Symmetric spaces, non-formal star products and Drinfel'd twists

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Introduction

These notes refer to a mini-course I gave at the occasion of the conference meeting "Applications of Non-Commutative Geometry to Gauge Theories, Field Theories, and Quantum Space-Time" to be held from 7 April to 11 April 2025 at the Centre International de Rencontres Mathématiques in Luminy. They consist in a review of a long standing work of mine and collaborators (see references therein) in the field of non-formal deformation quantization admitting a large group of symmetries. But they also contain new material and results.

More precisely, in a first part, I present a method (called the "Retract Method") to define quantizations/symbolic calculi and associated operator symbol composition formulae (non-formal deformations/star products) of symplectic symmetric spaces such as the hyperbolic plane SL(2,R)/SO(2) (Kahler) or symmetric co-adjoint orbits of the Poincaré group (non-metric). In a second part, I explain how to derive non-formal Drinfel'd twists for actions of non-Abelian solvable Lie groups (non-Abelian Universal Deformation Formulae) on C^* or Fréchet algebras from the non-formal noncommutative symmetric spaces defined in the first part.

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The motivations for this work is threefold. First, it concerns the solution of a problem raised by A. Weinstein in 1994 about geometrically describing the three-point Schwartz kernel of an invariant star-product on a symplectic symmetric space G/K in terms of symplectic areas of its geodesic triangles. The problem was first solved in [Bi02] for solvable symplectic symmetric spaces (i.e. G is a solvable Lie group). Later, a solution was given in [BDS09] in the case of the hyperbolic plane. The method used there is called the *retract method*. Chapter 2 presents a wide generalisation of that method. However, the question of interpreting these solutions as composition formulae in the context of a G-equivariant symbolic calculus ("quantization map") between classical observable defined as smooth functions on the phase space G/K and operators acting in a unitary irreducible representation of G remained unclear. In particular, so was the relation between our work and A. and J. Unterberger' s work in such a framework where $G = \mathrm{SL}(2, \mathbb{R})$ and $K = \mathrm{SO}(2)$.
Second, in the same years (1993-1994), M. Rieffel introduced a way to render *non-formal* the usual Abelian Moyal-Drinfel'd twist based on the vector group (or Lie algebra) $\mathbb{R}^{2n}$ ([Ri1]). In other words, Rieffel defined a Universal Deformation Formula which, when applied to $C^\star$ or Fréchet $\mathbb{R}^{2n}$ algebras, produces deformations of them *within the same topological class* as opposed to formal power series expansions. The question of defining such Universal Deformation Formulas for action of non-Abelian group was raised by Rieffel in [Ri2]. Based on the explicit non-formal star-product formulae of [Bi02], a solution to that problem was given in [BG15] for actions of the Iwasawa factor AN of every real simple Lie group of Hermitean type. In particular, this work provides an interpretation of the formulae of [Bi02] as symbolic composition formulae in a non-Abelian equivariant pseudo-differential calculus on a solvable Lie group or symmetric space. The fact that non-Abelian UDF's derive from symplectic areas of geodesic triangles in symplectic symmetric space is remarkable.
Third, the above question addressed in the solvable case remained open in the case of the hyperbolic plane. From both sides: in my context as well as in Unterberger's where no closed operator algebras were defined.

In Chapter 6 of the present review, I give a solution to that question by $\mathrm{SL}(2,\mathbb{R})$-equivariantly quantizing a Schwartz-type non-commutative function algebra on the hyperbolic plane by subalgebras of compact operators in the projective holomorphic discrete series of $\mathrm{SL}(2,\mathbb{R})$.

Chapter 1

Symplectic symmetric spaces

1.1 Symmetric spaces

Definition 1.1.1 *[Lo69] A* symmetric space *is a pair (M, s) where M is a connected differentiable manifold and*

$$s : M \times M \to M : (x, y) \mapsto s_x y$$

is a smooth map such that

(1) for every point x in M, the map $s_x : M \to M$ is involutive (i.e. $s_x^2 = \mathtt{id}_M$) and admits the point x as an isolated fixed point. The map s_x is called the symmetry *centered at x.*

(2) For all x and y in M, one has

$$s_x s_y s_x = s_{s_x y} .$$

Examples 1.1.1 (i) Every finite dimensional real vector space V is naturally a symmetric space for

$$s_x y := 2x - y \quad (x, y \in V) .$$

(ii) More generally, every Lie group G is canonically a symmetric space when equipped with the symmetries given by

$$s_x y := x y^{-1} x \quad (x, y \in G) .$$

(iii) Let B be a non-trivial scalar product (of arbitrary signature and possibly degenerated) on a finite dimensional real vector space V. Then every (non-null) sphere

$$S_{B,R} := \{x \in V \mid B(x, x) = R\} \quad (R \in \mathbb{R}_0)$$

is naturally a symmetric space when equipped with the "induced" symmetric structure. Indeed, for every x in the sphere, the orthogonal symmetry around the vector axis directed by x is well defined and stabilises the sphere. Its restriction to the sphere then defines the symmetry centered at x:

$$s_x y := \frac{2}{R} B(x, y) x - y \quad (y \in S_{B,R}) .$$

A symmetric space carries a canonical geometry. More precisely, there exists ([Lo69]) a symmetry-compatible covariant derivative on it, which admits a very simple explicit formula ([Bi95, BB11]):

Proposition 1.1.1 *Let (M, s) be a symmetric space. Let X, Y be smooth tangent vector fields on M and x be a point in M. Then,*

(i) the following formula

$$\nabla_X Y|_x := \frac{1}{2} [X , Y + s_{x\star} Y]_x \tag{1.1}$$

defines an affine connection ∇ on M.

(ii) ∇ *is the unique affine connection on* M *that is invariant under the symmetries.*

(iii) It is torsion-free and its Riemann curvature tensor is parallel.

(iv) The symmetries extend the local geodesic symmetries of ∇*. More precisely, if* x *is a point in* M*, and if we denote by* $\mathrm{Exp}^{\nabla}_x : U_x \subset T_xM \to M$ *the corresponding exponential map* (U_x *denotes a symmetric open neighborhood of zero in* T_xM *where the map is well defined), one has*

$$s_x \mathrm{Exp}^{\nabla}_x(v) \ = \ \mathrm{Exp}^{\nabla}_x(-v) \ .$$

Proof. First, we observe that

$$s_{x\star x} \ = \ -\mathtt{Id}_{T_xM} \ . \tag{1.2}$$

Indeed $s_{x\star x}$ is an involution of T_xM. Hence T_xM decomposes into the direct sum of ±1-eigenspaces. Because x is fixed isolately, the subspace of fixed vectors is trivial.

As a map that associates to two vector fields $X, Y \in \mathrm{Vec}(M)$ a third one $\nabla_X Y$ it is clearly $\mathbb{R}$-bilinear. Now let $f \in C^\infty(M)$ be a smooth function on M. We first have

$$\begin{aligned} \nabla_{fX} Y|_x \ &= \ \frac{1}{2} [fX \, , \, Y \ + \ s_{x\star} Y]_x \\ = \ & \frac{1}{2} \left(f\, [X \, , \, Y \ + \ s_{x\star} Y] - s_{x\star} Y(f) X - Y(f) X \right)_x \\ = \ & f \nabla_X Y|_x - \frac{1}{2} \left(s_{x\star} Y(f) X + Y(f) X \right)_x \end{aligned}$$

which equals $f\nabla_X Y|_x$ in view of Property (1.2).

Also, one has

$$\begin{aligned} \nabla_X (fY)|_x \ &= \ \frac{1}{2} [X \, , \, fY \ + \ s^\star_x f s_{x\star} Y]_x \\ = \ & \frac{1}{2} \left(f\, [X \, , \, Y] + (s^\star_x f)\, [X \, , \, s_{x\star} Y] + X(f) Y + X(s^\star_x f) s_{x\star} Y \right)_x \\ = \ & f \nabla_X Y|_x + \frac{1}{2} \left(X(f) Y + X(s^\star_x f) s_{x\star} Y \right)_x \ = \ f \nabla_X Y|_x \end{aligned}$$

as a consequence of Formula (1.2). This proves item (i).

Regarding item (ii), we first observe that for every $y \in M$, one has

$$\begin{aligned} \nabla_{s_{y\star} X} (s_{y\star} Y)|_x \ &= \ \frac{1}{2} \left(s_{y\star} \, [X \, , \, s_{x\star} s_{y\star} Y \ + \ Y] \right)_x \\ = \ & \frac{1}{2} \left(s_{y\star} \, [X \, , \, s_{y\star} s_{x\star} s_{y\star} Y \ + \ Y] \right)_x \ = \ \frac{1}{2} s_{y\star s_y x} \left[X \, , \, s_{s_y x\star} Y \ + \ Y \right] \\ = \ & \left(s_{y\star} \nabla_X (Y) \right)_x \ . \end{aligned}$$

Now consider another symmetry invariant connection ∇'. And form the tensorial quantity:

$$S(X, Y) \ = \ \nabla_X Y - \nabla'_X Y \ .$$

One then has:

$$\begin{aligned} S(X, Y)_x \ &= \ S(s_{x\star} X, s_{x\star} Y)_x \ = \ \nabla_{s_{x\star}} X s_{x\star} Y|_x - \nabla'_{s_{x\star} X} s_{x\star} Y|_x \\ = \ & (s_{x\star} S(X, Y))_x \ = \ -S(X, Y)_x \ = \ 0 \ . \end{aligned}$$

This prove item (ii).

Expanding all the terms involved in the torsion and in the covariant derivative of the Riemann tensor using (1.2), a long but straightforward computation yields item (iii).

Item (iv) follows from the fact that, since $s^2_{x\star_x} = \mathtt{id}_{T_xM}$, the tangent space at x splits into (±1)-eigenspaces of $s_{x\star_x}$. But, since s_x fixes x isolately, the sub-space of fixed vectors must be trivial. Indeed, considering a Riemannian metric preserved by s_x, every non-trivial geodesic (w.r.t. the metric) through x must be flipped under s_x because s_x fixes x isolately. Hence no non-trivial geodesic is tangent to the space of fixed vectors in T_xM. ∎

Remark 1.1.1 *The fact that the symmetries are globally defined then entails that the connection ∇ is geodesically complete.*

Definition 1.1.2 *A* morphism *between two symmetric spaces $(M_i, s^{(i)})$ $(i = 1, 2)$ is a smooth map $\phi : M_1 \to M_2$ that intertwines the symmetries:*

$$\phi \circ s^{(1)}_x \;=\; s^{(2)}_{\phi(x)} \circ \phi \quad (\forall x \in M_1)\;.$$

In the case of a diffeomorphism, one speaks about an isomorphism *between M_1 and M_2 and about an* automorphism *in the case $M_1 \;=\; M_2$.*

A symmetric space turns automatically into a homogeneous space. More precisely, one has

Proposition 1.1.2 *The group $\mathtt{Aut}(M, s)$ of automorphisms of a symmetric space is a Lie group of transformations of M that acts transitively on M.*

Proof. We will first prove that the group $\mathrm{Aff}(\nabla)$ of affine transformations of (M, ∇) equals $\mathtt{Aut}(M, s)$.
let ϕ be an automorphism of (M, s). Then one has

$$\begin{aligned} \phi_\star \nabla_X Y|_x \;&=\; \frac{1}{2}\phi_{\star_{\phi^{-1}(x)}}\left[X, Y + s_{\phi^{-1}(x)\star}Y\right] \\ =\;& \frac{1}{2}\phi_{\star_{\phi^{-1}(x)}}\left[X, Y + \phi_\star^{-1} s_{x\star}\phi_\star Y\right] \;=\; \frac{1}{2}\phi_\star\phi_\star^{-1}\left[\phi_\star X, \phi_\star Y + s_{x\star}\phi_\star Y\right]_x \\ =\;& \nabla_{\phi_\star X}\phi_\star Y|_x\;. \end{aligned}$$

The inclusion $\mathtt{Aut}(M, s) \subset \mathrm{Aff}(\nabla)$ then follows.
Regarding the other inclusion, we observe that any affine transformation ψ of (M, ∇) locally intertwines the geodesic symmetries around any point x of M:

$$\psi s_x(\mathrm{Exp}_x(v)) \;=\; \psi \mathrm{Exp}_x(-v) \;=\; \mathrm{Exp}_{\psi(x)}(-\psi_{\star x}(v)) \;=\; s_{\psi(x)}\psi(\mathrm{Exp}_x(v))\;.$$

We therefore have that there exists an open subset U of M where the affine transformations ψs_x and $s_{\psi(x)}\psi$ coincides. From the fact that the curvature and torsion tensors are parallel, one knows [KN69] [page 263 Theorem 7.7] that the manifold M is analytic with respect to the normal coordinates. Accordingly the affine mappings are then analytic mappings. Since the manifold is connected, ψs_x and $s_{\psi(x)}\psi$ must therefore coincide globally.
Since $\mathrm{Aff}(\nabla)$ is a Lie group of transformations of M [KN69] [Theorem 1.5 page 229], so is $\mathtt{Aut}(M, s)$.
Let us now prove the transitivity of the action. Let x, y be points in M and let $\gamma : [0, 1] \to M$ be a continuous path joining them. Since $\gamma[0, 1]$ is compact, there exists a finite number of points $\{x \;=\; x_0, ..., x_N \;=\; y\}$ such that $U_i \;:=\; \mathrm{Exp}_{x_i}(T_{x_i}M) \cap \mathrm{Exp}_{x_{i+1}}(T_{x_{i+1}}M) \neq \emptyset$. This yields a broken geodesic with $2N$ consecutive geodesic segments, $\{\gamma_j\}$ joining x to y. Denoting by m_j the midpoint of γ_j on has $s_{m_{2N}} \circ .. \circ s_{m_1}(x) \;=\; y$, proving the assertion of transitivity. ■

The affine symmetric space M is therefore a homogeneous space: fixing a base point $o \in M$, and denoting by H the stabilizer of o in $\mathtt{Aut}(M, s)$, one has the H-principal fibration:

$$\mathtt{Aut}(M, s) \to M \simeq \mathtt{Aut}(M, s)/H : g \mapsto g(o)\;.$$

We observe that the map

$$\tilde\sigma : \mathtt{Aut}(M, s) \to \mathtt{Aut}(M, s) : g \mapsto s_o g s_o$$

is an involutive automorphism of the Lie group $\mathtt{Aut}(M, s)$.
Denoting by $\mathfrak{aut}(M, s) \subset \mathrm{Vec}(M)$ the Lie algebra of $\mathtt{Aut}(M, s)$, one has the decomposition

$$\mathfrak{aut}(M, s) \;=\; \mathfrak{h} \oplus \mathfrak{p}$$

into ± 1-eigenspaces of

$$\sigma \;:=\; \tilde\sigma_{\star e} \;=:\; \mathtt{id}_{\mathfrak{h}} \oplus (-\mathtt{id}_{\mathfrak{p}})\;.$$

Proposition 1.1.3 *The Lie algebra of H equals $\mathfrak{h}$ and the vector space $\mathfrak{p}$ naturally identifies with $T_o(M)$.*

Proof. We let Z be an element of the Lie algebra $\tilde{\mathfrak{h}}$ of H. One has, for every $t \in \mathbb{R}$:

$$\exp(-tZ)s_o \exp(tZ) \;=\; s_{\exp(-tZ)o} \;=\; s_o \,.$$

Hence

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \exp(-t\sigma Z)\exp(tZ) \;=\; 0 \;=\; Z - \sigma Z \,.$$

that is; $Z \in \mathfrak{h}$.
For the other inclusion, one observes that if $X \in \mathfrak{h}$, one has

$$\exp(t\sigma X).o \;=\; s_o \exp(tX) s_o o \;=\; s_o \exp(tX).o \;=\; \exp(tX).o \,,$$

hence, since s_o fixes o isolately: $\exp(tX)o = o$ for t small i.e. $X \in \tilde{\mathfrak{h}}$.
Denoting by

$$\pi : \mathtt{Aut}(M,s) \to M : g \mapsto g.o \,,$$

on has that the restriction to $\mathfrak{p}$ of $\pi_{\star e} : \mathfrak{aut}(M,s) \to T_oM$

$$\pi_{\star e}|_{\mathfrak{p}} : \mathfrak{p} \to T_oM$$

is a linear isomorphism. ∎

Definition 1.1.3 *A pair $(\mathfrak{g},\sigma)$ where $\mathfrak{g}$ is a finite dimensional real Lie algebra and σ is an involutive automorphism of $\mathfrak{g}$ is called an* involutive Lie algebra *(briefly* iLa*).*

A morphism *between two iLa's $(\mathfrak{g}_i,\sigma^{(i)})$ $(i=1,2)$ is a homomorphism of Lie algebras $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ that intertwines the involutions:*

$$\psi \circ \sigma^{(1)} \;=\; \sigma^{(2)} \circ \psi \,.$$

When invertible, one speaks about an isomorphism.

Remark 1.1.2 *Given an involutive Lie algebra $(\mathfrak{g},\sigma)$, the vector space $\mathfrak{g}$ decomposes into a direct sum of the (± 1)-eigenspaces of σ:*

$$\mathfrak{g} \;=\; \mathfrak{k} \oplus \mathfrak{p} \qquad \sigma \;=\; \mathtt{id}_{\mathfrak{k}} \oplus (-\mathtt{id}_{\mathfrak{p}}) \,.$$

One then observes in this context the following relations:

$$\begin{array}{rcl} [\mathfrak{k},\mathfrak{k}] & \subset & \mathfrak{k} \\ [\mathfrak{k},\mathfrak{p}] & \subset & \mathfrak{p} \\ [\mathfrak{p},\mathfrak{p}] & \subset & \mathfrak{k}\,. \end{array}$$

In particular, $\mathfrak{k}$ turns into a Lie sub-algebra of $\mathfrak{g}$ that acts its canonical vector direct summand $\mathfrak{p}$.

Proposition 1.1.4 *The pair $(\mathfrak{aut}(M,s), \sigma := \tilde{\sigma}_{\star e})$ is an involutive Lie algebra.*

Given a Lie group, we denote by exp the associated exponential map.

Proposition 1.1.5 *(i) The subspace*

$$\mathfrak{g}(M,s) \;:=\; \mathfrak{k} \oplus \mathfrak{p} \quad \text{with} \quad \mathfrak{k} \;:=\; [\mathfrak{p},\mathfrak{p}]$$

is a Lie sub-algebra of $\mathfrak{aut}(M,s)$.

(ii) Denoting by $G(M,s)$ the connected Lie sub-group of $\mathtt{Aut}(M,s)$ tangent to $\mathfrak{g}(M,s)$ and by

$$K \;:=\; G(M,s) \cap H \,,$$

the mapping

$$\Psi : G(M,s)/K \to M : g \mapsto g.o$$

is a diffeomorphism.

Proof. We first prove that, for every $X \in \mathfrak{p}$, the mapping $\mathrm{Exp}_o : \mathbb{R} \to M : t \mapsto \exp(tX).o$ is the geodesic by o tangent to X:

$$\mathrm{Exp}_o(tX) \;=\; \exp(tX).o \quad \text{where} \quad \mathfrak{p} \;=\; T_oM\;.$$

To see this, we consider the vector field

$$X^\star_x \;:=\; \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \exp(-tX).x\;.$$

We have

$$\nabla_{X^\star}X^\star|_{\exp(tX).o} \;=\; \frac{1}{2}\left[X^\star, s_{\exp(tX).o_\star}X^\star\right]_{\exp(tX).o}\;.$$

But,

$$\begin{aligned}
-\left(s_{\exp(tX).o_\star}X^\star\right)_x \;&=\; s_{\exp(tX).o_\star s_{\exp(tX).o}(x)}X^\star \\
=\;& \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} s_{\exp(tX).o}\left(\exp(sX)s_{\exp(tX).o}(x)\right) \\
=\;& \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp(tX)s_o\exp(-tX)\exp(sX)\exp(tX)s_o\exp(-tX)(x) \\
=\;& \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp(tX)s_o\exp(sX)s_o\exp(-tX)(x) \\
=\;& \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp(tX)\exp(-sX)\exp(-tX)(x) \;=\; X^\star_x\;.
\end{aligned}$$

Hence the assertion.

Now let $x \in M$ and $\{\gamma_j\}_{j=0,\dots,N}$ be a finite sequence of consecutive geodesics joining o to x. At $\gamma_0(1) = \exp(X_0).o$ every geodesic γ_1 is a translate by $\exp(X_0)$ of a geodesic at o of the form $\exp(tY).o$ ($Y \in \mathfrak{p}$):

$$\gamma_1(t) \;=\; \exp(X_0)\exp(tY).o \;=\; \exp(t\mathtt{Ad}_{\exp X_0}Y)\exp(X_1).o\;.$$

In particular, there exists an element $X_1 \in \mathfrak{g}(M,s)$ such that

$$\gamma_1(t) = \exp(tX_1)\exp(X_0).o\;.$$

Iterating, we find elements $X_0, \dots, X_N \in \mathfrak{g}(M,s)$ such that

$$x \;=\; \exp(X_N)\dots\exp(X_0).o\;.$$

The group $G(M,s)$ therefore acts transitively on M. ∎

Definition 1.1.4 *The group $G(M,s)$ is called the* transvection group *of the symmetric space (M,s).*

Remark 1.1.3 The transvections generalize to symmetric spaces the notion of translation: think about the example of a vector space V, where the product of two Euclidean centered symmetries indeed defines a translation.

From all this, one encodes the geometry of the symmetric space M by the purely Lie theoretic notion of involutive Lie algebra. We describe this process below.

Definition 1.1.5 *A* transvection *iLa is an iLa $(\mathfrak{g},\sigma)$ enjoying the following two properties:*
(1) $[\mathfrak{p},\mathfrak{p}] = \mathfrak{k}$ and
(2) the representation $\mathfrak{k}\times\mathfrak{p}\to\mathfrak{p} : (X,Y)\mapsto[X,Y]$ is faithful.

Proposition 1.1.6 *Let (M,s) be a symmetric space. The pair $(\mathfrak{g}(M,s), \sigma := \tilde{\sigma}_{\star e}|_{\mathfrak{g}(M,s)})$ is a transvection iLa. In particular one has a correspondance between symmetric spaces and transvection iLa's:*

$$(M,s)\mapsto(\mathfrak{g}(M,s),\sigma)\;.$$

.

Proof. The only thing to prove is faithfulness, which follows immediately from the fact that $\mathtt{Aut}(M,s)$ is constituted of transformations of M which by definition do act faithfully. ∎

Theorem 1.1.1 *The correspondance described above that associates to every symmetric space a transvection iLa induces a bijection between the isomorphism classes of simply connected symmetric spaces and the isomorphism classes of transvection iLa's.*

Proof. We let (M, s) and (M', s') be two simply connected symmetric spaces that are isomorphic under the isomorphism $\phi : M \to M'$. We observe the isomorphism between their automorphism groups:

$$\hat{\phi} : \mathtt{Aut}(M, s) \to \mathtt{Aut}(M', s') : \varphi \mapsto \phi\varphi\phi^{-1} .$$

We choose a base point $o \in M$ and denote by $\phi(o) =: o'$ the corresponding one in M'. These choices define involutions $\tilde{\sigma}$ and $\tilde{\sigma}'$ where

$$\tilde{\sigma}' \ = \ \hat{\phi}\tilde{\sigma}\hat{\phi}^{-1} .$$

One also has the decompositions:

$$\mathfrak{aut}(M, s) \ = \ \mathfrak{h} \oplus \mathfrak{p} \quad \text{and} \quad \mathfrak{aut}(M', s') \ = \ \mathfrak{h}' \oplus \mathfrak{p}'$$

according respectively to

$$\sigma \quad \text{and} \quad \sigma' \ := \ \tilde{\sigma}'_{\star\mathtt{id}} \ = \ \hat{\phi}_{\star\mathtt{id}}\sigma\hat{\phi}^{-1}_{\star\mathtt{id}} .$$

Now, setting $\underline{\phi} \ := \ \hat{\phi}_{\star\mathtt{id}}$, one has

$$\mathfrak{p}' \ = \ \underline{\phi}(\mathfrak{p}) .$$

Hence, since $\underline{\phi}$ is an isomorphism of Lie algebras, its restriction to $\mathfrak{g}(M, s)$ realizes an isomorphism with $\mathfrak{g}(M', s')$ that intertwines the involutions.

We now revert the correspondance. We fix a transvection iLa $(\mathfrak{g} \ = \ \mathfrak{k} \oplus \mathfrak{p}, \sigma)$. We will associate to it a simply connected symmetric space. In order to do so, we consider the simply connected Lie group G whose Lie algebra is $\mathfrak{g}$ as well as the connected Lie subgroup K of G tangent to $\mathfrak{k}$. The long homotopy sequence of the fibration $K \hookrightarrow G \to G/K$ tells us that, since K is connected and G is simply connected, G/K is simply connected. This cosetspace is canonically a symmetric space. Indeed, considering $\tilde{\sigma}$, the involutive automorphism of G tangent to σ, one sets

$$s_{gK}(g'K) \ := \ g\tilde{\sigma}(g'g^{-1})K .$$

The above formula defines a structure of symmetric space on G/K. Indeed, denoting by $\pi : G \to G/K$ the canonical projection, we have for every $X \in \mathfrak{p}$:

$$s_{K_{\star K}}\pi_{\star e}X \ = \ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 s_K\pi(\exp(tX)) \ = \ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \pi\tilde{\sigma}(\exp(tX)) \ = \ -\pi_{\star e}X .$$

Hence $s_{K_{\star K}} \ = \ -\mathtt{id}_{T_K M}$ which implies by homogeneity that s_{gK} fixes gK isolately. The other axioms are easy to check.

Now, we consider the Lie group homomorphism:

$$\tau : G \to \mathtt{Aut}(G/K, s) : g \mapsto \tau_g \ := \ [g'K \mapsto gg'K] .$$

In general, this homomorphism is not injective. However, its differential is. Indeed, setting, at the level of $\mathtt{Aut}(G/K, s)$, $\hat{\sigma}(\varphi) \ := \ s_K\varphi s_K$, one has

$$\begin{aligned} \hat{\sigma}_{\star\mathtt{id}}\tau_{\star e}(X)|_{gK} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 s_K\tau_{\exp(tX)}s_K(gK) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \tilde{\sigma}(\exp(tX)\tilde{\sigma}(g)) \\ = \ \tau_{\star e}\sigma(X)|_{gK} . \end{aligned}$$

Which implies that, denoting by $\mathfrak{aut}(G/K, s) \ = \ \hat{\mathfrak{h}} \oplus \hat{\mathfrak{p}}$ the iLa decomposition, one has

$$\tau_{\star e}\mathfrak{p} \subset \hat{\mathfrak{p}} .$$

A dimensional argument on the dimension of the homogeneous space G/K implies that

$$\tau_{\star e}\mathfrak{p} = \hat{\mathfrak{p}} .$$

The fact that the action of $\mathfrak{k} \ = \ [\mathfrak{p}, \mathfrak{p}]$ on $\mathfrak{p}$ is faithful and the fact that $\tau_{\star e}$ is a homomorphism of Lie algebras imply that $\tau_{\star e}|_{\mathfrak{k}}$ is injective as well. Hence $\tau_{\star e}$ defines an isomorphism between $(\mathfrak{g}, \sigma)$ and the transvection iLa of $(G/K, s)$.

The rest is obvious. ∎

Example 1.1.1 Viewing a Lie group $\mathbb{L}$ as a symmetric space, one associates to it the so-called "exchange case" iLa. Denoting by $\mathfrak{L}$ the Lie algebra of $\mathbb{L}$, one sets

$$\mathfrak{L}^2 \ := \ \mathfrak{L} \oplus \mathfrak{L} \quad \text{and} \quad \sigma : \mathfrak{L}^2 \to \mathfrak{L}^2 : (X, Y) \mapsto (Y, X) \; .$$

The pair $(\mathfrak{L}^2, \sigma)$ is then an iLa with canonical eigensubspaces

$$\mathfrak{k}_2 \ = \ \{(X, X)\} \ \simeq \ \mathfrak{L}$$
$$\mathfrak{p}_2 \ := \ \{(X, -X)\} \; .$$

The action of $\mathfrak{k}_2$ on $\mathfrak{p}_2$ is generally not faithful and one generally does not have $[\mathfrak{p}_2, \mathfrak{p}_2] = \mathfrak{k}_2$. However, one may recover the transvection iLa by considering

$$\mathfrak{g} \ := \ \mathfrak{k} \oplus \mathfrak{p}$$

with

$$\mathfrak{k} \ := \ \mathfrak{L}'/\mathfrak{z} \quad \text{and} \quad \mathfrak{p} \ := \ \mathfrak{p}_2$$

where $\mathfrak{L}' \ := \ [\mathfrak{L}, \mathfrak{L}]$ denotes the first derivative of $\mathfrak{L}$ and where $\mathfrak{z}$ the centralizer of $\mathfrak{L}$ in $\mathfrak{L}'$ (w.r.t. the adjoint representation):

$$\mathfrak{z} \ := \ \mathfrak{L}' \cap \mathfrak{z}(\mathfrak{L}) \; .$$

Definition 1.1.6 *The symmetric space M is called* group type *when there exists a Lie subgroup $\mathbb{S}$ of G that acts simply transitively on M i.e. one has the $\mathbb{S}$-equivariant diffeomorphism:*

$$\mathbb{S} \to M : x \mapsto xK \; .$$

One then talk about a group type symmetric space on the abstract Lie group $\mathbb{S}$.

We now consider a Lie group $\mathbb{S}$ equipped with a smooth one-parameter family of group type symmetric space structure (hence left-invariant):

$$s^{(\kappa)} : \mathbb{S} \times \mathbb{S} \to \mathbb{S} \quad (\kappa \in I)$$

where I is a connected interval of $\mathbb{R}$.
We denote by $\mathfrak{g}_\kappa$ the associated family of transvection Lie algebras and by

$$\mathfrak{g}_\kappa \ = \ \mathfrak{k}_\kappa \oplus \mathfrak{p}_\kappa$$

the corresponding symmetric space decomposition. In that context we make the following definition.

Definition 1.1.7 *For κ and κ', we say that $s^{(\kappa')}$ is a* curvature contraction *of $s^{(\kappa)}$ if*

$$\dim \mathfrak{Z}_{\mathfrak{p}_{\kappa'}}(\mathfrak{k}_{\kappa'}) \ > \ \dim \mathfrak{Z}_{\mathfrak{p}_\kappa}(\mathfrak{k}_\kappa)$$

where we denote by $\mathfrak{Z}_{\mathfrak{p}}(\mathfrak{k})$ the centralizer of $\mathfrak{k}$ in $\mathfrak{p}$.

This essentially says that the curvature of $s^{(\kappa')}$ is obtained from the one of $s^{(\kappa)}$ by letting tend to zero part of it curvature components.

1.2 Symplectic symmetric spaces

Definition 1.2.1 *[Bi95, BCG96] A* symplectic symmetric space *(briefly SSS) is a triple (M, s, ω) where (M, s) is a symmetric space and where ω is a non-degenerate 2-form on M that is symmetry invariant:*

$$s_x^\star \omega \ = \ \omega \quad (\forall x \in M) \; .$$

Two such symplectic symmetric spaces are isomorphic *if there exists a symplectomorphism between them which is at the same time an isomorphism of symmetric spaces between the underlying symmetric spaces.*

The following observation is an immediate consequence of [KN69] (page ??)

Proposition 1.2.1 *The two form ω is parallel w.r.t. the canonical affine connection ∇:*

$$\nabla\omega \;=\; 0\;.$$

In particular, it is closed:

$$\mathrm{d}\omega \;=\; 0$$

hence symplectic.

Example 1.2.1 Hermitean symmetric spaces

We start with a simple Lie group G with finite center whose Lie algebra $\mathfrak{g}$ is absolutely simple (i.e. $\mathfrak{g}^{\mathbb{C}} \;:=\; \mathfrak{g}\otimes\mathbb{C}$ is simple). We consider a maxi lam compact subgroup $K\subset G$ whose Lie algebra $\mathfrak{k}$ is assumed to admit a non-trivial center $\mathfrak{z}(\mathfrak{k})$. Such a Lie group G is called of *Hermitean type*.
In that case, $\mathfrak{z}(\mathfrak{k})$ is known ([K65, Bi95]) to be one-dimensional and generated by an element Z_0 that acts as a complex structure on $\mathfrak{g}/\mathfrak{k}$.
More precisely, denoting by B the Killing form on $\mathfrak{g}$, the decomposition

$$\mathfrak{g} \;=\; \mathfrak{k}\oplus(\mathfrak{p}\;:=\;\mathfrak{k}^{\perp_B})$$

underlies an iLa $(\mathfrak{g},\theta)$ whose involution θ is a Cartan involution of $\mathfrak{g}$.
The action of Z_0 on $\mathfrak{p}$:

$$\mathcal{J}\;:\mathfrak{p}\to\mathfrak{p}:X\mapsto[Z_0,X]$$

is then a complex structure:

$$\mathcal{J}^2 \;=\; -\mathtt{id}_{\mathfrak{p}}\;.$$

One then gets the symplectic element $\Omega\in\bigwedge^2(\mathfrak{p}^\star)$ defined by

$$\Omega(X,Y)\;:=\;B(\mathcal{J}X,Y)\;,$$

which on the coset space $M\;:=\;G/K$ defines a G-invariant symplectic structure ω throughout the formula:

$$\omega_{gK}(X^\star,Y^\star)\;:=\;\Omega\left((Ad_{g^{-1}}X)_{\mathfrak{p}}\,,\,(Ad_{g^{-1}}Y)_{\mathfrak{p}}\right)\quad(X,Y\in\mathfrak{p}) \tag{1.3}$$

where we write for every $T\in\mathfrak{g}$

$$T \;=\; T_{\mathfrak{k}}+T_{\mathfrak{p}}$$

according to the Cartan decomposition.
As seen above, the homogeneous space M is canonically equipped with a symmetric space structure

$$s_{gK}g'K\;:=\;g\theta(g^{-1}g')K$$

where we (abusively) denote by θ the involutive automorphism of G that integrates the Cartan involution θ of $\mathfrak{g}$. The triple (M,s,ω) is then a symplectic symmetric space.

Remarks 1.2.1 *(i) The Iwasawa decomposition of the semi-simple Lie group $G\;=\;ANK$ implies that these Hermitean symmetric spaces are all of group type.*

(ii) In the above example, the formula:

$$J_{gK}(X^\star)\;:=\;\mathcal{J}(Ad_{g^{-1}}X)_{\mathfrak{p}}\quad(X\in\mathfrak{g})$$

defines a G-invariant complex structure on $M=G/K$. The element

$$\beta(X^\star,Y^\star)\;:=\;-\omega(\mathcal{J}X,Y)$$

then defines a G-invariant Kahler metric on M.

The infinitesimal structure, analogue to the one of iLa, that encodes the notion of *symplectic* symmetric space is the following one.

Definition 1.2.2 *A* symplectic triple *(briefly ST) is a triple* $(\mathfrak{g}, \sigma, \Omega)$ *where* $(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \sigma)$ *is an iLa and where* Ω *is a Chevalley 2-cocycle (w.r.t the trivial representation of* $\mathfrak{g}$ *on* $\mathbb{R}$*):*

$$\Omega \in \bigwedge^2(\mathfrak{g}^\star) : \delta\Omega \ = \ 0$$

that vanishes identically on $\mathfrak{k} \times \mathfrak{k}$ *and* $\mathfrak{k} \times \mathfrak{p}$ *and restricts to* $\mathfrak{p} \times \mathfrak{p}$ *as a non-degenerate 2-form.*

Two such symplectic triples are isomorphic *when there exists an isomorphism of iLa's between the underlying iLa's that intertwines the 2-cocycles.*

Remark 1.2.1 In the above definition, the Chevalley coboundary operator is defined as usual by

$$\delta\Omega(X, Y, Z) \ := \ \Omega([X, Y], Z) + \Omega([Z, X], Y) + \Omega([Y, Z], X) \ .$$

One then has

Theorem 1.2.1 *The correspondence between the symmetric spaces and the iLa's (c.f. Theorem 1.1.1) induces a bijection between the set of isomorphisms classes of simply connected SSS's and the set of isomorphism classes of transvection ST's.*

Proof. To a SSS (M, s, ω) one associates a ST in the obvious way: $(\mathfrak{g}, \sigma)$ is defined as the transvection iLa associated to the symmetric space (M, s), while, through the isomorphism

$$\pi_{\star_0}|_{\mathfrak{p}} : \mathfrak{p} \to T_o M$$

(with $o \in M$ such that $\sigma_{o_{\star e}} \ = \ \sigma$), one defines Ω to be the extension by zero on $\mathfrak{k} \times \mathfrak{k}$ and $\mathfrak{k} \times \mathfrak{p}$ of $(\pi_{\star_0}|_{\mathfrak{p}})^\star \omega_o$. One readily checks that the obtained triple $(\mathfrak{g}, \sigma, \Omega)$ is a SST.
Conversely, to a ST $(\mathfrak{g}, \sigma, \Omega)$ the proof of Theorem 1.1.1 associates a symmetric space modelled on G/K. The formula (1.3) then defines the symplectic structure needed to from a SSS $(G/K, s, \omega)$.
The rest is immediate following the lines of the proof of Theorem 1.1.1. ■

Due to an old result of B. Kostant, a symplectic symmetric space is, generally, locally a co-adjoint orbit. But in general not of its transvection group. However, in this book, we will be concerned with the following sub-class of SSS's.

Definition 1.2.3 *A SSS* (M, s, ω) *si called* Hamiltonian *when its transvection group* G *acts on* (M, ω) *is a Hamiltonian way, that is, when equipping* $C^\infty(M)$ *with the Poisson structure associated with the symplectic structure, there exists a* G*-equivariant homomorphism of Lie algebras*

$$\lambda : \mathfrak{g} \to C^\infty(M) : X \mapsto \lambda_X$$

such that

$$\mathrm{d}\lambda_X \ = \ i_{X^\star}\omega \ .$$

where i *denotes the usual interior product of differential forms by vector fields.*

Remark 1.2.2 *In that case, a general result due to Kostant about symplectic homogeneous spaces states that the map*

$$m : M \to \mathfrak{g}^\star : x \mapsto [X \mapsto \lambda_X(x)]$$

realizes a G*-equivariant symplectic covering of* M *over a co-adjoint orbit* $\mathcal{O}$ *of* G *in* $\mathfrak{g}^\star$*.*

Remark 1.2.3 *For* semi-simple *symplectic symmetric space i.e. when* G *is semi-simple, an easy application of Whitehead's lemmas implies that's it always the case.*

Remark 1.2.4 *To a Hamiltonian SSS* $(M, s., \omega)$ *corresponds such a transvection ST* $(\mathfrak{g}, \sigma, \Omega)$ *with* exact *symplectic element*

$$\Omega \ = \ \delta\xi \quad (\xi \in \mathfrak{g}^\star) \ .$$

Example 1.2.2 Contracting the Lobatchevsky plane
In this paragraph, we will express the geometry of the group $\mathbb{S} = ax+b$ as a curvature contraction of the hyperbolic plane.

We now consider the following realization of the connected component of 2-dimensional Lie group $\mathbb{S}$ of affine transformations of the real line. We consider the two dimensional Lie algebra $\mathfrak{s}$ generated by two elelments H and E, with Lie bracket:

$$[H,E] \;:=\; 2E\;.$$

The corresponding connected and simply connected Lie group $\mathbb{S}$ is then coordinated by the global chart

$$\mathfrak{s}\to\mathbb{S}: aH+nE\mapsto \exp(aH).\exp(nE) \;=:\; (a,n)$$

in which the group law is given by

$$(a,n).(a',n') \;:=\; (a+a'\,,\, e^{-2a'}n\,+\,n')$$

with unit

$$e \;:=\; (0,0)$$

and inverse

$$(a,n)^{-1} \;=\; (-a,-e^{2a}n)\;.$$

Within this chart, a left-invariant Haar volume is given by

$$\omega \;:=\; \mathrm{d}a\wedge\mathrm{d}n\;.$$

Within this context we have

$$\tilde{H} \;=\; \partial_a \,-\, 2n\partial_n \qquad \tilde{S} \;=\; \partial_n\;.$$

Proposition 1.2.2 *(i) Setting*

$$\begin{aligned} &s_e^{(\kappa)}:\mathbb{S}\to\mathbb{S}:(a,n)\mapsto(\,-a-\frac{1}{2}\log(1+\kappa n^2)\,,\,-n\,)\\ &\textit{and}\quad s_x^{(\kappa)} \;:=\; L_x s_e^{(\kappa)} L_{x^{-1}} \quad \kappa\in[0,\infty[\end{aligned}$$

defines a smooth family of symplectic symmetric spaces on $(\mathbb{S},\omega)$.

(ii) For every $\kappa>0$*, the symplectic symmetric space* $(\mathbb{S},s^{(\kappa)},\omega)$ *is isomorphic (as SSS) to the hyperbolic plane*

$$\mathbb{D} \;:=\; \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$$

with (normalized) Kahler symplectic form ω.

(iii) For $\kappa>0$*, the symplectic symmetric space* $(\mathbb{S},s^{(0)},\omega)$ *is a curvature contraction of the symplectic symmetric space* $(\mathbb{S},s^{(\kappa)},\omega)$.

(iv) As symmetric space, $(\mathbb{S},s^{(0)})$ *is isomorphic to the canonical Lie group symmetric space structure on* $\mathbb{S}$.

(v) The transvection Lie algebra $\mathfrak{g}_0$ *of* $(\mathbb{S},s^{(0)})$ *is isomorphic to the Lie algebra of the Poincaré group* $SO(1,1)\ltimes\mathbb{R}^2$.

(vi) As symplectic symmetric space, $(\mathbb{S},s^{(0)},\omega)$ *is canonically isomorphic to the symplectic symmetric space structure on a co-adjoint orbit of* $SO(1,1)\ltimes\mathbb{R}^2$ *in* $\mathfrak{g}_0^\star$.

Proof. We consider the family of Lie algebra structures $\{\mathfrak{g}_\kappa\}_{\kappa\in\mathbb{R}^+}$ over $\mathbb{R}^3 \;:=\; \mathtt{span}_{\mathbb{R}}\{H,E,F\}$ given by the Lie bracket table:

$$\begin{aligned} [H,E] &= 2E\\ [H,F] &= -2F\\ [E,F] &= \kappa H\;. \end{aligned}$$

When $\kappa \neq 0$, $\mathfrak{g}_\kappa$ is isomorphic to $\mathfrak{g}_0 \simeq \mathfrak{sl}(2,\mathbb{R})$.
There, the element $Z := E - F$ generates a line $\mathfrak{k}_\kappa$ that consists in a maximal compact Lie subalgebra of $\mathfrak{g}_\kappa$.
In basis $\{H, E, F\}$, the Killing form B_κ takes the following matrix form:

$$[B_\kappa] = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4\kappa \\ 0 & 4\kappa & 0 \end{pmatrix} .$$

The dual metric $B^\star_\kappa$ on $\mathfrak{g}^\star_\kappa$ defined by transporting B_κ from $\mathfrak{g}_\kappa$ to $\mathfrak{g}^\star_\kappa$ under the equivariant identification

$$\sharp : \mathfrak{g}^\star_\kappa \to \mathfrak{g}_\kappa : \Xi \mapsto {}^\sharp\Xi$$

$$B_\kappa({}^\sharp\Xi\,,\,X) := < \Xi, X >$$

has, within the dual basis $\{H^*, E^*, F^*\}$ of $\{H, E, F\}$, the following matrix form:

$$[B^\star_\kappa] = \frac{1}{4\kappa}\begin{pmatrix} \frac{\kappa}{2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

Rescaling, we therefore obtain of the dual $\mathfrak{g}^\star_\kappa$ a co-adjoint-invariant scalar product η_κ whose matrix form in basis $\{H^*, E^*, F^*\}$ is given by

$$[\eta_\kappa] := \begin{pmatrix} \frac{\kappa}{2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

The non-null spheres $S_{t,R}$ for η_κ are therefore symmetric spaces whose common transvection group is realized by the adjoint group G_κ of $\mathfrak{g}_\kappa$ (isomorphic to $\mathrm{PSL}(2,\mathbb{R})$) acting on $S_{\kappa,R}$ by the restricted co-adjoint action. Each $S_{\kappa,R}$ then turns into a SSS when equipped with its canonical symplectic form —the so-called *KKS form*— associated to its nature of co-adjoint orbit.
Observing that the element $o := E^* - F^*$ is fixed by K_κ (for $(E^* - F^*) \circ \mathtt{ad}_Z = 0$), we get that, within this picture, the hyperbolic plane $\mathbb{D}$ is realized by the connected component of $S_{\kappa,-2}$.
Setting

$$\begin{array}{rcl} \mathfrak{n} & := & \mathbb{R}E \\ \mathfrak{a} & := & \mathbb{R}H \\ A & := & \exp\mathfrak{a} \\ N & := & \exp\mathfrak{n} \\ \mathbb{S} & := & A.N \\ K_\kappa & := & \exp\mathfrak{k}_\kappa\ , \end{array}$$

the Iwasawa decomposition

$$G_\kappa = \mathbb{S}K_\kappa$$

yields a global diffeomorphism

$$\mathbb{S} \to \mathbb{D} \subset \mathfrak{g}^\star_\kappa : x \mapsto x.o\ .$$

The above diffeomorphism is $\mathbb{S}$-equivariant when endowing the source with the action of $\mathbb{S}$ on itself by left-translations.
Coordinating the Iwasawa factor $\mathbb{S}$ via

$$\mathbb{R}^2 \to \mathbb{S} : (a, n) \mapsto \exp(aH)\exp(nE)$$

or, equivalently, its orbit $\mathbb{D} \subset S_{\kappa,-2}$ under

$$\mathbb{R}^2 \to \mathbb{D} : (a, n) \mapsto \exp(aH)\exp(nE).o\ ,$$

a small computation realizes the symmetry s_o at the level of $\mathbb{S}$ as

$$\mathbb{S} \to \mathbb{S} : (a, n) \mapsto (\,-a - \frac{1}{2}\log(1 + \kappa n^2)\,,\,-n\,)\ .$$

The coordinate system (a, n) enjoys the property of being *Darboux*, in the sense that the following 2-form:

$$\omega \ := \ \mathrm{d}a \wedge \mathrm{d}n \ .$$

is left-invariant. It therefore corresponds to a normalization of both a Haar volume and the KKS form.

The limit $\kappa \to 0$ yields now a curvature contraction of the hyperbolic plane $(\mathbb{D}, \underline{s}, \omega)$ into a symplectic symmetric space denoted hereafter $(\mathbb{M}, s, \omega)$ whose transvection Lie algebra $\mathfrak{g}_0$ is isomorphic to the Poincaré Lie algebra $so(1,1) \ltimes \mathbb{R}^2$.
The spheres $S_{\kappa,-2}$ contract to spheres $S_{0,-2}$ in the inner product space $(\mathfrak{g}_0^\star, \eta_0)$. Warn the fact that in this limit $\kappa = 0$ the scalar product η_0 is now degenerated. As co-adjoint orbits in $\mathfrak{g}_0^\star$, these spheres $S_{0,-2}$ corresponds to "massive" orbits of the Poincaré group $SO(1,1) \ltimes \mathbb{R}^2$.
Regarding the symplectic symmetric space structure on $S_{\kappa,-2}$, in the limit $\kappa = 0$, one has the transvection iLa $(\mathfrak{g}_0, \sigma)$ with non-vanishing brackets:

$$\begin{array}{rcl} [H, E] & = & 2E \\ [H, F] & = & -2F \end{array}$$

and "contracted Cartan involution":

$$\sigma H = -H \quad \sigma E = -F \ .$$

The associated eigenspace decomposition is given by

$$\mathfrak{g}_0 \ = \ \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

with

$$\mathfrak{k}_0 \ = \ \mathtt{span}\{H \, , \, E + F\} \quad \mathfrak{p}_0 \ = \ \mathtt{span}\{Z \ := \ E - F\} \ .$$

At last, we observe that the transvection iLa $(\mathfrak{g}_0, \sigma)$ is isomorphic to the transvection iLa of the Lie group $\mathbb{S} \ = \ AN$ of affine transformations of the real line. Indeed, denoting by $\mathfrak{s} \ := \ \mathfrak{a} \oplus \mathfrak{n}$ its Lie algebra, from what was described above in Example 1.1.1, we have

$$\mathfrak{L} \ = \ \mathfrak{s} \quad , \mathfrak{z} \ = \ \{0\} \quad , \mathfrak{L}' \ = \ \mathbb{R}(E, E) \ .$$

The iLa isomorphism is then given by

$$\begin{array}{ccc} \mathfrak{L}' \oplus \mathfrak{p}_2 & \longrightarrow & \mathfrak{k}_0 \oplus \mathfrak{p}_0 \\ (E, E) & \mapsto & E - F \\ (E, -E) & \mapsto & E + F \\ (H, -H) & \mapsto & H \ . \end{array}$$

■

Chapter 2

The retract method in Formal Deformation Quantization

2.1 Star-products on symplectic manifolds

Definition 2.1.1 *Let (M, ω) be a symplectic manifold with associated symplectic Poisson bracket denoted by $\{\, ,\, \}^\omega$. Let us denote by $C^\infty(M)[[\nu]]$ the space of formal power series with coefficients in $C^\infty(M)$ in the formal parameter ν. A* star-product *on (M, ω) is a $\mathbb{C}[[\nu]]$-bilinear associative algebra structure on $C^\infty(M)[[\nu]]$:*

$$\star_\nu : C^\infty(M)[[\nu]] \times C^\infty(M)[[\nu]] \to C^\infty(M)[[\nu]]$$

such that, viewing $C^\infty(M)$ embedded in $C^\infty(M)[[\nu]]$ as the zero order coefficients, and writing for all $u, v \in C^\infty(M)$:

$$u \star_\nu v \;=: \sum_{k=0}^{\infty} \nu^k C_k(u, v) \;,$$

one has

(1) $C_0(u, v) \;=\; uv$ (pointwise multiplication of functions).

(2) $C_1(u, v) - C_1(v, u) \;=\; \{u, v\}^\omega$.

(3) For every k, the coefficient $C_k : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a bi-differential operator. These are called the cochains *of the star-product.*

A star-product is called natural *when C_2 is a second order bi-differential operator (cf. [Lo69]).*

Definition 2.1.2 *Two such star-products $\star^j_\nu$ $(j = 1, 2)$ are called* equivalent *if there exists a formal power series of differential operators of the form*

$$T \;=\; \mathtt{id} \;+\; \sum_{k=1}^{\infty} \nu^k T_k \;:\; C^\infty(M) \to C^\infty(M)[[\nu]] \tag{2.1}$$

such that for all $u, v \in C^\infty(M)$:

$$T(u) \star^2_\nu T(v) \;=\; T(u \star^1_\nu v) \;, \tag{2.2}$$

where we denote by T the $\mathbb{C}[[\nu]]$-linear extension of (2.1) to $C^\infty(M)[[\nu]]$.

Since the zeroth order of such an equivalence operator T is the identity, hence invertible, the followingl argument implies that T itself is invertible as a $\mathbb{C}[[\nu]]$-linear operator on $C^\infty(M)[[\nu]]$.

Lemma 2.1.1 *Let $(\mathbb{A}, I)$ be a unital associative algebra over $\mathbb{R}$ and $A \;:=\; I \;+\; \sum_{k=1}^{\infty} \nu^k A_k$ be an element of $\mathbb{A}[[\nu]]$. Then A admits an inverse A^{-1} in $\mathbb{A}[[\nu]]$.*

Proof. One readily check that $A^{-1} = I + \sum_{k=1}^{\infty} \nu^p A'_p$ where the A'_p are defined as solutions of the recurrence:

$$A'_p = -A_p - \sum_{n+k=p;\, n,k\geq 1} A_k A'_n .$$

■

One denotes its inverse by

$$T^{-1} : C^\infty(M)[[\nu]] \to C^\infty(M)[[\nu]] .$$

We the adopt the following notation.

Definition 2.1.3 *Let $\star^j_\nu$ $(j = 1, 2)$ be two star-products on (M, ω) that are equivalent to each other under an equivalence operator T as in Definition (2.1.2). We encode equation (2.2) by the notation*

$$\star^2_\nu = T(\star^1_\nu) .$$

Let now G be a group of transformations of M.

Definition 2.1.4 *A star-product $\star_\nu$ on (M, ω) is called G-*invariant *if its cochains are G-invariant bi-differential operators; for every $g \in G$, we write:*

$$g^\star(u \star v) = g^\star(u) \star_\nu g^\star(v) \quad (u, v \in C^\infty(M)) .$$

*Two G-invariant star-products $\star^j_\nu$ $(j = 1, 2)$ are called G-*equivariantly equivalent *if there exists a G-commuting equivalence T between them:*

$$T(\star^1_\nu) = \star^2_\nu \quad \text{and} \quad T(g^\star u) = g^\star T(u)$$

for all $g \in G$ and $u \in C^\infty(M)$.

Theorem 2.1.1 *Let (M, ω) be a symplectic manifold. Let G be a group of symplectic transformations of (M, ω) that preserves an affine connection on M. Then, the set of G-equivariant equivalence classes of G-invariant star-products on (M, ω) is canonically parametrized by the space of formal power series, $H^2_{dR}(M)^G[[\nu]]$, with coefficients in the G-invariant second de Rham cohomology space of M*

In the above statement, we mention the G-invariant de Rham complex. By this, in degree two, we mean the following. The de Rham differential in degree one restricts to the differentiable one-forms that are G-invariant and ranges in the closed two-forms that are also G-invariant. The cohomology space associated to this restriction defines $H^2_{dR}(M)$.

Remark 2.1.1 Theorem 2.1.1 summarizes a long series of works by many people in deformation quantization of symplectic manifolds. The existence of star-products on symplectic manifolds has essentially three sources : [DL83, OMY91, Fe94]. Where, to my opinion, the third one ([Fe94]) is the most constructive and suitable for applications. Classification under equivalence resulted from a combination of [Fe94], [G79] and [Be93]. Published later and independently, one finds it also in [NT95].
For the affine G-invariant case, see [BBG98].

We end this section with

Definition 2.1.5 *A* derivation *of a star-product $\star$ on (M, ω) is a formal series*

$$D = \sum_{k=0}^{\infty} \nu^k D_k$$

of differential operators $\{D_k\}$ on $C^\infty(M)$ such that for all $u, v \in C^\infty(M)$, one has

$$D(u \star v) = (Du) \star v + u \star (Dv) .$$

The (Lie algebra) of derivations of $\star$ is denoted by $\mathfrak{Der}(\star)$.

Proposition 2.1.1 *[GR99] When M is simply connected, every derivation D of $\star$ is* interior *in the sense that there exists an element $\lambda_D \in C^\infty(M)[[\nu]]$ such that*

$$\nu D u = [\lambda_D\, ,\, u]_\star \quad (\forall u)$$

where we set

$$[u, v]_\star = u \star v - v \star u .$$

2.2 Drinfel'd twists and symplectic Lie groups

We consider a Lie group $\mathbb{L}$ with Lie algebra $\mathfrak{L}$ whose we denote by $\mathcal{U}(\mathfrak{L})$ the enveloping algebra. The latter underlies a Hopf algebra structure whose we consider the associative product

$$. \; : \mathcal{U}(\mathfrak{L})\otimes\mathcal{U}(\mathfrak{L})\to\mathcal{U}(\mathfrak{L}) : X\otimes Y\to X.Y$$

and co-product

$$\Delta : \mathcal{U}(\mathfrak{L})\to\mathcal{U}(\mathfrak{L})\otimes\mathcal{U}(\mathfrak{L}) : X\mapsto\Delta(X)\;.$$

These structures naturally $\mathbb{C}[[\nu]]$-linearly extend to the space of formal power series $\mathcal{U}(\mathfrak{L})[[\nu]]$, endowing it with a bi-algebra structure.

Definition 2.2.1 *A* Drinfel'd twist based on $\mathfrak{L}$ *is an element* $F\in\mathcal{U}(\mathfrak{L})\otimes\mathcal{U}(\mathfrak{L})[[\nu]]$ *of the form*

$$F \;=\; 1\otimes 1 + \sum_{k=1}^{\infty}\nu^k F_k \tag{2.3}$$

such that the following cocycle relation *holds:*

$$(\Delta\otimes\mathtt{id})(F).(F\otimes 1) \;=\; (\mathtt{id}\otimes\Delta)(F).(1\otimes F)\;. \tag{2.4}$$

In the above relation (2.4), the "." stands for the algebra structure on $\mathcal{U}(\mathfrak{L})\otimes\mathcal{U}(\mathfrak{L})\otimes\mathcal{U}(\mathfrak{L})[[\nu]]$ naturally induced by the product on $\mathcal{U}(\mathfrak{L})$.
Another way to formulate the notion of Drinfel'd twist is the following one. Exactly along the same lines that a left-differential operator on $\mathbb{L}$ defines an element of $\mathcal{U}(\mathfrak{L})$, every element $F_k\in\mathcal{U}(\mathfrak{L})\otimes\mathcal{U}(\mathfrak{L})$ occurring in a series such as (2.3) may be interpreted as the value at the unit e of $\mathbb{L}$ of a uniquely associated left-invariant *bi-differential* operator

$$\widetilde{F}_k : C^\infty(\mathbb{L})\times C^\infty(\mathbb{L})\to C^\infty(\mathbb{L})\;.$$

To the series F is therefore uniquely associated a formal power series of left-invariant differential operators:

$$\widetilde{F} \;=\; m_0 + \sum_{k=1}^{\infty}\nu^k\widetilde{F}_k \tag{2.5}$$

where m_0 denotes the pointwise multiplication of functions on $C^\infty(\mathbb{L})$. One then has that the cocycle condition (2.4) exactly corresponds to associativity, namely one has

Proposition 2.2.1 *[Drinfel'd] An element* $F\in\mathcal{U}(\mathfrak{L})\otimes\mathcal{U}(\mathfrak{L})[[\nu]]$ *is a Drinfel'd twist if and only if the associated element* $\widetilde{F}$ *(cf. (2.5)) consists in the cochain series of a left-invariant formal star-product* $\tilde{\star}^F_\nu$ *on* $\mathbb{L}$ *i.e. an left-invariant associative algebra law on* $C^\infty(\mathbb{L})[[\nu]]$ *that satisfies conditions (1) and (3) of Definition 2.1.1.*

Given such a star-product, one readily checks that the associativity at order one in the deformation parameter ν corresponds to the property that the skewsymmetrization of the first order cochains is a *Poisson bracket*. Namely, one has in our case

Lemma 2.2.1 *The element*

$$\{\,,\,\}^F : C^\infty(\mathbb{L})\times C^\infty(\mathbb{L})\to C^\infty(\mathbb{L}) : (u,v)\mapsto\widetilde{F}_1(u,v)-\widetilde{F}_1(v,u)$$

defines a structure of Lie algebra on $C^\infty(\mathbb{L})$ *with the additional property that for every* $u\in C^\infty(\mathbb{L})$*, the element*

$$\Xi^F_u \;:=\; \{\,u\,,\,.\,\}^F$$

is a vector field on $\mathbb{L}$.

From left-invariance and Jacobi, one easily proves

Proposition 2.2.2 *The tangent distribution $\tilde{\mathfrak{s}}^F$ on $\mathbb{L}$ defined by*

$$x \mapsto \tilde{\mathfrak{s}}^F_x \ := \ \{\Xi^F_u|_x \ : \ u \in C^\infty(\mathbb{L})\}$$

is left-invariant i.e. preserved by the left-translations. In particular, it has constant rank r. Moreover, it is smooth and involutive *in the sense that the bracket of vector fields is internal.*

The distribution $\tilde{\mathfrak{s}}^F$ is therefore integrable in the sense that the group manifold $\mathbb{L}$ foliates into a collection of r-dimensional immersed leaves all tangent to $\tilde{\mathfrak{s}}^F$. From all this, one then immediately gets

Proposition 2.2.3 *(i) The leaf $\mathbb{S}^F$ tangent to $\tilde{\mathfrak{s}}^F$ through the unit e consists in an immersed Lie sub-group of $\mathbb{L}$.*

(ii) The Poisson bi-vector field

$$\tilde{w}^F(u,v) \ := \ \{u,v\}^F$$

when restricted to $\mathbb{S}^F$ is symplectic *in the sense that for every $x \in \mathbb{S}^F$, the map*

$$T^\star_x(\mathbb{S}^F) \to T_x(\mathbb{S}^F) : \alpha \mapsto \iota_\alpha \tilde{w}^F_x \ =: \ {}^\sharp\alpha \tag{2.6}$$

is a linear isomorphism.

In the above statement, the symbol ι denotes the interior product of bi-vectors by one-forms.

Remark 2.2.1 *The inverse of the above isomorphism (2.6) :*

$$T_x(\mathbb{S}^F) \to T^\star_x(\mathbb{S}^F) : v \mapsto {}^\flat v$$

corrseponds to an element ω^F_x of $T^\star_x(\mathbb{S}^F) \otimes T^\star_x(\mathbb{S}^F)$:

$$\omega^F_x(v,w) \ := < {}^\flat v \, , \, w > \ .$$

The field

$$\omega^F : x \mapsto \omega^F_x$$

turns out to be a symplectic 2-form on $\mathbb{S}^F$ which is invariant under the left-translations.

The above discussion motivates the following definition (see [LM88]) which singles out the class of Lie groups that are semi-classical limits of Drinfel'd twists.

Definition 2.2.2 *A* symplectic Lie group *is a pair $(\mathbb{S}, \omega)$ where $\mathbb{S}$ is a Lie group and ω is a symplectic structure on $\mathbb{S}$ that is invariant under the left-translations:*

$$L^\star_x \omega \ = \ \omega \quad (\forall x \in \mathbb{S}) \ .$$

One may wonder whether any symplectic Lie group comes from a Drinfel'd twist. This is indeed the case.

Proposition 2.2.4 *Every symplectic Lie group admits a natural left-invariant star-product.*

Proof. We start by considering the canonical affine symmetric connection $\nabla^{\mathbb{S}}$ on $\mathbb{S}$ associated with the structure of symmetric space of the Lie group $\mathbb{S}$. This connection is not compatible with ω in general in the sense that only when $\mathbb{S}$ is Abelian, the 2-form ω is parallel [Bi95]. But the usual trick (due to Lichnérowicz) produces a symplectic connection ∇ from the data of $\nabla^{\mathbb{S}}$:

$$\nabla_X Y \ := \ \nabla_X Y \ + \ \frac{1}{3}^\sharp \left(\iota_X \nabla^{\mathbb{S}}_Y \omega \ + \ \iota_Y \nabla^{\mathbb{S}}_X \omega\right) \ .$$

This connection ∇ is clearly left-invariant. Now, we use Fedosov's method to uniquely associate a star-product $\star_\nu$ to the data of (ω, ∇) [Fe94]. The constructed star-product then answers the question. ∎

As a consequence of the above proof, or more precisely, of the fact that every symplectic Lie group admits a left-invariant symplectic connection, we are in the situation where we can apply Theorem 2.1.1 to this specific situation:

Proposition 2.2.5 *On a simply connected symplectic Lie group $\mathbb{S}$ with Lie algebra $\mathfrak{s}$, the left-equivariant equivalence classes of left-invariant (symplectic) star-products are parametrized by the sequences of elements of the second Chevalley cohomology space $H^2_{Chev}(\mathfrak{s})$ associated to the trivial representation of $\mathfrak{s}$ on $\mathbb{R}$.*

2.3 Intertwining Drinfel'd twists – the Retract method

2.3.1 Symplectic twists and naturality

We consider a simply connected *solvable*[1] symplectic Lie group $(\mathbb{S},\omega)$ with Lie algebra $\mathfrak{s}$. The left-invariant symplectic structure ω determines a bilinear two-form

$$\Omega \;:=\; \omega_e$$

on $\mathfrak{s}$.

Definition 2.3.1 *A Drinfel'd twist F based on $\mathfrak{s}$ is said to be* based on $(\mathfrak{s},\Omega)$ *if its associated left-invariant Poisson structure on $\mathbb{S}$ coincides with the Poisson bracket associated to ω.*

Definition 2.3.2 *Let $F^{(j)}$ $(j=1,2)$ be two Drinfel'd twists based on $(\mathfrak{s},\Omega)$. We say that they are* equivalent *when their associated left-invariant formal star-products $\tilde{\star}_\nu^{F^{(j)}}$ live in the same $\mathbb{S}$-left-equivariant equivalence class.*

In the present section, we give a description of the equivalence classes of Drinfel'd twists.

Definition 2.3.3 *Let F be a Drinfel'd twist based on $(\mathfrak{s},\Omega)$ with associated left-invariant star-product $\tilde{\star}_\nu^F$. The* internal symmetry Lie algebra $\mathfrak{g}_F$ *of F is the Lie algebra of vector fields on $\mathbb{S}$ that derivate $\tilde{\star}_\nu^F$:*

$$\mathfrak{g}_F \;:=\; \Gamma^\infty(T(\mathbb{S}))\cap\mathfrak{Der}(\tilde{\star}_\nu^F)\;.$$

Remark 2.3.1 *One has the canonical inclusion*

$$\mathfrak{s}\hookrightarrow\mathfrak{g}_F : X\mapsto X^\star$$

with

$$X^\star_x \;:=\; \frac{\mathrm{d}}{\mathrm{d}t}|_0\exp(-tX)x\;.$$

Lemma 2.3.1 *When a twist F based on $(\mathfrak{s},\Omega)$ is natural, in the sense that $\tilde{\star}_\nu^F$ is natural, then its internal symmetry Lie algebra is finite dimensional.*

Proof. Such a natural star-product determines a an affine connection whose $\mathfrak{g}_F$ is a Lie sub-algebra of derivations [GR99]. ∎

2.3.2 The retract method I: describing equivalences as evolution solutions

We now consider, within a given equivalence class of Drinfeld's twists based on $(\mathfrak{s},\Omega)$, a natural element F (from what we know so far, such an element always exists). We denote by

$$\sharp \;:=\; \tilde{\star}_\nu^F$$

the associated left-invariant star-product on $\mathbb{S}$ and by

$$\mathfrak{g} \;:=\; \mathfrak{g}_F$$

its internal symmetry Lie algebra.
We also fix another Drinfel'd twist equivalent to F with associated left-invariant star-product denoted by $\star$, and no further requirement.

Proposition 2.3.1 [Relevance of the retract method] *Let $\mathtt{Hom}_{\mathfrak{s}}(\mathfrak{g},\mathfrak{Der}(\star))$ the space of Lie algebra homomorphisms from $\mathfrak{g}$ to the Lie algebra $\mathfrak{Der}(\star)$ of derivations of $(C^\infty(\mathbb{S})[[\nu]],\star)$ that are* $\mathfrak{s}$-relative *in the sense that every element D of $\mathtt{Hom}_{\mathfrak{s}}(\mathfrak{g},\mathfrak{Der}(\star))$ restricts to $\mathfrak{s}$ as the identity:*

$$D_X \;=\; X^\star\;.$$

Then, $\mathtt{Hom}_{\mathfrak{s}}(\mathfrak{g},\mathfrak{Der}(\star))$ is finite dimensional over the formal field $\mathbb{R}[[\nu]]$.

[1] This hypothesis of solvability is in fact not essential. However, it brings proofs that are more Lie theoretic.

Before passing to the proof, we observe

Lemma 2.3.2 *Let V be a finite dimensional real vector space. Consider $f \in C^\infty(\mathbb{S}) \otimes V$ and $A \in \mathrm{Hom}_\mathbb{R}(\mathfrak{s}, \mathtt{End}(V))$. Then, the space of solutions of the equation*

$$X^\star \underline{\lambda} \;=\; A(X)\underline{\lambda} \,+\, f$$

is finite dimensional.

Proof. Consider an element $0 \neq X_1 \in \mathfrak{s} \backslash [\mathfrak{s}, \mathfrak{s}]$ and a vector sub-space $\mathfrak{s}_2 \;<\; \mathfrak{s}$ such that

$$\mathfrak{s} \;=\; \mathbb{R}X_1 \oplus \mathfrak{s}_2 \quad \text{and} \quad [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}_2 \;.$$

The space $\mathfrak{s}_2$ is then automatically a Lie sub-algebra of $\mathfrak{s}$.
Denote by $\mathbb{S}_2$ the analytic sub-group of $\mathbb{S}$ with Lie algebra $\mathfrak{s}_2$ and consider the local coordinate system

$$\mathbb{R} \times \mathbb{S}_2 \to \mathbb{S} : (x^1, y) \mapsto \exp(-x^1 X_1).y \;.$$

Within this coordinate system, the equation

$$X_1^\star \underline{\lambda} \;=\; A(X_1)\underline{\lambda} \,+\, f$$

reads

$$\partial_{x^1} \underline{\lambda} \;=\; A_1 \underline{\lambda} + f \quad (A_1 \;:=\; A(X_1)) \;,$$

whose every solution is of the form

$$\underline{\lambda}(x^1, y) \;=\; \exp(x^1 A_1)\underline{\lambda}_2(y) \,+\, \mu_1(x^1, y)$$

with

$$\mu_1(x^1, y) \;:=\; \int_{x_0^1}^{x^1} \exp(-tA_1) f(\exp(tX_1), y)\, \mathrm{d}t$$

and $\underline{\lambda}_2 \in C^\infty(\mathbb{S}_2) \otimes V$ arbitrary.
In particular, in the case $m = 1$ the equation admits an affine space of solutions of dimension $\dim V$.
We now proceed by induction on $\dim \mathbb{S} \;=: m$.
The Lie sub-algebra $\mathfrak{s}_2$ being solvable as well, one fixes an element $0 \neq X_2 \in \mathfrak{s}_2 \backslash [\mathfrak{s}_2, \mathfrak{s}_2]$ and a vector sub-space $\mathfrak{s}_3$ of $\mathfrak{s}_2$ such that

$$\mathfrak{s}_2 \;=\; \mathbb{R}X_2 \oplus \mathfrak{s}_3 \quad \text{and} \quad [\mathfrak{s}_2, \mathfrak{s}_2] \subset \mathfrak{s}_3 \;.$$

Denote by $\mathbb{S}_3$ the analytic sub-group of $\mathbb{S}_2$ with Lie algebra $\mathfrak{s}_3$ and consider the local coordinate system

$$\mathbb{R}^2 \times \mathbb{S}_2 \to \mathbb{S} : (x^1, x^2, z) \mapsto \exp(-x^1 X_1).\exp(-x^2 \mathtt{Ad}_{x_1^{-1}} X_2).z \;,$$

where

$$x_1 \;:=\; \exp(-x^1 X_1) \;.$$

Within this coordinate system, setting

$$x_2 \;:=\; \exp(-x^2 \mathtt{Ad}_{x_1^{-1}} X_2) \;,$$

one has

$$X_2^\star \;=\; \partial_{x^2} \;.$$

Indeed:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \exp(-tX_2) x_1 x_2 z \;&=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 x_1 x_1^{-1} \exp(-tX_2) x_1 x_2 z \\ =\; L_{x_1 \star} &\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \exp(-t\mathtt{Ad}_{x_1^{-1}} X_2) \exp(-x^2 \mathtt{Ad}_{x_1^{-1}} X_2) z \\ =\; L_{x_1 \star} &\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \exp(-(t + x^2)\mathtt{Ad}_{x_1^{-1}} X_2) z \;. \end{aligned}$$

Therefore:

$$X_2^\star\underline{\lambda}(x^1,x^2,z) \;=\; \exp(x^1A_1)\partial_{x^2}\underline{\lambda}_2(x^2,z)+\mu_2(x^1,x^2,y)$$

with

$$\mu_2 \;:=\; \partial_{x^2}\mu_1\;.$$

This leads, with $A_2 \;:=\; A(X_2)$, to

$$A_2\underline{\lambda}+f \;=\; A_2\exp(x^1A_1)\underline{\lambda}_2+A_2\mu_2+f\;.$$

Hence, the equation with $X \;:=\; X_2$ reads:

$$\partial_{x^2}\underline{\lambda}_2 \;=\; A_2'\underline{\lambda}_2+f_2$$

with

$$A_2' \;:=\; \exp(-x^1A_1).A_2.\exp(x^1A_1) \quad\text{and}\quad f_2 \;:=\; \exp(-x_1A_1)(A_2\mu_2+f)\;.$$

The induction then yields the assertion. ■

Proof of the Proposition. Consider a vector sub-space W of $\mathfrak{g}$ supplementary to $\mathfrak{s}$:

$$\mathfrak{g} \;=\; \mathfrak{s}\oplus W$$

which is assumed to be non-trivial. Let $\{Y_a\}_{a\in\{1,\ldots,n\}}$ be a basis of W. Furthermore, we will consider a basis $\{X_j\}_{j\in\{1,\ldots,m\}}$ of $\mathfrak{s}$.
Denote by C and $\underline{C}$ the structure constants associated to the action of $\mathfrak{s}$:

$$[X_j,Y_a] \;=:\; C_{ja}^kX_k\;+\;\underline{C}_{ja}^bY_b\;.$$

Let $D\in\mathtt{Hom}_{\mathfrak{s}}(\mathfrak{g},\mathfrak{Der}(\star))$.
Since $\mathbb{S}$ is simply connected, we have functions $\{\lambda_j\}$ and $\{\underline{\lambda}_a\}$ such that

$$\nu X_j^\star u \;=\; [\lambda_j\,,\,u]_\star \quad\text{and}\quad \nu D_{Y_a}u \;=\; [\underline{\lambda}_a\,,\,u]_\star \quad (u\in C^\infty(\mathbb{S}))\;.$$

Moreover, the Lie algebra homomorphism condition on D implies that for all X_j and Y_a, the element

$$c_{ja} \;:=\; C_{ja}^k\lambda_k\;+\;\underline{C}_{ja}^b\underline{\lambda}_b-[\lambda_j\,,\,\underline{\lambda}_a]_\star$$

is central in $(C^\infty(\mathbb{S})[[\nu]],\star)$ hence a formal constant (because symplectic).
Therefore for every j, we have two matrices C_j and $\underline{C}_j$:

$$[C_j]_a^k \;:=\; -\nu^{-1}C_{ja}^k \quad\text{and}\quad -[\underline{C}_j]_a^b \;:=\; \nu^{-1}\underline{C}_{ja}^b$$

such that

$$X_j^\star\underline{\lambda} \;=\; -C_j\lambda\;+\;-\underline{C}_j\underline{\lambda}\;+\;c_j \tag{2.7}$$

where c_j is a vector in $\mathbb{R}^n[[\nu,\nu^{-1}]$.
The assertion is the an immediate consequence of the Lemma. ■

Now, let us discuss the intertwiners between $\star$ and $\sharp$. We start by observing

Lemma 2.3.3 *Because it commutes with the left-regular action, any (formally invertible) intertwiner T between $\star$ and $\sharp$ (i.e. $T(\star)=\sharp$) must be a convolution operator of the form*

$$T^{-1}\varphi(x) \;=<u\,,\,L_x^\star\varphi>$$

where $u\in\mathcal{D}'(\mathbb{S})[[\nu]]$ denotes the (formal) distributional kernel of T^{-1}.

Proof. By the kernel theorem of Schwartz, every differential operator A on $C^\infty(\mathbb{S})$ restricts to $\mathcal{D}(\mathbb{S})$ under the form

$$A\varphi \;=:\; [x\mapsto<u_A(x),\varphi>]$$

where $u_A\in C^\infty(\mathbb{S},\mathcal{D}'(\mathbb{S}))$. When left-invariant, one obtains

$$A\varphi(x) \;=<u\,,\,L_x^\star\varphi> \quad\text{with}\quad u \;:=\; u_A(e)\;.$$

This applies of course to the formal power series case. ■

A similar statement of course holds for T but we will use it only later on.
Back to T^{-1}, we will adopt the following notation in accordance with the distributions that are defined by locally summable functions:

$$T^{-1}\varphi(x) \ :=: < L^\star_{x^{-1}} u \, , \, \varphi > = \int_{\mathbb{S}} u(x^{-1}y) \, \varphi(y) \, \mathrm{d}y$$

where $\mathrm{d}y$ denotes a left-invariant Haar measure on $\mathbb{S}$.

Lemma 2.3.4 *Let $u \in \mathcal{D}'(\mathbb{S})$ and A be a differential operator on $C^\infty(\mathbb{S})$. Then, for every $\varphi \in \mathcal{D}(\mathbb{S})$, the map*

$$x \mapsto < L^\star_{x^{-1}} u, \varphi >$$

belongs to $C^\infty(\mathbb{S})$, and one has

$$< A_x[L^\star_{x^{-1}} u] \, , \, \varphi > = \ A_x(< [L^\star_{x^{-1}} u] \, , \, \varphi >) \; .$$

Proof. Set

$$\tau\varphi(x) \ := < [L^\star_{x^{-1}} u] \, , \, \varphi > = < u, L^\star_x \varphi > \; .$$

Consider the case where A is a vector field with local flow ϕ:

$$A_x \ =: \ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \phi_t(x) \; .$$

One has

$$\frac{1}{t}\left(< u \, , \, L^\star_{\phi_t(x)} \varphi > - < u \, , \, \varphi >\right) \ = < u \, , \, \frac{1}{t}\left(L^\star_{\phi_t(x)}\varphi - \varphi\right) \ > \; .$$

The limit $t \to 0$ of $\frac{1}{t}\left(L^\star_{\phi_t(x)}\varphi - \varphi\right)$ exists in $\mathcal{D}(\mathbb{S})$. Hence the limit $t \to 0$ of the right hand side to the above equality exists since $u \in \mathcal{D}'(\mathbb{S})$. Which proves, that $[x \mapsto \tau\varphi(x)]$ is smooth.
Iterating one gets the assertion for higher order differential operators. ∎

Proposition 2.3.2 *Consider such an intertwiner T with distributional kernel u. Then.*

(i) the map

$$D^T : \mathfrak{g} \to \mathfrak{Der}(\star) : X \mapsto T^{-1} X^\star T$$

belongs to $\mathtt{Hom}_{\mathfrak{s}}(\mathfrak{g}, \mathfrak{Der}(\star))$.

(ii) The kernel u is a (weak) joint solution of the following evolution equations:

$$-X^\star_y[u(x^{-1}y)] \ = \ D^T_X|_x[u(x^{-1}y)] \quad (\forall X \in \mathfrak{g}) \; .$$

Proof. With the notations introduced above and based on the previous lemmas, we have

$$\begin{aligned} T^{-1}X^\star(\varphi)(x) \ &= \ \int u(x^{-1}y) \, X^\star_y(\varphi) \, \mathrm{d}y \ = \ -\int X^\star_y[u(x^{-1}y)] \, \varphi(y) \, \mathrm{d}y \\ = \ D^T_X|_x \int u(x^{-1}y) \, \varphi(y) \, \mathrm{d}y \ &= \ \int D^T_X|_x u(x^{-1}y) \, \varphi(y) \, \mathrm{d}y \end{aligned}$$

because $X^\star$ is symplectic and thus preserves the (Liouville) Haar measure and because φ is compactly supported. ∎

2.3.3 The retract method II: the general context

In this section, we consider

- a solvable, simply connected symplectic Lie group $(\mathbb{S},\omega)$ with symplectic Lie algebra $(\mathfrak{s},\Omega)$.
- A fixed finite dimensional Lie algebra $\mathfrak{g}$ that contains $\mathfrak{s}$ as Lie sub-algebra.
- A fixed Drinfel'd twist based on $(\mathfrak{s},\Omega)$ or, equivalently, a left-invariant formal star-product $\star$ on $(\mathbb{S},\omega)$.

Lemma 2.3.5 *Let D be an element in $\mathtt{Hom}_{\mathfrak{s}}(\mathfrak{g},\mathfrak{Der}(\star))$.*

(i) For every $X\in\mathfrak{g}$, the map

$$X^\star_D : C^\infty(\mathbb{S})\to C^\infty(\mathbb{S}); f\mapsto D_X(f) \mod (\nu) \ =: \ X^\star_D f$$

is a symplectic vector field.

(ii) The map

$$\mathfrak{g}\to\Gamma^\infty(T(\mathbb{S})) : X\mapsto X^\star_D \tag{2.8}$$

is a homomorphism of Lie algebras.

(iii) For every $x\in\mathbb{S}$, we set

$$\mathfrak{k}_x \ := \ \{X\in\mathfrak{g}\mid \ X^\star_D|_x \ = \ 0\}$$

and, at the unit element $e\in\mathbb{S}$:

$$\mathfrak{k} \ := \ \mathfrak{k}_e \, .$$

Then, at every $x\in\mathbb{S}$, the stabilizer $\mathfrak{k}_x$ is a Lie sub-algebra of $\mathfrak{g}$ and one has

$$\mathfrak{k}_x \ = \ \mathtt{Ad}_x\mathfrak{k} \, .$$

Proof. As previously, we consider the map

$$\underline{\lambda} : \mathfrak{g}\to C^\infty(\mathbb{S})[[\nu]]; X\mapsto\underline{\lambda}_X$$

determined (up to additive formal constants) by the conditions:

$$D_X\varphi \ = \ \frac{1}{2\nu}\,[\underline{\lambda}_X,\varphi]_\star \ .$$

Taking zero orders then yields items (i) and (ii).
Item (iii) is obvious. ∎

Definition 2.3.4 *Let $(M,\mathrm{d}\mu)$ be a smooth manifold endowed with a volume form and denote by $L^2(M,\mathrm{d}\mu)$ the associated Hilbert space of square integrable function classes. Let A be a differential operator in $C^\infty(M)$. Denote by $A^\dagger$ its adjoint on $\mathcal{D}(M)$ with respect to the Hilbert structure:*

$$\int_M \overline{A^\dagger\varphi}\,\psi\,\mathrm{d}\mu \ := \ \int_M \overline{\varphi}\,A\psi\,\mathrm{d}\mu \quad (\varphi\,,\,\psi\in\mathcal{D}(M))\,.$$

We say that a distribution $u\in\mathcal{D}'(M)$ belongs to the weak kernel *of A when it satisfies the condition*

$$< u, A^\dagger\varphi > = \ 0$$

for every $\varphi\in\mathcal{D}(M)$.

Definition 2.3.5 *Let D be an element in $\mathtt{Hom}_{\mathfrak{s}}(\mathfrak{g},\mathfrak{Der}(\star))$.*
An element $u\in\mathcal{D}'(\mathbb{S})[[\nu]]$ such that

(a) it is a joint (weak) solution of

$$D_Z u \;=\; 0 \quad \forall Z \in \mathfrak{k} \;.$$

(b) Its singular support is reduced to the unit element:

$$\mathrm{supp}(u) \;=\; \{e\} \;.$$

(c) It satisfies the following semi-classical condition:

$$u \bmod \nu \;=\; \delta_e \;.$$

is called a D-retract.

The following statement is classical.

Proposition 2.3.3 *The space of left-invariant differential operators on* $\mathbb{S}$ *(canonically isomorphic to* $\mathcal{U}(\mathfrak{s})$*) identifies with the sub-space of distributions in* $\mathcal{D}'(\mathbb{S})$ *supported at the unit* e*.*

Proof. We first recall Peetre's theorem that realizes the space of differential operators on a manifold M as the space of endomorphisms of $C^\infty(M)$ that reduce the supports [P59].
Let A be an element in $\mathcal{U}(\mathfrak{s})$ and $\tilde{A}$ the corresponding left-invariant differential operator on $C^\infty(\mathbb{S})$. Denote by u_A its distributional kernel:

$$\tilde{A}_x\varphi \;=: < u_A \, , \, L^\star_x\varphi > \;\;.$$

Assume that $\mathrm{supp}\varphi$ does not contain the unit e. Then, since $\tilde{A}$ is differential, one has the inclusion $\mathrm{supp}\tilde{A}\varphi \subset \mathrm{supp}\varphi$. Hence

$$\tilde{A}_e\varphi \;=\; 0 \;= < u_A \, , \, \varphi > \;\;.$$

In other words, the support of u_A consists in the unit.
Reciprocally, assume u to be supported on the unit e only and consider the left-equivariant mapping $\tilde{U} : \mathcal{D}(\mathbb{S}) \to C^\infty(\mathbb{S})$ defined by

$$\tilde{U}\varphi(x) \;:= < u, L^\star_x\varphi > \;\;.$$

Observing, for every $x \in \mathbb{S}$, the equality

$$\mathrm{supp}L^\star_x\varphi \;=\; L_{x^{-1}}\mathrm{supp}\varphi \;,$$

one deduces that, for every point x belonging to $\mathrm{supp}\tilde{U}\varphi$:

$$\tilde{U}\varphi(x) \;\neq 0 \;\Rightarrow\; \mathrm{supp}L^\star_x\varphi \ni e \;\Rightarrow\; L_{x^{-1}}\mathrm{supp}\varphi \ni e \;\Rightarrow\; \mathrm{supp}\varphi \ni x \;,$$

hence the inclusion

$$\mathrm{supp}\tilde{U}\varphi \subset \mathrm{supp}\varphi$$

which implies differentiability by virtue of Peetre's theorem. ∎

Corollary 2.3.1 *Let* u *be a* D*-retract, and consider the modular function* $\Delta \in C^\infty(\mathbb{S})$ *defined by*

$$\mathrm{d}(y^{-1}) \;=:\; \Delta(y)\,\mathrm{d}y \;.$$

Then, the convolution operator

$$\tau_u\varphi(x) \;:= < u^\vee \, , \, L^\star_x\varphi > := < u \, , \, \Delta R^\star_{x^{-1}}(\varphi^\vee) > \quad (\varphi \in \mathcal{D}(\mathbb{S}))$$

is differential and satisfies the semi-classical condition:

$$\tau_u \;=\; \mathtt{Id} \bmod \nu \;.$$

It therefore extends to an invertible endomorphism

$$\tau_u : C^\infty(\mathbb{S})[[\nu]] \to C^\infty(\mathbb{S})[[\nu]]$$

called inverse retract operator*. Its formal inverse*

$$T_u \;:=\; \tau_u^{-1}$$

is called direct retract operator*.*

Proof. Within the same setting as in the proof of Proposition 2.3.3, we observe that

$$< u_A \, , \, \Delta R^\star_{x^{-1}}(\varphi^\vee) > = \; \tilde{A}_e \, (\Delta R^\star_{x^{-1}}(\varphi^\vee)) \; =: \; \tilde{A}^\dagger_x \varphi$$

which is a left-invariant differential operator as well. What remains is obvious. ■

Theorem 2.3.1 *(i) Let u be a D-retract with associated direct retract operator $T_u : C^\infty(\mathbb{S})[[\nu]] \to C^\infty(\mathbb{S})[[\nu]]$. Then,*

$$T_u(\star) \; =: \; \sharp^u$$

is a formal star-product on $(\mathbb{S}, \omega)$ that is $\mathfrak{g}$-invariant with respect to the infinitesimal action (2.8).

(ii) Given any symplectic action of $\mathfrak{g}$ on $(\mathbb{S}, \omega)$ that restricts to $\mathfrak{s}$ as the regular one, every $\mathfrak{g}$-invariant star-product that is $\mathbb{S}$-equivariantly equivalent to $\star$ is of the form $\sharp^u$ for a certain D-retract u.

Proof. The $\mathfrak{g}$-invariance is implied by the following condition on u:

$$D_X \tau_u \varphi \; = \; \tau_u X^\star_D \varphi$$

which, setting $v \; := \; u^\vee$, amounts to

$$< v \, , \, D_X|_x[L^\star_x \varphi] - L^\star_x X^\star_D \varphi > = \; 0 \, .$$

We now determine, for every x, the adjoint of the operator:

$$A_{X,x} : \varphi \mapsto D_X|_x[L^\star_x \varphi] - L^\star_x X^\star_D \varphi \, .$$

Let $w \in \mathcal{D}(\mathbb{S})$ and observe that left-invariance of the Haar measure and the fact that $X^\star_D$ is a symplectic vector field yield:

$$\begin{aligned} & \int w(y) \; (D_X|_x[L^\star_x \varphi](y) - L^\star_x X^\star_D \varphi(y)) \, \mathrm{d}y \\ = \quad & \int \big(D_X|_x[w(x^{-1}y)] + X^\star_D|_y[w(x^{-1}y)]\big) \; \varphi(y) \, \mathrm{d}y \, . \end{aligned}$$

Writing every X in $\mathfrak{g}$ as

$$X \; =: \; X_{\mathfrak{k}} + X_{\mathfrak{s}}$$

according to the decomposition

$$\mathfrak{g} \; = \; \mathfrak{k} \oplus \mathfrak{s} \, ,$$

we observe that

$$X^\star_D|_e \; = \; (X_{\mathfrak{s}})^\star_e \, .$$

Now, the $\mathbb{S}$-equivariance of D yields:

$$\begin{aligned} & D_X|_x[w(x^{-1}y)] + X^\star_D|_y[w(x^{-1}y)] \; = \; D_X|_x[w^\vee(y^{-1}x)] + X^\star_D|_y[w(x^{-1}y)] \\ = \quad & L^\star_{y^{-1}} \left(D_{\mathtt{Ad}_{y^{-1}}X} w^\vee\right)(x) + \left(\mathtt{Ad}_{y^{-1}} X_D\right)^\star|_e (L^\star_{x^{-1}y} w) \\ = \quad & L^\star_{y^{-1}} \left(D_{\mathtt{Ad}_{y^{-1}}X} w^\vee\right)(x) + \left(\mathtt{Ad}_{y^{-1}} X\right)_{\mathfrak{s}}{}^\star|_e (L^\star_{x^{-1}y} w) \\ = \quad & L^\star_{y^{-1}} \left(D_{\mathtt{Ad}_{y^{-1}}X} w^\vee\right)(x) + \left(\mathtt{Ad}_{y^{-1}} X\right)_{\mathfrak{s}}{}^\star|_{y^{-1}x} (w^\vee) \\ = \quad & L^\star_{y^{-1}} \left(D_{\mathtt{Ad}_{y^{-1}}X} w^\vee\right)(x) - D_{\left(\mathtt{Ad}_{y^{-1}}X\right)_{\mathfrak{s}}} w^\vee(y^{-1}x) \\ = \quad & L^\star_{y^{-1}} \left(D_{\left(\mathtt{Ad}_{y^{-1}}X\right)_{\mathfrak{k}}} w^\vee\right)(x) \; = \; \left(D_{\left(\mathtt{Ad}_{y^{-1}}X\right)_{\mathfrak{k}}} w^\vee\right)^\vee (x^{-1}y) \, . \end{aligned}$$

Hence, considering a basis $\{Z_j\}$ of $\mathfrak{k}$ and denoting by $[A(y)]$ the matrix of the linear map $\mathfrak{k} \to \mathfrak{k} : X \mapsto \left(\mathtt{Ad}_{y^{-1}} X\right)_{\mathfrak{k}}$, we get for every j:

$$D_{Z_j}|_x[w(x^{-1}y)] + {Z_j}^\star_D|_y[w(x^{-1}y)] \; = \; [A(y)]^k_j \, (D_{Z_k} w^\vee)^\vee \, (x^{-1}y) \, .$$

In other words, we have

$$< w, A_{Z_j,x}\varphi > = \sum_k < (D_{Z_k} w^\vee)^\vee \ , \ L_x^\star([A]_j^k \varphi) > \ .$$

We conclude that, whenever $u = v^\vee$ is in the weak kernel of D_{Z_k} for all k, the star product $\sharp^u$ is $\mathfrak{g}$-invariant, proving item (i).

Item (ii) follows from Proposition 2.3.2 (i) combined with the above discussion. ∎

Chapter 3

Admissible phases, Weinstein's and Ševera's areas

The question addressed in this section is : *given an orientable symmetric space M endowed with a symmetry invariant volume form μ, what are the conditions on a (symmetry invariant) three-point function*

$$S \in C^\infty(M \times M \times M, \mathbb{R})$$

which would guarantee associativity of the product

$$u \star v(x) = \int_{M\times M} e^{iS(x\,,\,.\,,\,.\,)} u \otimes v\, \mu \otimes \mu \quad ?$$

Above, by *symmetry invariant*, we mean:

$$S(s_w x, s_w y, s_w z) \;=\; S(x, y, z) \quad (\forall x, y, z, w \in M)\;.$$

When writing a (continuous) multiplication on functions via a kernel formula of the type :

$$u \star v(x) = \int_{M\times M} K(x, \,.\,, \,.\,) u \otimes v \quad ,$$

a computation shows that associativity for the multiplication $\star$ is (at least formally) equivalent to the following condition :

$$\int_M K(a,b,t)K(t,c,d)\mu(t) = \int_M K(a,\tau,d)K(\tau,b,c)\mu(\tau), \tag{3.1}$$

for every quadruple of points a, b, c, d in M. Equality (3.1) obviously holds if one can pass from one integrand to the other using a change a variables $\tau = \varphi(t)$. This motivates

Definition 3.0.1 *Let (M,μ) be a symmetric space endowed with a symmetry invariant volume form μ. A three-point kernel $K \in C^\infty(M \times M \times M)$ is called* **geometrically associative** *if for every quadruple of points a, b, c, d in M there exists a volume preserving diffeomorphism*

$$\varphi : (M,\mu) \to (M,\mu),$$

such that for all t in M :

$$K(a,b,t)K(t,c,d) = K(a,\varphi(t),d)K(\varphi(t),b,c).$$

We will prove in Proposition 3.0.1 that the following structure leads (under an additional condition) to a geometrically associative kernel.

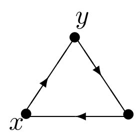
Definition 3.0.2 *Let (M,μ) be an oriented symmetric space. An* **admissible function** *$S : M \times M \times M \to \mathbb{R}$ is a smooth symmetry-invariant three-point function on M such that :*

(i) $S(x,y,z) = S(z,x,y) = -S(y,x,z)$;

(ii) for all $x \in M$, *one has:*

$$S(x,y,z) = -S(x, s_x(y), z) \qquad \forall y,z \in M.$$

Property (i) in Definition 3.0.2 naturally leads us to adopt the following "oriented graph" type notation for S :



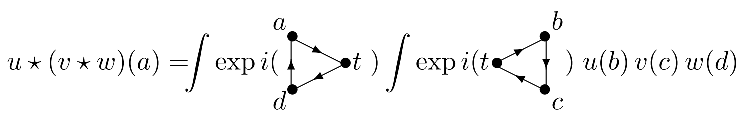
$$\stackrel{\text{def.}}{=} S(x,y,z).$$

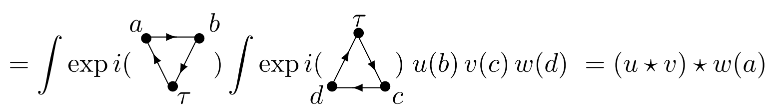
A change of orientation in such an "oriented triangle" leads to a change of sign of its value. However, the value represented by such a "triangle" does not depend on the way it "stands", only the data of the vertices and the orientation of the edges matters.

Now, given two compactly supported functions u and $v \in C_c^\infty(M)$, consider the following "product" :

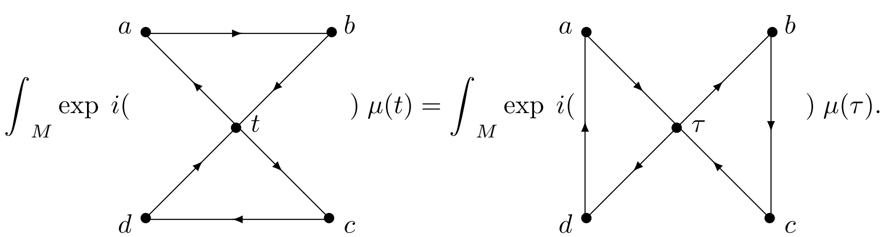
$$u \star v(x) := \int_{M\times M} u(y)v(z)e^{iS(x,y,z)}\mu(y)\,\mu(z).$$

With the above notation for S, associativity for $\star$ then formally reads as follows :





the μ-integration being taken over variables b, c, d, t and τ. This leads, for $K = e^{iS}$, to an equality between two "distribution valued functions" on $M \times M \times M \times M$ (cf. formula (3.1)): for every quadruple of points a, b, c, d in M associativity for $\star$ reads



(3.2)

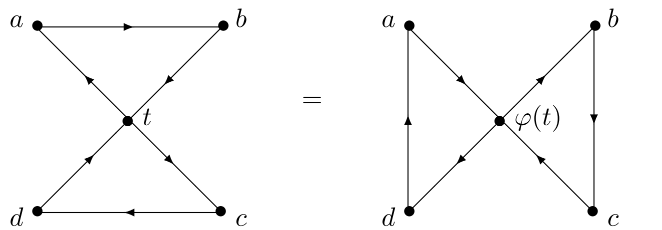
In the above formula, the diagram in the argument of the exponential in the LHS (respectively the RHS) stands for $S(a,b,t) + S(t,c,d)$ (respectively $S(a,d,\tau) + S(\tau,b,c)$).

Proposition 3.0.1 *Let S be admissible on M. Then, provided the following "cocycle condition" holds:*

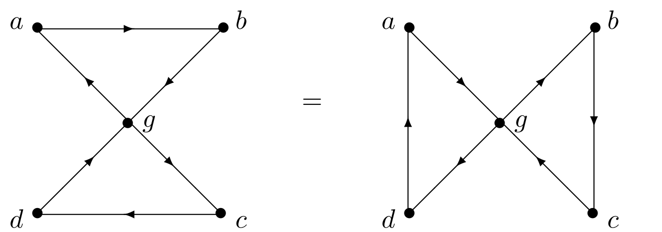
$$S(m,x,y) + S(m,y,z) + S(m,z,x) \;=\; S(x,y,z) \quad (\forall x,y,z,m \in M)\,,$$

the associated three-point kernel $K = e^{iS}$ is geometrically associative.

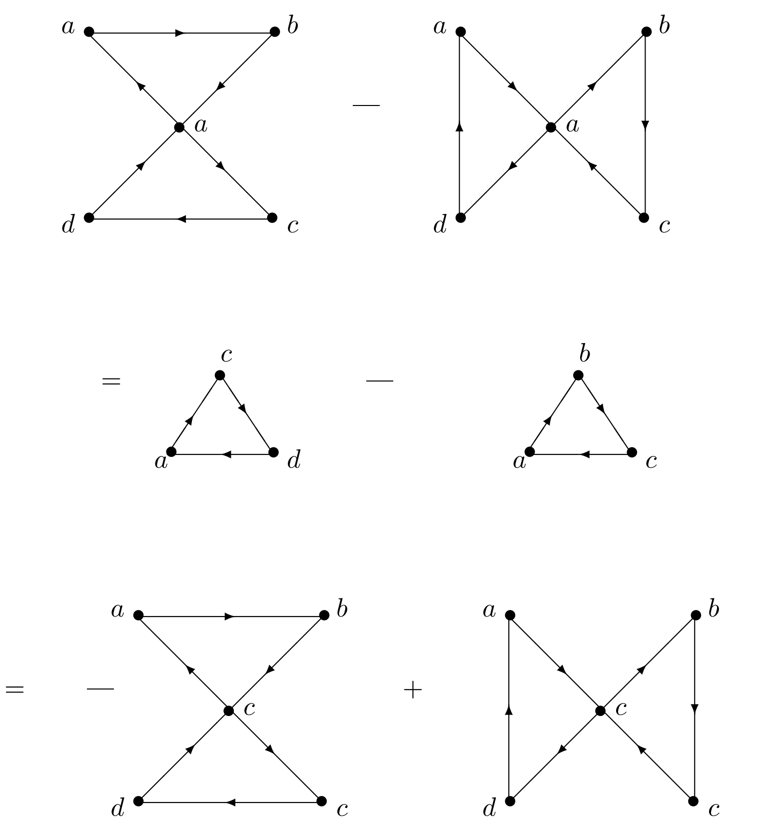
Proof. Fix four points a, b, c, d. Regarding Definition 3.0.1 and formula (3.2), one needs to construct our volume preserving diffeomorphism $\varphi : (M, \mu) \to (M, \mu)$ in such a way that for all t,



We first observe that the data of four points a, b, c, d determines what we call an "S-barycenter", that is a point $g = g(a, b, c, d)$ such that

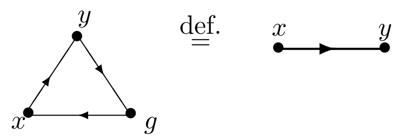


Indeed, since

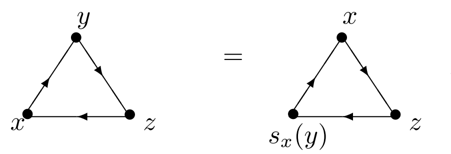


any continuous path joining a to c contains such a point g.

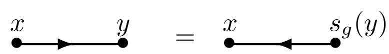
Now, we fix once for all such an S-barycenter g for $\{a, b, c, d\}$ and we adopt the following notation. For all x and y in M, the value of $S(g, x, y)$ is denoted by a "thickened arrow" :



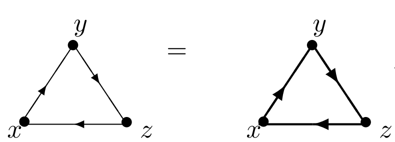
Again, a change of orientation in such an arrow changes the sign of its value. Also, property (ii) of admissibility (Definition 3.0.2) which reads



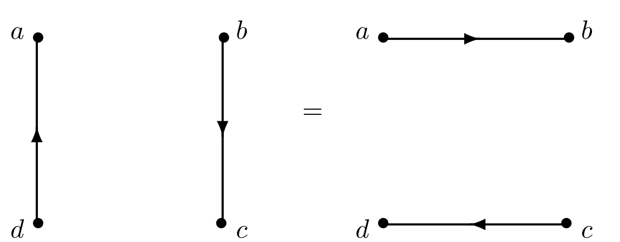
implies



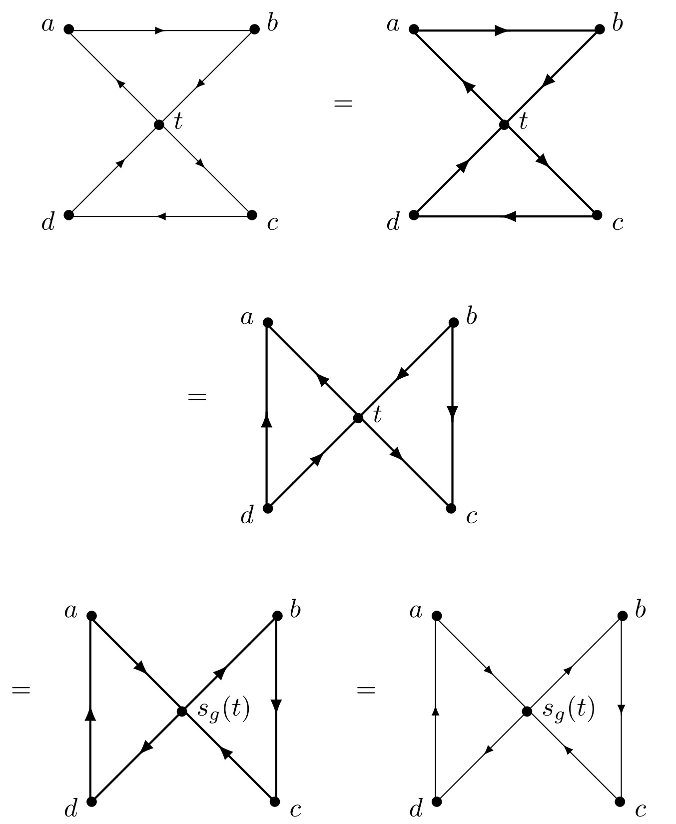
for all x and y in M. While, from the cocycle condition (with $m = g$), one gets



Moreover, the barycentric property of g reads



Hence



One can therefore choose our diffeomorphism φ as

$$\varphi = s_g.$$

■

Remark 3.0.1 In the case of the hyperbolic plane endowed with its natural structure of symmetric space, it is tempting to ask whether the three-point function defined by the symplectic area $SA(x, y, z)$ of the geodesic triangle Δxyz would satisfy properties (i)—(iii) in Definition 3.0.2. The answer is negative. Indeed, properties (ii) and (iii) would imply unboundedness of SA. This indicates that, when requiring some compatibility between S and the symmetries, property (iii) is somehow too strong.

3.1 The Weinstein area

In [We93], Weinstein proved that, in the case of a Hermitian symmetric space (M, ω, s) of the noncompact type, the phase function S_W, occurring in the expression of a given invariant WKB-quantization of (M, ω) defined by an oscillatory integral formula of the type

$$u \star v(x) = \int u(y)v(z)a_\hbar(x, y, z)e^{\frac{i}{\hbar}S_W(x,y,z)}dy\,dz, \tag{3.3}$$

must be as follows.

- Let x, y and z be three points in M such that the following equation admits a solution t :

$$t = s_x s_y s_z(t)$$

  (t is unique if it exists).

- Let Σ be a surface in M bounded by the geodesic triangle ΔtAB where $A = s_x(t)$ and $B = s_y(A) = s_y s_x(t) \quad (t = s_z(B))$.

Then, if for some formal amplitude of the form

$$a_\hbar(x, y, z) = a_0(x, y, z) + \hbar a_1(x, y, z) + \hbar^2 a_2(x, y, z) + \ldots \quad ,$$

the product (3.3) defines an invariant deformation quantization of (M, ω), the value of the "WKB"-phase function S_W on (x, y, z) is given by

$$S_W(x, y, z) = -\int_\Sigma \omega.$$

Practically, the function S_W is hard to compute explicitly; however, some two dimensional examples have been treated in [Qi96]. The problem of finding the amplitude $a_\hbar$ is open.

Definition 3.1.1 *Let (M, s) be a symmetric space with transvection group G. A two-point function $u \in C^\infty(M \times M, \mathbb{R})$ is called* admissible *iff the following two properties hold*

(a) there exists a point $o \in M$ such that the function u is invariant under the isotropy group $K_o \subset G$ at point o: for all $k \in K_o$ and $x, y \in M$:

$$u(k.x\ ,\ k.y) \ = \ u(x, y)\ .$$

(b) For all $x, y \in M$:

$$u(s_o x, y) \ = \ -u(x, y)\ .$$

(c) There exists a ∇-normal neighbourhood u_0 centered at point o in M such that for all $x \in u_0$ and $y \in M$:

$$u(x, s_{\frac{x}{2}} y) \ = \ -u(x, y)$$

where $\frac{x}{2}$ denotes the unique mid-point on the geodesic line between o and x in u_0.

the value of the "WKB"-phase function S_W on (x, y, z) is given by

$$S_W(x, y, z) = -\int_\Sigma \omega.$$

Practically, the function S_W is hard to compute explicitly; however, some two dimensional examples have been treated in [Qi96]. The problem of finding the amplitude $a_\hbar$ is open.

Proposition 3.1.1 *The function S_W is admissible.*

Proof. Let x, y and z be three points in M, and set

$$\begin{cases} Y &= s_z s_y s_x(Y) \qquad (*) \\ Z &= s_x(Y) \\ X &= s_y s_x(Y) = s_y(Z) \end{cases}$$

and

$$\begin{cases} \zeta &= s_{s_x(y)} s_z s_x(\zeta) \qquad (**) \\ \xi &= s_x(\zeta) \\ \eta &= s_z s_x(\zeta) = s_z(\xi), \end{cases}$$

(provided equations (*) and (**) admit solutions). The only thing we really need to show is that the function S_W satisfies property (iii); that is,

$$SA(X, Y, Z) = SA(\xi, \eta, \zeta),$$

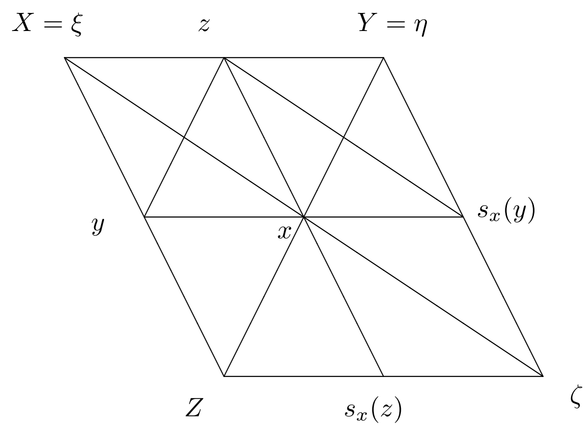
where the function SA is defined as follows. For a sequence of points $\{x_i\}_{0 \le i \le N}$, the expression $SA(x_0, ..., x_N)$ means $SA(\gamma) = \int_\gamma \alpha$, where γ is the piecewise geodesic path whose i^{th} geodesic segment starts from point x_i and ends at point $x_{i+1 \mathrm{mod}(N+1)}$, and where α is a 1-form such that $\omega = d\alpha$.

On the first hand, one has : $s_x s_{s_x(y)} s_z(X) = s_x s_x s_y s_x s_z(X) = s_y s_x s_z(X) = X$. Hence $X = \xi$ and $s_{s_x(z)}(Z) = \zeta$. On the second hand, the invariance of SA under the symmetries yields : $SA(\xi,\eta,\zeta) = SA(s_x(\xi), s_x(\eta), s_x(\zeta)) = SA(\zeta, Z, \xi)$. Moreover, $SA(\xi,\eta,\zeta) + SA(\zeta,Z,\xi) = SA(\xi,\eta,\zeta,Z)$, hence

$$S_W(x, s_x(y), z) = \frac{1}{2} SA(\xi,\eta,\zeta,Z).$$

Similarly, one gets

$$S_W(x, y, z) = -\frac{1}{2} SA(\xi,\eta,\zeta,Z).$$



■

Lemma 3.1.1 *Let $S \in C^\infty(M \times M \times M, \mathbb{R})$ be an admissible three-point function. Let o be a point in M. Then, the function $u \in C^\infty(M \times M, \mathbb{R})$ defined by*

$$u(x,y) \ := \ S(o,x,y)$$

is two-point admissible.

Proof. Let us first choose a normal neighbourhood u_0 centered at o with well-defined smooth mid-point map:

$$u_0 \to u_0 : x \mapsto \frac{x}{2} \ .$$

Then one has for all point $x \in u_0$ and $y \in M$

$$\begin{aligned} S(x,y,o) \ &= \ S(s_{\frac{x}{2}} o, y, o) \ = \ S(s_{\frac{x}{2}} s_{\frac{x}{2}} o, s_{\frac{x}{2}} y, s_{\frac{x}{2}} o) \ = \ S(o, s_{\frac{x}{2}} y, x) \\ = \ u(s_{\frac{x}{2}} y, x) \ &= \ S(o,x,y) \ = \ u(x,y) \ . \end{aligned}$$

The rest is obvious. ■

Reciprocally, one has

Proposition 3.1.2 *Let (M,s) be symmetric space. For simplicity, let us assume it to be a strictly geodesically convex i.e. any two points can be joined by a unique geodesic segment. Let $u \in C^\infty(M \times M, \mathbb{R})$ be two-point admissible. Then, the element $S \in C^\infty(M \times M \times M, \mathbb{R})$ defined by*

$$S(x,y,z) \ := \ u(s_{\frac{x}{2}} y, s_{\frac{x}{2}} z)$$

is three-point admissible.

Proof. We first note the s_o-invariance of u:

$$u(s_ox,s_oy) \;=\; -u(s_ox,s_os_oy) \;=\; u(y,s_ox) \;=\; -u(y,s_os_ox) = u(x,y)\;.$$

We continue by observing that for all point x and z in M, the element

$$s_os_{\frac{x}{2}}s_{\frac{z}{2}}s_{\left(\frac{s_{\frac{z}{2}}(x)}{2}\right)}$$

belongs to K_o.
Now,

$$\begin{aligned}
& S(x,y,z) \;=\; u(s_{\frac{x}{2}}(y),s_{\frac{x}{2}}(z)) \\
=\;& -u(s_{\frac{x}{2}}(z),s_{\frac{x}{2}}(y)) = -u\left(s_{\frac{x}{2}}s_{\frac{z}{2}}\left(s_os_{\frac{x}{2}}s_{\frac{z}{2}}s_{\left(\frac{s_{\frac{z}{2}}(x)}{2}\right)}\right)^{-1}(o),s_{\frac{x}{2}}(y)\right) \\
=\;& -u(s_{\frac{x}{2}}s_{\frac{z}{2}}s_{\left(\frac{s_{\frac{z}{2}}(x)}{2}\right)}s_{\frac{z}{2}}s_{\frac{x}{2}}(o),s_{\frac{x}{2}}(y)) \\
=\;& -u(s_{\frac{x}{2}}s_{\frac{z}{2}}s_{\left(\frac{s_{\frac{z}{2}}(x)}{2}\right)}s_{\frac{z}{2}}(x),s_{\frac{x}{2}}s_{\frac{z}{2}}s_{\left(\frac{s_{\frac{z}{2}}(x)}{2}\right)}s_{\left(\frac{s_{\frac{z}{2}}(x)}{2}\right)}s_{\frac{z}{2}}(y)) \\
=\;& -u(s_{\frac{z}{2}}(x),s_{\left(\frac{s_{\frac{z}{2}}(x)}{2}\right)}s_{\frac{z}{2}}(y)) = u(s_{\frac{z}{2}}(x),s_{\frac{z}{2}}(y)) \\
=\;& S(z,x,y) \;=\; -u(s_{\frac{z}{2}}(y),s_{\frac{z}{2}}(x)) \;=\; -S(z,y,x)\;.
\end{aligned}$$

Similarly, for every $x,w\in M$, the element

$$s_{\frac{x}{2}}s_ws_{\frac{s_wx}{2}}$$

stabilizes o. Hence:

$$\begin{aligned}
& S(s_wx,s_wy,s_wz) \;=\; u(s_{\frac{s_wx}{2}}s_wy,s_{\frac{s_wx}{2}}s_wz) \\
=\;& u(s_{\frac{x}{2}}s_ws_{\frac{s_wx}{2}}s_{\frac{s_wx}{2}}s_wy,s_{\frac{x}{2}}s_ws_{\frac{s_wx}{2}}s_{\frac{s_wx}{2}}s_wz) \\
=\;& u(s_{\frac{x}{2}x}y,s_{\frac{x}{2}x}z) \;=\; S(x,y,z)\;.
\end{aligned}$$

At last:

$$\begin{aligned}
& S(x,s_xy,z) \;=\; u(s_{\frac{x}{2}}s_xs_{\frac{x}{2}}s_{\frac{x}{2}}y,s_{\frac{x}{2}}z) \;=\; u(s_{s_{\frac{x}{2}}x}s_{\frac{x}{2}}y,s_{\frac{x}{2}}z) \\
=\;& u(s_os_{\frac{x}{2}}y,s_{\frac{x}{2}}z) \;=\; -u(s_{\frac{x}{2}}y,s_{\frac{x}{2}}z) \;=\; -S(x,y,z)\;.
\end{aligned}$$

■

We end this section by a remark and a consequent definition. The notion of 2-point admissibility on a function u as above entails that for every integer r:

$$u(x,y) \;=\; u(x,s_{\frac{x}{2}}s_oy) \;=\; u(x,\left(s_{\frac{x}{2}}s_o\right)^ry)$$

for all point x and y in M. Combining this fact with Lemma 3.1.1 and setting

$$x \;=\; \mathrm{Exp}_o^{\nabla}(X) \quad (X\in\mathfrak{p}\;:=\;\mathfrak{p}_o)\;,$$

we get

$$u(\exp(X).o\,,\,\exp(rX).y)$$

for every $y\in M$. This motivates the following slightly stronger notion of admissibility.

Definition 3.1.2 *Let (M,s) be a (strictly geodesically complete) symmetric space. A K_o-invariant skewsymmetric element $u\in C^\infty(M\times M,\mathbb{R})$ is called* regular admissible *whenever for every $X\in\mathfrak{p}$, every $t\in\mathbb{R}$ and $x\in M$:*

$$u(\exp(X).o\,,\,x) \;=\; -u(\exp(X)\,,\,s_ox) \;=\; u(\exp(X).o\,,\,\exp(tX).x)\;.$$

Examples 3.1.1 *(i) Given a real finite dimensional vector space V (viewed as a flat symmetric space) together with a bilinear 2-form $\Omega \in \bigwedge^2(V^\star)$, the element*

$$u_0(x,y) \ := \ \Omega(x,y) \quad (x,y \in V)$$

is regular admissible. The associated admissible three-point function

$$S_0(x,y,z) \ = \ \Omega(x,y) + \Omega(y,z) + \Omega(z,x)$$

satifies the cocycle condition of Proposition 3.0.1.

(ii) Given a non-compact irreducible Riemannian symmetric space $M = G/K$. One observes that the Iwasawa decomposition of its (necessarily reductive) transvection group $G = ANK$ implies that a regular admissible element $u \in C^\infty(M \times M, \mathbb{R})$ is entirely determined by the data of a smooth element $f \in C^\infty(M)$ through the formula

$$f(anK) \ := \ u(aK\,,\, nK)$$

where $a \in A$ and $n \in N$.

Based on this remark, we formulate

Definition 3.1.3 *We call A-*orbit *in M any path of the form*

$$t \mapsto \exp(tX).x$$

with $x \in M$ and $X \in \mathfrak{p}$.

3.2 The Ševera area

Let us consider a Hamiltonian SSS (M, s, ω) with transvection group G. We adopt the notations introduced in Section **??**. For every x in M, the linear injection

$$i_x : \mathfrak{p}_x \hookrightarrow \mathfrak{g}$$

into the transvection Lie algebra induces dually a K_x-equivariant linear projection

$$\mathfrak{g}^\star \to \mathfrak{p}_x^\star \ .$$

Composing the latter with the symplectic isomorphism $\mathfrak{p}_x^\star \to \mathfrak{p}_x$ induced by the ω_x and pre-composing with the moment map $m : M \to \mathfrak{g}^\star$, one gets the map:

$$p_x : M \to \mathfrak{p}_x \ . \tag{3.4}$$

Given two points y and z in M, the Ševera area is defined as the flux of ω_x through the Euclidean triangle with vertices $\{0\,,\, p_x y\,,\, p_x z\}$ in $\mathfrak{p}_x$. Formaly, one makes

Definition 3.2.1 *Given three points x, y and z in M, one denotes by*

$$\mathrm{Triang}(0\,,\, p_x y\,,\, p_x z) \tag{3.5}$$

the oriented Euclidean triangle with vertices $(0\,,\, p_x y\,,\, p_x z)$.
Choosing any primitive one-form $\alpha_x \in \Omega^1(\mathfrak{p}_x)$ for ω_x:

$$\mathrm{d}\alpha_x \ = \ \omega_x \ ,$$

the Ševera area *of x, y and z is defined by*

$$S_M(x,y,z) \ := \ \oint_{\mathrm{Triang}(0\,,\, p_x y\,,\, p_x z)} \alpha_x \ .$$

The smooth function

$$S_M : M \times M \times M \to \mathbb{R} : (x,y,z) \mapsto S_M(x,y,z)$$

is called the Ševera area of M.

Proposition 3.2.1 *Provided* $\dim M = 2$, *the Ševera area of* M *is regular admissible.*

Proof. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the transvection iLa of M corresponding to the choice of a base point $o \in M$. Since $\dim \mathfrak{p} = 2$ and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, one has $\dim \mathfrak{k} = 1$ (it cannot be zero by the Hamitonian condition) i.e.

$$\dim \mathfrak{g} = 3 .$$

Denote by $\mathcal{O}$ the coadjoint orbit in $\mathfrak{g}^\star$ image of the moment map m of M. For every $x \in M$ and every non-zero $X \in \mathfrak{p}$, one has, for every $t \in \mathbb{R}$:

$$< m(\exp(tX).x) \, , \, X > = < m(x) \, , \, \mathtt{Ad}_{\exp(-tX)} X > = < m(x) \, , \, X > \; .$$

In other words, the curve $\mathbb{R} \to \mathfrak{g}^\star : t \mapsto m(\exp(tX).x) - m(x)$ is valued in the 2-plane $X^{\perp\!\!\!\perp}$ orthodual to the element X.
Now, the kernel of the projection $i_o^\star : \mathfrak{g}^\star \to \mathfrak{p}^\star$ exactly consists in the space $\mathfrak{p}^{\perp\!\!\!\perp}$ orthodual to $\mathfrak{p}$. In particular, it is a vector line in $X^{\perp\!\!\!\perp}$.
Putting all this together, we conclude that the "A-orbit" $t \mapsto m(\exp(tX).x)$ projects under $i_o^\star$ into the straight line in $\mathfrak{p}^\star$ consisting in $i_o^\star m(x) + X^{\perp\!\!\!\perp} \cap \mathfrak{p}^\star$.

In other words, every A-orbit $t \mapsto \exp(tX).x$ projects under p_o into a straight line in $\mathfrak{p}$ that is *parallel* to the one which $\{p_o(\exp(tX).o)\}_{t\in\mathbb{R}}$ lives in.

Therefore, regarding assertion (i), since for all $v, w \in \mathfrak{p}$, the quantity $\Omega(u, v)$ (i.e. $\omega_o(u, v)$) computes the (signed) Euclidean area of the Euclidean triangle with vertices $0, v$ and w, we have for every t:

$$\begin{aligned} S_M(o, \exp(X).o, \exp(tX).x) \;&=\; \Omega(p_o(\exp(X).o), p_o(\exp(tX).x)) \\ =\; \Omega(p_o(\exp(X).o), p_o(x)) \;&=\; S_M(o, \exp(X).o, x) \; . \end{aligned}$$

We now prove that the element:

$$u_M(x, y) \;:=\; S_M(o, x, y)$$

satisfies the other conditions of admissibility.

K_o-invariance. The map

$$p_o : M \to \mathfrak{p}$$

is K_o-equivariant since the moment map m is G-equivariant by hypothesis and the injection $\mathfrak{p} \hookrightarrow \mathfrak{g}$ is K_o-invariant. The K_o-invariance of the element u_M then follows from the K_o invariance of the symplectic element $\Omega \;=\; \omega_o$ on $\mathfrak{p} \;=\; T_oM$.

Skew-symmetry. It is obvious from the skew-symmetry of Ω.

s_o-compatibility. It follows from the identity

$$p_o \circ s_o \;=\; -\mathtt{id}_{\mathfrak{p}} \; .$$

To conclude, we observe that the maps p_x $(x \in M)$ enjoy the property that for every $g \in G$:

$$p_{g.x}(g.y) \;=\; Ad_g p_x y \; .$$

The Ševera area S_M is therefore invariant under the diagonal action of G on $M \times M \times M$. Hence it is determined by u_M via $S_M(g.o, x, y) \;=\; u_M(g^{-1}.x, g^{-1}.y)$. ∎

3.3 The hyperbolic plane

In this section we treat the case of the hyperbolic plane. We prove in particular that admissible regular phases are far from being unique.

Let S be a three-point regular admissible function on the Lobatchevsky plane $\mathbb{D} = G/K$ ($G = \mathrm{SL}(2,\mathbb{R})$, $K = SO(2)$). It determines a regular admissible two-point function u through

$$u(x,y) \;:=\; S(o,x,y) \quad (o \;:=\; eK).$$

The two-point function u being K-invariant, it is itself determined by its restriction to $A.o \times \mathbb{D}$ where $G = ANK$ denotes the Iwasawa decomposition considered earlier. Moreover, Property (**??**) of regular admissibility that it is in fact completely determined by its restriction to $A.o \times N.o$ which naturally identifies with $\mathbb{D} \simeq \mathbb{S} = AN$. In other words, the element $f \in C^\infty(\mathbb{S})$ defined by

$$f(an) \;:=\; u(a.o, n.o) \;=\; S(o, a.o, n.o) \tag{3.6}$$

determines S.
Proposition **??** explains which 2-point functions on $\mathbb{D}$ come through this restriction procedure from regular admissible three-point phases on $\mathbb{D}$. Pushing one step further: what one-point functions f do come from two-point regular admissible ones? This is the question we answer to in the present paragraph.

We start with a few lemmata.

Lemma 3.3.1 *For every $n \in N$, there is a unique $a_n \in A$ and a unique $k_n \in K$ such that the following equality holds in $\mathbb{D}$:*

$$nK \;=\; k_n a_n K\;.$$

Definition 3.3.1 *We define the* dressing actions *of K on $\mathbb{S}$ and of $\mathbb{S}$ on K by*

$$K \times \mathbb{S} \to \mathbb{S} : (k,s) \mapsto s^k$$

and

$$\mathbb{S} \times K \to K : (s,k) \mapsto k^s$$

through the relations

$$ks \;=:\; s^k k^s \;\in G\;.$$

Also for every $g \in G$, denoting by

$$g \;=:\; g_A g_N g_K$$

its Iwasawa decomposition w.r.t. $G = ANK$, we define the map

$$\Phi : \mathbb{S} \to \mathbb{S} : an \mapsto a_n\left((a^{-1})^{k_n^{-1}}\right)_N\;.$$

Lemma 3.3.2

$$\Phi(a,n) \;=\; \left(\operatorname{arcsinh}(n/2)\,,\; \frac{\sinh(2a)}{\sqrt{1+n^2/4}}\right)\;.$$

Proof. We start by determining, for every $n \in N$, k_n and a_n. Which amounts to find $k_0 \in K$ such that

$$n \;=\; k_n a_n k_0$$

as dictated by the so-called "KA^+K-decomposition". Setting

$$k_n \;=:\; \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\;;\; k_0 \;=:\; \begin{pmatrix} \cos(\theta_0) & \sin(\theta_0) \\ -\sin(\theta_0) & \cos(\theta_0) \end{pmatrix}\;;$$

and

$$a_n \;=:\; \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix} \quad (r > 0)\,,$$

this amounts to the linear system

$$\begin{pmatrix} 1 & 0 & -e^r & 0 \\ 0 & 1 & 0 & -e^{-r} \\ n & 1 & 0 & -e^r \\ 1 & -n & -e^{-r} & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ \cos(\theta_0) \\ \sin(\theta_0) \end{pmatrix} = 0 \,.$$

The latter is equivalent under Gauss' algorithm to

$$\begin{pmatrix} 1 & 0 & -e^r & 0 \\ 0 & 1 & 0 & -e^{-r} \\ 0 & 0 & ne^r & -2\sinh(r) \\ 0 & 0 & 2\sinh(r) & -ne^{-r} \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ \cos(\theta_0) \\ \sin(\theta_0) \end{pmatrix} = 0 \,.$$

This implies

$$\det \begin{pmatrix} ne^r & -2\sinh(r) \\ 2\sinh(r) & -ne^{-r} \end{pmatrix} = 0$$

i.e.

$$n = \epsilon 2\sinh(r) \quad (\epsilon := \operatorname{sign}(n)) \,.$$

Using the above condition, the system is then solved by

$$\tan(\theta) = \epsilon e^{-r} \,.$$

Now the Iwasawa decomposition is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & 0 \\ 0 & \sqrt{c^2+d^2} \end{pmatrix} \begin{pmatrix} 1 & bd+ac \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{d}{\sqrt{c^2+d^2}} & \frac{-c}{\sqrt{c^2+d^2}} \\ \frac{c}{\sqrt{c^2+d^2}} & \frac{d}{\sqrt{c^2+d^2}} \end{pmatrix} .$$

Therefore, applying it to $g = k_n^{-1} a$ yields

$$(a^{k_n^{-1}})_N = -\sinh(2a)\sin(2\theta) = \frac{-\epsilon\sinh(2a)}{\cosh(r)} = \frac{-\epsilon\sinh(2a)}{\sqrt{1+n^2/4}} \,.$$

■

Proposition 3.3.1 *Let $f_0 : \mathbb{S} \to \mathbb{R}$ be a measurable function. Then, it is the one-point function associated to a (measurable) regular admissible function on the Lobatchevsky plane $\mathbb{D}$ (cf. (3.6)) if and only if it satisfies the two following properties:*

(a)

$$\Phi^\star f_0 = f_0 \,,$$

and

(b)

$$f_0(a,n) = -f_0(-a,n) \,.$$

Proof. Starting with any function f_0 it is easy to to define a two point function u_0 that is K-invariant and that possesses Property (?) of regular admissibility. Indeed, based on both $G = KA^+K$ polar decomposition and Iwasawa decomposition, one sets, for all points $x_j = a_j n_j \in \mathbb{S} \subset G$ $(j = 1, 2)$:

$$x_1 =: k_1 a_1' k_1' \quad \text{and} \quad a_2' n_2' k_2 := k_1^{-1} x_2 \,.$$

Then we define:

$$u_0(x_1.o, x_2.o) := f_0(a_1' n_2') \,.$$

The above element u_0 satisfies the requirement. Indeed, for every $k \in K$, one has

$$kx_1 = kk_1 a_1' k_1' \quad \text{and} \quad (kk_1)^{-1} kx_2 = a_2' n_2' k_2 \,.$$

Hence the K-invariance. Also, for every $a \in A$, one has

$$u_0(a_1'.o, ax_2.o) \;=\; u_0(a_1'.o, aa_2n_2.o) \;=\; f_0(a_1'n_2) \;=\; u_0(a_1'.o, x_2.o) \;.$$

Now, we address the question of skew symmetry for u_0. It amounts to the property that

$$u_0(x_1.o, a_2.o) \;=\; -u_0(a_2.o, x_1.o) \;.$$

Indeed:

$$u_0(x_2.o, x_1.o) \;=\; u_0(k_1^{-1}x_2.o, a_1'.o) \;=\; -u_0(a_1'.o, k_1^{-1}x_2.o) \;=\; -u_0(x_1.o, x_2.o) \;.$$

Now, write

$$x_1.o \;=:\; k_1a_1.o$$

as in Lemma 3.3.1. Skewsymmetry then amounts to

$$\begin{aligned} & u_0(k_1a_1.o, a_2.o) \;=\; u_0(a_1.o, (a_2^{k_1^{-1}})_N.o) \;=\; f_0(a_1(a_2^{k_1^{-1}})_N) \\ = \;\; & -u_0(a_2.o, k_1a_1.o) \;=\; -u_0(a_2.o, (a_1^{k_1})_N.o) \;=\; -f_0(a_2(a_1^{k_1})_N) \;. \end{aligned}$$

Now, expressing any $n \in N$ as in Lemma 3.3.1, we get

$$n.o \;=\; (a_n)_A^{k_n}(a_n)_N^{k_n}.o \;,$$

and the Iwasawa decomposition yields:

$$n \;=\; (a_n)_N^{k_n} \;.$$

Hence, from the above consideration, we get:

$$f_0(an) \;=\; f_0(a(a_n)_N^{k_n}) \;=\; -f_0(a_n(a_n)_N^{k_n^{-1}}) \;.$$

We then conclude from

$$f_0(an) \;=\; -f_0(s_oan) \;=\; f_0(a^{-1}n) \;.$$

■

We now give explicit formulae in the parametric models of $\mathbb{D}$ introduced in Example 1.2.2.

Proposition 3.3.2 *Consider the one-parameter family of Lie groups $\{G_\lambda\}_{\lambda\in[0,\infty)}$ introduced in Example 1.2.2.*
(i) Let $\lambda, R > 0$ and realize the Lobatchevsky plane $\mathbb{D}$ as the co-adjoint orbit $S_{\lambda,R}$ living in $(\mathfrak{g}_\lambda^\star, \eta_\lambda)$. Then, the hyperbolic Ševera area $S_{S_{\lambda,R}}$ is proportional to the regular admissible three-point phase $S_{\mathbb{D}}$ associated under (3.6) to the element $f_{\mathbb{D}} \in C^\infty(\mathbb{S})$ given by

$$f_{\mathbb{D}}(a,n) \;:=\; \sinh(2a)\,n \;.$$

(ii) The Ševera area $S_{\mathbb{S}}$ of the canonical symplectic symmetric space structure on the $ax+b$ group $\mathbb{S}$ is proportional to the regular admissible three-point phase $S_{\mathbb{S}}$ associated to the element $f_{\mathbb{S}} \in C^\infty(\mathbb{S})$ given by

$$f_{\mathbb{S}}(a,n) \;:=\; \sinh(2a)\,n \;.$$

Proof. Consider the element $Z = E - F$ as in Example 1.2.2. For every element $x \;=:\; an \;=\; \exp(aH)\exp(nE)$ in $\mathbb{S}$, one computes that

$$\mathtt{Ad}_xZ \;=\; -\lambda nH + \frac{1}{2}\left(e^{2a}(1+\lambda n^2) - e^{-2a}\right)H^\perp + \frac{1}{2}\left(e^{2a}(1+\lambda n^2) + e^{-2a}\right)Z$$

with

$$H^\perp \;:=\; E + F \;.$$

Therefore:

$$(\mathtt{Ad}_aZ)_{\mathfrak{p}} \;=\; \sinh(2a)H^\perp \quad\text{and}\quad (\mathtt{Ad}_nZ)_{\mathfrak{p}} \;=\; -\lambda nH + \frac{\lambda n^2}{2}H^\perp \;.$$

Hence, any skewsymmetric bilinear 2-form on $\mathfrak{p}$ evaluates on $((\mathtt{Ad}_aZ)_{\mathfrak{p}}\,,\,(\mathtt{Ad}_nZ)_{\mathfrak{p}})$ proportional to $f_{\mathbb{D}}(a,n) = \sinh(2a)n$. And the same holds for any multiple of Z.
Since the linear isomorphism $\sharp : \mathfrak{g}_\lambda^\star \to \mathfrak{g}^\lambda$ is G_λ-equivariant, one gets item (i). The other assertion then follows from the fact that the element $f_{\mathbb{D}}$ is λ-independant, hence it yields the same formula at the contraction limit $\lambda = 0$. ■

Remark 3.3.1 Using the explicit formula given in Lemma 3.3.2, for $\lambda = 1$, the invariance property $\Phi^\star f_{\mathbb{D}} = f_{\mathbb{D}}$ is manifest.

3.4 Elementary group type Hermitean spaces

We now pass to the particular case of a given $2d+2$-dimensional elementary normal **j**-group $\mathbb{S}$ with associated symplectic form $\omega^{\mathbb{S}}$. Let $a, t \in \mathbb{R}$ and $v \in V \simeq \mathbb{R}^{2d}$. The following identification will always be understood:

$$\mathbb{R}^{2d+2} \ni x := (a, v, t) \mapsto aH + v + tE \in \mathfrak{s}\,.$$

The following result is extracted from [BM01]:

Proposition 3.4.1 *(i) The map*

$$\mathfrak{s} \to \mathbb{S}\,, \quad (a, v, t) \mapsto \exp(aH)\exp(v + tE) = \exp(aH)\exp(v)\exp(tE)\,, \tag{3.7}$$

is a global Darboux chart on $(\mathbb{S}, \omega^{\mathbb{S}})$ *in which the symplectic structure reads:*

$$\omega^{\mathbb{S}} \; := \; 2\mathrm{d}a \wedge \mathrm{d}t \, + \, \omega_0\,.$$

(ii) Setting furthermore

$$s_{(a,v,t)}(a', v', t') \; := \; \big(2a - a', 2\cosh(a - a')v - v', 2\cosh(2a - 2a')t + \omega_0(v, v')\sinh(a - a') - t'\big)\,,$$

defines a symplectic symmetric space structure $(\mathbb{S}, s, \omega^{\mathbb{S}})$ *on the elementary normal* **j**-*group* $\mathbb{S}$.
(iii) The left action $L_x : \mathbb{S} \to \mathbb{S} : x' \mapsto x.x'$ *defines a injective Lie group homomorphism*

$$L : \mathbb{S} \to \mathtt{Aut}(\mathbb{S}, s, \omega^{\mathbb{S}})\,.$$

In the coordinates (3.7)*, we have*

$$x.x' \; = \; (a, v, t).(a', v', t') \; = \; \big(a + a', e^{-a'}v + v', e^{-2a'}t + t' + \tfrac{1}{2}e^{-a'}\omega_0(v, v')\big)\,.$$

and

$$x^{-1} \; = \; (a, v, t)^{-1} = (-a, -e^{a}v, -e^{2a}t)\,.$$

(iv) The action $\mathbf{R} \; : \; \mathrm{Sp}(V, \omega^0) \times \mathbb{S} \to \mathbb{S}$, $(A, (a, v, t)) \mapsto \mathbf{R}_A(a, v, t) := (a, Av, t)$ *by automorphisms of the normal* **j**-*group* $\mathbb{S}$ *induces an injective Lie group homomorphism:*

$$\mathbf{R} : \mathrm{Sp}(V, \omega^0) \to \mathtt{Aut}(\mathbb{S}, s, \omega^{\mathbb{S}})\,, \quad A \mapsto \mathbf{R}_A\,.$$

Note that in these coordinates the modular function of $\mathbb{S}$, $\Delta_{\mathbb{S}}$, reads $e^{(2d+2)a}$.
We now pass to the definition of the three-point phase on $\mathbb{S}$. For this we need the notion of "double geodesic triangle" as introduced by A. Weinstein [We93] and Z. Qian [Qi96].

Definition 3.4.1 *Let* (M, s, ω) *be a symplectic symmetric space. A* **midpoint map** *on* M *is a smooth map*

$$M \times M \to M\,, \quad (x, y) \mapsto \mathtt{mid}(x, y)\,,$$

such that, for all points x, y *in* M*:*

$$s_{\mathtt{mid}(x,y)}(x) \; = \; y\,.$$

Remark 3.4.1 Observe that in the case where the partial maps $s^y : M \to M$, $x \mapsto s_x(y)$ are global diffeomorphisms of M, a midpoint map exists and is given by:

$$\mathtt{mid}(x, y) \; := \; (\, s^x\,)^{-1}(y)\,.$$

Note that in this case, every $\varphi \in \mathtt{Aut}(M, s, \omega)$ intertwines the midpoints. Indeed, since for all $x, y \in M$ we have $\varphi(s_y(x)) = s_{\varphi(y)}\big(\varphi(x)\big)$, we get

$$\varphi\big(\mathtt{mid}(x, y)\big) \; = \; \mathtt{mid}\big(\varphi(x)\,,\, \varphi(y)\big)\,.$$

An immediate computation shows that a midpoint map always exists on the symplectic symmetric space attached to an elementary normal **j**-group:

Lemma 3.4.1 *For the symmetric space* $(\mathbb{S},s)$ *underlying an elementary normal* **j**-*group, the associated partial maps are global diffeomorphisms. In the coordinates* (3.7), *we have:*

$$\left(s^{(a_0,v_0,t_0)}\right)^{-1} : (a,v,t) \mapsto \Big(\frac{a+a_0}{2}, \frac{v+v_0}{2\cosh(\frac{a-a_0}{2})}, \frac{t+t_0}{2\cosh(a-a_0)} - \omega_0(v,v_0)\frac{\sinh(\frac{a-a_0}{2})}{4\cosh(a-a_0)\cosh(\frac{a-a_0}{2})}\Big) . \quad (3.8)$$

The following statement is proven in [Bi02].

Proposition 3.4.2 *(i) The affine space* $(\mathbb{S},\nabla)$ *is strictly geodesicaly complete, i.e. two points determine a unique geodesic arc.*
(ii) The "double triangle" three-point function

$$\Phi : \mathbb{S}^3 \to \mathbb{S}^3 , \quad (x_1,x_2,x_3) \mapsto \big(\mathtt{mid}(x_1,x_2), \mathtt{mid}(x_2,x_3), \mathtt{mid}(x_3,x_1)\big) ,$$

is a $\mathbb{S}$-*equivariant (under the left regular action) global diffeomorphism.*

Since our space $\mathbb{S}$ has trivial de Rham cohomology in degree two, any three points (x,y,z) define an oriented geodesic triangle $T(x,y,z)$ whose symplectic area is well-defined by integrating the two-form ω on any surface admitting $T(x,y,z)$ as boundary. With a slight abuse of notation, we set

$$\mathrm{Area}(x,y,z) := \int_{T(x,y,z)} \omega^{\mathbb{S}} .$$

Definition 3.4.2 *The* **canonical two-point phase** *associated to an elementary normal* **j**-*group is defined by*

$$S^{\mathbb{S}}_{\mathrm{can}}(x_1,x_2) := \mathrm{Area}\left(\Phi^{-1}(e,x_1,x_2)\right) \ \in \ C^\infty(\mathbb{S}^2,\mathbb{R}) ,$$

where $e:=(0,0,0)$ *denotes the unit element in* $\mathbb{S}$. *In the coordinates* (3.7), *one has the explicit expression:*

$$S^{\mathbb{S}}_{\mathrm{can}}(x_1,x_2) = \sinh(2a_1)t_2 - \sinh(2a_2)t_1 + \cosh(a_1)\cosh(a_2)\,\omega_0(v_1,v_2) . \quad (3.9)$$

The **canonical two-point amplitude** *associated to an elementary normal* **j**-*group is defined by*

$$A^{\mathbb{S}}_{\mathrm{can}}(x_1,x_2) := \mathrm{Jac}_{\Phi^{-1}}(e,x_1,x_2)^{1/2} \ \in \ C^\infty(\mathbb{S}^2,\mathbb{R}) .$$

In the coordinates (3.7), *it reads*

$$A^{\mathbb{S}}_{\mathrm{can}}(x_1,x_2) = \big(\cosh(a_1)\,\cosh(a_2)\,\cosh(a_1-a_2)\big)^d\big(\cosh(2a_1)\,\cosh(2a_2)\,\cosh(2a_1-2a_2)\big)^{1/2} . \quad (3.10)$$

3.5 Symplectic symmetric geometry of the group $ax+b$

We will denote by $\mathbb{S}$ the real two-dimensional connected simply connected Lie group whose Lie algebra $\mathfrak{s}$ is generated by two elements H and E with table

$$[H,E] \ = \ 2E .$$

We will refer to $\mathbb{S}$ as the *$ax+b$-group*.
Throughout the present book, the group $ax+b$ will be coordinated via the map

$$\mathbb{R}^2 \ \simeq \ \mathfrak{s} \longrightarrow \mathbb{S} : (a,n) \ := \ aH+nE \mapsto \exp(aH).\exp(nE) \ =: \ a.n .$$

The above coordinates have the advantage to constitute a global Darboux chart with respect to any left-invariant symplectic volume form $\omega^{(m)}$ on $\mathbb{S}$:

$$\omega^{(m)} \ = \ m\,\mathrm{d}a \wedge \mathrm{d}n \quad (m\in\mathbb{R}_0) .$$

In these coordinates, the group law reads

$$(a,n),(a',n') \;=\; (a+a'\,,\,e^{-2a}n+n')$$

with unit element

$$e \;=\; (0,0)$$

and inverse

$$(a,n)^{-1} \;:=\; (-a,-e^{2a}n)\;.$$

As a Lie group, $\mathbb{S}$ is canonically endowed with the symmetric space structure (cf. Definition 1.1.1):

$$\hat{s}:\mathbb{S}\times\mathbb{S}\to\mathbb{S}:(x,y)\mapsto\hat{s}_x y \;:=\; xy^{-1}x\;.$$

Remark 3.5.1 The Haar element $\omega^{(m)}$ is not invariant under the symmetries $\{\hat{s}_x\}_{x\in\mathbb{S}}$. However there is a structure of symplectic symmetric space on $\mathbb{S}$ canonically associated to its Lie group structure, which we now describe.

The above symmetric space is associated with the so called "exchange" iLg $(\mathbb{S}\times\mathbb{S},\sigma^e)$ with

$$\sigma^e(x,y) \;:=\; (y,x)\;;$$

the projection map being realized by

$$\pi:\mathbb{S}\times\mathbb{S}\to\mathbb{S}:(x,y)\mapsto xy^{-1}$$

whose fibers are the left lateral classes of the diagonal sub-group $K^e\simeq\mathbb{S}$ in $\mathbb{S}\times\mathbb{S}$.
At the infinitesimal level, it corresponds to the iLa $(\mathfrak{s}\oplus\mathfrak{s},\sigma^e)$ defined by

$$\sigma^e(X,Y) \;:=\; (Y,X)\;.$$

However, the above iLa does not realize as such the transvection iLa. Indeed, the isotropy Lie algebra in $\mathfrak{s}\oplus\mathfrak{s}$ consisting in the diagonal

$$\mathfrak{k}^e \;:=\; \{(X,X)\}$$

is isomorphic to $\mathfrak{s}$ while the holonomy Lie algebra $\mathfrak{k}$ is one-dimensional. The transvection Lie algebra may be viewed as a sub-iLa of the exchange iLa as follows. Considering the decomposition into σ^e-eigenspaces:

$$\mathfrak{s}\oplus\mathfrak{s} \;=\; \mathfrak{k}^e\oplus\mathfrak{p}$$

with $\mathfrak{p} \;:=\; \{(X,-X)\}$, one has

$$\mathfrak{k} \;=\; [\mathfrak{p},\mathfrak{p}]\;.$$

Hence, setting

$$\mathfrak{a} \;:=\; \mathbb{R}H \quad\text{and}\quad \mathfrak{n} \;:=\; \mathbb{R}E\;,$$

the transvection Lie algebra $(\mathfrak{g},\sigma)$ then identifies with following the semi-direct extension of the Abelian $\mathfrak{n}\oplus\mathfrak{n}$ by $\mathfrak{a}$:

$$\mathfrak{g} \;=\; \mathfrak{a}\ltimes(\mathfrak{n}\oplus\mathfrak{n})$$

with table

$$[H\,,\,E] \;=\; 2E \quad;\quad [H,F] \;:=\; -2F \quad\text{and}\quad [E\,,\,F] \;=\; 0$$

where, with some slight abuse of notation, we have set

$$H \;:=\; (H,-H) \quad;\quad E \;:=\; (E,0) \quad\text{and}\quad F \;:=\; (0,E)\;.$$

Within this presentation, the involution is given by

$$\sigma H \;=\; -H \quad,\quad \sigma E \;=\; F\;.$$

Remark 3.5.2 Note that the transvection Lie algebra $\mathfrak{g}$ is isomorphic to the Poincaré Lie algebra $\mathfrak{so}(1,1)\ltimes\mathbb{R}^2$.

The Lie algebra embedding

$$\mathfrak{s}\to\mathfrak{g}: aH+nE\mapsto a(H,-H)\;+\;n(E,0)$$

exponentiates to realizing the $ax+b$ group $\mathbb{S}$ into $\mathbb{S}\times\mathbb{S}$ under

$$\mathbb{S}\to\mathbb{S}\times\mathbb{S}: an\mapsto(a,a^{-1}).(n,e)\;=\;(an,a^{-1})\;.$$

This yields a simply transitive action of $\mathbb{S}$ on $(\mathbb{S}\times\mathbb{S})/K^e$ and associated diffeomorphism

$$\mathbb{S}\to\mathbb{S}:(a,n)\mapsto\pi(an,a^{-1})\;=\;ana\;. \tag{3.11}$$

Remark 3.5.3 The map (3.11) is not a group homomorphism.

As announced, we now have

Proposition 3.5.1 *Under the diffeomorphism (3.11), the group symmetric space structure $\hat{s}$ reads*

$$s_{(a,n)}(a',n')\;=\;(\,2a\;-\;a'\,,\,2\cosh(2(a-a'))n\;-\;n'\,)\;.$$

The Haar elements $\omega^{(m)}\;=\;m\,\mathrm{d}a\wedge\mathrm{d}n$ are the only 2-forms on $\mathbb{S}$ invariant under the transported symmetries $\{s_x\}_{x\in\mathbb{S}}$.

Proof. Denoting by $\mathbf{C}$ the conjugate action $\mathbf{C}_x y\;=\;xyx^{-1}$, one observes:

$$\hat{s}_{ana}(a'n'a')=anaa'^{-1}n'^{-1}a'^{-1}ana\;=\;a^2a'^{-1}\mathbf{C}_{a'a^{-1}}n.n'^{-1}.\mathbf{C}_{aa'^{-1}}n.a^2a'^{-1}\;.$$

The announced formula then follows from the fact that $\mathbf{C}_a n\;=\;(0,e^{-2a}n)$. ∎

In other words, the triple $(\mathbb{S},\omega^{(m)},s)$ is a symplectic symmetric space.

3.6 Symplectic fluxes trough geodesic triangles

Theorem 3.6.1 *(i) The symmetric space $(\mathbb{S},s)$ admits a* smooth mid-point map*, in the sense that given any two points x and y in $\mathbb{S}$, there exists a unique point $m(x,y)$ such that*

$$s_{m(x,y)}x\;=\;y$$

and the map $m:\mathbb{S}\times\mathbb{S}\to\mathbb{S}$ is C^∞.
(ii) The map

$$\Phi:\mathbb{S}\times\mathbb{S}\times\mathbb{S}\to\mathbb{S}\times\mathbb{S}\times\mathbb{S}:(x,y,z)\mapsto(\,m(x,y)\,,\,m(y,z)\,,\,m(z,x)\,)$$

is a diffeomorphism.

(iii) Setting $\Phi^{-1}(x,y,z)\;=:\;(x',y',z')$ and denoting by $x'\overset{\Delta}{y'}z'$ the oriented geodesic triangle with vertices x', y' and z', we consider the flux of $\omega^{(1)}$ through (any surface bounded by) $x'\overset{\Delta}{y'}z'$:

$$S(x,y,z)\;:=\;\int_{x'\overset{\Delta}{y'}z'}\omega^{(1)}\;.$$

One has the explicit formula:

$$S(x_0,x_1,x_2)\;=\;\sinh(2(a_0-a_1))n_2\;+\;\sinh(2(a_2-a_0))n_1\;+\;\sinh(2(a_1-a_2))n_0$$

where for every $j=0,1,2$, we set $x_j=(a_j,n_j)$.

Chapter 4

Oscillatory integrals on exponential Lie groups

4.0.1 Symbol spaces

We consider a centerless Lie group $\mathbb{L}$ with Lie algebra $\mathcal{L}$ and associated enveloping algebra $\mathcal{U}(\mathcal{L})$. For every X in $\mathcal{L}$, we denote by $\tilde{X}$ the associated left-invariant vector field on $\mathbb{L}$ and by $X^\star$ the associated right-invariant vector field:

$$\tilde{X}_x \;:=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 x\exp(tX)\,, \quad X^\star_x \;:=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \exp(-tX)x\ .$$

We use the same notations for the elements of the enveloping algebra $\mathcal{U}(\mathcal{L})$.

Definition 4.0.1 *Choosing a vector norm $|\;|$ on $\mathfrak{L}$, we will consider the element $\mathfrak{d}_{\mathbb{L}}\in C^\infty(\mathbb{L})$ defined by*

$$\mathfrak{d}_{\mathbb{L}}(x) \;:=\; \sqrt{1\,+\,|\mathtt{Ad}_x|^2\,+\,|\mathtt{Ad}_{x^{-1}}|^2}$$

where for every x in $\mathbb{L}$, $\mathtt{Ad}_x$ stands for the adjoint action of x in $\mathfrak{s}$ whose operator norm is denoted by $|\mathtt{Ad}_x|$.

The main property of the element $\mathfrak{d}_{\mathbb{L}}$ is

Lemma 4.0.1 *(i) There exists a positive C such that, for all $x,y\in\mathbb{L}$:*

$$\mathfrak{d}_{\mathbb{L}}(xy)\;<\;C\,\mathfrak{d}_{\mathbb{L}}(x)\,\mathfrak{d}_{\mathbb{L}}(y)\ .$$

(ii) For every $X\in\mathcal{U}(\mathfrak{L})$, there exists C such that

$$|\tilde{X}.\mathfrak{d}_{\mathbb{L}}|\;<\;C\,\mathfrak{d}_{\mathbb{L}}\quad \textit{and}\quad |\underline{X}.\mathfrak{d}_{\mathbb{L}}|\;<\;C\,\mathfrak{d}_{\mathbb{L}}\ .$$

Denoting by

$$\mathcal{U}(\mathfrak{L}) \;\;=\; \sum_{k\in\mathbb{N}}\mathcal{U}_k(\mathfrak{L})$$

the natural filtration, we make the

Definition 4.0.2 *Let $N\in\mathbb{R}$ and $p\in\mathbb{N}$. We set*

$$\mathcal{B}^N_p(\mathbb{L}) \;:=\; \{\,F\in C^p(\mathfrak{L})\,|\,\forall X\in\mathcal{U}_p(\mathfrak{L})\,\exists\, C>0\,:\,|\tilde{X}.F|\;<\;C\,\mathfrak{d}^N_{\mathbb{L}}\,\}\ .$$

Similarly, we define

$$\mathcal{B}^N(\mathbb{L}) \;:=\; \mathcal{B}^N_\infty(\mathbb{L}) \;:=\; \{\,F\in C^\infty(\mathfrak{L})\,|\,\forall X\in\mathcal{U}(\mathfrak{L})\,\exists\, C>0\,:\,|\tilde{X}.F|\;<\;C\,\mathfrak{d}^N_{\mathbb{L}}\,\}\,,$$

and

$$\mathcal{B}(\mathbb{L}) \;:=\; \mathcal{B}^0(\mathbb{L})\ .$$

Endowed with the pointwise multiplication of functions, these spaces becomes a commutative Fréchet algebras when equipped with the following seminorms:

$$|\varphi|_J^N \;:=\; \sup_{x\in\mathbb{S}}\{\,|\mathfrak{d}_{\mathbb{L}}^{-N}(x)\,\widetilde{\Xi_J}_x.\varphi|\,\} \tag{4.1}$$

where we fixed a basis $\{\Xi_j\}$ of $\mathfrak{L}$ and where $J=(j_1,j_2,...,j_d)\in\mathbb{N}^{\dim\mathbb{L}}$ is a multi-index such that $d\leq p$ to which we associate the Poincaré-Birkoff-Witt (PBW) basis element $\Xi_J \;:=\; \Xi_1^{j_1}...\Xi_d^{j_d}$ in $\mathcal{U}(\mathfrak{L})$.

Endowed with the seminorms (4.1), we observe that the $\mathcal{B}_p^N$'s become Banach spaces, with Fréchet projective limits, over p (N being fixed), $\mathcal{B}^N$ and $\mathcal{B}$.
Similarly as in the usual Abelian case of $\mathbb{L} \;=\; \mathbb{R}^n$, we have

Proposition 4.0.1 *(i) For all N,N' and p,p', the pointwise multiplcation of functions yields separately continuous bilinear maps:*

$$\mathcal{B}_p^N\times\mathcal{B}_{p'}^{N'}\to\mathcal{B}_{\min\{p,p'\}}^{N+N'}\;.$$

(ii) For every $X\in\mathcal{U}_k(\mathfrak{L})$, one has the continuous map

$$\mathcal{B}_p^N\to\mathcal{B}_{p-k}^N : F\mapsto\tilde{X}.F\;.$$

(iii)

$$\mathcal{B}_p^N \;=\; \mathfrak{d}_{\mathbb{L}}^N\mathcal{B}_p^0 \quad \textit{and} \quad \mathcal{B}^N \;=\; \mathfrak{d}_{\mathbb{L}}^N\mathcal{B}^0\;.$$

(iv) Denoting by $\mathrm{adh}_{\mathcal{B}_p^N}$ the topological closure in $\mathcal{B}_p^N$, one has, for all $N \;>\; N' \;\geq\; 0$ and $\infty \;\geq\; p' \;\geq\; p$:

$$\mathrm{adh}_{\mathcal{B}_p^N}\mathcal{D} \;\supset\; \mathcal{B}_{p'}^{N'}$$

where $\mathcal{D}=\mathcal{D}(\mathbb{L})$ denotes the test space of compactly supported, C^∞ functions.
(v) Let us denote by $C_b(\mathbb{L})$ the $C^\star$-algebra of (right-) uniformly continuous bounded functions on $\mathbb{L}$. This $C^\star$-alegbra is then strongly continuously and isometrically acted on by $\mathbb{L}$ under the right-regular representation. The (Fréchet) subalgebra of smooth vectors in $C_b(\mathbb{L})$ consists in the space $\mathcal{B}(\mathbb{L})$.
In particular, the space $\mathcal{B}(\mathbb{L})$ is a smooth left $\mathbb{L}$-module under R.

If $(\mathfrak{F},\{|\;|\}_{j\in\mathbb{N}})$ is a Fréchet space whose topology is defined a fixed set of semi-norms $\{|\;|\}_{j\in\mathbb{N}}$, there is an obvious $\mathfrak{F}$-valued version version of the above defined spaces $\mathcal{B}_p^N$. However, it turns out that a slightly more sophisticated definition is actually more relevant in the applications:

Definition 4.0.3 *We set*

$$\mathcal{B}_p^N(\mathbb{L},\mathfrak{F}) := \{F\in C^\infty(\mathbb{L},\mathfrak{F})\mid \forall j\in\mathbb{N}, X\in\mathcal{U}_p(\mathfrak{L}) \;:\exists C>0\;: |\tilde{X}.F| \;<\; C\,\mathfrak{d}_{\mathbb{L}}^N\}.$$

As in the complex valued case, these spaces are Fréchet spaces in a natural way and the obvious analogue of Proposition 4.0.1 holds. Namely,

Proposition 4.0.2 *(i) For all N and p,p', the pointwise multiplcation of functions yields separately continuous bilinear maps:*

$$\mathcal{B}_p^N(\mathbb{L})\times\mathcal{B}_{p'}^N(\mathbb{L},\mathfrak{F})\to\mathcal{B}_{\min\{p,p'\}}^{2N}(\mathbb{L},\mathfrak{F})\;.$$

(ii) For every $X\in\mathcal{U}_k(\mathfrak{L})$, one has the continuous map

$$\mathcal{B}_p^N\to\mathcal{B}_{p-k}^N : F\mapsto\tilde{X}.F\;.$$

(iii) Denoting by $\mathrm{adh}_{\mathcal{B}_p^N}$ the topological closure in $\mathcal{B}_p^N$, one has, for all $N'\prec N$ and $\infty \;\geq\; p' \;\geq\; p$:

$$\mathrm{adh}_{\mathcal{B}_p^N}\mathcal{D} \;\supset\; \mathcal{B}_{p'}^{N'}$$

where $\mathcal{D}=\mathcal{D}(\mathbb{L},\mathfrak{F})$ denotes the test space of F-valued compactly supported, C^∞ functions.

4.0.2 Oscillatory integrals

In this section, we define a notion of "improper integral" that will allow us to define our non-formal deformation product on $\mathcal{B}$-type spaces. We start with the following rather technical definitions, but which will be basic in the sequel.

From now on, we consider our Lie group $\mathbb{L}$ to be solvable of exponential type.

Definition 4.0.4 *The pair (G, S) is called* **tempered** *if the map*

$$\phi : G \to \mathfrak{g}^\star : x \mapsto \left[\mathfrak{g} \to \mathbb{R} : X \mapsto \mathrm{d}S_x(\widetilde{X}) = \big(\widetilde{X}.S\big)(x) \right], \tag{4.2}$$

is a global diffeomorphism.

Given a tempered pair (G, S), with $\mathfrak{g}$ the Lie algebra of G, we now consider a vector space decomposition:

$$\mathfrak{g} = \bigoplus_{n=0}^{N} V_n \,, \tag{4.3}$$

and for every $n = 0, \dots, N$, an ordered basis $\{e_j^n\}_{j=1,\dots,\dim(V_n)}$ of V_n. We get global coordinates on G:

$$x_n^j := \big(\widetilde{e_j^n}.S\big)(x)\,, \qquad n = 0, \dots, N\,, \quad j = 1, \dots, \dim(V_n)\,. \tag{4.4}$$

We choose a scalar product on each V_n and let $|.|_n$ be the associated Euclidean norm. We will always identify the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ with the symmetric algebra $\mathfrak{S}(\mathfrak{g})$ of $\mathfrak{g}$, through the Poincaré-Birkhoff-Witt linear isomorphism. Given an element $A \in \mathcal{U}(\mathfrak{g})$, we let $\widetilde{A}^*$ the formal adjoint of the left-invariant differential operator $\widetilde{A}$, with respect to the inner product of $L^2\big(G, \mathrm{d}_G\big)$. We make the obvious observation that $\widetilde{A}^*$ is still left-invariant. Indeed, for $\psi, \varphi \in C_c^\infty(G)$ and $g \in G$, we have

$$\langle L_g^\star \widetilde{A}^* \psi, \varphi \rangle = \langle \psi, \widetilde{A} L_{g^{-1}}^\star \varphi \rangle = \langle \psi, L_{g^{-1}}^\star \widetilde{A} \varphi \rangle = \langle \widetilde{A}^* L_g^\star \psi, \varphi \rangle \,.$$

Moreover, we make the following requirement of compatibility of the adjoint map on $L^2(G, \mathrm{d}_G)$ with respect to the ordered decomposition (4.3):

$$\forall\, n = 0, \dots, N, \quad \forall\, A \in \mathfrak{S}(V_n)\,, \quad \exists\, B \in \prod_{k=0}^{n} \mathfrak{S}(V_k) \quad \text{such that} \quad \widetilde{A}^* = \widetilde{B}\,. \tag{4.5}$$

We now pass to regularity assumptions regarding the function S.

Definition 4.0.5 *([BG15] Definition 2.24 page 26) Set*

$$\mathbf{E} := \exp\{iS\}\,. \tag{4.6}$$

A tempered pair (G, S) is called **admissible***, if there exists a decomposition (4.3) with associated coordinate system (4.4), such that for every $n = 0, \dots, N$, there exists an element $X_n \in \mathfrak{S}(V_n) \subset \mathcal{U}(\mathfrak{g})$ whose associated multiplier α_n, defined as*

$$\widetilde{X}_n \mathbf{E} =: \alpha_n \mathbf{E}\,, \tag{4.7}$$

satisfies the following properties:
(i) There exist $C_n > 0$ and $\rho_n > 0$ such that:

$$|\alpha_n| \geq C_n \big(1 + |x_n|_n^{\rho_n}\big)\,,$$

where $x_n := (x_n^j)_{j=1,\dots,\dim(V_n)}$.
(ii) For all $n = 0, \dots, N$, there exists a tempered function $0 < \mu_n \in C^\infty(G)$ such that:

(ii.1) For every $A \in \prod_{k=0}^{n} \mathfrak{S}(V_k) \subset \mathcal{U}(\mathfrak{g})$ there exists $C_A > 0$ such that:

$$\left|\widetilde{A}\,\alpha_n\right| \;\leq\; C_A\;|\alpha_n|\;\mu_n\;. \tag{4.8}$$

(ii.2) The function μ_n is independent of the variables $\{x_r^j\}_{j=1,\ldots,\dim(V_r)}$, for all $r \leq n$:

$$\frac{\partial \mu_n}{\partial x_r^j} \;=\; 0\,, \quad \forall r \leq n\,,\;\; \forall j = 1, \ldots, \dim(V_r)\;. \tag{4.9}$$

From the Proposition 2.28 page 30 and its proof in [BG15], we have

Proposition 4.0.3 *For every $N \in \mathbb{N}$, there exists a differential operator $\mathbf{D}$ and an integer K such that for every M and $F \in \mathcal{B}_K^M(\mathbb{L})$, one has*
(i)

$$|\mathbf{D}^\star F| \;\leq\; |F|_{K,M}\;\mathfrak{d}^{-N}$$

and
(ii)

$$\mathbf{D}\left(e^{iS}\right) \;=\; e^{iS}\;.$$

Letting $(\mathfrak{B}, ||\;||)$ be a Banach space, we deduce:

Proposition 4.0.4 *For every $N \in \mathbb{N}$, there exists a differential operator $\mathbf{D}$ and an integer K such that for every M and $F \in \mathcal{B}_K^M(\mathbb{L}, \mathfrak{B})$, one has*
(i)

$$||\mathbf{D}^\star F|| \;\leq\; ||F||_{K,M}\;\mathfrak{d}^{-N}$$

and
(ii)

$$\mathbf{D}\left(e^{iS}\right) \;=\; e^{iS}\;.$$

Corollary 4.0.1 *Let $\mu \in \mathcal{B}^N(\mathbb{L})$ and K sufficiently large. Then, for every $M \in \mathbb{N}$, the map*

$$\mathcal{D}(\mathbb{L}, \mathfrak{B}) \to \mathfrak{B} : \varphi \mapsto \int_{\mathbb{L}} \mu\, e^{iS}\, \varphi \tag{4.10}$$

is continuous w.r.t. the topology induced on $\mathcal{D}$ by $\mathcal{B}_K^{M+\epsilon}(\mathbb{L}, \mathfrak{B})$.
Therefore, for every K sufficiently large, the map

$$\int \mu\, e^{iS} \;:\; \mathcal{D}(\mathbb{L}, \mathfrak{B}) \longrightarrow \mathfrak{B} : F \mapsto \langle \int \mu\, e^{iS}\,,\, F\,\rangle\;.$$

extends in a unique way as a continuous linear map:

$$\widetilde{\int \mu\, e^{iS}} \;:\; \mathcal{B}_K^M(\mathbb{L}, \mathfrak{B}) \longrightarrow \mathfrak{B} : F \mapsto \langle \widetilde{\int \mu\, e^{iS}}\,,\, F\,\rangle\;.$$

.

Proof. One has, for every large N':

$$\left|\left|\int_{\mathbb{L}} \mu e^{iS} \varphi\right|\right| \;=\; \left|\left|\int_{\mathbb{L}} \mu \mathbf{D}(e^{iS}) \varphi\right|\right| \;=\; \left|\left|\int_{\mathbb{L}} e^{iS} \mathbf{D}^\star(\mu\varphi)\right|\right| \;\leq\; \int_{\mathbb{L}} ||\mathbf{D}^\star(\mu\varphi)||$$
$$\leq\;\; ||\mu\varphi||_{K,N'} \int \mathfrak{d}^{-N'} \;=\; C'||\mu\varphi||_{K,N'}\;.$$

Now, one has

$$||\mu\varphi||_{K,N'} \;=\; \sup \mathfrak{d}^{-N'} ||\tilde{X}(\mu\varphi)|| \;=\; \sup \mathfrak{d}^{-N'} || \sum_{(X)} \tilde{X}_{(1)}(\mu) \tilde{X}_{(2)}(\varphi)||$$
$$\leq\;\; \sum_{(X)} \sup \mathfrak{d}^{-N'} \mathfrak{d}^{N} C_{(1)} ||\tilde{X}_{(2)}(\varphi)|| \;\leq\; C''||\varphi||_{K,M+\epsilon} \sup \mathfrak{d}^{N-N'+M+\epsilon}$$

which is bounded as soon as $N' > N + M + \epsilon$. This implies that for every ϵ, one has

$$\left|\left|\int_{\mathbb{L}} \mu e^{iS}\varphi\right|\right| \;\leq\; C||\varphi||_{K,M+\epsilon}\;.$$

■

4.0.3 Deformations

We consider now $\mathfrak{B}$ to be a Banach algebra.

Definition 4.0.6 *We consider the following continuous bilinear map*

$$\begin{aligned} \mathcal{R}\;:\quad C(\mathbb{L},\mathfrak{B}) \quad &\times \quad C(\mathbb{L},\mathfrak{B}) \to C(\mathbb{L}\times\mathbb{L}, C(\mathcal{F},\mathfrak{B}))\;: && (4.11)\\ (F_1,F_2) \quad &\mapsto [\,(x_1,x_2)\mapsto\,[x\mapsto F_1(xx_1).F_2(xx_2)\;= && (4.12)\\ &\qquad \mu_{\mathfrak{B}}(R^\star\otimes R^\star_{(x_1,x_2)}(F_1\otimes F_2))(x)\,]\,] && (4.13)\end{aligned}$$

$\mu_{\mathfrak{B}}$ *denotes the mutiplication in* $\mathfrak{B}$*.*

Lemma 4.0.2 *The restriction of the map (4.11) to* $\mathcal{B}^N_p(\mathbb{L},\mathfrak{B})\times\quad\mathcal{B}^N_p(\mathbb{L},\mathfrak{B})$ *is separately continuously valued in* $\mathcal{B}^{q+N,q+N}_{p-q,p-q}(\mathbb{L}\times\mathbb{L},\mathcal{B}^{2N}_q(\mathbb{L},\mathfrak{B}))$*:*

$$\begin{aligned} \mathcal{R}\;:\quad \mathcal{B}^N_p(\mathbb{L},\mathfrak{B}) \quad &\times \quad \mathcal{B}^N_p(\mathbb{L},\mathfrak{B}) \to \mathcal{B}^{q+N,q+N}_{p-q,p-q}(\mathbb{L}\times\mathbb{L},\mathcal{B}^{2N}_q(\mathbb{L},\mathfrak{B}))\;: && (4.14)\\ (F_1,F_2) \quad &\mapsto [\,(x_1,x_2)\mapsto\,[x\mapsto F_1(xx_1).F_2(xx_2)\,]\,] && (4.15)\end{aligned}$$

for every $q\leq p$*.*

Proof. The generic semi-norm on $\mathcal{B}^{N_1}_{q_1}(\mathbb{L},\mathfrak{B})\ni f$ is given by ($X\in\mathcal{U}_{q_1}(\mathfrak{L})$):

$$||f||^{N_1}_{q_1}\;:=\;\sup_x\{\mathfrak{d}^{-N_1}_{\mathbb{L}}(x)||\tilde{X}_x(f)||\}\;.$$

The generic semi-norm on $\mathcal{B}^{N_2,N_2'}_{q_2,q_2'}(\mathbb{L}\times\mathbb{L},\mathcal{B}^{N_1}_{q_1}(\mathbb{L},\mathfrak{B}))\ni\Phi$ is therefore given by ($Y\in\mathcal{U}_{q_2+q_2'}(\mathfrak{L}\oplus\mathfrak{L})$):

$$||\Phi||^{N_2,N_2'}_{q_2,q_2'}\;:=\;\sup_{(x_1,x_2)}\{\mathfrak{d}^{-N_2,-N_2'}_{\mathbb{L}\times\mathbb{L}}(x_1,x_2)||\tilde{Y}_{(x_1,x_2)}(\Phi)||^{N_1}_{q_1}\}\;.$$

We now consider the element $\mathcal{R}(F_1,F_2)\in C^{q_2,q_2'}(\mathbb{L}\times\mathbb{L},C^{q_1}(\mathbb{L},\mathfrak{B}))$. We want to estimate the quantity:

$$\sup_{(x_1,x_2)}\{\mathfrak{d}^{-N_2,-N_2'}_{\mathbb{L}\times\mathbb{L}}(x_1,x_2)||\tilde{Y}_{(x_1,x_2)}(\mathcal{R}(F_1,F_2))||^{N_1}_{q_1}\}\;.$$

is bounded. We start by observing that for every $\vec{x}\in\mathbb{L}\times\mathbb{L}$ and $\mathcal{R}(F_1,F_2)\in C^q(\mathbb{L}\times\mathbb{L},\mathfrak{B})$:

$$\tilde{Y}_{\vec{x}}\mathcal{R}(F_1,F_2)\;=\;\mathcal{R}(\tilde{Y}(F_1,F_2))(\vec{x})\;.$$

Hence:

$$||\tilde{Y}_{(x_1,x_2)}(\mathcal{R}(F_1,F_2))||^{N_1}_{q_1}\;=\;||\mathcal{R}(\tilde{Y}(F_1,F_2))(x_1,x_2)||^{N_1}_{q_1}$$

which equals

$$\sup_x\{\mathfrak{d}^{-N_1}_{\mathbb{L}}(x)||\tilde{X}_x\left(\mathcal{R}(\tilde{Y}(F_1,F_2))(x_1,x_2)\right)||\;.$$

Considering Y of the form $Y\;=\;Y_1\otimes Y_2$, observe that

$$\tilde{X}_x\left(\mathcal{R}(\tilde{Y}(F_1,F_2))(x_1,x_2)\right)\;=\;\left((\mathtt{Ad}_{x_1^{-1}}X)Y_1\otimes(\mathtt{Ad}_{x_2^{-1}}X)Y_2\right)^{\sim}_{\vec{x}}\mathcal{R}(F_1,F_2)(x)\;.$$

Hence

$$\begin{aligned}
&\left|\left|\tilde{X}_x\left(\mathcal{R}(\tilde{Y}(F_1,F_2))(x_1,x_2)\right)\right|\right| \leq \\
&\left|\left|C\mathfrak{d}_{\mathbb{L}\times\mathbb{L}}^{|X|,|X|}(x_1,x_2)\mathcal{R}((X\otimes X.Y)(F_1,F_2))(x_1,x_2)(x)\right|\right| \leq \\
&C'\left|\left|\mathfrak{d}_{\mathbb{L}\times\mathbb{L}}^{q_1,q_1}(x_1,x_2)\mathcal{R}((X\otimes X.Y)(F_1,F_2))(x_1,x_2)(x)\right|\right| \leq \\
&C'\mathfrak{d}_{\mathbb{L}\times\mathbb{L}}^{q_1,q_1}(x_1,x_2)\,\mathfrak{d}_{\mathbb{L}}^{N}(xx_1)\,\mathfrak{d}_{\mathbb{L}}^{N}(xx_2)\,||F_1||_{q_1+q_2}^{N}||F_2||_{q_1+q_2'}^{N} \leq \\
&C"\mathfrak{d}_{\mathbb{L}\times\mathbb{L}}^{q_1+N,q_1+N}(x_1,x_2)\,\mathfrak{d}_{\mathbb{L}}^{2N}(x)\,||F_1||_{q_1+q_2}^{N}||F_2||_{q_1+q_2'}^{N}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&|\tilde{Y}_{(x_1,x_2)}(\mathcal{R}(F_1,F_2))|_{q_1}^{N_1} \\
\leq\ &C'''||F_1||_{q_1+q_2}^{N}||F_2||_{q_1+q_2'}^{N}\mathfrak{d}_{\mathbb{L}\times\mathbb{L}}^{q_1+N,q_1+N}(x_1,x_2)\ \sup_x \mathfrak{d}_{\mathbb{L}}^{2N-N_1}(x)
\end{aligned}$$

which leads to the estimate:

$$\begin{aligned}
&\sup_{(x_1,x_2)}\{\mathfrak{d}_{\mathbb{L}\times\mathbb{L}}^{-N_2,-N_2'}(x_1,x_2)|\tilde{Y}_{(x_1,x_2)}(\mathcal{R}(F_1,F_2))|_{q_1}^{N_1}\} \\
\leq\ &C_j||F_1||_{q_1+q_2}^{N}||F_2||_{q_1+q_2'}^{N}\times \\
\times\ &\sup_{(x_1,x_2)}\{\mathfrak{d}_{\mathbb{L}\times\mathbb{L}}^{(-N_2+q_1+N,-N_2'+q_1+N)}(x_1,x_2)\}\ \sup_x \mathfrak{d}_{\mathbb{L}}^{2N-N_1}(x)
\end{aligned}$$

which leads to the condition of finiteness:

$$\begin{array}{rcl}
q_1+q_2 & \leq & p \\
q_1+q_2' & \leq & p \\
q_1+N & \leq & N_2 \\
q_1+N & \leq & N_2' \\
2N & \leq & N_1
\end{array}$$

which setting

$$q\ :=\ \min\{q_2,q_2'\}$$

yields the assertion. ■

We then obtain

Corollary 4.0.2 *Let $A\in\mathcal{B}^{N_0}(\mathbb{L}^2,\mathbb{C})$. Let $q\in\mathbb{N}$. Then, provided p is large enough, one has that, to all F_1,F_2 in $\mathcal{B}_p^N(\mathbb{L},\mathfrak{B})$, the oscillating integral associates an element of $\mathcal{B}_q^{2N}(\mathbb{L},\mathfrak{B})$:*

$$\star_S\ =\ \widetilde{\int_{\mathbb{L}\times\mathbb{L}} A}\,e^{-iS}:\mathcal{B}_p^N(\mathbb{L},\mathfrak{B})\times\mathcal{B}_p^N(\mathbb{L},\mathfrak{B})\to\mathcal{B}_q^{2N}(\mathbb{L},\mathfrak{B}):(F_1,F_2)\mapsto F_1\star_S F_2\,.$$

Moreover, the product $\star_S$ is bilinear and separately continuous.
In particular, for $p=\infty$, one has

$$\star_S\ =\ \widetilde{\int_{\mathbb{L}\times\mathbb{L}} A}\,e^{-iS}:\mathcal{B}_\infty^0(\mathbb{L},\mathfrak{B})\times\mathcal{B}_\infty^0(\mathbb{L},\mathfrak{B})\to\mathcal{B}_\infty^0(\mathbb{L},\mathfrak{B})\,.$$

Definition 4.0.7 *We say that the product $\star_S$, given in Corollary 4.0.2, is* **weakly associative** *when for all $\psi_1,\psi_2,\psi_3\in\mathcal{D}(\mathbb{L},\mathfrak{B})$, one has $(\psi_1\star_S\psi_2)\star_S\psi_3=\psi_1\star_S(\psi_2\star_S\psi_3)$ in $\mathcal{B}_\infty^0(\mathbb{L},\mathfrak{B})$.*

Proposition 4.0.5 *Weak associativity implies strong associativity in the sense that, when weakly associative, for all elements $(F,F',F'')\in\mathcal{B}_p^N(\mathbb{L},\mathfrak{B})\times\mathcal{B}_p^N(\mathbb{L},\mathfrak{B})\times\mathcal{B}_p^N(\mathbb{L},\mathfrak{B})$, one has the equality $(F\star_S F')\star_S F''=F\star_S(F'\star_S F'')$ in $\mathcal{B}_q^{4N+1}(\mathbb{L},\mathfrak{B})$ for some q.*

Proof. This follows by density of the test functions in $\mathcal{B}$-type spaces. Indeed, let $\{\phi_n\}_{n\in\mathbb{N}}$, $\{\phi'_{n'}\}_{n'\in\mathbb{N}}$ and $\{\phi''_{n''}\}_{n''\in\mathbb{N}}$ be sequences of test functions converging to F, F' and F'' respectively in $\mathcal{B}_p^{N+\epsilon}(\mathbb{L},\mathfrak{B})$. By separate continuity, we have that

$$F \star_S F' \;=\; \lim_{n,n'}(\phi_n \star_S \phi_{n'})$$

where the limit on the RHS is taken in $\mathcal{B}_q^{2(N+\epsilon)}(\mathbb{L},\mathfrak{B})$ with $q << p$. Now, since

$$\mathcal{B}_p^N(\mathbb{L},\mathfrak{B}) \;\subset\; \mathcal{B}_q^{2(N+\epsilon)}(\mathbb{L},\mathfrak{B})\;,$$

the convergence $\lim_{n''}\phi_{n''} \;=\; F''$ holds in $\mathcal{B}_q^{2(N+\epsilon)}(\mathbb{L},\mathfrak{B})$ as well. Hence, for every $q' << q << p$, one has, again by separate continuity

$$(F\star_S F')\star_S F'' \;=\; \lim_{n,n',n''}(\phi_n\star_S\phi'_{n'})\star(\phi''_{n''})$$

where the limit in the RHS is taken in $\mathcal{B}_{q'}^{4(N+\epsilon)}(\mathbb{L},\mathfrak{B})$.
The same argument for $F\star_S(F'\star_S F'')$, yields

$$F\star_S(F'\star_S F'') \;=\; \lim_{n,n',n''}\phi_n\star_S(\phi'_{n'}\star\phi''_{n''})\;.$$

And we conclude by weak associativity. ∎

We now introduce a notion of Schwartz space which will be of importance in the sequel.

Definition 4.0.8 *We define the* Schwartz space *of* $\mathbb{L}$ *as the following function space:*

$$\mathcal{S}(\mathbb{L}) \;=\; \{\varphi\in C^\infty(\mathbb{L}) \mid \forall X\in\mathcal{U}(\mathfrak{L}), N\in\mathbb{R}:\; |\mathfrak{d}^N\tilde{X}(\varphi)|<\infty\}\;.$$

Of course the Schwartz space is a Fréchet space for the following semi-norms:

$$|\varphi|_{J,N}| \;:=\; |\varphi|_J^{-N}$$

with the notation of Definition 4.1. Note that N is variable here, which was not the case for the spaces of $\mathcal{B}$-type.
Note also that

$$\mathcal{S}(\mathbb{L}) \;=\; \cap_{N,q\in\mathbb{N}}\,\mathcal{B}_q^{-N}(\mathbb{L})$$

and that one has an obvious $\mathfrak{B}$-valued analogue.

Proposition 4.0.6 *The Schwartz space $\mathcal{S}(\mathbb{L})$ is a two-sided ideal in $(\mathcal{B}^0_\infty(\mathbb{L}),\star_S)$.*

Proof. The proof is similar to the one of Formula 4.14. We consider $F\in\mathcal{B}^0_\infty(\mathbb{L})$, which we view as an element of $\mathcal{B}^0_(\mathbb{L})$, and $\varphi\in\mathcal{S}(\mathbb{L})$. As previously, we consider basic elements $Y\in\mathcal{U}_{q_1}(\mathfrak{L})\otimes\mathcal{U}_{q_2}(\mathfrak{L})$ and $X\in\mathcal{U}_q(\mathfrak{L})$. We then bound the quantity:

$$\sup_{(x_1,x_2)\in\mathbb{L}^2}\{\left|\tilde{Y}_{(x_1,x_2)}\mathcal{R}(F\otimes\varphi)\right|_{X,N}\mathfrak{d}_{\mathbb{L}}^{-N_1}(x_1)\mathfrak{d}_{\mathbb{L}}^{-N_2}(x_2)\} \tag{4.16}$$

where we have first observed that $\mathcal{R}(F\otimes\varphi)$ belongs to $\mathcal{S}(\mathbb{L})$.
Again, following the same lines as before, we have that

$$\tilde{X}_x\left(\tilde{Y}_{(x_1,x_2)}\mathcal{R}(F\otimes\varphi)\right) \;=\; \mathcal{R}\left[(\mathtt{Ad}_{x_1^{-1}}X\otimes\mathtt{Ad}_{x_2^{-1}}X).Y\right]^{\sim}_{(xx_1,xx_2)}(F\otimes\varphi)\;.$$

The module of the above expression is therefore bounded by elements of the form

$$C\mathfrak{d}^q_{\mathbb{L}}(x_1)\mathfrak{d}^q_{\mathbb{L}}(x_2)|F|_{q+q_1}|\varphi|_{q+q_2,N'}\mathfrak{d}_{\mathbb{L}}^{-N'}(x_2)\mathfrak{d}_{\mathbb{L}}^{-N'}(x)$$

for any $N'\in\mathbb{N}_0$.
In order to bound (4.16), we therefore need to bound expressions of the form

$$\mathfrak{d}_{\mathbb{L}}^{-N_1+q}(x_1)|F|_{q+q_1}|\varphi|_{q+q_2,N}\;.$$

We therefore have that the restriction of $\mathcal{R}$ to $\mathcal{B}^0_p(\mathbb{L})\times\mathcal{S}(\mathbb{L})$ is valued in

$$\mathcal{B}^{q,0}_{p-q}(\mathbb{L}^2,\mathcal{B}_{q'}^{-N'}(\mathbb{L}))$$

for all $q,q',N'\in\mathbb{N}_0$ (N' large enough). ∎

Chapter 5

Left-invariant non-formal star-products and Drinfel'd-Iwasawa twists

5.1 Unitary representations of symmetric spaces

Definition 5.1.1 *Let (M, s) be a symmetric space with transvection involutive Lie group (G, σ). A* **unitary representation** *of (M, s) is a triple $(\mathcal{H}, \mathbf{U}, \mathbf{\Sigma})$ where $\mathcal{H}$ is a separable Hilbert space, $\mathbf{U}$ is a unitary (strongly continuous) representation $\mathbf{U}$ of the Lie group G on $\mathcal{H}$:*

$$\mathbf{U} : G \to GL(\mathcal{H}) \; ,$$

and where $\mathbf{\Sigma}$ is a unitary involution:

$$\mathbf{\Sigma} : \mathcal{H} \to \mathcal{H}$$

such that, for every $g \in G$:

$$\mathbf{U}(\sigma g) \;=\; \mathbf{\Sigma}\, \mathbf{U}(g)\, \mathbf{\Sigma} \; .$$

The representation $(\mathcal{H}, \mathbf{\Omega})$ is called **irreducible** *whenever the underlying group representation $(\mathcal{H}, \mathbf{U})$ is.*

Proposition 5.1.1 *Let $(\mathcal{H}, \mathbf{U}, \mathbf{\Sigma})$ be a unitary representation of the symmetric space (M, s). Let K be isotropy subgroup in G of a base point $o \in M$, and consider the G-homogeneous spaces isomorphism:*

$$G/K \to M : gK \mapsto g.o \; .$$

Then,

(i) The map:

$$\mathbf{\Omega} : G \to U(\mathcal{H}) : g \mapsto \mathbf{\Omega}(g) \;:=\; \mathbf{U}(g)\mathbf{\Sigma}\mathbf{U}(g^{-1})$$

is constant each left-lateral class of K in G. It therefore descends to $M = G/K$ as a map again denoted by $\mathbf{\Omega}$:

$$\mathbf{\Omega} : M \to U(\mathcal{H}) : g.o \mapsto \mathbf{\Omega}(g) \; .$$

(ii) The map $\mathbf{\Omega}$ satisfies the following properties:

$$\mathbf{\Omega}(x)\mathbf{\Omega}(y)\mathbf{\Omega}(x) \;=\; \mathbf{\Omega}(\mathbf{s}_x y)$$

for all $x, y \in M$.

5.1.1 Representations of group type symmetric spaces

The notion of representation of a symmetric space is defined in a natural way: it is a representation of its transvection group that is compatible with the symmetric space structure as homogeneous space.

Definition 5.1.2 *A* square integrable irreducible unitary representation *of a group type symmetric space M is a triple $(\mathcal{H}, U, \Sigma)$ where*

- *$\mathcal{H}$ is a separable Hilbert space,*
- *U is a unitary representation of G on $\mathcal{H}$,*
- *There exists a G-invariant dense Fréchet subspace $\mathcal{H}^\infty$ of $\mathcal{H}$ and an automorphism Σ of $\mathcal{H}^\infty$ that commutes with $U(K)$.*

such that

(i) the following map

$$\Omega : G/K \to \mathtt{Aut}(\mathcal{H}^\infty) : gK \mapsto U(g)\Sigma U(g^{-1}) \; .$$

is continuous and injective.

(ii) The restriction of the representation U to $\mathbb{S}$ is irreducible and square integrable.

Remark 5.1.1 *Note that the map Ω is necessarily G-equivariant.*

Given such a represented group type symmetric space M, since Ω is continuous, we consider the map

$$\Omega : \mathcal{D}(M) \to \mathcal{B}(\mathcal{H}^\infty) : f \mapsto \int_{\mathbb{S}} f(x)\, \Omega(x)\, \mathrm{d}x$$

where the measure on M is the transportation of the left-invariant Haar measure on $\mathbb{S}$ and where $\mathcal{B}(\mathcal{H}^\infty)$ denotes the algebra of continuous linear operators in $\mathcal{H}^\infty$.

Definition 5.1.3 *The representation is called* tracial *if the map Ω extends to a unitary (G-equivariant) isomorphism from $L^2(M)$ onto the algebra $\mathcal{L}^2(\mathcal{H})$ of Hilbert-Schmidt operators on $\mathcal{H}$:*

$$\Omega : L^2(M) \xrightarrow{\sim} \mathcal{L}^2(\mathcal{H}) \; .$$

Remark 5.1.2 *In the tracial case, the inverse σ_Ω of Ω is given by*

$$\sigma_\Omega(A)(x) \;=\; \mathrm{Tr}(A\Omega(x))$$

understood as a weak trace distribution.

5.1.2 Retract pairs

The notion of retract pair concerns a pair of group type symmetric spaces that share a common symmetry. Generally, we think of the situation where one of them is a curvature contraction of the other. They then form a retract pair when they share this common symmetry at the representation level as well.
In the sequel, we will use the notion of retract pair to somehow invert the contraction level at the functional level.

Let $M_1 = G_0/K_1$ and $M_2 = G_1/K_2$ be group type symmetric spaces on the same abstract Lie group $\mathbb{S}$. Let $(\mathcal{H}_j, U_j, \Sigma_j)$ $(j = 1, 2)$ be a square integrable irreducible unitary representation of M_j.

Lemma 5.1.1 *Let $\phi_e^{(1)} \in \mathcal{H}_1\backslash\{0\}$ and $\phi_e^{(2)} \in \mathcal{H}_2\backslash\{0\}$ be both in the domains of the square-root of the Duflo-Moore operators respectively associated to both representation. Set for every $x \in \mathbb{S}$:*

$$\phi_x^{(1)} := U_1(x)\phi_e^{(1)} \quad \phi_x^{(2)} := U_2(x)\phi_e^{(2)} .$$

Then, for all $\varphi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_2$, the function

$$\mathbb{S} \to \mathbb{C} : x \mapsto < \varphi, \phi_x^{(1)} >_1 < \phi_x^{(2)}, \psi >_2 \tag{5.1}$$

(where $<\, ,\, >_j$ denotes the inner product on $\mathcal{H}_j$) is integrable w.r.t. a left-invariant Haar measure $\mathrm{d}x$ on $\mathbb{S}$.

Proof. For square-integrability of the representations (restricted to $\mathbb{S}$), Cauchy-Schwartz inequality yields the assertion. ■

We denote by T the map from $\mathcal{H}_2$ to $\mathcal{H}_1$ defined by the above matrix coefficients, which wihtin the bra-cket notations is expressed by the weak integrals:

$$[T : \mathcal{H}_2 \to \mathcal{H}_1] := \int_{\mathbb{S}} |\phi_x^{(1)} >< \phi_x^{(2)}| \, \mathrm{d}x .$$

Proposition 5.1.2 *The map T is either trivial either a unitary equivalence of $\mathbb{S}$-representations (up to a nonzero constant factor).*

Proof. Consider the map

$$[T^* : \mathcal{H}_1 \to \mathcal{H}_2] := \int_{\mathbb{S}} |\phi_x^{(2)} >< \phi_x^{(1)}| \, \mathrm{d}x .$$

One readily checks that T and T^* are intertwiners:

$$U_1 T = T U_2 \quad \text{and} \quad U_2 T^* = T^* U_1$$

Therefore we have

$$T^*TU_2 = T^*U_1T = U_2T^*T .$$

The self-adjoint element T^*T therefore commutes with the irreducible representation U_2. Hence Schur's lemma concludes. ■

Definition 5.1.4 *The represented group type symmetric spaces M_1 and M_2 constitute a* retract pair (M_1, M_2) *if the restrictions of the representations U_0 and U_1 of $\mathbb{S}$ are unitarily equivalent.*
Based on Proposition 5.1.2, which provides an explicit intertwiner, we will in this case therefore assume that

$$\mathcal{H}_1 = \mathcal{H}_2 =: \mathcal{H} .$$

We will see that in many cases the geometry of a group type symmetric space is quite well encoded by a representation. The idea of the retract pair then, as we will see later on, is that from a "nice" group type symmetric space, one can reach the geometry of a richer and more complicated one. By "reaching", we mean finding some kind of equivalence, at the operator algebraic level, that passes from the "nice"one to the other one.

5.2 Equivariant quantization of Iwasawa factors $\mathbb{S} = AN$

The results exposed in this section are extracted from [Bi02] and [BG15], except for the first subsection which is classical.

5.2.1 Weyl's quantization

We consider the plane $\mathbb{R}^2 = \{x = (q,p)\}$. For every $\tilde{u} \in \mathcal{D}(\mathbb{R})$, we set

$$\Omega_W(x)\tilde{u}(q_0) = e^{2i/\theta(q-q_0).p}\tilde{u}(2q-q_0)$$

where μ is a non-zero real parameter. For every $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we consider

$$\Omega_W(\varphi)\tilde{u}(q_0) := \int \varphi(x)\Omega_W(x)\tilde{u}(q_0)\mathrm{d}x = \frac{1}{2}\int e^{i/\theta(q-q_0).p}\varphi(\frac{q_0+q}{2},p)\,\tilde{u}(q)\,\mathrm{d}p\,\mathrm{d}q\,. \tag{5.2}$$

We denote by $\mathcal{S}(\mathbb{R}^2)$ the usual Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^2$. One then has the following classical result:

Theorem 5.2.1 *Let $\theta \in \mathbb{R}_0$. Then,*
(i) Formula (5.2) defines a unitary isomorphism (up to constant):

$$\Omega_W : L^2(\mathbb{R}^2) \to \mathcal{L}^2(L^2(\mathbb{R})) : \varphi \mapsto \Omega_W(\varphi)\,.$$

(ii) For all $u, v \in \mathcal{S}(\mathbb{R}^2)$, one has the symbol composition formula:

$$\left(\Omega^W_\theta\right)^*(\Omega^W_{\theta,\lambda}(u)\Omega^W_{\theta,\lambda}(v))(x) =: u\star^W_\theta v(x) = \frac{1}{\theta^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} e^{i/\theta S_W(x,y,z)}u(y)\,v(z)\,\mathrm{d}y\,\mathrm{d}z$$

where, setting $\omega := \mathrm{d}p\wedge\mathrm{d}q$:

$$S_W(x,y,z) := \omega(x,y)+\omega(y,z)+\omega(z,x)\,.$$

(iii) The product $\star^W_\theta$ closes on the Schwartz space $\mathcal{S}(\mathbb{R}^2)$. The algebra $(\mathcal{S}(\mathbb{R}^2),\star^W_\theta)$ is then Fréchet.

5.2.2 Quantization of $\mathbb{S}$

We now consider the affine group $\mathbb{S}$ as realized in Section 1.2.2.

Notation 5.2.1 *Let $\mathbf{m} \in \mathcal{B}^N_\infty(\mathbb{S},\mathbb{R})$, $\mathbf{m} > 0$ and $\tilde{E}\mathbf{m} = 0$ (i.e. $\mathbf{m}$ depends on the variable $a \in A$ only).*
1. We denote by

$$\mathcal{F}\varphi(a,\xi) := \int_N e^{-in\xi}\varphi(a,n)\,\mathrm{d}n$$

the partial Fourier transform in the N-variable.
2. We consider the one-parameter family of diffeomorphisms of $\mathbb{R}^2_{(a,\xi)}$:

$$\phi_\theta(a,\xi) := (a,\frac{2}{\theta}\sinh(\frac{\theta}{2}\xi))\,.$$

3. We set

$$\mathbf{m}_\theta(\xi) := \mathbf{m}(\frac{\theta}{2}\mathrm{arcsinh}(2\xi))\,.$$

4. For every $\varphi \in \mathcal{D}(\mathbb{S})$, we define

$$T_{\theta,\mathbf{m}}\varphi := \mathcal{F}^{-1}\circ\mathbf{m}_\theta^{-1}\circ(\phi_\theta^{-1})^\star\circ\mathcal{F}(\varphi)\,.$$

5. We denote by $L^2_{\theta,\mathbf{m}}(\mathbb{S})$ the Hilbert completion of $\mathcal{D}(\mathbb{S})$ under the inner product:

$$< u,v >_{\theta,\mathbf{m}} := \int(\phi^\star_\theta\mathbf{m}_\theta\mathcal{F}\overline{u})\,.\,(\phi^\star_\theta\mathbf{m}_\theta\mathcal{F}v)\,\mathrm{d}\xi\,\mathrm{d}a\,.$$

6. For every $\lambda \in \mathbb{R}_0$, we set

$$\mathcal{H}_\lambda := L^2(A, e^{2\lambda a}\,\mathrm{d}a)\,.$$

Lemma 5.2.1 *Let $r \in \mathbb{R}$. Then, for all $k \geq 2(1+r)$ and $N \in \mathbb{Z}$, the operator*

$$\mathbf{A}_r \ := \ \mathcal{F}^{-1} \circ \left(1 + \frac{\theta^2 \xi^2}{4}\right)^r \circ \mathcal{F}$$

extends from $\mathcal{D}(\mathbb{S})$ to a continuous linear map:

$$\mathbf{A}_r \ : \ \mathcal{B}_k^N(\mathbb{S}) \to \mathcal{B}_{k-2(1+r)}^{N+2}(\mathbb{S}).$$

In particular, the map $\mathbf{A}_r$ continuously preserves the Schwartz space $\mathcal{S}(\mathbb{S})$.

Proof. We have, for $\varphi \in \mathcal{D}(\mathbb{S})$:

$$\mathbf{A}_r \varphi(a_0, n_0) \ := \ \int e^{in_0\xi} \left(1 + \frac{\theta^2 \xi^2}{4}\right)^r e^{-in\xi} \varphi(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi \ .$$

Observing that, for every $m \in \mathbb{N}$, one has:

$$(1 - \partial_n^2)^m \frac{1}{(1+\xi^2)^m} \, e^{-in\xi} \ = \ e^{-in\xi} \ ,$$

an integration by parts yields:

$$\mathbf{A}_r \varphi(a_0, n_0) \ := \ \int e^{in_0\xi} \chi_m(\xi) \, e^{-in\xi} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi$$

with

$$\chi_m(\xi) \ = \ \frac{\left(1 + \frac{\theta^2 \xi^2}{4}\right)^r}{(1+\xi^2)^m} \quad \text{and} \quad \varphi_m \ := \ ((1 - \tilde{E}^2)^m \varphi) \ .$$

Since

$$\tilde{H} \ = \ \partial_{a_0} - 2n_0 \partial_{n_0} \ ,$$

one has

$$\begin{aligned}
\tilde{H} \mathbf{A}_r \varphi(a_0, n_0) \quad &= \int e^{in_0\xi} \chi_m(\xi) \, e^{-in\xi} \partial_{a_0} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi - 2in_0 \int \xi \, e^{in_0\xi} \chi_m(\xi) \, e^{-in\xi} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi \\
&= \int e^{in_0\xi} \chi_m(\xi) \, e^{-in\xi} \partial_{a_0} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi + 2n_0 \int e^{in_0\xi} \chi_m(\xi) \, \partial_n(e^{-in\xi}) \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi \\
&= \int e^{in_0\xi} \chi_m(\xi) \, e^{-in\xi} \partial_{a_0} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi - 2n_0 \int e^{in_0\xi} \chi_m(\xi) \, (e^{-in\xi}) \tilde{E} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi \\
&= \int e^{in_0\xi} \chi_m(\xi) \, e^{-in\xi} \partial_{a_0} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi + 2i \int \partial_\xi (e^{in_0\xi}) \chi_m(\xi) \, e^{-in\xi} \tilde{E} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi \\
&= \int e^{in_0\xi} \chi_m(\xi) \, e^{-in\xi} \partial_{a_0} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi - 2i \int e^{in_0\xi} \chi'_m(\xi) \, e^{-in\xi} \tilde{E} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi \\
&\quad -2 \int e^{in_0\xi} \chi_m(\xi) \, n e^{-in\xi} \tilde{E} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi \\
&= \int e^{in_0\xi} \chi_m(\xi) \, e^{-in\xi} \tilde{H} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi - 2i \int e^{in_0\xi} \chi'_m(\xi) \, e^{-in\xi} \tilde{E} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi \ .
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
\tilde{E} \mathbf{A}_r \varphi(a_0, n_0) \quad &= i \int \xi \, e^{in_0\xi} \chi_m(\xi) \, e^{-in\xi} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi \\
&= - \int e^{in_0\xi} \chi_m(\xi) \, \partial_n(e^{-in\xi}) \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi \\
&= \int e^{in_0\xi} \chi_m(\xi) \, e^{-in\xi} \tilde{E} \varphi_m(a_0, n) \, \mathrm{d}n \, \mathrm{d}\xi \ .
\end{aligned}$$

Iterating, we conclude that for every $X \in \mathcal{U}_k(\mathfrak{s})$, $\tilde{X}\mathbf{A}_r\varphi$ is a linear combination of elements of the form:

$$\Xi(a_0, n_0) \;:=\; \int e^{in_0\xi}\chi_{_{X'}}(\xi)\, e^{-in\xi}\,\tilde{X}'\varphi(a_0, n)\,\mathrm{d}n\,\mathrm{d}\xi$$

with

$$|X'| = k + 2m$$

and $\chi_{_{X'}}$ of the form

$$\chi_{_{X'}}(\xi) \;=\; \frac{b_{_{X'}}(\xi)}{(1+\xi^2)^{m-r}}$$

where $b_{_{X'}} \in \mathcal{B}^0_\infty(\mathbb{R})$.
Now, we estimate the growth in the variable n. Since

$$(1-\partial_\xi^2)^j\,\frac{1}{(1+n^2)^j}\,e^{-in\xi} \;=\; e^{-in\xi}\;,$$

we have for every $j \in \mathbb{N}$:

$$\Xi(a_0, n_0) \;:=\; \int (1-\partial_\xi^2)^j\,\left(e^{in_0\xi}\chi_{_{X'}}(\xi)\right)\,\frac{1}{(1+n^2)^j}\,e^{-in\xi}\,\tilde{X}'\varphi(a_0, n)\,\mathrm{d}n\,\mathrm{d}\xi$$

which is a linear combination of elements of the form:

$$\Xi_\ell(a_0, n_0) \;:=\; n_0^\ell \int e^{in_0\xi}\chi_{_{X'}}(\xi)\,\frac{1}{(1+n^2)^j}\,e^{-in\xi}\,\tilde{X}'\varphi(a_0, n)\,\mathrm{d}n\,\mathrm{d}\xi$$

where

$$\ell \leq 2j$$

and where $b_{_{X'}}$ in $\chi_{X'}$ has possibly been redefined. One concludes observing that

$$\mathfrak{d}_{\mathbb{S}}(a, n) \;\sim\; \sqrt{n^2 + \cosh(4a)}\;.$$

■

Lemma 5.2.2 *(i) Let $\varphi \in \mathcal{H}_\lambda$. The formula*

$$(U_{\theta,\lambda}(a, n)\varphi)(a_0) \;:=\; e^{-\lambda a + \frac{i}{\theta}e^{2(a-a_0)}n}\varphi(a_0 - a)$$

defines an irreducible unitary representation of $\mathbb{S}$ on $\mathcal{H}_\lambda$.
(ii) Let

$$\Lambda(a) \;:=\; e^{-\lambda a}\;.$$

Then the multiplication operator by Λ:

$$m_\lambda : \varphi \mapsto \Lambda\varphi$$

defines a unitary equivalence of $\mathbb{S}$-representation:

$$m_\lambda : \mathcal{H}_0 \to \mathcal{H}_\lambda$$
$$m_\lambda U_{\theta,0} m_\lambda^{-1} \;=\; U_{\theta,\lambda}\;.$$

(iii) The formula

$$\Sigma_{\mathbf{m},\lambda}\varphi(a_0) \;:=\; \mathbf{m}(a_0)\Lambda^2(a_0)\varphi(-a_0) \;=\; \mathbf{m}(a_0)e^{-2\lambda a_0}\varphi(-a_0)$$

defines a (possibly unbounded) invertible operator:

$$\Sigma_{\mathbf{m},\lambda} : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$$

that underlies, together with $(\mathcal{H}_\lambda, U_{\theta,\lambda})$, a square integrable unitary irreducible representation:

$$\Omega_{\theta,\mathbf{m},\lambda} : \mathbb{M} \to U(\mathcal{H}_\lambda)$$

of the symmetric space $\mathbb{M}$.

The following proposition describes

Proposition 5.2.1 *(i) The map $T^{-1}_{\theta,\mathbf{m}}$ stabilises the Schwartz space:*

$$T^{-1}_{\theta,\mathbf{m}}\ \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R}^2)\ .$$

(ii) The map

$$\varphi \mapsto \Omega_{\theta,\mathbf{m},\lambda}(\varphi)\ :=\ m_\Lambda\left(\Omega^W_\theta T^{-1}_{\theta,\mathbf{m}}(\varphi)\right) m_{\Lambda^{-1}}$$

extends from $\mathcal{D}(\mathbb{S})$ to an $\mathbb{S}$-equivariant unitary isomorphism between $L^2_{\theta,\mathbf{m}}(\mathbb{S})$ and the Hilbert-Schmidt operators on $\mathcal{H}_\lambda$:

$$\Omega_{\theta,\mathbf{m},\lambda} : L^2_{\theta,\mathbf{m}}(\mathbb{S}) \to \mathcal{L}^2(\mathcal{H}_\lambda)\ .$$

(iii) In terms of Lemma (6.1.1) (iii), one has for every $\varphi \in \mathcal{D}(\mathbb{S})$:

$$\Omega_{\theta,\mathbf{m},\lambda}(\varphi)\ =\ \int_{\mathbb{S}} \varphi(x)\,\Omega_{\theta,\mathbf{m},\lambda}(x)\,\mathrm{d}x\ .$$

We now have analogous results as in the flat Heisenberg case.

Proposition 5.2.2 *Set*

$$\mathbf{m}^0(a,n)\ :=\ \sqrt{\cosh(2a)}\ .$$

Then,

(i)

$$L^2_{\theta,\mathbf{m}^0}(\mathbb{S})\ =\ L^2(\mathbb{S})\ .$$

(ii) For every $u, v \in \mathcal{D}(\mathbb{S})$, the function

$$\Omega^{-1}_{\theta,\mathbf{m},\lambda}(\Omega_{\theta,\mathbf{m},\lambda}(u)\Omega_{\theta,\mathbf{m},\lambda}(v))$$

does not depend on λ.

(iii) Setting

$$u \star_{\theta,\mathbf{m}} v\ :=\ \Omega^{-1}_{\theta,\mathbf{m},\lambda}(\Omega_{\theta,\mathbf{m},\lambda}(u)\Omega_{\theta,\mathbf{m},\lambda}(v))\ ,$$

one has

$$u \star_{\theta,\mathbf{m}} v(x)\ =\ \frac{1}{\theta^2}\int_{\mathbb{L}} A_{\mathbf{m}}(x,y,z)\,e^{\frac{1}{\theta}S_{\mathbb{M}}(x,y,z)}u(y)\,v(z)\,\mathrm{d}y\,\mathrm{d}z$$

where

$$A_{\mathbf{m}}((a_0,n_0),(a_1,n_1),(a_2,n_2))\ :=\ \mathbf{m}(a_0-a_1)\frac{(\mathbf{m}^0)^2(a_1-a_2)}{\mathbf{m}(a_1-a_2)}\mathbf{m}(a_2-a_0)$$

and

$$S_{\mathbb{M}}((a_0,n_0),(a_1,n_1),(a_2,n_2))\ :=\ \sinh(a_0-a_1)n_2+\sinh(a_2-a_0)n_1+\sinh(a_1-a_2)n_0\ .$$

(iv) The product $\star_{\theta,\mathbf{m}}$ closes on the Schwartz space $\mathcal{S}(\mathbb{S})$. The algebra $(\mathcal{S}(\mathbb{S}),\star_{\theta,\mathbf{m}})$ is then Fréchet.

Notation 5.2.2 *The case $\mathbf{m} = 1$ is of particular importance for us. We therefore adopt the following notations:*

$$\begin{aligned} \Sigma_{1,\lambda} &=: \Sigma_\lambda \\ \Omega_{\theta,1,\lambda} &=: \Omega_{\theta,\lambda} \\ \star_{\theta,1} &=: \star_\theta\ . \end{aligned}$$

We have now three results concerning extensions of $\star_{\theta,\mathbf{m}}$ to $\mathcal{B}$-type spaces. The following proposition directely follows from the proofreading of Theorem 2.43 and Proposition 2.47 of [BG15].

Proposition 5.2.3 *Let $t, s \geq 0$, $p \in \mathbb{N} \cup \{\infty\}$ with $p > 4 + 2N + t + s$ ($\mathbf{m} \in \mathcal{B}^N_\infty(\mathbb{S})$). Then,*

(i) the product $\star_{\theta,\mathbf{m}}$ canonically extends form $\mathcal{D}(\mathbb{S}) \times \mathcal{D}(\mathbb{S})$ to a bi-linear separately continuous mapping:

$$\star_{\theta,\mathbf{m}} : \mathcal{B}^t_p(\mathbb{S}) \times \mathcal{B}^s_p(\mathbb{S}) \to \mathcal{B}^{t+s}_{p-4-2N-2t}(\mathbb{S}) \; .$$

(ii) The above extension is associative in the sense that for all $F_1, F_2, F_3 \in \mathcal{B}^t_\infty(\mathbb{S})$, one has

$$(F_1 \star_{\theta,\mathbf{m}} F_2) \star_{\theta,\mathbf{m}} F_3 \;=\; F_1 \star_{\theta,\mathbf{m}} (F_2 \star_{\theta,\mathbf{m}} F_3)$$

in $\mathcal{B}^{3t}_\infty(\mathbb{S})$.

Lemma 5.2.3 *The product $\star_\theta$ associated with $\mathbf{m} = 1$ corresponds to*

$$T_\theta \varphi \;:=\; \mathcal{F}^{-1} \circ (\phi_\theta^{-1})^\star \circ \mathcal{F}(\varphi) \; .$$

One then has

$$\left(T_\theta^{-1}\right)^* T_\theta^{-1} \;=\; \mathbf{A}_{-1/2}.$$

Proof. From [Bi02], one has

$$\mathrm{Jac}_{\phi_\theta^{-1}}(\xi) \;=\; (1 + \frac{\theta^2 \xi^2}{4})^{-1/2} \; .$$

The lemma then follows from a direct computation. ∎

Corollary 5.2.1 *Setting*

$$\mathbf{S}_0 \;:=\; \mathbf{A}_{1/4} \; ,$$

the product law on $\mathcal{S}(\mathbb{S})$

$$u \star^2_\theta v \;:=\; \mathbf{S}_0^{-1}(\mathbf{S}_0 u \star_\theta \mathbf{S}_0 v)$$

is tracial in the sense that

$$\int \overline{u} \star^2_\theta v \;=\; \int \overline{u} v \; .$$

Proof. We have

$$\mathrm{Tr}(\Omega_{\theta,\lambda}(\varphi_1)^* \Omega_{\theta,\lambda}(\varphi_2)) \;=\; \mathrm{Tr}(\Omega_W T_\theta^{-1}(\varphi_1)^* \Omega_W T_\theta^{-1}(\varphi_2)) \;=\; \int T_\theta^{-1}(\overline{\varphi_1}) \star_W T_\theta^{-1}(\varphi_2)$$
$$=\; \int T_\theta^{-1}(\overline{\varphi_1}) T_\theta^{-1}(\varphi_2) \;=\; \int \left(T_\theta^{-1}\right)^* T_\theta^{-1}(\overline{\varphi_1})\varphi_2 \;=\; \int \mathbf{A}_{-1/2}(\overline{\varphi_1})\varphi_2 \; .$$

Setting

$$\Omega^{(2)}_{\theta,\lambda} \;:=\; \Omega_{\theta,\lambda}\mathbf{S}_0$$

one then has

$$\mathrm{Tr}(\Omega^{(2)}_{\theta,\lambda}(\varphi_1)^* \Omega^{(2)}_{\theta,\lambda}(\varphi_2)) \;=\; \int \overline{\varphi_1} \star^2 \varphi_2 \;=\; \mathrm{Tr}(\Omega_{\theta,\lambda}\mathbf{S}_0(\varphi_1)^* \Omega_{\theta,\lambda}\mathbf{S}_0(\varphi_2))$$
$$=\; \int \overline{\mathbf{S}_0\varphi_1} \star_1 \mathbf{S}_0\varphi_2 \;=\; \int \mathbf{A}_{-1/2}(\overline{\mathbf{S}_0\varphi_1})\mathbf{S}_0\varphi_2 \;=\; \int \mathbf{S}_0^*\mathbf{A}_{-1/2}\mathbf{S}_0(\overline{\varphi_1})\mathbf{S}_0\varphi_2 \;=\; \int \overline{\varphi_1}\varphi_2$$

because $\mathbf{A}$ hence $\mathbf{S}_0$ are self-adjoint. ∎

At last, we have a non-Abelian version [BG15] of the classical Calderòn-Vaillancourt theorem.

Theorem 5.2.2 *There exists $p_0 \in \mathbb{N}$ such that the map $\Omega_{\theta,\mathbf{m},\lambda}$ canonically extends from $\mathcal{D}(\mathbb{S})$ to a continuous mapping from $\mathcal{B}^0_{p_0}(\mathbb{S})$ valued in the bounded operators:*

$$\Omega_{\theta,\mathbf{m},\lambda} : \mathcal{B}^0_{p_0}(\mathbb{S}) \to \mathcal{B}(\mathcal{H}_\lambda) \; .$$

Proposition 5.2.4 *The mapping*

$$\Omega_{\theta,\mathbf{m},\lambda} : \mathcal{B}^0_{p_0}(\mathbb{S}) \to \mathcal{B}(\mathcal{H}_\lambda) \ .$$

is injective.

Proof. For every $F \in \mathcal{B}^0_{p_0}(\mathbb{S})$ and every $\varphi \in \mathcal{S}(\mathbb{S})$, the operator $\Omega_{\theta,\mathbf{m},\lambda}(F)\Omega_{\theta,\mathbf{m},\lambda}(\varphi)$ is Hilbert-Schmidt. If $\Omega_{\theta,\mathbf{m},\lambda}(F) = 0$, one therefore has $F \star_{\theta,\mathbf{m}} \varphi = 0$. Differentiating w.r.t θ around zero then yields $F\varphi = 0$ for every $\varphi \in \mathcal{S}(\mathbb{S})$. Hence $F = 0$. ■

Combining these two last results, one gets

Proposition 5.2.5 *[BG15] Denoting by $||\ .\ ||_\lambda$ the operator norm on $\mathcal{H}_\lambda$, the map*

$$||\ .\ ||_{\theta,\mathbf{m},\lambda} : \mathcal{B}^0_{p_0}(\mathbb{S}) \to \mathbb{R} : F \mapsto ||\Omega_{\theta,\mathbf{m},\lambda}(F)||_\lambda$$

restricts to a $C^\star$-norm on $(\mathcal{B}^0_\infty(\mathbb{S}), \star_{\theta,\mathbf{m}})$.

In fact, in terms of the regular action one has:

Theorem 5.2.3 *The regular action*

$$\mathcal{B}^0_\infty(\mathbb{S}) \times \mathcal{S}(\mathbb{S}) \to \mathcal{S}(\mathbb{S}) : (F, \varphi) \mapsto F \star_{\theta,\mathbf{m}} \varphi \ =: \ L_{\star_{\theta,\mathbf{m}}}(F)\varphi$$

extends to a bounded action of $\mathcal{B}^0_\infty(\mathbb{S})$ on $L^2_{\theta,\mathbf{m}}(\mathbb{S})$, and one has

$$||L_{\star_{\theta,\mathbf{m}}}(F)||_{\theta,\mathbf{m}} \ = \ ||\Omega_{\theta,\mathbf{m},\lambda}(F)||_\lambda$$

where $||\ .\ ||_{\theta,\mathbf{m}}$ denotes the operator norm of operators on $L^2_{\theta,\mathbf{m}}(\mathbb{S})$. In particular, as expected, $||\ .\ ||_{\theta,\mathbf{m},\lambda}$ does not depend on λ.

Chapter 6

Quantization of the hyperbolic plane and Drinfel'd twists

We start from the Moyal $\star$-product on the plane $\mathbb{R}^2 = \{(a,n)\}$:

$$u \star^0_\hbar v \quad := \quad u.v + i\frac{\hbar}{2}\{u,v\} + \sum_{k=2}^{\infty} \frac{(i\,\hbar/2)^k}{k!} \Omega^{i_1 j_1} \cdots \Omega^{i_k j_k} \partial_{i_1 \cdots i_k} u \partial_{j_1 \cdots j_k} v \quad , \tag{6.1}$$

$$= \sum_{k=0}^{\infty} \frac{(i\,\hbar/2)^k}{k!} \sum_{p=0}^{k} (-1)^p \frac{k!}{p!(k-p)!} \partial_a^{k-p} \partial_n^p u \, \partial_a^p \partial_n^{k-p} v \quad , \tag{6.2}$$

where

$$\Omega_{ik}\Omega^{kj} = -\delta_i^k \qquad , \qquad \{u,v\} = \Omega^{ij}\partial_i u \partial_j v := \partial_a u \partial_n v - \partial_n u \partial_a v \quad . \tag{6.3}$$

It turns out that in coordinates (a,n) the Moyal product (6.1) is $\mathfrak{sl}_2(\mathbb{R})$-covariant with respect to the hyperbolic action of $G := SL_2(\mathbb{R})$ on the hyperbolic plane $\mathbb{D} = \mathbb{S}$ [Bi95]. Precisely, presenting the Lie algebra $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{R})$ as generated over $\mathbb{R}$ by H, E and F satisfying:

$$[H,E] = 2E \quad , \quad [H,F] = -2F \quad , \quad [E,F] = H, \tag{6.4}$$

the moment map associated with the action of G on $\mathbb{D} = G/K = \mathbb{S}$ reads:

$$\lambda_H = \sqrt{\kappa}\, n \,;\; \lambda_E = \frac{\sqrt{\kappa}}{2} e^{-2a} \,;\; \lambda_F = -\frac{\sqrt{\kappa}}{2} e^{2a}(1+n^2) \,. \tag{6.5}$$

The associated fundamental vector fields are given by:

$$H^\star = -\partial_a;\; E^\star = -e^{-2a}\partial_n;\; F^\star = e^{2a}(n\partial_a - (\kappa + n^2)\partial_n) \,. \tag{6.6}$$

At last, the representation of $\mathfrak{g}$ by derivations of $\star^0_\hbar$ admits the expression:

$$\rho_\hbar(H) \quad = \quad -\partial_a \tag{6.7}$$

$$\rho_\hbar(E) \quad = \quad -\frac{e^{-2a}}{\hbar} \sin(\hbar\partial_n) \tag{6.8}$$

$$\rho_\hbar(F) \quad = \quad e^{2a}\left(\frac{\hbar}{4}\sin(\hbar\partial_n)\partial_a^2 + n\cos(\hbar\partial_n)\partial_a - (\kappa+n^2)\frac{\sin(\hbar\partial_n)}{\hbar}\right) \,. \tag{6.9}$$

We now consider the partial Fourier transform in the n-variable

$$\mathcal{F}(\varphi)(a,\zeta) = \int_{-\infty}^{\infty} e^{-i\zeta n}\varphi(a,n)\mathrm{d}n \quad , \tag{6.10}$$

and denote by $\tilde{\mathbb{S}} := \{(a,\zeta)\}$ the space where the Fourier transformed $\mathcal{F}(g)$ is defined on. Defining the following one-parameter family of diffeomorphisms of $\tilde{\mathbb{S}}$:

$$\phi_\hbar(a,\zeta) = \left(a, \frac{\sinh(\hbar\,\zeta)}{\hbar}\right) \quad , \tag{6.11}$$

and denoting by $\tilde{\mathcal{S}}$ (resp. $\mathcal{S}$) the space of Schwartz test functions $\mathcal{S}(\tilde{\mathbb{S}})$ (resp. $\mathcal{S}(\mathbb{S})$) on $\tilde{\mathbb{S}} = \{(a,\zeta)\}$ (resp. on $\mathcal{S}(\mathbb{S})$ of $\mathbb{S} = \{(a,n)\}$), one observes the following inclusions:

$$\phi_\hbar^\star\,\tilde{\mathcal{S}} \subset \tilde{\mathcal{S}} \text{ and } \tilde{\mathcal{S}} \subset (\phi_\hbar^{-1})^\star\tilde{\mathcal{S}} \subset \tilde{\mathcal{S}}' \,. \tag{6.12}$$

Therefore, every data of (reasonable) one parameter smooth family of invertible functions $\mathcal{P}_\hbar = \mathcal{P}_\hbar(b)$ yields an operator on the Schwartz space $\mathcal{S}(\mathbb{S})$ of $\mathbb{S} = \{(a,n)\}$:

$$\mathbf{T}^{-1} : \mathcal{S}(\mathbb{S}) \longrightarrow \mathcal{S}(\mathbb{S}) \tag{6.13}$$

defined as

$$\mathbf{T}^{-1}\varphi(a_0,n_0) = \frac{1}{2\,\pi}\int e^{i\zeta n_0}\,\mathcal{P}_\hbar(\zeta)\,e^{\frac{-i}{\hbar}n\sinh(\hbar\zeta)}\,\varphi(a_0,n)\,\mathrm{d}n\,\mathrm{d}\zeta \,. \tag{6.14}$$

More generally, denoting by $\mathcal{M}_f$ the pointwise multiplication operator by f, the operator:

$$\mathbf{T} : \mathcal{S}(\mathbb{S}) \longrightarrow \mathcal{S}'(\mathbb{S}) \tag{6.15}$$

defined as

$$\mathbf{T} \;:=\; \mathcal{F}^{-1} \circ (\phi_\hbar^{-1})^\star \circ \mathcal{M}_{\frac{1}{\mathcal{P}}} \circ \mathcal{F} \tag{6.16}$$

is a left-inverse of T^{-1}. Intertwining Moyal-Weyl's product by $\mathbf{T}$ yields the above product as $\star_{\hbar,\mathcal{P}} = \mathbf{T}(\star_\hbar^0)$. Observe that the latter closes on the range space $\mathcal{E}_\hbar := \mathbf{T}\mathcal{S}$ yielding a (non-formal) one parameter family of associative function algebras: $(\mathcal{E}_\hbar, \star_{\hbar,\mathcal{P}})$. Asymptotic expansions of these non-formal products produce genuine Poincaré-invariant formal star products on $\mathbb{M}$ [Bi02].

For generic $\mathcal{P}$ the above oscillatory integral product defines, for all real *value* of $\hbar$, an associative product law on some function space (as opposed to formal power series space) on $\mathbb{M}$. The most remarkable case being probably the one where $\mathcal{P}$ is pure phase. In the latter case, the above product formula extends (when $\hbar \neq 0$) to the space $L^2(\mathbb{M})$ of square integrable functions as a Poincaré invariant Hilbert associative algebra. The star product there appears to be *strongly closed*: for all u and v in $L^2(\mathbb{M})$, $u \star_{\hbar,\mathcal{P}} v$ belongs to $L^1(\mathbb{M})$, and one has:

$$\int_{\mathbb{M}} u \star_{\hbar,\mathcal{P}} v \;=\; \int_{\mathbb{M}} u\,v \,.$$

Combining [BG15] page 111 and [BDS09] Section 3, one finds that the relation with the star-product defined in 5.2.2 is given by

$$A_{\mathrm{can}}(a_1,a_2,a_3)\,\frac{\mathcal{P}(a_1-a_2)\,\mathcal{P}(a_3-a_1)}{\mathcal{P}(a_2-a_3)} \;=\; \mathbf{m}(a_1-a_2)\frac{\mathbf{m}_0^2(a_2-a_3)}{\mathbf{m}(a_2-a_3)}\,\mathbf{m}(a_3-a_1) \,.$$

Now we consider the following derivation D_0 of $\star := \mathbf{T}(\star^0)$: the operator

$$D_0 \;:=\; \mathbf{T} \circ \rho_\hbar(F) \circ \mathbf{T}^{-1} \,. \tag{6.17}$$

The particular choice of

$$\mathcal{P}(\zeta) \;\equiv 1 \tag{6.18}$$

yields the following expression for $i\,\square := \mathcal{F} \circ D_0 \circ \mathcal{F}^{-1}$ that appears to be a second order differential operator:

$$\square_{(a,\zeta)} = e^{2a}\left[\frac{\hbar^2}{4}\,\zeta\,\partial_a^2 + \zeta\,(1+\hbar^2\,\zeta^2)\partial_\zeta^2 + (1+\hbar^2\,\zeta^2)\partial_a\partial_\zeta + \hbar^2\,\zeta\,\partial_a + (2+3\hbar^2\,\zeta^2)\partial_\zeta - \zeta\,(k-\hbar^2)\right] \,. \tag{6.19}$$

Proposition 6.0.1 *Let $\star := \star_{\hbar,\mathcal{P}}$ be a (formal) invariant star-product on $\mathbb{M}$ whose we denote by $\mathfrak{Der}(\star)$ the algebra of derivations. Under the condition*

$$\mathcal{P}'(0) \;=\; 0\;,$$

(i) the classical moment

$$D^1 : \mathfrak{g} \to \mathfrak{Der}(\star) : X \mapsto \frac{1}{i\hbar}[\lambda_X\;,\;.\,]_\star$$

is a Lie map.
(ii) The restriction of D^1 to $\mathfrak{s}$ coincides with the geometric action by fundamental vector fields:

$$D^1_X \;=\; X^\star \quad \forall X \in \mathfrak{s}\;.$$

(iii) Every Lie map

$$D : \mathfrak{g} \to \mathfrak{Der}(\star) : X \mapsto D_X$$

that extends $D^1|_{\mathfrak{s}}$ is of the form

$$D_F \;=\; D^1_F \;+\; c_\hbar\,[e^{2a}\;,\;.\,]_\star$$

where $c_\hbar \in \mathbb{C}[\hbar^{-1},\hbar]]$ is a formal constant.

Proof. We seek for a Lie algebra homomorphism

$$D : \mathfrak{g} \to \mathfrak{Der}(\star)$$

that extends the mapping $\mathfrak{s} \to \Gamma^\infty(T\mathbb{M}) : X \mapsto X^\star$.
Since $\star$ is formally equivalent to Moyal's star product, every of its derivation must be interior. In particular, there exists a linear map

$$\Lambda : \mathfrak{g} \to C^\infty(\mathbb{M})[\hbar^{-1},\hbar]] : X \mapsto \Lambda_X$$

such that

$$D_X\varphi \;=:\; [\Lambda_X,\varphi]_\star \qquad (\forall\varphi \in C^\infty(\mathbb{M}))\;.$$

Denoting

$$\mathcal{A}_\hbar \;:=\; (C^\infty(\mathbb{M})[\hbar^{-1},\hbar]],\star)\;,$$

Jacobi's identity implies that the homomorphism defect

$$\Gamma_\hbar : \bigwedge^2 \mathfrak{g} \to \mathcal{A}_\hbar : X \wedge Y \mapsto \Lambda_{[X,Y]} - [\Lambda_X,\Lambda_Y]_\star$$

is actually valued in the center of $\mathcal{A}_\hbar$ i.e. in the formal constants $\mathbb{C}[\hbar^{-1},\hbar]]$. Observing that it is also a Chevalley two-cocycle, it must be exact for $\mathfrak{g}$ is simple:

$$\Gamma_\hbar \;=:\; \delta\gamma_\hbar \qquad \gamma_\hbar \in \mathfrak{g}^\star \otimes \mathbb{C}[\hbar^{-1},\hbar]]\;.$$

Possibly modifying it by subtracting $\gamma_\hbar$, we may therefore assume that the map $\Lambda : \mathfrak{g} \to \mathcal{A}_\hbar$ is Lie.
Using the condition $\mathcal{P}'(0) = 0$, the expression (6.16) yields

$$\mathbf{T}\lambda_X \;=\; \lambda_X$$

for every $X \in \mathfrak{s}$. The map

$$\Lambda^0 : \mathfrak{s} \to \mathcal{A}_\hbar : X \mapsto \frac{1}{i\hbar}\lambda_X \;=:\; \Lambda^0_X$$

is therefore Lie. Moreover, by a similar computation as for $T(n) = n$, we observe that there is a (generally non-zero) formal constant

$$\kappa_\hbar \;:=\; \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{\cosh(\hbar\xi)}{\mathcal{P}(\xi)}\right)|_{\xi=0}$$

such that

$$T(n^2) \;=\; \kappa_\hbar + n^2\;.$$

Since the operators T and T^{-1} commute with every multiplication operator $\mathcal{M}_f$ with $\partial_n f \equiv 0$, we get

$$T\lambda_F \;=\; \lambda_F - \frac{\sqrt{\kappa}}{2}\kappa_\hbar\, e^{2a} \;.$$

Setting

$$\Lambda^0_F \;:=\; \frac{1}{i\hbar}\left(\lambda_F - \frac{\sqrt{\kappa}}{2}\kappa_\hbar\, e^{2a}\right) \;,$$

we therefore get the Lie maps

$$\Lambda^0 : \mathfrak{g} \to \mathcal{A}_\hbar$$

and

$$D^0 : \mathfrak{g} \to \mathfrak{Der}(\star) : X \mapsto D^0_X \;:=\; [\Lambda^0_X \;,\; .\,]_\star \;.$$

Given another extension D, we consider the defect

$$D_F - D^0_F \;=\; [\gamma_F \;,\; .\,]_\star \quad \text{with} \quad \gamma_F \;:=\; \Lambda_F - \Lambda^0_F \;.$$

Since for every $X \in \mathfrak{s}$, we have $D_X \;=\; D^0_X \;=\; X^\star$, we observe:

$$D_X\gamma_F \;=\; [\Lambda_X, \Lambda_F]_\star - [\Lambda^0_X, \Lambda^0_F]_\star \;=\; \Lambda_{[X,F]} - \Lambda^0_{[X,F]} \;=\; X^\star\gamma_F \;.$$

Hence $E^\star\gamma_F \;=\; 0 \quad \text{and} \quad H^\star\gamma_F \;=\; -2\gamma_F$. ■

Remark 6.0.1 *Note that varying D only corresponds to redefining the curvature element κ.*

Proposition 6.0.2 *Let U be a $\mathbb{S}$-equivariant equivalence between $\star$ and a G-invariant $\star$-product $\sharp$. Then,*
(i) There exists a formal distribution $v \in \mathcal{D}'(\mathbb{S})[[\hbar]]$ whose associated left-invariant convolution operator on $\mathbb{S}$ coincides with U^{-1}:

$$U^{-1} \;=\; \int_{\mathbb{S}} v(y)\, R^\star_y \,\mathrm{d}y \;.$$

(ii) The Fourier transformed convolution kernel $\hat{v} \;:=\; \mathcal{F}v$ is a weak solution of second order PDE:

$$\Box(P\hat{v}) \;=\; \zeta(P\hat{v})$$

where the multiplicative factor P is defined by

$$P \;:=\; \left(\phi_\hbar^{-1}\right)^\star \mathcal{P} \;.$$

(iii) The $\star$-products $\star_{\hbar,\mathcal{P}}$ for which there exists a ζ-independent solution $\hat{v}$ correspond, up to a formal constant multiple, either to

$$\mathcal{P} \;=\; 1 \quad \text{or} \quad \mathcal{P} \;=\; \cosh(\hbar\zeta) \;.$$

6.1 Retracting the hyperbolic plane: non formal context

In this section, we will first prove that the pair $(\mathbb{M} \;:=\; (\mathbb{S}, s^{(0)}, \omega) \;,\; \mathbb{D})$ constitute a retract pair on the group manifold $\mathbb{S} = AN$. We will then describe how both symmetric space structures lift at the representation level as "phase" symmetries. At last we will express an operator that realizes a passage (a curvature deformation process) from one phase symmetry to the other and vice versa. This last operator will define the intertwiner $\mathfrak{W}$ (see the Introduction) that will transport the star product on $\mathbb{M}$ onto a star product on $\mathbb{D}$ with full $\mathrm{SL}(2,\mathbb{R})$-symmetry.

Notation 6.1.1 *For every $\lambda \in \mathbb{R}_0$, we set*

$$\mathcal{H}_\lambda \;:=\; L^2(A, e^{2\lambda a}\,\mathrm{d}a) \;.$$

From direct computations, we obtain

Proposition 6.1.1 *(i) Let θ be a positive real. Consider the upper half-plane*

$$\mathbb{D} \ := \ \{ z \ := \ p + iq \in \mathbb{C} \ : \ q > 0 \} \ .$$

and let

$$\mathcal{D}_{\left(\frac{\theta+2}{\theta}\right)} \ := \ \{h \in \mathrm{hol}(\mathbb{D}) \mid \int_{\mathbb{D}} \frac{1}{q^{2-2\left(\frac{\theta+2}{\theta}\right)}} |h(z)|^2 \mathrm{d}q\mathrm{d}p < \infty\}$$

where $\mathrm{hol}(\mathbb{D})$ *stands for the space of holomorphic functions on* $\mathbb{D}$.
Denote by V_θ the usual projective holomorphic discrete series of $\mathrm{SL}(2,\mathbb{R})$ *on* $\mathcal{D}_{\left(\frac{\theta+2}{\theta}\right)}$.
Then,

(i) The expression

$$T_{\theta,\lambda}\tilde{u}(z_0) \ := \ \int_{\mathbb{R}} e^{(\lambda - 2\left(\frac{\theta+2}{\theta}\right))q_0} e^{\frac{i}{\theta} e^{-2q_0} z_0} \, \tilde{u}(q_0) \, \mathrm{d}q_0$$

defines a unitary isomorphism

$$T_{\theta,\lambda} \ : \ \mathcal{H}_\lambda \to \mathcal{D}_{\left(\frac{\theta+2}{\theta}\right)}$$

that intertwines $V_\theta|_{\mathbb{S}}$ and $U_{\theta,\lambda}|_{\mathbb{S}}$.
We denote by $\mathbf{U}_{\theta,\lambda}$ the corresponding representation of $\mathrm{SL}(2,\mathbb{R})$ *on $\mathcal{H}_\lambda$:*

$$\mathbf{U}_{\theta,\lambda} \ := \ T^{-1}_{\theta,\lambda} V_\theta T_{\theta,\lambda} \ .$$

(ii) Consider J_μ the first type Bessel function *with parameter μ. Then the formula:*

$$\vartheta_{\theta,\lambda}\varphi(a_0) \ := \ \frac{4}{\theta} e^{-2\lambda a_0 + i\pi\left(\frac{\theta+2}{\theta}\right)} \int_{-\infty}^{\infty} J_{2\left(\frac{\theta+2}{\theta}\right)-1}\left(\frac{2}{\theta} e^{-2a}\right) e^{2(\lambda-1)a} \varphi(2a - a_0) \, \mathrm{d}a$$

defines a unitary involution:

$$\vartheta_{\theta,\lambda} : \mathcal{H}_\lambda \to \mathcal{H}_\lambda \ .$$

(iii) The triple $(\mathcal{H}_\lambda, \mathbf{U}_{\theta,\lambda}, \vartheta_{\theta,\lambda})$ is a square integrable unitary irreducible representation of the hyperbolic plane.
We denote by

$$\mathbf{\Omega}_{\theta,\lambda} : \mathbb{D} \to \mathtt{Aut}(\mathcal{H}_\lambda)$$

the corresponding mapping (see Definition 5.1.2).

Proof. The restriction to $\mathbb{S}$ of the holomorphic discrete series reads:

$$V_\theta(a_0, n_0)h(z) \ := \ e^{-2\left(\frac{\theta+2}{\theta}\right)a_0} h((a_0, n_0)^{-1}.z)$$

where

$$(a_0, n_0).z \ := \ e^{2a_0}(n_0 + z) \ .$$

We will now write an explicit expression for an intertwiner T from $(\mathcal{H}_0, U_{\theta,0})$ $(\theta \neq 0)$ to $(\mathcal{D}_{\left(\frac{\theta+2}{\theta}\right)}, V_{\left(\frac{\theta+2}{\theta}\right)})$ of the form (cf. Proposition 5.1.2):

$$T\tilde{u} \ := \ \int_{\mathbb{S}} < U_{\theta,0}(g)\tilde{u}^0, \tilde{u} > \, V_\theta(g)h^0 \, \mathrm{d}g$$

where $\tilde{u}^0 \in \mathcal{H}_0$ and $h^0 \in \mathcal{D}_{\left(\frac{\theta+2}{\theta}\right)}$ are fixed non-zero elements in the domain of the Duflo-Moore operator. This amounts to:

$$T\tilde{u}(z_0) = \int e^{-2\left(\frac{\theta+2}{\theta}\right)a + \frac{i}{\theta} e^{2(a-q_0)} n} \, h^0(e^{-2a} z_0 - n) \, \overline{\tilde{u}^0}(q_0 - a)\tilde{u}(q_0) \, \mathrm{d}q_0 \mathrm{d}a \mathrm{d}n \ .$$

Choosing

$$h^0(z) := (i + z)^{-2\left(\frac{\theta+2}{\theta}\right)} \ ,$$

the formulae

$$\underline{\mathcal{F}}(u)(\tau) := \int_{\mathbb{R}} u(t)\, e^{-it\tau}\mathrm{d}t \quad \text{and} \quad \underline{\mathcal{F}}^{-1}\left((\alpha+i\tau)^{-k}\right)(t) = \frac{t^{k-1}}{(k-1)!}e^{-\alpha t}H(t) \tag{6.20}$$

($H = H(t)$ denotes the characteristic function of $]0,\infty[$ and $\Re(\alpha) > 0$) lead to the following expression:

$$T\tilde{u}(z_0) = \frac{(-i/\theta)^{2\left(\frac{\theta+2}{\theta}\right)-1}}{2\pi(\Gamma\left(2\left(\frac{\theta+2}{\theta}\right)-1\right)}\int e^{-2\left(\frac{\theta+2}{\theta}\right)a}\, e^{-\frac{1}{\theta}(1-iz_0e^{-2a})e^{2(a-q_0)}}\, e^{2(2\left(\frac{\theta+2}{\theta}\right)-1)(a-q_0)}\,\overline{\tilde{u}^0}(q_0-a)\tilde{u}(q_0)\,\mathrm{d}q_0\mathrm{d}a$$

or

$$T\tilde{u}(z_0) = \frac{(-i/\theta)^{2\left(\frac{\theta+2}{\theta}\right)-1}}{2\pi(\Gamma\left(2\left(\frac{\theta+2}{\theta}\right)-1\right)}\left(\int_{\mathbb{R}} e^{2(1-\left(\frac{\theta+2}{\theta}\right))q-\frac{1}{\theta}e^{-2q}}\,\overline{\tilde{u}^0}(q)\,\mathrm{d}q\right)\int_{\mathbb{R}} e^{-2\left(\frac{\theta+2}{\theta}\right)q_0}e^{+\frac{i}{\theta}e^{-2q_0}z_0}\,\tilde{u}(q_0)\,\mathrm{d}q_0\ .$$

In terms of the Laplace transform

$$\mathcal{L}[f](w) \;:=\; \int_0^\infty e^{-rw}\, f(r)\,\mathrm{d}r \quad (\Re(w)>0)\ ,$$

we have:

$$T\tilde{u}(z_0) \;=\; C_{\theta,\tilde{u}^0}\,\mathcal{L}\left[r^{\left(\frac{\theta+2}{\theta}\right)-1}\tilde{v}(r)\right](-\frac{i}{\theta}z_0)$$

where

$$\tilde{v}(e^{-2q_0}) := \tilde{u}(q_0) \quad \text{and} \quad C_{\theta,\tilde{u}^0} := \frac{(-i/\theta)^{2\left(\frac{\theta+2}{\theta}\right)-1}}{4\pi(\Gamma\left(2\left(\frac{\theta+2}{\theta}\right)-1\right)}\int_{\mathbb{R}} e^{2(1-\left(\frac{\theta+2}{\theta}\right))q-\frac{1}{\theta}e^{-2q}}\,\overline{\tilde{u}^0}(q)\,\mathrm{d}q\ .$$

The symmetry $s_K = -1/z$ centered at i naturally "quantizes" or lifts to the state space $\mathcal{D}_{\left(\frac{\theta+2}{\theta}\right)}$ as (see also [BG15]):

$$\vartheta(h)(z) \;:=\; i^{2\left(\frac{\theta+2}{\theta}\right)}z^{-2\left(\frac{\theta+2}{\theta}\right)}h(s_K(z))\ .$$

Using the intertwiner T, we can therefore transport it at the level of $\mathcal{H}_0$:

$$\vartheta_{\theta,0}(\tilde{u}) \;:=\; T^{-1}\vartheta\left(T(\tilde{u})\right)\ .$$

In order to do so, we start by observing that setting

$$h_{r_0}(z) \;:=\; e^{\frac{i}{\theta}r_0z} \quad (z\in\mathbb{D})\ ,$$

we have

$$\vartheta(h_{r_0})(z) \;:=\; i^{2\left(\frac{\theta+2}{\theta}\right)}z^{-2\left(\frac{\theta+2}{\theta}\right)}e^{\frac{-ir_0}{\theta z}}\ ,$$

which we view as a function η_{r_0} on the domain $\Re(w) > 0$:

$$\eta_{r_0}(w) \;:=\; i^{2\left(\frac{\theta+2}{\theta}\right)}(-\frac{i}{\theta})^{2\left(\frac{\theta+2}{\theta}\right)}w^{-2\left(\frac{\theta+2}{\theta}\right)}e^{-\frac{r_0}{\theta^2 w}} \quad \text{with} \quad w \;:=\; \frac{-i}{\theta}z\ .$$

From the formula

$$\frac{1}{2\pi i}\int_{\epsilon+i\mathbb{R}} e^{Z-\frac{\zeta^2}{4Z}}Z^{-\mu-1}\,\mathrm{d}Z = \left(\frac{2}{\zeta}\right)^{\mu} J_\mu(\zeta) \quad (\epsilon,\mu,\zeta>0)\ ,$$

we get

$$\mathcal{L}^{-1}(\eta_{r_0})(s) \;=\; \frac{1}{\theta}\left(\frac{s}{r_0}\right)^{-\frac{-2\left(\frac{\theta+2}{\theta}\right)+1}{2}} J_{-(1-2\left(\frac{\theta+2}{\theta}\right))}\left(\frac{2}{\theta}\sqrt{r_0 s}\right) \quad (s>0)\ .$$

We now express

$$T \;=\; D_{\frac{-i}{\theta}}\circ\mathcal{L}\circ\mathcal{M}_{Cr^{\left(\frac{\theta+2}{\theta}\right)-1}}\circ\psi^{-1\star} \quad \text{and} \quad T^{-1} \;=\; \psi^\star\mathcal{M}_{\frac{1}{C}r^{1-\left(\frac{\theta+2}{\theta}\right)}}\mathcal{L}^{-1}D_{i\theta} \quad (C \;:=\; C_{\theta,\tilde{u}^0})$$

where D stands for the variable dilation, $\mathcal{M}$ for functional multiplication and $\psi(q) := e^{-2q}$. Observing that

$$T\tilde{u}(z) \;=\; C_{\theta,\tilde{u}^0} \int_0^\infty r_0^{\left(\frac{\theta+2}{\theta}\right)-1} \tilde{u}(\psi^{-1}(r_0)) h_{r_0}(z) \mathrm{d}r_0 \;,$$

a tedious computation yields:

$$\vartheta_{\theta,0}(\tilde{u})(q) \;=\; \int_0^\infty J_{2\left(\frac{\theta+2}{\theta}\right)-1}(\sigma)\, \tilde{u}\left(\ln\left(\frac{2}{\theta\sigma}\right) \;-\; q\right) \mathrm{d}\sigma \;.$$

The case $\lambda \neq 0$ is obtained from the above one by considering the unitary equivalence mentioned in (ii) of Lemma 6.1.1. ■

As corollary of the proof of the above proposition, we have

Lemma 6.1.1 *Explicitly, one has for every $\varphi \in \mathcal{H}_\lambda$:*

$$\vartheta_{\theta,\lambda}\Sigma_\lambda\varphi(a_0) \;=\; \frac{4e^{i\pi\left(\frac{\theta+2}{\theta}\right)}}{\theta} \int_{-\infty}^\infty J_{2\left(\frac{\theta+2}{\theta}\right)-1}\left(\frac{2}{\theta}e^{-2a}\right) e^{2(\lambda-1)a} \left(U_{\theta,\lambda}((2a,0))\varphi\right)(a_0)\, \mathrm{d}a \;.$$

Formally, setting

$$\mathcal{J}_{\theta,\lambda}(a) \;:=\; \frac{4e^{i\pi\left(\frac{\theta+2}{\theta}\right)}}{\theta}\, J_{2\left(\frac{\theta+2}{\theta}\right)-1}\left(\frac{2}{\theta}e^{-2a}\right) e^{2(\lambda-1)a}$$

and

$$(a,0)^2 \;=:\; a^2 \;\in\; A$$

one writes

$$\vartheta_{\theta,\lambda}\Sigma_\lambda \;=\; \int_A \mathcal{J}_{\theta,\lambda}(a)\, U_{\theta,\lambda}(a^2)\, \mathrm{d}a \;.$$

We will now give a rigorous meaning to the above formula and extend the notion of retract to any, sufficiently regular, action of the group $A \simeq \mathbb{R}$ on a Fréchet space.
We let $\mathfrak{F}$ be a Fréchet space equipped with an increasing family of semi-norms $\{|\;|\}_{j\in\mathbb{N}}$. We consider a sub-isometric strongly continuous representation ρ of A on $\mathfrak{F}$ and equip the space $\mathfrak{F}_\infty$ of smooth vectors with the usual refined topology associated with the action that turns it into a smooth Fréchet A-module for the restriction of ρ to $\mathfrak{F}_\infty$. Our aim in a first place, is to define the following "improper integral" which to every $v \in \mathfrak{F}_\infty$, associates the quantity (in $\mathfrak{F}_\infty$):

$$\Xi^{(\rho)}_{\theta,\lambda}(v) \;:=\; \int_A \mathcal{J}_{\theta,\lambda}(a)\, \rho(a)v\, \mathrm{d}a \;.$$

For this, we observe the following proposition whose proof is postponed to the proof of Proposition 6.2.1.

Proposition 6.1.2 *Let $s \in \mathbb{R}$ and $\left(\frac{\theta+2}{\theta}\right) > 0$. Then there exists $\lambda_{\theta,s}$ such that the element*

$$\mathcal{D}(A,\mathfrak{F}_\infty) \to \mathfrak{F}_\infty : \phi \mapsto \int_A \mathcal{J}_{\theta,\lambda_{\theta,s}}(a)\, \phi(a)\, \mathrm{d}a$$

continuously extends from $\mathcal{D}(A,\mathfrak{F}_\infty)$ to $\mathcal{B}^s_\infty(A,\mathfrak{F}_\infty)$:

$$\widetilde{\int_A} \mathcal{J}_{\theta,\lambda_{\theta,s}} \;:\; \mathcal{B}^s_\infty(A,\mathfrak{F}_\infty) \to \mathfrak{F}_\infty \;.$$

Corollary 6.1.1 *Let $v \in \mathfrak{F}_\infty$, then the element*

$$A \to \mathfrak{F}_\infty : a \mapsto \rho(a)v$$

belongs to $\mathcal{B}^0_\infty(A,\mathfrak{F}_\infty)$. In particular, Proposition 6.2.1 implies that the integral

$$\Xi^{(\rho)}_{\theta,\lambda}(v) \;:=\; \int_A \mathcal{J}_{\theta,\lambda}(a)\, \rho(a)v\, \mathrm{d}a \;.$$

is a well-defined element of $\mathfrak{F}_\infty$.

Proof. The first assertion is obvious. For the second one, we use the fact that the closure of $\mathcal{D}(\mathbb{S},\mathfrak{F}_\infty)$ in $\mathcal{B}^\epsilon_\infty(A,\mathfrak{F}_\infty)$ contains $\mathcal{B}^0_\infty(A,\mathfrak{F}_\infty)$ (c.f. [BG15]). ■

Proposition 6.1.3 *For all $\lambda \in \mathbb{R}$ and $x \in \mathbb{S}$, one has*

$$\mathbf{\Omega}_{\theta,\lambda}(x) \;=\; \int_{-\infty}^{\infty} \mathcal{J}_{\theta,\lambda}(a)\,\Omega_{\theta,\lambda}(xa)\,\mathrm{d}a$$

when acting in $\mathcal{H}_\lambda^\infty$.
Also, for every $\phi \in \mathcal{D}(\mathbb{S})$, one has

$$\mathbf{\Omega}_{\theta,\lambda}(\phi) \;=\; \Omega_{\theta,\lambda}\left(\Xi^{(R^\vee)}_{\theta,\lambda+1}(\phi)\right)$$

where $R^\vee_a\phi(x) \;:=\; \phi(xa^{-1})$.

Proof. We will use the fact that

$$\vartheta_{\theta,\lambda}U_{\theta,\lambda}(a)\vartheta_{\theta,\lambda} \;=\; \Sigma_\lambda U_{\theta,\lambda}(a)\Sigma_\lambda \;=\; U_{\theta,\lambda}(a^{-1})\ .$$

One has

$$\begin{aligned}
\mathbf{\Omega}_{\theta,\lambda}(x) &= U_{\theta,\lambda}(x)\vartheta_{\theta,\lambda}U_{\theta,\lambda}(x^{-1})\\
&= U_{\theta,\lambda}(x)\vartheta_{\theta,\lambda}\Sigma_\lambda^2 U_{\theta,\lambda}(x^{-1})\\
&= U_{\theta,\lambda}(x)\left(\int_A \mathcal{J}_{\theta,\lambda}(a)\,U_{\theta,\lambda}(a^2)\mathrm{d}a\right)\Sigma_\lambda U_{\theta,\lambda}(x^{-1})\\
&= \int_A \mathcal{J}_{\theta,\lambda}(a)\,U_{\theta,\lambda}(xa)\Sigma_\lambda^2 U_{\theta,\lambda}(a)\Sigma_\lambda U_{\theta,\lambda}(x^{-1})\,\mathrm{d}a\\
&= \int_A \mathcal{J}_{\theta,\lambda}(a)\,U_{\theta,\lambda}(xa)\Sigma_\lambda U_{\theta,\lambda}(a^{-1}x^{-1})\,\mathrm{d}a\ ,
\end{aligned}$$

which proves the first assertion noticing that Σ_λ acts as a continuous linear operator on $\mathcal{H}_\lambda^\infty$.
The second assertion is a direct application of Lemma 6.1.3:

$$\begin{aligned}
\mathbf{\Omega}_{\theta,\lambda}(\phi) &= \int_{\mathbb{S}} \phi(y)\mathbf{\Omega}_{\theta,\lambda}(y)\,\mathrm{d}y\\
&= \int_{\mathbb{S}\times A} \phi(y)\mathcal{J}_{\theta,\lambda}(a)\,\Omega_{\theta,\lambda}(ya)\,\mathrm{d}a\,\mathrm{d}y\\
&= \int_{\mathbb{S}\times A} \Delta(a)\phi(ya^{-1})\mathcal{J}_{\theta,\lambda}(a)\,\Omega_{\theta,\lambda}(y)\,\mathrm{d}a\,\mathrm{d}y
\end{aligned}$$

where

$$\Delta(a)\mathrm{d}(ya) \;:=\; \mathrm{d}y\ .$$

Observing that $\Delta(a) \;=\; e^{2a}$, we find

$$\begin{aligned}
\mathbf{\Omega}_{\theta,\lambda}(\phi) &= \frac{4e^{i\pi(\frac{\theta+2}{\theta})}}{\theta}\int_{\mathbb{S}\times A} (R^\star_{a^{-1}}\phi)\,(y)\,e^{2\lambda a}\,J_{2(\frac{\theta+2}{\theta})-1}(\frac{2}{\theta}e^{-2a})\,\Omega_{\theta,\lambda}(y)\,\mathrm{d}a\,\mathrm{d}y\\
&= \int_{\mathbb{S}\times A} (R^\star_{a^{-1}}\phi)\,(y)\,\mathcal{J}_{\theta,\lambda+1}(a)\,\Omega_{\theta,\lambda}(y)\,\mathrm{d}a\,\mathrm{d}y\ .
\end{aligned}$$

■

Definition 6.1.1 *Let $C_b(\mathbb{S})$ denote the space of right-uniformly continuous functions on $\mathbb{S}$. Denoting by $C_b(\mathbb{S})^\infty$ the space of C^∞-vectors for the action $R^\vee$ of A on $C_b(\mathbb{S})$, we consider the following continuous linear operator on $C_b(\mathbb{S})^\infty$:*

$$\frac{-1}{4}e^{\frac{2i}{\theta}\pi}\,\Xi^{R^\vee}_{\theta,\lambda+1} \;=:\; \mathfrak{W}_{\theta,\lambda} \;:=\; \frac{4e^{i\pi(\frac{\theta+2}{\theta})}}{\theta}\int_A e^{2\lambda a}\,J_{2(\frac{\theta+2}{\theta})-1}(\frac{2}{\theta}e^{-2a})\,R^\star_{a^{-1}}\,\mathrm{d}a\ .$$

We call the above element the inverse retract operator*.*

Lemma 6.1.2 *We have*

$$\left(\Xi^{R^\vee}_{\theta,\lambda+1}\right)^{-1} \;=\; -e^{\frac{2i}{\theta}\pi}\,\Xi^{R}_{\theta,1-\lambda} \;=\; -\frac{4e^{i\pi\left(\frac{\theta+4}{\theta}\right)}}{\theta}\int_A e^{-2\lambda a}\,J_{2\left(\frac{\theta+2}{\theta}\right)-1}\big(\frac{2}{\theta}e^{-2a}\big)\,R^\star_a\,\mathrm{d}a\;.$$

Proof. We start with the well known inversion formula:

$$\int\int J_\mu(\eta)\,J_\mu(\xi)\;\phi(\frac{\eta}{\xi})\,\mathrm{d}\eta\,\mathrm{d}\xi \;=\; \phi(1)\;.$$

In other words, for every x, the operator:

$$A\phi(x) \;:=\; \int J_\mu(\xi)\,\phi(x\xi^{-1})\,\mathrm{d}\xi$$

has inverse:

$$A^{-1}\psi(x) \;=\; \int J_\mu(\eta)\,\psi(x\eta)\mathrm{d}\eta\;.$$

Now we consider the operators

$$A_\lambda\phi(x) \;:=\; \int J_\mu(\xi)\,\xi^\lambda\phi(x/\xi)\mathrm{d}\xi$$

and

$$B_\tau\phi(x) \;:=\; \int J_\mu(\eta)\eta^\tau\phi(x\eta)\mathrm{d}\eta\;.$$

We have then

$$B_\tau A_\lambda(\phi)(x) \;=\; \int J_\mu(\eta)J_\mu(\xi)\eta^\tau\xi^\lambda\phi(x\eta/\xi)\mathrm{d}\eta\,\mathrm{d}\xi\;.$$

Setting $\Psi(\eta/\xi) \;:=\; (\eta\xi^{-1})^{-\lambda}\phi(x\eta/\xi)$ and $\tau = -\lambda$, the above inversion formula leads to

$$B_{-\lambda}A_\lambda(\phi)(x) \;=\; \Psi(1) \;=\; \phi(x)\;.$$

The lemma then follows when setting, for $x \in \mathbb{S}$:

$$\xi \;:=\; \frac{2}{\theta}\,e^{-2a} \quad\text{and}\quad \phi(\xi^{-1}) \;=:\; \varphi_x(\xi^{-1}) \;:=\; \varphi(x.\xi^{-1})\;.$$

■

6.2 Fundamental property of the retract

We now pass to extensions of the retract operators to spaces of type $\mathcal{B}$. The following proposition involves the parameter λ. It is the only reason why this parameter appears in this paper.

Proposition 6.2.1 *Let θ be such that $\left(\frac{\theta+2}{\theta}\right) > 0$.*

(i) For all s and $k \in \mathbb{N}$, there exists $\lambda_{\theta,k,s}$ such that the inverse retract operator $\mathfrak{W}_{\theta,\lambda_{\theta,s,k}}$ extends from $\mathcal{D}(\mathbb{S})$ to $\mathcal{B}^s_\infty(\mathbb{S})$ as a continuous injection $\mathcal{B}^s_\infty(\mathbb{S})$ into $\mathcal{B}^s_k(\mathbb{S})$:

$$\mathfrak{W}_{\theta,\lambda_{\theta,k,s}} : \mathcal{B}^s_\infty(\mathbb{S}) \to \mathcal{B}^s_k(\mathbb{S})\;.$$

(ii) And similarly for the direct retract: for every $p \in \mathbb{N}$ with $2\left(\frac{\theta+2}{\theta}\right) - 1/2 \le p \le k$, the direct retract operator associated with the same parameter $\lambda_{\theta,k,s}$ as in (i) continuously injects $\mathcal{B}^s_p(\mathbb{S})$ into $\mathcal{B}^s_p(\mathbb{S})$:

$$\mathfrak{W}^{-1}_{\theta,\lambda_{\theta,k,s}} : \mathcal{B}^s_p(\mathbb{S}) \to \mathcal{B}^s_p(\mathbb{S})\;.$$

Proof. Recall that considering $x \in \mathbb{S}$ and $a := (a, 0) \in A$, we have for every k:

$$\tilde{E}^k_x(R^\star_{a^{-1}}\varphi) = e^{2ka}\tilde{X}_{x.a^{-1}}(\varphi) \quad \text{and} \quad \tilde{E}^k_x(R^\star_a\varphi) = e^{-2ka}\tilde{X}_{x.a}(\varphi) . \tag{6.21}$$

We will use an integration by parts technique in order to estimate the retracts. For convenience, we will express the group A under its multiplicative form:

$$A = \mathbb{R}^+_0 \quad \text{through} \quad \zeta := e^{-2a}$$

For every $\varphi \in \mathcal{D}(\mathbb{S})$, the inverse retract essentially turns into the form

$$\mathfrak{W}\varphi(x) := \int_0^\infty \zeta^{-\lambda} J_\mu(\zeta)\,\varphi(x.a^{-1})\,\mathrm{d}\zeta$$

where we adopt the same notations as in the proof of Lemma 6.1.2 and where we redefined $\lambda \leftarrow \lambda + 1$ (due to the change of variable $\zeta := e^{-2a}$, $\mathrm{d}a$ becomes $\zeta^{-1}\mathrm{d}\zeta$).
Since

$$\tilde{H}|_{C^\infty(A)} = \partial_a := -2\zeta\partial_\zeta ,$$

we observe:

$$e^{-\frac{\zeta^2}{4Z}} = \frac{1}{1+\zeta^2}\,(1 + Z\,\tilde{H})e^{-\frac{\zeta^2}{4Z}} .$$

The adjoint operator w.r.t. ζ (and not to a):

$$\mathbf{O} := \left[\frac{1}{1+\zeta^2}\,(1 + Z\,\tilde{H})\right]^* = (1{+}2Z - Z\tilde{H}) \circ \frac{1}{1+\zeta^2}$$

applied to $\psi \in \mathcal{D}(\mathbb{S})$ involves multiplications by $\frac{1}{1+\zeta^2}$ and by Z of derivatives of ψ. An induction yields

$$\mathbf{O}^N\psi = \frac{1}{[1+\zeta^2]^N}\sum_{n,m=0}^N b_{m,n}(\zeta)\,Z^n\,(\tilde{H})^m.\psi$$

where the $b_{m,n}$ are smooth bounded functions with all derivatives bounded i.e. belonging to $\mathcal{B}^0_\infty(A)$.
We now use the expression:

$$\frac{1}{2\pi i}\int_{\epsilon+i\mathbb{R}} Z^{n-(1+\mu)}e^{Z-\frac{\zeta^2}{4Z}}\,\mathrm{d}Z = \left(\frac{2}{\zeta}\right)^{\mu-n} J_{\mu-n}(\zeta) .$$

An integration by parts then yields:

$$\mathfrak{W}\varphi(x) = \sum_{m,n}^N \int_0^\infty \int_{\epsilon+i\mathbb{R}} \frac{Z^{n-(1+\mu)}e^{Z-\frac{\zeta^2}{4Z}}b_{m,n}(\zeta)}{[1+\zeta^2]^N}\,(\tilde{H})^m_\zeta\left(\zeta^{\mu-\lambda}L^\star_x\varphi\right)\,\mathrm{d}Z\,\mathrm{d}\zeta .$$

Observing that $\zeta^{\mu-\lambda}$ is an eigen-function of $\tilde{H}$, we are led to bound expressions of the following type:

$$\mathcal{I} = \int_0^\infty \int_{\epsilon+i\mathbb{R}} \frac{\zeta^{\mu-\lambda}\,Z^{n-(1+\mu)}e^{Z-\frac{\zeta^2}{4Z}}b(\zeta)}{[1+\zeta^2]^N}\,L^\star_x\left[(\tilde{H})^m\varphi\right]\,\mathrm{d}Z\,\mathrm{d}\zeta .$$

The latter entails for every $X \in \mathcal{U}_k(\mathfrak{s})$:

$$\tilde{X}_x.\mathfrak{W}\varphi = \sum_{m,n}^N \int_0^\infty \frac{\zeta^{n-\lambda}\,b^{(2)}_{m,n}(\zeta)}{[1+\zeta^2]^N}\,J_{\mu-n}(\zeta)\,\left(\mathtt{Ad}_{\zeta^{-1}}(X)\,H^m\right)^\sim_{xa^{-1}}.\varphi\,\mathrm{d}\zeta \quad (b^{(2)}_{m,n} \in \mathcal{B}^1_\infty(A)) .$$

Therefore, we get positive constants $\{c_{m,n}\}$ and $X' \in \mathcal{U}_k(\mathfrak{s})$ (at worse c.f. (6.21)) such that, for every $s \in \mathbb{R}$:

$$\sup\{\frac{1}{\mathfrak{d}^s}|\tilde{X}(\mathfrak{W}\varphi)|\} \leq \sum_{m,n=0}^N c_{m,n}\sup_x\left\{\frac{1}{\mathfrak{d}^s(x)}\int_0^\infty \frac{\zeta^{n-k-\lambda}}{[1+\zeta^2]^N}\,|J_{\mu-n}(\zeta)|\,\left|\widetilde{H^m X'}_{x.a^{-1}}(\varphi)\right|\,\mathrm{d}\zeta\right\} .$$

Hence, because $\mathfrak{d}|_A(\zeta) = (\zeta + 1/\zeta)$ and submultiplicativity of the weight $\mathfrak{d}$, one has the following expression in terms of the input semi-norms associated to $s' \in \mathbb{R}$:

$$|\mathfrak{W}\varphi|_{k,s} \leq \sum_{m,n=0}^{N} c_{m,n}\, |\varphi|_{k+m,s'} \sup_x \left\{ \frac{1}{\mathfrak{d}^{s-s'}(x)} \right\} \int_0^\infty \frac{\zeta^{n-k-\lambda}}{[1+\zeta^2]^N}\, |J_{\mu-n}(\zeta)|\, (\zeta + 1/\zeta)^{s'}\, \mathrm{d}\zeta \ .$$

Now,

- for ζ near zero the integrand behaves, up to multiplicative constant as

$$\frac{\zeta^{n-k-\lambda}}{[1+\zeta^2]^N}\, |J_{\mu-n}(\zeta)|\, (\zeta + 1/\zeta)^{s'} \sim \zeta^{\mu-k-s'-\lambda} \ ,$$

while

- for ζ large, we have

$$\frac{\zeta^{n-k-\lambda}}{[1+\zeta^2]^N}\, |J_{\mu-n}(\zeta)|\, (\zeta + 1/\zeta)^{s'} \sim \zeta^{n-k-2N-1/2+s'-\lambda} \ .$$

For given k and s, finiteness of $|\mathfrak{W}\varphi|_{k,s}$ therefore amounts to satisfying the following inequalities:

$$\left\{ \begin{array}{ccc} s - s' & \geq & 0 \\ 0 & < & \mu - k - s' + 1 - \lambda \\ n - k - 2N - 1/2 + s' - \lambda & < & -1 \ . \end{array} \right.$$

In other words ($n \leq N$):

$$(1) \quad \left\{ \begin{array}{ccc} s' \leq & & s \\ k - 1 + s' + \lambda & < & \mu \\ -k - N + s' - \lambda & < & -1/2 \ . \end{array} \right.$$

The inequality $k - 1 + s' + \lambda < \mu$ imposes a strong condition on λ: in order to hold, one is constrained to choose λ smaller than the limiting quantity $\lambda_{\theta,k,s'}$ given by

$$\lambda_{\theta,k,s'} := \mu - s' - k + 1 \ .$$

Which for large μ is generally negative.
The function $\mathfrak{W}_{m,\left(\frac{\theta+2}{\theta}\right),\lambda_{\theta,k,s'}}\varphi$ is therefore C^k.
Now we analyse the situation with the order of derivation ℓ is smaller than k:

$$\ell \leq k \ .$$

The analyse therefore leads us to ($\left(\frac{\theta+2}{\theta}\right)$ being large):

$$(1') \quad \ell - 1 + s' + \lambda_{\theta,k,s'} \leq \mu \ .$$

Which amounts to

$$\ell - 1 + s' + \mu - s' - k + 1 \leq \mu \quad \text{i.e.} \quad \ell - k \leq 0$$

which holds. Therefore the element $\mathfrak{W}_{m,\left(\frac{\theta+2}{\theta}\right),k,s',\lambda_{\theta,k,s'}}\varphi$ belongs to $\mathcal{B}^s_k(\mathbb{S})$.

Concerning the assertion relative to the direct retract and within obvious notations omitting indices, we have the expression

$$\mathfrak{W}^{-1}\varphi(x) := \int_0^\infty \zeta^\lambda J_\mu(\zeta)\, \varphi(x.a)\, \mathrm{d}\zeta \quad (\varphi \in \mathcal{D}(\mathbb{S})) \ .$$

The same discussion as for the inverse retract leads to the following estimate:

$$|\mathfrak{W}\varphi|_{k,s,p} \leq \sum_{m,n=0}^{N} c_{m,n}\, |\varphi|_{k+m,s'} \sup_x \left\{ \frac{1}{\mathfrak{d}^{s-s'}(x)} \right\} \int_0^\infty \frac{\zeta^{n+k+\lambda}}{[1+\zeta^2]^{N'}}\, |J_{\mu-n}(\zeta)|\, (\zeta + 1/\zeta)^{s'}\, \mathrm{d}\zeta \ .$$

Now,

- for ζ near zero the integrand behaves, up to multiplicative constant as

$$\frac{\zeta^{n+k+\lambda}}{[1+\zeta^2]^{N'}}\,|J_{\mu-n}(\zeta)|\,(\zeta+1/\zeta)^{s'}\sim\zeta^{\mu+k-s'+\lambda}\;,$$

while

- for ζ large, we have

$$\frac{\zeta^{n+k+\lambda}}{[1+\zeta^2]^{N'}}\,|J_{\mu-n}(\zeta)|\,(\zeta+1/\zeta)^{s'}\sim\zeta^{n+k-2N-1/2+s'+\lambda}\;.$$

For given k and s, finiteness of $|\mathfrak{W}\varphi|_{k,s}$ therefore amounts to satisfying the following inequalities:

$$\left\{\begin{array}{ccc} s-s' & \geq & 0\\ 0 & < & \mu+k-s'+1+\lambda\\ k-N'-1/2+s'+\lambda & < & -1\;. \end{array}\right.$$

In other words

$$(2)\quad\left\{\begin{array}{ccc} s' & \leq & s\\ -k-1+s'-\lambda & < & \mu\\ k-N'+s'+\lambda & < & -1/2\;. \end{array}\right.$$

Which, together with the system (1) gives

$$\left\{\begin{array}{ccc} s'\leq & & s\\ k-1+s'+\lambda & < & \mu\\ -k-N+s'-\lambda & < & -1/2\\ -k-1+s'-\lambda & < & \mu\\ k-N'+s'+\lambda & < & -1/2 \end{array}\right.$$

where N and N' are independant. They can therefore be chosen such that the inequalities $-k-N+s'-\lambda<-1/2$ and $k-N'+s'+\lambda<-1/2$ are satisfied for any choice of λ. These conditions in the limit case $\lambda=\lambda_{\mu,k,s'}-\epsilon$ amount to

$$\left\{\begin{array}{ccc} s' & \leq & s\\ s'-1+\epsilon/2 & < & \mu \end{array}\right.$$

But now the function φ is only C^p with possibly $p\leq k$. Which leaves us with an output $\mathfrak{W}^{-1}_{\theta,\lambda_{\theta,k,s'}}\varphi$ at best in $C^{p-N'}$, with $N'>\mu+3/2$ i.e.

$$\mu+3/2<N'\leq p\;.$$

Imposing

$$k\geq p\geq\mu+3/2\;.$$

Now, the set of inequalities (2), using the density of $\mathcal{D}$-type symbols in $\mathcal{B}$, leads to the fact that $\mathfrak{W}^{-1}$ sends $\mathcal{B}^s_\infty(\mathbb{S})$ continuously into $\mathcal{B}^s_\infty(\mathbb{S})$, for every $s\in\mathbb{R}$.
While the second one (2) implies the same result for $\mathfrak{W}^{-1}$ since N' is arbitrarily large.

A similar argument (but much simpler) as above applies in the case of the improper integral involved in the statement of Proposition 6.1.2. ■

As a direct consequence of the above discussion we have

Theorem 6.2.1 *The retract operators stabilize the Schwartz space $\mathcal{S}(\mathbb{S})=\mathcal{S}(\mathbb{M})=\mathcal{S}(\mathbb{D})$ (c.f. Definition 4.0.8).*

6.3 Explicit formula for the non-formal star-product and Drinfel'd twist

We realize the hyperbolic plane as an adjoint orbit in $\mathfrak{g}$ of a compact element Z_0. The dual picture would be more intrinsical but the computations are simpler within the adjoint orbit setting. We consider the moment map

$$J : \mathbb{S} \to \mathfrak{g} : x \mapsto \mathtt{Ad}_x Z_0$$

where $\mathbb{S}$ is realized as a sub-group of G.
At every $x \in \mathbb{S}$, we realize the tangent space as the sub-space of $\mathfrak{g}$ Killing-orthogonal to the line, denoted hereafter $\mathfrak{k}_x$, supported by $J(x)$:

$$\mathfrak{p}_x \ := \ J_{\star x}(T_x\mathbb{S}) \ = \ J(x)^{\perp} \ .$$

The (partial) **Ševera map** is then defined at x as

$$\sigma_x : \mathbb{S} \to \mathfrak{p}_x \ \subset \ \mathfrak{g} : y \mapsto \pi_x(J(y))$$

where $\pi_x : \mathfrak{g} \to \mathfrak{p}_x$ denotes the Killing-orthogonal projection. We then observe

Lemma 6.3.1 *The Ševera map is $\mathbb{S}$-equivariant:*

$$\sigma_{gx} \ = \ \mathtt{Ad}_g \circ \sigma_x \circ L_{g^{-1}} \ .$$

For $Z_0 \ := \ \frac{1}{2}(E - F)$, one has

$$\sigma_e(y) \ = \ [Z_0, [\mathtt{Ad}_y Z_0, Z_0]]$$

and more generally:

$$\sigma_x(y) \ = \ [J(x), [J(y), J(x)]] \ .$$

A simple computation then yields

Proposition 6.3.1 *Let us normalize the Killing form β on $\mathfrak{g}$ in such a way that*

$$\beta(Z_0, Z_0) \ = -1$$

and endow the vector space $\mathfrak{p} := \mathfrak{p}_e$ with the symplectic structure

$$\omega_0(X, Y) \ := \ \beta(Z_0, [X, Y]) \ .$$

Then the vectors

$$e_q \ := \ \frac{1}{2}(E + F) \qquad e_p \ := \ \frac{1}{2}H$$

constitute an orthnormal symplectic basis of $\mathfrak{p}$ ($\omega_0(e_q, e_p) = 1$) and through the symplectic identification

$$\mathbb{R}^2 \to \mathfrak{p} : (q, p) \mapsto q e_q + p e_p$$

the Ševera map reads

$$\sigma_e(\exp(aH)\exp(nE)) \ = \ \Psi^{-1} Z(a, n)$$

where Z is the natural map

$$Z \ : \ \mathbb{S} \to \ \Pi \ : \ an \mapsto \ an(1) \ .$$

Definition 6.3.1 *The* **canonical area** *around point x is defined as*

$$S(x, y, z) \ := \ \beta(J(x), [\sigma_x(y), \sigma_x(z)]) \ .$$

We set

$$S(x, y) \ := \ S(e, x, y) \ .$$

Remark 6.3.1 *We note the relevance of the function*

$$\hat{S} : \mathbb{L} \times \mathbb{S} \to \mathfrak{g} : (x, y, z) \mapsto [\sigma_x(y), \sigma_x(z)] \, .$$

which, in the present two-dimensional and normalized setting, is related to the canonical area through the identity:

$$\hat{S}(x, y, z) \;=\; S(x, y, z)\, J(x) \, .$$

We now give an geometrical explicit formula (i.e. coordinate free) for the $\mathrm{SL}_2(\mathbb{R})$-invariant star product on the hyperbolic plane.

Theorem 6.3.1 *Let*

$$\mathbb{D} \;:=\; SL_2(\mathbb{R})/SO(2)$$

be the hyperbolic plane.
Consider the following special function of the real variable:

$$\mathbf{E}_\lambda(t) \;:=\; \int_0^\infty s^2 e^{-\,is\,t}\; J_\lambda(s)\,\mathrm{d}s \;=\; -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}(J_\lambda)(-it) \, .$$

For a suitable choice of a constant c_θ, the product

$$\varphi\sharp_\theta\psi(x) \;:=\; c_\theta \int_{\mathbb{D}^2} A(x,y,z)\, \mathbf{E}_{2\frac{\theta+2}{\theta}-1}\left(S(x,y,z)\right)\, \varphi(y)\, \psi(z)\,\mathrm{d}y\,\mathrm{d}z$$

where

1. $\mathrm{d}x$ *denotes (a normalisation of) the $SL_2(\mathbb{R})$-invariant measure on $\mathbb{D}$,*
2. *$S(x,y,z)$ denotes the Ševera area on $\mathbb{D}$ (c.f. 3.5),*
3. *and A is defined in terms of the Jacobians of the Ševera projections (c.f. 3.4) as*

$$A(x,y,z) \;:=\; Jac_{\sigma_x}(y)\, Jac_{\sigma_x}(z)$$

is such that

(i) it extends from $C^\infty_c(\mathbb{D})$ to the Schwartz space $\mathcal{S}(\mathbb{D})$ (c.f. Definition 4.0.8) as an $SL_2(\mathbb{R})$-equivariant associative Fréchet algebra structure.

(ii) It admits an asymptotic expansion as an $SL_2(\mathbb{R})$-invariant formal star-product on the hyperbolic plane equipped with (a normalization) of its $SL_2(\mathbb{R})$-invariant symplectic Kahler two-from.

(iii) Identifying the hyperbolic plane $\mathbb{D}$ with the Iwasawa factor $\mathbb{S} := AN$ of $SL_2(\mathbb{R}) = ANK$, the formula

$$\mathcal{F}_\theta \;:=\; c_\theta \int_{\mathbb{S}^2} Jac_{\sigma_e}(y)\, Jac_{\sigma_e}(z)\, \mathbf{E}_{2\frac{\theta+2}{\theta}-1}\left(S(y,z)\right)\, R^\star_y \otimes R^\star_z\,\mathrm{d}y\,\mathrm{d}z$$

defines a Drinfel'd twist based on the Iwasawa factor $\mathbb{S} := AN$.

Proof. We first notice that

$$\begin{aligned}
&\int_{\mathfrak{p}^2} \mathcal{G}(\varphi_1)(x')\,\mathcal{G}(\varphi_2)(x'')\,\int_0^\infty J_\lambda(s)\; e^{i/s\,\omega^{\mathfrak{p}}(x'\,,\,x'')}\,\mathrm{d}x'\,\mathrm{d}x'' \\
=\;& \int_0^\infty J_\lambda(s)\int_{\mathfrak{p}^4}\varphi_1(x_1)\,\varphi_2(x_2)\; e^{i\,(\omega^{\mathfrak{p}}(x_1,x')\,+\,\omega^{\mathfrak{p}}(x_2,x''))}\; e^{i/s\,\omega^{\mathfrak{p}}(x'\,,\,x'')}\,\mathrm{d}s\,\mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
=\;& \int_{\mathfrak{p}^4}\int_0^\infty e^{i\,(\omega^{\mathfrak{p}}(x_1,x')\,+\,\omega^{\mathfrak{p}}(x_2,x'')\,+\,1/s\,\omega^{\mathfrak{p}}(x'\,,\,x''))}\; J_\lambda(s)\varphi_1(x_1)\,\varphi_2(x_2)\,\mathrm{d}s\,\mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
=\;& \int_{\mathfrak{p}^4}\int_0^\infty e^{i\,(\omega^{\mathfrak{p}}(x_1\,-\,1/s\,x'',x')\,+\,\omega^{\mathfrak{p}}(x_2,x''))}\; J_\lambda(s)\varphi_1(x_1)\,\varphi_2(x_2)\,\mathrm{d}s\,\mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
=\;& \int_{\mathfrak{p}^2}\int_0^\infty e^{i\,\omega^{\mathfrak{p}}(x_2,x'')}\; J_\lambda(s)\varphi_1(1/s\,x'')\,\varphi_2(x_2)\,\mathrm{d}s\,\mathrm{d}x_2\,\mathrm{d}x'' \\
=\;& \int_{\mathfrak{p}^2}\int_0^\infty s^2 e^{-\,is\,\omega^{\mathfrak{p}}(x_1,x_2)}\; J_\lambda(s)\;\varphi_1(x_1)\,\varphi_2(x_2)\,\mathrm{d}s\,\mathrm{d}x_1\,\mathrm{d}x_2 \, .
\end{aligned}$$

The above Proposition 6.3.1 shows that the symplectic structure $\omega^{\mathfrak{p}}$ implied in the symplectic Fourier transform $\mathcal{G}$ is the Ševera area element at e. Therefore, setting

$$x_j \;=\; \Psi^{-1}(\zeta_j)\;,$$

we have (with a slight abuse of notation : identifying $\mathbb{S}$ with Π through the map Z):

$$\omega^{\mathfrak{p}}(x_1,x_2) \;=\; S(o\,,\,\zeta_1\,,\,\zeta_j)\;.$$

Hence

$$\varphi_1\bullet\;\varphi_2(0) \;=\; \int_{\Pi^2}\!\left(\int_0^\infty s^2 e^{-\,is\,S(\zeta_1,\zeta_2)}J_\lambda(s)\;\mathrm{d}s\right)\varphi_1(\Psi^{-1}(\zeta_1))\,\varphi_2(\Psi^{-1}(\zeta_2))\,\mathrm{Jac}_{\Psi^{-1}}(\zeta_1)\,\mathrm{Jac}_{\Psi^{-1}}(\zeta_2)\,\mathrm{d}\zeta_1\mathrm{d}\zeta_2$$

where $\mathrm{d}\zeta$ denotes the hyperbolic measure on the hyperbolic plane Π (realized as a Poincaré half plane).
Therefore, for all smooth and compactly supported fonctions f_1 and f_2 on the hyperbolic plane Π, we have

$$(\Psi^{-1\,\star})(f_1)\bullet(\Psi^{-1\,\star})(f_2)(0) = \int_{\Pi^2}\!\left(\int_0^\infty s^2 e^{-\,is\,S(\zeta_1,\zeta_2)}J_\lambda(s)\mathrm{d}s\right)\mathrm{Jac}_{\Psi^{-1}}(\zeta_1)\mathrm{Jac}_{\Psi^{-1}}(\zeta_2)f_1(\zeta_1)f_2(\zeta_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2$$

where $\mathrm{Jac}_{\Psi^{-1}}(\zeta)$ denotes the Jacobian of the map Ψ^{-1} relating the symplectic (Lebesgue) measure on $\mathfrak{p}$ with the hyperbolic measure on Π.
Now we observe that for every $\zeta \;=\; x \;+\; i\xi$ in the half plane Π, the change of coordinates

$$\begin{cases} x &=& e^{2a}\\ \xi &=& -\,e^{2a}n\end{cases}$$

identifies the product $f\bullet g$ with the product $\star$ defined by

$$(Z^\star(f)\star Z^\star(g))\,(Z(x_0)) := \int_{\mathbb{S}\times\mathbb{S}}\cosh(2(a_0\;-\;a_2)\cosh(2(a_1\;-\;a_0)\,e^{i\,S(x_0,x_2,x_2)}f(Z(x_1))\,g(Z(x_2))\mathrm{d}x_1\mathrm{d}x_2.$$

Within this setting, we also observe that for every $x\;+\;i\xi\;=\in\Pi$ and $t := e^{2b}\in A$, one has

$$tx\;+\;i\xi\;=\;Z(anb)\;.$$

Therefore Unterberger's map G_λ is a solution of the retract. Precisely we have

$$G_\lambda g(Z(an)) \;=\; 2\pi\int_A g(Z(anb))\;J_\lambda(4\pi\,e^{2b})\;e^{2b}\;\mathrm{d}b\;,$$

or equivalently:

$$G_\lambda\circ Z^\star \;=\; 2\pi\int_A J_\lambda(4\pi\,e^{2a})\;e^{2a}R^\star_a\,\mathrm{d}a\;.$$

That is

$$G_{2\frac{\theta+2}{\theta}-1} \;=\; \frac{1}{4}\,e^{-\,i\,\pi\,\frac{\theta+2}{\theta}}\,\mathfrak{W}_{-1\,,\,\theta}\;\circ\;R^\star_{-\frac{1}{2}\log(2\pi\theta)}\;.$$

Note that since the star-product $\star$ is AN-invariant, the A-translation $R^\star_{-\frac{1}{2}\log(2\pi\theta)}$ won't be seen when intertwining $\bullet$ with $G^{-1}_\lambda(\bullet)$.
As already observed by A. and J. Unterberger, the product $G^{-1}_\lambda(\bullet)$ is therefore $\mathrm{SL}_2(\mathbb{R})$-equivariant.
Again identifying $\mathbb{S}$ with Π through the map Z, we have that

$$\sigma_o(\eta) \;=\; \Psi^{-1}(\eta)\quad(\eta\in\Pi)\;.$$

Hence, by the $\mathrm{SL}_2(\mathbb{R})$-equivariance of the Ševera map, we get, for every g in $\mathrm{SL}_2(\mathbb{R})$:

$$\begin{aligned} f_1\sharp f_2(g.o) &:= \int_{\Pi^2}\!\left(\int_0^\infty s^2 e^{-\,is\,S(\zeta_1,\zeta_2)}J_\lambda(s)\mathrm{d}s\right)\mathrm{Jac}_{\Psi^{-1}}(\zeta_1)\mathrm{Jac}_{\Psi^{-1}}(\zeta_2)f_1(g.\zeta_1)f_2(g\zeta_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2\\ &= \int_{\Pi^2}\!\left(\int_0^\infty s^2 e^{-\,is\,S(g^{-1}\zeta_1,g^{-1}\zeta_2)}J_\lambda(s)\mathrm{d}s\right)\mathrm{Jac}_{\Psi^{-1}}(g^{-1}\zeta_1)\mathrm{Jac}_{\Psi^{-1}}(g^{-1}\zeta_2)f_1(\zeta_1)f_2(\zeta_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2\\ &= \int_{\Pi^2}\!\left(\int_0^\infty s^2 e^{-\,is\,S(g.o\,,\,\zeta_1,\zeta_2)}J_\lambda(s)\mathrm{d}s\right)\mathrm{Jac}_{\sigma_{g.o}}(\zeta_1)\mathrm{Jac}_{\sigma_{g.o}}(\zeta_2)f_1(\zeta_1)f_2(\zeta_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2\end{aligned}$$

i.e., for every ζ in Π:

$$f_1\sharp f_2(\zeta)=\int_{\Pi^2}\left(\int_0^\infty s^2e^{-is\,S(\zeta\,,\zeta_1,\zeta_2)}J_\lambda(s)\mathrm{d}s\right)\mathrm{Jac}_{\sigma_\zeta}(\zeta_1)\mathrm{Jac}_{\sigma_\zeta}(\zeta_2)\;f_1(\zeta_1)\;f_2(\zeta_2)\;\mathrm{d}\zeta_1\mathrm{d}\zeta_2\;.$$

We conclude by Theorem 6.2.1. ■

Remarks 6.3.1 *(i) In Theorem 6.3.1 item (iii), despite the fact that the formula is non-formal, it is not clear that the Drinfel'd twist defines a strict universal deformation formula (in the sense of Rieffel) for actions of AN on Fréchet AN-module algebras. Indeed, an analysis analogue to what has been done in Chapter 4 still needs to be done. So, in the present state, the Drinfel'd twist $\mathcal{F}_\theta$ should be considered as formal (through its asymptotic expansion).*

Appendix: André and Julianne Unterberger's symbolic calculus

In [UU88], A. and J. Unterberger consider the open half plane

$$\Pi\;:=\;\{x+i\xi\mid x>0\;;\;\xi\in\mathbb{R}\}\;\subset\;\mathbb{C}$$

equipped with the action of the group $\mathrm{SL}_2(\mathbb{R})$ given by

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}.\zeta\;:=\;\frac{a\zeta\;-\;ib}{ic\zeta\;+\;d}\;.$$

This is an equivalent realization of the hyperbolic plane equipped with the natural isometric action of $\mathrm{SL}_2(\mathbb{R})$. The so-called "composition formula for Fuchs symbols" $\mathbf{f}$ and $\mathbf{g}\in C^\infty_c(\Pi)$ is expressed as

$$\begin{aligned}\mathbf{f}\bullet\mathbf{g}(x+i\xi)\quad:=\quad&\int_{\Pi\times\Pi}\mathbf{f}(y'+i\eta')\,\mathbf{g}(y"+i\eta")\\&\exp-2i\pi\left\{\left(\frac{y'}{y"}-\frac{y"}{y'}\right)\frac{\xi}{x}+\left(\frac{y"}{x}-\frac{x}{y"}\right)\frac{\eta'}{y'}+\left(\frac{x}{y'}-\frac{y'}{x}\right)\frac{\eta"}{y"}\right\}\\&\left(\frac{x}{y'}+\frac{y'}{x}\right)\left(\frac{x}{y"}+\frac{y"}{x}\right)y'^{-2}\,y"^{-2}\,\mathrm{d}y'\,\mathrm{d}\eta'\,\mathrm{d}y"\,\mathrm{d}\eta"\;.\end{aligned}$$

Setting, for every $\lambda>0$:

$$G_\lambda\mathbf{g}(y+i\eta)\;:=\;4\pi\int_0^\infty\mathbf{g}(ty+i\eta)\,J_\lambda(4\pi\,t)\,\mathrm{d}t\;,$$

they prove

Theorem 6.3.2

$$G_\lambda^{-1}\left(G_\lambda(\mathbf{f})\bullet G_\lambda(\mathbf{g})\right)(1)\quad=\quad\int_{\mathbb{R}^2\times\mathbb{R}^2}\mathcal{G}(\mathbf{f}\circ\Psi)(q',p')\,\mathcal{G}(\mathbf{g}\circ\Psi)(q",p")\tag{6.22}$$

$$4\pi\int_0^\infty J_\lambda(4\pi\,s)\,\exp\frac{i\pi}{s}(-q'p"+q"p')\,\mathrm{d}s\tag{6.23}$$

$$\mathrm{d}q'\,\mathrm{d}p'\,\mathrm{d}q"\,\mathrm{d}p"\tag{6.24}$$

where

$$\Psi:\mathbb{R}^2\to\Pi:(q,p)\mapsto\frac{\sqrt{1+q^2+p^2}+q}{1-ip}$$

and

$$\mathcal{G}\varphi(\alpha,\beta)\;:=\;\int_{\mathbb{R}^2}\varphi(q,p)\,e^{2i\pi(-\alpha p+\beta q)}\,\mathrm{d}q\,\mathrm{d}p\quad(\varphi\in C^\infty_c(\mathbb{R}^2))\;.$$

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