{\mathbf{r}_1[1_{P_{n\setminus\{i\}}}]} = 0$$

Since $[(\rho_i P)_{n-1}] = [p_i] = \iota^1 \mathbf{r}_1 [p_i]$, one finds

$$\begin{array}{rcl} [(\rho_i P)_{n-1}] \cap \partial(\rho_i P) & = & \iota^1([P_n] \cap \mathsf{Ker}\,(p_i)) \cap \partial(\rho_i P) \\ & = & \iota^1(([P_n] \cap \mathsf{Ker}\,(p_i)) \cap \bigcap_{j=0, j\neq i}^{n-1} \mathsf{Ker}\,(p_j)) \\ & = & \iota^1([P_n] \cap \partial P) \end{array}$$

and the result is proved. □

Theorem 3.2.4. **[31]** *Let $\Gamma_{\mathbf{r}}$ be a Galois structure as above and B in $\mathcal{E}^-$. Then $\Delta_n P \cong \Delta_n Q$ for any $\mathcal{P}$-projective n-fold presentations P and Q of B.*

Proof. One proceeds by induction. The case where $n = 1$, has already been proved in theorem 3.2.2. So let $n \geqslant 2$ and let us suppose that for every Galois structure of the type above and $k < n$, the theorem holds. Let B in $\mathcal{P}^-$, P and Q be $\mathcal{P}$-projective n-fold presentations of B. $\rho_0 P$ and $\rho_1 Q$ are $\mathcal{P}^1$-projective $n-1$-fold presentations of $(\rho_0 P)_0 = p_0^1$ and $(\rho_1 Q)_0 = q_0^{\{1\}}$, respectively. Since $\mathcal{A}_{\mathcal{P}}$ has enough projective objects w.r.t. $\mathcal{P}$, we can produce a double presentation of B

$$\begin{array}{ccc} R_1 & \xrightarrow{r} & P_1 \\ {\scriptstyle r'}\downarrow & & \downarrow{\scriptstyle p_0^1} \\ Q_{\{1\}} & \xrightarrow[q_0^{\{1\}}]{} & B. \end{array}$$

This double presentation of B can be extended to a $\mathcal{P}$-projective n-fold presentation R of B (see Theorem 1.4.21) such that $r_1^2 = r$, $r_{\{1\}}^2 = r'$, $r_0^1 = p_0^1$ and $r_0^{\{1\}} = q_0^{\{1\}}$. So, we have that $(\rho_0 P)_0 = (\rho_0 R)_0$ and $(\rho_1 Q)_0 = (\rho_1 R)_0$. Then, the induction hypothesis allows us to infer that $\Delta_{n-1}(\rho_0 P) \cong \Delta_{n-1}(\rho_0 R)$ and $\Delta_{n-1}(\rho_1 Q) \cong \Delta_{n-1}(\rho_1 R)$. One concludes by Lemma 3.2.3 that $\Delta_n P \cong \Delta_n R \cong \Delta_n Q$. □

If one starts from a composite adjunction of the type (2.4), then the descriptions of the fundamental groups can be refined, as shown in the following theorem (see also **[29]**).

Theorem 3.2.5. *Let B be an object of $\mathcal{A}_{\mathcal{P}}$ and P a $\mathcal{P}$-projective 1-fold presentation of B. Then*

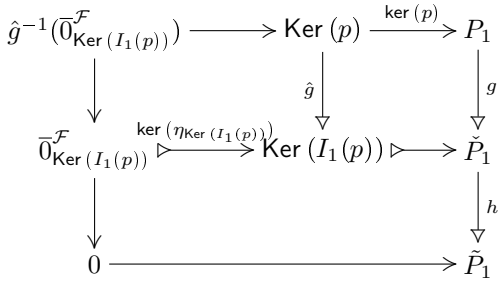
$$\pi_1(B) \cong \frac{\overline{([P_1]_{0,\mathcal{B}})}^{\mathcal{F}}_{P_1} \cap \partial P}{\overline{([P]_{1,\mathcal{B}})}^{\mathcal{F}}_{\partial P}}.$$

Proof. One decomposes the formula from Theorem 3.2.2. Clearly, in our case

$$[P_1]_{0,\mathcal{F}} = \overline{([P_1]_{0,\mathcal{B}})}^{\mathcal{F}}_{P_1}.$$

Furthermore, by Lemma 2.3.2, the arrow ρ_p^1 in the proof of Theorem 3.2.2, the cokernel of $[P]_{1,\mathcal{F}}$, can be factorized as $h \circ g$ where g is the cokernel of $[P]_{1,\mathcal{B}}$ and h the cokernel of $\overline{0}^{\mathcal{F}}_{\mathsf{Ker}\,(I_1(p))}$. Let us consider the following commutative

diagram



in which all rectangles are pullbacks (we write here $\check{P}_1$ for $I_1[p]$). Then one sees that there is an isomorphism

$$\mathsf{Ker}\,(\rho_p^1) = \mathsf{Ker}\,(h \circ g) \cong \hat{g}^{-1}(\overline{0}^{\mathcal{F}}_{\mathsf{Ker}\,(I_1(p))})$$

between the domains of the normal monomorphisms

$$\mathsf{ker}\,(h \circ g) \quad \text{and} \quad \hat{g}^{-1}(\mathsf{ker}\,(\eta_{\mathsf{Ker}\,(I_1(p))})),$$

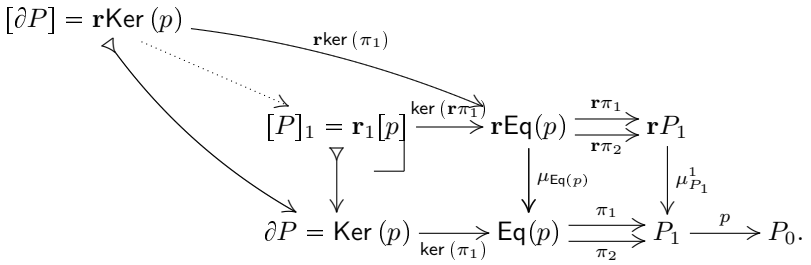
respectively. Since $\hat{g}$ is in $\underline{\mathcal{E}}$, and the closure operator corresponding to the reflection $F \circ I \colon \mathcal{A} \to \mathcal{F}$ is an $\underline{\mathcal{E}}$-closure operator, one has the following equalities:

$$\begin{aligned}
\hat{g}^{-1}(\overline{0}^{\mathcal{F}}_{\mathsf{Ker}\,(I_1(p))}) &= \overline{(\hat{g}^{-1}(0))}^{\mathcal{F}}_{\mathsf{Ker}\,(p)} \\
&= \overline{(\mathsf{Ker}\,(\hat{g}))}^{\mathcal{F}}_{\mathsf{Ker}\,(p)} \\
&= \overline{(\mathsf{Ker}\,(g))}^{\mathcal{F}}_{\mathsf{Ker}\,(p)} \\
&= \overline{([P]_{1,\mathcal{B}})}^{\mathcal{F}}_{\mathsf{Ker}\,(p)}.
\end{aligned}$$

□

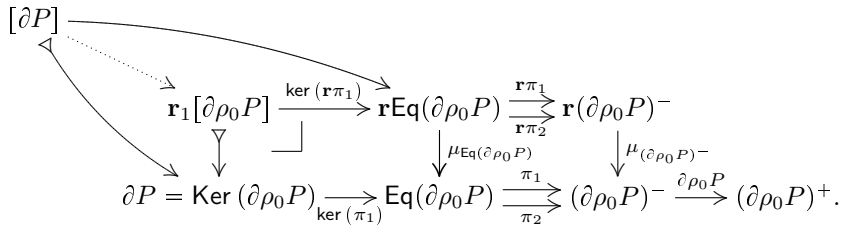
Lemma 3.2.6. *Let P be an $\mathcal{E}$-exact n-cube. If $n \geqslant 1$, then $[P]_n \geqslant [\partial P]$. If $n \geqslant 2$, then $[P]_n \geqslant \mathbf{r}_1[\partial\rho_0 P]$.*

Proof. One proceeds by induction. Let $n = 1$. Then one sees that $[P]_1 \geqslant [\partial P]$ using the diagram



For $n = 2$, it is also easily seen (using the same diagram with $p = \delta_0(P)$) that $[P]_n \geqslant \mathbf{r}_1[\partial\rho_0 P]$. Let us suppose now, that $n \geqslant 2$ and that the inequalities hold for any $k < n$ and Galois structure as above. Then one has, using Theorem 2.2.22, that $\iota^1[P]_n = [\rho_0 P]_{n-1} \geqslant [\partial\rho_0 P] = \iota^1 \mathbf{r}_1[\partial\rho_0 P]$ and $[P]_n \geqslant \mathbf{r}_1[\partial\rho_0 P]$.

Let us consider now the following diagram



Then one has of course $[P]_n \geqslant [\partial P]$. □

Theorem 3.2.7. *Let us suppose that* $\mathsf{QNExCub}^n_{\mathcal{B}}\mathcal{A}$ *is closed under quotients in* $\mathsf{ExCub}^n\mathcal{A}$ *for all* n. *For any* $n \geqslant 1$, $B \in \mathcal{A}_{\mathcal{P}}$ *and a* $\mathcal{P}$*-projective* n*-fold presentation* P *of* B, *one has*

$$\pi_n(B) \cong \frac{\overline{([P_n]_{0,\mathcal{B}})}^{\mathcal{F}}_{P_n} \cap \partial P}{\overline{([P]_{n,\mathcal{B}})}^{\mathcal{F}}_{\partial P}}$$

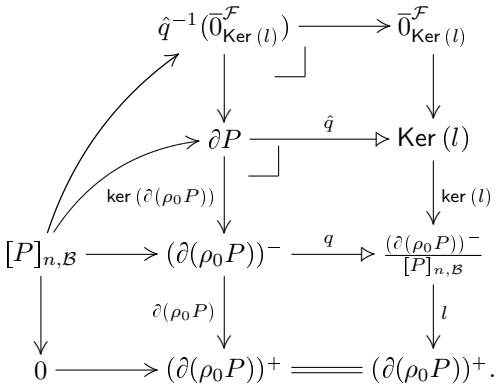
Proof. The invariance has already been proved. One proceeds by induction in order to obtain the above description of the denominator. The case $n = 1$ has been proved above. Let $n \geqslant 2$ and suppose that for $1 \leqslant k \leqslant n-1$ and any Galois structures as above the isomorphism holds. Then, in particular,

$$\Delta_{n-1}(\rho_0 P) \cong \frac{\overline{([(\rho_0 P)_{n-1}]_{0,\mathcal{B}_1})}^{\mathcal{F}_1}_{(\rho_0 P)_{n-1}} \cap \partial(\rho_0 P)}{\overline{([\rho_0 P]_{n-1,\mathcal{B}_1})}^{\mathcal{F}_1}_{\partial(\rho_0 P)}}.$$

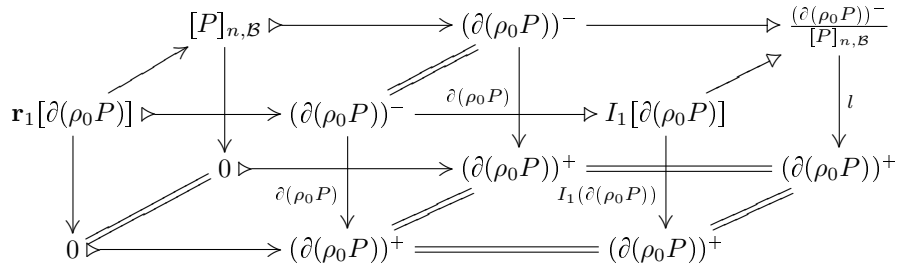
It is clear, from the previous description (3.2), that the numerator is

$$\iota^1(\overline{([P_n]_{0,\mathcal{B}})}^{\mathcal{F}}_{P_n} \cap \partial P).$$

Since $\iota^1[P]_{n,\mathcal{B}} = [\rho_0 P]_{n-1,\mathcal{B}_1}$, we have to figure out what is the closure of $\iota^1[P]_{n,\mathcal{B}}$ in $\partial(\rho_0 P)$ with respect to $\mathcal{F}_1$. Let us consider the following commutative diagram



Using Lemma 3.2.6, one sees that l is $\mathcal{B}$-quasi-normal since one has a commutative diagram



in $\mathsf{ExCub}_{\mathcal{E}}\mathcal{A}$ and $\mathsf{QNExCub}_{\mathcal{E},\mathcal{B}}\mathcal{A}$ is closed under quotients in $\mathsf{ExCub}_{\mathcal{E}}\mathcal{A}$. This shows that $\overline{0}^{\mathcal{F}}_{\mathsf{Ker}\,(l)}$ is in fact $\mathbf{r}_{1,\mathcal{F}}[l]$ (see Lemma 2.3.2). Since $\overline{\cdot}^{\mathcal{F}}$ is an $\underline{\mathcal{E}}$-closure operator,

$$\hat{q}^{-1}(\overline{0}^{\mathcal{F}}_{\mathsf{Ker}\,(l)}) = \overline{\hat{q}^{-1}(0)}^{\mathcal{F}}_{\partial P} = \overline{[P]_{n,\mathcal{B}}}^{\mathcal{F}}_{\partial P}$$

and

$$\overline{([\rho_0 P]_{n-1,\mathcal{B}_1})}^{\mathcal{F}_1}_{\partial(\rho_0 P)} = \overline{(\iota^1[P]_{n,\mathcal{B}})}^{\mathcal{F}_1}_{\partial(\rho_0 P)} = \iota^1\overline{[P]_{n,\mathcal{B}}}^{\mathcal{F}}_{\partial P}.$$

By Lemma 3.2.3, $\iota^1\Delta_n P = \Delta_{n-1}(\rho_0 P)$, the result holds. □

Remark 3.2.8. We assumed in the previous theorem that $\mathsf{QNExCub}^n_{\mathcal{B}}\mathcal{A}$ is closed under quotients in $\mathsf{ExCub}^n\mathcal{A}$ for all n. This assumption is satisfied for any Galois structure of the type (2.4) if I is a Birkhoff reflector or the identity functor, for instance.

CHAPTER 4

Examples

We start this chapter with a recollection of some results on projective objects which are important in order to obtain the existence of weakly universal quasi-normal extensions (see the beginning of Section 3.2). We then recall some classical notions, such as the one of variety of Ω-groups and the one of Huq-commutator. Finally, we give some particular instances of the situation (2.4), and provide the corresponding descriptions of the fundamental groups obtained in these cases.

4.1. Projective objects

Proposition 4.1.1. *Let $(F, U, \eta, \epsilon) : \mathcal{A} \rightharpoonup \mathcal{C}$ be an adjunction,*

$$\mathbb{G} = (G = F \circ U, \epsilon \colon G \to 1_{\mathcal{A}}, \delta \colon G \to G^2)$$

the induced comonad on $\mathcal{C}$, and $\mathcal{P}$ the class of morphisms in $\mathcal{C}$ which are sent by U on split epimorphisms in $\mathcal{A}$. Then $\mathcal{C}$ has enough $\mathcal{P}$-projective objects.

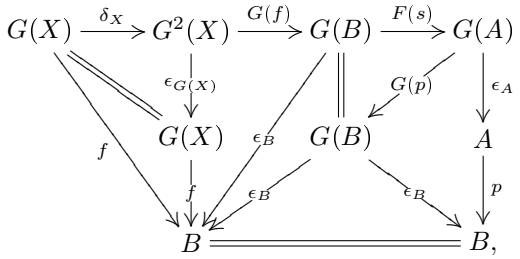
Proof. First, let us recall that by the triangular laws of the adjunction $U(\epsilon_X)$ is a split epimorphism for any X in $\mathcal{A}$ and so ϵ_X is in $\mathcal{P}$. Thus, it remains to prove that $G(X)$ is $\mathcal{P}$-projective. Let us consider a diagram

$$\begin{array}{ccc} & & G(X) \\ & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[p \in \mathcal{P}]{} & B. \end{array}$$

If we apply U to this diagram, we find a commutative diagram

$$\begin{array}{ccc} & & UG(X) \\ & {\scriptstyle s \circ U(f)} \swarrow & \downarrow {\scriptstyle U(f)} \\ U(A) & \underset{U(p)}{\overset{s}{\leftrightarrows}} & U(B) \end{array}$$

where s is a section of $U(p)$. So, since each triangle in the following diagram is commutative



$G(X)$ is $\mathcal{P}$-projective. □

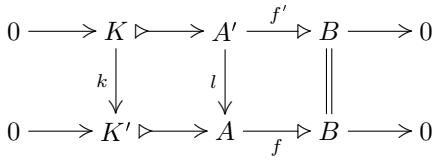
Corollary 4.1.2. *Algebraic varieties have enough projective objects with respect to the class of surjective homomorphisms.*

Proof. This follows from the fact that they are monadic over Set and because Set satisfies the axiom of choice: every surjective function (epimorphism) is a split epimorphism. □

The following is a generalisation of a result in **[33]** (see Example 3.19).

Corollary 4.1.3. *For $\mathbb{T}$ a semi-abelian theory, the category $\mathbb{T}(\mathsf{Top})$ has enough $\mathcal{P}$-projective objects where $\mathcal{P}$ is the class of morphisms in $\mathbb{T}(\mathsf{Top})$ which are split in Top and is a class of extensions in $(\mathbb{T}(\mathsf{Top}), \mathcal{P}^-)$ (with $\mathcal{P}^-$ the class of topological $\mathbb{T}$-algebras). One has also $\mathcal{P} \subset \mathsf{RegEpi}(\mathbb{T}(\mathsf{Top}))$.*

Proof. First, let us recall that the forgetful functor $U\colon \mathbb{T}(\mathsf{Top}) \to \mathsf{Top}$ is monadic ([**8**],[**75**]) whenever $\mathbb{T}$ is a semi-abelian theory. So by Proposition 4.1.1, $\mathbb{T}(\mathsf{Top})$ has enough $\mathcal{P}$-projective objects with respect to $\mathcal{P}$ and $\mathcal{P}^-$ is the whole class of objects of $\mathbb{T}(\mathsf{Top})$. Now, (A1)-(A4) follows easily from the definition of $\mathcal{P}$ and from the fact that the forgetful functor to Top is a right adjoint. We now use Theorem 1.5.40 to show that (A5) is satisfied. Let us consider a diagram



with $f', f, k \in \mathcal{P}$ with sections s, $t = l \circ s$ and u in Top, respectively. Then every element a in A can be written as

$$a = \theta(\alpha_1(a, (t \circ f)(a)), \ldots, \alpha_n(a, (t \circ f)(a)), (t \circ f)(a))$$

with $\alpha_i(a, (t \circ f)(a)) \in \mathsf{Ker}\,(f)$, since

$$f(\alpha_i(a, (t \circ f)(a)) = \alpha_i(f(a), (f \circ t \circ f)(a)) = \alpha_i(f(a), f(a)) = 0,$$

and $(t \circ f)(a)$ in the image of t. The function

$$v\colon A \to A'\colon a \mapsto \theta(u(\alpha_1(a, (t \circ f)(a))), \ldots, u(\alpha_n(a, (t \circ f)(a))), (s \circ f)(a))$$

is continuous and a section of l. Now, a morphism in $\mathcal{P}$ is a homomorphism which is surjective since it has a section as a continuous function. Indeed, for any a in A

$$\begin{aligned} & (l \circ v)(a) \\ = \; & l\left(\theta(u(\alpha_1(a, (t \circ f)(a))), \ldots, u(\alpha_n(a, (t \circ f)(a))), (s \circ f)(a))\right) \\ = \; & \theta\left((l \circ u)(\alpha_1(a, (t \circ f)(a))), \ldots, (l \circ u)(\alpha_n(a, (t \circ f)(a))), (l \circ s \circ f)(a)\right) \\ = \; & \theta\left(\alpha_1(a, (t \circ f)(a)), \ldots, \alpha_n(a, (t \circ f)(a)), (t \circ f)(a)\right) \\ = \; & a. \end{aligned}$$

Obviously, $(A6')$ is satisfied, so that it only remains to show that the morphisms in $\mathcal{P}$ are regular epimorphisms in $\mathbb{T}(\mathsf{Top})$. Let $f\colon A \to B$ be in $\mathcal{P}$. The existence of a section of f in Top implies that the topology on the algebra B is the quotient topology. One concludes by Proposition 21 in [**8**]. □

Corollary 4.1.4. *For $\mathbb{T}$ a semi-abelian theory, the category $\mathbb{T}(\mathsf{HComp})$ of compact Hausdorff $\mathbb{T}$-algebras has enough projective objects with respect to the class of regular epimorphisms.*

Proof. In [**66**], it was proved that $\mathbb{T}(\mathsf{HComp})$ is in fact monadic over Set. □

Proposition 4.1.5. [**34**] *Let $\mathcal{A}$ be a regular Mal'tsev category with finite colimits. Then $\mathsf{Gpd}(\mathcal{C})$, the category of internal groupoids in $\mathcal{C}$, has enough projective objects w.r.t. the class of regular epimorphisms as soon as $\mathcal{A}$ has enough projective objects w.r.t. the class of regular epimorphisms.* □

4.2. Ω-groups

Let us recall that an algebraic variety is called a variety of Ω-groups [**44**] if among the operations and identities which define the variety they are those of groups (we shall use the additive notation although we do not assume the commutativity of the group operation) and if each n-ary operation $w \in \Omega$ satisfies the idempotence law $w(0, 0, \ldots, 0) = 0$. In the sequel, $\mathbf{m}$, $\mathbf{n}$, $\mathbf{x}$, ... denotes sequences $(m_1, \ldots, m_k)$, $(n_1, \ldots, n_k)$, ... and $\mathbf{m}+\mathbf{n}$ the sequence $(m_1 + n_1, \ldots, m_k + n_k)$. Let us also recall some notation and facts from universal algebra (we adopt here the notation from [**21**]). For $\mathbb{V}$ a variety of Ω-groups:

- $T(\mathbb{N})$ is the algebra of terms on a countable set of variables associated with Ω;
- $\Phi_{\mathbb{V}} = \{\theta \in \mathsf{Eq}(T(\mathbb{N})) \,|\, T(\mathbb{N})/\theta \in \mathbb{V}\}$;
- $\theta_{\mathbb{V}} = \bigcap \Phi_{\mathbb{V}}$;
- For $p, q \in T(\mathbb{N})$

$$\left(\forall A \in \mathbb{V}, \forall \mathbf{a} \in A \,:\, p^A(\mathbf{a}) = q^A(\mathbf{a})\right) \Leftrightarrow (p/\theta_{\mathbb{V}} = q/\theta_{\mathbb{V}}),$$

  so that for $v = p/\theta_{\mathbb{V}}$ and $A \in \mathbb{V}$, one can define $v^A = p^A$.
- There exists an adjunction

$$\mathbb{V} \underset{U}{\overset{F_{\mathbb{V}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Set},$$

  where U is the forgetful functor;

- $F_{\mathbb{V}}(\mathbb{N}) = T(\mathbb{N})/\theta_{\mathbb{V}}$ is the free Ω-group (in $\mathbb{V}$) on a countable set of variables.

For $\mathcal{A}$ a variety of Ω-groups (a semi-abelian category by Theorem 1.5.40), and a choice of a subvariety $\mathcal{B}$ (a Birkhoff subcategory of $\mathcal{A}$), one has an adjunction

$$\mathcal{B} \underset{\subseteq}{\overset{I}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{A}.$$

One has also $\theta_{\mathcal{A}} \subseteq \theta_{\mathcal{B}}$ and for $\theta_{\mathcal{B}}/\theta_{\mathcal{A}} = \{(p/\theta_{\mathcal{A}}, q/\theta_{\mathcal{A}}) \in (F_{\mathcal{A}}(\mathbb{N}))^2 \mid (p,q) \in \theta_{\mathcal{B}}\}$, $I(F_{\mathcal{A}}(\mathbb{N})) = F_{\mathcal{A}}(\mathbb{N})/(\theta_{\mathcal{B}}/\theta_{\mathcal{A}}) \cong F_{\mathcal{B}}(\mathbb{N})$. Let us define

$$W_{\mathcal{B}} = \{v(\mathbf{x}) \in F_{\mathcal{A}}(\mathbb{N}) \mid \forall B \in \mathcal{B}, \forall \mathbf{b} \in B \,:\, v^B(\mathbf{b}) = 0\}.$$

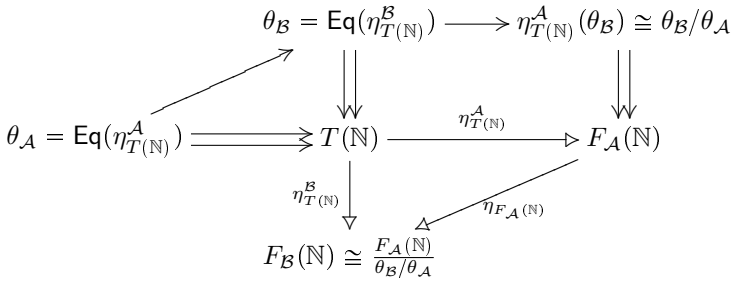
It is in fact the 0 class for the congruence $\theta_{\mathcal{A}}/\theta_{\mathcal{B}}$ (and then the kernel of $\eta_{F_{\mathcal{A}}(\mathbb{N})}$) as we now prove. For $v = p/\theta_{\mathcal{A}}$ in $F_{\mathcal{A}}(\mathbb{N})$

$$\begin{array}{ll} & v \in 0_{(\theta_{\mathcal{B}}/\theta_{\mathcal{A}})} \\ \Leftrightarrow & (p/\theta, 0/\theta_{\mathcal{A}}) \in (\theta_{\mathcal{B}}/\theta_{\mathcal{A}}) \\ \Leftrightarrow & (p,0) \in \theta_{\mathcal{B}} \\ \Leftrightarrow & \forall B \in B, \forall \mathbf{b} \in B \,:\, p^B(\mathbf{b}) = 0^B(\mathbf{b}) \\ \Leftrightarrow & \forall B \in B, \forall \mathbf{b} \in B \,:\, v^B(\mathbf{b}) = 0 \\ \Leftrightarrow & v \in W_{\mathcal{B}}. \end{array}$$

Categorically speaking, we have adjunctions

$$(F_{\mathcal{B}}, U_{\mathcal{B}}, \eta^{\mathcal{B}}, \epsilon^{\mathcal{B}}) : \mathsf{Set} \rightharpoonup \mathcal{B} \quad (F_{\mathcal{A}}, U_{\mathcal{A}}, \eta^{\mathcal{A}}, \epsilon^{\mathcal{A}}) : \mathsf{Set} \rightharpoonup \mathcal{A} \quad (I, \subseteq, \eta, \epsilon) : \mathcal{A} \rightharpoonup \mathcal{B}$$

with $F_{\mathcal{B}} = I \circ F_{\mathcal{A}}$, $U_{\mathcal{B}} = U_{\mathcal{A}} \circ \subseteq$ and a diagram



With respect to choices of variety $\mathcal{A}$ and $\mathcal{B}$ as above, one can define a *relative binary commutator* $[-,-]_{\mathcal{B}}$ (see [**30**]). For M and N ideals (normal subobjects) of an Ω-group A, $[M,N]_{\mathcal{B}}$ is the ideal of $M \vee N$ (the supremum of M and N in the lattice of ideals of A) generated by the set

$$\{v^A(\mathbf{m}+\mathbf{n}) - v^A(\mathbf{n}) - v^A(\mathbf{m}), v^A(\mathbf{p}) \mid \mathbf{m} \in M, \mathbf{n} \in N, \mathbf{p} \in M \cap N, v \in W_{\mathcal{B}}\}.$$

The basic properties of this kind of commutator are listed in the following

Proposition 4.2.1. [**30**] *For any Ω-group A in $\mathcal{A}$ and ideals M, N, N' of A, one has*

(1) $[A,A]_{\mathcal{B}} = \{0\} \Leftrightarrow A \in \mathcal{B}$;
(2) $[M,N]_{\mathcal{B}} = [N,M]_{\mathcal{B}}$;
(3) $[M,N]_{\mathcal{B}} \leqslant M \cap N$;
(4) $N \leqslant N' \Rightarrow [M,N]_{\mathcal{B}} \leqslant [M,N']_{\mathcal{B}}$;

(5) *Universal property:* $[M,N]_{\mathcal{B}}$ *is the smallest ideal* I *of* $M \vee N$ *such that*

$$[q_I(M), q_I(N)]_{\mathcal{B}} = \{0\}$$

for $q_I \colon M \vee N \to \frac{M \vee N}{I}$ *the canonical quotient.* □

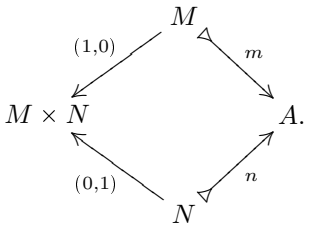
Let us recall that a surjective homomorphism $f \colon A \to B$ is called a $\mathcal{B}$-central extension in the sense of Frölich if $v^{\mathcal{A}}(\mathbf{k}+\mathbf{a}) - v^{\mathcal{A}}(\mathbf{a}) = 0$ for all $v \in W_{\mathcal{B}}$, $k \in \mathsf{Ker}\,(f)$ and $a \in A$.

Proposition 4.2.2. **[30]** *$f \colon A \to B$ in $\mathcal{A}$ is a $\mathcal{B}$-central extension in the sense of Frölich if and only if* $[\mathsf{Ker}\,(f), A]_{\mathcal{B}} = \{0\}$. □

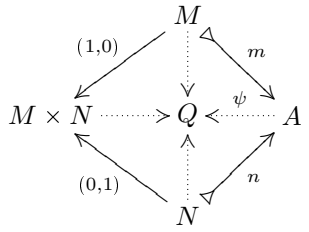
Remark 4.2.3. In fact, central extensions in the sense of Frölich correspond to surjective homomorphisms which are normal extensions in the sense of Galois theory (see **[51]** page 158).

4.3. The Huq commutator

Let $\mathcal{A}$ be a semi-abelian category and $m \colon M \to A$, $k \colon N \to A$ be two normal subobjects of an object A in $\mathcal{A}$. Taking the colimit of the diagram



one finds a commutative diagram



where ψ is a cokernel (since it is an extremal epimorphism by Corollary 1.3.9). One defines $[M,N]_{\mathsf{Huq}}$, the *Huq-commutator* of M and N, as the kernel of this morphism ψ and says that M and N *commute in the sense of Huq* if $[M,N]_{\mathsf{Huq}} = 0$ (see **[46, 14]**). $[M,N]_{\mathsf{Huq}}$ is in fact the smallest normal subobject I of A such that $[q_I(H), q_I(K)] = 0$ as one can easily check. An object A of $\mathcal{A}$ is called *abelian* if $[A,A]_{\mathsf{Huq}} = 0$ and in the present context, an object is abelian if and only if it can be equipped with (a unique) structure of internal abelian group. One can show that $\mathsf{Ab}(\mathcal{A})$ the full subcategory of abelian objects in $\mathcal{A}$ is a Birkhoff subcategory of $\mathcal{A}$ which is abelian (the reflection of an object A is $A/[A,A]_{\mathsf{Huq}}$). For a variety of Ω-groups $\mathcal{A}$, which is semi-abelian (see Theorem 1.5.40), and $\mathcal{B} = \mathsf{Ab}(\mathcal{A})$, one has

$$[M,N]_{\mathsf{Huq}} = [M,N]_{\mathcal{B}}.$$

Notice that an Ω-group is abelian if the corresponding group operations are homomorphisms of Ω-groups [**6**]. There are also a lot of other commutators in the semi-abelian context whose mutual relations have been studied in [**37, 41**], for instance.

4.4. Groups with coefficient functor into torsion-free abelian groups

As a first example, we consider the adjunction

$$\mathsf{Ab}_{t.f.} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Ab} \underset{H}{\overset{\mathsf{ab}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Gp}$$

where Gp is the category of groups, Ab the category of abelian groups and $\mathsf{Ab}_{t.f.}$ the category of torsion-free abelian groups. If we choose $\mathcal{E} = \mathsf{RegEpi}(\mathcal{A})$, we have an instance of the situation (2.4), since the reflector $F\colon \mathsf{Ab} \to \mathsf{Ab}_{t.f.}$ is an additive functor. Note that the kernel of the A-component of the unit η of this adjunction at an abelian group A is given by

$$T(A) = \mathsf{Ker}\,(\eta_A) = \overline{0}_A^{\mathsf{Ab}_{t.f.}} = \{a \in A \,|\, \exists n \in \mathbb{N}_0 : a^n = 1\}.$$

Now, let K be a normal subgroup of a group A with quotient map $q_K\colon A \to A/K$ such that $K \geqslant \overline{0}_A^{\mathsf{Ab}} = [A,A]_{\mathsf{Ab}} = [A,A]$ (here the commutator is the group-theoretic one, thus the quotient group A/K is abelian). One has that

$$\begin{aligned} \overline{K}_A^{\mathsf{Ab}_{t.f.}} &= q_K^{-1}(\overline{0}_{A/K}^{\mathsf{Ab}_{t.f.}}) \\ &= \{a \in A \,|\, \exists n \in \mathbb{N}_0 : (Ka)^n = K\} \\ &= \{a \in A \,|\, \exists n \in \mathbb{N}_0 : a^n \in K\}. \end{aligned}$$

Let us recall the well-known characterisation of Ab-normal $\mathcal{E}$-exact n-cube.

Proposition 4.4.1. [**36**] *Let P an $\mathcal{E}$-exact n-cube in Gp. Then P is Ab-normal if and only if*

$$[P]_{n,\mathsf{Ab}} = \prod_{I \subseteq n} [\bigcap_{i \in I} \mathsf{Ker}\,(p_i), \bigcap_{i \notin I} \mathsf{Ker}\,(p_i)] = 0.$$

Recalling Lemma 3.2.6, our description in Theorem 3.2.7 here becomes

Proposition 4.4.2. *For P a n-fold presentation of a group B*

$$\begin{aligned} \pi_n(B) &\cong \frac{\overline{[P_n, P_n]_{(P_n)}}^{\mathsf{Ab}_{t.f.}} \cap \partial P}{\overline{\prod_{I \subseteq n} [\bigcap_{i \in I} \mathsf{Ker}\,(p_i), \bigcap_{i \in n \setminus I} \mathsf{Ker}\,(p_i)]}^{\mathsf{Ab}_{t.f.}}_{(\partial P)}} \\ &= \frac{\{k \in \partial P \,|\, \exists m \in \mathbb{N}_0 : k^m \in [P_n, P_n]\}}{\{k \in \partial P \,|\, \exists m \in \mathbb{N}_0 : k^m \in \prod_{I \subseteq n} [\bigcap_{i \in I} \mathsf{Ker}\,(p_i), \bigcap_{i \in n \setminus I} \mathsf{Ker}\,(p_i)]\}}. \end{aligned}$$

□

Let us recall, from [**25**] (Example 7.2) for instance, that the hereditary torsion theories in Ab are completely classified. They are in bijection with radicals of the form

$$\mathbf{t}_{\mathscr{P}}(A) = \bigvee_{p \in \mathscr{P}} \mathbf{t}_p(A)$$

where $\mathscr{P}$ is a set of prime numbers and

$$\mathbf{t}_p(A) = \{a \in A \,|\, \exists n \in \mathbb{N} : \mathsf{ord}(a) = p^n\}$$

(here $\mathsf{ord}(a)$ denotes the order of a). $\mathbf{t}_{\mathscr{P}}(A)$ corresponds to $T(A)$ when $\mathscr{P}$ is the set of all prime numbers and to 0 when $\mathscr{P}$ is the empty set. Let us fix a set $\mathscr{P}$ and let $\mathcal{F}_{\mathscr{P}}$ be the associated torsion-free subcategory of Ab. One can prove that the corresponding closure of a normal subgroup K of a group A such that $K \geqslant [A, A]$ is

$$\overline{K}_A^{\mathcal{F}_{\mathscr{P}}} \quad = \quad \{a \in A \,|\, \exists m \in \langle \mathscr{P} \rangle : a^m \in K\}. \tag{4.1}$$

where $\langle \mathscr{P} \rangle$ is the ideal generated by $\mathscr{P}$ in the (commutative) monoid $(\mathbb{N}_0, \cdot, 1)$.

Proposition 4.4.3. *Let $q\colon A \to B$ be a surjective homomorphism with B abelian. One has*

$$q^{-1}(\mathbf{t}_{\mathscr{P}}) = \{a \in A \,|\, \exists m \in \langle \mathscr{P} \rangle : q(a)^m = 1\}.$$

$$\begin{array}{ccc} q^{-1}(\mathbf{t}_{\mathscr{P}}) & \longrightarrow & \mathbf{t}_{\mathscr{P}}(B) \\ \downarrow & \lrcorner & \downarrow \\ A & \xrightarrow[q]{} & B \end{array}$$

Proof. ($\subseteq$:) Let a be in $q^{-1}(\mathbf{t}_{\mathscr{P}})$, that is $a \in A$ such that

$$q(a) = \prod_{i=1}^{k} b_i$$

for some $b_i \in \mathbf{t}_{p_i}(B)$ and $p_i \in \langle \mathscr{P} \rangle$. Then, there exist $l_i \in \mathbb{N}$ such that $b_i^{p_i^{l_i}} = 1$, and for $m = \prod_{i=1}^{k} p_i^{l_i} (\in \langle \mathscr{P} \rangle)$,

$$q(a)^m = \prod_{i=1}^{k} b_i^m = \prod_{i=1}^{k} (b_i^{p_i^{l_i}})^{\prod_{j \neq i} p_j^{l_j}} = 1.$$

($\supseteq$:) We prove by induction that for all m in $\langle \mathscr{P} \rangle$,

$$\{a \in A \,|\, q(a)^m = 1\} \subseteq q^{-1}(\mathbf{t}_{\mathscr{P}}(B)).$$

Since $\mathbf{t}_{\mathscr{P}}(B)$ is a normal subgroup of B, the inclusion holds for $m = 1$. Now, let $m > 1$ be in $\langle \mathscr{P} \rangle$ and let us decompose it as

$$m = m' \cdot p^l$$

with $p \in \mathscr{P}$, $l \in \mathbb{N}$ and $m' \in \langle \mathscr{P} \backslash \{p\} \rangle (\subseteq \langle \mathscr{P} \rangle)$ such that $1 \leqslant m' < m$. One knows from Bezout's theorem that one can find some $c', c \in \mathbb{Z}$ such that

$$c' \cdot m' + c \cdot p^l = 1.$$

For $a \in A$ such that $q(a)^m = 1$, one has

$$(q(a)^{m'})^{p^l} = q(a)^{m' \cdot p^l} = 1$$

and

$$q(a^{p^l})^{m'} = (q(a)^{p^l})^{m'} = q(a)^{m' \cdot p^l} = q(a)^m = 1,$$

so that, by definition,

$$q(a)^{m'} \in \mathbf{t}_p(B) \subseteq \mathbf{t}_{\mathscr{P}}(B)$$

and, by induction,

$$q(a^{p^l}) \in \mathbf{t}_{\mathscr{P}}(B).$$

Hence,

$$q(a) = q(a)^{c'\cdot m' + c\cdot p^l} = \underbrace{(q(a)^{m'})^{c'}}_{\in \mathbf{t}_{\mathscr{P}}(B)} \cdot \underbrace{(q(a)^{p^l})^{c}}_{\in \mathbf{t}_{\mathscr{P}}(B)} \in \mathbf{t}_{\mathscr{P}}(B)$$

and a is in $q^{-1}(\mathbf{t}_{\mathscr{P}}(B))$. □

Finally, for an adjunction of type

$$\mathcal{F}_{\mathscr{P}} \underset{\subseteq}{\overset{F_{\mathscr{P}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Ab} \underset{\subseteq}{\overset{\mathsf{ab}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Gp},$$

the formulae in Theorem 3.2.7 become

$$\pi_n(B) \cong \frac{\{k \in \partial P \,|\, \exists m \in \langle \mathscr{P} \rangle : k^m \in [P_n, P_n]\}}{\{k \in \partial P \,|\, \exists m \in \langle \mathscr{P} \rangle : k^m \in \prod\limits_{I \subseteq n} [\bigcap\limits_{i \in I} \mathsf{Ker}\,(p_i),\ \bigcap\limits_{i \in n \setminus I} \mathsf{Ker}\,(p_i)]\}}.$$

For $\mathscr{P} = \varnothing$, the Brown-Ellis (see **[20]**) formulae for the integral homology of a group are recovered:

$$\mathsf{H}_{n+1}(B, \mathbb{Z}) \cong \frac{[P_n, P_n] \cap \partial P}{\prod\limits_{I \subseteq n} [\bigcap\limits_{i \in I} \mathsf{Ker}\,(p_i),\ \bigcap\limits_{i \in n \setminus I} \mathsf{Ker}\,(p_i)]} \cong \pi_n(B).$$

4.5. Rings with coefficient functor into commutative reduced rings

Let Rng be the semi-abelian category of (not necessarily unitary) rings and CRng its subvariety of (not necessarily unitary) commutative rings: we denote the corresponding reflection

$$\mathsf{CRng} \underset{H}{\overset{I}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Rng}.$$

Let then $\mathsf{RedCRng}$ be the full replete subcategory of CRng having no (non-zero) nilpotent element: the rings in $\mathsf{RedCRng}$ satisfy all implications of the form $x^n = 0 \Rightarrow x = 0$ (with $n \geqslant 1$). This yields a regular epi-reflective subcategory

$$\mathsf{RedCRng} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{CRng}$$

where $\mathsf{RedCRng}$ is the torsion-free part of $(\mathsf{CRng}_{Nil}, \mathsf{RedCRng})$, a hereditary torsion theory whose torsion part is the subcategory CRng_{Nil} of nilpotent commutative rings (see **[25]**, for instance). The homological closure operator associated with this last reflection can be described explicitely, and it actually gives the well known notion of *nilradical* of an ideal. Indeed, for any ideal I of a commutative ring A, its closure in A is its nilradical in A, written $\sqrt{I}_{(A)}$:

$$\begin{aligned} \overline{I}_A^{\mathsf{RedCRng}} &= q_I^{-1}(\overline{0}_{A/I}^{\mathsf{RedCRng}}) \\ &= \{a \in A \,|\, \exists n \in \mathbb{N}_0 : (I + a)^n = I\} \\ &= \{a \in A \,|\, \exists n \in \mathbb{N}_0 : a^n \in I\} \\ &= \sqrt{I}_{(A)}. \end{aligned}$$

One can then consider the following composite adjunction

$$\mathsf{RedCRng} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{CRng} \underset{H}{\overset{I}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Rng}$$

where the reflector $F\colon \mathsf{CRng} \to \mathsf{RedCRng}$ is indeed protoadditive, as one can easily see by using the fact that the corresponding torsion theory is hereditary. This adjunction is then an example of situation (2.4) if we choose $\mathcal{E} = \mathsf{RegEpi}(\mathsf{Rng})$.

Here one can prove that $W_{\mathsf{CRng}} = \langle [m,n] = mn - nm \rangle$ so that $[M,N]_{\mathsf{CRng}} = \langle [m,n] \,|\, m \in M, n \in N \rangle$.

Proposition 4.5.1. *Let $n \geqslant 1$. Then for P an $\mathcal{E}$-exact n-cube in* Rng*, one has*

$$[P]_{n,\mathsf{CRng}} = \bigvee_{I \subseteq n} [\bigcap_{i \in I} \mathsf{Ker}\,(p_i),\ \bigcap_{i \in n \setminus I} \mathsf{Ker}\,(p_i)].$$

Proof. First, let us prove that the equality holds for $n = 1$, i.e. that $[P]_{1,\mathcal{B}} = [\mathsf{Ker}\,(p), P_1]$ for any $\mathcal{E}$-exact 1-cube P (this is not really useful in view of Remark 4.2.3 but we give this proof here in order to facilitate the understanding of the case $n \geqslant 2$). For this, it suffices to show that $[P]_{1,\mathcal{B}} = 0$ if and only if $[\mathsf{Ker}\,(p), P_1] = 0$. Let $k \in \mathsf{Ker}\,(p)$ and $a \in A$ and suppose that $[\pi_1^p]_{\mathsf{CRng}} = [\pi_2^p]_{\mathsf{CRng}}$. Since $(k,0)$ and (a,a) are in $\mathsf{Eq}(p)$, one has of course that

$$[(k,0),(a,a)] = (k,0)(a,a) - (a,a)(k,0) = (ka - ak, 0)$$

is in $[\mathsf{Eq}(p)]_{\mathsf{CRng}}$. Then

$$[k,a] = ka - ak = [\pi_1^p]_{\mathsf{CRng}}(ka - ak, 0) = [\pi_2^p]_{\mathsf{CRng}}(ka - ak, 0) = 0$$

and $[\mathsf{Ker}\,(p), A]_{\mathsf{CRng}} = 0$. Conversely, let us suppose that $ka - ak = 0$ for all $k \in \mathsf{Ker}\,(p)$ and $a \in A$. A generator of $[\mathsf{Eq}(p)]_{\mathsf{CRng}}$ can be written

$$\begin{array}{ll} & [(a, a+k), (a', a'+k')] \\ = & (a, a+k)(a', a'+k') - (a', a'+k')(a, a+k) \\ = & (aa', aa' + ka' + ak' + kk') - (a'a, aa' + k'a + a'k + k'k) \\ = & (aa' - a'a, aa' - a'a + (ka' - a'k) + (ak' - k'a) + (kk' - k'k)) \\ = & (aa' - a'a, aa' - a'a) \end{array}$$

with $a, a' \in A$ and $k, k' \in K$ so that of course $[\pi_1^p]_{\mathsf{CRng}} = [\pi_2^p]_{\mathsf{CRng}}$. So let P be an $\mathcal{E}$-exact n-cube for $n \geqslant 2$ and let us suppose that the equality holds for $0 \leqslant j \leqslant n-1$. Let us recall that we write $\delta_{n-1}P = p\colon P_1 \to P_0$. So $(P_1)_{n-1} = P_n$, $(P_0)_{n-1} = P_{n \setminus \{n-1\}}$ and $(\mathsf{Eq}(p))_{n-1} = \mathsf{Eq}(p_{n-1})$ and we have to show that the restrictions of $\pi_1^{p_{n-1}}$ and $\pi_2^{p_{n-1}}$ to

$$[\mathsf{Eq}(p)]_{n-1} = \bigvee_{I \subseteq n-1} [\bigcap_{i \in I} \mathsf{Ker}\,((\mathsf{Eq}(p))_i),\ \bigcap_{i \in n-1 \setminus I} \mathsf{Ker}\,((\mathsf{Eq}(p))_i)]$$

are equal if and only if

$$\bigvee_{I \subseteq n} [\bigcap_{i \in I} \mathsf{Ker}\,(p_i),\ \bigcap_{i \in n \setminus I} \mathsf{Ker}\,(p_i)] = 0.$$

So let $I \subseteq n$ and $[k,k'] \in [\bigcap_{i\in I} \mathsf{Ker}\,(p_i), \bigcap_{i\in n\setminus I} \mathsf{Ker}\,(p_i)]$ and let us suppose (without any loss of generality) that $n-1 \in I$ so that $[(k,0),(k',k')] \in \mathsf{Eq}(p_{n-1})$. If one shows that

$$([k,k'],0) = [(k,0),(k',k')] \in [\bigcap_{i\in I\setminus\{n-1\}} \mathsf{Ker}\,((\mathsf{Eq}(p))_i), \bigcap_{i\notin I\setminus\{n-1\}} \mathsf{Ker}\,((\mathsf{Eq}(p))_i)],$$

then

$$[k,k'] = \pi_1^{p_{n-1}}([k,k'],0) = \pi_2^{p_{n-1}}([k,k'],0) = 0.$$

So let us show that $(k,0)$ is in $\mathsf{Ker}\,((\mathsf{Eq}(p))_i)$ for any $i \in I\setminus(n-1)$. But $\mathsf{Ker}\,((\mathsf{Eq}(p))_i) = \mathsf{Ker}\,((\mathsf{Eq}(p))^{n-1}_{(n-1)\setminus\{i\}}) = \mathsf{Ker}\,(\tilde{p}_i)$ where $\tilde{p}_i$ is the morphism in the following commutative diagram

$$\begin{array}{ccccc}
\mathsf{Eq}(p)_{n-1} = \mathsf{Eq}(p_{n-1}) & \overset{\pi_1^{p_{n-1}}}{\underset{\pi_2^{p_{n-1}}}{\rightrightarrows}} & (P_1)_{n-1} = P_n & \xrightarrow{p_{n-1}} & (P_0)_{n-1} = P_{n\setminus\{n-1\}} \\
\tilde{p}_i\downarrow & & \downarrow p_i & & \downarrow \\
\mathsf{Eq}(p)_{(n-1)\setminus\{i\}} & \rightrightarrows & (P_1)_{(n-1)\setminus\{i\}} = P_{n\setminus\{i\}} & \longrightarrow & (P_0)_{(n-1)\setminus\{i\}}.
\end{array}$$

and $\tilde{p}_i(k,0) = (p_i(k), p_i(0)) = (0,0)$. One can prove in a similar way that $(k',k') \in \bigcap_{i\notin I\setminus\{n-1\}} \mathsf{Ker}\,((\mathsf{Eq}(p))_i)$. Conversely, let $I \subseteq n-1$ and

$$[(k,\overline{k}),(k',\overline{k}')] \in [\bigcap_{i\in I} \mathsf{Ker}\,((\mathsf{Eq}(p))_i), \bigcap_{i\notin I} \mathsf{Ker}\,((\mathsf{Eq}(p))_i)].$$

One can write $[(k,\overline{k}),(k',\overline{k}')] = [(k,k+l),(k',k'+l')]$ for some $l,l' \in \mathsf{Ker}\,(p_{n-1})$ since in particular $(k,\overline{k})$ and $(k',\overline{k}')$ are in $\mathsf{Eq}(p_{n-1})$. One has in fact $k \in \bigcap_{i\in I} \mathsf{Ker}\,(p_i)$, $l \in \bigcap_{i\in I\cup\{n-1\}} \mathsf{Ker}\,(p_i)$, $k' \in \bigcap_{i\in(n-1)\setminus I} \mathsf{Ker}\,(p_i)$ and $l' \in \bigcap_{i\in n\setminus I} \mathsf{Ker}\,(p_i)$. Finally

$$\begin{array}{rl}
& \pi_1^{p_{n-1}}[(k,k+l),(k',k'+l')] \\
= & \pi_1^{p_{n-1}}([k,k'],[k+l,k'+l']) \\
= & [k,k'] \\
= & [k,k'] + [k,l'] + [l,k'] + [l,l'] \\
= & [k+l,k'+l'] \\
= & \pi_2^{p_{n-1}}[(k,k+l),(k',k'+l')]
\end{array}$$

since

$$\bigvee_{I\subseteq n} [\bigcap_{i\in I} \mathsf{Ker}\,(p_i), \bigcap_{i\notin I} \mathsf{Ker}\,(p_i)] = 0. \qquad \square$$

Theorem 3.2.7 and the previous proposition give

Corollary 4.5.2. *For P an $\mathcal{E}$-projective n-fold presentation of B*

$$\pi_n(B) \cong \frac{\sqrt{[P_n,P_n]}_{(P_n)} \cap \partial P}{\sqrt{\bigvee_{I\subseteq n} [\bigcap_{i\in I} \mathsf{Ker}\,(p_i), \bigcap_{i\notin I} \mathsf{Ker}\,(p_i)]}_{(\partial P)}}$$

$\square$

4.6. Compact groups with coefficient functor into abelian profinite groups

We now consider the following composite adjunction:

$$\mathsf{Ab}(\mathsf{Prof}) \underset{\subseteq}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Ab}(\mathsf{HComp}) \underset{\subseteq}{\overset{\mathsf{ab}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Gp}(\mathsf{HComp}).$$

Here $\mathsf{Gp}(\mathsf{HComp})$ is the semi-abelian category (see Theorem 50 in [**8**]) of compact Hausdorff groups, $\mathsf{Ab}(\mathsf{HComp})$ is its Birkhoff subcategory of abelian compact Hausdorff groups, and $\mathsf{Ab}(\mathsf{Prof})$ its Birkhoff subcategory of profinite abelian groups (compact and totally disconnected abelian groups). We choose $\mathcal{E} = \mathsf{RegEpi}(\mathsf{Gp}(\mathsf{HComp}))$. The left adjoint ab sends a compact group G to the quotient $G/\overline{[G,G]}^{\mathsf{top}}$ of G by the normal subgroup $\overline{[G,G]}^{\mathsf{top}}$, the *topological closure* in G of the derived subgroup $[G,G]$ of G. The left adjoint F sends an abelian compact group A to the quotient $A/\Gamma_0(A)$ of A by the connected component $\Gamma_0(A)$ of the neutral element 0 of A. Now, since the category $\mathsf{Ab}(\mathsf{HComp})$ is abelian, the reflector $F\colon \mathsf{Ab}(\mathsf{HComp}) \to \mathsf{Ab}(\mathsf{Prof})$ is (proto)additive and we have another special instance of the situation (2.4). We shall need a characterisation of the (quasi-)normal extensions with respect to the adjunction

$$\mathsf{Ab}(\mathsf{HComp}) \underset{\subseteq}{\overset{\mathsf{ab}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Gp}(\mathsf{HComp}).$$

As explained in [**70**], a semi-abelian category which is *strongly protomodular* satisfies the *commutator condition* (CC) [**70**]: i.e. a n-extension P (an $\mathcal{E}$-exact n-cube) is normal with respect to the abelianisation functor if and only if it is H-central which means that for all $I \subseteq n$

$$[\bigcap_{i\in I} \mathsf{Ker}\,(p_i),\ \bigcap_{i\in n\setminus I} \mathsf{Ker}\,(p_i)]_{\mathsf{Huq}} = 0.$$

Now, let us recall that a semi-abelian category $\mathcal{A}$ is *strongly protomodular* if and only if given a commutative diagram in $\mathcal{A}$ of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \overset{k}{\rhd\!\!\longrightarrow} & A & \overset{f}{\longrightarrow\!\!\rhd} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \| & & \\ 0 & \longrightarrow & K' & \underset{k'}{\rhd\!\!\longrightarrow} & A' & \underset{f'}{\longrightarrow\!\!\rhd} & B & \longrightarrow & 0, \end{array}$$

the morphism $k' \circ \alpha$ is necessarily a normal monomorphism (see Proposition 5.3 and Proposition 5.4 in [**68**]). Using this fact, one can check that the category of groups is strongly protomodular, which implies that the category $\mathsf{Gp}(\mathsf{HComp})$ is strongly protomodular. Also, the Huq commutator $[M,N]_{\mathsf{Huq}}$ corresponds to $\overline{[M,N]}^{\mathsf{Top}}$ the topological closure of the classical commutator of M and N in Gp (see [**35**] for instance), so that

$$[P]_{n,\mathsf{Ab}(\mathsf{HComp})} = \prod_{I\subseteq n} \overline{[\bigcap_{i\in I} \mathsf{Ker}\,(p_i),\ \bigcap_{i\in n\setminus I} \mathsf{Ker}\,(p_i)]}^{\mathsf{Top}}$$

for P any $\mathcal{E}$-exact n-cube in $\mathsf{Gp}(\mathsf{HComp})$. Now, we can also give an explicit description of the closure operator in this situation. In order to do this, let us also consider the Birkhoff subcategory $\mathsf{Gp}(\mathsf{Prof})$ of $\mathsf{Gp}(\mathsf{HComp})$ of profinite groups. The reflection of a compact group G in this subcategory is given by the quotient $G/\Gamma_0(G)$, with $\Gamma_0(G)$ the connected component of the neutral element of G. Notice that the internal product $K \cdot L$ of two closed normal subgroups of a compact group G is necessarily closed, so that it is the supremum $K \vee L$ in the lattice of normal subobjects of A. Using that $\mathsf{Ab}(\mathsf{Prof}) \leqslant \mathsf{Gp}(\mathsf{Prof})$, as well as Lemma 1.6.15(2), we find, for any normal subobject K of G such that $K \geqslant \overline{[G,G]}^{\mathsf{top}}$—which implies that $G/(K \cdot \Gamma_0(G)) \in \mathsf{Ab}(\mathsf{Prof})$—that

$$\overline{(K)}_G^{\mathsf{Ab}(\mathsf{Prof})} \leqslant \overline{(K \cdot \Gamma_0(G))}_G^{\mathsf{Ab}(\mathsf{Prof})} = K \cdot \Gamma_0(G) = \overline{(K)}_G^{\mathsf{Gp}(\mathsf{Prof})} \leqslant \overline{(K)}_G^{\mathsf{Ab}(\mathsf{Prof})},$$

hence

$$\overline{(K)}_G^{\mathsf{Ab}(\mathsf{Prof})} = K \cdot \Gamma_0(G).$$

Let P be an $\mathcal{E}$-exact n-cube. By Lemma 3.2.6, one has

$$[P]_{n,\mathsf{Ab}(\mathsf{HComp})} \geqslant \overline{[\partial P, \partial P]}^{\mathsf{Top}},$$

so that

$$\begin{aligned} \overline{[P]_{n,\mathsf{Ab}(\mathsf{HComp})\,\partial P}}^{\mathsf{Ab}(\mathsf{Prof})} &= [P]_{n,\mathsf{Ab}(\mathsf{HComp})} \cdot \Gamma_0(\partial P) \\ &= \coprod_{I \subseteq n} \overline{\left[\bigcap_{i \in I} \mathsf{Ker}\,(p_i),\ \bigcap_{i \in n \setminus I} \mathsf{Ker}\,(p_i)\right]}^{\mathsf{Top}} \cdot \Gamma_0(\partial P). \end{aligned}$$

By Theorem 3.2.7 and the above considerations, one obtained for P an $\mathcal{E}$-projective n-fold presentation of a compact Hausdorff group B

$$\pi_n(B) \cong \frac{\overline{[P_n, P_n]}^{\mathsf{Top}} \cdot \Gamma_0(P_n) \cap \partial P}{\coprod_{I \subseteq n} \overline{\left[\bigcap_{i \in I} \mathsf{Ker}\,(p_i),\ \bigcap_{i \in n \setminus I} \mathsf{Ker}\,(p_i)\right]}^{\mathsf{Top}} \cdot \Gamma_0(\partial P)} \cong H_{n+1}(B, F \circ \mathsf{ab}).$$

Let us notice that this Galois structure has already been studied and the above formulae given in [**35**]. If $\mathbb{T}$ is a semi-abelian theory which is strongly protomodular (in the sense that its category of models in Set is), one can propose the same kind of formulae for an adjunction

$$\mathsf{Ab}(\mathbb{T}(\mathsf{Prof})) \underset{\subseteq}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Ab}(\mathbb{T}(\mathsf{HComp})) \underset{\subseteq}{\overset{\mathsf{ab}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{T}(\mathsf{HComp})$$

(see [**8**] for more informations on semi-abelian topological algebras).

4.7. Topological groups with coefficient functor into Hausdorff abelian groups

We consider here the adjunction

$$\mathsf{Ab}(\mathsf{Haus}) \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Ab}(\mathsf{Top}) \underset{H}{\overset{\mathsf{ab}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Gp}(\mathsf{Top})$$

and choose $\mathcal{E} = \mathsf{Epi}(\mathsf{Gp}(\mathsf{Top}))$. Here, $\mathsf{Gp}(\mathsf{Top})$ denotes the $\mathcal{E}$-semi-abelian category of topological groups, $\mathsf{Ab}(\mathsf{Top})$ its $\mathcal{E}$-Birkhoff subcategory of abelian topological groups, and $\mathsf{Ab}(\mathsf{Haus})$ the category of Hausdorff abelian topological

groups. The functor ab sends a topological group G on $G/[G,G]$ with the quotient topology and F sends an abelian topological group G on $G/\overline{\{0\}}_G$ where $\overline{\{0\}}_G$ is the topological closure of the trivial subgroup. Since, $\mathsf{Ab}(\mathsf{Top})$ is an additive category the reflector F is protoadditive and we are in a position to apply our Theorem 3.2.7. For K a normal subobject of a topological group A such that $K \geqslant [A,A]$, one finds that

$$\overline{K}_A^{\mathsf{Ab}(\mathsf{Haus})} = \overline{K}^{\mathsf{Top}},$$

the topological closure of K in A. Obviously, the inequality

$$\overline{K}^{\mathsf{Top}} \leqslant q_{A/K}^{-1}(\overline{0}_{A/K}^{\mathsf{Ab}(\mathsf{Haus})}) = \overline{K}_A^{\mathsf{Ab}(\mathsf{Haus})}$$

holds since $q_{A/K}$ is continuous. The converse inequality

$$q_{A/K}^{-1}(\overline{0}_{A/K}^{\mathsf{Ab}(\mathsf{Haus})}) \leqslant \overline{K}^{\mathsf{Top}}$$

is also valid since the map $q_{A/K}$ is open (Proposition 21 in [**8**]). Let $\mathcal{P}$ be the class of morphisms in $\mathsf{Gp}(\mathsf{Top})$ which are split in the category Top. Then for P a $\mathcal{P}$-projective n-fold presentation of a topological group, we have

$$\pi_n(B) \cong \frac{\overline{[P_n, P_n]}^{\mathsf{Top}} \cap \partial P}{\overline{\coprod_{I \subseteq n} [\bigcap_{i \in I} \mathsf{Ker}\,(p_i), \bigcap_{i \notin I} \mathsf{Ker}\,(p_i)]}^{\mathsf{Top}}}.$$

4.8. Torsion theories

Let us recall from [**10**] the following

Definition 4.8.1. A *protosplit monomorphism* in a pointed protomodular category $\mathcal{A}$ is a normal monomorphism $k\colon K \to A$ that is the kernel of a split epimorphism.

Now let $\mathcal{M}$ be a class of monomorphisms and $(\mathcal{T}, \mathcal{F})$ be a torsion theory in a category $\mathcal{A}$ with zero object, cokernels and kernels. One says that the torsion theory is $\mathcal{M}$-*hereditary* if $\mathcal{T}$ is closed under $\mathcal{M}$-subobjects. It was proved in [**35**] that

Theorem 4.8.2. *For $(\mathcal{T}, \mathcal{F})$ a torsion theory in a homological category $\mathcal{A}$, the following conditions are equivalent:*

(1) *the torsion subcategory $\mathcal{T}$ is $\mathcal{M}$-hereditary, for $\mathcal{M}$ the class of protosplit monomorphisms;*
(2) *the reflector $F\colon \mathcal{A} \to \mathcal{F}$ is protoadditive.* □

As shown in [**18**], any torsion theory $(\mathcal{T}, \mathcal{F})$ in a homological category $\mathcal{A}$ determines also a *semi-left-exact* [**24**] reflector $F\colon \mathcal{A} \to \mathcal{F}$ to the torsion-free subcategory $\mathcal{F}$ of $\mathcal{A}$: F preserves any pullback of the form

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \phi} \\ B & \xrightarrow[\eta_B]{} & I(B) \end{array}$$

where $X \in \mathcal{F}$. In particular, the corresponding Galois structure (with $\mathcal{E}$ the class of all morphisms in $\mathcal{A}$) is admissible (see [**40, 43**]). Thus, a class of examples of the situation (2.4) is given by any adjunction

$$\mathcal{F} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{A} \tag{4.2}$$

with $\mathcal{A}$ a homological category which satisfies (INN) and $\mathcal{F}$ the torsion free-part of an $\mathcal{M}$-hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ with $\mathcal{M}$ the class of protosplit monomorphisms and $\mathcal{E}$ the whole class of morphisms in $\mathcal{A}$. For an example of such a situation, one can think of any torsion theory in a Raïkov semi-abelian category (since in this case the reflector is necessarily additive). One knows from Corollary 2.2.19 that the category of quasi-normal extensions is reflective in the category of extensions. Moreover, if $\mathcal{A}$ has enough projective objects with respect to a class $\mathcal{P}$ of extensions in $\mathcal{A}$ and P is a $\mathcal{P}$-projective n-fold presentation of an object B of $\mathcal{A}$, one obtains:

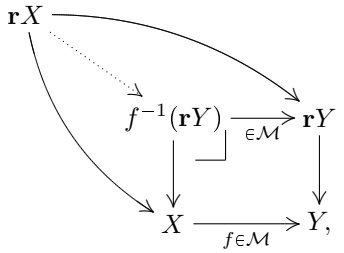
$$\begin{aligned} \pi_n(B) &\cong \frac{\overline{0}^{\mathcal{F}}_{P_n} \cap \partial P}{\overline{0}^{\mathcal{F}}_{\partial P}} \\ &= \frac{T(P_n) \cap \partial P}{T(\partial P)}. \end{aligned}$$

This expression becomes trivial (0) when the torsion subcategory is closed under normal subobjects. In fact, if $\mathcal{M}$ is a pullback stable class of monomorphisms containing the normal monomorphisms, one has

Proposition 4.8.3. *For* $\mathbf{r}$ *a preradical of* $\mathcal{A}$*, the following conditions are equivalent:*

(1) $f^{-1}(\mathbf{r}Y) = \mathbf{r}X$ *for every morphism* $f\colon X \to Y$ *in* $\mathcal{M}$*;*
(2) $\mathbf{r}$ *is idempotent and* $\mathsf{T}_{\mathbf{r}}$ *is closed under* $\mathcal{M}$*-subobjects.*

Proof. (1) $\Rightarrow$ (2): Taking $f = \mathbf{r}X \to X$ and applying condition (1) immediately gives that $\mathbf{r}$ is idempotent. Now if one takes a morphism $f\colon X \to Y$ in $\mathcal{M}$ such that Y in $\mathsf{T}_{\mathbf{r}}$, then $\mathbf{r}Y \cong Y$ gives $\mathbf{r}X \cong X$. (2) $\Rightarrow$ (1): If one starts from a morphism $f\colon X \to Y$ in $\mathcal{M}$ and considers the following commutative diagram



one sees that $\mathbf{r}Y \in \mathsf{T}_{\mathbf{r}}$ and $f^{-1}(\mathbf{r}Y) \in \mathsf{T}_{\mathbf{r}}$ by hypothesis, so that $f^{-1}(\mathbf{r}Y) \leqslant X$ implies $f^{-1}(\mathbf{r}Y) \leqslant \mathbf{r}X$ and then $f^{-1}(\mathbf{r}Y) \cong \mathbf{r}X$. □

The previous proposition (a slightly modified version of Proposition 3.3 (1) in [**25**]) shows that many torsion theories do not give interesting invariants.

However, not all torsion-free subcategories are closed under normal monomorphism or have a protoadditive reflector. The following example comes from [**35**]. Let us consider the following adjunction

$$\mathsf{Gp}_{t.f.} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Gp}.$$

where $\mathsf{Gp}_{t.f.}$ is the full subcategory of Gp whose objects are the torsion-free groups: G is torsion-free if there is only one element $g \in G$ such that g^n for some $n \in \mathbb{N}_0$. The torsion part of the corresponding torsion theory is the full subcategory of groups which are generated by elements of finite order. One can consider the following split short exact sequence in Gp:

$$0 \longrightarrow \mathbb{Z} \longrightarrow C_2 \ltimes \mathbb{Z} \rightleftarrows C_2 \longrightarrow 0$$

where the infinite dihedral group $C_2 \ltimes \mathbb{Z}$ is the group whose underlying set is $C_2 \times \mathbb{Z}$ and multiplication is given by

$*$	$(1, z')$	(c, z')
$(1, z)$	$(1, z + z')$	$(c, z + z')$
(c, z)	$(c, z - z')$	$(1, z - z')$

for $z, z' \in \mathbb{Z}$ and $C_2 = \{1, c\}$. Since $C_2 \ltimes \mathbb{Z}$ is generated by the set $\{(c, z) \mid z \in \mathbb{Z}\}$ and every element of this set has order 2, one sees that the above short exact sequence is sent on the following sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \rightleftarrows 0 \longrightarrow 0$$

which is of course not exact. Now, let us consider an adjunction of type

$$\mathbb{T}(\mathsf{Prof}) \underset{\subseteq}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{T}(\mathsf{HComp})$$

where $\mathbb{T}$ is a semi-abelian theory. $\mathbb{T}(\mathsf{Prof})$ is in fact the torsion-free part of the torsion theory $(\mathbb{T}(\mathsf{ConnComp}), \mathbb{T}(\mathsf{Prof}))$ where $\mathbb{T}(\mathsf{ConnComp})$ is the category of connected compact Hausdorff ($\mathbb{T}$-)algebras [**18**]. The torsion part of the theory is in fact closed under protosplit monomorphisms [**35**] so that the reflector F (which sends an algebra A on $A/\Gamma_0(A)$ [**8**]) is protoadditive. We recall here the arguments. Let A be in $\mathbb{T}(\mathsf{ConnComp})$ and let us consider a split short exact sequence

$$0 \longrightarrow K \overset{k}{\rightarrowtriangle} A \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0.$$

We have to show that K is connected. For this, it suffices to consider the continuous function

$$g \colon A \to K \colon a \mapsto \sigma(a, s(p(a)))$$

where $\sigma(x, y) = \theta(\alpha_1(x, y), \ldots, \alpha_n(x, y), 0)$ is a *subtraction* [**74**], i.e. satisfies

$$\sigma(x, 0) = x \quad \text{and} \quad \sigma(x, x) = 0.$$

Since one has $g \circ k = 1_K$, g is surjective and $K = g(A)$ is connected. So $\mathcal{T}$ is closed under protosplit monomorphisms. Considering the normal subobject $\langle e^{i\frac{\pi}{2}} \rangle$ of S^1 in $\mathsf{Gp}(\mathsf{HComp})$, one sees that $\mathcal{T} = \mathsf{Gp}(\mathsf{ConnComp})$ is not closed under normal subobjects. So this gives an example of adjunction of type (4.2) which

gives (a priori) interesting invariants. Now, the description from Theorem 3.2.7 becomes for P a RegEpi-projective n-fold presentation of a compact Hausdorff $\mathbb{T}$-algebras B

$$\pi_n(B) \cong \frac{\Gamma_0(P_n) \cap \partial P}{\Gamma_0(\partial P)}.$$

The last example we shall consider is an adjunction of type

$$\mathcal{C} \underset{\subseteq}{\overset{\pi_0}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Gpd}(\mathcal{C})$$

where $\mathcal{C}$ is a semi-abelian category. It was proved in [**17**] that $\mathsf{Gpd}(\mathcal{C})$ (the category of internal groupoids in $\mathcal{C}$) is also semi-abelian. Let us explain why this is a particular instance of the situation (2.4). First, we have to recall that an internal groupoid $\mathbf{X}$ is *discrete* if it is a discrete internal equivalence relation, i.e. in

$$X_1 \times_{X_0} X_1 \underset{p_2}{\overset{p_1}{\underset{m}{\rightrightarrows}}} X_1 \overset{\sigma}{\circlearrowright} \underset{d_2}{\overset{d_1}{\underset{s}{\rightleftarrows}}} X_0$$

d_1, d_2 and s are isomorphisms. One has an isomorphism $D\colon \mathcal{C} \to \mathsf{Dis}(\mathcal{C})$ which sends an object C of $\mathcal{C}$ to the discrete groupoid such that $d_1 = d_2 = s = 1_C$. It is well-known that the category $\mathsf{Dis}(\mathcal{C})$ of discrete groupoids is in fact a Birkhoff subcategory of $\mathsf{Gpd}(\mathcal{C})$. The reflector π_0 sends an internal groupoid $\mathbf{X}$ to the discrete groupoid whose objects are the connected components of X (so $\pi_0(\mathbf{X})_0 = \pi_0(\mathbf{X})$ is the coequalizer of d_1 and d_2). An internal groupoid is *connected* if $\langle d_1, d_2\rangle\colon X_1 \to X_0 \times X_0$ is a regular epimorphism. One can observe that for every internal groupoid $\mathbf{X}$ one has a short exact sequence

$$0 \longrightarrow \Gamma_{\mathbf{X}}(0) \longrightarrow \mathbf{X} \longrightarrow D(\pi_0\mathbf{X}) \longrightarrow 0$$

where $\Gamma_{\mathbf{X}}(0)$ denotes the connected component of 0 in $\mathbf{X}$ and that a morphism $f = (f_1, f_0)\colon \mathbf{X} \to \mathbf{Y}$ with $\mathbf{X}$ connected and $\mathbf{Y}$ discrete is necessarily a zero morphism. So, one has a torsion theory $(\mathsf{ConnGpd}(\mathcal{C}), \mathsf{Dis}(\mathcal{C}))$. Moreover, π_0 is protoadditive [**34**] and so, Theorem 3.2.7 gives for P a $\mathsf{RegEpi}(\mathsf{Gpd}(\mathcal{C}))$-projective n-fold presentation of an internal groupoid $\mathbf{X}$ in $\mathcal{C}$

$$\pi_n(\mathbf{X}) \cong \frac{\Gamma_0(P_n) \cap \partial P}{\Gamma_0(\partial P)}.$$

Note that this invariant can already be found in [**34**].

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