

Atmospheric turbulence modelling and monitoring for satellite-to-ground optical communications

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*The mind is not a vessel to be filled,
but a fire to be kindled.*

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Abstract

Free-space optical communication (FSOC) between satellites and the ground will revolutionize space communications, boosting global connectivity and enabling increased data exchange for Earth observation, deep space probes and human space exploration. As this revolution is ongoing, the propagation of optical communication signals in the atmosphere remains a significant challenge, namely due to atmospheric turbulence.

The influence of atmospheric turbulence on optical waves is known as optical turbulence (OT), which is the focus of this thesis. While numerous OT models and monitoring instruments have been developed for astronomical sites, facing similar issues affecting the quality of observations, their application to FSOC sites is limited.

By reviewing existing astronomical OT models and instruments, this thesis highlights their limitations at FSOC sites. It also presents novel OT models based on meteorological parameters derived from radiosonde observations or numerical weather prediction simulations. These new models enable continuous OT prediction, for both daytime and nighttime operations, with a particular focus on the atmospheric boundary layer. Furthermore, several approaches to obtain OT measurements from wavefront sensor observations are explored in this work. The integration of OT measurements with prediction models to improve short-term OT forecast is presented, paving the way for the future development and operation of robust FSOC networks.

This thesis is composed of two parts. The first part focuses on OT modelling, based on the statistical description of atmospheric turbulence. It introduces the refractive index structure parameter C_n^2 and presents various C_n^2 models, inspired by the literature as well as novel approaches. The second part is oriented towards electromagnetic wave propagation and OT monitoring. It details the quantities associated with OT and presents the Shack-Hartmann Image Motion Monitor (SHIMM) instrument, before discussing autoregressive models for OT forecast.

This thesis contributes to the field by offering a comprehensive review of C_n^2 modelling and OT, developing new C_n^2 models for satellite-to-ground optical communications, and studying OT monitoring instruments with applications to short-term forecast.

List of Publications

Journal Publications

1. F. Quatresooz, D. Vanhoenacker-Janvier, and C. Oestges. “Computation of optical refractive index structure parameter from its statistical definition using radiosonde data”. In: *Radio Science* 58.1 (2023), e2022RS007624. DOI: 10.1029/2022RS007624.
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Conference Publications

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 8. N. Védrenne, A. Montmerle-Bonnefois, C. Petit, E. Chalali, Y. Lai-Tim, L. Krafft, F. Gustave, K. Caillault, A. Bedoya, D. Barbon-Dubosc, A. Vincent-Randonnier, J. Henrion, F. Quatresooz, G. Kermarrec, J.-C. Richard, S. Poulenard, C. Laurent, and T. Anfray. “First results of ground to GEO optical links channel monitoring with FEELINGS”. in: *International Conference on Space Optics—ICSO 2024*. 2024.

Presentations and seminars

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3. “Communications optiques : scintillation”, *Boost Aero & Space, LiEU, SkyWin*, Gembloux (Belgium), October 2021.
4. “Modelling of optical refractive index structure parameter using radiosonde data”, *IEEE International Conference on Space Optical Systems and Applications (ICSOS)*, remote presentation, March 2022.
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6. “Scintillation effects on satellite-to-ground optical communications”, *IEEE Benelux COM/VT Chapter and ST Engineering iDirect seminar*, Sint-Niklaas (Belgium), December 2022.
7. “Use of numerical weather prediction models for continuous atmospheric optical turbulence forecast”, *12èmes Journées Recherche Innovation en Optique Adaptative (JRIOA)*, Palaiseau (France), November 2023.

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Acronyms

1D	One-Dimensional xxviii, 45–47, 54, 58, 59, 63, 134, 135, 137, 245, 287,
2D	Two-Dimensional xxvii, xxix, 47, 48, 192, 196, 241–245, 249, 281, 287, 382,
3D	Three-Dimensional xxvii, xxviii, 32, 38, 43, 45, 46, 48, 53, 177–179, 185, 189, 190, 192, 194, 196, 224, 244, 371,
AFGL	Air Force Geophysical Laboratory 73, 75, 83,
AFRL	Air Force Research Laboratory 84, 85, 149, 375,
AMOS	Air Force Maui Optical Station 73–75, 77,
AO	Adaptive Optics 13–15, 255, 275,
AR	Autoregressive 16–18, 183, 345–347, 349–351, 353–356, 358–360, 362–364, 367,
ARIMA	Autoregressive Integrated Moving Average 346, 347, 349,
ARMA	Autoregressive Moving Average 349,
ARW	Advanced Research WRF 185, 186, 189,
BKB	Bendersky-Kopeika-Blaunstein 98, 99,
BL	Boundary Layer 7, 11, 14, 17, 35, 39, 71, 72, 77, 80, 83, 88, 93–95, 100, 105–108, 110, 112, 119, 169, 175, 176, 179–181, 183, 191, 196, 198, 201, 203, 206–208, 210, 211, 214, 216, 218, 219, 221, 223, 224, 254, 257, 265, 312, 320, 347, 351, 353, 356, 359, 360, 368–370,
CCSDS	Consultative Committee for Space Data Systems 1, 5,
CDF	Cumulative Distribution Function 40, 219,
CLEAR	Critical Laser Enhancing Atmospheric Research 74, 75, 84,
CO-SLIDAR	Coupled SLODAR SCIDAR 265, 330,
DIMM	Differential Image Motion Monitor 183, 221, 259, 260, 262–266, 274, 348,
DNS	Direct Numerical Simulations 38, 39, 178, 179,

| Acronyms

ECMWF	European Centre for Medium-Range Weather Forecasts xxiv, 177, 184, 190,
EM	Electromagnetic 8–10, 16–18, 26–28, 66, 154, 233, 234, 248, 266, 367, 368,
EMD	Empirical Mode Decomposition 347,
ERA5	ECMWF Reanalysis v5 177, 190, 194,
ESA	European Space Agency 4, 5, 79, 193, 217,
ESO	European Southern Observatory 183, 259,
EVD	Eigenvalue Decomposition 318,
FA	Free Atmosphere 7, 71, 72, 80, 93, 94, 140, 143, 175, 176, 178, 180, 181, 198, 199, 203,
FFI	Norwegian Defence Research Establishment 159,
FFT	Fast Fourier Transform 59, 287, 288,
FSO	Free-Space Optics 1, 13,
FSOC	Free-Space Optical Communication 1–5, 9, 12, 14–18, 170, 223, 256, 341, 367–372,
FWHM	Full Width at Half Maximum 11, 12, 253,
GCM	General Circulation Model 178, 183,
HAP	Hufnagel/Andrews/Phillips 80, 107, 140, 170,
HV	Hufnagel-Valley 76, 77, 79–81, 119, 140–143, 145–148, 153, 170, 247, 257, 259,
ITU	International Telecommunication Union 28, 77, 154, 161,
LAMOST	Large sky Area Multi-Object fiber Spectroscopic Telescope 346, 348,
LBT	Large Binocular Telescope 183, 347,
LES	Large Eddy Simulations 38, 39, 132, 178, 179, 224, 368, 370,
LGS	Laser Guide Star 13,
LS	Least Squares 298–300, 302–305, 308, 311, 313, 319, 323, 341, 350,
LSTM	Long Short-Term Memory 347, 348,
MASS	Multi-Aperture Scintillation Sensor 183, 265, 266, 348,

MEP	Multi-model Ensemble Pattern 92, 110,
Meso-NH	Mesoscale Non-Hydrostatic 182–184, 198, 347,
ML	Machine Learning 72, 100, 109–111, 119, 345–348, 368, 369, 371,
MM5	Fifth-Generation Penn State/National Center for Atmospheric Research Mesoscale Model 181, 183,
MOST	Monin-Obukhov Similarity Theory 100, 101, 103–106, 108, 175, 184, 198, 201, 223,
MRE	Mean Relative Error 114, 305, 306, 308, 315, 328, 332, 337, 359, 360,
NAVSLaM	Navy Vertical Surface Layer Model 105, 109,
NNLS	Non-Negative Least Squares 299, 300, 302–305, 308, 311, 312, 314, 319, 332, 341,
NOAA	National Oceanic and Atmospheric Administration 88, 184,
NWP	Numerical Weather Prediction 16–18, 37, 71, 91, 93, 105, 119, 129, 132, 167, 169–171, 175–184, 208, 214, 217, 221, 223, 224, 255, 340, 345–347, 351, 353, 356, 359, 360, 362, 367–372, 375, 376, 385,
OGS	Optical Ground Station 2, 9, 10, 12–15, 17, 72, 79, 119, 176, 221, 223, 224, 273, 362, 367–371,
OT	Optical Turbulence xi, 9–18, 72, 77, 79, 105, 116, 119, 129, 176–181, 183, 191, 198, 203, 204, 209, 223, 224, 233, 234, 241, 247, 248, 257, 259, 264–266, 273, 275, 276, 297, 340, 341, 345–351, 359, 362, 363, 367–372, 385,
OTF	Optical Transfer Function 249–253, 257,
PCA	Principal Component Analysis 318–320, 322, 332, 335, 337, 339,
PDF	Probability Density Function 40, 89, 135, 154, 346,
PSF	Point Spread Function 12, 14, 249, 253,
RANS	Reynolds-Averaged Navier-Stokes 38, 39,
RF	Radio-Frequency 1, 3, 4, 130, 153, 154, 162, 164, 170, 224, 368, 370,
RINGSS	Ring-Image Next Generation Scintillation Sensor 266,
RMS	Root-Mean-Square 76, 90, 254,
RMSE	Root-Mean-Square Error 113, 117, 149, 150, 164, 205–208, 214, 216, 219, 221, 254, 305, 306, 308, 313, 315, 318, 319, 321, 322, 328, 331, 332, 337, 347, 348, 355, 356, 358–360,

| Acronyms

SALTO	Smart Adaptive-optics and Laser guide-star for medium-size Telescopes and Optical communications 193, 221, 260, 263,
SCIDAR	SCIntillation Detection And Ranging xxiii, 72, 87, 183, 263, 264, 266, 274, 275, 294, 296, 297, 304–306, 312–315, 318, 320, 321, 325–327, 331, 335, 337, 339, 371,
SHIMM	Shack-Hartmann Image Motion Monitor 16–18, 209, 247, 248, 259, 265, 266, 273–277, 280, 285, 287, 290–292, 297, 299, 300, 302–305, 308, 310, 311, 318, 319, 330, 335, 337, 339–341, 345, 349–351, 363, 367, 371, 372, 381, 385,
SHWFS	Shack-Hartmann Wavefront Sensor 274–277, 285, 330,
SL	Surface Layer 7, 35, 96, 100–102, 104, 105, 108, 109, 119, 198, 214,
SLC	Submarine Laser Communication 73,
SLODAR	Slope Detection and Ranging xxiii, 264, 275, 330, 371,
SSM	Solar Scintillation Monitor 217, 221,
std	Standard Deviation 112, 138, 139,
T-REX	Terrain-induced Rotor Experiment 149, 176, 180, 181, 201, 203–205, 207, 208, 224, 375, 376,
TKE	Turbulent Kinetic Energy xxvii, 34–36, 91, 105, 175, 176, 180–184, 191, 196, 198, 199, 203, 206–208, 210, 211, 214, 216, 218, 219, 221, 223, 351, 353, 354, 356, 359, 360,
TV	Trinquet-Vernin 76, 93, 181, 182,
UAV	Unmanned Aerial Vehicle 132, 167, 169, 170, 370,
UTC	Coordinated Universal Time 138, 160, 169, 194, 196, 199, 201, 203, 209, 210, 218, 292, 294, 335, 351, 353,
VHF	Very High Frequency 88, 154,
VLT	Very Large Telescope 183, 259, 346, 347,
WPS	WRF Preprocessing System 185, 186, 189, 194,
WRF	Weather Research and Forecasting xxiii, xxvi, 17, 169, 175, 176, 181–194, 196, 199, 201, 203–211, 214, 217, 218, 221, 223, 224, 347, 351,
WSPT	Wind Speed and Potential Temperature 92, 140,
WSTG	Wind Shear and Temperature Gradient 86, 92,

Symbols and Notations

$B_f(\tau), B_f(\rho)$	Covariance functions for stationary stochastic processes $f(t)$ or homogeneous and isotropic random fields $f(\mathbf{r})$
χ	Log-amplitude , log-amplitude fluctuations, defined in (6.9)
C_n^2	Refractive index structure parameter
C_q^2, C_e^2	Specific humidity structure parameter , water vapor pressure structure parameter
C_T^2, C_θ^2	Temperature structure parameter , potential temperature structure parameter
$C_{Tq}, C_{\theta q}, C_{Te}$	Temperature-specific humidity structure parameter , potential temperature-specific humidity structure parameter, temperature-water vapor pressure structure parameter
$D_f(\tau), D_f(\rho)$	Structure functions for stationary (with increment) stochastic processes $f(t)$ or (locally) homogeneous and isotropic random fields $f(\mathbf{r})$
e	Partial pressure of water vapor
ε	Viscous dissipation of turbulent kinetic energy
$\epsilon, \epsilon_r, \epsilon_0$	Dielectric permittivity , relative permittivity, vacuum permittivity, presented in (2.1)
$\mathcal{E}_0$	Seeing , defined in (6.35)
$\mathbf{f}, f$	2D spatial frequency vector in [1/m] having for components $\mathbf{f} = (f_x, f_y)^T$, magnitude of spatial frequency vector
$f(\cdot)$	Generic function notation primarily used to denote stochastic processes and random fields
g	Gravitational acceleration
$\boldsymbol{\kappa}, \kappa$	3D spatial frequency vector in [rad/m] having for components $\boldsymbol{\kappa} = (\kappa_x, \kappa_y, \kappa_z)^T$, magnitude of spatial frequency vector
k_ϵ	Turbulent kinetic energy (TKE) , defined in (2.16)

| Symbols and Notations

L_0	Outer scale of turbulence
l_0	Inner scale of turbulence
λ	Wavelength
n	Refractive index
N	Refractivity , defined in (2.2)
p	Pressure
$\Phi_f(\kappa), V_f(\kappa)$	3D power spectral density , 1D power spectral density, for (locally) homogeneous and isotropic random fields $f(\mathbf{r})$
φ	Phase , phase fluctuations, defined in (6.9)
q	Specific humidity
$\mathbf{r}$	3D position vector , having for components $\mathbf{r} = (x_1, x_2, x_3)^T = (x, y, z)^T$
r_0	Fried parameter , coherence length, defined in (6.32)
Re	Reynolds number , defined in (2.18)
RH	Relative humidity
ϱ	Density
ρ, ρ	Spacing difference vector , magnitude of spacing difference vector (for covariance and structure functions)
Ri, Ri_f, Ri_b	Richardson number (gradient) , flux Richardson number, bulk Richardson number, defined in (2.21) and (2.22)
σ_I^2	Scintillation index , defined in (6.38)
σ_χ^2	Variance of the log-amplitude
T	Temperature
τ_0	Coherence time , defined in (6.37)
τ	Temporal difference for covariance and structure functions (except in Chapter 7, used for a regularization parameter)
θ_0	Isoplanatic angle , defined in (6.36)
$\vartheta, \tilde{\vartheta}$	Elevation angle measured from the ground (ϑ) or from zenith ($\tilde{\vartheta}$, zenith angle)

θ, θ_v	Potential temperature , virtual potential temperature, defined in (2.19) and (2.20)
t	Time
μ, μ_r, μ_0	Magnetic permeability , relative permeability, vacuum permeability, presented in (2.1)
$\mathbf{u}$	Wind velocity vector , having for components $\mathbf{u} = (u_1, u_2, u_3)^T = (u, v, w)^T$
ω	Pulsation , temporal frequency in [rad/s] (except in Chapter 4, parameter of the C_n^2 statistical definition model)
$W_f(\omega)$	Power spectral density for stationary (with increment) stochastic processes $f(t)$
$\mathbf{x}$	2D position vector , having for components $\mathbf{x} = (x, y)^T$

Only the notations shared across all chapters are listed above; other notations are defined within their respective chapters.

Mathematical notations

$a, \mathbf{a}, \mathbf{A}$	Scalar number a , vector $\mathbf{a}$, matrix $\mathbf{A}$
$\bar{a}, \mathbf{a}'$	Mean of a , fluctuations of a
a^*	Complex conjugate of a
$\mathbf{a}^T$	Transposition of vector $\mathbf{a}$
$ \mathbf{a} , \ \mathbf{a}\ _2$	L2-norm of vector $\mathbf{a}$
$*$	Convolution operation
∇	Gradient operator
$\nabla \cdot$	Divergence operator
∇^2	Laplacian operator
$\frac{d}{dt}, \frac{\partial}{\partial t}$	Total derivative , partial derivative, with respect to variable t
Δ	Differencing , e.g., between two levels
δ_{ij}	Kronecker delta , equal to 1 if $i = j$ and 0 otherwise
$\delta(x)$	Dirac delta function
$\langle \cdot \rangle, \mathbb{E}[\cdot]$	Ensemble average , expectation
$e, \mathbf{e}$	Exponential , number e
$\mathcal{F}\{\cdot\}(f)$	Fourier transform , expressed with variable f
j	Imaginary unit ($j^2 = -1$)

In the coming decades, the demand for high-speed and reliable communications is expected to boom. This is true for cellular wireless networks on the ground, but also applies to communications in space, where an increasing number of satellites are being deployed [Toy21].

The previous revolution in communication capabilities arose from the adoption of fiber-based optical communications, i.e., using light in a guided medium to transfer information. This transformation enabled to connect Earth's continents and facilitated the deployment of the internet and associated services. However, installing fiber-optics requires significant investments, and remote areas may remain unreachable. As an alternative, using light to convey information wirelessly; either through the vacuum of space or through Earth's atmosphere; will soon be a viable alternative to fiber-optic networks and Radio-Frequency (RF) wireless networks. This technology, referred to as Free-Space Optical Communication (FSOC) or Free-Space Optics (FSO), promises to enhance connectivity for everyone, boost achievable data rates, and offer strategic alternatives to fiber-optic networks.

Figure 1.1 presents possible application scenarios for future FSO-based communication networks, involving ground terminals linked by fiber-optics, relay satellites with FSO links, Earth-to-satellite and satellite-to-Earth optical links, FSO terrestrial links, etc. Most likely, FSOC networks will be intertwined with current fiber-optic and RF wireless networks, and serve as core or backhaul networks [Liu+23]. Space applications also motivate the development of space-to-ground communication links, e.g., for transmitting large amount of data from Earth observation satellites or deep space probes.

Nevertheless, the deployment of new FSOC technologies faces numerous challenges, especially due to the propagation of light in Earth's atmosphere. So far, numerous demonstrations of optical links have been conducted by worldwide space agencies, research institutes and companies, but fully commercial networks have yet to be developed. While the technology may be mature, its reliability in diverse atmospheric conditions and on a network-scale remains to be proven. As a result, there is a pressing need for atmospheric characterization at potential FSOC sites, as demonstrated by ongoing measurements campaigns [Gri+24; Zia+23] and recent reports from the Consultative Committee for Space Data Systems (CCSDS) [CCS17; CCS22]. The work presented in this thesis is part of this effort of improving atmospheric monitoring and modelling at FSOC sites.

The modelling of atmospheric effects on the propagation of optical waves has

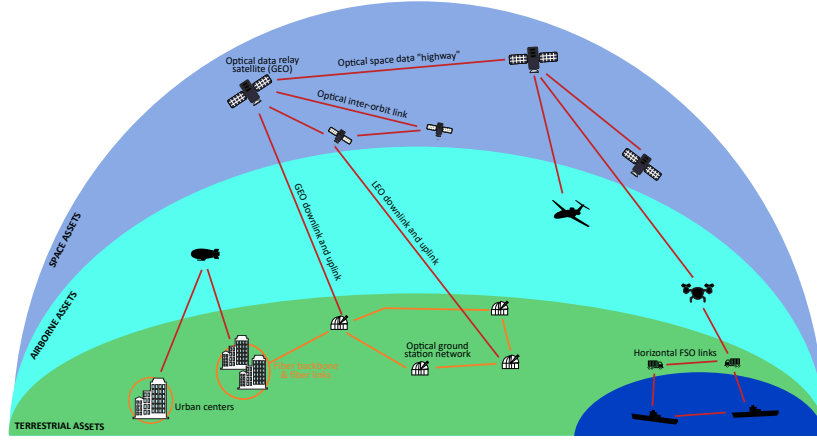


Figure 1.1 – Illustration of possible free-space optical communication networks (adapted from [Can18]).

already been developed for astronomical applications in the past decades. Indeed, it has been and continues to be particularly important to identify optimal locations on Earth for installing astronomical observatories, that suffer from the lowest atmospheric perturbations as possible (along with other impairments such as light pollution for example). This process of site-selection for astronomical optical telescopes is also meaningful for future Optical Ground Station (OGS) locations. Moreover, mitigation techniques (e.g., adaptive optics) for atmospheric degradations have been designed for astronomical telescopes, and could be applied to optical communication terminals.

There are therefore strong synergies between atmospheric characterization and correction for astronomical and optical communication applications. Nevertheless, astronomical sites are situated in remote areas with little atmospheric impairments, such as at the top of mountains, ideal for observations of good quality. Such sites are not of interest for optical communications, as OGSs need to be located close to the users, i.e., in urban locations. Moreover, optical communication systems should be able to operate in daytime and nighttime, while astronomical observations are mostly conducted during nighttime. These major differences explain the requirements for further research in this field, and this work focuses on the modelling and monitoring of atmospheric propagation of optical waves at FSOC sites, extending previous results obtained for astronomical sites.

The primary atmospheric effect examined in this thesis is atmospheric turbulence. It is, for example, responsible for the twinkling of stars, i.e., the apparent change of brightness or position of stars when observed with the naked eye under a clear night sky. This phenomenon is well-known by astronomers and comes from local fluctuations of the air refractive index along the propagation of light through Earth's atmosphere, caused by atmospheric turbulence.

This chapter presents the concepts necessary to understand the context of this

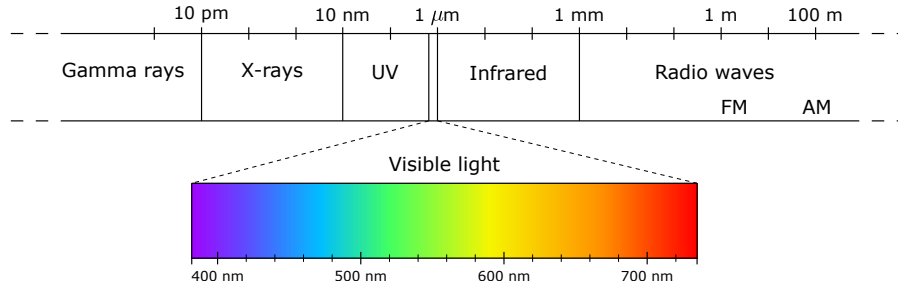


Figure 1.2 – Electromagnetic spectrum.

work and its major contributions. It starts with the description of FSOC in the context of satellite-to-ground links, also highlighting the motivations behind this technology (Section 1.1). Then, the atmospheric effects on light propagation are deeper introduced in Section 1.2, with a particular focus on atmospheric turbulence consequences. Sections 1.3 and 1.4 finally present the objectives of the thesis, its main contributions, as well as its structure.

1.1 Free-space optical communication

Free-space optical communication is a form of optical wireless communication and usually refers to long range (outdoor) communication [KK16]. It relies on the use of light to convey an information signal wirelessly, i.e., without needing a support such as an optical fiber to guide the signal. Hence, FSOC takes place in vacuum, air or even water, making use of visible or near-infrared light.

The electromagnetic spectrum is depicted in Figure 1.2, where the visible region is highlighted. However, for many FSOC systems designed for space applications, the considered wavelengths are in the near- and short-infrared regions, namely at 850 nm, 1064 nm and 1550 nm, that correspond to wavelengths used in fiber-optic networks and for which components are already available [KK16; Pla+12]. Atmospheric absorption is also limited for those wavelengths (Section 1.2.2) [CCS17].

1.1.1 Motivations for optical communications

The motivations behind FSOC come from its advantages compared to RF communications. These advantages are listed in the paragraphs below, based on [KK16]. They are mostly related to the higher frequencies, or equivalently the smaller wavelengths, of optical waves compared to RF waves.

Large modulation bandwidth Optical frequencies are in the order of hundreds of THz. Using them as carrier frequencies for communications offer larger modulation bandwidths than RF systems. The larger the modulation bandwidth, the more data can be transmitted per second, hence increasing the data rate.

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Narrow beam divergence and a high directivity Given the smaller wavelengths of optical waves compared to RF waves, the communication beams are narrower and high directivity can be achieved.

Unlicensed spectrum Compared to RF communications, where licensing is required to avoid interferences between systems using on the same frequencies, optical communications are unlicensed. This licensing is prevented by the high directivity of optical waves.

Reduced power consumption, weight and size Given the small beam divergence of optical waves, optical systems can be equipped with smaller antennas than RF systems while still obtaining a sufficient signal intensity for the communication. This is especially of interest for space applications.

Increased security Signal interception is made difficult by the high directivity, and would require eavesdroppers to be in the line of sight.

Quantum key distribution FSOC systems could enable the development of Quantum Key Distribution (QKD), i.e., a cryptographic protocol used for sharing secret keys to encrypt and decrypt communications. The protocol is based on quantum mechanics, and requires the capability to measure quantum states of particles such as photons (e.g., BB84 protocol given in [BB14]).

1.1.2 Satellite-to-ground optical communication

As depicted in Figure 1.1, FSOC links can be of various types, either terrestrial or in space. For space applications, optical links can be grouped in three distinct categories [KK16]:

- Deep space links: for communications with spacecrafts exploring the solar system [Kar+24];
- Inter-satellite links: for data relay between satellites in orbit around the Earth, such as the European Data Relay Satellite System (EDRS) developed by the European Space Agency (ESA);
- Earth-to-space and space-to-Earth links: also named feeder links, they enable to bring data collected by satellites back to Earth, and vice-versa, i.e., send information from the ground to the satellites in orbit.

This work focuses on Earth-to-space and space-to-Earth links, also named ground-to-satellite and satellite-to-ground links. In the following chapters, the terminology *satellite-to-ground* is used for both referring to ground-to-satellite and satellite-to-ground links. Hence, both uplink (i.e., communication link from the ground towards the satellite) and downlink (i.e., communication link from the satellite to the ground) scenarios are considered, except if specified.

Associated challenges The major challenges associated to satellite-to-ground optical communications come from the atmospheric propagation. They are introduced in Section 1.2. Other challenges include the accuracy of satellite acquisition, tracking and pointing, made difficult by the high directivity of optical beams [KK16].

Demonstrations and future missions Several experiments and demonstrations of satellite-to-ground optical links have been conducted in the past twenty years, mostly by space agencies. A detailed list of those missions can be found in [Toy21; Can18; KK16].

Since the publication of these works, the National Aeronautics and Space Administration (NASA) has successfully demonstrated a deep space optical communication link with the Psyche spacecraft with a maximum data-rate of 267 Mb/s at 55 million kilometers [Bis+24]. The agency is also planning optical links with the Orion spacecraft during the Artemis II mission, that will bring astronauts back to the Moon's orbit [Rob+22].

Furthermore, ESA is working on the High throughput Optical Network (HyDRON) project, part of the Secure and Laser communication technology (ScyLight) program, that aims to demonstrate an optical multi-orbit network in space, extending terrestrial fiber-based networks [Aca+23]. The Advanced Laser guide star Adaptive optics for Satellite Communication Assessments (ALASCA) project is also ongoing, which goal is to demonstrate pre-compensation of atmospheric turbulence effects for uplink communications [Bia+23].

Regarding the National Institute of Information and Communications Technology (NICT) and the Japan Aerospace Exploration Agency (JAXA), they conducted their first experimental demonstration of optical feeder links with the Laser Utilizing Communication System (LUCAS) onboard a geostationary satellite in 2021 [Kot+23].

1.2 Atmospheric effects on satellite-to-ground optical communications

Earth's atmosphere can lead to increased attenuations and fluctuations on transmitted optical signals, jeopardizing the reliability of optical communications. As illustrated in Figure 1.3, inspired from the CCSDS reports [CCS17; CCS22], the three major causes of atmospheric impairments on FSOC are:

1. Clouds, leading to important atmospheric attenuation and even blockage of optical signals;
2. Atmospheric turbulence, inducing fluctuations of phase and intensity of the signals;
3. Aerosols (e.g., dust), responsible for complementary attenuation.

The impacts of clouds and aerosols is presented in Section 1.2.2, while the impact of atmospheric turbulence is given in Section 1.2.3. The structure of the atmosphere is

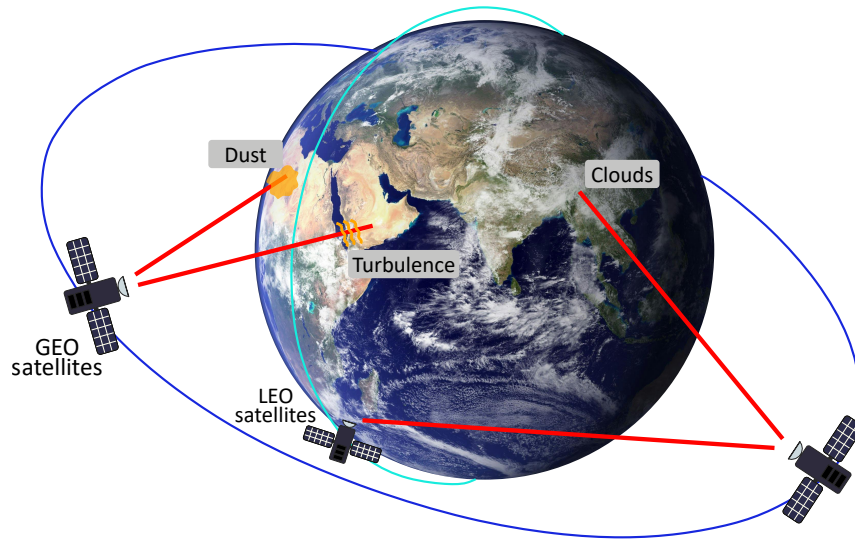


Figure 1.3 – Illustration of atmospheric effects on satellite-to-ground optical communications (adapted from [CCS22]).

first introduced in Section 1.2.1. In Figure 1.3, *GEO satellites* stands for Geostationary Orbit satellites (orbit altitudes of approximately 36000 km), and *LEO satellites* is for Low Earth Orbit satellites (orbit altitudes below 2000 km).

1.2.1 Description of the atmosphere

The atmosphere is a gaseous envelope surrounding the Earth, principally made of nitrogen and oxygen. It extends up to several hundreds of kilometers above the surface and is usually divided into four layers based on temperature variations, as shown in Figure 1.4, inspired by [AP05, p. 10].

The first atmospheric layer is the *troposphere*, starting at the surface and extending up to 12 km of altitude on average, with important seasonal and latitude variations. In this layer, the temperature decreases progressively, with an average temperature lapse rate of $-6.5^{\circ}\text{C}/\text{km}$, varying with the location. It contains 75% of Earth's atmospheric mass [AP05, p. 10]. It is limited by the *tropopause*, which is an isothermal layer with an average temperature close to -55°C .

Above the tropopause, the *stratosphere* extends up to approximately 50 km of altitude. The temperature increases progressively with the altitude, due to the absorption of ultraviolet light by ozone molecules transformed into heat. It is limited by the *stratopause* that is an isothermal layer with an average temperature close to 0°C .

Above the stratopause, the *mesosphere* spans from approximately 50 to 80 km of altitude. It ends at the *mesopause*, which is the coldest layer in the atmosphere, at roughly -90°C .

The *thermosphere* starts from the mesopause and can reach 600 km of altitude. It is then replaced by the *exosphere*, the last atmospheric region where particle density

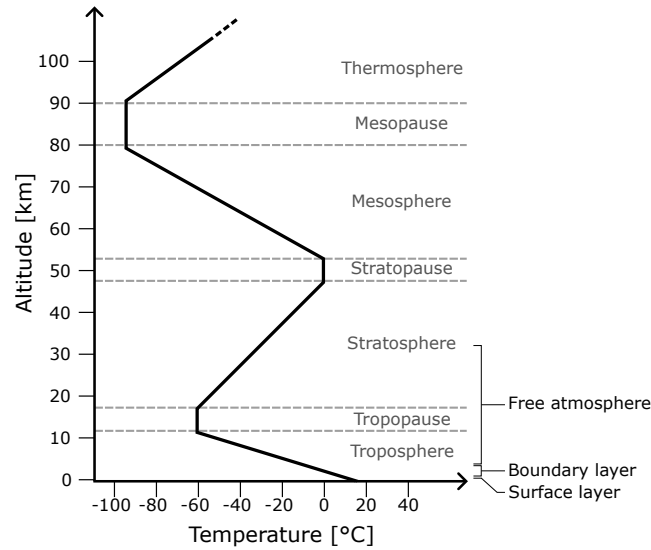


Figure 1.4 – Atmospheric layers based on temperature variations, and terminology.

is very small such that atoms are considered to move freely.

Not depicted in Figure 1.4 is the *ionosphere*, that is an atmospheric layer located in the thermosphere and exosphere, ranging from approximately 80 km to 1000 km altitude. It contains gases ionized by solar radiation, such that the concentration of free electrons is important. This influences the propagation of radio signals, for example Global Navigation Satellite System (GNSS) signals used for navigation.

In this work, the layers with the most interest are the troposphere (and tropopause) and part of the stratosphere, as those layers contain most of the atmospheric mass. The air is the densest, and its fluctuations caused by atmospheric turbulence strongly influences the propagation of optical signals, especially between the ground and 20 to 25 km of altitude.

In the context of optical wave propagation, the troposphere and stratosphere are usually separated into two distinct layers:

- the *Boundary Layer* (BL), ranging from the ground to about 0.5 to 2 km altitude. It corresponds to the part of the atmosphere influenced by its contact with the ground. [FN08, p. 1]
- the *Free Atmosphere* (FA), i.e., the part of the atmosphere where the ground influence is marginal.

Finally, the lowest part (i.e., the lowest 5-10%) of the BL is referred to as the *Surface Layer* (SL). It contains most of the energy exchange in the atmosphere, from solar heating and wind surface friction.

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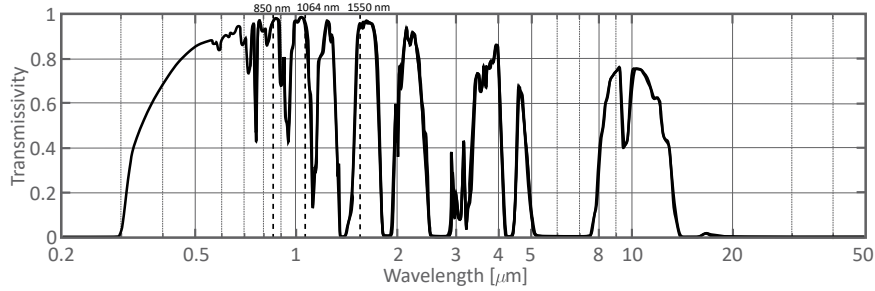


Figure 1.5 – Spectral transmissivity of the atmosphere from sea level to space, at zenith (reproduced from [GS22]).

1.2.2 Impact of absorption, clouds, aerosols and water particles

Even in the absence of clouds and aerosols, atmospheric molecular absorption and scattering (Rayleigh scattering) introduce partial or total attenuation of Electromagnetic (EM) waves [AP05, p. 11]. Indeed, the atmospheric composition defines clear spectral windows in which the propagation of EM waves is possible, i.e., where the atmospheric transmittance is close to one. The wavelengths classically used for optical communications fall within several of these windows, as shown in Figure 1.5.

Aerosols Aerosols contribute to the scattering (Mie scattering) of optical signals [AP05]. They are particles larger than molecules that can remain suspended in the atmosphere for long periods, such as desert dust, volcanic ash, or sea spray. Their impact on optical waves is limited to 1 dB of attenuation most of the time, but punctual events could lead to larger attenuations [CCS17]. It is also the case of fog and haze, as described in the paragraph associated to water particles.

Details about aerosol and molecular attenuations for optical satellite-to-ground links can be found in [GS22; Ric+06] and the references therein. Moreover, several software are available for modelling the absorption and scattering effects, as listed in [AP05, p. 12]. They are not studied in this work.

Water particles The impact of snow and rain on the propagation of optical waves is rather limited (typically on the order of 3 dB/km for rain), at the exception of heavy rain event. It is a consequence of the particle sizes that is much larger than the considered wavelengths [KU14].

On the contrary, for fog and haze, the particle diameter is on the order of the wavelength, such that scattering is very high. In this case, large attenuations are expected, such as 90% of loss over a propagation distance of 50 m in moderate fog [KU14].

More details about the effects of water particles can be found in [KK16; KU14] and the references therein.

Clouds Clouds are the largest source of atmospheric attenuation of optical signals, often leading to blockage of optical links. Depending on cloud types and thicknesses, as well as on the operating wavelengths, the attenuation can reach tens to hundreds of dB, namely for stratus and stratocumulus clouds [Mol09]. For thin cirrus clouds, the attenuation can remain lower than 3 dB, but cloud inhomogeneities lead to signal fluctuation that is also damaging for the communication [CCS17].

This cloud blockage has an important impact on the availability of satellite-to-ground optical communications. A practical solution to this issue is to rely on ground station diversity, i.e., to have a network of OGSs located in different regions. The OGS locations are chosen such that they have uncorrelated atmospheres, and it is unlikely that clouds will prevent communications with all ground stations at the same time. This requires the OGSs to be connected together in order to establish a ground network that can relay the information to the desired location. As an example, [Ben+19] presents multiple locations across France or Europe where OGSs should be located to maximize the availability of optical links. Similar studies have been conducted for other countries and world regions, e.g., [Mol09; CCS17; Kry+21; Bir+22].

1.2.3 Atmospheric turbulence

Even under clear sky conditions, i.e., in the absence of clouds, aerosols or rains, the atmospheric propagation of optical waves is degraded, namely by atmospheric turbulence.

Atmospheric turbulence refers to the chaotic motion of air in the atmosphere. It is associated to local inhomogeneities in air density, created by variations in pressure, temperature, humidity, etc. Such fluctuations in density modify the local atmospheric refractive index, hence impacting the propagation of EM waves.

In the context of optical astronomy and FSOC, the impact of atmospheric turbulence on light propagation is referred to as *Optical Turbulence* (OT). As introduced above, it is responsible for the twinkling of stars. It also drastically impacts the quality of ground-based astronomical observations, leading to blurred images. Its impacts on FSOC are presented in Section 1.2.3.1, while its characterization is briefly discussed in Section 1.2.3.2. Some mitigation techniques, mostly developed for astronomical applications, are introduced in Section 1.2.3.3.

1.2.3.1 Optical turbulence impacts on FSOC channels

The impacts of OT on FSOC are multiple and include phase and intensity fluctuations, beam spreading and beam wandering [KK16].

Phase and intensity fluctuations Fluctuations of the atmospheric refractive index introduce phase variations on optical waves propagating through the atmosphere, leading to *wavefront distortion*. The imaging of deformed wavefronts through a telescope pupil results in speckle patterns, where the light intensity is distributed over a wider region. An example of this phenomenon is depicted in Figure 1.6a,

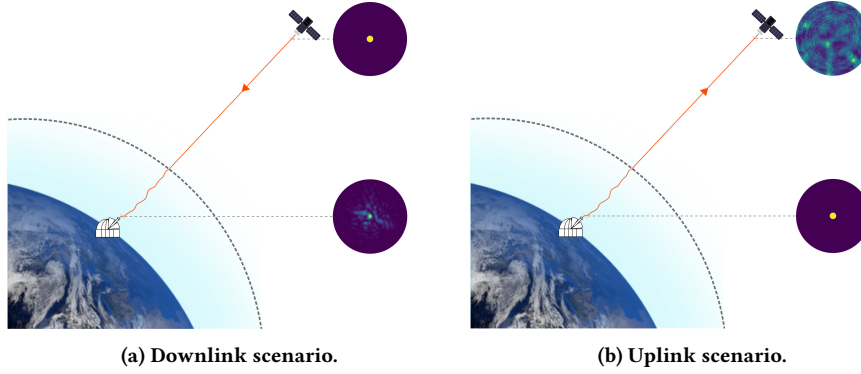


Figure 1.6 – Illustration of intensity fluctuations observed at the receiver for downlink and uplink Earth-space optical communications.

showing the propagation of a Gaussian beam, i.e., a laser, through the atmosphere. For the downlink scenario, the light propagates in vacuum from the satellite to the upper part of the atmosphere, with pure diffraction. Then, atmospheric propagation introduces wavefront distortions, resulting in a speckle pattern in the OGS telescope. This pattern is random and dynamically varying, due to the dynamics of atmospheric turbulence, and determines the communication signal intensity collected at the receiver.

On top of phase variations leading to *angle-of-arrival fluctuations*, intensity variations of the propagating waves are also introduced by atmospheric turbulence. It is namely the case for strong OT or for OT arising at high altitudes, in downlink scenarios. Indeed, in such cases, the phase variations caused by OT progressively lead to intensity fluctuations during the propagation of the damaged wavefronts. This results in *scintillation*, that is, fluctuations of the received amplitude (or equivalently, intensity) of an EM wave.

In uplink scenarios, atmospheric degradations are introduced close to the transmitter, located on Earth's surface. Then, the distorted wavefronts propagate for hundreds or thousands of kilometers before reaching the satellite, leading to stronger intensity fluctuations and intensity redistribution at the receiver. This is depicted in Figure 1.6b.

Such variations of the received signal intensity are referred to as fading in the field of telecommunication. As the temporal evolution of intensity fluctuations are on the order of 1 to 10 milliseconds, and since the communication data rate is several Gbps, the communication channel is said to be *slow fading* [Can18, p. 47],[KU14]. In this case, signal losses that can occur when the received signal intensity is lower than the detection threshold will lead to burst of errors in the communication. Such errors can be limited by the use of coding and interleaving techniques [Can18]. Increasing the size of the receiver aperture also limits the amount of scintillation on the received signal (aperture averaging) [Ric+06].

Beam spreading *Beam spreading* originates from the natural diffraction of the wave as it propagates, and reduces the amount of energy within the original beam diameter in the case of Gaussian beams. The resolution of Maxwell equations for Gaussian beams leads to mathematical expressions enabling its quantification in free space propagation scenarios [AP05, p. 8, 91].

However, due to atmospheric turbulence, the observed beam spreading is larger than the one induced purely by diffraction. As a result, extra power losses are measured at the receiver, since it has a finite aperture size. The turbulence-induced beam spreading is associated to small scale turbulent eddies, smaller than the beam size [KK16].

Beam wandering Wavefront distortions arising from OT also leads to *beam wandering*, i.e., random variations of the beam centroid positions at the receiver [AP05, p. 15, 201]. It is caused by turbulent eddies of sizes larger than the beam size, and results in random deflections of the beam that can lead to link failures [KK16]. It can be mitigated by relying on fast tracking systems of the communication beam.

1.2.3.2 Characterization of optical turbulence

Under several hypotheses described in Chapter 2, OT is mainly characterized by one physical quantity named the *refractive index structure parameter*, noted C_n^2 with units $[m^{-2/3}]$. It is a statistical parameter that describes how strong the refractive index fluctuations are: the larger C_n^2 , the larger the fluctuations of the refractive index, and thus the larger the OT. Typical values of C_n^2 are between 10^{-17} and $10^{-13} m^{-2/3}$, respectively associated to weak and strong turbulence [AP05, p. 65].

C_n^2 profiles Similarly to other physical quantities in the atmosphere (e.g., pressure, temperature, etc.), C_n^2 varies in space (location and altitude) and time. The altitude dependency of C_n^2 is important, with large C_n^2 values observed close to the ground (in the BL) and sometimes at the tropopause. On the contrary, horizontal variations of C_n^2 are limited such that, for practical applications studied in the context in this thesis, the quantity of interest is a C_n^2 *profile* of the altitude z , noted $C_n^2(z)$. Regarding the temporal variations of C_n^2 , they include seasonal and diurnal dependencies, with typically stronger values during daytime close to the ground.

C_n^2 profiles are usually obtained from classical meteorological parameters (e.g., pressure, temperature, wind speed, etc.) using various models that are available in the literature. A review of these C_n^2 models is given in Chapter 3, while new approaches developed in the framework of this thesis are presented in Chapters 4 and 5. Measurements of C_n^2 profiles are made possible by dedicated instruments, introduced in Chapter 6 and 7.

Seeing C_n^2 influences numerous quantities associated to the propagation of optical waves in the atmosphere, as described in Chapter 6. It is namely the case of the *seeing* $\mathcal{E}_0$, an integrated parameter of the C_n^2 profile giving the Full Width at

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Half Maximum (FWHM) of the Point Spread Function (PSF) of a point source (i.e., a star) imaged through atmospheric turbulence, measured in radians or arcseconds. In strong OT conditions, the seeing is large (several arcseconds) while in weak turbulence, such as the one observed for astronomical sites, seeing is small (below one, or even 0.5 arcsecond). In a broader sense, seeing refers to the quality of the atmospheric conditions for the propagation of optical waves [Tok23], and namely the imaging of astronomical objects. In this case, one talks about bad seeing (i.e., strong OT), or good seeing (i.e., low OT).

Scintillation index The *scintillation index* σ_I^2 , is also an integrated quantity of the C_n^2 profile, giving the normalized variance of the received intensity I . It is useful to characterize, and parametrize, probability distributions of the received communication signal intensity $p_I(i)$, for which various distributions have been suggested in the literature. Under weak turbulence conditions, the received intensity is often assumed to follow a log-normal distribution. On the contrary, in strong turbulence conditions, K distributions or Gamma-Gamma distributions are often used [KU14; KK16; HW10]. Knowledge of the received intensity distribution is important for FSOC, as it strongly influences the probability of fade and the communication bit error rate [Osb+21].

1.2.3.3 Mitigation techniques

The significant impacts of OT on optical communications and astronomy necessitate the implementation of mitigation techniques to eliminate or at least reduce these effects.

High altitude As far back as 1703, Sir Isaac Newton had recognized that [New04, p. 111]

“They [Telescopes] cannot be so formed as to take away that confusion of the Rays which arises from the Tremors of the Atmosphere. The only Remedy is a most serene and quiet Air, such as may perhaps be found on the tops of the highest Mountains above the grosser Clouds.”

Indeed, on mountain tops, the air is thinner and the propagation path through the atmosphere is reduced. Consequently, OT is weaker. This explains why most astronomical observatories are built in remote areas at high altitudes, with other motivations such as lower cloud coverage and reduced light pollution. Alternatively, space telescopes, such as the Hubble Space Telescope, are ideal solutions to avoid OT as well as atmospheric absorption, despite increased costs and maintenance difficulties.

However, gaining altitude or relying on space telescopes are not viable solutions for satellite-to-ground optical communications since OGSs must be situated near populated areas, such as cities, where the majority of communication needs are concentrated.

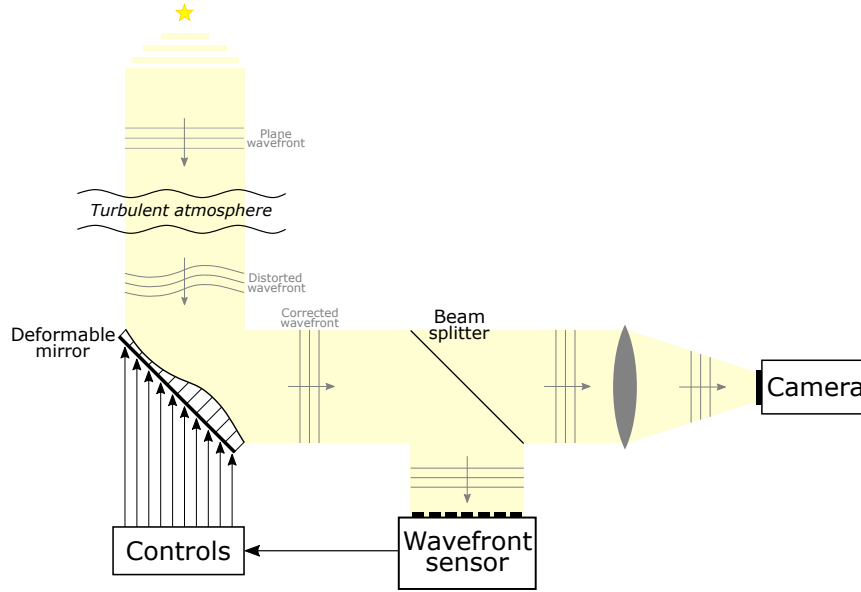


Figure 1.7 – Illustration of adaptive optics.

Adaptive optics As an alternative to space telescopes, large ground-based telescopes are often equipped with *Adaptive Optics* (AO) systems. Such systems enable to partially correct OT impacts on the observed objects, hence improving image quality.

The underlying principle of AO is to (mechanically) compensate the wavefront distortions prior to imaging, using a deformable mirror. This is illustrated in Figure 1.7: the perturbed light from a star is reflected by a mirror in such a way that atmospheric turbulent effects are corrected (i.e., going from a distorted wavefront to a planar wavefront). In order to determine how the mirror should be deformed to ensure correction, the received wavefront is analysed in real-time by a wavefront sensor. A closed-loop system then ensures that commands to apply on the deformable mirror are computed from the received wavefront analysis. Such systems typically work at a rate of a few kilohertz. More details about AO can be found in [TF22], and a historical perspective on the development of this technology is available in [DF09].

For wavefront sensing, a natural star close the astronomical object of interest can be used as a reference, and is named guide star. Alternatively, an artificial guide star can be created by a laser at high altitudes in the atmosphere, and is known as a *Laser Guide Star* (LGS).

The combination of AO with LGS is now at the forefront of modern astronomical observation capabilities, and applications to Earth-satellite FSO communications are ongoing. AO is a practical solution to correct for wavefront distortion at the OGS for downlinks [Can18; Gla+22], and much efforts are being deployed for uplink precompensation [Ved+21; Bia+23; Mar23; Osb+21]. It consists in purposely deforming the

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transmitted wavefront prior to its propagation through the atmosphere, ensuring it becomes planar when it reaches the satellite. It raises several challenges regarding the sensing of OT along the uplink and the angular coherence of OT.

FSOC signals and terminals design The impairments introduced by OT can also be managed at the communication level, through engineering of the transmitted signals and design of optical terminals. This includes, for example, the selection of modulation schemes, the implementation of interleaving and forward error correction coding, or the choice of the aperture size. Other system considerations must also be taken into account at this stage, such as tracking errors, noise, etc. [Can18; Art+23; SA24; GKF23]

1.2.4 Specific challenges for FSOC

As previously discussed, OT has been extensively studied for astronomical applications, and necessary mitigation techniques have been developed. However, FSOC are associated to new challenges, motivating further research in the field. In summary, these challenges include:

- **Stronger optical turbulence:** As OGSs will be located near urban areas, at low altitudes, they will face stronger turbulence, particularly due to contributions from the BL. Accurate characterization of OT within the BL remains a challenge. Additionally, potential daytime operations will lead to even stronger OT conditions. The increased strength of OT has significant consequences, as it amplifies scintillation (i.e., intensity fluctuations). This results in more severe signal fades and complicates wavefront estimation for AO control loops, with potential issues like low flux in wavefront sensors and branch cuts [Osb+21].
- **Fiber coupling and received power:** The ultimate aim of an optical communication terminal is to couple the received signal into a fiber (e.g., a single-mode fiber) for subsequent demodulation and information retrieval. However, phase perturbations on the incoming wave degrade the PSF, thereby reducing fiber coupling efficiency [Can18]. This issue justifies AO correction on the downlink to correct the wavefront prior to fiber coupling. Moreover, smaller telescopes are often used in FSOC compared to astronomy for cost efficiency, which decreases the collected signal intensity.
- **Satellite communications:** Communications with satellites often involve operating at low elevation angles, which increases the propagation distance through the atmosphere and thus amplifies OT effects. Furthermore, satellite motion, particularly for low-Earth-orbit satellites, adds extra constraints on AO control loops, requiring faster operational speeds (up to 10 kHz) [CM20].
- **Uplink precompensation and point-ahead angle:** Due to the satellite motion and the finite speed of light, the uplink beam requires a point-ahead an-

gle, i.e., it must be directed to a position slightly ahead of the satellite's current location (relative to the downlink beam). Consequently, OT sensed on the downlink differs from that on the uplink due to reduced angular coherence of OT. This introduces added challenges in estimating OT in the uplink direction [Osb+21].

1.3 Objectives of the thesis and summary of contributions

The goal of this work is to characterize OT at FSOC sites, through its modelling and its monitoring. It aims to provide insights into the following research question:

“How can we achieve accurate characterization of OT to ensure the operations of future satellite-to-ground optical communications?”

This question implicitly assumes that state of the art OT models developed for astronomical sites are used as a starting point for this work. As a result, synergies between optical astronomy and optical communications are explored. Indeed, for several decades, OT characterization has been critical for optical astronomy, as it enables:

- **Site characterization and selection**, providing useful information for choosing locations for building future observatories [OS18].
- **Flexible scheduling** of astronomical observations, based on required OT conditions. Not all science observations require the same conditions and by rationalizing their scheduling, most sensitive experiments could be executed in optimum conditions, thus reducing the cost of out-of-constraints observations [Mil+19; MLF13; OS18].
- **Instrument design and optimization**, based on expected OT at the site of interest [OS18].

Similarly, OT characterization at FSOC sites is paramount for *designing of optical communication systems* and serves several purposes, such as [CCS17]:

- **Site characterization and selection**, using long-term atmospheric statistics in order to identify ideal locations for building future OGSs.
- **Real-time decisions** regarding optical communication links, including their scheduling and handovers to other OGSs within the network when atmospheric conditions deteriorate.
- **Prediction of FSOC performance**, such as link budget estimation.
- **Development of mitigation techniques**, including AO, modulation, coding and diversity, etc.

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Achieving accurate OT characterization at FSOC sites, as expressed in the main research question of this work, raises numerous sub-questions explored in the various chapters:

- *How is atmospheric turbulence modelled and why is OT characterized by C_n^2 profiles?* (Chapter 2)
- *What is the current state of the art of C_n^2 modelling?* (Chapter 3)
- *Can high-resolution macroscale meteorological parameters collected by radiosonde measurements be used for obtaining C_n^2 profiles?* (Chapter 4)
- *Can C_n^2 profiles be estimated from Numerical Weather Prediction (NWP) simulations, for long-term forecast?* (Chapter 5)
- *What is the impact of OT on EM quantities, how can it be monitored?* (Chapter 6)
- *How do OT monitoring instruments measure C_n^2 profiles and is it possible to improve their vertical resolution?* (Chapter 7)
- *Can accurate short-term forecasts of OT be obtained by relying on Autoregressive (AR) approaches?* (Chapter 8)

These questions set the basis structure of this work, detailed in Section 1.4.

1.3.1 Thesis objectives

The main goal of this thesis leads to two distinct objectives, regarding OT modelling and OT monitoring, as well as other secondary objectives.

OT modelling OT modelling refers to the use of mathematical, physical, or even numerical models to describe OT based on local conditions and parameters that can easily be measured (e.g., temperature, wind speed, etc.). It also consists in the forecast of C_n^2 profiles from weather simulations.

The primary objective of this work, in terms of OT modelling, is to study and develop C_n^2 models that describes optical wave propagation in atmospheric turbulence, specifically for FSOC sites. It includes the application and the evaluation of C_n^2 models derived at astronomical sites, highlighting their limitations for FSOC locations. It also encompasses the derivation of C_n^2 profiles depending on local meteorological quantities, and the study of the feasibility of C_n^2 profile predictions. Validation of the applied C_n^2 models with simultaneous measurements is conducted.

OT monitoring The monitoring of OT is usually conducted using dedicated instruments, in order to obtain direct measurements of C_n^2 profiles. This work aims at studying the working principle of one of such instruments, the Shack-Hartmann Image Motion Monitor (SHIMM). It also explores the integration of OT measurements in real-time with long-term forecasts to improve short-term predictions.

Secondary objectives Secondary objectives of this work feature the study of C_n^2 models for optical frequencies and radiofrequencies, the quantification of OT on EM wave propagation, and a study of the relevant literature on C_n^2 models.

1.3.2 Summary of contributions

Three key contributions are presented in this work:

1. **Comprehensive review of C_n^2 modelling and optical turbulence:** A thorough investigation of the fundamental concepts required to describe OT and its impact on EM wave propagation is conducted. This work presents the principles essential for understanding OT from an engineering perspective, beginning with atmospheric turbulence basics and the introduction of the refractive index structure parameter C_n^2 (Chapter 2). An up-to-date and exhaustive overview of existing C_n^2 models in the literature is presented, with a particular focus on their applicability to OGS sites (Chapter 3). Additionally, the behaviour of EM propagation through turbulent media is detailed (Chapter 6). Hence, this work highlights the multidisciplinary nature of OT modelling and monitoring, offering a comprehensive entry point while providing numerous references for deeper exploration in areas such as atmospheric turbulence modelling, OT prediction from NWP simulations, and OT monitoring instrumentation.
2. **Development of C_n^2 models for satellite-to-ground optical communications:** A significant part of this thesis is dedicated to modelling C_n^2 profiles at FSOC locations. This motivates the development of new C_n^2 models for FSOC, such as a novel approach for estimating C_n^2 from high-density radiosonde profiles of meteorological parameters (Chapter 4). This approach can be applied universally, wherever radiosonde measurements are available. Alternatively, C_n^2 models based on NWP simulations are presented and applied to forecast OT at various locations, covering both daytime and nighttime conditions (Chapter 5). These models rely on the Weather Research and Forecasting (WRF) software and highlight the influence of the atmospheric BL at OGS sites, addressing the additional challenges posed by daytime OT prediction.
3. **Study of optical turbulence monitoring instruments and application to short-term forecast:** In terms of OT monitoring, this work explores the operational principles of the SHIMM instrument and present novel approaches for reconstructing C_n^2 vertical profiles from its observations. It provides a complete understanding of the instrument, supporting its use at FSOC sites (Chapter 7). Moreover, AR models are introduced for improving short-term OT prediction. These models are validated for both astronomical and optical communications sites under continuous operation, covering both daytime and nighttime conditions (Chapter 8).

Each contribution are further detailed in their associated chapters.

1.4 Thesis structure

This thesis is divided into two parts. Part I, made of Chapters 2 to 5, focuses on the modelling of C_n^2 profiles, namely at FSOC sites. Part II, consisting of Chapters 6 to 8, explores the impacts of OT on the propagation of EM waves and its monitoring.

The first part (including Chapter 6) is summarized by the schematic shown in Figure 1.8, which serves as a common thread throughout Chapters 2 to 6. First, Chapter 2 introduces atmospheric turbulence, with a focus on the refractive index and its structure parameter, C_n^2 . Chapter 3 then provides a comprehensive review of C_n^2 models from the literature, including both theoretical and empirical approaches that utilize meteorological parameters to describe C_n^2 . In Chapter 4, a method for deriving C_n^2 profiles based on its statistical definition and high-density radiosonde measurements is presented. Chapter 5 focuses on the application of NWP software for predicting C_n^2 profiles, making use of numerical models and exploiting meteorological quantities.

The second part of the thesis shifts the focus to EM propagation. It begins with Chapter 6, which examines the propagation of EM waves through turbulent media and the derivation of key OT integrated parameters. Chapter 7 follows with a presentation of the SHIMM instrument and several methods for reconstructing C_n^2 profiles. Finally, Chapter 8 integrates the work on OT modelling (notably from Chapter 5) and OT monitoring (Chapter 7) to develop short-term OT forecasts using AR models.

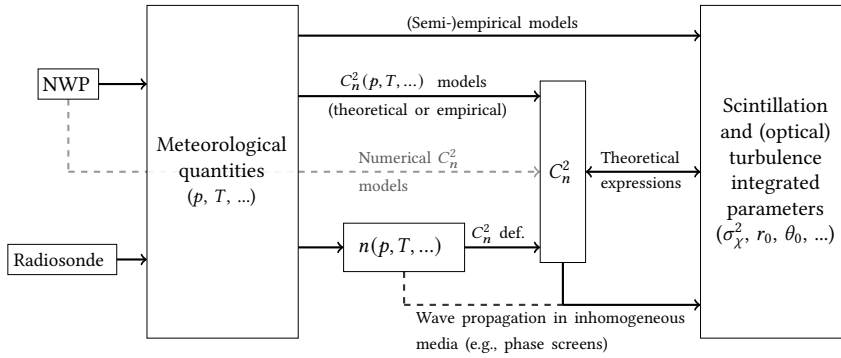


Figure 1.8 – Schematic of the first part of this work (adapted from [QVO23]).

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Part I

**Modelling of the refractive index
structure parameter profile $C_n^2(z)$**

Statistical description of refractive index fluctuations due to atmospheric turbulence

2

“Turbulence is the most important unsolved problem of classical physics.”

This quote from Richard Feynman, first presented in 1964, still holds true today. Despite recent advances in turbulence understanding and computational capabilities, the inherent complexity and seeming unpredictability of turbulent flows continue to pose significant challenges, particularly in the context of atmospheric turbulence.

The essential tools for defining, characterizing, and modelling atmospheric turbulence are presented in this chapter, covering both numerical and statistical approaches. It lays the foundation for this work, introducing the refractive index n and the refractive index structure parameter C_n^2 , as illustrated in Figure 2.1. Specifically, it provides:

- An overview and the contextual background of various approaches used in atmospheric turbulence modelling, justifying the methods chosen in this work;
- A clear identification of the hypotheses and relevant assumptions necessary for applying atmospheric turbulence model;
- A detailed parametric analysis of the statistical description approach of atmospheric turbulence, further explored in Chapter 4.

The foundation of this work is based on the refractive index structure parameter C_n^2 , which is derived from the statistical description of atmospheric turbulence. This statistical description is therefore thoroughly explored in this chapter, first by presenting the underlying mathematical concepts and the necessary physical principles, and then by applying these concepts and principles to the atmospheric refractive index.

The chapter begins with a description of the atmospheric refractive index in Section 2.1. Atmospheric turbulence is introduced in Section 2.2, highlighting its complexity and the relevant parameters. The numerical description of atmospheric turbulence is discussed in Section 2.3, while its statistical description is given in Section 2.4. Finally, the implications of atmospheric turbulence on the refractive index are detailed and illustrated in Section 2.5.

2 | Statistical description of refractive index fluctuations due to atmospheric turbulence

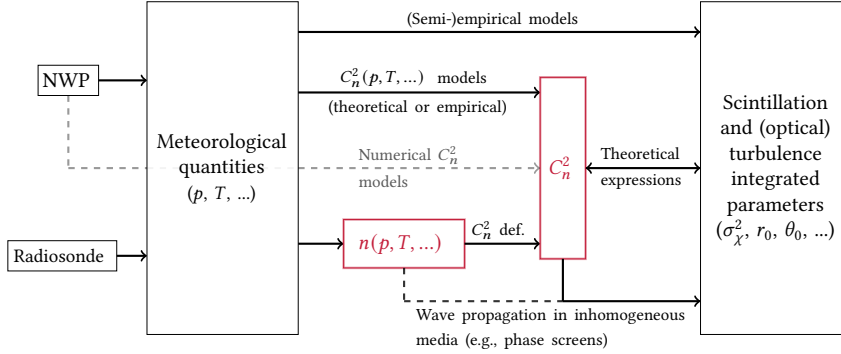


Figure 2.1 – Schematic of the thesis structure, highlighting the concepts detailed in Chapter 2.

2.1 Atmospheric refractive index

In order to fully characterize the propagation of EM waves in the atmosphere, knowledge about the atmospheric refractive index is required. Its definition is first reminded, before being particularized to the atmosphere. Refraction effects related to large-scale variations of the refractive index are also briefly introduced.

2.1.1 Definition of the refractive index

The propagation of EM waves in a medium, or a material, is impacted by its electric and magnetic properties. These properties are given by two physical quantities: the dielectric permittivity ϵ , and the magnetic permeability μ . Equivalently, they can be expressed using the medium relative permittivity ϵ_r and relative permeability μ_r , thanks to

$$\epsilon = \epsilon_r \epsilon_0 \quad \text{and} \quad \mu = \mu_r \mu_0, \quad (2.1)$$

with $\epsilon_0 = 8.854 \times 10^{-12}$ F/m the vacuum permittivity, and $\mu_0 = 4\pi \times 10^{-7}$ H/m the vacuum permeability.

The *refractive index* n of a medium, also named index of refraction, is defined by $n = \sqrt{\epsilon_r \mu_r}$ and is a dimensionless number. For non-magnetic media ($\mu_r = 1$), it is given by $n = \sqrt{\epsilon_r}$. Besides this purely mathematical definition, the refractive index has important physical motivations:

- It is the numerical factor by which the speed c_m of EM waves is decreased in a medium, i.e., $c_m = c/n$, with $c = 299792458$ m/s the speed of light in vacuum;
- It also affects the wavelength λ_m of an EM wave in this medium, following $\lambda_m = \lambda/n$, where λ is the wavelength in vacuum;
- It determines the ability of a medium to bend the propagation of EM waves at its interfaces, according to the famous Snell's law, also known as Snell-

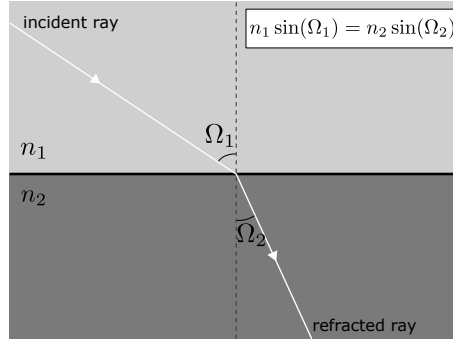


Figure 2.2 – Snell's law of refraction at the interface between two media.

Descartes law:

$$n_1 \sin(\Omega_1) = n_2 \sin(\Omega_2).$$

The refractive indices of the media are noted n_1 and n_2 , while the incident angle is Ω_1 and the refracted angle is Ω_2 , as depicted in Figure 2.2.

This latter interpretation is of particular interest for this work, as it relates refractive index variations to changes in the direction of propagating EM waves.

In the air, the refractive index is very close to unity such that, in order to identify small variations of the refractive index, the *refractivity* N has been defined (units: ppm, parts per million). It is directly related to the index of refraction using

$$N = (n - 1) \times 10^6. \quad (2.2)$$

2.1.2 Empirical expressions of atmospheric refractive index

Direct measurements of refractive indices can be achieved using dedicated tools, such as refractometers, but are not always practical [BD68]. This is especially true for obtaining vertical profiles of atmospheric refractive index, varying with the altitude.

Instead, indirect approaches relying on meteorological quantities, such as the pressure p , the temperature T , or the humidity, have been designed. They provide the refractive index of air as a function f of meteorological quantities, i.e., $n = f(\lambda, p, T, \dots)$. Expressions for f have been measured empirically, and depend on the wavelength of the considered EM wave [And88].

2.1.2.1 Optical refractive index

At optical frequencies, the refractive index is given by [AP05; Whe01]

$$n = 1 + 10^{-6} \frac{77.6 p}{T} \left(1 + \frac{7.52 \times 10^{-3}}{\lambda^2} \right), \quad (2.3)$$

2 | Statistical description of refractive index fluctuations due to atmospheric turbulence

with p the pressure expressed in hectopascal [hPa], T the temperature in kelvin [K], and λ the wavelength in micrometer [μm]. Equation (2.3) is valid for wavelengths between 0.36 and 3 μm according to [And88], following the work of [Owe67; HL86]. It is often approximated by its value for $\lambda = 0.5 \mu\text{m}$, leading to

$$n = 1 + 10^{-6} \frac{80 p}{T}. \quad (2.4)$$

More complex expressions of the optical refractive index can be found in the literature, namely in [McK16] or in International Telecommunication Union (ITU) recommendation ITU-R P.1621-2 [ITU15]. Most of those expressions rely on the work of Edlén [Edl66] or Owens [Owe67], with some recent updates [BD93; BD94].

Based on (2.3) and (2.4), one can notice that the atmospheric refractive index at optical frequencies does not depend on the humidity. Indeed, only a small dependence with the humidity is observed [Kun96], such that it is very often neglected. This is not true for infrared frequencies, where the effect of humidity must be taken into account. Expressions of the atmospheric refractive index in the infrared region can be found in [And88; HL86; Tun03].

2.1.2.2 Radiofrequency refractive index

For radiofrequencies, it is commonly accepted that the refractive index of air is given by [Whe01; AP05; Ish78]

$$n = 1 + 10^{-6} \frac{77.6}{T} \left(p + \frac{4810 e}{T} \right), \quad (2.5)$$

where p is the atmospheric pressure in hectopascal [hPa], T is the temperature in kelvin [K], and e is the partial pressure of water vapor in hectopascal [hPa]. Equation (2.5) is valid for frequencies ranging from 1 MHz to 30 GHz [Whe01], and in the temperature range from -50 to 40°C. In fact, (2.5) is an approximation of a more general expression presented in [BD68; SW53; ITU19]

$$n = 1 + 10^{-6} \left(77.6 \frac{p}{T} - 5.6 \frac{e}{T} + 3.75 \times 10^5 \frac{e}{T^2} \right). \quad (2.6)$$

Both expressions (2.5) and (2.6) do not depend on the frequency of the EM wave that propagates in the atmosphere.

2.1.3 Refraction effects in the atmosphere

Large-scale (slow) variations of the atmospheric refractive index namely originate from the height dependency of meteorological quantities, while small-scale (fast) fluctuations are related to atmospheric turbulence (discussed in Section 2.2). These large-scale variations are responsible for refraction effects in the atmosphere, bending the propagation of EM waves.

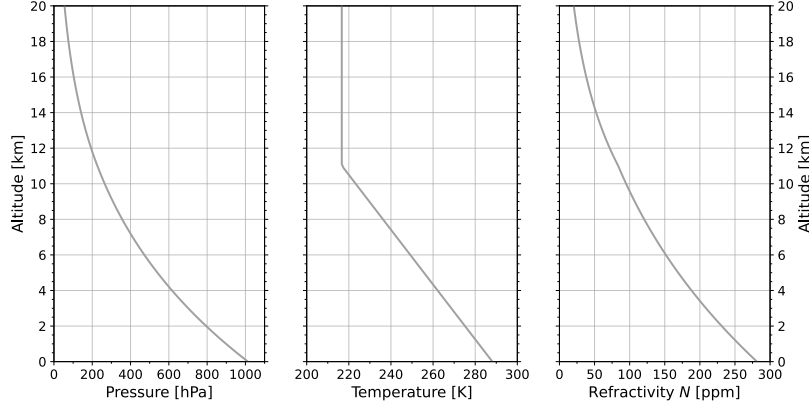


Figure 2.3 – Illustration of standard atmosphere pressure, temperature and refractivity profiles.

As an example, Figure 2.3 presents the vertical profile of the refractivity obtained using the optical refractive index equation (2.4) with pressure and temperature profiles from the International Civil Aviation Organization (ICAO) Standard Atmosphere [Int93]. It shows that refractivity is the largest close to ground (reaching 280 ppm), and decreases with the altitude.

Atmospheric refraction can impact the required pointing on Earth to communicate with satellites, as well as modify the propagation delay of the waves [MW15]. It is also responsible for the angular displacement of astronomical objects from their true position, a phenomenon often referred to as astronomical refraction [McK16]. This phenomenon is illustrated in Figure 2.4a, where R_e is the Earth radius, H_a is the atmosphere height, and δ_d is the deviation angle, i.e., the difference between real geometrical angle and the observed angle¹. In order to determine this deviation, approaches considering a vertically or spherically stratified atmosphere have been presented [BD68; McK16].

In the following, a spherically stratified atmosphere is assumed, and a methodology to obtain an approximation of the deviation angle is presented, such that its order of magnitude can be determined. For spherically stratified media, the refractive invariant is derived from the application of Snell's law [You06], as depicted in Figure 2.4b. The radii of each layer interface are denoted r_i , with $i \in \mathbb{N}$. As an example, r_0 can be taken as the Earth radius. The angles Λ_i are the incident angles at each interface, while the angles Ω_i are the refracted angles after the interface (in this case, Ω_0 is the elevation angle ϑ measured from zenith at the ground). Application of Snell's law to interfaces 1 and 2 leads to

$$n_0 \sin(\Lambda_1) = n_1 \sin(\Omega_1) \quad \text{and} \quad n_1 \sin(\Lambda_2) = n_2 \sin(\Omega_2). \quad (2.7)$$

¹In other words, due to atmospheric refraction, an observer locates the star following the dashed line in Figure 2.4a, instead of getting the star true position (plain line).

2 | Statistical description of refractive index fluctuations due to atmospheric turbulence

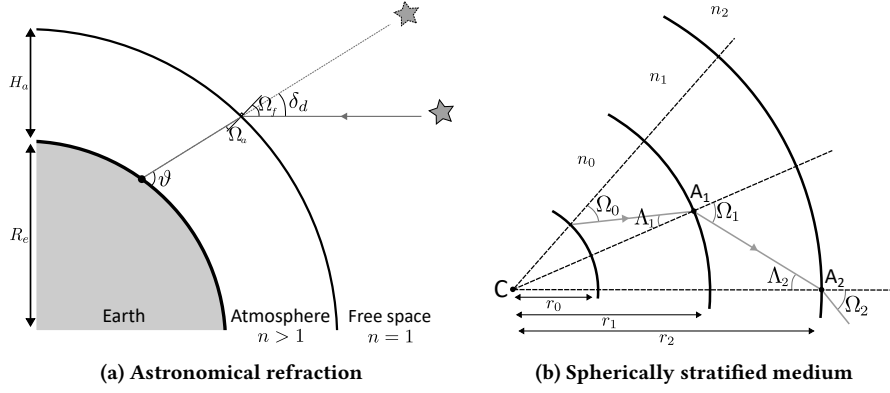


Figure 2.4 – Illustrations of astronomical refraction and of a spherically stratified medium.

Moreover, the law of sines applied to the triangle CA_1A_2 gives

$$\frac{\sin(\pi - \Omega_1)}{r_2} = \frac{\sin(\Lambda_2)}{r_1} \quad \Leftrightarrow \quad r_1 \sin(\Omega_1) = r_2 \sin(\Lambda_2). \quad (2.8)$$

Using (2.7) and (2.8), one obtains

$$n_1 r_1 \sin(\Omega_1) = n_2 r_2 \sin(\Omega_2) \quad \Rightarrow \quad n_i r_i \sin(\Omega_i) = \text{constant} \quad \forall i.$$

The generalization of this observation to a continuous atmosphere with a refractive index $n(r)$ leads to the refractive invariant

$$n(r)r \sin(\Omega) = \text{constant}.$$

Vertical integration of this invariant, as presented in [You06; BA23], enables to accurately obtain the deviation angle. Alternatively, a simpler model considering that the atmosphere is made of a single layer of thickness H_a and refractive index n_{atm} can be used, as in [ITU15]. In this case, the propagation in the atmosphere is a straight line, and the deviation angle is obtained thanks to the application of Snell's law and the refractive invariant,

$$\delta_d = \Omega_f - \Omega_a \quad \text{with} \quad \begin{cases} n_{\text{atm}} \sin(\Omega_a) = \sin(\Omega_f), \\ n_{\text{atm}} R \sin(\frac{\pi}{2} - \vartheta) = (R_e + H_a) \sin(\Omega_f), \end{cases}$$

where Ω_f and Ω_a are the incident and refracted angles at the interface between the atmosphere and free space (see Figure 2.4a). This gives

$$\delta_d = \arcsin\left(\frac{n_{\text{atm}} R_e \cos(\vartheta)}{R_e + H_a}\right) - \arcsin\left(\frac{R_e \cos(\vartheta)}{R_e + H_a}\right). \quad (2.9)$$

Despite the simplicity of the model, (2.9) provides the order of magnitude of the deviation angle that can be expected. For example, assuming a single atmospheric layer of height $H_a = 10$ km, with a fixed refractive index $n_{\text{atm}} = 1.000250$, and the Earth radius R_e being 6371 km, one obtains a deviation angle $\delta_d = 0.0143^\circ$ for an elevation angle $\vartheta = 45^\circ$. This result is similar to the values presented in [McK16] that uses a vertically stratified atmosphere (e.g., a deviation of 0.0160° is obtained for $\vartheta = 45^\circ$ and for a wavelength $\lambda = 486$ nm). For $\vartheta = 90^\circ$, i.e., at zenith, the deviation angle given by (2.9) is zero, as expected.

For optical communication, deviation angles are not an issue, since they are corrected by satellite tracking. However, they start to have an influence when wavelengths used for tracking and communication are different. In this case, both wavelengths suffer from their own deviation angles, and the difference between those angles can be as large as $60 \mu\text{rad}$ ($= 0.00344^\circ$) when considering an elevation angle of 5° , a pointing wavelength of 500 nm, and a communication wavelength of 1550 nm [BA23].

Finally, it is worth mentioning that more complex approaches have been designed to compute refraction effects in the atmosphere. Most of them rely on ray tracing propagation and on the associated eikonal equation, derived from geometrical optics [Whe01]. For example, this enables to use numerical modelling of the atmosphere at a particular location and a given time, compute the atmospheric refractive index in the modelled volume, and then compute the ray trajectories in this volume [Nun+15].

2.2 Atmospheric turbulence

Atmospheric turbulence originates from wind shear and convective motions due to temperature (or humidity) gradients in the atmosphere. It can be observed in the daily life, for example looking at the motion of flags, at plume patterns rising from stacks (chimneys), or at the irregular shapes of clouds [Kun96].

Even though those phenomena are quite common, the turbulence itself is difficult to define, as explained in Section 2.2.1. It is assumed to follow Navier-Stokes equations presented in Section 2.2.2, and only arises based on the conditions presented in Section 2.2.3. It can be described by relying on numerical or statistical approaches (Sections 2.3 and 2.4).

2.2.1 Definition of turbulence

Turbulence is associated with the chaotic motion of a fluid, leading to irregular fluctuations in velocity, pressure, and other quantities of the flow. This vague definition arises from the nature of turbulence itself and from our current understanding of the phenomenon. Indeed, it is still considered as one of the greatest unsolved problem in classical physics, as we cannot explain in details why turbulence occurs, nor predict it with sufficient reliability in any conditions [McD07].

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Throughout the years, turbulence has first been considered as a completely random phenomenon, that can be studied statistically. More recently, new research directions suggesting that turbulence can be a deterministic, though chaotic, solution to the Navier-Stokes equations have been explored [CT85]. These statistical and deterministic considerations of turbulent flows are indicative of the difficulty of defining turbulence, and lead to different approaches and models to describe it. Nowadays, the deterministic description of turbulence is associated to numerical models, solving the Navier-Stokes equations on a numerical Three-Dimensional (3D) grid. It is presented in Section 2.3. However, the main approach of interest for this work is the statistical one, relying on Kolmogorov theory, that is introduced in Section 2.4.

Nevertheless, whatever the description chosen (statistical or deterministic), common features have been established for turbulent flows. They are found to exhibit disorganized and chaotic 3D behaviours, that are sensitive to the initial conditions. They also span over a large range of time and length scales, as well as enhance diffusion and dissipation [McD07].

2.2.2 Navier-Stokes equations and Reynolds decomposition

The Navier-Stokes equations express the conservation of mass and the conservation of momentum for Newtonian fluids. For viscous and incompressible fluids, they are [LK16, p. 23]

$$\nabla \cdot \mathbf{u} = 0 \quad (2.10)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \mathcal{F}, \quad (2.11)$$

where $\mathbf{u} = (u, v, w)^T$ is the velocity vector, depending on the spatial coordinates (x, y, z) . The pressure is denoted by p , while $\mathcal{F}$ represents an external force acting on the fluid (per unit of mass), e.g., gravitation. The density of the fluid is ρ , and ν its kinematic viscosity. The different operators used are: ∇ for the gradient, $\nabla \cdot$ for the divergence, and ∇^2 for the Laplacian operator. Equation (2.10) is associated with the conservation of the mass of the flow, while (2.11) comes from its conservation of momentum.

Defining $\mathbf{u} = (u_1, u_2, u_3)^T = (u, v, w)^T$ and $\mathbf{r} = (x_1, x_2, x_3)^T = (x, y, z)^T$, and using the Einstein notation for the summation over repeated indices ($u_i u_i := \sum_{i=1}^3 u_i u_i$ for example), (2.10) and (2.11) can be rewritten

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (2.12)$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mathcal{F}_i, \quad i = 1, 2, 3. \quad (2.13)$$

In the atmosphere, the external force $\mathcal{F}$ (or $\mathcal{F}_i$) includes the gravitational force and the Coriolis force [Stu88, p. 77].

2.2.2.1 Reynolds decomposition

In order to study the average motion of flows and turbulence, Reynolds decomposition is often used. In this decomposition, each quantity of interest is separated into a mean (time-averaged) component, and a fluctuating component. As an example, the velocity vector is now expressed as

$$\mathbf{u}(\mathbf{r}, t) = \bar{\mathbf{u}}(\mathbf{r}) + \mathbf{u}'(\mathbf{r}, t),$$

where $\bar{\mathbf{u}}$ is the mean component, and $\mathbf{u}'$ depicts the fluctuations.

By substituting all quantities in Navier-Stokes equations (2.10) and (2.11) with their Reynolds decomposition, and by averaging (over time) the resulting equations, one obtains [McD07]

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \quad (2.14)$$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial \overline{u'_i u'_j}}{\partial x_j} + \bar{\mathcal{F}}_i. \quad (2.15)$$

The second equation (momentum equation) is often expressed using the mean rate strain tensor $\bar{S}_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right)$, leading to

$$\rho \frac{\partial \bar{u}_i}{\partial t} + \rho \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left[-\bar{p} \delta_{ij} + 2\nu \bar{S}_{ij} - \rho \overline{u'_i u'_j} \right] + \rho \bar{\mathcal{F}}_i,$$

with δ_{ij} the Kronecker delta, equal to 1 if $i = j$ and 0 otherwise. The derivative with respect to time $\frac{\partial \bar{u}_i}{\partial t}$ is 0 by definition of the mean component, but this is often ignored to keep the temporal evolution of the mean components [McD07]. The dynamic viscosity of the fluid is ν , linked to its kinematic viscosity ν thanks to $\nu = \frac{\nu}{\rho}$.

Equation (2.15) is similar to (2.13), the only difference being the addition of a new turbulent term. What is of particular interest is to notice that, due to the Reynolds decomposition, *six new unknowns* are introduced in the equations: $\overline{u'_i u'_j}$, the so-called *turbulent stresses*. This means that, *even for forecasting mean quantities of the flow, turbulence must be considered*. Hence, an additional model for describing the turbulent stresses is required to be able to solve the equations, i.e., having as many equations as unknowns. This is referred to as the *turbulence closure problem*. A similar development can be applied to equations describing the conservation of heat and passive scalars (e.g., the temperature T , the humidity, etc.), again introducing new unknowns, named *turbulent fluxes* (e.g., $\overline{u'_i T'}$) [Stu88, p. 90-92], [Saa11].

In the literature, numerous turbulence closure models have been developed. They are classified based on the additional partial differential equations they include, and on the order of the statistical moments they model [Stu88, Chapter 6], [Saa11; Han04]. Indeed, (2.15) involves the second statistical moments $\overline{u'_i u'_j}$. If one tries to obtain prognostic equations for those turbulent stresses (or fluxes), they will involve the

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third statistical moments, e.g., $\overline{u'_i u'_j u'_k}$. Therefore, extra unknowns are added, that also need new models or equations to predict them. One can keep going indefinitely, the turbulence closure problem will get more and more complicated to solve, and never be closed.

As an example, the gradient transport theory, or K-theory, is a local closure model of first order [Stu88, p. 204]. Turbulent stresses and fluxes are parameterized assuming that they are proportional to their gradients, i.e.,

$$\overline{u'_i u'_j} = -K \frac{\partial \bar{u}_j}{\partial x_i},$$

with K a scalar parameter in $[\text{m}^2\text{s}^{-1}]$, often determined empirically.

Turbulence closure models of order two seeks for parametrization of the third statistical moments, while models of order three require parameterization of the fourth statistical moments, etc. In general, it is assumed that the higher the order of the closure model, the more accurate the predictions on the lower-order variables. However, high-order moments are hard to parametrize accurately, and even harder to measure in the atmosphere. A review of existing turbulence closure models can be found in [Stu88, Chapter 6].

2.2.2.2 Turbulent kinetic energy

Some turbulent closure models involve the Turbulent Kinetic Energy (TKE), denoted k_ϵ in this work. It is defined as

$$k_\epsilon = \frac{1}{2} \overline{u'_i u'_i} = \frac{1}{2} \left(\overline{(u')^2} + \overline{(v')^2} + \overline{(w')^2} \right), \quad (2.16)$$

hence involving the sum of the variances of the velocity components. From the Navier-Stokes equations and using the Reynolds decomposition, it is possible to obtain a prognostic equation for the TKE that can be added to the turbulence closure model. In the case of the atmosphere, the TKE equation is [Stu88, p. 152],[FN08]

$$\frac{\partial k_\epsilon}{\partial t} + \bar{u}_j \frac{\partial k_\epsilon}{\partial x_j} = \underbrace{\delta_{i3} \frac{g}{\bar{\theta}_v} \left(\overline{u'_i \theta'_v} \right)}_{\mathcal{P}_I} - \underbrace{\overline{u'_i u'_j} \frac{\partial \bar{u}_i}{\partial x_j}}_{\mathcal{P}_{II}} - \underbrace{\frac{\partial \left(\overline{u'_j k'_\epsilon} \right)}{\partial x_j}}_{\mathcal{D}_I} - \underbrace{\frac{1}{\rho} \frac{\partial \left(\overline{u'_i p'} \right)}{\partial x_i}}_{\mathcal{D}_{II}} - \epsilon, \quad (2.17)$$

with θ_v the virtual potential temperature, defined in (2.19) in Section 2.2.3, and g the gravitational acceleration. The first term on the right side of (2.17), $\mathcal{P}_I$, corresponds to the buoyancy production (consumption) term, while $\mathcal{P}_{II}$ is the mechanical shear production (loss) term. $\mathcal{D}_I$ denotes the turbulent transport of TKE, and $\mathcal{D}_{II}$ describes how TKE is redistributed by pressure fluctuations. Finally, ϵ represents the viscous dissipation of TKE, i.e., its conversion to heat by viscous forces (see Section 2.4.2).

Closure of the TKE equation is achieved thanks to extra models of these different terms, sometimes introducing empirical coefficients evaluated empirically or with

direct numerical simulations (see Section 2.3.1) [Han04]. More details about the TKE and the turbulent fluxes in the BL are given in Chapter 5 of [Stu88]. The major advantage of relying on the TKE is that, thanks to the TKE equation, one can balance TKE production and loss terms to determine the future increase or decay of turbulence in the BL.

2.2.3 Conditions for turbulence

Turbulent conditions are dependent on the Reynolds and Richardson numbers, leading to the combined criterion presented in Section 2.2.3.3.

2.2.3.1 Reynolds number

Traditionally, the Reynolds number is used to determine whether a flow is turbulent. It is defined as the ratio between the kinetic and viscous forces in the Navier-Stokes equations. Indeed, defining a characteristic velocity U and a length scale L , the magnitude of the kinetic term in (2.11), i.e., $\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$, is given by $\frac{U^2}{L}$. The magnitude of the viscous term $\nu \Delta \mathbf{u}$ is $\nu \frac{U}{L^2}$, such that the Reynolds number Re is

$$Re = \frac{\frac{U^2}{L}}{\nu \frac{U}{L^2}} = \frac{UL}{\nu} = \frac{\rho UL}{\mu}. \quad (2.18)$$

The Reynolds number sets the limit between laminar and turbulent flows. Flows become turbulent at high Reynolds numbers, where kinetic effects are dominant. On the contrary, for low Reynolds numbers, viscous effects stabilize the flow such that it remains laminar. The critical value, based on experimental work, is around 2000-2100 [Ver02], [Stu88, p. 353].

In the atmosphere, Reynolds numbers are typically in the range of 10^6 to 10^8 . For example, in the SL, wind shear U of 5 m/s are typical, within a distance L of 100 m. Since the kinematic viscosity of air is $1.5 \times 10^{-5} \text{ m}^2/\text{s}$, the Reynolds number is 3×10^7 [Stu88, p. 93].

2.2.3.2 Richardson number

The Reynolds number is not enough to characterize the turbulence of the atmosphere, as it is stratified. In this case, the Richardson number must also be taken into account.

Stratification of the atmosphere arises due to buoyancy, coming from the variable density between different parcels of air. It is associated to the notion of *static stability*, i.e., not depending on the wind. Static stability of the atmosphere, and especially the SL, is studied thanks to the vertical profile (and gradient) of the potential temperature.

The potential temperature θ is defined as the temperature that an air parcel initially at temperature T and pressure p would reach if moved adiabatically to a standard reference pressure p_0 (usually, $p_0 = 1000 \text{ hPa}$). Mathematically,

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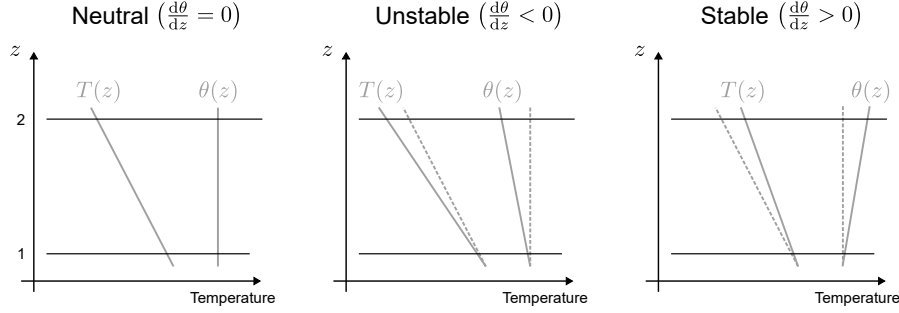


Figure 2.5 – Static stability criterion: illustration of three cases, with dashed lines corresponding to the neutral conditions (adapted from [Ver02]).

$$\theta = T \left(\frac{p_0}{p} \right)^{R/c_p}, \quad (2.19)$$

where θ and T are in kelvin, R is the gas constant of air, and c_p is the specific heat capacity at constant pressure. The ratio $\frac{R}{c_p}$ for air is close to 0.286.

The potential temperature is a conservative and passive additive², meaning that it keeps its property when moved in the atmosphere [Ver02]. As such, an air parcel of potential temperature θ_1 that is moved upwards to an area of the atmosphere with potential temperature θ_2 , such that $\theta_2 > \theta_1$, will tend to sink down afterward, due to buoyancy. In this case, the atmosphere is stable. On the contrary, if $\theta_2 < \theta_1$, then the air parcel will keep rising, leading to (static) instability. This is summarized in Figure 2.5, inspired from [Ver02], and valid for a dry atmosphere. In order to take into account humidity effects in the stability, the virtual potential temperature θ_v must be used. It is defined by

$$\theta_v = \theta(1 + 0.61 q_v), \quad (2.20)$$

for unsaturated air with mixing ratio q_v [Stu88, p. 7].

Furthermore, even in the case of static stability of the air, wind shears can generate turbulence dynamically. This is expressed with the concept of *dynamic stability*, directly depending on the wind [Stu88, p. 172].

The Richardson number enables to link both notions of stability, i.e., static and dynamic, based on the TKE equation (2.17). Rigorously, one defines the *flux Richardson number* as the ratio between the buoyancy production term $\mathcal{P}_I$ and the mechanical

²According to [Rod81, p. 286], an additive is said to be passive if it does not affect the dynamics of turbulence, and conservative if it does not disappear by chemical reaction.

shear production term $\mathcal{P}_{II}$, that is,

$$\text{Ri}_f = \delta_{i3} \frac{\frac{g}{\theta_v} \left(\overline{u'_i \theta'_v} \right)}{\overline{u'_i u'_j} \frac{\partial \bar{u}_i}{\partial x_j}} = \frac{\frac{g}{\theta_v} \left(\overline{w' \theta'_v} \right)}{\overline{u' w'} \frac{\partial \bar{u}}{\partial z} + \overline{v' w'} \frac{\partial \bar{v}}{\partial z}}, \quad (2.21)$$

where the second equality is obtained by using the definition of u_i and assuming horizontal homogeneity and neglecting the subsidence (mean vertical velocity $\bar{w}$) [Stu88, p. 175]. It follows that if buoyancy effects are stronger than wind shear effects ($\text{Ri}_f > 1$), the atmosphere is dynamically (and statically) stable. If this is not the case ($0 < \text{Ri}_f < 1$), then it is considered dynamically unstable. Finally, if the flux Richardson number is negative, the atmosphere is statically unstable.

The definition of the flux Richardson number is difficult to use in practice, since it involves second order moments (the turbulent stresses and fluxes). Hence, they are often substituted based on the gradient transport theory (introduced above), leading to the *gradient Richardson number*, denoted Ri :

$$\text{Ri} = \frac{\frac{g}{\theta_v} \frac{\partial \bar{\theta}_v}{\partial z}}{\left[\left(\frac{\partial \bar{u}}{\partial z} \right)^2 + \left(\frac{\partial \bar{v}}{\partial z} \right)^2 \right]}. \quad (2.22)$$

It is usually referred to as the Richardson number, without specifying that it is the gradient expression [Stu88, p. 176]. Sometimes, the vertical gradients are approached using differences between discrete height intervals, e.g., $\frac{\partial \bar{\theta}_v}{\partial z} = \frac{\Delta \bar{\theta}_v}{\Delta z}$, leading to the so-called *bulk Richardson number* Ri_b . Moreover, some authors also define the Richardson number with the temperature T or the potential temperature θ , instead of the virtual potential temperature θ_v [FN08; Ver02].

With the (gradient) Richardson number, a flow is said to be turbulent if $\text{Ri} < 0.25$, while it is laminar (stabilized by buoyancy) if $\text{Ri} > 1$. Between those two values, flow characteristic depends on its history: it will remain laminar if it comes from a region where $\text{Ri} > 1$, or remains turbulent in the contrary case.

Finally, two important remarks can be made about the Richardson number [Stu88, p. 177]:

- It does not provide any information about the intensity of the turbulence, only about its presence or absence;
- The critical values of 1 and 0.25 given above only applies to local gradients. When finite differences across thick layers are used, gradients are smoothed and the occurrence of turbulence cannot be predicted accurately.

This second remark explains why it is difficult to predict Richardson number based on radiosonde measurements or NWP simulations, that inherently have a limited vertical resolution [MM10; Mar+17].

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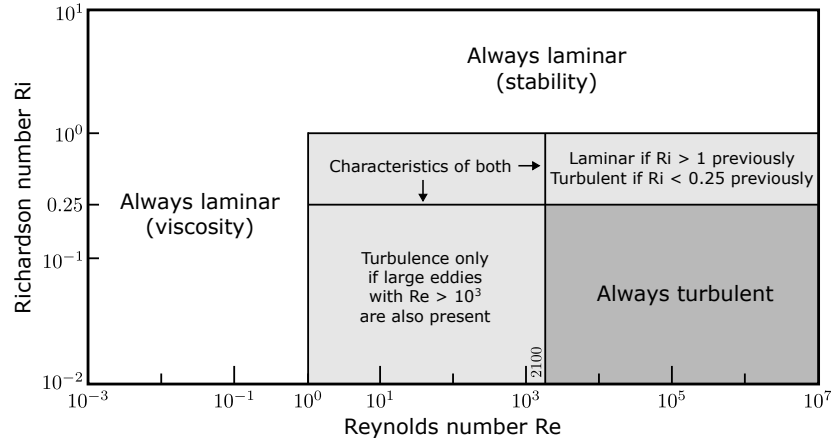


Figure 2.6 – Stability parameter relationships (reproduced from [Stu88, p. 186]).

2.2.3.3 Combined criterion

In conclusion, the occurrence of turbulence in the atmosphere is neither solely based on the Reynolds number, nor on the Richardson number. A combination of both numbers is required to account for stabilization effects arising from the viscosity or from the buoyancy. Figure 2.6 summarizes these results, highlighting the different critical values [Stu88, p. 186],[Woo69].

2.3 Numerical description of turbulence

Numerical models of turbulence belongs to the field of Computational Fluid Dynamics (CFD), aiming at solving fluid motions thanks to numerical analysis. Three categories of models are of interest for modelling atmospheric turbulence: Direct Numerical Simulations (DNS), Reynolds-Averaged Navier-Stokes (RANS) equations, and Large Eddy Simulations (LES). They are all based on the Navier-Stokes equations presented in Section 2.2.2.

2.3.1 Direct numerical simulations

In DNS, the Navier-Stokes equations are directly solved on a 3D mesh, including all length scales of the turbulence (from the largest scales up to the dissipation scale). The benefit of DNS is in its accuracy of modelling turbulent flows, since they do not involve extra models to represent the turbulence. However, they are very computationally expensive. It can be shown that the number of arithmetic operations of DNS scales with the third power of the Reynolds number (i.e., Re^3), making such simulations unpractical for most applications [McD07]. Hence, with current computer capabilities, DNS can only be used for research in understanding the physics of turbulence in simple geometries, at relatively low Reynolds numbers.

Nevertheless, DNS have been recently used to model the turbulent atmospheric BL, and obtain parameterization of important quantities of the turbulence [He16; BDH21]. Future improvements of computational power and memory storage will also promote the development of more complex DNS.

2.3.2 Reynolds-averaged Navier-Stokes equations

RANS equations are based on the Reynolds decomposition of the flow quantities, given in Section 2.2.2, corresponding to (2.14) and (2.15). As already introduced, due to the turbulence closure problem, methods based on RANS equations require modelling of the turbulent stresses and fluxes. However, they do not suffer from computational limitations, since their arithmetic is nearly independent of the Reynolds number [McD07].

2.3.3 Large Eddy Simulations

LES is an alternative and hybrid approach to DNS and RANS. It applies a low-pass spatial filter to the Navier-Stokes equations, such that large-scale motions (e.g., large eddies) are computed directly, similar to DNS, while small-scale motions are modelled, as in RANS [McD07]. This allows LES to fully capture large eddies, which contain most of the turbulent energy and are responsible for the majority of momentum transfer and turbulent mixing, while small-scale motions, which tend to be more isotropic and homogeneous, are approximated by models [Zhi15]. The derivation of LES equations can be found in [McD07].

2.4 Statistical description of turbulence

As an alternative to numerical models of atmospheric turbulence, statistical models exploit the intrinsic stochastic behaviour of turbulence. Such models are based on the mathematical formalism of stochastic processes and random fields, that is first reminded in Section 2.4.1.1. Then, its application to the field of turbulence is presented, relying on the energy cascade theory (Section 2.4.2) and the Kolmogorov power spectrum (Section 2.4.3).

2.4.1 Stochastic processes and random fields

Turbulence is stochastic in time t and in space $\mathbf{r}$, making it a space-time random field. In the following, time variations are first considered. Next, spatial fluctuations are detailed separately, before considering temporal and spatial variations jointly. This is the approach often used in textbooks describing atmospheric turbulence for wave propagation [Tat71; Ish78; AP05]. This section is quite theoretical, but provides all the necessary equations and hypotheses used in the following of this work, summarized in Section 2.4.1.5.

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2.4.1.1 Stochastic processes

Given $f(t)$ a function of the time t , f is said to be a stochastic process, or random process, if $f(t)$ is a random variable for each particular value of t . Equivalently, a stochastic process can be seen as a collection of different realisations of time functions $f(t)$, each of them being associated to a probability [AP05, p. 37].

At a given instant t , the Probability Density Function (PDF) of $f(t)$ is given by $p_f(f, t)$, while its Cumulative Distribution Function (CDF) is $F_f(f, t) = \Pr[f(t) \leq f]$, where $\Pr[.]$ denotes the probability. These quantities are the first-order PDF and CDF, respectively. The stochastic process is fully characterized if one has access to the m -th order joint distribution where m tends to infinity,

$$\lim_{m \rightarrow \infty} p_f(f(t_1), f(t_2), \dots, f(t_m)),$$

with $t_1, \dots, t_m$ being a sequence of sampled times.

The *mean* of such a process, expressed as an *ensemble average*, is a deterministic function of t , that is,

$$\bar{f}(t) = \langle f(t) \rangle = \int_{-\infty}^{+\infty} f(t) p_f(f, t) df,$$

the bracket notation $\langle \cdot \rangle$ depicting this ensemble average. Any stochastic process can be written as $f(t) = \bar{f}(t) + f'(t)$, where $f'(t)$ denotes the fluctuations of $f(t)$ around its mean, such that $\langle f'(t) \rangle = 0$.

The *variance* of $f(t)$ is also a deterministic function of t ,

$$\sigma_f^2(t) = \langle (f(t) - \bar{f}(t)) (f(t) - \bar{f}(t))^* \rangle = \langle f'(t) (f'(t))^* \rangle.$$

Moreover, the *covariance function* $B_f(t_1, t_2)$ (or autocovariance function) between two instants t_1 and t_2 is given by

$$B_f(t_1, t_2) = \langle (f(t_1) - \bar{f}(t_1)) (f(t_2) - \bar{f}(t_2))^* \rangle.$$

By definition, $B_f(t, t) = \sigma_f^2(t)$. The *correlation function* (or autocorrelation function) is defined by

$$R_f(t_1, t_2) = \langle f(t_1) f^*(t_2) \rangle.$$

Stationarity A stochastic process is said to be *stationary* (wide sense stationarity) if:

1. its mean is constant with time, i.e., $\bar{f}(t) = \bar{f} = \text{const}$;
2. its covariance function only depends on the time difference, that is,

$$B_f(t_1, t_2) = B_f(t_1 - t_2, 0) = B_f(\tau) = \langle (f(t + \tau) - \bar{f})(f(t) - \bar{f})^* \rangle,$$

with $\tau = t_1 - t_2$ the time difference.

It follows that the correlation function is also a function of the time difference τ , and is given by

$$R_f(\tau) = B_f(\tau) + \bar{f}^2.$$

Stationarity with increments Very often in turbulence, the mean values can be considered constant only for short periods of time and the process is not stationary. However, the stochastic function $f(t_1+t) - f(t_1)$ can behave like a stationary process considering its mean is slowly varying, even though $f(t)$ is not stationary [AP05, p. 40]. For such processes, assuming they are real, the *structure function* $D_f(t_1, t_2)$ is defined by

$$\begin{aligned} D_f(t_1, t_2) &= \langle (f(t_1) - f(t_2))^2 \rangle \\ &= \langle (\bar{f}(t_1) - f'(t_1) - \bar{f}(t_2) + f'(t_2))^2 \rangle \\ &= \langle (\bar{f}(t_1) - \bar{f}(t_2))^2 \rangle + \langle (f'(t_2) - f'(t_1))^2 \rangle \\ &\quad + 2 \langle (\bar{f}(t_1) - \bar{f}(t_2))(f'(t_1) - f'(t_2)) \rangle \\ &= \langle (\bar{f}(t_1) - \bar{f}(t_2))^2 \rangle + \langle (f'(t_2) - f'(t_1))^2 \rangle \\ &\approx \langle (f'(t_2) - f'(t_1))^2 \rangle, \end{aligned}$$

since $\bar{f}(t)$ is not a random function and $\langle f'(t) \rangle = 0$. The final approximation is only valid if the mean is slowly varying, and motivates the use of structure functions [Tat71, p. 16].

Based on this definition of the structure function $D_f(t_1, t_2)$, the process $f(t)$ is said to be *stationary with increment* if [Ish78, p. 521]:

1. the ensemble average $\langle f(t+\tau) - f(t) \rangle$ is independent of t and depends only on τ . In other words, the mean is slowly varying, linearly with time, e.g., at first order using Taylor series [Tat71, p. 15];
2. the structure function depends only on the time difference $\tau = t_1 - t_2$:

$$D_f(t_1, t_2) = D_f(t_1 - t_2, 0) = D_f(\tau) = \langle (f(t+\tau) - f(t))^2 \rangle.$$

In this case, the structure function describes the fluctuations having a period less than or comparable with τ [Tat71, p. 16].

As stated in [AP05, p. 40], a *stationary process* is a particular case of a process with stationary increments. It follows from the definitions, for real processes:

$$\begin{aligned} D_f(\tau) &= \langle (f(t+\tau) - f(t))^2 \rangle \\ &= \langle f^2(t+\tau) \rangle + \langle f^2(t) \rangle - 2 \langle f(t+\tau) f(t) \rangle \\ &= 2 (B_f(0) - B_f(\tau)). \end{aligned} \tag{2.23}$$

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Ergodicity Ergodic stochastic processes are such that ensemble averages can be substituted by temporal averages, that is,

$$\bar{f}(t) = \langle f(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt,$$

with T the duration of the time-window during which the process is observed. Ergodicity is usually assumed for stationary processes [AP05, p. 42].

Power spectral density For a stationary process, the Fourier transform of the covariance function $B_f(\tau)$ is named *power spectral density* (or *power spectrum*), denoted $W_f(\omega)$, with ω the pulsation, in [rad/s]. Their Fourier transform relationships are, following the convention in [AP05, p. 43]

$$W_f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B_f(\tau) e^{-j\omega\tau} d\tau,$$

$$B_f(\tau) = \int_{-\infty}^{\infty} W_f(\omega) e^{j\omega\tau} d\omega.$$

These relations are known as the Wiener-Khinchin theorem. They may be transformed to cosine transforms if $f(t)$ is real:

$$W_f(\omega) = \frac{1}{\pi} \int_0^{\infty} B_f(\tau) \cos(\omega\tau) d\tau, \quad (2.24)$$

$$B_f(\tau) = 2 \int_0^{\infty} W_f(\omega) \cos(\omega\tau) d\omega. \quad (2.25)$$

$W_f(\omega)$ provides a frequency domain representation of the temporal correlations of the stochastic process. Assuming ergodicity, it can be estimated using time averages, and interpreted as the power distribution, or power density, over the different frequencies.

Proof

In a general case, a stationary random process $f(t)$, with mean value zero^a, may not satisfy the absolutely integrable property, i.e., $\int_{-\infty}^{+\infty} |f(t)| dt < \infty$. Therefore, its Fourier transform may not exist. However, it is possible to represent it with a Fourier-Stieltjes integral, see [AP05, p. 44] and [Ish78, p. 513]:

$$f(t) = \int_{-\infty}^{\infty} e^{j\omega t} d\nu(\omega),$$

where $dv(\omega)$ is a random complex amplitude, function of ω . Using such integrals, the covariance function can be expressed as:

$$B_f(t_1, t_2) = \langle f(t_1)f^*(t_2) \rangle = \iint_{-\infty}^{\infty} e^{j(\omega_1 t_1 - \omega_2 t_2)} \langle dv(\omega_1)dv^*(\omega_2) \rangle.$$

Since the process is stationary, the covariance function must be a function of the time difference $\tau = t_1 - t_2$, which is only possible if:

$$\langle dv(\omega_1)dv^*(\omega_2) \rangle = \delta(\omega_2 - \omega_1)W_f(\omega_1)d\omega_1d\omega_2$$

where $\delta(\omega_2 - \omega_1)$ is the Dirac delta function. Using its multiplication property when substituting $\langle dv(\omega_1)dv^*(\omega_2) \rangle$ in the integral, one obtains that the inverse Fourier transform of $W_f(\omega)$ (replacing ω_1 by ω) must be equal to the covariance function $B_f(\tau)$.

^aIf $f(t)$ does not have a zero mean, it can be replaced by the fluctuations $f(t) - \bar{f}(t)$.

The power spectral density $W_f(\omega)$ is also related to the structure function $D_f(\tau)$, through (for real processes)

$$D_f(\tau) = 2 \int_{-\infty}^{\infty} [1 - \cos(\omega\tau)] W_f(\omega)d\omega. \quad (2.26)$$

Proof

By substituting (2.25) in (2.23), the proof is immediate [Tat71, p. 17].

One of the benefit of relying on the structure function is the handling of processes having a singularity at the origin in their power spectrum, i.e., having infinite power in the low-frequency region. Indeed, in those cases, the integral in (2.26) converges, on the contrary of (2.25) [Tat71, p. 17],[Tat61, p. 12].

Furthermore, (2.25) and (2.26) illustrate the main motivation of using the power spectral density $W_f(\omega)$. Indeed, it exists both for stationary and stationary with increments processes, having the same interpretation in both cases. Hence, there is no need the distinct the two hypotheses, except at the end of the computation, when one wants to obtain $B(\tau)$ or $D(\tau)$ [Tat71, p. 18].

2.4.1.2 Random fields

Similarly to stochastic processes, random fields are random functions of the spatial position $\mathbf{r}$. The treatment of random fields is very comparable to the one of stochastic processes, except for the nomenclature of the hypotheses, and the handling of the 3D dependency of the fields.

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A random field is denoted $f(\mathbf{r})$, having a mean $\bar{f}(\mathbf{r}) = \langle f(\mathbf{r}) \rangle$, and fluctuations $f'(\mathbf{r})$. Its spatial (auto)covariance function is defined by

$$B_f(\mathbf{r}_1, \mathbf{r}_2) = \langle (f(\mathbf{r}_1) - \bar{f}(\mathbf{r}_1)) (f(\mathbf{r}_2) - \bar{f}(\mathbf{r}_2))^* \rangle = \langle f'(\mathbf{r}_1) (f'(\mathbf{r}_2))^* \rangle$$

The variance is obtained thanks to

$$\sigma_f^2(\mathbf{r}) = B_f(\mathbf{r}, \mathbf{r}) = \langle (f(\mathbf{r}) - \bar{f}(\mathbf{r})) (f(\mathbf{r}) - \bar{f}(\mathbf{r}))^* \rangle = \langle f'(\mathbf{r}) (f'(\mathbf{r}))^* \rangle$$

Homogeneity Homogeneity in space, for random fields, is the counterpart of stationarity in time for stochastic processes [Tat71, p. 22]. Hence, a random field is said to be statistically homogeneous if it has a constant mean with the position

$$\bar{f}(\mathbf{r}) = \bar{f} = \text{const}$$

and the covariance function depends only on the spatial separation $\boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_2$:

$$B_f(\mathbf{r}_1, \mathbf{r}_2) = B_f(\mathbf{r}_1 - \mathbf{r}_2, 0) = B_f(\boldsymbol{\rho}) = \langle (f(\mathbf{r} + \boldsymbol{\rho}) - \bar{f}) (f(\mathbf{r}) - \bar{f})^* \rangle.$$

Local homogeneity Similarly to stationarity with increments, a random field is said to be statistically locally homogeneous if it can be decomposed into a slowly varying mean $\bar{f}(\mathbf{r})$ and statistically homogeneous fluctuations $f'(\mathbf{r})$. Again, the structure function is used to characterize such fields, and is defined as (for real processes)

$$\begin{aligned} D_f(\mathbf{r}_1, \mathbf{r}_2) &= \langle (f(\mathbf{r}_1) - f(\mathbf{r}_2))^2 \rangle \\ &= \langle (\bar{f}(\mathbf{r}_1) - f'(\mathbf{r}_1) - \bar{f}(\mathbf{r}_2) + f'(\mathbf{r}_2))^2 \rangle \\ &= \langle (\bar{f}(\mathbf{r}_1) - \bar{f}(\mathbf{r}_2))^2 \rangle + \langle (f'(\mathbf{r}_2) - f'(\mathbf{r}_1))^2 \rangle \\ &\quad + 2 \langle (\bar{f}(\mathbf{r}_1) - \bar{f}(\mathbf{r}_2))(f'(\mathbf{r}_1) - f'(\mathbf{r}_2)) \rangle \\ &= (\bar{f}(\mathbf{r}_1) - \bar{f}(\mathbf{r}_2))^2 + \langle (f'(\mathbf{r}_2) - f'(\mathbf{r}_1))^2 \rangle \\ &\approx \langle (f'(\mathbf{r}_2) - f'(\mathbf{r}_1))^2 \rangle \end{aligned} \tag{2.27}$$

since $\bar{f}(\mathbf{r})$ is not a random field and $\langle f'(\mathbf{r}) \rangle = 0$. The final approximation is only valid if the mean is slowly varying in space.

With the locally homogeneous assumption, the ensemble average $\langle n(\mathbf{r} + \boldsymbol{\rho}) - n(\mathbf{r}) \rangle$ is independent of $\mathbf{r}$ and depends only on $\boldsymbol{\rho}$. Moreover, the structure function depends only on the spatial difference $\boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_2$:

$$D_f(\mathbf{r}_1, \mathbf{r}_2) = D_f(\mathbf{r}_1 - \mathbf{r}_2, 0) = D_f(\boldsymbol{\rho}) = \langle (f(\mathbf{r} + \boldsymbol{\rho}) - f(\mathbf{r}))^2 \rangle.$$

For *statistically homogeneous* random fields, the following holds:

$$D_f(\boldsymbol{\rho}) = 2 (B_f(0) - B_f(\boldsymbol{\rho})). \tag{2.28}$$

Isotropy Adding the isotropy hypothesis, all previous dependencies in $\boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_2$ are now substituted by the absolute distance between $\mathbf{r}_1$ and $\mathbf{r}_2$, i.e., by $\rho = |\boldsymbol{\rho}| = |\mathbf{r}_1 - \mathbf{r}_2|$. Therefore, the covariance and the structure functions are now given by

$$\begin{aligned} B_f(\rho) &= \langle (f(\mathbf{r} + \boldsymbol{\rho}) - \bar{f}) (f(\mathbf{r}) - \bar{f})^* \rangle \\ D_f(\rho) &= \langle (f(\mathbf{r} + \boldsymbol{\rho}) - f(\mathbf{r}))^2 \rangle. \end{aligned}$$

Spatial power spectra In the case of random fields, power spectra of different dimensions can be defined. They assume $f(\mathbf{r})$ to be zero-mean.

- **One-dimensional spatial power spectrum:** for statistically *homogeneous* and *isotropic* random fields, the One-Dimensional (1D) power spectrum $V_f(\kappa)$ is the Fourier transform of $B_f(\rho)$ [AP05, p. 46]:

$$V_f(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B_f(\rho) e^{-j\kappa\rho} d\rho = \frac{1}{\pi} \int_0^{\infty} B_f(\rho) \cos(\kappa\rho) d\rho \quad (2.29)$$

$$B_f(\rho) = \int_{-\infty}^{\infty} V_f(\kappa) e^{j\kappa\rho} d\kappa = 2 \int_0^{\infty} V_f(\kappa) \cos(\kappa\rho) d\kappa, \quad (2.30)$$

with κ the spatial frequency in [rad/m]. The second equalities (cosine transforms) only hold if $f(\mathbf{r})$ is real.

Proof

The proof is similar to the one given for the power spectral density $W_f(\omega)$ [AP05, p. 46], [Tat71, p. 23].

- **Three-dimensional spatial power spectrum:** for statistically *homogeneous* random fields, $\Phi_f(\boldsymbol{\kappa})$ is the 3D spatial power spectrum, linked to the covariance function thanks to

$$\Phi_f(\boldsymbol{\kappa}) = \left(\frac{1}{2\pi}\right)^3 \iiint_{-\infty}^{\infty} B_f(\boldsymbol{\rho}) e^{-j\boldsymbol{\kappa} \cdot \boldsymbol{\rho}} d^3\boldsymbol{\rho} = \left(\frac{1}{2\pi}\right)^3 \iiint_{-\infty}^{\infty} B_f(\boldsymbol{\rho}) \cos(\boldsymbol{\kappa} \cdot \boldsymbol{\rho}) d^3\boldsymbol{\rho} \quad (2.31)$$

$$B_f(\boldsymbol{\rho}) = \iiint_{-\infty}^{\infty} \Phi_f(\boldsymbol{\kappa}) e^{j\boldsymbol{\kappa} \cdot \boldsymbol{\rho}} d^3\boldsymbol{\kappa} = \iiint_{-\infty}^{\infty} \Phi_f(\boldsymbol{\kappa}) \cos(\boldsymbol{\kappa} \cdot \boldsymbol{\rho}) d^3\boldsymbol{\kappa}, \quad (2.32)$$

with $d^3\boldsymbol{\rho} = d\rho_x d\rho_y d\rho_z$. The second equalities only apply if $f(\mathbf{r})$ is real.

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Proof [AP05, p. 47], [Tat71, p. 23]

Using Fourier-Stieltjes integrals,

$$f(\mathbf{r}) = \iiint_{-\infty}^{\infty} e^{j\boldsymbol{\kappa} \cdot \mathbf{r}} dV(\boldsymbol{\kappa}).$$

The covariance function is then expressed with

$$\begin{aligned} B_f(\mathbf{r}_1, \mathbf{r}_2) &= \langle f(\mathbf{r}_1) f^*(\mathbf{r}_2) \rangle \\ &= \iiint_{-\infty}^{\infty} \iiint_{-\infty}^{\infty} e^{j(\boldsymbol{\kappa}_1 \cdot \mathbf{r}_1 - \boldsymbol{\kappa}_2 \cdot \mathbf{r}_2)} \langle dV(\boldsymbol{\kappa}_1) dV^*(\boldsymbol{\kappa}_2) \rangle. \end{aligned}$$

Because the random field is assumed homogeneous, the following equality holds

$$\langle dV(\boldsymbol{\kappa}_1) dV^*(\boldsymbol{\kappa}_2) \rangle = \delta(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2) \Phi_f(\boldsymbol{\kappa}_1) d^3\boldsymbol{\kappa}_1 d^3\boldsymbol{\kappa}_2,$$

and substitution in the above expression gives the 3D Fourier transforms.

For real, statistically *homogeneous* and *isotropic* random fields, the following relations hold ($\kappa = |\boldsymbol{\kappa}|$):

$$\Phi_f(\kappa) = \frac{1}{2\pi^2\kappa} \int_0^{+\infty} B_f(\rho) \sin(\kappa\rho) \rho d\rho \quad (2.33)$$

$$B_f(\rho) = \frac{4\pi}{\rho} \int_0^{+\infty} \Phi_f(\kappa) \sin(\kappa\rho) \kappa d\kappa. \quad (2.34)$$

Proof

From (2.31), adding the isotropy hypothesis and using spherical coordinates to integrate over the angular variables lead to the presented result [Tat71, p. 24], [Ish78, p. 517].

In this case, the link between 1D and 3D power spectra is:

$$\Phi_f(\kappa) = -\frac{1}{2\pi\kappa} \frac{dV_f(\kappa)}{d\kappa} \quad (2.35)$$

Proof

This result comes from the derivation of (2.29) with respect to κ , and its comparison with (2.33), as detailed in [Tat71, p. 24] and [Ish78, p. 517].

- **Two-dimensional spatial power spectrum:** in homogeneous and isotropic random medium, it is sometimes of interest to characterize the fluctuations in a plane perpendicular to a direction of propagation (for example, for studying wave propagation in random media). In this case, the Two-Dimensional (2D) spectrum between transverse planes separated by a distance z , denoted $F_f(\kappa_x, \kappa_y, 0; z)$, is given by [AP05, p. 48]

$$F_f(\kappa_x, \kappa_y, 0; z) = \int_{-\infty}^{+\infty} \Phi_f(\kappa_x, \kappa_y, \kappa_z) \cos(z\kappa_z) d\kappa_z$$

$$\Phi_f(\kappa_x, \kappa_y, \kappa_z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F_f(\kappa_x, \kappa_y, 0; z) \cos(z\kappa_z) dz,$$

using the fact that $\Phi_f(\boldsymbol{\kappa})$ is an even function.

Proof

The proof can be found in [Ish78, p. 516]. Two-dimensional spatial power spectra are presented for the sake of completeness, but are not used in this work.

The different spatial power spectra are also related to structure functions. The relevant equation, for real and locally homogeneous random fields, is

$$D_f(\rho) = 2 \iiint_{-\infty}^{\infty} \Phi_f(\boldsymbol{\kappa}) (1 - \cos(\boldsymbol{\kappa} \cdot \boldsymbol{\rho})) d^3\boldsymbol{\kappa}.$$

Proof

The proof is available in [Tat71, p. 26] and [Ish78, p. 524].

If the random field is also locally isotropic, with $\rho = |\boldsymbol{\rho}|$, then [AP05, p. 49]

$$D_f(\rho) = 8\pi \int_0^{\infty} \kappa^2 \Phi_f(\kappa) \left(1 - \frac{\sin(\kappa\rho)}{\kappa\rho}\right) d\kappa \quad (2.36)$$

$$\Phi_f(\kappa) = \frac{1}{4\pi^2\kappa^2} \int_0^{\infty} \frac{\sin(\kappa\rho)}{\kappa\rho} \frac{d}{d\rho} \left(\rho^2 \frac{d}{d\rho} D_f(\rho) \right) d\rho. \quad (2.37)$$

Proof

The proof is presented in [Tat71, p. 26] and [Ish78, p. 524].

Relationships with the 1D power spectrum can be found in [Tat71, p. 28] and

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[Ish78, p. 525], such as

$$D_f(\rho) = 2 \int_{-\infty}^{\infty} (1 - \cos(\kappa\rho)) V_f(\kappa) d\kappa \quad (2.38)$$

$$V_f(\kappa) = \frac{1}{2\pi\kappa} \int_0^{\infty} \sin(\kappa\rho) \frac{d}{d\rho} D_f(\rho) d\rho. \quad (2.39)$$

Finally, expansions with 2D power spectra are available in [Tat61, p.23], [Tat71, p. 31], and [Ish78, p. 525].

2.4.1.3 Space-time random fields

Expressions and hypotheses from Sections 2.4.1.1 and 2.4.1.2 can be combined to describe space-time random fields, having temporal and spatial fluctuations. Denoting such a field by $f(\mathbf{r}, t)$, if it is assumed to be *stationary in time* and *homogeneous in space*, its covariance function is given by [Tat71, p. 33], [Ish78, p. 518]

$$B_f(\boldsymbol{\rho}, \tau) = \langle (f(\mathbf{r} + \boldsymbol{\rho}, t + \tau) - \bar{f}) (f(\mathbf{r}, t) - \bar{f})^* \rangle.$$

For a real space-time random field that is *stationary with increments* and *locally homogeneous*, its structure function is

$$D_f(\boldsymbol{\rho}, \tau) = \langle (f(\mathbf{r} + \boldsymbol{\rho}, t + \tau) - f(\mathbf{r}, t))^2 \rangle.$$

In both cases, if the field is isotropic, then one can access to $B_f(\rho, \tau)$ and $D_f(\rho, \tau)$, with $\rho = |\boldsymbol{\rho}|$.

Defining the four-dimensional power spectrum $U_f(\boldsymbol{\kappa}, \omega)$, the following relations hold [Tat71, p. 33]:

$$\begin{aligned} B_f(\boldsymbol{\rho}, \tau) &= \iiint_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j(\boldsymbol{\kappa} \cdot \boldsymbol{\rho} - \omega \tau)} U_f(\boldsymbol{\kappa}, \omega) d^3 \boldsymbol{\kappa} d\omega, \\ \Phi_f(\boldsymbol{\kappa}) &= \int_{-\infty}^{\infty} U_f(\boldsymbol{\kappa}, \omega) d\omega, \\ W_f(\omega) &= \iiint_{-\infty}^{\infty} U_f(\boldsymbol{\kappa}, \omega) d^3 \boldsymbol{\kappa}. \end{aligned}$$

These equations express the temporal and 3D spatial power spectra thanks to the four-dimensional one.

Frozen field (Taylor hypothesis) A well-known hypothesis in the field of turbulence is the Taylor frozen field hypothesis [Tat71, p. 34], [Ish78, p. 519], [AP05, p. 72]. It considers that the time variations of the space-time random field $f(\mathbf{r}, t)$ are solely caused by a translation of the spatial variations, with a velocity $\mathbf{v}$, that is,

$$f(\mathbf{r}, t) = f(\mathbf{r} - \mathbf{v}t, 0).$$

Then, one has, respectively for stationary and homogeneous fields, or for stationary with increments and locally homogeneous fields:

$$\begin{aligned} B_f(\boldsymbol{\rho}, \tau) &= B_f(\boldsymbol{\rho} - \mathbf{v}\tau) \\ D_f(\boldsymbol{\rho}, \tau) &= D_f(\boldsymbol{\rho} - \mathbf{v}\tau), \end{aligned}$$

and, in both cases, the following holds: $U_f(\boldsymbol{\kappa}, \omega) = \Phi_f(\boldsymbol{\kappa}) \delta(\omega - \boldsymbol{\kappa} \cdot \mathbf{v})$ [Tat71, p. 34].

If the field is also isotropic (or locally isotropic):

$$\begin{aligned} W_f(\omega) &= \frac{2\pi}{v} \int_{|\omega|/v}^{+\infty} \Phi_f(\kappa) \kappa \, d\kappa \\ \Phi_f(\kappa) &= -\frac{v^2}{2\pi\kappa} \left[\frac{d}{d\omega} W_f(\omega) \right]_{\omega=v\kappa} \end{aligned}$$

choosing a velocity vector $\mathbf{v}$ of magnitude v and aligned along the z -direction, using spherical coordinates. These last equations relate space and time characterization of space-time random fields, under the frozen field hypothesis.

Proof

The proof is available in [Tat71, p. 35] and [Ish78, p. 519].

2.4.1.4 Vector random fields

Finally, vector random fields $\mathbf{f}(\mathbf{r}) = (f_1(\mathbf{r}), f_2(\mathbf{r}), f_3(\mathbf{r}))$ are also encountered in practice, e.g., velocity fields. In those cases, the description of scalar fields presented above applies, except that cross-covariance and cross-structure functions are needed for describing correlations between the different components of the vector fields. One can define the covariance and structure tensors:

$$\begin{aligned} B_{f,ij}(\boldsymbol{\rho}) &= \langle (f_i(\boldsymbol{\rho} + \mathbf{r}) - \bar{f}_i) (f_j(\mathbf{r}) - \bar{f}_j)^* \rangle \\ D_{f,ij}(\boldsymbol{\rho}) &= \langle (f_i(\boldsymbol{\rho} + \mathbf{r}) - f_i(\mathbf{r})) (f_j(\boldsymbol{\rho} + \mathbf{r}) - f_j(\mathbf{r})) \rangle, \end{aligned}$$

with i, j denoting the components of the vector (1, 2, or 3). The temporal dependency can also be included in the equations above.

Adding the isotropy hypothesis, only two components remain (longitudinal and

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transverse), that are linked by [Tat71, p. 41]:

$$B_{f,t}(\rho) = \frac{1}{2\rho} \frac{d}{d\rho} (\rho^2 B_{f,l}(\rho)) \quad \text{and} \quad D_{f,t}(\rho) = \frac{1}{2\rho} \frac{d}{d\rho} (\rho^2 D_{f,l}(\rho)), \quad (2.40)$$

where the subscript t stands for the transverse component, and l for the longitudinal component. Thanks to these equations, there is only one independent component to characterize.

Proof

The proof and much more details are available in [Tat71, p. 38].

2.4.1.5 Summary

This section presented the required theoretical background to describe space-time random fields. Several assumptions and hypotheses have been introduced, and are summarized below:

- For *stationary stochastic processes*, the covariance function is used and depends only on the time difference τ , i.e., $B_f(\tau)$;
- For *stationary with increments stochastic processes*, one should rely on their structure functions, that depend only on the time difference, i.e., $D_f(\tau)$;
- For *homogeneous random fields*, the covariance function is used and depends only on the spatial separation ρ , i.e., $B_f(\rho)$. For isotropic fields, it can be expressed as $B_f(\rho)$, with $\rho = |\rho|$.
- For *locally homogeneous random fields*, this is the structure function that is used and that depends only on the spatial separation ρ , i.e., $D_f(\rho)$. For isotropic fields, it can be expressed as $D_f(\rho)$, with $\rho = |\rho|$.
- In the case of *space-time random fields*, the Taylor frozen field hypothesis enables to relate the temporal and spatial dependencies, thanks to $B_f(\rho - v\tau)$ and $D_f(\rho - v\tau)$.
- In all cases, those quantities are associated to power spectra in the temporal and spatial frequency domains. Those spectra are computed based on centred quantities, i.e., after removing the mean $\bar{f}$.

Covariance and structure functions are mathematical tools, important to characterize space and time random fields. However, at this stage, physical inputs particular to the process considered (here, atmospheric turbulence) are missing. They are presented in the next section, providing a closed form for the power spectrum.

2.4.2 Energy cascade theory

Turbulence spans a large range of length scales, and the transfer of energy between those scales (i.e., from large scales to small scales) is described by the energy cascade

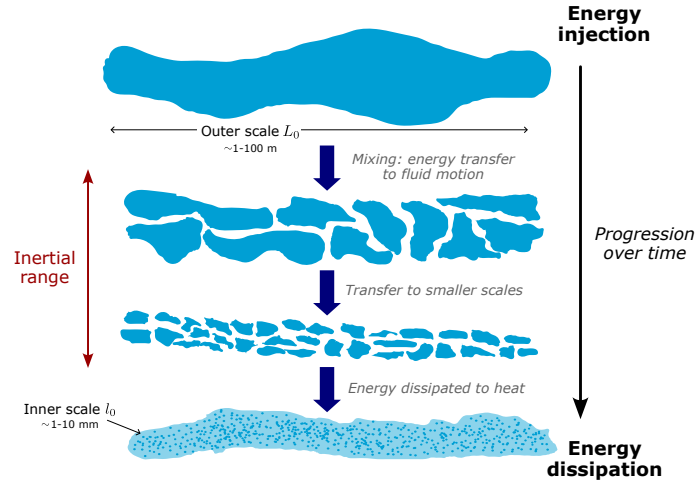


Figure 2.7 – Illustration of energy cascade theory (adapted from [TF04; Can18]).

theory. This theory has first been presented by Richardson with the following verses [Ric22, p. 66]:

“Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity.”

Richardson’s whirls correspond to turbulent eddies, each of them having an associated length scale. Figure 2.7 graphically presents this cascade of energy. First, energy is injected at large scales, either from wind shear or convection [AP05, p. 59]. It is then transferred from large scales to small scales under inertial forces, up to reaching a scale at which the energy is dissipated by viscous forces. As a result, there is a continuum of scales, i.e., of eddy sizes, for the transfer of energy. This continuum is named the *inertial range*, and is delimited by an upper scale, called the *outer scale* of turbulence L_0 , and a lower scale, named the *inner scale* of turbulence l_0 . Typical values for the outer scale L_0 ranges from meters to tens of meters, or even hundreds of meters, and vary with the altitude, as detailed in Section 3.2.2.2. Regarding the inner scale l_0 , it is on the order of millimeters close to the ground, and centimeters or more in the troposphere and stratosphere [AP05, p. 60].

As a summary, the cascade of energy is a “*transfer of energy from large, macroscopic scales of motion [...] through successively smaller scales, ending with viscous dissipation and conversion to heat*” [McD07, p. 21]. The hypotheses made in the different ranges of scales are important to statistically describe the turbulence. They have been presented by Kolmogorov and Obukhov in 1941 [Hic14, p. 3], and are referred to as Kolmogorov’s hypotheses in [Tat71, p. 47, 53]:

- In the inertial range, the flow is assumed to be statistically isotropic and locally homogeneous. On the contrary, for scales larger than L_0 , the flow is non-isotropic [AP05, p. 60].

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- The small-scale structure (i.e., the inertial and dissipation ranges) is uniquely determined by the kinematic viscosity ν [m²/s] and the average rate of dissipation $\bar{\varepsilon}$ [m²/s³].
- In the inertial range, the turbulence structure is only determined by $\bar{\varepsilon}$.

As previously stated, ε represents the quantity of energy converted into heat by viscous forces, per unit of mass and time. The mathematical definition of its average is presented in [Tat71, p. 48] for example:

$$\bar{\varepsilon} = \frac{\nu}{2} \overline{\left[\frac{\partial u'_i}{\partial x_k} + \frac{\partial u'_k}{\partial x_i} \right]^2}.$$

From those hypotheses and using dimensional analysis, Kolmogorov showed that the structure function of the velocity must be proportional to $(\bar{\varepsilon}\rho)^{2/3}$, i.e.,

$$D_v(\rho) \propto (\bar{\varepsilon}\rho)^{2/3} \quad \Rightarrow \quad D_v(\rho) = C_v^2 \rho^{2/3} \quad \text{for} \quad l_0 \ll \rho \ll L_0, \quad (2.41)$$

where C_v^2 , named *velocity structure constant*, aggregates the product between a dimensionless constant and $\bar{\varepsilon}^{2/3}$, having therefore units of [m^{4/3}/s²]. This results is known as the *two-thirds law* of Kolmogorov and Obukhov [Ish78, p. 531],[AP05, p. 60].

In (2.41), $D_v(\rho)$ actually corresponds to the longitudinal component of the velocity structure function, as detailed in [Tat71, p. 54]. Using (2.40), the transverse structure function is also proportional to $\rho^{2/3}$, such that the distinction between longitudinal and transverse velocity structure functions is not always made.

A sketch of the dimensional analysis conducted to obtain (2.41) is presented in [Ish78, p. 530]. It is based on the observation that the kinetic energy per unit of mass and time of eddies of scale L is on the order of V^3/L [m²/s³=J/kg/s], V being the velocity fluctuation associated with those eddies, while the energy dissipation per unit of mass and time is on the order of $\nu V^2/L^2$ [m²/s³=J/kg/s]. For large Reynolds number, the dissipation is negligible, and almost all kinetic energy is transferred from large eddies to small eddies, until eddies of size l_0 are reached, where the energy dissipation is comparable to the kinetic energy. By defining a range of eddy scales between l_0 and L_0 given by $l_0 < L_{N_e} < L_{N_e-1} < \dots < L_2 < L_1 < L_0$, with $N_e \in \mathbb{Z}$, and the associated velocities $V_{l_0} < V_{N_e} < V_{N_e-1} < \dots < V_2 < V_1 < V_0$, one has [Ish78, p. 530]

$$\frac{V_0^3}{L_0} \approx \frac{V_1^3}{L_1} \approx \frac{V_2^3}{L_2} \approx \dots \approx \frac{V_{N_e-1}^3}{L_{N_e-1}} \approx \frac{V_{N_e}^3}{L_{N_e}} \approx \frac{V_{l_0}^3}{l_0} \approx \nu V_{l_0}^2 / l_0^2 \approx \bar{\varepsilon}. \quad (2.42)$$

This reasoning shows that the velocity fluctuation of eddies of size L is proportional to $(\bar{\varepsilon}L)^{1/3}$, i.e., $V \propto (\bar{\varepsilon}L)^{1/3}$. As a result, the structure function must be proportional to $(\bar{\varepsilon}\rho)^{2/3}$ in the inertial range, leading to (2.41). Furthermore, in the dissipation range ($\rho \ll l_0$), the last equality in (2.42) enables to obtain that $D_v(\rho) \propto \frac{\bar{\varepsilon}}{\nu} \rho^2$.

For eddies of size l_0 , both expressions of the structure function should hold, such that

$$D_v(l_0) \propto \frac{\bar{\varepsilon}}{\nu} l_0^2 \propto (\bar{\varepsilon} l_0)^{2/3},$$

and the inner scale l_0 should be on the order of $l_0 \propto \left(\frac{\nu^3}{\bar{\varepsilon}}\right)^{1/4} = \eta_K$, η_K being the Kolmogorov length scale [AP05, p. 60], [McD07, p. 51]. In [HC78], a ratio l_0/η_K equal to 7.4 for the air has been computed.

Finally, ensuring the continuity in l_0 of the velocity structure functions by choosing the proportionality constant appropriately, the asymptotic expressions of the velocity structure function are [AP05, p. 61]

$$D_v(\rho) = \begin{cases} C_v^2 l_0^{-4/3} \rho^2, & 0 \leq \rho \ll l_0 \text{ (dissipation range)} \\ C_v^2 \rho^{2/3}, & l_0 \ll \rho \ll L_0 \text{ (inertial range)}. \end{cases}$$

2.4.3 Kolmogorov power spectrum

The two-thirds law on the velocity structure function $D_v(\rho)$ has implications on the velocity 3D power spectrum $\Phi_v(\kappa)$. Indeed, the spectral equivalent of the two-thirds law is the Kolmogorov power law of minus 11/3, referred to as the *Kolmogorov power spectrum*. From the structure function in (2.41), it can be obtained by computing the integral in (2.37), i.e.,

$$\begin{aligned} \Phi_v(\kappa) &= \frac{1}{4\pi^2 \kappa^2} \int_0^\infty \frac{\sin(\kappa\rho)}{\kappa\rho} \frac{d}{d\rho} \left(\rho^2 \frac{d}{d\rho} D_v(\rho) \right) d\rho \\ &= \frac{1}{4\pi^2 \kappa^2} \int_0^\infty \frac{\sin(\kappa\rho)}{\kappa\rho} \frac{d}{d\rho} \left(\rho^2 \frac{d}{d\rho} C_v^2 \rho^{2/3} \right) d\rho \\ &= \frac{\frac{2}{3} \frac{5}{3}}{4\pi^2} C_v^2 \frac{1}{\kappa^2} \int_0^\infty \frac{\sin(\kappa\rho)}{\kappa\rho} \rho^{2/3} d\rho \\ &= \frac{\frac{2}{3} \frac{5}{3}}{4\pi^2} C_v^2 \frac{1}{\kappa^2} \frac{\pi}{\Gamma(1/3) \kappa^{5/3}} \\ &= 0.033 C_v^2 \kappa^{-11/3}. \end{aligned} \tag{2.43}$$

These developments involve the Gamma function $\Gamma(x)$, with $\Gamma(1/3) \approx 2.6789$. The upper and lower bound of the inertial range have been neglected in the integration, considering that $D_v(\rho)$ is proportional to $\rho^{2/3}$ from $\rho = 0$ to $\rho \rightarrow \infty$. Hence, the obtained result in (2.43) is assumed to be valid only for spatial frequencies κ inside the inertial range. The complete mathematical expression of the Kolmogorov power spectrum is therefore [AP05, p. 62]

$$\Phi_v(\kappa) = 0.033 C_v^2 \kappa^{-11/3} \quad \text{for} \quad \frac{2\pi}{L_0} \ll \kappa \ll \frac{2\pi}{l_0}.$$

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Using (2.39), the 1D power spectrum is given by

$$V_v(\kappa) = \frac{1}{2\pi\kappa} \int_0^\infty \sin(\kappa\rho) \frac{d}{d\rho} D_v(\rho) d\rho = 0.1244 C_v^2 \kappa^{-5/3},$$

where a power trend in $-5/3$ appears. This result fulfils (2.35).

Several extensions to the Kolmogorov power spectrum have been presented in the literature. They are detailed in Section 2.5 for power spectra associated to the refractive index n .

2.5 Application to the atmospheric refractive index

Based on the work of Kolmogorov, it has been shown that the statistical description of turbulence presented in the previous section can be applied to any conservative passive additive [Rod81, p. 286], [Whe01, p. 31]. Hence, in the following, it will be applied to the temperature (Section 2.5.1), and to the refractive index (Section 2.5.2).

2.5.1 Temperature structure function and power spectrum

Strictly, the temperature is not a conservative quantity. Indeed, when a parcel of air moves up (respectively down), it cools down (resp. warms up) adiabatically [Bel93, p. 169]. On the contrary, the potential temperature θ , defined in (2.19), is a passive and conservative quantity, such that its structure function follows [Bel93, p. 170]

$$D_\theta(\rho) = \langle (\theta(\mathbf{r} + \boldsymbol{\rho}) - \theta(\mathbf{r}))^2 \rangle = \begin{cases} C_\theta^2 l_0'^{-4/3} \rho^2, & 0 \leq \rho \ll l_0' \\ C_\theta^2 \rho^{2/3}, & l_0' \ll \rho \ll L_0, \end{cases}$$

with C_θ^2 the potential temperature structure parameter, in $[K^2 m^{-2/3}]$. It follows that the Kolmogorov power spectrum for the potential temperature is $\Phi_\theta(\kappa) = 0.033 C_\theta^2 \kappa^{-11/3}$ in the inertial range. This inertial range is actually slightly different than the one for the velocity fluctuations, as, for (potential) temperature fluctuations, the inner scale l_0' is proportional to $\left(\frac{\tilde{\chi}}{\varepsilon}\right)^{1/4}$, where $\tilde{\chi}$ is the thermal diffusivity $[m^2/s]$ [Tat71, p. 65]. Nevertheless, since the thermal diffusivity $\tilde{\chi}$ is on the same order as the kinematic viscosity ν (the ratio $\nu/\tilde{\chi}$ is approximately equal to 0.72 for air [Bel93, p. 171]), the difference between l_0 and l_0' is often neglected, as assumed in the following³.

The potential temperature primarily varies with the altitude z . Based on this observation, Tatarskii suggested to use a pseudopotential temperature $\tilde{\theta}$, defined by

$$\tilde{\theta} = T + \gamma_a z, \quad (2.44)$$

³Some models take into account this difference, since it can lead to a bump in the spectrum at small scales, that can have some implications for the propagation of optical waves [HC78], [Bel93, p. 171].

where $\gamma_a = \frac{g}{c_p}$ is the adiabatic lapse rate of dry air [Tat71, p. 74], [Bel93, p. 170], [CB13]. It is equal to 9.8 K/km in the troposphere.

For small vertical separations, or for horizontal separations, the potential temperature θ , the pseudo-potential temperature $\tilde{\theta}$, and the temperature T are similar. In this case, the temperature structure function $D_T(\rho)$ is assumed to follow the $\rho^{2/3}$ in the inertial range, and

$$D_T(\rho) = \langle (T(\mathbf{r} + \boldsymbol{\rho}) - T(\mathbf{r}))^2 \rangle = \begin{cases} C_T^2 l_0'^{-4/3} \rho^2, & 0 \leq \rho \ll l_0' \\ C_T^2 \rho^{2/3}, & l_0' \ll \rho \ll L_0, \end{cases} \quad (2.45)$$

with C_T^2 the temperature structure parameter [$\text{T}^2 \text{m}^{-2/3}$], and $\Phi_T(\kappa) = 0.033 C_T^2 \kappa^{-11/3}$.

In the literature, $D_T(\rho)$ and C_T^2 are very often used, due to the simplicity of temperature measurements. Some of the implications of this simplification are presented in [CB13].

2.5.2 Refractive index structure function and power spectrum

The refractive index, being also a conservative passive additive⁴, has similar statistical properties than the (potential) temperature and the velocity fluctuations. These properties are detailed in this section, along with some extensions and illustrations.

2.5.2.1 Kolmogorov power spectrum for refractive index fluctuations

In the inertial and dissipation ranges, the structure function of refractive index fluctuations follows [AP05, p. 64]

$$D_n(\rho) = \langle (n(\mathbf{r} + \boldsymbol{\rho}) - n(\mathbf{r}))^2 \rangle = \begin{cases} C_n^2 l_0^{-4/3} \rho^2, & 0 \leq \rho \ll l_0 \\ C_n^2 \rho^{2/3}, & l_0 \ll \rho \ll L_0, \end{cases} \quad (2.46)$$

with C_n^2 the refractive index structure parameter, in [$\text{m}^{-2/3}$]. Equation (2.46) is associated to the Kolmogorov power spectrum $\Phi_n(\kappa)$ in the inertial range, given by [Whe01, p. 32], [AP05, p. 67]

$$\Phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3} \quad \text{for} \quad \frac{2\pi}{L_0} \ll \kappa \ll \frac{2\pi}{l_0}. \quad (2.47)$$

It is illustrated in Figure 2.8, along with several extensions including inner and outer scale effects, described below.

⁴See (2.4) and (2.6) that can also be expressed only with conservative and passive quantities, as done in Section 2.5.4.

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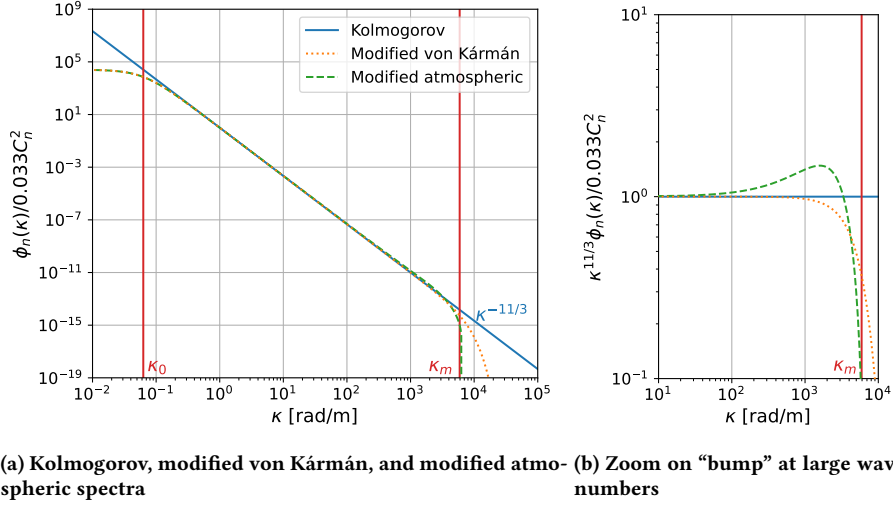


Figure 2.8 – Illustration of power spectra of refractive index fluctuations.

2.5.2.2 Extensions to Kolmogorov power spectrum for refractive index fluctuations

The Kolmogorov power spectrum is only valid for spatial frequencies, also named wave numbers, larger than $2\pi/L_0$ and lower than $2\pi/l_0$. Several extensions for small wave numbers and large wave numbers have been suggested, either for modelling small scale and large scale effects, or for mathematical convenience.

Tatarskii spectrum For small scales, i.e., large spatial frequencies, the Kolmogorov power spectrum must be truncated. One way to achieve such a truncation has been suggested by Tatarskii [Tat71, p. 76], and relies on a Gaussian decrease of the power spectrum at large wave numbers [AP05, p. 67], [Whe01, p. 34], [Bel93, p. 174]

$$\Phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3} e^{-\frac{\kappa^2}{\kappa_m^2}} \quad \text{for} \quad \frac{2\pi}{L_0} \ll \kappa, \quad (2.48)$$

with $\kappa_m = 5.92/l_0$, found by ensuring the continuity of the structure function in l_0 [Whe01, p. 35]. This model is simple to use mathematically but is not an accurate physical description of the dissipation range [Whe01, p. 35].

von Kármán spectrum Kolmogorov and Tatarskii spectra suffer from a singularity at $\kappa = 0$, such that it is not possible to compute a covariance function $B_n(\rho)$ from those spectra [AP05, p. 67]. Modelling of large scale effects, i.e., small spatial frequencies, while avoiding a singularity at the origin, could be achieved by using the von Kármán spectrum [AP05, p. 68], [Whe01, p. 41], [Bel93, p. 174]

$$\Phi_n(\kappa) = 0.033 C_n^2 (\kappa^2 + \kappa_0^2)^{-11/6} \quad \text{for} \quad 0 \leq \kappa \ll \frac{2\pi}{l_0}, \quad (2.49)$$

with $\kappa_0 = 2\pi/L_0$.

Modified von Kármán spectrum The combination of the Tatarskii spectrum and the von Kármán spectrum leads to the so-called modified von Kármán spectrum, valid for all spatial frequencies κ [AP05, p. 68], [Bel93, p. 174]

$$\Phi_n(\kappa) = 0.033 C_n^2 (\kappa^2 + \kappa_0^2)^{-11/6} e^{-\frac{\kappa^2}{\kappa_m^2}} \quad \text{for} \quad 0 \leq \kappa < \infty, \quad (2.50)$$

with $\kappa_m = 5.92/l_0$ and $\kappa_0 = 2\pi/L_0$.

Exponential spectrum An alternative model considering the outer scale is the exponential spectrum [AP05, p. 68], [Whe01, p. 45]

$$\Phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3} \left(1 - e^{-\frac{\kappa^2}{\kappa_0^2}} \right) \quad \text{for} \quad 0 \leq \kappa \ll \frac{2\pi}{l_0}, \quad (2.51)$$

with $\kappa_0 = C_0/L_0$, where C_0 is a scaling constant depending on the application [AP05, p. 68].

Modified atmospheric spectrum All power spectra presented so far, in (2.47) to (2.51), are mathematically tractable models that can be used to compute theoretical expressions of structure functions and covariances, or to study wave propagation. They are in agreement with the physical behaviour of atmospheric turbulence in the inertial range, under the hypotheses stated in Section 2.4.2. However, their description of the dissipation range and large scale effects is questionable [AP05, p. 68].

Nevertheless, more complex models based on physical insights outside the inertial range exist, such as the modified atmospheric spectrum presented in [AP05, p. 69]. It aims at including the bump in the temperature power spectrum observed by [HC78] in the model, for which an analytical approximation has been suggested by [And92],

$$\Phi_n(\kappa) = 0.033 C_n^2 \left[1 + 1.802 \left(\frac{\kappa}{\kappa_l} \right) - 0.254 \left(\frac{\kappa}{\kappa_l} \right)^{7/6} \right] \frac{e^{-\frac{\kappa^2}{\kappa_l^2}}}{(\kappa^2 + \kappa_0^2)^{11/6}}, \quad (2.52)$$

valid for $0 \leq \kappa < \infty$, with $\kappa_l = 3.3/l_0$ and $\kappa_0 = 2\pi/L_0$.

The Kolmogorov, the modified von Kármán, and the modified atmospheric spectra are depicted in Figure 2.8. They use an inner scale l_0 of 1 mm, and an outer scale L_0 of 100 m. This gives the values indicated on the plot for the wave numbers κ_0 and κ_m . Power spectra are normalised by $0.033 C_n^2$ (left graph), and also multiplied by $\kappa^{11/3}$ (right graph). The agreement of all spectra in the inertial range (between κ_0 and κ_m) is noticeable. The bump in the modified atmospheric spectrum is also highlighted.

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Besides the models presented in this section, other models exist and the interested reader can find them, as well as their consequences on the refractive index structure functions, in [Whe01, p. 33-46].

2.5.2.3 Temporal fluctuations of the refractive index

Temporal fluctuations of the refractive index are usually expressed from their spatial fluctuations, using the Taylor frozen turbulence hypothesis introduced in Section 2.4.1.3. It neglects the dynamics of turbulent eddies, considering they are frozen in space and advected by the mean wind. As a result, in a reference frame moving with the mean flow, the refractive index $n(\mathbf{r}, t)$ does not depend on time. On the contrary, in a fixed reference frame, the time dependence is [Bel93, p. 175]

$$n(\mathbf{r}, t) = n(\mathbf{r} - \mathbf{v}_\perp t, 0), \quad (2.53)$$

with $v_\perp$ the mean wind speed transverse to the direction of observation, i.e., the direction of propagation of the wave.

The validity of Taylor hypothesis in the case of wave propagation is discussed in [Whe01, p. 242-250]. It usually fails when the mean wind speed $v_\perp$ is smaller than the velocity fluctuations arising from the turbulence itself [AP05, p. 73].

2.5.3 Illustration of refractive index fluctuations

Thanks to the knowledge of the power spectrum Φ_n , synthetic fluctuations of the refractive index can be generated. This is achieved in this section in order to highlight the influence of the refractive index structure parameter C_n^2 , the inner scale l_0 and the outer scale L_0 , on the refractive index variations.

2.5.3.1 Generation of one-dimensional fluctuations

Generation of fluctuations can be achieved by filtering a white noise sequence with a filter designed to ensure that the output random sequence has the desired power spectral density (white noise coloring). Here, 1D fluctuations are generated, such that the refractive index is considered to be a function of the x -position only, i.e., $n(x)$. Its fluctuations are denoted by $n'(x)$, its mean being $\bar{n}(x)$.

Starting from a Gaussian white noise sequence $w(x)$, it is filtered by a filter having for frequency response $H(f_x)$ in order to obtain $n'(x)$. This approach is illustrated in Figure 2.9. The sequence $w(x) \sim N(0, \sigma_w^2)$ follows a normal distribution of variance σ_w^2 and fulfils the homogeneity hypothesis, such that its covariance function and 1D power spectrum are

$$\begin{aligned} B_w(\rho) &= \mathbb{E}[w(x + \rho)w^*(x)] = \sigma_w^2 \delta(\rho), \\ V_w(f_x) &= \int_{-\infty}^{+\infty} B_w(\rho) e^{-j2\pi f_x \rho} d\rho = \sigma_w^2, \end{aligned}$$

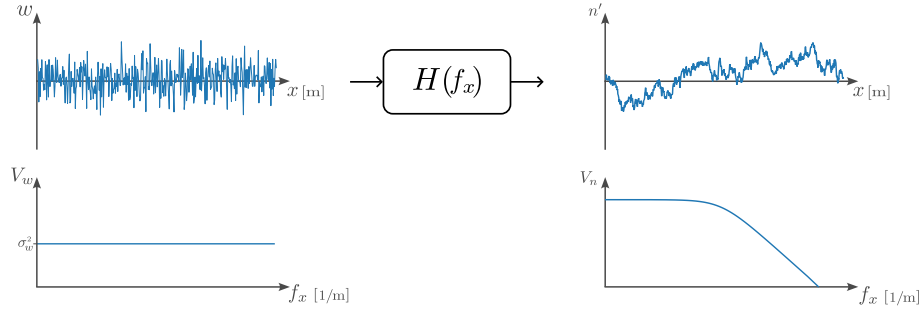


Figure 2.9 – Generation of synthetic refractive index fluctuations from a Gaussian white noise sequence.

where the spatial frequency f_x for the x -direction is used explicitly in the Fourier transform ($\kappa_x = 2\pi f_x$).

The desired 1D power spectrum of $n'(x)$ is the von Kármán spectrum, chosen for its simplicity of implementation. In 1D, it is expressed by

$$V_n(f_x) = 0.1244 C_n^2 \left(f_x^2 + \left(\frac{1}{L_0} \right)^2 \right)^{-5/6} \quad \text{for} \quad 0 \leq f_x \ll \frac{1}{l_0}, \quad (2.54)$$

following (2.35). It does not consider the influence of the inner scale, that is set to 0 such that (2.54) is supposed to be valid for all spatial frequencies f_x .

In order to obtain $n'(x)$ from $w(x)$, the white noise $w(x)$ is filtered by $h(x)$, having $H(f_x)$ for frequency response. According to the Wiener-Khinchin theorem, the power spectrum of the filtered signal is given by

$$V_n(f_x) = |H(f_x)|^2 V_w(f_x).$$

This expression enables to obtain one possible form for $H(f_x)$, given that $V_n(f_x)$ and $V_w(f_x)$ are known:

$$|H(f_x)|^2 = \frac{V_n(f_x)}{\sigma_w^2} \quad \Rightarrow \quad H(f_x) = \sqrt{\frac{V_n(f_x)}{\sigma_w^2}}. \quad (2.55)$$

The filtering can then be conducted in the frequency domain. Figure 2.10 presents one possible realisation of $n'(x)$, for an outer scale L_0 equal to 100 meters and $C_n^2 = 10^{-16} \text{ m}^{-2/3}$.

2.5.3.2 One-dimensional power spectrum, covariance and structure functions

The power spectrum computed from the generated fluctuations is depicted in Figure 2.11. It is either approximated by averaging the Fourier transforms of 100 realisations, i.e., the Fast Fourier Transform (FFT) approach, or by using the Welch's

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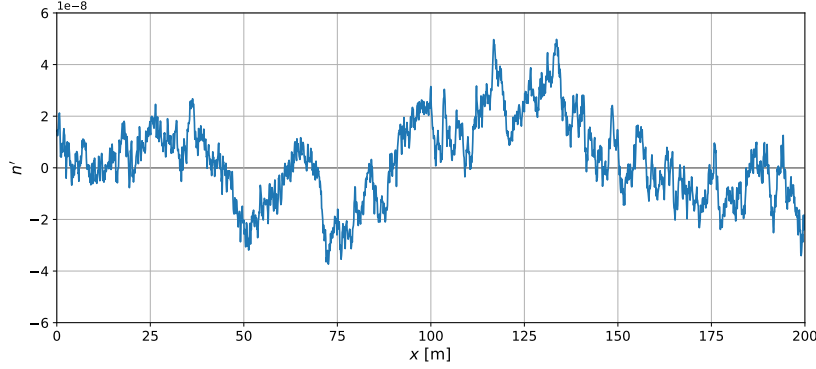


Figure 2.10 – Synthetic refractive index fluctuations for $C_n^2 = 10^{-16} \text{ m}^{-2/3}$ and $L_0 = 100 \text{ m}$.

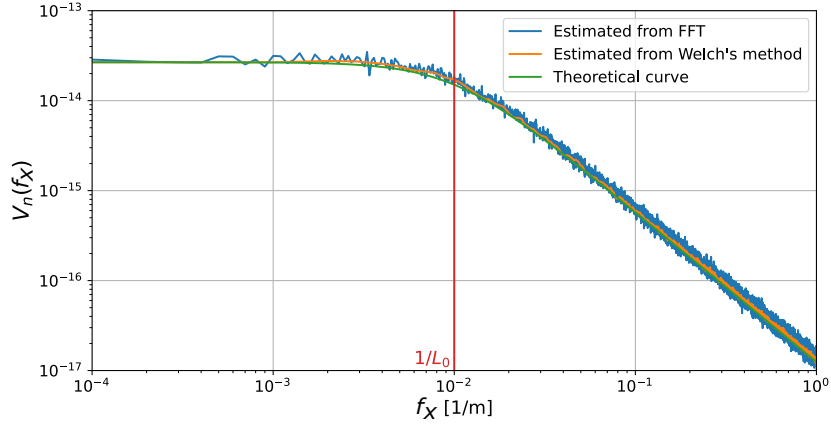


Figure 2.11 – Power spectrum of synthetic refractive index fluctuations for $C_n^2 = 10^{-16} \text{ m}^{-2/3}$ and $L_0 = 100 \text{ m}$.

method [Wel67]. It is in agreement with the theoretical expression (2.54), used for generating the fluctuations, hence validating the generation method.

Figure 2.12a shows the associated structure function $D_n(\rho)$, either computed from the generated fluctuations of $n'(x)$, or from its theoretical expression. The estimation of $D_n(\rho)$ from the fluctuations $n'(x)$ makes use of its statistical definition

$$D_n(\rho) = \langle ((n(x + \rho) - \bar{n}) - (n(x) - \bar{n}))^2 \rangle = \langle (n'(x + \rho) - n'(x))^2 \rangle, \quad (2.56)$$

evaluated for several distances ρ , multiples of the simulation sampling distance. The averaging is conducted spatially over the whole simulated distance.

The theoretical expression of $D_n(\rho)$ relies on (2.36), with the modified von Kármán spectrum used for Φ_n , as it enables to integrate over spatial frequencies ranging from 0 to infinity. Integration is conducted numerically, with an inner scale l_0 equal to 1 mm.

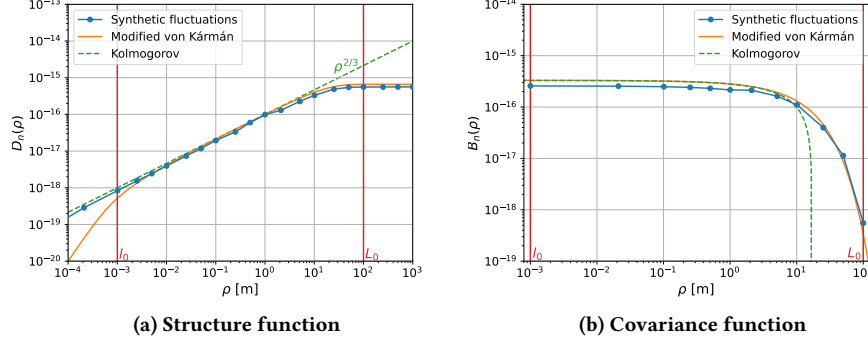


Figure 2.12 – Structure function and covariance computed from the generated fluctuations, compared to their theoretical expressions.

The Kolmogorov trend ($D_n(\rho) \propto \rho^{2/3}$) is also depicted in Figure 2.12a. Interestingly, it can be observed that the generated fluctuations follows the Kolmogorov trend up to the outer scale L_0 . Above L_0 , the structure function starts to saturate. The dissipation range ($\rho \ll l_0$) is not included in the power spectrum used to generate the fluctuations $n'(x)$, explaining why there is a discrepancy between the theoretical model of $D_n(\rho)$ using the modified von Kármán spectrum and the estimated values.

Finally, the definition of the covariance function

$$B_n(\rho) = \langle (n(x + \rho) - \bar{n})(n(x) - \bar{n}) \rangle = \langle n'(x + \rho) n'(x) \rangle \quad (2.57)$$

can be used to estimate it from the generated fluctuations in Figure 2.10, for different spacings ρ . The averaging is conducted over the whole simulated distance, and the estimated covariance function is depicted in Figure 2.12b.

Alternatively, the theoretical expression of the covariance comes from (2.34). Relying on numerical integration, the modified von Kármán spectrum is again used to obtain the theoretical curve of $B_n(\rho)$, plotted in Figure 2.12b. The Kolmogorov trend is also depicted, estimated from the $\rho^{2/3}$ trend in the structure function. Indeed, using (2.28), the covariance function can be obtained from $B_n(\rho) = B_n(0) - \frac{1}{2}D_n(\rho)$, with $B_n(0)$ coming from numerical integration of (2.34). It can be observed that close to the outer scale, the covariance decreases slower than the Kolmogorov trend. Again, inner scale effects are not included in the synthetic fluctuations, and cannot be observed in this figure.

2.5.3.3 Interpretation of main turbulence parameters C_n^2 , L_0 , and l_0

By observing the sequence of generated refractive index fluctuations and the associated statistical quantities, insights about the main turbulence parameters can be obtained.

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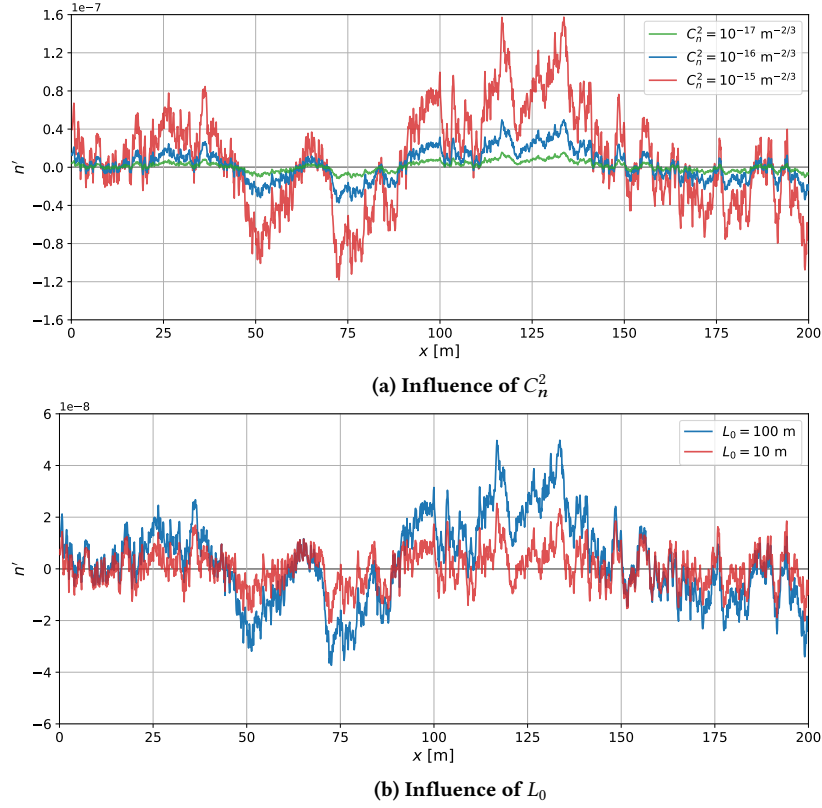


Figure 2.13 – Illustrations of the influence of C_n^2 and L_0 on the generated fluctuations.

Refractive index structure parameter C_n^2 The influence of C_n^2 on the sequence of generated fluctuations is presented in Figure 2.13a. It represents the same realisation of the white Gaussian noise, filtered by $H(f_x)$ specified in (2.55), with $L_0 = 100$ m and three different C_n^2 values given in the legend. The sequence with $C_n^2 = 10^{-16} \text{ m}^{-2/3}$ is the same as in Figure 2.10. From Figure 2.13a, it is clear that C_n^2 impacts the amplitudes of the fluctuations, such that the larger C_n^2 , the larger the fluctuations. This is a direct consequence of the multiplicative effect of C_n^2 on the power spectrum, and on the covariance and structure functions. Even though this observation is straightforward, it highlights the main interpretation of the refractive index structure parameter: C_n^2 is a statistical parameter describing the intensity of the refractive index fluctuations. It is similar to a variance, giving how strong the fluctuations around their average are.

Outer scale L_0 Modifying L_0 changes the corner frequency in the refractive index power spectrum (assuming that a power spectrum taking it into account has been considered), as shown in Figure 2.11. It also influences the distance at which fluctuations can be considered as correlated, see Figure 2.12. Finally, L_0 has an impact on

the amplitudes of the fluctuations. This effect is illustrated in Figure 2.13b. Again, the same white Gaussian noise realisation has been filtered with $H(f_x)$ having a C_n^2 of $10^{-16} \text{ m}^{-2/3}$, with an outer scale length of 10 m or 100 m. It shows that larger outer scales lead to larger fluctuations. This can be explained by observing the level of the power spectrum in its plateau region, i.e., for spatial frequencies lower than $\frac{1}{L_0}$. Indeed, for $f_x = 0$, the magnitude of the 1D von Kármán spectrum is

$$V_n(0) = 0.1244 C_n^2 \left(\left(\frac{1}{L_0} \right)^2 \right)^{-5/6} = 0.1244 C_n^2 L_0^{5/3}.$$

With $C_n^2 = 10^{-16} \text{ m}^{-2/3}$ and $L_0 = 100 \text{ m}$, it gives a numerical value of $V_n(0) = 2.68 \times 10^{-14}$, in agreement with Figure 2.11. Therefore, the multiplication of the power spectrum by L_0 explains the increase of the intensity of the fluctuations when L_0 grows. An alternative, and intuitive, explanation makes use of the correlation distance: since the correlation distance is longer for larger L_0 , upwards (or downwards) trends can last for longer distances, hence leading to higher fluctuations.

Inner scale l_0 The impact of the inner scale has not been studied on the generated 1D fluctuations. Nevertheless, l_0 influences high spatial frequencies, as illustrated in Figure 2.8. Its effect on the structure function is visible in Figure 2.12a.

2.5.4 Relationships between temperature, humidity, and refractive index structure parameters

Since turbulent fluctuations of any conservative passive additive can be described using Kolmogorov theory (see Section 2.5), several structure parameters are defined, such as the temperature structure parameter C_T^2 , the refractive index structure parameter C_n^2 , etc. The aim of this section is to explore the relationships between those structure parameters, namely for C_n^2 .

2.5.4.1 Radiofrequency refractive index

Starting from the expression of the refractive index of air for radiofrequencies given in (2.5), reminded below,

$$n = 1 + 10^{-6} \frac{77.6}{T} \left(p + \frac{4810 e}{T} \right), \quad (2.5)$$

fluctuations of the refractive index are related to fluctuations of temperature, water vapor and pressure. Indeed, the differential of n is

$$dn = \frac{\partial n}{\partial T} dT + \frac{\partial n}{\partial e} de + \frac{\partial n}{\partial p} dp, \quad (2.58)$$

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where the following quantities are defined

$$a = \frac{\partial n}{\partial T} = -10^{-6} \frac{77.6}{T^2} \left(p + \frac{9620e}{T} \right), \quad (2.59)$$

$$b = \frac{\partial n}{\partial e} = 10^{-6} \frac{3.73 \times 10^5}{T^2}, \quad (2.60)$$

$$c = \frac{\partial n}{\partial p} = 10^{-6} \frac{77.6}{T}. \quad (2.61)$$

By denoting the deviations from the mean by n' , T' , e' and p' , the fluctuations of n are given by [Gos77]

$$n' = aT' + be' + cp'. \quad (2.62)$$

Under the hypotheses presented in Section 2.4.2, the definition of the refractive index structure function $D_n(\rho)$ is

$$D_n(\rho) = \langle (n'(\mathbf{r} + \rho) - n'(\mathbf{r}))^2 \rangle, \quad (2.63)$$

in which fluctuations n' are substituted by (2.62). This substitution enables to relate the refractive index structure function to the structure function of the temperature $D_T(\rho)$, the partial pressure of water vapor structure function $D_e(\rho)$, the pressure structure function $D_p(\rho)$, as well as to some cross-structure functions (e.g., $D_{Te}(\rho)$, $D_{Tp}(\rho)$, etc.). The definitions of all those functions is similar to (2.63). As a result, the relationships between the different structure functions is given by [Gos77]

$$\begin{aligned} D_n(\rho) = & a^2 D_T(\rho) + b^2 D_e(\rho) + c^2 D_p(\rho) \\ & + 2ab D_{Te}(\rho) + 2ac D_{Tp}(\rho) + 2bc D_{ep}(\rho). \end{aligned} \quad (2.64)$$

In the inertial range, $D_n(\rho)$ is equal to $C_n^2 \rho^{2/3}$. Assuming a similar form for the other structure functions and using (2.64), C_n^2 can be expressed as

$$C_n^2 = a^2 C_T^2 + b^2 C_e^2 + c^2 C_p^2 + 2ab C_{Te} + 2ac C_{Tp} + 2bc C_{ep}. \quad (2.65)$$

This last result is presented in [Gos77], and, as already pointed out by [Moe03], is valid as long as the dependency in ρ is the same for all structure functions (hence, not necessarily $\rho^{2/3}$).

Since pressure fluctuations tend to be one order of magnitude smaller than temperature or humidity fluctuations [Gos77; Whe01], they are usually neglected in (2.65). This leads to

$$C_n^2 = a^2 C_T^2 + b^2 C_e^2 + 2ab C_{Te}.$$

However, the temperature T is not a conservative additive, as stated in Section 2.5.1. Following the recommendations in [CB13], it must be substituted by the potential temperature θ . Moreover, the specific humidity q [kg/kg] is used instead of the partial pressure of water vapor e [Hil+88]. These two quantities are linked by $e \approx 1.62 pq$ [CB13], the accurate expression being given in [FN08, p. 41].

Equation (2.58) now reads

$$dn = \frac{\partial n}{\partial \theta} d\theta + \frac{\partial n}{\partial q} dq + \frac{\partial n}{\partial p} dp,$$

and (2.62) is adapted accordingly

$$n' = A\theta' + Bq' + cp',$$

with new quantities A and B defined by

$$A = \frac{\partial n}{\partial \theta} = \frac{\partial n}{\partial T} \frac{dT}{d\theta} = a \left(\frac{p}{p_0} \right)^{R/c_p}, \quad (2.66)$$

$$B = \frac{\partial n}{\partial q} = \frac{\partial n}{\partial e} \frac{de}{dq} = 1.62 pb. \quad (2.67)$$

Using the same approach as above, and neglecting pressure fluctuations, C_n^2 at radiofrequencies is found to be related to C_θ^2 , C_q^2 and $C_{\theta q}$ by [Bel93, p. 172]

$$C_n^2 = A^2 C_\theta^2 + B^2 C_q^2 + 2AB C_{\theta q}. \quad (2.68)$$

In fact, (2.68) is general and sometimes used for the refractive index in the infrared region, also dependent on the humidity (see Section 2.1.2). In this case, the constants A and B must be recomputed, based on the refractive index expression as a function of the temperature and the humidity. It has also been applied by [Hil+88] for comparing C_n^2 at millimetre waves with optical C_n^2 . It is used in Section 4.5, highlighting the different contributions of C_θ^2 , C_q^2 and $C_{\theta q}$ in the final C_n^2 profile.

2.5.4.2 Optical refractive index

The expression of the optical refractive index as a function of the temperature, pressure, and wavelength is given in (2.3), reminded here (p is in hectopascal and λ in μm)

$$n = 1 + 10^{-6} \frac{77.6 p}{T} \left(1 + \frac{7.52 \times 10^{-3}}{\lambda^2} \right). \quad (2.3)$$

Since humidity does not influence the optical refractive index, and neglecting pressure fluctuations, C_n^2 for optical frequencies is directly related to C_T^2 by [Whe01, p. 54]

$$C_n^2 = \left(\frac{\partial n}{\partial T} \right)^2 C_T^2 = \left[\frac{77.6 \times 10^{-6} p}{T^2} \left(1 + \frac{7.52 \times 10^{-3}}{\lambda^2} \right) \right]^2 C_T^2. \quad (2.69)$$

For a wavelength $\lambda = 0.5 \mu\text{m}$, the relationship between C_n^2 and C_T^2 is [AP05, p. 65], [Rod81, p. 287]

$$C_n^2 = \left(\frac{80 \times 10^{-6} p}{T^2} \right)^2 C_T^2. \quad (2.70)$$

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This is a key result, very often used in the literature. As an example, its application to thermosonde measurements is presented in Chapter 4, Section 4.4. Equation (2.69) also enables to compute the ratio of C_n^2 at two different optical frequencies (e.g., λ_1 and λ_2 , in μm), highlighting the (small) frequency dependency of C_n^2 :

$$\frac{C_{n,\lambda_1}^2}{C_{n,\lambda_2}^2} = \frac{\left(1 + \frac{7.52 \times 10^{-3}}{\lambda_1^2}\right)^2}{\left(1 + \frac{7.52 \times 10^{-3}}{\lambda_2^2}\right)^2}.$$

In terms of potential temperature, one has

$$C_n^2 = \left(\frac{80 \times 10^{-6} p}{T^2}\right)^2 \left(\frac{\partial T}{\partial \theta}\right)^2 C_\theta^2 = \left(\frac{80 \times 10^{-6} p}{T \theta}\right)^2 C_\theta^2, \quad (2.71)$$

that corresponds to the result presented in [CB13], and that should be more appropriate since it does not assume that the temperature T is conservative. In (2.70) and (2.71), the pressure p is in hectopascal.

2.6 Chapter summary

The propagation of EM waves in the atmosphere is influenced by the atmospheric refractive index along the propagation path. This refractive index, which varies with altitude, is linked to macroscale meteorological quantities (e.g., pressure, temperature, etc.) using (2.3) to (2.6).

Atmospheric turbulence, driven by wind shear and convection, causes fluctuations in the refractive index. Predicting turbulence remains a complex problem, and while numerical simulations have become more feasible, they are not yet practical for accurately modelling the atmosphere across all spatial scales. Instead, this work exploits the statistical description of atmospheric turbulence.

The statistical description of atmospheric turbulence is based on Kolmogorov's hypotheses, as discussed in Section 2.4.2. It leads to the definition of the refractive index structure parameter C_n^2 , somehow representing the variance of the refractive index fluctuations caused by turbulence. Extensions of the presented framework to non-Kolmogorov turbulence are not explored in this work [SWR95; Gol+06].

In the following (Chapters 3 to 5), variations of C_n^2 with altitude are studied, exploring different models that relate this parameter to meteorological quantities. The consequences of refractive index fluctuations on EM waves (i.e., phase fluctuations and scintillation) are explored in Chapters 6 and 7.

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Modelling of C_n^2 profiles

3

Following the introduction and motivations for the refractive index structure parameter C_n^2 presented in Chapter 2, this chapter delves into various C_n^2 models available in the literature. These models describe how C_n^2 varies with altitude, referred to as C_n^2 (vertical) profiles, and are particularly relevant for the application of this work (i.e., satellite-to-ground optical communications).

A classification of literature-based C_n^2 profiles is proposed, based on the parameters involved in the models and the atmospheric layers they target (e.g., BL, FA). Models that use macroscale meteorological quantities such as the pressure, temperature, humidity, etc., to model C_n^2 profiles are of practical interest, as they can be adapted to local conditions. These models enable to translate meteorological quantities into C_n^2 profiles and are referred to as parametric model, following the terminology in [Goo+88; Bel93]. They are highlighted in Figure 3.1, presenting the focus of this chapter within the framework of this thesis. The input meteorological data for parametric models can originate from radiosonde (balloon) observations, which record vertical profiles of pressure, temperature, wind speed, and other variables as they ascend. Alternatively, data from NWP models can also be used.

C_n^2 profiles can be classified in various ways, depending on factors such as their origins (e.g., empirical, theoretical, numerical), the specific locations for which they are designed (e.g., high-altitude observatories), or the time of the day (daytime or nighttime models) [Qua+23]. Therefore, all of these factors must carefully be considered when applying any C_n^2 model and comparing them. Furthermore, it is essential

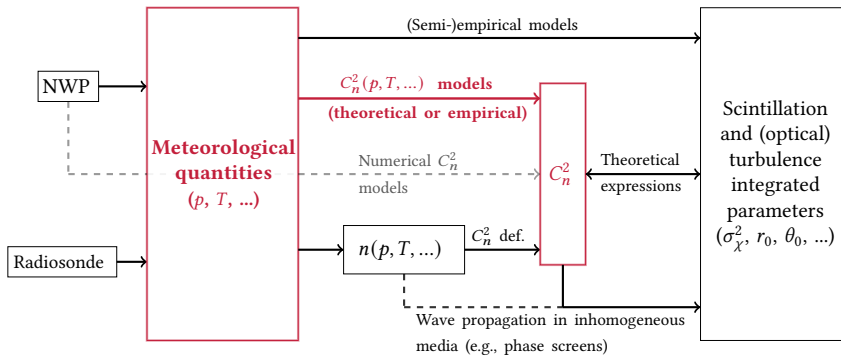


Figure 3.1 – Schematic of the thesis structure, highlighting the C_n^2 models detailed in Chapter 3.

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to validate C_n^2 models against actual C_n^2 measurements, which can be obtained using instruments such as thermosonde (i.e., balloons measuring temperature fluctuations, discussed in Section 4.4), scintillometers, SCIntillation Detection And Ranging¹ (SCIDAR), etc.

This chapter contributions are:

- An up-to-date state of the art of C_n^2 models available in the literature, complementing past reviews in [Goo+88; Bel93; Ric+06; Can14]. An update is necessary as most existing reviews are outdated and focused primarily on astronomical applications.
- A clear classification of C_n^2 models, highlighting the motivations behind the designs of the models, as well as their limitations. This is particularly important as models initially developed for astronomical sites are increasingly being applied to OGS sites, which may fall beyond their intended scope.
- An overview of key quantities for modelling C_n^2 in the FA and in the BL, such as temperature gradients or wind shear. These quantities are thus important to measure at OGS sites for possible application of the C_n^2 models.
- A strong focus on C_n^2 modelling in the BL, including the modelling of diurnal variations, essential for OGS sites.

The structure of this chapter is the following: first, Section 3.1 presents non-parametric C_n^2 models, derived from empirical observations. Then, Section 3.2 details the available parametric models, while Section 3.3 focuses on atmospheric BL models. Recent developments in Machine Learning (ML)-based models are presented in Section 3.4. Finally, Section 3.5 discusses relevant metrics used to validate and compare C_n^2 models.

3.1 Non-parametric models

Non-parametric models describe C_n^2 profiles solely as a function of altitude, i.e., $C_n^2(z) = f(z)$, where $f(\cdot)$ is a generic function that varies depending on the models. These models are simple to use and are typically derived from the averaging of measurement sets collected at a specific location during a particular season. As such, they are considered empirical models, dependent on both location and season. Moreover, due to this averaging, these models cannot capture the stratification in C_n^2 profiles or account for temporal variations. Nevertheless, despite these limitations, their application to obtain median or averaged OT integrated quantities (defined in Section 6.3), or to compare different sites, still makes them relevant [Goo+88, p. 171].

¹SCIDAR instruments are introduced in Section 6.4 of Chapter 6.

3.1.1 SLC model

The Submarine Laser Communication (SLC) model is an empirical profile derived from multiple measurements (e.g., scintillation, acoustic soundings, microthermal probes) above the Air Force Maui Optical Station (AMOS), Haleakalā, Hawaii [MZ78]. It was initially a nighttime model, but a daytime extension has been developed [Bel93]. Defining z as the altitude above the ground (being approximately at 3 km above the mean sea level at this location), the nighttime SLC model is given by [Bel93, p. 218]

$$C_n^2(z) = \begin{cases} 8.40 \times 10^{-15} & \text{for } z \leq 18.5 \text{ m} \\ 2.87 \times 10^{-12} z^{-2} & \text{for } 18.5 < z \leq 110 \text{ m} \\ 2.50 \times 10^{-16} & \text{for } 110 < z \leq 1500 \text{ m} \\ 8.87 \times 10^{-7} z^{-3} & \text{for } 1500 < z \leq 7200 \text{ m} \\ 2.00 \times 10^{-16} z^{-0.5} & \text{for } 7200 < z \leq 20000 \text{ m.} \end{cases}$$

The daytime SLC model only differs for the altitudes below 1500 m, and is obtained from

$$C_n^2(z) = \begin{cases} 1.70 \times 10^{-14} & \text{for } z \leq 18.5 \text{ m} \\ 3.13 \times 10^{-13} z^{-1} & \text{for } 18.5 < z \leq 240 \text{ m} \\ 1.30 \times 10^{-15} & \text{for } 240 < z \leq 880 \text{ m} \\ 8.87 \times 10^{-7} z^{-3} & \text{for } 880 < z \leq 7200 \text{ m} \\ 2.00 \times 10^{-16} z^{-0.5} & \text{for } 7200 < z \leq 20000 \text{ m.} \end{cases}$$

Comments about the model derivation and its limitations can be found in [Bel93, p. 218].

3.1.2 AFGL AMOS models

Based on more high-resolution balloon measurements above Mount Haleakalā, the nighttime SLC model has been refined, leading to the Air Force Geophysical Laboratory (AFGL) AMOS model [Bel93, p. 219], [Goo+88, p. 173]. As the SLC models, it is based on piecewise fitting of the measured profiles. The nighttime AFGL AMOS model is given by [Bel93, p. 219]

$$\log_{10}(C_n^2(z)) = \begin{cases} -12.412 - 0.4713 z - 0.0906 z^2 & \text{for } 3.052 < z \leq 5.2 \text{ km} \\ -17.1273 - 0.0301 z - 0.0010 z^2 + 0.5061 e^{-0.5 \left(\frac{z-15.0866}{3.2977} \right)^2} & \text{for } 5.2 < z \leq 30 \text{ km} \end{cases},$$

with z expressed in kilometres, above the mean sea level (the ground altitude is 3.038 km at Mount Haleakalā). The differences between the nighttime SLC and AFGL AMOS models are discussed in [Bel93, p. 219].

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A daytime version of the model is presented in [Goo+88, p. 173],

$$\log_{10}(C_n^2(z)) = \begin{cases} 0.0482 - 2.3416 z - 0.7211 z^2 & \text{for } 3.052 < z \leq 3.54 \text{ km} \\ -17.4778 + 0.0320 z + 0.0078 z^2 + 1.5066 e^{-0.5\left(\frac{z-4.4603}{0.0968}\right)^2} & \text{for } 3.54 < z \leq 5.1 \text{ km} \\ -16.5589 - 0.1424 z + 0.0030 z^2 + 1.5694 e^{-0.5\left(\frac{z-16.6300}{4.8757}\right)^2} & \text{for } 5.1 < z \leq 30 \text{ km} \end{cases},$$

still assuming that the model starts from 3.052 km of altitude, up to 30 km. A sunrise version is also available [Goo+88, p. 173],

$$\log_{10}(C_n^2(z)) = \begin{cases} -14.0245 - 0.4809 z - 0.0144 z^2 & \text{for } 3.052 < z \leq 5.78 \text{ km} \\ -16.7545 + 0.0259 z - 0.0022 z^2 - 0.6693 e^{-0.5\left(\frac{z-7.0330}{2.8558}\right)^2} & \text{for } 5.78 < z \leq 30 \text{ km} \end{cases}.$$

3.1.3 CLEAR I model

Using a similar approach as the AFGL AMOS models, the Critical Laser Enhancing Atmospheric Research (CLEAR) I model has been derived from measurements in the Tularosa Basin, New Mexico, during the summer [Bel93, p. 220], [TOV06]. The CLEAR I nighttime model is given by

$$\log_{10}(C_n^2(z)) = \begin{cases} -10.7025 - 4.3507 z + 0.8141 z^2 & \text{for } 1.23 < z \leq 2.13 \text{ km} \\ -16.2897 + 0.0335 z - 0.0134 z^2 & \text{for } 2.13 < z \leq 10.34 \text{ km} \\ -17.0577 - 0.0449 z - 0.0005 z^2 + 0.6181 e^{-0.5\left(\frac{z-15.5617}{3.4666}\right)^2} & \text{for } 10.34 < z \leq 30 \text{ km} \end{cases},$$

with z the altitude above mean sea level, in km (the ground altitude is 1.216 km in the location where the model was obtained). A revision of the CLEAR I model can be found in [TOV06, p. 35].

3.1.4 Greenwood model

Another model based on nighttime C_n^2 measurements at the AMOS is the Greenwood model [Gre77] [RWH96, p. 62]

$$C_n^2(z) = (2.2 \times 10^{-13} (z + 10)^{-1.3} + 4.3 \times 10^{-17}) e^{-z/4000}, \quad (3.1)$$

where z is the altitude above the ground. The model has initially been presented with a dependency to the zenith angle, substituting the altitude z in (3.1) by $l \cos(\tilde{\theta})$, with $\tilde{\theta}$ the zenith angle and l the distance from the telescope aperture (or from the ground station in the context of optical communication). It is considered as a “poor

seeing” model in [Goo+88, p. 179], while its “good seeing” version is given by

$$C_n^2(z) = (2.2 \times 10^{-12} (z + 10)^{-1.3} + 7 \times 10^{-16}) e^{-z/2000}.$$

3.1.5 Gracheva and Gurvich models

Gracheva and Gurvich models, presented in [Ric+06], provide some fitting of the logarithm in base 10 of the C_n^2 profile, similar to the AFGL AMOS and CLEAR I models. A separation between weak and strong turbulence conditions is made, leading to the weak turbulence model

$$\begin{aligned} \log_{10} \left(C_{n,\min}^2(z) - 5.19 \times 10^{-16} \times 10^{-0.00086 z} \right) \\ = -18.34 + 2.9 \times 10^{-4} z - 2.84 \times 10^{-8} z^2 + 7.43 \times 10^{-13} z^3, \end{aligned}$$

and the strong turbulence model

$$\begin{aligned} \log_{10} \left(C_{n,\max}^2(z) - 9.5 \times 10^{-14} \times 10^{-0.00209 z} \right) \\ = -14.39 + 1.7 \times 10^{-4} z - 3.48 \times 10^{-8} z^2 + 9.59 \times 10^{-13} z^3. \end{aligned}$$

A model for average atmospheric conditions is derived from the average of the weak and strong turbulence models, i.e.,

$$\log_{10} (C_{n,\text{av}}^2(z)) = \frac{1}{2} \left[\log_{10} (C_{n,\min}^2(z)) + \log_{10} (C_{n,\max}^2(z)) \right].$$

3.1.6 Maritime C_n^2 models

The maritime C_n^2 models, presented in [MNJ00] and inspired from the Hufnagel-Valley model (see Section 3.2.1.1), aim at providing a standard profiles for maritime conditions. They have been derived from a collection of measurements near or over the ocean, and use the generic form

$$C_n^2(z) = c_1 + c_2 e^{-z/c_3} + c_4 e^{-z/c_5}. \quad (3.2)$$

The altitude z is in meters and spans from 0 to 6000 m. The values of the different coefficients c in (3.2) depend on the considered cases (worst, median, or best conditions) and are given in Table 3.1.

Table 3.1 – Coefficients in maritime C_n^2 models, in (3.2) [MNJ00].

Case	c_1 [$\text{m}^{-2/3}$]	c_2 [$\text{m}^{-2/3}$]	c_3 [m]	c_4 [$\text{m}^{-2/3}$]	c_5 [m]
“Best”	9.8286×10^{-18}	7.1609×10^{-17}	100	1.9521×10^{-17}	1500
“Median”	9.8583×10^{-18}	4.9877×10^{-16}	300	2.9228×10^{-16}	1200
“Worst”	9.2002×10^{-18}	9.4387×10^{-15}	800	6.7328×10^{-16}	1000

3.2 Parametric models

Unlike non-parametric models, parametric models of C_n^2 profiles account for location and seasonal dependencies. These models are based on the general expression $C_n^2(z) = f(z, t, p, T, \dots)$, where C_n^2 is related to macroscale meteorological parameters (e.g., pressure p , temperature T), time of day t , and other factors. Parametric models can have a theoretical basis, such as Tatarskii-based models presented in Section 3.2.2, or an empirical origin, like the Hufnagel-Valley (HV) model or the Trinquet-Vernin (TV) model given in Sections 3.2.1 and 3.2.3.

3.2.1 Hufnagel-Valley model and associated models

The origin of the HV model, along with its commonly used extensions in the literature, are presented in this section. These are simple models involving few parameters, and when the values of these parameters are known, they can be treated as non-parametric models (e.g., the HV-5/7 model described in Section 3.2.1.2).

3.2.1.1 Hufnagel-Valley model

Early C_n^2 models, referenced in [Goo+88, p. 170], proposed an exponential decrease of C_n^2 with altitude. Later observations revealed that this average exponential decrease was primarily valid in the troposphere, and that the models needed to be extended to account for contributions from the tropopause [Buf73]. These considerations are apparent in the Hufnagel model, which was derived from stellar observations and thermosonde measurements [Buf75]. Its simplest form is given by [Huf74]

$$C_n^2(z) = 2.7 \times 10^{-16} e^{-z/1500} + 8.2 \times 10^{-56} \tilde{w}^2 z^{10} e^{-z/1000}, \quad (3.3)$$

where z is the altitude in meters above the sea level, and $\tilde{w}$ is the Root-Mean-Square (RMS) wind speed, averaged from 5 km to 20 km of altitude. It is only valid above the first strong inversion layer, setting its limits from 3 km to 24 km of altitude [Goo+88, p. 174]. Interestingly, (3.3) contains two terms, the first one being an exponential decrease associated with the troposphere, and the second one representing a bump in the C_n^2 profile at the tropopause and its stratospheric decrease. The origin of the coefficient 2.7 in front of the exponential decrease is briefly detailed in [Goo+88, p. 175], while the RMS wind speed can be computed from the wind speed profile $v(z)$, with

$$\tilde{w}^2 = \frac{1}{15000} \int_{5000}^{20000} v^2(z) dz. \quad (3.4)$$

This explains why the Hufnagel model belongs to the parametric models, as it depends on local meteorological quantities (wind speed profile) through the parameter $\tilde{w}$.

In [Huf74], a value of 27 m/s is suggested for the parameter $\tilde{w}$ in order to obtain a fixed model. Random models can also be generated by considering $\tilde{w}$ as a Gaussian random variable, having a mean value of 27 m/s and a standard deviation of 9 m/s.

This particular value of 27 m/s led to the rewriting of the Hufnagel model with [AP05, p. 481]

$$C_n^2(z) = 2.7 \times 10^{-16} e^{-z/1500} + 0.00594 \left(\frac{\tilde{w}}{27} \right)^2 \left(\frac{z}{10^5} \right)^{10} e^{-z/1000}.$$

Later, according to [Goo+88, p. 175] and [Bel93, p. 222], Ulrich [Ulr88] extended the Hufnagel model to include a BL term, following a suggestion of Valley [Val80]. This led to the well-known Hufnagel-Valley model, given by

$$C_n^2(z) = A e^{-z/100} + 2.7 \times 10^{-16} e^{-z/1500} + 0.00594 \left(\frac{\tilde{w}}{27} \right)^2 \left(\frac{z}{10^5} \right)^{10} e^{-z/1000}. \quad (3.5)$$

In (3.5), the BL term is assumed to be exponential and involves the parameter A , associated to the ground C_n^2 value [$\text{m}^{-2/3}$]. As a result, the HV model has two parameters, A and $\tilde{w}$, that can be adjusted to model C_n^2 variations at high altitudes and near the ground. This offers a certain freedom to adapt the model to different geographical locations [And+09], even though it remains mostly suited for mid-latitude regions [Goo+88, p. 175]. The ITU Recommendation P.1621-2 [ITU15] suggests using the HV model to calculate C_n^2 profiles for frequencies between 20 THz and 375 THz. Figure 3.2 shows C_n^2 profiles obtained using the HV model with different parameters A and $\tilde{w}$. These profiles are compared with the HV-5/7 model ($A = 1.7 \times 10^{-14} \text{ m}^{-2/3}$ and $\tilde{w} = 21 \text{ m/s}$), described below. In the legend, parameters not specified correspond to those of the HV-5/7 model. The figure highlights that the parameter A primarily affects C_n^2 in the BL, while $\tilde{w}$ influences C_n^2 in the tropopause and stratosphere.

3.2.1.2 Applications of Hufnagel-Valley model

The adaptability of the HV model, along with its simplicity, explain its widespread use for practical and theoretical studies. Some of them are detailed in this section.

Hufnagel-Valley 5/7 model The most well-known parametrization of the HV model is the Hufnagel-Valley 5/7 model, denoted HV-5/7. The parameters are: $A = 1.7 \times 10^{-14} \text{ m}^{-2/3}$ and $\tilde{w} = 21 \text{ m/s}$ [Bel93, p. 222].

The HV-5/7 model is named after the values of some OT integrated parameters that it yields. Indeed, vertical integration of the HV-5/7 model using (6.32) and (6.36) (detailed in Section 6.3) gives a coherence length of 5 cm, and an isoplanatic angle of $7 \mu\text{rad}$, assuming a wavelength of 500 nm.

Hufnagel-Valley night model A parameterization of the HV model compatible with observations at AMOS is presented in [PS94], that is,

$$C_n^2(z) = 1.9 \times 10^{-15} e^{-z/100} + 3.02 \times 10^{-17} e^{-z/1500} + 8.16 \times 10^{-54} z^{10} e^{-z/1000},$$

with z the altitude in meters. This modified version of the HV model yields a coherence length of 20 cm and an isoplanatic angle of $20 \mu\text{rad}$, for a wavelength of 550 nm,

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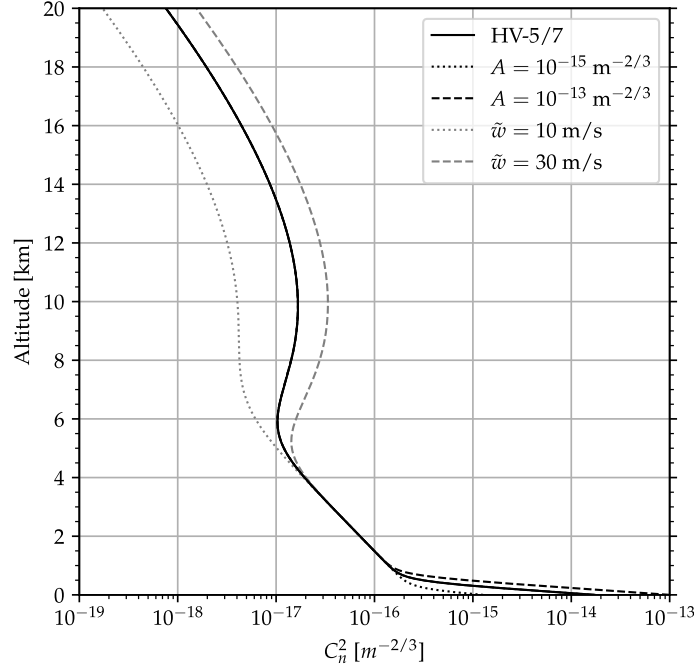


Figure 3.2 – Hufnagel-Valley model variations with $\tilde{w}$ and A parameters (reproduced from [QVO23]).

at zenith.

Models of wind speed profile The determination of the $\tilde{w}$ parameter can be achieved using (3.4), which requires the knowledge of the wind speed profile. In cases where this profile cannot be observed, the Bufton wind model (or Greenwood model, according to [RB11]) is often used in the literature. It is inspired by simultaneous measurements of C_n^2 , temperature, and wind velocity collected by Bufton [Buf73] [Har98, p. 87], and is presented in [Gre77]

$$v(z) = v_G + 30 e^{-\left(\frac{z-9400}{4800}\right)^2},$$

and includes a constant mean value for the wind profile associated to the ground wind speed v_G (usually set to 8 m/s [Gre77], or 5 m/s [PS94]), on which a Gaussian is added to represent the jet stream (at the altitude of the tropopause, here 9400 m above the ground). Initially, the model was given with a dependency to the elevation angle, as already presented in Section 3.1.4 for the Greenwood C_n^2 model.

The Bufton model is derived from rawinsonde measurements in Hawaii, and assumes z to be the altitude above the ground (located at 3048 m above the mean sea level). Hence, Roberts recommends to add the ground altitude to the tropopause term (9400 m) when applying the model to other locations, even though it does not seem

to be suited to other locations than Hawaii [RB11]. The generalization of this model is given by [RB11]

$$v(z) = v_G + v_T e^{-\left(\frac{z-h_T}{L_T}\right)^2}, \quad (3.6)$$

where v_G is the ground (or low altitude) wind speed, v_T is the tropopause wind speed, h_T is the tropopause altitude, and L_T is the thickness of the tropopause layer. For ground-to-satellite communications with moving satellites, an extra term is added to the model to take into account this motion as an apparent wind [AP05, p. 481], [MR10, p. 73]

$$v(z) = \omega_s z + v_G + v_T e^{-\left(\frac{z-h_T}{L_T}\right)^2},$$

with ω_s the satellite slew rate [rad/s].

Modified Hufnagel-Valley model for optical communications In order to obtain more realistic simulations of optical communication scenarios, at the ESA OGS site at Tenerife for example, the modified HV model has been suggested in [Gig05; Cal+14]

$$C_n^2(z) = A e^{-h_{GS}/700} e^{-(z-h_{GS})/100} + 2.7 \times 10^{-16} e^{-z/1500} + 0.00594 \left(\frac{\tilde{w}}{27}\right)^2 \left(\frac{z}{10^5}\right)^{10} e^{-z/1000}.$$

Its main feature is the presence of h_{GS} , that corresponds to the ground station altitude (above mean sea level). Hence, the model is only suited for altitudes z (above mean sea level) that are larger than the ground station altitude.

Parametrization from OT integrated parameter measurements If OT integrated parameters are measured at a location, a probable C_n^2 profile can be obtained from the HV model by fitting its parameters A and $\tilde{w}$. This approach requires knowledge of at least two integrated parameters, such as the coherence length r_0 and the isoplanatic angle θ_0 , as presented in [Tys96].

3.2.1.3 Generalized Hufnagel-Valley model

A generalization of the Hufnagel-Valley model is presented in [Har98, p. 85], that is convenient to particularize the HV model to different locations, as well as to add localized turbulent layers. The generalized HV form is

$$C_n^2(z) = A e^{-\frac{z}{h_A}} + B e^{-\frac{z}{h_B}} + C z^{10} e^{-\frac{z}{h_C}} + D e^{-\frac{(z-h_D)^2}{2d^2}}. \quad (3.7)$$

In (3.7), A is the ground turbulence strength and h_A is the altitude associated to its $1/e$ decay, while B and h_B have similar interpretation but for the troposphere. The tropopause is modelled with C and h_C , and the last term actually enables to add localized turbulent layers, modelled by a Gaussian located at an altitude h_D , having a standard deviation (i.e., thickness) d , and a C_n^2 magnitude of D . This last term can be

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repeated several times with different coefficients to model multiple isolated layers.

The benefit of this generalized model is that it can be easily adapted to any location where C_n^2 measurements are available, while being easily integrated to give optical turbulence integrated parameters (e.g., coherence length, isoplanatic angle). As an example, the values of the different coefficients involved for applying (3.7) to Mauna Kea, Hawaii, are given in [Har98, p. 86].

3.2.1.4 Hufnagel/Andrews/Phillips model and extensions

The Hufnagel/Andrews/Phillips (HAP) model aims at improving the modelling of the ground layer given by the HV model. Indeed, numerous measurements in the atmospheric BL showed a decrease of the C_n^2 profile following a $z^{-4/3}$ trend instead of the exponential decrease from the HV model (see Section 3.3.2). Hence, the BL term in (3.5) should be substituted with a term proportional to $z^{-4/3}$, as suggested in the HAP model [And+09]

$$C_n^2(z) = C_n^2(h_0) \left(\frac{h_0}{z} \right)^{4/3} + 2.7 \times 10^{-16} e^{-z/1500} + 0.00594 \left(\frac{\tilde{w}}{27} \right)^2 \left(\frac{z}{10^5} \right)^{10} e^{-z/1000}, \quad (3.8)$$

for altitudes z (in meters, above the mean sea level) higher than the ground altitude, given by h_0 . Expressing z as the altitude above the ground requires to substitute $\left(\frac{h_0}{z} \right)^{4/3}$ by $\left(\frac{h_0}{z+h_0} \right)^{4/3}$ in (3.8). $C_n^2(h_0)$ is the ground C_n^2 value, hence corresponding to the A parameter in the HV model in (3.5). Due to this $z^{-4/3}$ trend, the C_n^2 in the BL decreases faster with increasing altitudes in the HAP model than in the HV model. However, for altitudes higher than 1 km, both models are similar.

Later revisions of the HAP model suggest the addition of a multiplicative factor $\tilde{M}$ for the FA (troposphere + stratosphere) part, as well as a parametric exponent α instead of the fixed 4/3 exponent in the BL [And+12], leading to

$$C_n^2(z) = C_n^2(h_0) \left(\frac{h_0}{z} \right)^\alpha + \tilde{M} \left[2.7 \times 10^{-16} e^{-\frac{z+h_s}{1500}} + 0.00594 \left(\frac{\tilde{w}}{27} \right)^2 \left(\frac{z+h_s}{10^5} \right)^{10} e^{-\frac{z+h_s}{1000}} \right], \quad (3.9)$$

where h_s is the altitude of the ground above the mean sea level, and h_0 is the height above ground of the ground station or telescope. By explicitly taking h_s into account in (3.9), z now corresponds to the altitude above the ground surface, higher than h_0 . Besides these two altitude parameters (h_s and h_0), the revision of the HAP model includes four other parameters: $\tilde{M}$ and α that are new additions to the model, and $C_n^2(h_0)$ and $\tilde{w}$, similar to the previous HAP and HV models.

The parameter $\tilde{M}$ comes from an average value of the random background turbulence above 1 km, and is typically close to unity [And19, p. 88]. In [Can14], a seasonal variation of $\tilde{M}$ is presented based on one-year analysis of radiosonde observations near Munich, Germany. Modifying $\tilde{M}$ can also be useful to model percentiles of the turbulence statistics, i.e., modulate the amplitude of the HAP profile (or of the HV profile, if $\tilde{M}$ is added in (3.5) for example) in order to fit observed distributions of the C_n^2 profile above a given location. This is the approach followed in [SA21] with the

HV model, using turbulence statistics collected in Korea, in 1999. A factor $\tilde{M} = 1$ is found for the median profile (50%-percentile).

Regarding the exponent α , it varies between $4/3$ during daytime and $2/3$ at night. The following model has been suggested for daytime (between sunrise and sunset)[And+12]

$$\alpha = \begin{cases} -0.11(12 - \text{TH})^2 + 1.83(12 - \text{TH}) - 6.22, & 0.75 < \text{TH} < 3.5 \\ 1.45 - 0.02(\text{TH} - 6)^2, & 3.5 < \text{TH} < 8.5 \\ -0.048 \text{TH}^2 + 0.68 \text{TH} - 1.06, & 8.5 < \text{TH} < 11.25 \end{cases}, \quad (3.10)$$

that requires the knowledge of the sunrise and sunset times, as well as the computation of the temporal hour TH. The latter is given by equally dividing the time between sunrise (at t_{sunrise}) and sunset (at t_{sunset}) in 12 hours, and computing [SK92]

$$\text{TH}(t) = \frac{t - t_{\text{sunrise}}}{\frac{t_{\text{sunset}} - t_{\text{sunrise}}}{12}}, \quad (3.11)$$

with t the local time. For nighttime (between sunset and sunrise), α is set to $2/3$. The model for α in (3.10) has been obtained in Hollister, CA, and may require to be adapted to other locations [And+12].

For the parameter $C_n^2(h_0)$, corresponding to the C_n^2 value at the altitude h_0 , i.e., at or near the ground, it is usually measured or estimated using ground layer C_n^2 models (see Section 3.3).

Finally, the wind speed parameter $\tilde{w}$ is the same as for the HV model, and can be computed using (3.4) with measured or modelled wind profiles. Recent works also explored its substitution by the wind speed profile $v(z)$ [SA23; Mah+22].

3.2.2 Tatarskii-based models

In his pioneering work, Tatarskii proposed the following equation to relate the temperature structure parameter C_T^2 to macroscale atmospheric quantities [Tat71, p. 74]

$$C_T^2 = a^2 \tilde{\alpha} L_0^{4/3} \left(\frac{d\bar{T}}{dz} \right)^2, \quad (3.12)$$

where a^2 is an empirical constant, $\tilde{\alpha} = K_H/K_M$ is the ratio between the exchange coefficient for heat and momentum [CB13], L_0 is the outer scale of turbulence, and $\bar{T} = \langle T \rangle$.

For the refractive index structure parameter C_n^2 , (3.12) is

$$C_n^2 = a^2 \tilde{\alpha}' L_0^{4/3} M^2,$$

with $M^2 = \left(\frac{dn}{dz} \right)^2$ and $\tilde{\alpha}'$ similar to $\tilde{\alpha}$. The latter is close to one [Van+78] and usually

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neglected in the literature, such that the Tatarskii equation for C_n^2 is

$$C_n^2 = a^2 L_0^{4/3} M^2, \quad (3.13)$$

with a value of 2.8 often used for a^2 [Bel93, p. 173].

In (3.13), M is the vertical gradient of the mean (non-fluctuating) refractive index [Bel93, p. 173], which can be computed directly from the refractive index definitions in (2.3) to (2.6) and the derivative chain rule. For example, using the approximate optical refractive index definition in (2.4), M is given by

$$M = \frac{dn}{dz} = \frac{\partial n}{\partial T} \frac{dT}{dz} = -\frac{80 \times 10^{-6} p}{T^2} \frac{dT}{dz},$$

which agrees with (2.70), relating C_n^2 and C_T^2 .

Other expressions for M can be found in the literature depending on the conservative variables used (refer to Section 2.5.4), as discussed in [CB13]. Using the pseudopotential temperature $\tilde{\theta}$ in (2.44), M is given by [Tat71; CB13]

$$M = -\frac{80 \times 10^{-6} p}{T^2} \left(\frac{dT}{dz} + \gamma_a \right),$$

while using the potential temperature θ leads to [CB13]

$$M = -\frac{80 \times 10^{-6} p}{T\theta} \frac{d\theta}{dz}. \quad (3.14)$$

For radiofrequencies, the contributions of the humidity in the refractive index should also be included in M , for which the developments can be found in [Tat71, p. 75], [CB13].

In addition to defining M , applying the Tatarskii equation (3.13) necessitates a parametrization of the outer scale L_0 . This section reviews the various models for L_0 that have been proposed in the literature.

3.2.2.1 Dewan model and extensions (outer scale models using wind shear and temperature gradient)

Among the Tatarskii-based models, the Dewan model, and its extensions, are very often used in the literature thanks to their simplicity of application. Indeed, they can easily be applied to meteorological quantities coming from radiosonde observations [Xu+22; Wu+20], or from numerical weather predictions [Bus+02; Qin+20; Jum+07; Fre19]. All are based on the following form of the Tatarskii equation [Dew93, p. 2]

$$C_n^2 = 2.8 L_0^{4/3} M^2, \quad (3.15)$$

where the outer scale L_0 comes from different parametrizations. Those parametrizations are derived from empirical observations and fitting, involving the horizontal

wind shear S and the vertical gradient of temperature $\frac{dT}{dz}$, that is,

$$L_0^{4/3} = f\left(S, \frac{dT}{dz}\right).$$

The distinction between the Dewan model and its different extensions is in the expression of the function $f\left(S, \frac{dT}{dz}\right)$. Several expressions are presented in the paragraphs below.

As detailed in [Dew93, p. 1], Dewan models are only suited for modelling the troposphere and stratosphere, and not the BL.

Dewan model The Dewan model, initially referred to as the AFGL model, has been presented in 1993 in [Dew93]. It relies on (3.15), using the following expression for M ,

$$M^2 = \left[\left(\frac{79 \times 10^6 p}{T^2} \right) \left(\frac{dT}{dz} + \gamma_a \right) \right]^2,$$

where T is the absolute atmospheric temperature in kelvin, p is the pressure in mb, γ_a is the adiabatic lapse rate (9.8 K/km), and z is the altitude above the ground.

In order to model the outer scale of turbulence L_0 , a rule of thumb, inspired by the literature at that time, is used: L_0 is assumed to be of the order of 0.1 times the thickness of the turbulent layer [Dew93, p. 2]. As a result, knowledge of the turbulent layer thickness enables to determine L_0 .

Relying on high resolution (vertical resolution of 10 meters) stratospheric velocity profiles obtained by rocket laid smoke trails, the turbulent layer thickness is estimated from the Richardson number Ri . The expression used by Dewan is similar to (2.22), except that the absolute temperature is used:

$$Ri = \frac{\tilde{N}^2}{S^2},$$

where $\tilde{N}^2 = \frac{g(\frac{dT}{dz} + \gamma_a)}{T}$ is the “buoyancy frequency” (Brunt-Väisälä frequency), and $S = \sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2}$ is the wind shear.

Indeed, as stated in Section 2.2.3, a layer is considered turbulent if the Richardson number is smaller than a threshold, usually set to 0.25. In [Dew93], a value of 0.5 is chosen, and the turbulent layers can be identified from the velocity measurements and the criterion $S^2 > 2\tilde{N}^2$. Then, their thickness is computed. This approach is further detailed in [Dew93], and has been extended in [DG07].

Finally, the thickness of the different turbulent layers is associated to the large scale shears measured by radiosonde observations (vertical resolution of 300 meters). A statistical relationship is determined by averaging the layer thicknesses over 300-

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meter slabs, leading to the following fit:

$$Y = \begin{cases} 1.64 + 42.0 S_{\text{raw}} & (\text{troposphere}) \\ 0.506 + 50.0 S_{\text{raw}} & (\text{stratosphere}), \end{cases} \quad (3.16)$$

where Y corresponds to the logarithm in base 10 of the average turbulent layer thicknesses. S_{raw} is the wind shear S as measured by the radiosondes (referred to as raw-sondes in the original paper), having a 300-meter vertical resolution. Hence, the outer scale is modelled by

$$L_0^{4/3} = 0.1^{4/3} 10^Y. \quad (3.17)$$

Equations (3.16) and (3.17) are simple to use, but it is important to consider the conditions under which they have been obtained, highlighting their limitations:

- First, the fitting coefficients were obtained by using radiosonde observations with a 300-meter vertical resolution. As a result, this same resolution should be used when computing the wind shear to insert in (3.16) [Dew93, p. 3, 28].
- Second, (3.16) requires the knowledge of the tropopause altitude, separating the troposphere and the stratosphere. Moreover, only stratospheric velocity profiles have been measured, hence the tropospheric part of the model may not be very accurate [Dew93, p. 7].
- Finally, some upper limits on the computed meteorological quantities from the radiosonde observations are used: wind shear larger than 0.04 s^{-1} are set to 0.04 s^{-1} in the model, and tropospheric temperature gradients are set to 10^{-3} K/m whenever they exceed zero [Dew93, p. 8].

Those limitations are rarely taken into account while applying the Dewan model to new observations at other locations.

Validation of the Dewan model is presented in the original paper [Dew93], comparing the model predictions with thermosonde measurements from the CLEAR-1 and CLEAR-3 programs in New Mexico (USA), and from the Hawaiian program in 1984.

Other values for the coefficients in (3.16) fitted specifically for the Terskol Observatory (Caucasus, Russia) can be found in [Shi+24].

Jackson In 2004, a report from the Air Force Research Laboratory (AFRL), written by Jackson, presented some modified-Dewan parametrizations for the outer scale. The approach used to obtain the new parametrizations is different than the one previously followed by Dewan, and relies on statistical regressions.

Indeed, from C_n^2 measurements obtained by thermosondes, one can invert the Dewan model and isolate the Y parameter, called Y_{obs} in [Jac04, p. 4], that is,

$$Y_{\text{obs}} = \log_{10} \left(\frac{C_{n,\text{meas}}^2}{2.8M^2 0.1^{4/3}} \right).$$

From scatter plots of Y_{obs} against the wind shear, measured during the Holloman Spring 1999 campaign, little relationship was found between these two parameters. However, scatter plots involving the vertical gradient of temperature showed stronger relationships, motivating its use for predicting L_0 . Consequently, several regressions involving linear functions of the wind shear and up to fourth-order polynomials of the temperature vertical gradient have been derived and presented in [Jac04]. Special care has been devoted to the lower troposphere (from 1572 to 5472 meters at Holloman, New Mexico), establishing valid parametrizations for this part of the atmosphere². All parametrizations can be found in [Jac04, p. 13, 16, 19]. Only the most complex parametrizations are presented below, using the fourth-order polynomials of the temperature vertical gradient:

$$Y = \begin{cases} 2.9767 + 27.9804 S + 2.9012 \frac{dT}{dz} + 1.1843 \left(\frac{dT}{dz}\right)^2 + 0.1741 \left(\frac{dT}{dz}\right)^3 + 0.0086 \left(\frac{dT}{dz}\right)^4 \\ 0.7152 + 30.6024 S + 0.0003 \frac{dT}{dz} - 0.0057 \left(\frac{dT}{dz}\right)^2 - 0.0016 \left(\frac{dT}{dz}\right)^3 + 0.0001 \left(\frac{dT}{dz}\right)^4 \\ 0.6763 + 8.1569 S - 0.0536 \frac{dT}{dz} + 0.0084 \left(\frac{dT}{dz}\right)^2 - 0.0007 \left(\frac{dT}{dz}\right)^3 + 0.0002 \left(\frac{dT}{dz}\right)^4 \end{cases} \quad (3.18)$$

with the first equality valid in the lower troposphere, the second in the troposphere, and the last in the stratosphere. The expressions in (3.18), transcribed from Jackson's report, uses the temperature vertical gradient expressed in [$^{\circ}\text{C}/\text{km}$], and not in [K/m]. A reasonable fit for both the troposphere and the stratosphere is also presented in [Jac04, p 21]. All fits were obtained from the thermosonde observations of the AFRL Holloman Spring 1999 campaign. As in the original model from Dewan, a vertical resolution of 300 meters is chosen.

The modified-Dewan models have been applied to 8 thermosonde launches conducted during the Holloman Spring 1998 campaign (launches HMNSP104, HMNSP107, HMNSP108, and HMNSP111 to HMNSP115). Most improvements compared to the Dewan model have been found to be located in the lower troposphere part. Nevertheless, the model has been developed and tested for only one location (Holloman, New Mexico), and for one season (spring), hence its performance for other locations and seasons are uncertain [Jac04, p. 61].

HMNSP99 model The HMNSP99 model, attributed to [RD02], is another parametrization of L_0 inspired by the Dewan model. It makes use of linear relationships of the wind shear and the temperature vertical gradient, that is, [Qin+20; Wu+20]

$$Y = \begin{cases} 0.362 + 16.728 S - 192.347 \frac{dT}{dz} & (\text{troposphere}) \\ 0.757 + 13.819 S - 57.784 \frac{dT}{dz} & (\text{stratosphere}), \end{cases} \quad (3.19)$$

²The distinction between the different parts of the atmosphere, used in [Jac04], is the following: lower troposphere from 1572 to 5472 meters of altitude, then the troposphere up to the tropopause, and above the stratosphere. The tropopause height is found using the World Meteorological Organization definition, defining the tropopause height as “the lowest height in the atmosphere where the lapse rate [i.e., the temperature vertical gradient] decreases to an average of $20^{\circ}\text{C}/\text{km}$ for a 2 km layer.” [Jac04, p. 6]

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with $\frac{dT}{dz}$ in [K/m]. This model is applied in numerous recent papers [Qin+20; Wu+20; Xu+22]. The influence of the vertical resolution of radiosonde observations used with the HMNSP99 model is briefly studied in [Hu+23b].

WSTG Based on radiosonde and thermosonde observations collected from November to December 2017 in Sanya (Hainan island, China), the Wind Shear and Temperature Gradient (WSTG) model has been developed [Xu+22]. As the HMNSP99 model, it depends linearly with the wind shear and the vertical temperature gradient. However, the model originality lies on the boundary between the different parametrization, that is dependent on the values of S and $\frac{dT}{dz}$, and not on the altitude. Indeed, the outer scale is modelled by (3.17), with

$$Y = \begin{cases} 0.835 - 37.164S - 306.034\frac{dT}{dz}, & S < 0.016 \cap \frac{dT}{dz} < 0 \\ 0.825 + 66.9S - 52.783\frac{dT}{dz}, & S < 0.016 \cap \frac{dT}{dz} > 0 \\ 0.715 + 52.907S - 102.515\frac{dT}{dz}, & S > 0.016 \cap \frac{dT}{dz} < 0 \\ 2.215 - 9.882S - 101.666\frac{dT}{dz}, & S > 0.016 \cap \frac{dT}{dz} > 0. \end{cases}$$

The symbol $\cap$ denotes the intersection, meaning that both conditions should be fulfilled. This model derives from observed increases in the correlations of statistical operators when wind shears are larger than 0.016 m/s, and coefficients have been obtained by minimizing the residual sum of squares between measured values and estimations. A vertical resolution of 100 meters is used.

3.2.2.2 Altitude-dependent outer scale models

As an alternative to more complex models of the outer scale depending on meteorological parameters, simple altitude-dependent models have been suggested in the literature. All of them are derived from C_n^2 (or C_T^2) measurements, fitting for the values of L_0 giving C_n^2 profiles as close as possible to the measurements when using the Tatarskii model in (3.13). Alternatively, the observed outer scale profile can be computed by isolating L_0 in the Tatarskii model, i.e.,

$$L_{0,\text{meas}} = \left(\frac{C_{n,\text{meas}}^2}{a^2 M^2} \right)^{3/4}. \quad (3.20)$$

The vertical resolution chosen to compute the outer scale from the measurements is important. Indeed, coarse vertical resolution can lead to average L_0 values over large atmospheric slabs, that may not be representative of the physical outer scale (underestimation). A possible scaling argument is presented in [BB88], relying on an estimation of the fractional turbulent volume inside the atmospheric slabs. For example, if 5% of the volume is considered turbulent, the obtained L_0 values should be scaled by 20 [Goo+88]. Despite this simple approximation, the effect of various vertical resolutions for the determination of L_0 requires further research, and it is recommended to use the same resolution as for the design of the different models

when applying them.

Coulman model and extensions The Coulman model has been obtained from SCIDAR (nighttime) observations at the Observatoire de Haute-Provence (France), and at the McDonald Observatory (Texas, USA) in 1985 [Cou+88]. SCIDAR observations provided measurements of C_n^2 profiles with a resolution of ~ 800 m, while nearby radiosonde launches enabled to compute M^2 with a vertical spacing on the order of 1000 m. By segregating the data over 1-km slabs from 2 km to 17 km of altitude, the following fit have been computed in order to model the observed altitude dependency in L_0 :

$$L_0(z) = \frac{4}{1 + \left(\frac{z-8500}{2500}\right)^2}, \quad (3.21)$$

with z the altitude in meters ($2000 < z < 17000$).

Later application of this model to SCIDAR observations at the European Southern Observatory, La Silla (Chile), in 1986, showed a similar altitude dependency [Cou+88]. Fitting these new observations gave

$$L_0(z) = \frac{5}{1 + \left(\frac{z-7500}{2000}\right)^2}, \quad (3.22)$$

that is similar to (3.21). Both (3.21) and (3.22) retrieves outer scale no larger than 5 meters.

In [Luk05], an updated Coulman profile is presented for the Paranal observatory (Chile), following the Paranal Seeing Campaign in March 1992. For an altitude greater than the mountain summit ($z > z_0 = 2636$ meters), L_0 is modelled by

$$L_0 = \frac{2}{3} \left(\frac{4}{1 + \left(\frac{z-8500}{2500}\right)^2} + 8 e^{\frac{z_0-z}{1500}} \right).$$

This fit follows observations of large outer scale values at the ground level.

Furthermore, a revised version of the Coulman model for the Observatorio del Teide (Tenerife, Spain) is also presented in [Mon+17], following SCIDAR observations with a 300-meter resolution:

$$L_0(z) = \frac{5.2}{1 + \left(\frac{z-7526}{2404}\right)^2}.$$

Beland & Brown model The Beland & Brown model is a deterministic C_n^2 model making use of the Tatarskii equation with the following parametrization for L_0 [BB88]

$$L_0(z) = 0.307 - 0.0324 (z - 17) + 0.00167 (z - 17)^2 + 0.000476 (z - 17)^3,$$

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with z the altitude in kilometres, ranging from 17 to 30 km. It has been derived from thermosonde and radiosonde measurements during nighttime in New Mexico (8 flights) and Hawaii (7 flights), having a vertical resolution of 20 meters. It only describes the lower stratosphere, and has been obtained using a 500-meter window average for computing the outer scale L_0 , and by fitting a cubic polynomial to all the data. It gives values of L_0 between 0.25 m and 1 m.

Boundary layer outer scale models Some simple outer scale models in the BL can be found in the literature. This is the case of the Tatarskii expression [Goo+88, p. 177]

$$L_0(z) = \begin{cases} 0.4 & \text{for } z < 1 \\ 0.4 z & \text{for } 1 < z \leq 25 \end{cases},$$

with z in meters (altitude above the ground). The completion of the Tatarskii outer scale expression with the Fried model [Fri67], given by

$$L_0(z) = 2\sqrt{z},$$

for z ranging from the surface to 10 km, can be found in [Luk05].

Alternatively, a fitted L_0 profile in the BL from 0 to 1 km above the ground has been presented in [Aba+04]. It relies on radiosonde and thermosonde measurements at several astronomical sites, recording an average L_0 profile (with a 25-meter vertical resolution) given by

$$L_0(z) = 3.21 z^{-0.11},$$

where z is the altitude above the ground expressed in meters and lower than 1000 m.

3.2.2.3 VanZandt model (probabilistic approach)

The VanZandt model, from the National Oceanic and Atmospheric Administration (NOAA) is a probabilistic model based on the Tatarskii equation. It has been derived from Very High Frequency (VHF) Doppler radar measurements, and provides an estimation of C_n^2 from radiosonde observations with a vertical resolution of 150 m (at best) [Van+78; WVG85]. This model is suited both for radiofrequency C_n^2 [dMM93; Vas99] and optical C_n^2 [Bel93, p. 223], [Goo+88, p. 176], [Cou+88]. The main reference presenting the model and its necessary hypotheses is the NOAA Technical Report from Warnock and VanZandt, published in 1985 [WV85].

VanZandt model makes use of (3.13) with a^2 approximated by 2.8 and M given by [WV85, p. 9]

$$M = -77.6 \times 10^{-6} \frac{p}{T} \frac{\tilde{N}^2}{g} \left[1 + 15500 \frac{q}{T} - \frac{15500}{2} \frac{q'g}{\tilde{N}^2 T} \right],$$

where $\tilde{N}^2 = g \frac{d(\ln \theta)}{dz}$, $q' = \frac{dq}{dz}$, and p is in [hPa]. Since the radiosonde observations used in [WV85] have a resolution of 150 meters (or coarser), the atmosphere is con-

sidered to be split in slabs of 150 meter thickness (or greater). Hence, C_n^2 cannot be computed for each fine-scale turbulent layer (i.e., layers being 10-meter thick or thinner), and only an estimation of the mean C_n^2 over the slab is available, denoted $\bar{C}_n^2$. This lack of data availability led Warnock and VanZandt to use a probabilistic approach, expressing the $\bar{C}_n^2$ estimate as [WV85, p. 9]

$$\bar{C}_n^2 = \int_{L_{\min}}^{L_{\max}} dL \iint_{\text{Ri} < \text{Ri}_c} dS d\tilde{N}^2 \int_{-\infty}^{\infty} dq' p_{L,S,\tilde{N}^2,q'} C_n^2, \quad (3.23)$$

where the dependency of $p_{L,S,\tilde{N}^2,q'}$ and C_n^2 in $(L, S, \tilde{N}^2, q')$ is implicit. Equation (3.23) involves the Richardson number $\text{Ri} = \tilde{N}^2/S^2$, the critical value of the Richardson number Ri_c chosen to be 0.25, and the multivariate PDF $p_{L,S,\tilde{N}^2,q'}$ of the outer scale L , the wind shear S , $\tilde{N}^2$, and q' .

Assuming that q' and $\tilde{N}^2$ are fully (anti)correlated, and that the PDFs of the outer scale, the wind shear, and $\tilde{N}^2$ are independent (approximation), the estimate $\bar{C}_n^2$ can be computed from [WV85, p. 13]

$$\bar{C}_n^2 = 2.8 M_0^2 \tilde{R} \int_{L_{\min}}^{L_{\max}} L^{4/3} p_L dL \int_0^{\infty} p_S dS \int_{-\infty}^{S^2 \text{Ri}_c} (N^2)^2 p_{N^2} dN^2.$$

The factor M_0 is equal to $M_0 = -77.6 \times 10^{-6} \frac{p}{T} \frac{1}{g}$, while $\tilde{R}$ is

$$\tilde{R} = \left[1 + 15500 \frac{q}{T} - \frac{15500}{2} \frac{q'g}{\tilde{N}^2 T} \right]^2.$$

For optical C_n^2 estimates, the humidity variations are neglected and $\tilde{R}$ reduces to 1 [WV85, p. 14]. The probability distributions are assumed to be the following: uniform distribution for L , Gaussian distribution for $\tilde{N}^2$, and Gaussian distributions for each components of the velocity vector, giving a Rice distribution for S [WV85, p. 13]. The distribution parameters are estimated from the radiosonde measurements over the atmospheric slabs. A detailed numerical implementation of the model can be found in [WV85].

3.2.2.4 Alternative length scales

The outer scale L_0 , defined as the upper limit of the inertial range (see Section 2.4.2), can be related to various length scales available in the literature. As detailed in [BDH21], commonly used outer scales are the Ozmidov length scale L_{OZ} and the Corrsin length scale L_C , given by [Ozm99; Cor58]

$$L_{\text{OZ}} = \sqrt{\frac{\varepsilon}{\tilde{N}^3}} \quad \text{and} \quad L_C = \sqrt{\frac{\varepsilon}{S^3}},$$

where ε is the mean dissipation rate of turbulent energy. From those definitions, turbulent eddies smaller than the Ozmidov length scale L_{OZ} are not influenced by

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the buoyancy, while eddies smaller than the Corrsin length scale L_C are not impacted by the shear. As a result, eddies smaller than L_{OZ} and L_C are considered isotropic [BDH21]. The ratio L_C/L_{OZ} is directly related to the Richardson number since $L_C/L_{OZ} = \left(\tilde{N}/S\right)^{3/2} = \text{Ri}^{3/4}$.

Practical computations of L_{OZ} or L_C , for example from radiosonde observations, is made difficult by the presence of ε in their definitions. Nevertheless, substitutes for these quantities, and hence for the outer scale L_0 , have been suggested in the literature, even though they do not have the same interpretation as L_0 . When used for L_0 in (3.13), they enable to model C_n^2 .

Thorpe scale The Thorpe scale L_T is often used as a substitute for the Ozmidov scale L_{OZ} in stably stratified fluids (see [Gav+05; CK08] and references therein for the existing proportionality relationships between L_T and L_{OZ}). Initially, it has been presented by Thorpe in 1977 to analyze the turbulence in oceans and lakes from water density and potential temperature profiles [Tho77]. Since then, it has been demonstrated to also apply to the atmosphere [CK08].

The approach is empirical and estimates the scales of overturns in the measured profiles, caused by the turbulence. Indeed, it relies on the assumption that, under stable conditions, density or potential temperature profiles should be monotonic. Hence, inversions in these profiles come from turbulent mixing [Mar+17].

In order to estimate L_T , observed profiles are rearranged into monotonic profiles containing no inversion, i.e., sorting the different measured samples at each sampled altitude (or depth, in water). The required displacement to create the stable profiles is recorded at each altitude (respectively, each depth), the Thorpe scale L_T being simply the RMS of this displacement [CK08]. Instead of the RMS of the displacement, some authors use its absolute value [Mon+17; Hu+23b; Zha+23].

From the measured Thorpe scale L_T , the temperature structure parameter C_T^2 can be estimated using [Gav+05; Bas15]

$$C_T^2 = c_T L_T^{4/3} \left(\frac{d\theta_s}{dz} \right)^2, \quad (3.24)$$

with θ_s the sorted potential temperature profile (used as an estimation of the average profile), and c_T a proportionality constant depending on a^2 in (3.13) and the relationship between L_{OZ} and L_T . From the C_T^2 profile, the C_n^2 profile is obtained using the expressions in Section 2.5.4.

The literature behind the Thorpe length scale is rich, with applications both in oceanography and atmospheric sciences [Bas15; Mar+17; Nat+10; Hu+23a]. The influence of instrumental noise in the estimation of L_T is rather important, and must be handled appropriately [WDL11].

Ellison scale Similarly to the Thorpe scale, the Ellison scale L_E is sometimes used to model C_n^2 , with a similar equation as (3.24) where L_T is substituted by L_E [BDH21].

The Ellison scale has been originally suggested by Ellison in 1957 and corresponds to the ratio between the standard deviation of the density fluctuations and the vertical gradient of the density [Ell57]. In terms of potential temperature, it is expressed by [BDH21]

$$L_E = \frac{\sigma_\theta}{\left(\frac{d\theta}{dz}\right)},$$

with σ_θ^2 the variance of the potential temperature fluctuations.

Applications of the Ellison scale to radiosonde observations can be found in [Wu+20; Zha+23]. In [Wu+20], the Ellison scale is estimated by

$$L_E = \left| \frac{\Delta\theta}{\left(\frac{d\theta_s}{dz}\right)} \right|,$$

where $\Delta\theta(z) = \theta(z) - \theta_s(z)$, i.e., the difference between the measured potential temperature and its sorted profile.

Deardoff length scale In [MVB99], the Deardoff length L_D is used in Tatarskii equation for stable layers. It is given by [Dea80],

$$L_D = \sqrt{\frac{2k_\epsilon}{\frac{g}{\theta_v} \frac{\partial\theta_v}{\partial z}}}. \quad (3.25)$$

Equation (3.25) involves the TKE depicted by k_ϵ , the gravity of Earth g , and the virtual potential temperature θ_v defined in (2.20). These different quantities are obtained from NWP simulations in [MVB99], and this model is further described in Section 5.3.1.

Louis and Tjernström mixing length In [Tje93], the mixing length of sensible heat is related to the gradient Richardson number Ri , following the work of [Lou82],

$$L_H = \tilde{l}_0 \left(1 + 15 Ri \sqrt{1 + 5 Ri}\right)^{-1/2}. \quad (3.26)$$

The variable $\tilde{l}_0$ represents an asymptotic mixing length, equal to 450 m in [Lou82], and 23 m in [Tje93].

Equation (3.26) is used in [Spr+21; Ull+23] for the parametrization of L_0 in the Tatarskii equation.

Temperature-based mixing scale In [Raf+22], the outer scale L_0 is estimated from the mixing scale Δz_0 at which the equality between temperature systematic difference and the temperature random difference is reached [Tat61], i.e.,

$$[\langle T(z) \rangle - \langle T(z + \Delta z_0) \rangle]^2 = \langle [T(z) - T(z + \Delta z_0)]^2 \rangle.$$

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The outer scale is then obtained from

$$L_0(z) \approx \left(2.8 \left(1 + \frac{Y_a}{\frac{dT}{dz}} \right)^2 \right)^{-3/4} \Delta z_0.$$

3.2.2.5 Combination of length scales

As shown above, many models for the outer scale L_0 , or for alternative scales, are available. Some of them are empirically-based, others require some physical understanding of atmospheric turbulence. Recently, new models have been suggested by combining the previous models, either with new analytical expressions, or with weights computed numerically.

WSPT model The Wind Speed and Potential Temperature (WSPT) model is inspired by the Thorpe scale and the Dewan model, combining the meteorological quantities involved in each model. Indeed, the WSPT scale L_W is defined by [Wu+21]

$$L_W = \sqrt{\frac{\Delta\theta}{\frac{d\theta_s}{dz}} \left(\frac{u v}{S^2} \right)^{1/2}},$$

where all quantities have been defined previously. It comes from the product between a temperature scale given by $\frac{\Delta\theta}{\frac{d\theta_s}{dz}}$, and a wind speed scale obtained from $\left(\frac{u v}{S^2} \right)^{1/2}$. C_T^2 is then estimated using

$$C_T^2 = c_w L_W^{4/3} \left(\frac{d\theta_s}{dz} \right)^2, \quad (3.27)$$

with c_w a proportionality constant, calculated to ensure agreement between the measured C_T^2 and its estimation from (3.27).

The WSPT model has been validated with thermosonde measurements from Rongcheng and Maoming, in China. A vertical resolution of 100 m is used. A median c_w of 0.3 has been found for Rongcheng for example.

Multi-model ensemble pattern The Multi-model Ensemble Pattern (MEP) approach presented in [Zha+23] aggregates seven different outer scale models into a single C_n^2 estimation. The different models are: Hufnagel-Valley 5/7, Dewan, HMNSP99, Thorpe, Ellison, WSPT, and WSTG. For each model, a weighting coefficient (between 0 and 1, with their summation equal to 1) is computed empirically, based on the different model performance on predicting the measured C_n^2 profiles, ensuring that larger coefficients will be given to models offering more accurate predictions. Then, the MEP final prediction comes from the summation of each model prediction, weighted by the coefficients previously computed. Details about the combination approach can be found in [Zha+23], where the MEP approach has been applied to radiosonde and thermosonde soundings in China.

3.2.3 Trinquet-Vernin model

The Trinquet-Vernin model [TV07], published in 2007, relies on the potential temperature vertical gradient and the wind-shear to predict the temperature structure parameter C_T^2 profile. A C_n^2 profile can then be computed, for example using (2.70).

Defining the vertical gradient of the potential temperature by $\chi = \frac{d\theta}{dz}$ and the wind shear by $S = \sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2}$, the core equation of the TV model is [TV07]

$$\frac{C_T^2(z)}{\langle C_T^2(z) \rangle_m} = \left(\frac{\chi(z)}{\langle \chi(z) \rangle_m} \right)^a \left(\frac{S(z)}{\langle S(z) \rangle_m} \right)^b, \quad (3.28)$$

where $\langle C_T^2(z) \rangle_m$ represents the median C_T^2 profile, $\langle \chi \rangle_m$ is the median potential temperature profile, and $\langle S \rangle_m$ is the median wind profile. The exponents a and b have been deduced from the analysis of balloon measurements of C_T^2 and meteorological parameters at diverse locations (mostly astronomical sites). Thanks to least-squares fits between the measured and modelled C_T^2 profiles, a value of 1 has been found for a , while b is set to 1/2 (see [TV07] for the standard deviations of those parameters depending on the atmospheric layer and on the vertical resolution). The exponent $a = 1$ for the potential temperature gradient is in disagreement with Tatarskii equation (3.13), and potentially highlights the limitation of Tatarskii-based models to describe the turbulence at macroscopic scales [TV07].

For practical use, the TV model in (3.28) is usually rewritten with the function $\phi(z)$, defined as [Gio+21]

$$\phi(z) = \frac{\langle C_T^2(z) \rangle_m}{\langle \chi(z) \rangle_m \langle S(z) \rangle_m^{1/2}},$$

leading to

$$C_T^2(z) = \phi(z) \chi(z) S(z)^{1/2}. \quad (3.29)$$

The function $\phi(z)$ is deduced from the balloon measurements, with an example given in Table 3.2, specifying its values in the BL and in the FA [Gio+14]. However, recent works suggest that $\phi(z)$ is site-dependent, and should be empirically “learned” at the site of interest, i.e., using balloon measurements at this site [Gio+21]. A diurnal variation of $\phi(z)$ has also been found in [Gio+21].

A major limitation of the TV model is related to the unitary exponent a of the potential temperature gradient in (3.29). Indeed, this prevents the model to be applied to convective cases, i.e., cases where the potential temperature gradient is negative [TV07]. Convective regimes, which occur mostly during daytime and close to the ground, are thus excluded. A possible solution to circumvent this limitation is to use an alternative C_n^2 model for the convective regions, such as a Tatarskii-based model, as presented in [Gio+21].

Recent applications of the TV model with NWP software are detailed in Section 5.1.3.2 and can be found in [Gio+21; Qia+21; Raf+22].

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Table 3.2 – Values of the function $\phi(z)$ deduced from balloon measurements (table reproduced from [Gio+14]).

Boundary layer		Free atmosphere	
Altitude z [m]	$\phi(z)$	Altitude z [km]	$\phi(z)$
5	2.8350	1.5	0.2202
55	0.7826	2.5	0.1233
105	0.2851	3.5	0.1221
155	0.2248	4.5	0.1117
205	0.2334	5.5	0.0796
255	0.2369	6.5	0.0766
305	0.1394	7.5	0.0947
355	0.1698	8.5	0.0824
405	0.1351	9.5	0.0856
455	0.1152	10.5	0.0796
505	0.1202	11.5	0.0596
555	0.1242	12.5	0.0445
605	0.1528	13.5	0.0453
655	0.1258	14.5	0.0386
705	0.1038	15.5	0.0492
755	0.0960	16.5	0.0455
805	0.0832	17.5	0.0459
855	0.1062	18.5	0.0397
905	0.0947	19.5	0.0413
955	0.1022		

3.3 Boundary and surface layer models

Most of the non-parametric and parametric models presented in Sections 3.1 and 3.2 are valid for the FA, i.e., the part of the troposphere that is not influenced by the Earth’s surface. Nevertheless, the atmospheric BL, defined in [Stu88, p. 2] as “*the part of the troposphere that is directly influenced by the presence of the Earth’s surface, and responds to surface forcing with a timescale of about an hour or less*”, plays a major role in the modelling of optical turbulence.

Indeed, the atmospheric BL can have important diurnal variations impacting atmospheric turbulence and the associated C_n^2 values. As an example, Figure 3.3 shows the measured variations of ground C_n^2 in spring 1977 in the Tularosa basin (desert), New Mexico (USA) [WK81]. C_n^2 is measured at 9 m above ground, and the figure depicts the seasonal average, as well as its standard deviation. A daytime trend is clearly visible and is dominated by convection induced by solar irradiance. C_n^2 therefore peaks around noon, i.e., when solar heating of the ground is the largest. Minima occur near sunrise and sunset, associated to neutral events during which the soil and air temperature are similar. The residual turbulence is not driven by convection anymore, but arises from mechanical turbulence and incomplete mixing. Finally, night-time measurements show more variations, arising from the delicate balance between

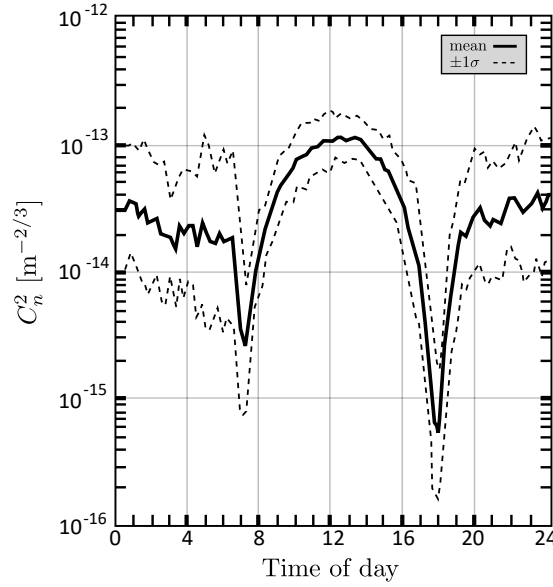


Figure 3.3 – Illustration of diurnal variations of ground C_n^2 (reproduced from [WK81]).

wind shear and stable temperature gradients in the desert [Can14, p. 18] [WK81].

The example provided in Figure 3.3 is useful to get some insights about turbulence in the BL. However, it is highly site-dependent, as illustrated in various models and measurements from the literature [Qin+16a; Ull+19]

The atmospheric BL over land and in high pressure regions has a well-defined structure showing a diurnal cycle, given in Figure 3.4. It is separated in [Stu88, p. 10]:

- the *surface layer*, that is, the layer the closest to the ground, where “turbulent fluxes and stress vary by less than 10% of their magnitude,”
- the *mixed layer*, occurring in daytime and mostly driven by convection, such as heat transfer from the warm ground surface,
- the *residual layer*, arising from the decay of the mixed layer in nighttime,
- the *stable (nocturnal) boundary layer*, that is, “the bottom portion of the residual layer, transformed by its contact with the ground into a stable BL”.

The identification of the different parts of the BL can usually be achieved with the knowledge of virtual potential temperature profiles, and is important as different C_n^2 models have been developed according to the layer type. Surface layer C_n^2 models are introduced in Section 3.3.1, while daytime and nighttime profiles over the whole BL are presented in Section 3.3.2.

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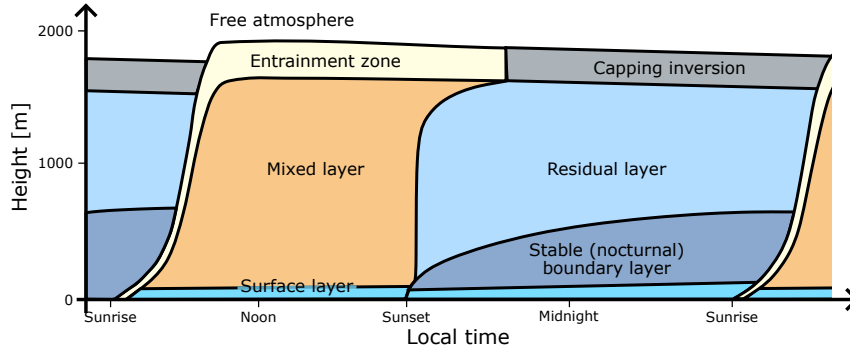


Figure 3.4 – Schematic of atmospheric boundary layer diurnal variations (adapted from [Stu88, p. 11]).

3.3.1 Surface layer and ground C_n^2 value

Modelling of SL C_n^2 can be achieved by regressions models based on empirical observations (Section 3.3.1.1), or from physical insights from the similarity theory (Section 3.3.1.2).

3.3.1.1 Regressions of macroscale meteorological parameters

C_n^2 models based on regressions of macroscale meteorological parameters give access to the surface C_n^2 value from measurements of surface meteorological parameters, such as the temperature, relative humidity, wind speed. They are designed based on simultaneous measurements of C_n^2 and meteorological parameters at a given location, using empirical fitting of the collected data.

Sadot-Kopeika models The Sadot-Kopeika models have been presented in 1992 and provide an estimation of the C_n^2 ground value based on regressions [SK92]. The models have been established from angle-of-arrival and amplitude fluctuation measurements over a laser horizontal path in the Israeli desert (Beer-Sheva). In total, over 70 experiments have been conducted during three months in 1990 (late winter, spring, and early summer), in day and night conditions, with recording of meteorological parameters. Based on the collected measurements, two polynomial regression models have been suggested, where their coefficients have been found by minimizing the least-square error.

The first model is based on the temperature T [K], the relative humidity RH [%], and the wind speed U [m/s]. It also involves the temporal hour weight $W(TH)$ given in Table 3.3, making use of the temporal hour TH as defined in (3.11). It is given by

$$C_n^2 = a_1 W + b_1 T + c_1 RH + c_2 RH^2 + c_3 RH^3 + d_1 U + d_2 U^2 + d_3 U^3 + e_1, \quad (3.30)$$

with the coefficients $a_1, b_1, \dots, e_1$ given in Table 3.4. The model in (3.30) is suited for practical use, as it is simple and involves standard meteorological parameters.

Table 3.3 – Temporal hour weights for Sadot-Kopeika models (table reproduced from [SK92]).

Temporal hour (TH) interval		Relative weight (W)
sunrise	until -4	0.11
	-4 to -3	0.11
	-3 to -2	0.07
	-2 to -1	0.08
	-1 to 0	0.06
	0 to 1	0.05
	1 to 2	0.1
	2 to 3	0.51
	3 to 4	0.75
	4 to 5	0.95
	5 to 6	1.00
	6 to 7	0.90
	7 to 8	0.80
	8 to 9	0.59
sunset	9 to 10	0.32
	10 to 11	0.22
	11 to 12	0.10
	12 to 13	0.08
	over 13	0.13

Table 3.4 – Regression coefficients for the Sadot-Kopeika model in (3.30).

a_1	b_1	c_1	c_2	c_3
3.8×10^{-14}	2.0×10^{-15}	-2.8×10^{-15}	2.9×10^{-17}	-1.1×10^{-19}
d_1	d_2	d_3	e_1	
-2.5×10^{-15}	1.2×10^{-15}	-8.5×10^{-17}	-5.3×10^{-13}	

Table 3.5 – Regression coefficients for the Sadot-Kopeika model in (3.31).

A_1	B_1	C_1	C_2	C_3	D_1
5.9×10^{-15}	1.6×10^{-15}	-3.7×10^{-15}	6.7×10^{-17}	-3.9×10^{-19}	-3.7×10^{-15}
D_2	D_3	E_1	F_1	F_2	G
1.3×10^{-15}	-8.2×10^{-17}	2.8×10^{-14}	-1.8×10^{-14}	1.4×10^{-14}	-3.9×10^{-13}

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Table 3.6 – Regression coefficients for the TCSA model in (3.32).

$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_3$	$\tilde{a}_4$
9.69×10^{-4}	-2.76×10^{-5}	4.86×10^{-7}	-4.48×10^{-9}
$\tilde{a}_5$	$\tilde{b}$	$\tilde{c}$	$\tilde{d}$
1.66×10^{-11}	6.26×10^{-3}	-1.34×10^{-5}	7.3×10^{-3}

A second model is also suggested in [SK92] that offers improved performance when compared with the measurements. It extends the model in (3.30) by including the solar flux SF [Cal/cm²/min] and the total cross-sectional area of particles per cubic meter TCSA [cm²/m³]. Indeed, the solar flux is expected to influence C_n^2 through its contribution to the temperature gradient. Particle scattering should not influence C_n^2 , but it impacts the beam scintillation. Hence, the *effective* C_n^2 estimated from the measurements may include a contribution from the aerosol scattering [SK92]. With those new parameters, the model is

$$C_n^2 = A_1 W + B_1 T + C_1 RH + C_2 RH^2 + C_3 RH^3 + D_1 U + D_2 U^2 + D_3 U^3 + E_1 SF + F_1 TCSA + F_2 TCSA^2 + G, \quad (3.31)$$

with the different coefficients $A_1, B_1, \dots, G$, given in Table 3.5.

Hence, this second Sadot-Kopeika model is of interest if the solar flux and the TCSA of particles are known. An estimation of the TCSA in [m⁻¹] from macroscale meteorological parameters at Beer-Sheva is available in [GK91; Sad+95]. It makes use of the following regression:

$$TCSA = \tilde{a}_1 RH + \tilde{a}_2 RH^2 + \tilde{a}_3 RH^3 + \tilde{a}_4 RH^4 + \tilde{a}_5 RH^5 + \tilde{b} \ln(RH) + \tilde{c} SF + \tilde{d} \quad (3.32)$$

where all coefficients $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{d}$, are given in Table 3.6.

The validation of the two Sadot-Kopeika models has been conducted with new measurements in the Negev desert in Sde-Boker (May 1990) and at Beer-Sheva (in winter time). Correlation coefficients between 0.86 and 0.92 were obtained [SK92].

Comments related to the coefficients given in Tables 3.4 and 3.5 can be found in the original paper [SK92]. The meteorological parameter ranges over which the regressions are valid are: from 9 to 35°C for the temperature, from 14 to 92% of relative humidity, and for wind speed between 0 and 10 m/s.

Several publications making use of the Sadot-Kopeika models are: [YDK97; BDN12; HW17].

BKB models The Bendersky-Kopeika-Blaunstein (BKB) models, referred to as BKB models in [Lec+10], extend Sadot-Kopeika models, namely by including the type of land through the albedo value A . They have been obtained from angle-of-arrival fluctuations along horizontal propagation paths, in the Negev desert and the Golan Heights (Israel) [BKB04]. For temperatures ranging from 9 to 35°C, for wind

speed values between 8 and 17 m/s, and for a relative humidity between 30 and 70%, the BKB daytime C_n^2 model gives C_n^2 at an altitude of 2.5 meter above ground from

$$C_n^2 = 3.8 \times 10^{-14}W + \frac{A}{e^T} 10^{-4} + f(U) + f(RH) - 4.45 \times 10^{-14},$$

with W the temporal hour weight, as given in Table 3.3. For rocky terrain (rocky desert, Negev), an albedo value A equals to 0.35 is used, while the wind speed and humidity functions are [BKB04]

$$\begin{aligned} f(U) &= 8 \times 10^{-16}U - 4 \times 10^{-18}U^2, \\ f(RH) &= -8 \times 10^{-16}RH + 5 \times 10^{-18}RH^2. \end{aligned}$$

For vegetation covered land (Golan), A is set to 0.5 and the functions $f(U)$ and $f(RH)$ are

$$\begin{aligned} f(U) &= 2.58 \times 10^{-14}U \\ f(RH) &= -6.797 \times 10^{-15}RH. \end{aligned}$$

In case of nighttime and for vegetation covered land, the following model is suggested [BKB04]

$$C_n^2 = 3 \times 10^{-17}T + 1.2 \times 10^{-14}U - 7.5 \times 10^{-16}RH - 1.9 \times 10^{-14}, \quad (3.33)$$

that is only valid for $5 \text{ m/s} \leq U \leq 10 \text{ m/s}$ and $92\% \leq RH \leq 100\%$. The parametrization in (3.33) results from the observation that, in nighttime and for high relative humidity and wind speed values, the temporal hour weight W and the albedo value A slightly influence the ground C_n^2 value [BKB04].

Ryznar and Bartlo model The Ryznar and Bartlo C_n^2 model is a regression of the wind speed S [m/s], measured at 10 meters, and the solar irradiance SF [Cal/cm²/min]. It is derived from scintillometer measurements (height of 4 meters) collected at the Boulder Atmospheric Observatory, CO, USA, and is given by [RB85]

$$C_n^2 = (a_1 + b_1SF + b_2SF^2 + b_3SF^3 + c_1S + c_2S^2 + c_3S^3)^2, \quad (3.34)$$

with the regression coefficients in Table 3.7. The Ryznar and Bartlo model is valid for solar irradiance between 0.07 and 1.49 Cal/cm²/min and wind-speed ranging from 0.3 to 19 m/s.

Other parametrizations Other models involving regressions of macroscale meteorological parameters can be found in the literature. For example, a recently published empirical C_n^2 model for communication links over a river can be found in [Car+23]. It involves quantities such as the temperature difference between the air and the water, the dew point, as well as the ratio between the width of the river and

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Table 3.7 – Regression coefficients for the Ryznar and Bartlo model in (3.34).

a_1	b_1	b_2	b_3
0.12724×10^{-6}	0.17314×10^{-6}	0.62238×10^{-6}	-0.39727×10^{-6}
c_1	c_2	c_3	
-0.33899×10^{-7}	0.10264×10^{-7}	-0.53824×10^{-9}	

the total length of the link. The Wang model [Wan+15] and the Raj model [AAD15] are other examples, using similar meteorological quantities as the Sadot-Kopeika models.

Regressions involving Machine Learning (ML) approaches also exist in the recent literature and are detailed in Section 3.4.

3.3.1.2 Monin-Obukhov similarity theory models

The description of BLs often relies on similarity theory, among which the Monin-Obukhov Similarity Theory (MOST) is well-known and used for modelling the atmospheric SL. Naturally, some ground C_n^2 models have been derived from the MOST and this section presents the main relationships behind these models.

Similarity theory Similarity theory is based on BL experiments showing repeatable characteristics that cannot always be derived or explained from our current knowledge of physics. Instead, empirical relationships are established based on these experiments, enabling to gain some insights about BL processes by linking the identified quantities of interests. [Stu88, p. 347]

Similarity theory starts with the identifications of relevant (meteorological) quantities, or variables, to describe a given situation. They are then organized in dimensionless groups with the aid of the Buckingham Pi theorem, before conducting experiments in order to establish empirical relationships between groups. The hope is that, with appropriate dimensionless groups, universal relationships could be obtained, i.e., a set of curves showing similar shapes, hence the name of similarity theory. [Stu88, p. 347]

Similarity theory is often used in steady-state situations, i.e., not including time in the relevant variables, giving mean variables and turbulence statistics. It is a type of zero-order closure as the established empirical relationships enable to obtain mean wind, temperature, humidity, etc., profiles without requiring other turbulence closure models (see Section 2.2.2.1). [Stu88, p. 348]

Monin-Obukhov similarity theory MOST, also called surface-layer similarity, is applied to the atmospheric SL. Such layer is defined as the part of the BL where fluxes are assumed to be constant, i.e., they vary by less than 10% in magnitude with height [Stu88, p. 357].

The fluxes of interest are the vertical momentum flux $\overline{u'w'}$, the kinematic heat flux $\overline{w'\theta'}$ (or $\overline{w'T'}$ if the temperature is used instead of the potential temperature), and the kinematic moisture flux $\overline{w'q'}$, using the notations introduced in Section 2.2.2 and [Stu88, p. 52]. From those constant surface fluxes, several scales are defined: the friction velocity u_* , the temperature scale θ_* (or T_*), and the humidity scale q_* , defined by [And88]

$$u_*^2 = -\overline{u'_L w'}, \quad (3.35)$$

$$u_* \theta_* = -\overline{w' \theta'} \quad (\text{or} \quad u_* T_* = -\overline{w' T'}), \quad (3.36)$$

$$u_* q_* = -\overline{w' q'}, \quad (3.37)$$

with u'_L the horizontal velocity fluctuations ($u_L = \sqrt{u^2 + v^2}$, streamwise wind component).

From these scales, the Obukhov length L_{Ob} , i.e., a length scale, constant with height can be defined. Several definitions exist, that depend on the choice of the temperature variable in the temperature scale: either the temperature T , or the potential temperature θ , or even the virtual potential temperature θ_v . In [Stu88, p. 181], the Obukhov length definition is

$$L_{\text{Ob}} = \frac{-\bar{\theta}_v u_*^3}{k_a g (\overline{w' \theta'_v})_s},$$

with k_a the von Kármán constant ($k_a = 0.4$) and where $(\overline{w' \theta'_v})_s$ denotes the kinematic heat flux at the surface. A similar definition is used in [WIC71], with the temperature T instead of the virtual potential temperature θ_v . In [And88], L_{Ob} is given by

$$L_{\text{Ob}}^{-1} = \frac{g k_a}{u_*^2 \bar{T}} \left(T_* + \frac{0.61 \bar{T}}{\varrho + 0.61 \bar{q}} q_* \right). \quad (3.38)$$

Following MOST, any surface-layer meteorological quantity properly scaled by u_* , T_* , q_* , L_{Ob} and the height z , is a universal function of the stability parameter $\zeta = z/L_{\text{Ob}}$. For example, the vertical gradient of the average wind speed is given by [And88]

$$\frac{\partial u_L}{\partial z} = \frac{u_*}{k_a z} \phi_m(\zeta), \quad (3.39)$$

where the similarity function $\phi_m(\zeta)$ is found empirically [And88]

$$\phi_m(\zeta) = \begin{cases} (1 - 16\zeta)^{-1/4} & \text{for } \zeta < 0 \quad (\text{unstable conditions}), \\ 1 + 7\zeta & \text{for } \zeta > 0 \quad (\text{stable conditions}). \end{cases}$$

Equation (3.39) shows that MOST is indeed a zero-order turbulence closure model for the SL, as it directly gives the mean wind speed profile. Similarly, the vertical

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gradients of the average temperature and humidity are given by [And88]

$$\frac{\partial T}{\partial z} = \frac{T_*}{k_a z} \phi_h(\zeta), \quad (3.40)$$

$$\frac{\partial q}{\partial z} = \frac{q_*}{k_a z} \phi_w(\zeta), \quad (3.41)$$

with

$$\phi_h(\zeta) = \phi_w(\zeta) = \begin{cases} (1 - 16\zeta)^{-1/2} & \text{for } \zeta < 0 \quad (\text{unstable conditions}), \\ 1 + 7\zeta & \text{for } \zeta > 0 \quad (\text{stable conditions}). \end{cases}$$

Of particular interest for this work are the similarity functions associated with the SL parameters C_T^2 , C_q^2 , and C_{Tq} . Their scaling is given in [And88],

$$C_T^2 = \frac{T_*^2}{z^{2/3}} g_T(\zeta), \quad (3.42)$$

$$C_q^2 = \frac{q_*^2}{z^{2/3}} g_q(\zeta), \quad (3.43)$$

$$C_{Tq} = \frac{T_* q_*}{z^{2/3}} g_{Tq}(\zeta), \quad (3.44)$$

where $g_T(\zeta)$, $g_q(\zeta)$, and $g_{Tq}(\zeta)$ are similarity functions³. The expression of $g_t(\zeta)$ for C_T^2 is found in [WIC71], and updated in [And88]

$$g_T(\zeta) = \begin{cases} 4.9(1 - 6.1\zeta)^{-2/3} & \text{for } \zeta \leq 0, \\ 4.9(1 + 2.2\zeta^{2/3}) & \text{for } \zeta \geq 0. \end{cases} \quad (3.45)$$

This expression of $g_T(\zeta)$ is not unique and different numerical values are available in the literature [Fre+00].

Based on several experiments, detailed in [And88], the similarity functions $g_T(\zeta)$, $g_q(\zeta)$, and $g_{Tq}(\zeta)$ are assumed to be equal. As a result,

$$\frac{z^{2/3} C_T^2}{T_*^2} = \frac{z^{2/3} C_q^2}{q_*^2} = \frac{z^{2/3} C_{Tq}}{T_* q_*} = g(\zeta),$$

where $g(\zeta) = g_T(\zeta) = g_q(\zeta) = g_{Tq}(\zeta)$. The refractive index structure parameter C_n^2 can then be obtained from (2.68) for example, where the temperature T is used instead of the potential temperature θ , that is

$$C_n^2 = A^2 C_T^2 + B^2 C_q^2 + 2AB C_{Tq}. \quad (3.46)$$

The SL C_n^2 follows thus the same similarity function $g(\zeta)$, and can be computed

³In [Fre+00; Dav+81], a temperature-humidity correlation coefficient r_{Tq} is added in the expression of C_{Tq} given by (3.44), i.e., $C_{Tq} = r_{Tq} \frac{T_* q_*}{z^{2/3}} g_{Tq}(\zeta)$. It is often assumed to be equal to 0.8 under unstable conditions.

from measurements or estimations of T_* , q_* and L_{Ob} . Different approaches to obtain these required parameters are presented in the remainder of this section. More details about MOST can also be found in [Wyn10, p. 217] and [Fok06].

Bulk models Bulk models are based on the differences in the mean values of meteorological parameters between two different heights (typically, a reference height and the surface) to estimate u_* , T_* , q_* and ζ [Dav+81]. Indeed, the integration of (3.39), (3.40) and (3.41), as conducted in [Pan63; Pau70], leads to [And88; Dav+81; Fre+00]

$$u_L(z) = \frac{u_*}{k_a} \left[\ln \left(\frac{z}{z_u} \right) - \Psi_m(\zeta) \right], \quad (3.47)$$

$$T(z) - T_s = \frac{T_*}{k_a} \left[\ln \left(\frac{z}{z_T} \right) - \Psi_h(\zeta) \right], \quad (3.48)$$

$$q(z) - q_s = \frac{q_*}{k_a} \left[\ln \left(\frac{z}{z_q} \right) - \Psi_w(\zeta) \right], \quad (3.49)$$

where T_s and q_s are the surface temperature and humidity, such that the left parts of equations (3.47) to (3.49) correspond to differences Δu_L , ΔT and Δq between a level z and the surface (the wind speed u_L is 0 at the surface), as anticipated for bulk models. The other side of the equality sign involves the roughness lengths of the wind speed z_u , of the temperature z_T , and of the humidity z_q . Those lengths are associated to the altitude levels where the different quantities u_L , T , and q , reach their surface values. Finally, the functions $\Psi_m(\zeta)$, $\Psi_h(\zeta)$, and $\Psi_w(\zeta)$ are the integrated forms of the functions $\phi_m(\zeta)$, $\phi_h(\zeta)$, and $\phi_w(\zeta)$, respectively. More precisely, they correspond to the following integration, given for $\Psi_m(\zeta)$ for example,

$$\Psi_m(\zeta) = \int_0^\zeta \frac{1 - \phi_m(\zeta')}{\zeta'} d\zeta',$$

as explained in [Pan63; Pau70]. The analytical expressions resulting from the integration can be found in [Pau70; Fre+00].

From (3.47) to (3.49), it should be possible to isolate u_* , T_* and q_* , that can then be fed in (3.42) to (3.44) and (3.46) to get C_n^2 ,

$$u_* = k_a u_L(z) \left[\ln \left(\frac{z}{z_u} \right) - \Psi_m(\zeta) \right]^{-1}, \quad (3.50)$$

$$T_* = k_a (T(z) - T_s) \left[\ln \left(\frac{z}{z_T} \right) - \Psi_h(\zeta) \right]^{-1}, \quad (3.51)$$

$$q_* = k_a (q(z) - q_s) \left[\ln \left(\frac{z}{z_q} \right) - \Psi_w(\zeta) \right]^{-1}. \quad (3.52)$$

From these equations, it seems that the knowledge of mean meteorological quantities at a level z and at the surface enables to find u_* , T_* and q_* . However, the roughness

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lengths z_u , z_T and z_q , are unknown, and usually parameterized as a function of u_* . Moreover, the stability parameter $\zeta = z/L_{Ob}$ itself is also dependent on u_* , T_* and q_* , as given in (3.38) for example. Hence, closed-form expressions for u_* , T_* and q_* are not possible.

To overcome this issue, two different approaches can be found in the literature:

1. The first approach is iterative, and aims at iteratively solving (3.47) to (3.49), with the definition of ζ and the parameterizations of the roughness lengths as given in [Fre+00] and the references therein. This is the approach presented in [Dav+81; Fre+00; AM85], and recently applied in [Qin+16b]. More details can also be found in [Kun96].
2. The second approach starts from rewriting (3.50) to (3.52) with the drag coefficients c_D , c_T , and c_q [Dav+81]

$$\begin{aligned} u_* &= c_D^{1/2} u_L(z), \\ T_* &= c_T^{1/2} (T(z) - T_s), \\ q_* &= c_q^{1/2} (q(z) - q_s). \end{aligned}$$

From a theoretical point of view, the drag coefficients are simply equal to the expression they substitute, e.g., $c_D^{1/2} = k_a \left[\ln \left(\frac{z}{z_u} \right) - \Psi_m(\zeta) \right]^{-1}$. However, some empirical relationships have also been established, and only depend on mean meteorological quantities [Fri77]. Hence, closed-form expressions are possible in this case, as successfully applied in [Qin+17; Qin+19].

Finally, alternative approaches making use of two heights z_1 and z_2 instead of a reference height z and the surface have also been developed, such as the algorithm MARIAH [Tun03; RTH95] or crude gradient approximations [WIC71] for example. The Thiermann model is also based on MOST [TG92; TK88; Wan+15].

ARL model The American Army Research Laboratory (ARL) model, presented in [TOV06], is also inspired by MOST for obtaining C_n^2 in the SL. It involves some of the equations presented above for the bulk models, and requires only three measured parameters as inputs: the values of wind speed, temperature and C_n^2 at a single reference height. The determination of the stability conditions is also necessary, as well as the knowledge of the roughness length z_u . The procedure is then iterative to obtain u_* , T_* and L_{Ob} , and finally a SL C_n^2 profile. More details can be found in [TOV06; Can14].

PAMELA model The PAMELA model is another model relying on MOST, described in several reports from the U.S. Naval Research Laboratory and the Army Research Laboratory [Dos+04; Ric+06; Oh+04a; Oh+04b]. A complete description of the model is available in [Ric+06]. It makes use of (3.50) to compute the friction velocity u_* , where the roughness length is tabulated and depends on the surface

conditions, while the Monin-Obukhov length is determined from the average wind speed at the roughness height (using the Pasquill stability parameter). Regarding the temperature scale T_* , it is estimated directly from the sensible heat flux H in $[\text{W/m}^2]$ [Ric+06]

$$T_* = \frac{-H}{c_p \rho u_*}, \quad (3.53)$$

with c_p the specific heat at constant pressure and ρ the atmospheric density. An estimation of H is provided from the latitude, longitude, time of day, and cloud cover. As a result, the PAMELA model circumvents the need for an iterative solution.

Kunkel and Walters model The Kunkel and Walters model is also based on MOST [KW83] [Bel93, p. 213]. By solving a time-dependent energy balance equation for the SL, it provides an estimation of the sensible and latent heat fluxes, that can then be used in MOST equations to get C_n^2 . As inputs, this model requires the wind speed, the latitude, the time of the year, as well as some soil parameters and the soil moisture content. It assumes a clear sky (no cloud) and a surface with little vegetation, and has been validated with scintillometer measurements in New Mexico, USA [KW83].

NAVSLaM model The Navy Vertical Surface Layer Model (NAVSLaM) is another example of surface C_n^2 model based on MOST to model OT in maritime environments [Mah+20; Bar+22]. As inputs, it requires measured or modelled meteorological quantity (wind speed, air temperature, and humidity) in the atmospheric SL, at one or two height levels.

Use of numerical weather prediction software Current NWP software can embed a MOST-based description of the BL, as well as other parametrizations involving a variety of turbulence closure models. By using such software to model macroscale meteorological quantities, alternative quantities can be retrieved in the BL, e.g., the TKE, the friction velocity, kinematic fluxes, etc. As a result, from an user point-of-view, the BL quantities predicted by the software can be plugged in the equations presented above, to retrieve the ground C_n^2 value.

This approach is illustrated in [Che+11] for example, where surface fluxes and roughness lengths are extracted from the weather simulations to predict local C_n^2 at the surface. A similar approach is used in [Ull+19], where the friction velocity u_* , the sensible latent heat flux H and the latent heat flux E are retrieved from NWP simulations. The link between those fluxes and the kinematic fluxes $\overline{w'T'}$ and $\overline{w'q'}$ involved in (3.36) and (3.37) is [Stu88, p. 428], [Bro78; RGB78]

$$H = \bar{\rho} \overline{w'(c_p T)'},$$

with $\bar{\rho}$ the average density of air and c_p the specific heat for air at constant pressure,

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that approximately varies with the humidity according to

$$c_p = c_{p,d}(1 + 0.84q),$$

where $c_{p,d}$ is the specific heat of dry air at constant pressure. Consequently, the sensible heat flux is given by [Stu88, p. 428]

$$H \approx \bar{\rho} c_{p,d} \left(\overline{w'T'} + 0.84 \bar{T} \overline{w'q'} \right),$$

neglecting higher-order terms. Equation (3.53) is actually an approximation of this result, neglecting the influence of humidity, while [Ull+19] uses

$$T_* = \frac{-H}{c_p(1 + 0.84q)\bar{\rho}u_*}. \quad (3.54)$$

Regarding the kinematic moisture flux $\overline{w'q'}$, it is related to the latent heat flux E with [Stu88, p. 428]

$$E = \bar{\rho} l_v \overline{w'q'},$$

with l_v the latent heat of vaporization of water. Hence, q_* is estimated from [Ull+19]

$$q_* = \frac{-E}{u_* \bar{\rho} l_v}. \quad (3.55)$$

3.3.2 Atmospheric boundary layer

Modelling of the atmospheric BL is challenging and depends on the stability. Usually, unstable and free convective conditions occur during daytime, while stable conditions are found in nighttime, as illustrated in Figure 3.4. Nevertheless, these simple rules are sensitive to the local conditions and may not apply to all locations. For example, snow-covered terrains with warm air over cold ground experience stable conditions during daytime. As a result, it is advised to rely on the potential temperature lapse rate when concluding about local stability condition (see Figure 2.5 and the associated discussion) [Bel93, p. 213].

According to [RB85, p. 13] and numerous works referenced therein, height variations of C_T^2 and C_n^2 in the atmospheric BL fit within the framework of MOST. Hence, the extension of (3.42) and (3.45) to the BL, remembering that $\zeta = z/L_{Ob}$, gives an altitude dependency in $z^{-4/3}$ of C_T^2 for unstable conditions, while stable conditions lead to a dependency in $z^{-2/3}$. C_n^2 is sometimes assumed to follow the same altitude dependency as C_T^2 , considering that the pressure-temperature term p/T^2 in (2.70) is self-compensating with an increase in altitude (within 10% at 1 km) [WK81]. Otherwise, C_T^2 can be determined before computing C_n^2 from (2.70) or from the more advanced equations given in Section 2.5.4.2.

The following of this section presents some BL C_n^2 models based on these two trends: $z^{-4/3}$ in daytime and $z^{-2/3}$ in nighttime. Such results are often found in the

literature, for example in [TOV06, p. 14], [AP05, p. 480], [Bel93, p. 213], etc.

3.3.2.1 Convective boundary layer (daytime)

Two common models used for describing C_n^2 in the convective (unstable) BL are the Kaimal model and the Kukharets and Tsvang model [Bel93, p. 214], [TOV06, p. 20], [RB85, p. 14]. Both include a $z^{-4/3}$ dependency that inspired the BL term in the HAP model presented in Section 3.2.1.4. They provide profiles for average turbulence conditions at the locations where they have been developed.

Kaimal model The Kaimal model has been developed based on C_T^2 measurements in the BL in New Mexico, USA [Kai+76]. It originally made use of the surface heat flux, but has been later extended to rely on the knowledge of C_T^2 (or C_n^2) at a given altitude near the ground [WK81]. The Kaimal model expressed in terms of C_T^2 can be found in [TOV06, p. 20], while its C_n^2 -based form assuming that C_n^2 has the same dependency as C_T^2 is given by [WK81] [RB85, p. 16]

$$\frac{C_n^2(z)}{C_n^2(z_0)} = \begin{cases} \left(\frac{z}{z_0}\right)^{-4/3} & \text{for } z_0, z \leq 0.5z_i, \\ \left(\frac{0.5z_i}{z_0}\right)^{-4/3} & \text{for } 0.5z_i \leq z \leq 0.7z_i, \\ 2.9 \left(\frac{0.5z_i}{z_0}\right)^{-4/3} \left(\frac{z}{z_i}\right)^3 & \text{for } 0.7z_i \leq z \leq z_i, \end{cases}$$

where z_0 is the near-ground altitude where C_n^2 is known (e.g., measured), and z_i is the altitude of the inversion layer, associated to the top of the atmospheric BL. Three segments can be identified: the mixed layer up to $0.5z_i$ where C_n^2 scales as $z^{-4/3}$, a transition layer where C_n^2 is constant, and an interface layer between the BL and the troposphere.

Extensions of the Kaimal model have been developed in the High Energy Laser Handbook Empirical Model (HELHEM) for example [TOV06, p. 20].

Kukharets and Tsvang model The Kukharets and Tsvang model also applies to convective BLs, and has been derived from measurements in the Soviet Union, over steppe and forested surfaces [KT80]. In comparison with the Kaimal model, it enables to represent increased turbulence arising at the top of the BL, using a Gaussian shape in the profile. Moreover, it avoids the segmentation of the profile in different regions.

The Kukharets and Tsvang model has initially been presented in terms of $C_T^2(\xi)$, where $\xi = z/z_i$ is a normalized altitude, with z_i identified as the height where an abrupt decrease on the turbulence energy dissipation rate is observed. The main equation of the model is [RB85, p. 16]

$$\frac{C_T^2(\xi)}{C_T^2(0.1)} = k_1 \xi^{-4/3} + k_2 e^{-k_3(\xi-1.1)^2},$$

with $k_1 = 0.046$, $k_2 = 0.6$ and $k_3 = 12$. Hence, the C_T^2 profile is normalized with C_T^2 at

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an altitude $z/z_i = 0.1$. It has a $z^{-4/3}$ variation for altitudes below z_i , and a Gaussian-shaped layer at altitude z_i which magnitude is given by k_2 and standard deviation influenced by k_3 .

The model has later been modified by [MDS86] to allow for any reference altitude, and applied to C_n^2 . The updated model is [MDS86][Bel93, p. 215]

$$\frac{C_n^2(z)}{C_n^2(z_0)} = \frac{k_1 \left(\frac{z}{z_i}\right)^{-4/3} + k_2 e^{-k_3 \left(\frac{z}{z_i} - 1.1\right)^2}}{k_1 \left(\frac{z_0}{z_i}\right)^{-4/3}}.$$

In [TOV06, p. 21], the model has been further modified to account for measurements taken in a desert basin (New Mexico) [KWE81], while removing the $z^{-4/3}$ trend above the inversion layer:

$$\log_{10}(C_T^2(z)) = \log_{10}(C_T^2(0)) + 0.775 e^{-\frac{(z-0.95z_i)^2}{2(z_i/7)^2}} + \left(\frac{1}{3} + \frac{4}{3} \log_{10}\left(\frac{z}{z_i}\right)\right) \cdot \sigma\left(\frac{z_i - z}{z_i/8}\right),$$

where $\sigma(z) = \frac{1}{1+e^z}$, such that $\lim_{z \rightarrow \infty} \sigma(z) = 0$ and $\lim_{z \rightarrow -\infty} \sigma(z) = 1$. $C_T^2(0)$ is the surface C_T^2 value, as given by a MOST-based model for example.

3.3.2.2 Stable boundary layer (nighttime)

Fewer models are available for describing the altitude dependency of C_n^2 in stable atmospheric BLs. This may be explained by the lack of profile measurements in this layer. Moreover, energy sources responsible for atmospheric turbulence are smaller in the absence of convection (e.g., wind driven mixing, gravity waves), hence less predictable and more localized [TOV06, p. 38].

From MOST, a $z^{-2/3}$ trend is expected in the SL, but does not seem to extend very far in the BL. Decreases lower than $z^{-2/3}$ have been suggested in the literature [WIC71], [RB85, p. 17], as well as a “z-less” dependency for altitudes larger than L_{Ob} , only associated to the local wind and temperature gradients at each level [KF94][TOV06, p. 38]. A large range between $z^{-0.5}$ and z^{-2} has also been observed, but requires further validation [RB85], [Bel93, p. 217].

Nevertheless, nighttime BL C_n^2 profiles are of particular interest for astronomical observations. According to [Aba+04], and based on 168 radiosonde flights taken during nighttime at nine astronomical sites, the mean and median profiles in the BL, assumed to extend from 0 to 1 km above the surface, are given by

$$\begin{aligned} C_{n_{\text{median}}}^2(z) &= 1.39 \times 10^{-13} z^{-1.45}, \\ C_{n_{\text{mean}}}^2(z) &= 1.24 \times 10^{-12} z^{-1.53}. \end{aligned}$$

The altitude dependency and amplitude of these profiles have been fitted based on the measurements, and question the validity of the $z^{-2/3}$ trend, or the stability conditions of the BL.

3.4 Machine learning-based C_n^2 models

In recent years, ML has developed considerably, driven by new advances in computational resources and the abundance of data that can be collected and processed. ML refers to computer systems able to learn and improve based on data they process, without following explicit instructions from a programmer. It can be applied to diverse fields of human knowledge, and science has seen an increase in ML-based models. This is also the case of C_n^2 modelling, and this section presents some of the new models published in the past 5 to 10 years⁴, as well as highlighting the motivations behind ML approaches.

3.4.1 Overview of models

Focus is given here to ML models directly providing C_n^2 from meteorological quantities, either for surface C_n^2 (Section 3.4.1.1) or for vertical C_n^2 profiles (Section 3.4.1.2).

3.4.1.1 Surface C_n^2 models

Several ML-based C_n^2 models for the ground (surface) C_n^2 are available, generalizing or complementing the models presented in Section 3.3. They usually rely on simultaneous measurements of surface meteorological parameters and C_n^2 in order to learn (non)linear relationships between them.

One of the pioneering model is the Wang and Basu model, published in 2016 [WB16]. It uses a multilayer perceptron artificial neural network with only one hidden layer to predict ground C_n^2 time series based on five variables: 2-meter temperature, 2-meter relative humidity, 2-meter pressure, 15-meter potential temperature gradient and 15-meter wind shear. The model has been trained with and validated on data at Mauna Loa Observatory, Hawaii (USA), capturing the temporal evolution of ground C_n^2 and enabling its prediction in very stable conditions. The application of this model and its extension to Antarctica are given in [Su+20]. A similar approach is used in [But+17] to predict the integrated C_n^2 in the SL at Paranal Observatory.

Various ML models have been explored in [Rud+18] and showed improvements by including the solar flux as an input. Performance are also improved when giving the Wyngaard model predicted C_n^2 as an input to the ML algorithms.

Other ML models have been presented in [Lio+22; Lio+23; Cam+23; Skl21], comparing the performance with the NAVSLaM model in maritime environments. The models in [Lio+23] use the following macroscopic meteorological parameters as inputs: wind speed and direction, air temperature, air pressure, relative humidity, solar radiation, dew point and rainfall rate. A classifier between weak and strong turbulence is also introduced.

⁴The list is not exhaustive but presents the main approaches used for ML-based C_n^2 models.

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In [Jel+20], twelve different input quantities are given to the ML models in order to predict surface C_n^2 in a near-maritime environment. The relative importance of the inputs has been established, suggesting that air temperature, pressure, air-water temperature difference, temporal hour weight and time of the year are important parameters to accurately predict C_n^2 . The model is further extended in [Jel+21], and compared with regression-based approaches (see Section 3.3.1.1).

It is worth highlighting that the use of ML does not prevent access to analytical expressions. Indeed, in [BAS21], a group method of data handling neural network is presented to predict surface C_n^2 at the Baikal Astrophysical Observatory, providing “learned” analytical equations. These equations involve various powers of the ground temperature, wind speed, pressure and humidity.

Finally, hybrid models have been introduced, namely in [Jel+23]. They rely on a macro-meteorological model, similar the ones presented in Section 3.3.1.1, and referred to as the baseline model in [Jel+23]. Then, a ML model is trained on the baseline model residual errors, i.e., the differences between the baseline model predictions and the actual measurements, in order to learn and correct those errors. This approach can be of particular interest to include local effects and learn features specific to a given environment, while using a baseline model validated at another location.

3.4.1.2 C_n^2 profile models

The ML algorithms for obtaining C_n^2 profile models are similar to the algorithm presented for surface C_n^2 . They involve macroscale meteorological parameters, with the altitude z added as an input to the ML algorithm. This is the case for the neural networks presented in [Su+21; Bi+22] for example, validated with C_n^2 profile measurements over the Tibetan Plateau and in a marine environment (Haikou, China). A comparable approach is presented in [Xu+21], focussing only on the BL profile.

Alternatively, models combining ML approaches and physical models have been developed, such as the MEP approach from [Zha+23], presented in Section 3.2.2.5.

3.4.2 Motivations for machine learning-based models

Based on the various ML-based references consulted, several motivations behind the use of ML models to predict C_n^2 can be highlighted.

Site dependency For the moment, ML approaches are rather site-specific, as they are trained based on measurements collected at a given location. However, the architecture of ML models (e.g., the choice of using a neural network with a given number of layers and requiring specified macro-meteorological quantities as inputs) is generic and can easily be applied to new measurements at other locations. Hence, provided that local measurements are available, a ML model can easily be (re)trained to provide accurate predictions at this location.

Furthermore, in the future, location-agnostic models could be developed, if trained with large amounts of data coming from various environments [Cam+23].

Hybrid models Relying on ML does not necessarily imply to get rid of all C_n^2 models developed in the past and based on empirical observations of physical insights. Indeed, hybrid models, as introduced in [Jel+23], are useful to combine ML with other “classical” C_n^2 models, while the outputs of “classical” models can also be used as inputs to ML models [Rud+18].

Model extensions Some of the models presented in this chapter rely on hypotheses that may not be fulfilled, e.g., unstable conditions. On the contrary, with appropriate training in all conditions, ML models can circumvent this issue [WB16].

Physical insights Last but not least, ML models can provide physical insights about the underlying processes by identifying the most important features and input parameters. This is illustrated in [PSB23], where various non-dimensional groups are constructed based on surface meteorological parameters prior to using a ML approach. Then, groups leveraging most accurate predictions can be identified, as well as the relevant input parameters.

3.5 Metrics for C_n^2 profile comparisons

In this chapter, many models for C_n^2 profiles have been introduced. The validity and quality of these models depend on their agreement with measurements and observations of C_n^2 profiles. Different metrics have been suggested in order to quantitatively evaluate the agreement between models and measurements, and are presented in this section. The wording used in this section always refers to a modelled profile and a measured one, but the metrics could also be used to compare profiles directly coming from different models (i.e., substituting “measured profiles” by “profiles coming from another model”). Some metrics have initially been suggested to compare vertical profiles of meteorological quantities, such as temperature, pressure, wind speed, etc., and then been applied for comparing C_n^2 profiles.

3.5.1 Vertical profiles

The evaluation of vertical profiles, i.e., functions of the altitude z and not single scalar values, is often conducted using classical statistical metrics described below. Alternative metrics are also briefly introduced. These metrics are used in Chapters 4 and 5 to validate the developed C_n^2 models with available measurements.

3.5.1.1 Common statistical metrics

Most statistical metrics can be used with vertical profiles of any quantities (e.g., temperature profiles $T(z)$, C_n^2 profiles, etc.). Hence, the following notations are introduced: $X(z)$ refers to the observed profile of a specified quantity (T , C_n^2 , etc.), while $Y(z)$ denotes the modelled profile of the same quantity. Since several observations are usually collected at a given location for different times, an index i is added to

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refer to the i -th observation, i.e., $X_i(z)$. Similarly, the modelled profile corresponding to this observation is $Y_i(z)$. Averaging can then be conducted over the different observations i , with N_{obs} being the maximal number of collected observations.

For C_n^2 profile comparisons, $X(z)$ and $Y(z)$ are very often chosen as the logarithms in base 10 ($\log_{10}$) of measured and modelled C_n^2 , given that C_n^2 can span several orders of magnitude. It is often required for $X(z)$ and $Y(z)$ to have the same vertical sampling for the altitudes. If it is not the case, reinterpolation of the individual profiles can be used.

Mean, variance and standard deviation profiles From a set of measured vertical profiles $\{X_i(z)\}_{1,\dots,N_{\text{obs}}}$ and a set of modelled profiles $\{Y_i(z)\}_{1,\dots,N_{\text{obs}}}$, the *average (or mean) measured profile* $\bar{X}(z)$ and the *average (or mean) modelled profile* $\bar{Y}(z)$ are obtained by

$$\bar{X}(z) = \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} X_i(z) \quad \text{and} \quad \bar{Y}(z) = \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} Y_i(z). \quad (3.56)$$

Fluctuations around the average profiles are measured by the *variance profiles* $\sigma_X^2(z)$ and $\sigma_Y^2(z)$, where $\sigma_X(z)$ is the *Standard Deviation (std)* profile of the measured profiles, while $\sigma_Y(z)$ is the one associated to the modelled profiles. They are computed using

$$\sigma_X^2(z) = \frac{1}{N_{\text{obs}} - 1} \sum_{i=1}^{N_{\text{obs}}} (X_i(z) - \bar{X}(z))^2 \quad \text{and} \quad \sigma_Y^2(z) = \frac{1}{N_{\text{obs}} - 1} \sum_{i=1}^{N_{\text{obs}}} (Y_i(z) - \bar{Y}(z))^2.$$

In cases where the logarithm of C_n^2 is used for $X_i(z)$ and $Y_i(z)$, e.g., $\log_{10}(C_n^2(z))$, the mean computed from (3.56) tends to the median profile of C_n^2 [Bel93, p. 218]. This follows the assumption of a lognormal distribution for C_n^2 , as briefly introduced in the next paragraph.

Median, quartiles and distributions Based on multiple samples of C_n^2 profiles, the *distributions* of possible C_n^2 values at a given altitude can be computed. The *median profile* can then be obtained, as well as any *quartile profile*. For example, in [Raf+20], the median, first quartile, and third quartile profiles are computed and compared between the measured and modelled C_n^2 profiles.

In [TV07], the distributions are obtained for C_T^2 , for four different parts of the atmosphere (BL, troposphere, tropopause, and stratosphere). It is shown that the logarithm in base 10 of C_T^2 , normalized by its median value, follows a Gaussian distribution (chi-square test, estimation of skewness and kurtosis). As a result, C_T^2 (and equivalently, C_n^2) is distributed according to a lognormal distribution, and the average of $\log_{10}(C_n^2)$ tends to the median values.

Bias, RMSE and bias-corrected RMSE The definitions of the bias, Root-Mean-Square Error (RMSE) and bias-corrected RMSE presented below are inspired from, and used in, [MLF13; TMF17; Qin+20; Bi+20; Xu+22].

Given the sets of measured vertical profiles $\{X_i(z)\}_{1,\dots,N_{\text{obs}}}$ and modelled profiles $\{Y_i(z)\}_{1,\dots,N_{\text{obs}}}$, the *bias*, also named *Mean Bias Error (MBE)* in [Raf+22], is defined as

$$\text{BIAS}(z) = \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} (Y_i(z) - X_i(z)) = \bar{Y}(z) - \bar{X}(z). \quad (3.57)$$

In (3.57), the altitude dependency of the bias is made explicit, since it is indeed a vertical profile of the bias that is obtained. The *absolute bias*, or *Mean Absolute Error (MAE)*, has a similar definition, except that the absolute value of the error (i.e., the difference between the modelled and the measured profiles) is used,

$$\text{ABSOLUTE BIAS}(z) = \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} |Y_i(z) - X_i(z)|. \quad (3.58)$$

Similarly, the *RMSE* profile is defined by

$$\text{RMSE}(z) = \sqrt{\frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} (Y_i(z) - X_i(z))^2}. \quad (3.59)$$

The bias enables to identify systematic errors between the models and the measurements, while the RMSE includes systematic and statistical errors. To remove the systematic error from the RMSE, the *bias-corrected RMSE*, also named *centred RMSE* in [Xu+22], is used, given by

$$\begin{aligned} \text{CRMSE}(z) &= \sqrt{\frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} [(Y_i(z) - \bar{Y}(z)) - (X_i(z) - \bar{X}(z))]^2} \\ &= \sqrt{\frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} [(Y_i(z) - X_i(z)) - (\bar{Y}(z) - \bar{X}(z))]^2} \\ &= \sqrt{\text{RMSE}^2(z) - \text{BIAS}^2(z)}. \end{aligned}$$

These metrics are very often used, also for comparing profiles of meteorological quantities. In the case of wind direction, slight modifications are necessary to account for its periodicity (angular direction), as stated in [LMF15].

Vertical averaging The error metrics previously defined in this section, i.e., the (absolute) bias, RMSE and bias-corrected RMSE, are profiles of the altitude. In order to reduce them to single scalar values, averaging over the altitudes is sometimes conducted, considering that discrete sampled altitudes $\{z_m\}_{m=1,\dots,N_s}$ are available

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[Hu+23a]. For example, the altitude-averaged bias is

$$\overline{\text{BIAS}} = \frac{1}{N_s} \sum_{m=1}^{N_s} \text{BIAS}(z_m) = \frac{1}{N_s} \sum_{m=1}^{N_s} \text{BIAS}(m\Delta z), \quad (3.60)$$

with N_s the number of sampled altitudes and Δz the sampling distance between the altitudes (assumed constant, i.e., $z_m = m\Delta z$). Equation (3.60) can be generalized for the other metrics.

The main drawback of altitude-averaging is the loss of information about the profile that enables to determine which part of the atmosphere are better (or worse) modelled. However, this could be mitigated by averaging for different parts of the atmosphere, for example for the ground layer, the troposphere, and the stratosphere [Ull+23]. In this case, three scalar values summarize the profile metrics.

As a complementary remark, some references conduct the averaging only on the altitudes and not over a set of different measured profiles. Hence, one scalar metric per observed profiles is obtained [Wu+20; Xu+22], e.g.,

$$\text{BIAS}_i = \frac{1}{N_s} \sum_{m=1}^{N_s} (Y_i(m\Delta z) - X_i(m\Delta z)).$$

Relative errors The error between the measured $X_i(z)$ and the modelled $Y_i(z)$, as used in (3.57) and (3.58), can be divided by the measured value $X_i(z)$ in order to obtain the relative error. It is then averaged over the multiple sampled profiles, yielding the *average (absolute) relative error profile* [Ull+23]. It is referred to as the *Mean Relative Error (MRE)* in [Raf+22], defined by

$$\text{MRE}(z) = \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} \frac{|Y_i(z) - X_i(z)|}{|X_i(z)|}.$$

Averaging over different atmospheric regions can then be performed [Ull+23].

Correlation coefficients Correlation coefficients R_{XY} measure statistical relationships between two variables. The *Pearson correlation coefficient*, noted R_{XY} , measures the linear relationship between the variables, and is defined as the ratio between the covariance of the two variables and the product of their standard deviation.

An altitude profile of the correlation coefficient could be defined as

$$\begin{aligned} R_{XY}(z) &= \frac{\sigma_{XY}(z)}{\sigma_X(z) \sigma_Y(z)} \\ &= \frac{\sum_{i=1}^{N_{\text{obs}}} (X_i(z) - \bar{X}(z)) (Y_i(z) - \bar{Y}(z))}{\sqrt{\sum_{i=1}^{N_{\text{obs}}} (X_i(z) - \bar{X}(z))^2} \sqrt{\sum_{i=1}^{N_{\text{obs}}} (Y_i(z) - \bar{Y}(z))^2}}, \end{aligned}$$

where $\sigma_{XY}(z)$ is the covariance profile between $X(z)$ and $Y(z)$, i.e.,

$$\sigma_{XY}(z) = \frac{1}{N_{\text{obs}} - 1} \sum_{i=1}^{N_{\text{obs}}} (X_i(z) - \bar{X}(z)) (Y_i(z) - \bar{Y}(z)).$$

In practice, the correlation coefficient is averaged over the altitudes to get one scalar value, using

$$\bar{R}_{XY} = \frac{1}{N_s} \sum_{m=1}^{N_s} R_{XY}(m\Delta z).$$

Alternatively, the coefficient can be averaged over only a given part of the atmosphere, as in [Raf+22].

Finally, in [Wu+20], the coefficient is computed for each measured profile, averaging over the altitudes:

$$R_{XY,i} = \frac{\sum_{m=1}^{N_s} (X_i(m\Delta z) - \bar{X}_i) (Y_i(m\Delta z) - \bar{Y}_i)}{\sqrt{\sum_{m=1}^{N_s} (X_i(m\Delta z) - \bar{X}_i)^2} \sqrt{\sum_{m=1}^{N_s} (Y_i(m\Delta z) - \bar{Y}_i)^2}},$$

with $\bar{X}_i$ (resp. $\bar{Y}_i$) the altitude-average of the i -th measured (resp. modelled) profile.

3.5.1.2 Alternative metrics

In addition to the common statistical metrics presented above, custom metrics are sometimes used for comparing C_n^2 profiles. Some of them are presented in this section.

Relative error of integrated C_n^2 In [TV07], the relative error of the integrated C_n^2 over 1 km thick layers is defined, noted $\text{EIC}_{n,i}^2$. As this metric is computed for each pair of measured and modelled profiles, it can be indexed by i , using the notations introduced previously. Hence, for a layer centred at an altitude z_m (in meter) and for the i -th measured/modelled profile, its $\text{EIC}_{n,i}^2(z_m)$ is defined by

$$\text{EIC}_{n,i}^2(z_m) = \frac{\left| \log_{10} \left(\int_{z_m-500}^{z_m+500} Y_i(z) dz \right) - \log_{10} \left(\int_{z_m-500}^{z_m+500} X_i(z) dz \right) \right|}{\log_{10} \left(\int_{z_m-500}^{z_m+500} X_i(z) dz \right)},$$

where $X_i(z)$ is the i -th measured C_n^2 profile, and $Y_i(z)$ is the associated modelled C_n^2 profile. The logarithm is taken after the integration, $X_i(z)$ and $Y_i(z)$ are thus directly the C_n^2 values, and not their logarithms. As a generalization, the width of the layer, i.e., the bounds for the integration over the altitudes, can be modified.

The average profile of the relative errors of integrated C_n^2 , computed over all profiles indexed by i , estimates the static error of the models, while its standard deviation shows the gain of reliability in C_n^2 profile prediction from the models [TV07].

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Index of agreement Inspired by [WRM12] and the references therein, [Raf+22] makes use of the statistical index of agreement IoA to compare measured and modelled profiles of meteorological quantities. It is defined by

$$\text{IoA}(z) = 1 - \frac{\sum_{i=1}^{N_{\text{obs}}} (Y_i(z) - X_i(z))^2}{\sum_{i=1}^{N_{\text{obs}}} (|Y_i(z) - \bar{Y}(z)| + |X_i(z) - \bar{X}(z)|)^2},$$

and varies between 0 (total disagreement) and 1 (perfect agreement). Alternative definitions as well as more explanations behind the origin of this index can be found in [WRM12]. A weighted (linear) combination of indexes of agreement for several meteorological quantities in order to define a new merit function is also introduced in [Raf+22].

Customized figures of merit If deemed relevant, any metric presented previously can be combined together to define a custom figure of merit. For example, a simple linear combination can be made, or more complex expressions can be used (e.g., see the definition of the index of agreement).

3.5.2 Integrated parameters and surface values

Instead of comparing full vertical profiles of C_n^2 with the metrics presented in Section 3.5.1, the comparison can be conducted based on scalar values associated to the C_n^2 profiles. Those values can come either from a subset of the C_n^2 profile (such as choosing only C_n^2 at the ground for example), or from vertical integration of the profile as conducted for *OT integrated parameters* (see Section 6.3 of Chapter 6). They have important interpretations discussed in Section 6.3, but in this section, they are simply considered as a set of scalar values associated to vertical C_n^2 profiles.

In this case where scalar values are compared, there is no altitude dependency. Hence, in the following, X refers to the measured quantity, while Y is the modelled one. Again, an index i is added to refer to the i -th measurement or model, with $i = 1, \dots, N_{\text{obs}}$. Some metrics presented in this section are used in Chapters 4, 5, 7 and 8.

3.5.2.1 Common statistical metrics

Most of the metrics presented for vertical profiles in Section 3.5.1 are also valid for scalar quantities, by removing the altitude dependency in the definitions of the metrics. As an example, the mean is given by

$$\bar{X} = \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} X_i \quad \text{and} \quad \bar{Y} = \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} Y_i,$$

and the RMSE is

$$\text{RMSE} = \sqrt{\frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} (Y_i - X_i)^2}.$$

Similar changes are used to define the standard deviations σ_X^2 and σ_Y^2 , the (absolute) bias, the centred RMSE, the average (absolute) relative error, and the Pearson correlation coefficient R_{XY} . Median values and distributions are also often used for comparing the measurement and predictions.

A non-exhaustive list of references making use of such metrics for integrated parameters is given by [Mas+17; Gio+13; Gio+21; MTF23]. In [Qin+16b; Qin+16a], the different metrics are used to validate ground C_n^2 values, while [TMF17] relies on these same metrics to compare surface values of meteorological parameters.

3.5.2.2 Contingency tables

Rather than comparing directly the measured and modelled values X_i and Y_i , one might be interested in determining whether they belong to the same category, i.e., to the same interval of values. Such analysis can be made by using contingency tables, as explained in [LMF15] for example.

The first step is to separate the data into different categories, or classes, which are defined in advance. Typically, they correspond to intervals of values, separated by thresholds that can be related to percentiles of the measured distributions (e.g., tertiles in cases where three categories are used), or important values related to the application (e.g., maximal value for a parameter that sets a practical limit. An example is the ground wind speed, where a maximal value above which the dome of the telescope must be closed can be identified). It is common to define the same categories for the observations and the models such that, in the following, the different categories are referred to intervals $\mathcal{Z}_\beta$, with β an integer between 1 and the total number of category N_c .

Then, a $N_c \times N_c$ table is constructed, where categories associated to the measurements are presented horizontally and categories related to the modelled values are given vertically. Finally, for each pair of (X_i, Y_i) , the categories for X_i and Y_i are identified independently and counted in the appropriate entry of the table. As a result, for a set of N_{obs} observations, the contingency table gathers the distribution into the different categories of all pairs (X_i, Y_i) , counting the number of times a pair fell into a given entry in the table.

Table 3.8 presents a generic 3×3 table, similar to [LMF15]. Observations and modelled values are separated into the three categories (intervals) $\mathcal{Z}_1$ to $\mathcal{Z}_3$, and the variables a to i are the number of occurrences in each entry of the table. For example, a represents the number of times an observation X_i belonging to $\mathcal{Z}_1$ has an associated modelled value Y_i also found in $\mathcal{Z}_1$. The other variables are defined similarly. The total number of observations is such that $N_{\text{obs}} = a + b + c + d + e + f + g + h + i$.

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Table 3.8 – Generic 3×3 contingency table, inspired from [LMF15].

		Observations ($X_i \in \dots$)			Total (model)
		$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	
Model ($Y_i \in \dots$)	$\mathcal{Z}_1$	a	b	c	$a + b + c$
	$\mathcal{Z}_2$	d	e	f	$d + e + f$
	$\mathcal{Z}_3$	g	h	i	$g + h + i$
Total (obs.)		$a + d + g$	$b + e + h$	$c + f + i$	N_{obs}

Clearly, one would like to have a perfect agreement (called a “hit”) between the intervals for the modelled and observed values, such that only a , e , and i should be non-zero in Table 3.8. The percentage of correct detections PC [%] can be defined as

$$\text{PC} = \frac{a + e + i}{N_{\text{obs}}} \times 100,$$

and should be close to 100%. However, in practice, some modelled values may not belong to the same interval as the measured ones (over- or underestimation), such that b , c , d , f , g , or h , may be different from zero, reducing the percentage of correct detections.

The probability of detection POD, or “hit rate”, is also an important quantity that relates to all observations belonging to a given interval $\mathcal{Z}_\beta$. In the 3×3 case, they are given by (in percent)

$$\text{POD}(\mathcal{Z}_1) = \frac{a}{a + d + g} \times 100,$$

$$\text{POD}(\mathcal{Z}_2) = \frac{e}{b + e + h} \times 100,$$

$$\text{POD}(\mathcal{Z}_3) = \frac{i}{c + f + i} \times 100.$$

A perfect modelling would give PODs of 100% for all intervals of the observations. Finally, the extremely bad detection EBD metric is sometimes used to assess the number of times observations and modelled values belongs to the two most different intervals, i.e., $\mathcal{Z}_1$ and $\mathcal{Z}_3$. Its definition, in percent, is

$$\text{EBD} = \frac{c + g}{N_{\text{obs}}} \times 100.$$

It should be as close as possible to 0, and is exactly equal to zero for a perfect model.

When assessing the different metrics related to a contingency table, it is always relevant to compare with a purely random case where categories are given randomly to the modelled values, assuming there is exactly $N_{\text{obs}}/3$ observations in each category. In this case, $a = b = \dots = i = N_{\text{obs}}/9$ and $\text{PC} = 33.33\%$. All PODs are also equal to 33.33%, while the EBD is 22.22%. This provides a good reference to evaluate the performance of the model. Typical values achieved in practice depends on the pa-

parameters studied, the threshold values used to define the $\mathcal{Z}_\beta$, and the models. They can be found in some of the references cited in the next paragraphs.

Applications of contingency tables to evaluate predicted classical meteorological quantities (e.g., ground temperature, wind speed and direction) in the SL can be found in [LMF15; TMF17; Qin+17]. The use of contingency tables with integrated parameters is presented in [Mas+17; MTF23]. In [Qin+16a; Qin+17; Qin+19], contingency tables are applied to the logarithm of surface C_n^2 .

3.5.3 Interests of multiple metrics

The various metrics presented in this Section 3.5 are useful to validate developed models with real observations. Furthermore, they can also be of interest to get insights about the model parameters, e.g., by monitoring the evolution of the metrics when tuning some parameters of the models (as conducted in Chapter 8 example).

It is always a good practice to monitor several metrics to assess the quality of a model and use them in a complementary way.

3.6 Chapter summary

This chapter has provided a rare and extensive overview of the numerous C_n^2 models found in the literature, some of which date back to the 1970s and are still in use today (e.g., HV models). There is ongoing development of new models for different sites, seasons, and applications, namely using new ML approaches.

The literature review highlighted the main parameters commonly used in C_n^2 models, such as temperature gradients and wind shear, having a significant impact on atmospheric turbulence. It also revealed the necessity of carefully handling the BL to capture its diurnal variations and its large C_n^2 values. Moreover, it showed that several literature-based C_n^2 models only provide average C_n^2 profiles, based on measurements at one location. They are therefore not suited for depicting localized OT layers, which are expected to induce most of OT and vary with time [Ver02; Mar+17].

This literature study also enabled to reach the following conclusion: for optical communication applications, the ideal C_n^2 models should be location- and season-independent, rely on large-scale meteorological quantities, be straightforward to apply, and avoid excessive empiricism or allow for easy calibration at specific sites. Such models, falling into the category of parametric models, could potentially incorporate inputs from NWP or make use of ML approaches.

The goals of Chapters 4 and 5 is to further develop C_n^2 models at OGS sites, using several of the models presented in this chapter, as well as the insights provided by this review (e.g., parametric model, separate handling of BL, importance of temperature gradient and wind shear, etc.).

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3 | Modelling of C_n^2 profiles

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Computation of C_n^2 based on its statistical definition

4

The two previous chapters, i.e., Chapters 2 and 3, respectively introduced the theory behind OT and the different C_n^2 models available in the literature. The aim of Chapter 4 is to merge both fields by presenting a new C_n^2 model based on its statistical definition, inspired by the theoretical work in Chapter 2.

As described in Chapter 3, there exist numerous approaches to develop parametric C_n^2 models that can translate meteorological quantities into a C_n^2 value or profile. Here, the model developed in this chapter is another alternative, that relies on the computation of the refractive index using the knowledge of meteorological quantities (namely the pressure, the temperature, and the humidity) before the calculation of C_n^2 . This is the approach depicted in red in Figure 4.1, using radiosonde observations as the primary source of meteorological quantities. Indeed, such measurements have an improved spatial resolution in the vertical direction compared to NWP simulation data for example, that is of major interest for the work conducted in this chapter.

Compared to most models from Chapter 3, the C_n^2 model detailed below provides time- and location-dependent C_n^2 profiles, as long as radiosonde observations are available at the desired time at a given location. It can therefore be applied to any place in the world where radiosonde measurements are collected and is not limited to astronomical sites, justifying its interest for optical communication sites. In addition, it gives access to C_n^2 at optical frequencies as well as C_n^2 at radiofrequencies by simply modifying the refractive index definition used (see Sections 2.1.2.1 and 2.1.2.2). It led

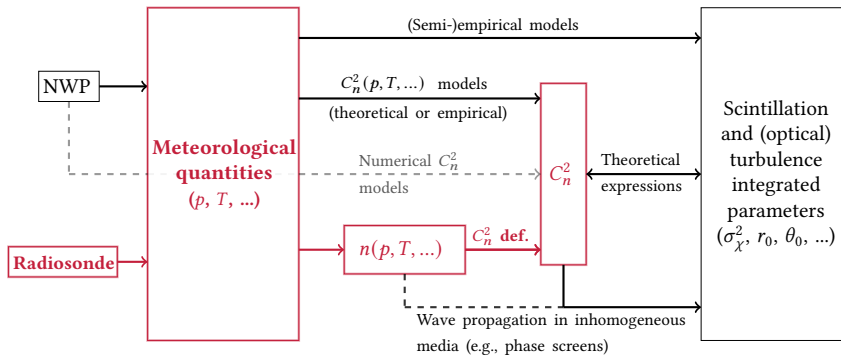


Figure 4.1 – Schematic of the thesis structure, highlighting the C_n^2 model presented in Chapter 4.

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to several publications in the framework of this thesis, in both fields of optical wave and radiofrequency propagation:

- [QVO23a]: F. Quatresooz, D. Vanhoenacker-Janvier, and C. Oestges. “Computation of optical refractive index structure parameter from its statistical definition using radiosonde data”. In: *Radio Science* 58.1 (2023), e2022RS007624. DOI: 10.1029/2022RS007624.
- [QVO23b]: F. Quatresooz, D. Vanhoenacker-Janvier, and C. Oestges. “Modelling of Scintillation at Radio and Optical Frequencies from Radiosonde Observations”. In: *2023 17th European Conference on Antennas and Propagation (EuCAP)*. 2023, pp. 1–5. DOI: 10.23919/EuCAP57121.2023.10132985.
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The contributions in this chapter are:

- The presentation and validation of a C_n^2 model relying on the refractive index statistical definition, highlighting its advantages and limitations with respect to other C_n^2 models;
- A multi-site comparison of C_n^2 profiles predicted by the model, demonstrating its location-independence;
- The application of the same C_n^2 model to estimate both optical and RF C_n^2 profiles, showing the relationship between both profiles.

This chapter is based on the publications listed above, with additional explanations, applications, and validations of the model. The C_n^2 model is first presented in Section 4.1, along with the relevant equations. Then, it is validated with synthetic data in Section 4.2 and applied to various radiosonde measurements in Section 4.3. Its validation with measurements of optical C_n^2 profiles is conducted in Section 4.4, and its adaptation and validation for radiofrequencies is detailed in Section 4.5. Finally, Section 4.6 discusses the limitations of the model based on its main hypotheses, and offers possible alternatives.

4.1 Statistical definition of C_n^2 and associated model

The new C_n^2 model presented in this section is referred to as *the C_n^2 model based on its statistical definition*, or in short *the statistical definition C_n^2 model*. Section 4.1.1 motivates this name, and highlights the relevant equations and hypotheses already presented in Chapter 2. Section 4.1.2 presents the model, with a step by step approach to simplify its application.

4.1.1 Physics-based equations

In all generality, the refractive index n is considered as a space-time random field, denoted $n(\mathbf{r}, t)$, with t the time and $\mathbf{r} = (x, y, z)$ the position in space (see Section 2.4). It can be separated into two parts [AP05, p. 63]:

$$n(\mathbf{r}, t) = \bar{n}(\mathbf{r}, t) + n'(\mathbf{r}, t),$$

where $\bar{n}(\mathbf{r}, t) = \langle n(\mathbf{r}, t) \rangle$ is the mean value of $n(\mathbf{r}, t)$, and $n'(\mathbf{r}, t) = n(\mathbf{r}, t) - \bar{n}(\mathbf{r}, t)$ corresponds to the fluctuation of the refractive index around its mean. By definition, $\langle n'(\mathbf{r}, t) \rangle = 0$.

The time variation of the refractive index field is ignored in the following, such that $n(\mathbf{r}) = \bar{n}(\mathbf{r}) + n'(\mathbf{r})$. Hence, the theory about space random fields, described in Section 2.4.1.2, can be applied. The space variation of the refractive index can directly be related to the variation of meteorological quantities (e.g., pressure, temperature, etc.) through the empirical expressions presented in Section 2.1.2. For example, making the dependency in $\mathbf{r}$ explicit in (2.3), the optical refractive index is given by

$$n(\mathbf{r}) = 1 + 10^{-6} \frac{77.6 p(\mathbf{r})}{T(\mathbf{r})} \left(1 + \frac{7.52 \times 10^{-3}}{\lambda^2} \right), \quad (4.1)$$

and its approximation in (2.4) is

$$n(\mathbf{r}) = 1 + 10^{-6} \frac{80 p(\mathbf{r})}{T(\mathbf{r})}. \quad (4.2)$$

Based on (2.27), the structure function associated with the refractive index fluctuations between two positions in space ($\mathbf{r}_1$ and $\mathbf{r}_2$) is [Tat71, p. 25]

$$D_n(\mathbf{r}_1, \mathbf{r}_2) = \langle (n'(\mathbf{r}_1) - n'(\mathbf{r}_2))^2 \rangle.$$

Assuming local homogeneity, as presented in Section 2.4.1.2, the structure function depends only on the position difference $\boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_2$. Adding the isotropy hypothesis, only the absolute distance difference $\rho = |\boldsymbol{\rho}|$ matters and not its orientation. With these two assumptions, the structure function is now expressed as [Ish78, p. 526]

$$D_n(\rho) = \langle (n'(\mathbf{r} + \boldsymbol{\rho}) - n'(\mathbf{r}))^2 \rangle. \quad (4.3)$$

The application of Kolmogorov theory of turbulence to the refractive index field in Section 2.5.2 leads to a structure function proportional to $\rho^{2/3}$ in the inertial range. The proportionality constant is the refractive index structure parameter C_n^2 , while the initial range is defined between the inner scale l_0 and the outer scale L_0 . As a result, using (2.46), the structure function is given by

$$D_n(\rho) = \langle (n'(\mathbf{r} + \boldsymbol{\rho}) - n'(\mathbf{r}))^2 \rangle = C_n^2 \rho^{2/3} \quad \text{for} \quad l_0 \ll \rho \ll L_0. \quad (4.4)$$

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Equation (4.4) provides a simple approach to obtain C_n^2 by evaluating the structure function for one or several known distances ρ in the inertial range, that is,

$$C_n^2 = \frac{\langle (n'(\mathbf{r} + \boldsymbol{\rho}) - n'(\mathbf{r}))^2 \rangle}{\rho^{2/3}} \quad \text{for} \quad l_0 \ll \rho \ll L_0. \quad (4.5)$$

This result is the main equation for the model in this chapter and links C_n^2 with the statistical definition of the structure function, hence the name of *statistical definition C_n^2 model*.

The use of (4.5) to estimate C_n^2 is not new, and has already been applied in [Van+17], [Qui20, p. 143] to obtain complementary cumulative distribution functions for microwave scintillation. In these works, meteorological data used to compute the refractive index and then C_n^2 come from NWP simulations.

More generally, (4.3) is often used to estimate the structure function from the knowledge of the local refractive index. For example, this is the case for the illustrations presented in Section 2.5.3, using a 1D-formulation of the structure function in (2.56) and leading to the estimation of $D_n(\rho)$ in Figure 2.12a.

In the literature, this approach is referred to as the *direct method*, as opposed to spectral or wavelet approaches that also enable to estimate structure functions [Bra+16; Pla+17; GF20]. In [WF12] and [Wai+15], a similar approach is applied to LES to obtain the structure functions in the x -, y -, and z -directions. The averaging, necessary to estimate $D_n(\rho)$, is conducted over space and time. The limits of the inertial range are also clearly visible in the obtained structure functions (Figure 2 in [WF12] and Figure 6 in [Wai+15]). In [GF20], the velocity structure function is computed with this direct method, from LES simulations.

This approach has also been applied to Unmanned Aerial Vehicles (UAVs) to obtain C_T^2 and C_n^2 estimations [Van+12; Bey+12]. These works are further detailed in Section 4.6.3.

4.1.2 Suggested model

The computation of (4.5) involves several steps outlined below and illustrated in Figure 4.2.

Initially, one assumes access to vertical profiles of meteorological quantities, e.g., the pressure p and the temperature T . These profiles, on the left part of Figure 4.2, are typically obtained from high-density radiosonde observations. The vertical z -direction corresponds to altitude, with black points representing locations where pressure and temperature measurements are available. In the case of radiosondes, the vertical sampling may not be equispaced. The data is then resampled to a constant vertical spacing denoted Δz through linear interpolation. This interpolation is essential in the last step of the C_n^2 computation, which requires a fixed vertical spacing between points, as outlined in (4.6) detailed below.

Subsequently, the optical refractive index is calculated using (4.1) or (4.2), de-

Statistical definition of C_n^2 and associated model | 4.1

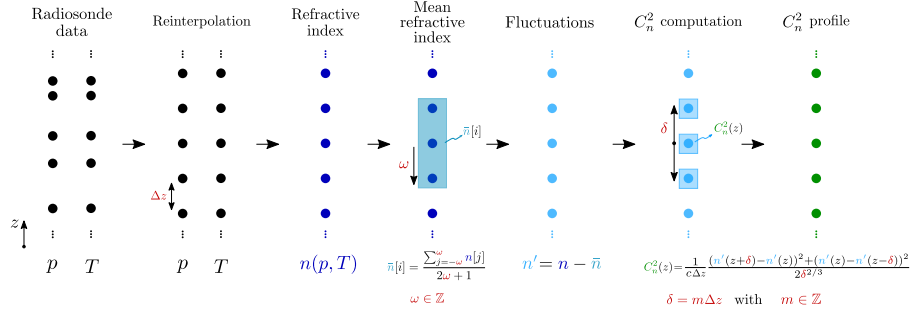


Figure 4.2 – Followed approach to extract C_n^2 from its statistical definition (adapted from [QVO23a]).

pending on the wavelength information and the decision to consider it¹. This yields a vertical profile of the refractive index, constituting the third step in Figure 4.2.

In (4.5), only the fluctuations n' of the refractive index are involved, not directly the refractive index n itself. Therefore, it is necessary to extract the local mean refractive index $\bar{n}$ and subsequently obtain the fluctuations from $n' = n - \bar{n}$. The local mean is computed based on neighbouring points, using a window average of size $2\omega + 1$, with $\omega \in \mathbb{Z}$ (steps 4 and 5 in Figure 4.2).

Eventually, the C_n^2 profile is computed using

$$C_n^2(z) = \frac{1}{c \Delta z} \frac{(n'(z+\delta) - n'(z))^2 + (n'(z) - n'(z-\delta))^2}{2 \delta^{2/3}}. \quad (4.6)$$

The ensemble average in (4.5) is replaced by a three-point average between points spaced by a vertical spacing δ , that is,

$$D_n(\rho) = \langle (n'(\mathbf{r} + \boldsymbol{\rho}) - n'(\mathbf{r}))^2 \rangle \approx \frac{(n'(z+\delta) - n'(z))^2 + (n'(z) - n'(z-\delta))^2}{2},$$

with $\rho = \delta$. This vertical spacing δ is a multiple of the vertical resolution Δz , i.e., $\delta = m \Delta z$ with $m \in \mathbb{Z}$ (see Figure 4.2). The substitution of the ensemble average by a spatial average makes use of the ergodicity hypothesis, while this spatial average is computed exclusively from measurements along the vertical direction, assuming isotropy.

Furthermore, due to the vertical sampling, a factor Δz arises at the denominator to ensure the (local) homogeneity of (4.6). This is justified in the next section. Finally, c is a calibration parameter named *scale factor*. Its origin is motivated in Sections 4.2 and 4.3.1, and, as shown in the next sections, this parameter does not seem to depend on the location.

In this approach, three parameters require careful consideration:

1. The vertical spacing Δz used for resampling the data;

¹This approach allows for modelling variations of optical C_n^2 with the wavelength, but also to obtain C_n^2 at radiofrequencies by changing the expression used for the refractive index (Section 4.5).

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2. The integer ω depicting the window size used for the computation of the mean refractive index;
3. The vertical spacing $\delta = m \Delta z$ with $m \in \mathbb{Z}$ used for computing the average in the C_n^2 expression.

These three parameters (Δz , ω , and δ) are depicted in red in Figure 4.2.

The current approach exhibits certain limitations:

- Determining the local refractive index mean $\bar{n}$ for extracting fluctuations is challenging. Indeed, the window for computing the mean must not be too large as the refractive index mean value also varies with altitude, due to variations of the mean pressure and temperature.
- The available vertical resolution of the input meteorological data can impose constraints on the choice of Δz . A low resolution may result in a computed mean that does not accurately represent the local refractive index mean $\bar{n}$. Additionally, the choice of Δz should be related to the vertical extent of the turbulent structures to be identified. For example, large Δz 's (e.g., 400 m) lead to average profiles smoothed out of noisy artefacts, but do not enable to detect thin turbulent layers.
- The first point of the profile should ideally be located at an altitude $\delta + \omega \Delta z$ for computing the mean refractive index and C_n^2 according to Figure 4.2. Otherwise, the absence of neighbouring points hinders the computations, making the model less suitable for depicting turbulence close to the ground, particularly with large Δz 's.
- The presented approach is valid only between the inner scale l_0 and outer scale L_0 . While the inner scale is typically on the order of millimetres or centimetres [AP05, p. 60], the outer scale, ranging from meters to hundreds of meters, can limit the computation (see Sections 4.2 and 4.6).

4.1.2.1 Justification of sampling factor $\frac{1}{\Delta z}$

The C_n^2 profile obtained from (4.6) corresponds to the conventional C_n^2 profile, which is defined in a continuous domain with respect to z . However, the observations collected from radiosonde measurements are acquired at discrete points, then resampled with a spacing of Δz . Hence, the C_n^2 profile in the continuous domain is computed from observations in a discrete domain, and the proof below shows that a factor $\frac{1}{\Delta z}$ should be added in the C_n^2 estimation. It corresponds to the $\frac{1}{\Delta z}$ in (4.6).

For simplicity, a sketch of the proof is given in 1D for the vertical direction z [QVO23a]. Assuming that the refractive index fluctuations $n'(z)$ are sampled in the vertical direction, the corresponding sampled field $n'_s(z)$ is obtained as follows:

$$n'_s(z) = n'(z) \sum_{k=-\infty}^{\infty} \delta(z - k\Delta z - z_0) = \sum_{k=-\infty}^{\infty} \underbrace{n'(k\Delta z + z_0)}_{n'[k]} \delta(z - k\Delta z - z_0),$$

where $\delta(z)$ is the Dirac delta function, Δz is the sampling distance and z_0 is the initial position where sampling is started ($z_0 \in [0, \Delta z[$). To ensure homogeneity in the sampled refractive index field, the initial position z_0 is assumed to be random [Leo94]. The discrete refractive index field is denoted $n'[k]$.

Under the assumptions of homogeneity and 1D fluctuations, the structure function in the inertial range is given by (2.56), here expressed for the z -direction,

$$D_n(\rho) = \langle (n'(z + \rho) - n'(z))^2 \rangle = 2(B_n(0) - B_n(\rho)) = C_n^2 \rho^{2/3}, \quad (4.7)$$

where $B_n(\rho)$ represents the 1D covariance function, defined using (2.57)

$$B_n(\rho) = \langle n'(z + \rho) n'(z) \rangle.$$

The covariance function of the sampled field is

$$B_{n_s}(\rho) = \left\langle \left(\sum_{k=-\infty}^{\infty} n'[k] \delta(z + \rho - k\Delta z - z_0) \right) \left(\sum_{k'=-\infty}^{\infty} n'[k'] \delta(z - k'\Delta z - z_0) \right) \right\rangle.$$

Assuming that z_0 is a known and independent quantity of $n'(z)$, the expectation can be initially computed on n' :

$$B_{n_s}(\rho|z_0) = \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \underbrace{\langle n'[k] n'[k'] \rangle}_{B_n[k-k']} \delta(z + \rho - k\Delta z - z_0) \delta(z - k'\Delta z - z_0),$$

with $B_n[k - k']$ the discrete covariance function. Subsequently, the expectation over z_0 is considered, provided that z_0 follows a uniform distribution between 0 and Δz . Its PDF $p_{Z_0}(z_0)$ is

$$p_{Z_0}(z_0) = \begin{cases} \frac{1}{\Delta z} & \text{if } 0 \leq z_0 < \Delta z, \\ 0 & \text{otherwise.} \end{cases}$$

This leads to the following integration, according to the definition of the expectation:

$$\begin{aligned} B_{n_s}(\rho) &= \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} B_n[k - k'] \int_0^{\Delta z} \frac{1}{\Delta z} \delta(z + \rho - k\Delta z - z_0) \delta(z - k'\Delta z - z_0) dz_0 \\ &= \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \frac{B_n[k - k']}{\Delta z} \int_{k'\Delta z - z}^{(k'+1)\Delta z - z} \delta(\rho - (k - k')\Delta z - s) \delta(s) ds \\ &= \sum_{k=-\infty}^{\infty} \frac{B_n[k - \hat{k}']}{\Delta z} \delta(\rho - (k - \hat{k}')\Delta z) \\ &= \sum_{i=-\infty}^{\infty} \frac{B_n[i]}{\Delta z} \delta(\rho - i\Delta z), \end{aligned}$$

using the change of variable $s = k'\Delta z + z_0 - z$ and recognizing that there exists a

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unique value of k' (denoted as $\hat{k}'$) for which $s = 0$ is within the integration domain and $\int_{\hat{k}'\Delta z - r}^{(\hat{k}'+1)\Delta z - r} \delta(s) ds = 1$. Finally, the change of variable $i = k - \hat{k}'$ is applied.

The structure function given in (4.7) becomes

$$D_{n_s}(\rho) = 2 \left(\sum_{i=-\infty}^{\infty} \frac{B_n[i]}{\Delta z} \delta(i\Delta z) - \sum_{i'=-\infty}^{\infty} \frac{B_n[i']}{\Delta z} \delta(\rho - i'\Delta z) \right) = C_n^2 \rho^{2/3},$$

that is equalized to expected C_n^2 in the continuous domain. As $D_{n_s}(\rho)$ is available only for distances ρ that are multiples of the sampling distance, i.e., $\rho = m\Delta z$ with $m \in \mathbb{Z}$:

$$\begin{aligned} D_{n_s}(m\Delta z) &= 2 \left(\sum_{i=-\infty}^{\infty} \frac{B_n[i]}{\Delta z} \delta(i\Delta z) - \sum_{i'=-\infty}^{\infty} \frac{B_n[i']}{\Delta z} \delta((m - i')\Delta z) \right) \\ &= 2 \left(\frac{B_n[0]}{\Delta z} - \frac{B_n[m]}{\Delta z} \right) = C_n^2 (m\Delta z)^{2/3}, \end{aligned} \quad (4.8)$$

using the properties of the Dirac delta function.

The computation of the structure function in the discrete domain gives

$$\begin{aligned} D_n[m] &= \langle (n'[k+m] - n'[k])^2 \rangle \\ &= \langle n'^2[k+m] \rangle - 2 \langle n'[k+m] n'[k] \rangle + \langle n'^2[k] \rangle \\ &= 2 (B_n[0] - B_n[m]). \end{aligned} \quad (4.9)$$

From (4.8) and (4.9), the estimation of C_n^2 based on the discrete structure function is

$$C_n^2 = \frac{1}{\Delta z} \frac{D_n[m]}{(m\Delta z)^{2/3}},$$

corresponding to (4.6). In (4.6), the ergodicity hypothesis is also used to compute the structure function $D_n[m]$ from a spatial average on three points.

4.2 Validation with synthetic data

In order to validate the statistical definition approach presented in Section 4.1.1 and assess the influence of some parameters (namely ω and L_0), a preliminary analysis is conducted with synthetic data. A specific focus is placed on studying limitations associated with the outer scale L_0 , as it is frequently observed to be smaller than 10 meters in practical scenarios [BB88; Cou+88; Aba+04]. However, the statistical definition model is applied to radiosonde observations having a vertical resolution on the order of 10 meters, with a resampling distance Δz between 25 and 400 meters. Thus, the assumption about the inertial range, required in (4.5), may not hold, and the consequences are introduced here.

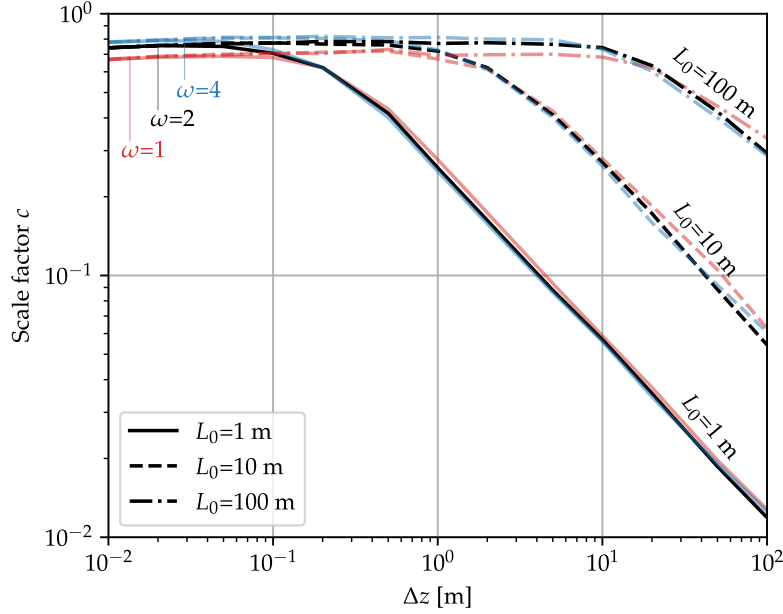


Figure 4.3 – Scale factor c for C_n^2 correction on synthetic data (reproduced from [QVO23a]).

Synthetic data are generated following the approach detailed in Section 2.5.3, using the 1D von Kármán spectrum from (2.54). Figures 2.10 and 2.12a in Chapter 2 already presented the generated data and the associated structure function (blue curve - synthetic fluctuations), for $C_n^2 = 10^{-16} \text{ m}^{-2/3}$ and $L_0 = 100$ meters.

From the synthetic data, C_n^2 is estimated along the direction using (4.6), for different spatial resolutions Δz . This “profile” of C_n^2 is then averaged over the whole simulation range (typically 10 km to 100 km), giving a single scalar value that should correspond to C_n^2 used for generating the data. The ratio between the estimated C_n^2 (recovered using the statistical definition method on simulated data) and the actual C_n^2 (applied as input in the generation of synthetic data) is defined as the *scale factor* c . This factor is equal to one if C_n^2 is correctly estimated.

Figure 4.3 shows the scale factor c for $C_n^2 = 10^{-16} \text{ m}^{-2/3}$, considering an outer scale L_0 that ranges from 1 to 100 meters. The x-axis corresponds to the spatial resolution Δz , with $\delta = \Delta z$ and varying window sizes ω for the mean estimation.

For resolutions significantly smaller than the outer scale (roughly one order of magnitude), the estimated C_n^2 closely aligns with the simulation value as the factor is slightly lower than one. This indicates a small underestimation of C_n^2 , primarily arising from an underestimation of the fluctuations n' when extracting the mean. Indeed, the correlation among neighbouring fluctuations implies that the extracted mean not only comprises $\bar{n}$ but also a non-zero mean of adjacent fluctuations n' . As a result, fluctuations tend to be underestimated, leading to a lower estimation

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of C_n^2 . Increasing the window size ω enables to consider more points that are also farther apart, with less correlated fluctuations. This improves the extracted mean $\bar{n}$ and consequently reduces the underestimation of fluctuations.

For spacing Δz close to or larger than L_0 , the factor begins to decrease, following a minus two-third power trend. This decrease is attributed to the diminishing validity of (4.5), as observed in Figure 2.12a, where the structure function tends to saturate. Since this structure function is the numerator in (4.5), only the influence of its denominator remains, corresponding precisely to a minus two-third power trend. In the context of real-world measurements, a decline in the scale factor c would suggest reaching the outer scale L_0 . Section 4.3.1 outlines a potential approach to determine the factor c for C_n^2 profiles based on radiosonde measurements, in practical scenarios.

4.3 Application to radiosonde measurements and calibration

In this section, the C_n^2 statistical definition model is applied to various locations in the world where high-density radiosondes are available. These radiosondes offer a vertical resolution of approximately 10 meters along the vertical profile, enabling the application of the C_n^2 statistical definition model.

Since 2018, a collection of high-density radiosonde measurements is publicly accessible from the University of Wyoming Atmospheric Science Radiosonde Archive.² The main measured meteorological quantities include pressure [hPa], geopotential height [m], temperature [°C], relative humidity [%], wind direction [°], and wind speed [m/s]. Dewpoint temperature [°C], mixing ratio [g/kg], potential temperature [K], equivalent potential temperature [K], and virtual potential temperature [K] are also recorded. Depending on the location, soundings take place once or twice a day, typically at 00h00 and/or 12h00 Coordinated Universal Time (UTC).

Four different locations with radiosonde measurements for the year 2020 are considered in this section. They are summarized in Tables 4.1 and 4.2, based on their station ID, the identification (ID) number corresponding to the five-digit World Meteorological Organization (WMO) number. Latitude (lat.), longitude (long.) and altitude above the mean sea level (alt.) are also provided in Table 4.1, as well as the launch times of the daily soundings³.

Table 4.2 gives further details, presenting the average (av.) vertical resolution (i.e., sampling distance) for all soundings made during the year 2020 at the location of interest. This sampling distance is relatively constant with the altitude, as shown by the associated standard deviations (std) given in parentheses. Similarly, the average

² Accessible at the following link: <http://weather.uwyo.edu/upperair/bufrfraob.shtml>. Last accessed on 30 August 2024.

³ Radiosonde measurements at the launch times specified in Table 4.1 are used for the analysis conducted in Sections 4.3.2 to 4.3.4. However, for some seasons or particular days, soundings may be available for other times (e.g., 00h, 06h, 12h, and 18h UTC), or not available at all (e.g., if the weather conditions made the radiosonde launch impossible).

Table 4.1 – Locations of radiosonde stations used for the application of the C_n^2 statistical definition model (part I)

Station ID	Location (country)	Lat., long., alt.	Launch times [UTC]
07145	Trappes (France)	48.773°N, 2.012°E, 168 m	00h and 12h
07510	Bordeaux (France)	44.831°N, 0.691°W, 45 m	00h and 12h
91285	Hilo, HI (USA)	19.717°N, 155.049°W, 10 m	00h and 12h
01004	Svalbard (Norway)	78.923°N, 11.922°E, 8 m	12h

Table 4.2 – Locations of radiosonde stations used for the application of the C_n^2 statistical definition model (part II)

Station ID	Av. vertical resolution ($\pm$ std) [m]	Av. maximal altitude ($\pm$ std) [km]
07145	10.74 ($\pm$ 3.52)	33.34 ($\pm$ 4.38)
07510	10.11 ($\pm$ 2.14)	27.61 ($\pm$ 3.37)
91285	5.32 ($\pm$ 0.93)	31.88 ($\pm$ 4.36)
01004	4.72 ($\pm$ 1.17)	15.62 ($\pm$ 1.20)

maximal altitudes reached by the radiosondes over the year 2020 are computed, along with their standard deviation.

In the following, Section 4.3.1 presents the approach used to calibrate the C_n^2 model for practical scenarios, i.e., to determine the value of the scale factor c . Then, Sections 4.3.2 to 4.3.4 detail the results obtained when applying the model to the four locations listed in Table 4.1.

For comparison, a Tatarskii-based model (Section 3.2.2) is also applied in Sections 4.3.2 to 4.3.4. It is based on (3.13), using $a^2 = 2.8$ and (3.14). The outer scale L_0 model is the HMNSP99 presented in Section 3.2.2.1, relying on (3.17) and (3.19), leading to

$$C_n^2 = 2.8 \left(0.1^{4/3} 10^Y \right) \left(\frac{80 \times 10^{-6} p}{T\theta} \frac{d\theta}{dz} \right)^2,$$

with

$$Y = \begin{cases} 0.362 + 16.728 S - 192.347 \frac{dT}{dz} & (\text{troposphere}) \\ 0.757 + 13.819 S - 57.784 \frac{dT}{dz} & (\text{stratosphere}) \end{cases}. \quad (3.19)$$

Finally, for all scenarios considered in this section, radiosonde drifting due to wind has been neglected. As a result, all C_n^2 profiles obtained are considered at zenith, while in practice they are obtained along the trajectories of the radiosondes. The spatial extent of this drifting is rather limited: for instance, in 2020 at Trappes, an average drift of 50 km (std of 28 km) has been computed at the altitude of 20 km, following [LS13] with a mean ascent rate of 6 m/s. Moreover, it is not expected to influence the vertical resolution given that the radiosonde records the altitude above ground directly rather than the distance covered by the balloon. Additional

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comments about the drifting are provided in Section 4.5.3.

4.3.1 Model calibration

The calibration of the statistical definition model aims to determine the value of the scale factor c in practical scenarios. A simple approach is presented, that relies on the HV model presented in Section 3.2.1.1 in (3.5):

$$C_n^2(z) = A e^{-z/100} + 2.7 \times 10^{-16} e^{-z/1500} + 0.00594 \left(\frac{\tilde{w}}{27} \right)^2 \left(\frac{z}{10^5} \right)^{10} e^{-z/1000}, \quad (3.5)$$

with A the ground turbulence strength [$\text{m}^{-2/3}$] and $\tilde{w}$ the root-mean-square high altitude wind speed [m/s]. The altitude z is referenced above the sea level.

As seen in Figure 3.2, there is a range of altitude in the HV model that is independent of the parameters A and $\tilde{w}$. It corresponds to the range of altitudes between 1 and 4 km, its origin being the second term in (3.5) that has a coefficient set to a fixed value (2.7×10^{-16}). This coefficient corresponds to the constant slope of the profile of the logarithm of C_n^2 . For low values of $\tilde{w}$, this slope persists as a trend at higher altitudes, extending up to 6 km in the HV-5/7 profile for example.

This interesting feature is likely associated with the relatively constant temperature and pressure mean lapse rates in the troposphere. When applying the statistical definition C_n^2 model on radiosonde data, a similar slope (on a semilogarithmic representation) has been observed in the same region (1 to 4 km of altitude). Consequently, this characteristic can be exploited to calibrate the obtained C_n^2 profiles, ensuring a good alignment between the modelled profiles and the HV-5/7 profile within this specific altitude range. This is further detailed and illustrated in Section 4.3.2 for radiosonde measurements in Trappes.

The existence of a calibration coefficient c is not particularly surprising, as it is commonly employed in existing literature models. For instance, revisions of the HAP model in (3.9) involve the coefficient $\tilde{M}$ as a multiplicative factor in front of the FA part of the profile. Similarly, the WSPT model includes a proportionality constant to ensure agreement with the measurements (Section 3.2.2.5).

4.3.2 Application to Trappes and Bordeaux (France)

This section presents the application of the C_n^2 statistical definition model at Trappes and Bordeaux, using radiosonde measurements collected for the year 2020. Details about each location are gathered in Tables 4.1 and 4.2. They show that radiosondes are launched twice a day at both locations, and have a similar vertical resolution of approximately 10 meters. The average maximum altitude reached by Trappes' radiosondes is higher than in Bordeaux (33.3 km compared to 27.6 km).

Trappes Following the approach detailed in Section 4.1, individual C_n^2 profiles have been computed at Trappes. Only radiosonde observations where the balloon reached an altitude of 30 km have been exploited, leading to a total of 620 distinct C_n^2 profiles

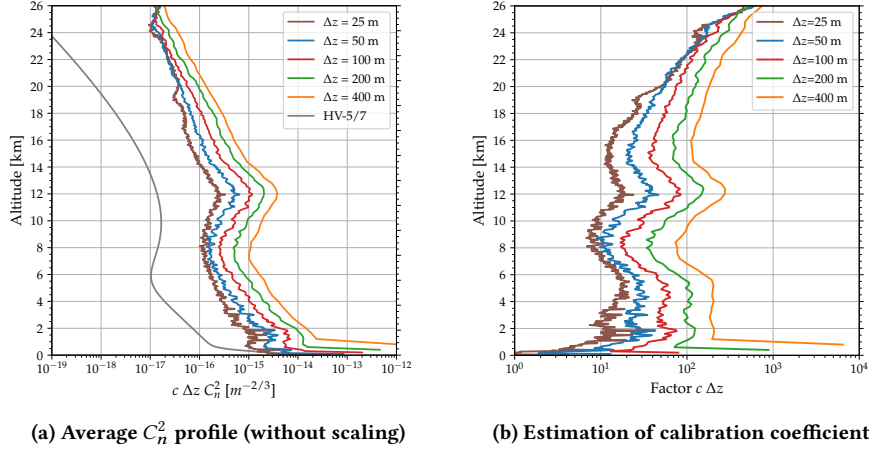


Figure 4.4 – Average C_n^2 profile at Trappes prior to calibration (left) and estimated calibration coefficient (right).

for the entire year of 2020. These profiles have been obtained for different vertical spacings Δz , ranging from 25 m to 400 m. Their average for the year 2020 is depicted in Figure 4.4a, using $\omega = 2$ and a δ equals to the Δz -spacing. The approximate refractive index model from (4.2) has been used in the C_n^2 statistical definition model. In the figure, the altitude corresponds to the altitude above ground, and the HV-5/7 profile is also depicted for comparison.

The average profiles presented in Figure 4.4a are given prior to calibration and scaling, corresponding to $c \Delta z C_n^2$ in (4.6). It clearly shows that the magnitude of $c \Delta z C_n^2$ depends on the resolution Δz , as larger Δz -spacings lead to larger C_n^2 values. Moreover, all profiles seem to experience the same slope (on the semilogarithmic representation) as the HV-5/7 profile. This is further illustrated in Figure 4.4b that presents the ratios between modelled C_n^2 and C_n^2 from the HV-5/7 model, computed at each altitude. For a given resolution, these ratios are indeed constant in the 2 to 6 km range of altitude, motivating the calibration procedure introduced in Section 4.3.1 and the estimation of the scale factor c . Table 4.3 gathers the different values suggested for the scale factor c , obtained as the average (over the altitudes) of the ratios in Figure 4.4b between 2 and 6 km. The standard deviation is also given, as well as the values for other parameters ω and δ .

The scale factors in Table 4.3 are independent of the altitude and the initial radiosonde vertical resolution. They experience small variations with the window size parameter ω and the vertical spacing δ . Surprisingly, the obtained values deviates from the expected $\Delta z^{-2/3}$ trend observed in the simulation values depicted in Figure 4.3 for Δz exceeding the outer scale L_0 . This observation suggests that the outer scale might not have been reached yet, such that constraints on ρ in (4.4) could potentially be relaxed, or, at the very least, are partially obscured by the calibration

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Table 4.3 – Calibration coefficients (scale factors) at Trappes: mean value and standard deviation.

Δz [m]	Scale factors c		
	$\omega = 1, \delta = \Delta z$	$\omega = 2, \delta = \Delta z$	$\omega = 2, \delta = 2\Delta z$
25	0.47 ± 0.13	0.60 ± 0.13	0.58 ± 0.11
50	0.37 ± 0.07	0.53 ± 0.08	0.55 ± 0.08
100	0.36 ± 0.05	0.53 ± 0.06	0.56 ± 0.05
200	0.35 ± 0.03	0.51 ± 0.04	0.54 ± 0.04
400	0.35 ± 0.02	0.49 ± 0.01	0.51 ± 0.02

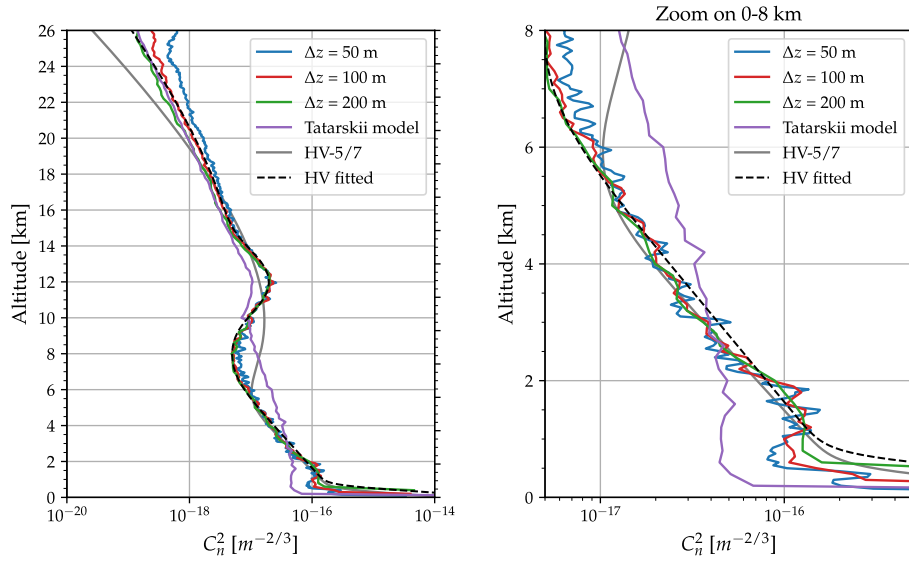


Figure 4.5 – Average C_n^2 profile at Trappes after calibration.

with the scale factor c .

The default parameters used for the subsequent analysis in this section are $\omega = 2$ and $\delta = \Delta z$, for all choice of Δz . In this case, the scale factors c seems to be independent of Δz and is set to 0.5. The application of this factor to the profiles in Figure 4.4a leads to a superimposition of the C_n^2 profiles from the statistical definition C_n^2 model. This is illustrated in Figure 4.5 for Δz equals to 50, 100 and 200 m, with the right part of the figure zooming in on the altitudes near the ground. The C_n^2 profiles from the Tatarskii-based model, using a tropopause altitude of 10 km in (3.19), and the HV-5/7 model are also presented for reference.

Statistical definition C_n^2 profiles in Figure 4.5 exhibit a large ground C_n^2 value and a consistent slope from 1 to 6 km of altitude. This slope precisely corresponds to the same slope as observed in the HV-5/7 model, while the slope in this region for the Tatarskii model differs slightly. Subsequently, there is a noticeable bump in the C_n^2 profiles around 12 km altitude, related to the tropopause's altitude and the associated constant temperature and large wind shears. This bump is also evident in the

Tatarskii model. Beyond the tropopause, C_n^2 gradually decreases, introducing some fluctuations attributed to the reinterpolation process. These fluctuations primarily arise from small variations in atmospheric quantities at these altitudes, which may not be precisely captured by radiosondes, along with the uneven and larger vertical sampling distance of radiosondes at high altitudes.

Above 6 km, the shapes of the statistical definition C_n^2 profiles and the HV-5/7 model start to diverge. This discrepancy arises from the FA component of the HV model that is sensitive to the tropopause altitude, which, in turn, depends on the geographical location and the season. As an alternative, a generic (generalized) HV model can be computed, where the C_n^2 magnitudes and turbulent layer altitudes can be fitted from the statistical definition models. The generalized HV model equation is given by (3.7),

$$C_n^2(z) = A e^{-\frac{z}{H_A}} + B e^{-\frac{z}{H_B}} + C z^{10} e^{-\frac{z}{H_C}} + D e^{-\frac{(z-H_D)^2}{2d^2}}, \quad (3.7)$$

where A , B , C , and D have units of C_n^2 [$\text{m}^{-2/3}$], and H_A , H_B , H_C , and H_D that are expressed in [m]. The first term in (3.7) is associated with the surface turbulence, the second term depicts the troposphere turbulence whereas the third term defines the turbulence at the tropopause. The fourth term represents isolated layers of turbulence with a Gaussian shape of thickness d , and is repeated in cases of several turbulent layers.

Fitting of the generalized HV model in (3.7) leads to the black dashed line in Figure 4.5, named HV fitted. The fitting process involves minimizing the least square error with the statistical definition C_n^2 model for $\Delta z = 200$ m. The fitted coefficients are given in Table 4.4, in the column associated to Trappes. The parameters highlighted in blue were determined prior to the least square fitting.

As depicted in Figure 4.5, the effective parameterization of the generalized HV profile allows for the analytical modelling of the mean C_n^2 profile at a specific location, such as Trappes in this instance. This results in simple expressions for average C_n^2 profiles that can be generated for various locations and compared, which may be of interest to conduct site selection for example. Table 4.4 also gathers the fitted HV coefficients for other locations, based on the results presented below.

Bordeaux Similarly to the analysis conducted for Trappes, radiosonde measurements in 2020 at Bordeaux have been exploited. As the average maximal altitude reached by radiosondes at Bordeaux is lower than in Trappes, the altitude limit below which measured profiles are neglected is set to 25 km. In total, 650 individual profiles have been computed, and their average is depicted in Figure 4.6 based on the default configuration of the statistical definition model, i.e., $\omega = 2$, $\delta = \Delta z$, and $c = 0.5$ for the plots. The precise scale factors computed from the average ratios between 2 and 6 km of altitude are given in Table 4.5.

Obtained C_n^2 profiles at Bordeaux are very similar to the ones of Trappes, as indicated by visual inspection of Figures 4.5 and 4.6, comparison of the scale factors given

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Table 4.4 – Parameters for generalized HV model in (3.7).

Parameters	Trappes	Bordeaux	Hilo	Svalbard
$A [m^{-2/3}]$	1.32×10^{-13}	1.11×10^{-13}	4.66×10^{-14}	9.70×10^{-14}
$B [m^{-2/3}]$	2.7×10^{-16}	2.7×10^{-16}	2.7×10^{-16}	2.7×10^{-16}
$C [m^{-2/3}]$	2.07×10^{-4}	2.93×10^{-5}	3.96×10^{-5}	1.08×10^{-3}
$D [m^{-2/3}]$	1.37×10^{-17}	1.91×10^{-17}	4.67×10^{-18}	1.51×10^{-17}
$H_A [m]$	100	100	100	100
$H_B [m]$	1645	1750	2006	1612
$H_C [m]$	1200	1365	1340	1019
$H_D [m]$	12000	12000	17000	10500
$d [m]$	1200	1200	1700	1000
$E [m^{-2/3}]$	/	/	1.59×10^{-16}	4.18×10^{-17}
$H_E [m]$	/	/	2200	1500
$e [m]$	/	/	300	300

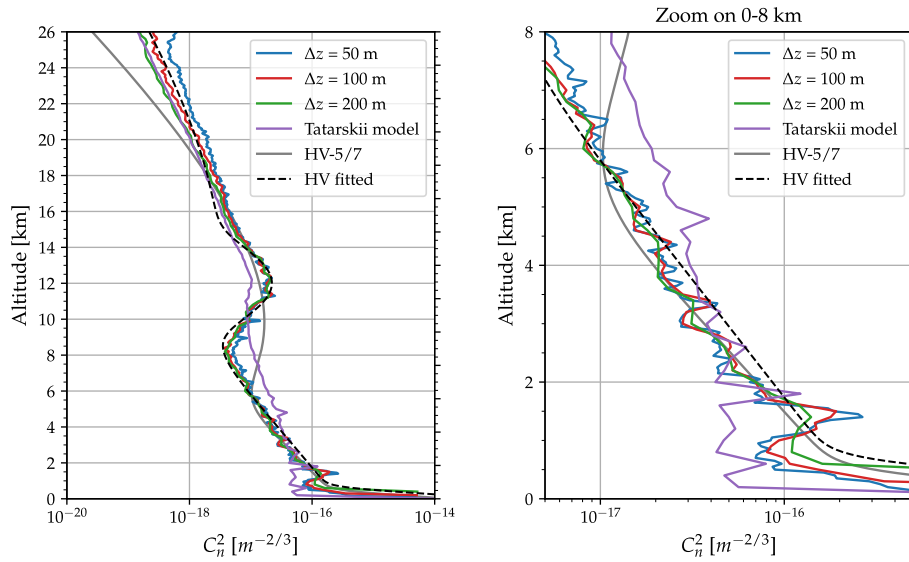


Figure 4.6 – Average C_n^2 profile at Bordeaux after calibration.

Table 4.5 – Calibration coefficients (scale factors) at Bordeaux: mean value and standard deviation.

Δz [m]	Scale factors c		
	$\omega = 1, \delta = \Delta z$	$\omega = 2, \delta = \Delta z$	$\omega = 2, \delta = 2\Delta z$
25	0.49 $\pm$ 0.15	0.63 $\pm$ 0.18	0.61 $\pm$ 0.15
50	0.38 $\pm$ 0.09	0.55 $\pm$ 0.11	0.58 $\pm$ 0.10
100	0.37 $\pm$ 0.07	0.55 $\pm$ 0.09	0.58 $\pm$ 0.08
200	0.37 $\pm$ 0.05	0.53 $\pm$ 0.07	0.56 $\pm$ 0.07
400	0.35 $\pm$ 0.05	0.50 $\pm$ 0.06	0.52 $\pm$ 0.05

in Tables 4.3 and 4.5, and comparison of the fitted coefficients for the generalized HV model, presented in Table 4.4. This result is not unexpected since Trappes and Bordeaux are both located in France, less than 500 km apart. Nevertheless, there appears to be an extra turbulent layer close to the ground in Bordeaux, around 1.5 km of altitude, that is more important than in Trappes (see Figure 4.6). The origin of this layer for coastal areas is further discussed in the next section, presenting the application of the C_n^2 statistical definition model to Hawaii.

4.3.3 Application to Hilo, Hawaii (USA)

The application of the C_n^2 statistical definition to Hilo, Hawaii (USA), is presented in this section. The selection of Hilo is motivated by its difference in location compared to Trappes and Bordeaux, and by the accessibility of high-density profiles on the University of Wyoming website. Details about the radiosounding location are given in Tables 4.1 and 4.2. A total of 656 radiosonde profiles collected in 2020 were used in the analysis, excluding those with a maximum altitude below 30 km.

Figure 4.7 presents the mean C_n^2 profiles computed with the default configuration as for Trappes and Bordeaux ($\omega = 2$, $\delta = \Delta z$, and $c = 0.5$ for the plots). Consistent agreement is observed for the profiles at different Δz after the application of the Trappes (and Bordeaux) scale factors, suggesting the location independence of these factors. Indeed, aside from an isolated turbulent layer at 2 km altitude (whose origin is discussed below), the slope of the statistical definition C_n^2 profiles matches that of the Hufnagel-Valley profile ($B = 2.7 \times 10^{-16}$), and quantitative agreement with HV-5/7 is nearly achieved with the utilization of Trappes scale factors. Additionally, in Figure 4.7, the generalized HV model is fitted, introducing an extra term to model an isolated turbulent layer. Detailed information on all fitted parameters can be found in Table 4.4.

Compared to the average C_n^2 profiles at Trappes and Bordeaux shown in Figures 4.5 and 4.6, two important differences are noticeable:

- The tropopause and its associated turbulent layer are situated at a higher altitude, approximately 17 km, as indicated by the parameter H_D in Table 4.4. The deviation from the classical HV-5/7 model is considerable, highlighting the model's dependence on geographical factors.

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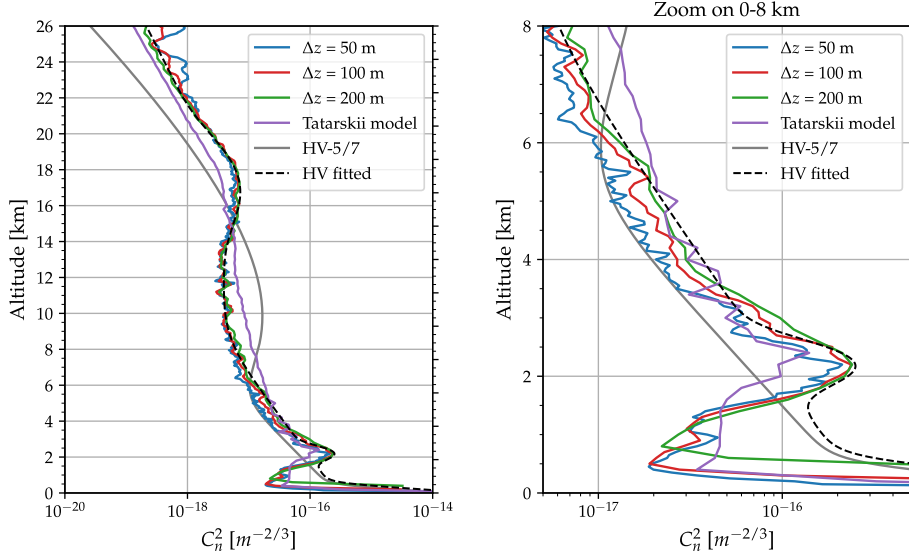


Figure 4.7 – Average C_n^2 profile at Hilo, Hawaii, after calibration.

- In addition to the tropopause Gaussian turbulent layer (parameters D , H_D , and d), there exists a second isolated layer near the ground (at 2 km altitude) characterized by the parameters E (magnitude), H_E (altitude of the middle point), and e (thickness). While the generalized HV model can effectively capture this turbulent layer, it is unable to represent the reduced turbulence strength beneath the layer.

This second layer is also identified in the Tatarskii model, which has been applied with a troposphere's limit set at 16 km in (3.19). Its origin is justified in Figure 4.8, where C_n^2 profiles computed from the statistical definition model with $\Delta z = 200$ m are presented for two different days in August.

In the left graph, a turbulent layer is evident at an altitude of 2 km, with C_n^2 reaching nearly $10^{-15} \text{ m}^{-2/3}$. This layer correlates with a tropospheric temperature inversion occurring at this altitude. Indeed, the temperature, represented in red, exhibits an initial increase followed by a decrease. Sharp fluctuations in temperature can significantly impact the calculation of C_n^2 using its statistical definition, as temperature plays a key role in determining the refractive index. On the contrary, in the right graph of Figure 4.8, no temperature inversion is observed, and consequently, no turbulent layer is identified. Previous studies have also established connections between regions of high C_n^2 and temperature inversions [Cou+86; Buf73] or large gradients of (potential) temperature [CVF95]. Given the prevalence of temperature inversions near the ground in Hilo, it is understandable why this turbulent layer remains important in the mean profile.

The turbulent layer at 2 km raises challenges for the estimation of the scale factors. Using a similar approach as for Trappes and Bordeaux, i.e., the computation of

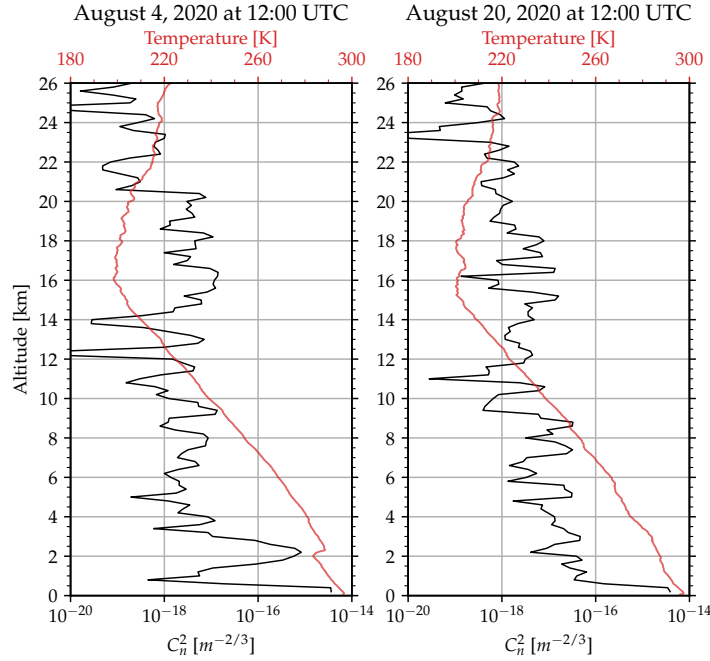


Figure 4.8 – Two examples of day C_n^2 profiles at Hilo, Hawaii, using the C_n^2 statistical definition model with $\Delta z = 200$ m.

the scale factors as the averages of the ratios between the modelled C_n^2 and HV-5/7 C_n^2 for the altitudes ranging from 2 to 6 km, gives the different values presented in Table 4.6. They differ from the values in Tables 4.3 and 4.5, and experiences large variations as seen from their standard deviations. This is a direct consequence of the turbulent layer that arises exactly in the region where the slope matching with the HV-5/7 model is conducted. Instead, a visual estimation of the scale factors is preferred, and shows that similar scale factors as for Trappes and Bordeaux apply quite well to Hilo (e.g., $c = 0.5$ in Figure 4.7).

Table 4.6 – Calibration coefficients (scale factors) at Hilo, Hawaii: mean value and standard deviation.

Δz [m]	Scale factors c		
	$\omega = 1, \delta = \Delta z$	$\omega = 2, \delta = \Delta z$	$\omega = 2, \delta = 2\Delta z$
25	0.27 $\pm$ 0.05	0.47 $\pm$ 0.11	0.57 $\pm$ 0.19
50	0.39 $\pm$ 0.14	0.70 $\pm$ 0.28	0.84 $\pm$ 0.34
100	0.57 $\pm$ 0.22	0.93 $\pm$ 0.37	1.06 $\pm$ 0.41
200	0.67 $\pm$ 0.23	1.07 $\pm$ 0.37	1.18 $\pm$ 0.38
400	0.77 $\pm$ 0.24	1.10 $\pm$ 0.30	1.16 $\pm$ 0.23

4 | Computation of C_n^2 based on its statistical definition

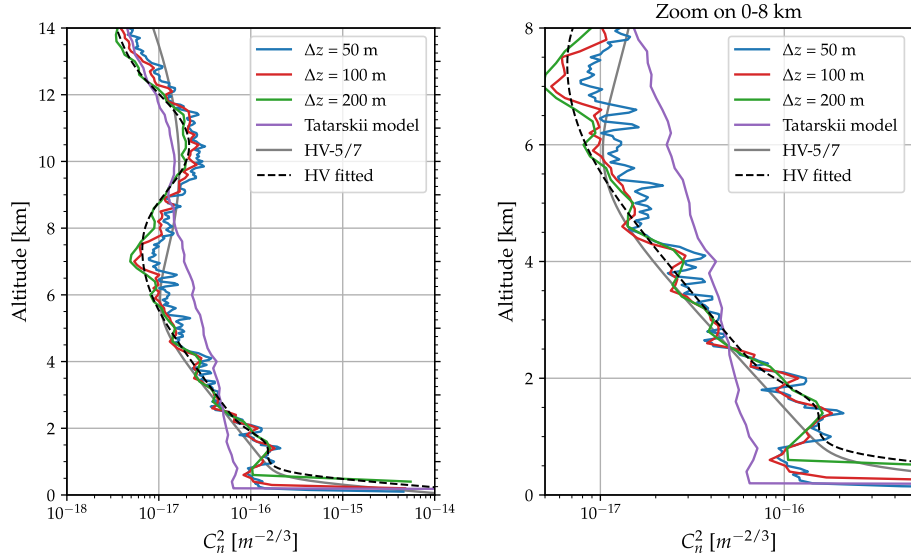


Figure 4.9 – Average C_n^2 profile at Svalbard after calibration.

Table 4.7 – Calibration coefficients (scale factors) at Svalbard: mean value and standard deviation.

Δz [m]	Scale factors c		
	$\omega = 1, \delta = \Delta z$	$\omega = 2, \delta = \Delta z$	$\omega = 2, \delta = 2\Delta z$
25	0.87 ± 0.19	1.00 ± 0.19	0.86 ± 0.17
50	0.50 ± 0.10	0.67 ± 0.12	0.65 ± 0.10
100	0.40 ± 0.06	0.57 ± 0.08	0.59 ± 0.08
200	0.39 ± 0.06	0.57 ± 0.08	0.60 ± 0.07
400	0.39 ± 0.03	0.54 ± 0.04	0.57 ± 0.07

4.3.4 Application to Svalbard (Norway)

As a last example, the C_n^2 statistical definition model has been applied to radiosonde measurements collected in 2020 in Svalbard, Norway (see Tables 4.1 and 4.2). Observations are made once a day, and using only measured profiles with a maximal altitude higher than 15 km, 282 C_n^2 profiles were computed. Their average is depicted in Figure 4.9, for the default configuration ($\omega = 2$, $\delta = \Delta z$, and $c = 0.5$ for the plots).

The coefficients for the HV fitted model are given in Table 4.4. The tropopause altitude is slightly lower than at Trappes and Bordeaux. Moreover, as for Hilo, there is a strong turbulent layer arising from temperature inversions below 2 km altitude. This layer is not in the altitude range used for estimating the scale factors, such that the values obtained for Svalbard have smaller standard deviations than at Hilo, and are comparable to the values at Trappes and Bordeaux. They are given in Table 4.7. This tends to show that the scale factors are indeed independent of the location where the C_n^2 statistical definition is applied.

4.4 Validation with T-REX thermosonde measurements

Validation of the C_n^2 statistical definition model is conducted using thermosonde measurements from the Terrain-induced Rotor Experiment (T-REX) campaign of the AFRL [Jum06]. It consisted in the launches of several radiosondes equipped with thermosondes in Sierra Nevada Mountains (Three Rivers, California, USA), from 20 March 2006 to 6 April 2006. Details about the locations and schedules of the flights are presented in Appendix A. Data is accessible online from the Earth Observing Laboratory data archive [Jum07].

The combination of radiosondes and thermosondes enables to measure macroscale meteorological parameters (e.g., pressure, temperature, etc.) as well as the C_n^2 profile. Indeed, thermosondes compute the temperature structure function $D_T(\rho)$ based on its definition in (2.45) and using measurements from two temperature sensors spaced by a known distance ρ ($\rho = 1$ m for the T-REX thermosondes). A temporal average is substituted to the ensemble average in the definition of $D_T(\rho)$, and C_T^2 is extracted from

$$D_T(\rho) = \langle (T(\mathbf{r} + \boldsymbol{\rho}) - T(\mathbf{r}))^2 \rangle = C_T^2 \rho^{2/3} \quad \text{for } l'_0 \ll \rho \ll L_0. \quad (2.45)$$

Then, C_n^2 is usually obtained from C_T^2 with (2.70), in the case of optical C_n^2 :

$$C_n^2 = \left(\frac{80 \times 10^{-6} p}{T^2} \right)^2 C_T^2, \quad (2.70)$$

using the measured pressure p [hPa] and temperature T [K] by the radiosondes.

In total, radiosonde and thermosonde measurements from 15 different flights have been exploited. One flight (T-REX039) was excluded due to the presence of intense turbulence, reaching up to two orders of magnitude larger than observed in other flights. For each flight, the C_n^2 statistical definition model and the Tatarskii-based model presented in Section 4.3 have been applied to the radiosonde measurements, and compared with the thermosonde profiles. The statistical definition model was parameterized with its default configuration ($\omega = 2$, $\delta = \Delta z$, and $c = 0.5$), hence using the scaling factors identified previously, with a vertical resolution $\Delta z = 200$ m. The thermosonde measurements were grouped into bins and averaged over the same distance Δz . Regarding the Tatarskii model, the troposphere's limit was set to 10 km.

C_n^2 profiles obtained for all flights are presented in Figures A.1 and A.2 in Appendix A. The average profiles can be seen on the left part of Figure 4.10, where solid lines are modelled C_n^2 based on radiosonde measurements while the dashed line corresponds to thermosonde measurements.

In the central and right panels of this figure, two metrics were introduced to assess the performance of both C_n^2 models: the absolute bias and the RMSE. These metrics are computed on $\log_{10}(C_n^2)$, and their definitions are given in Section 3.5 of

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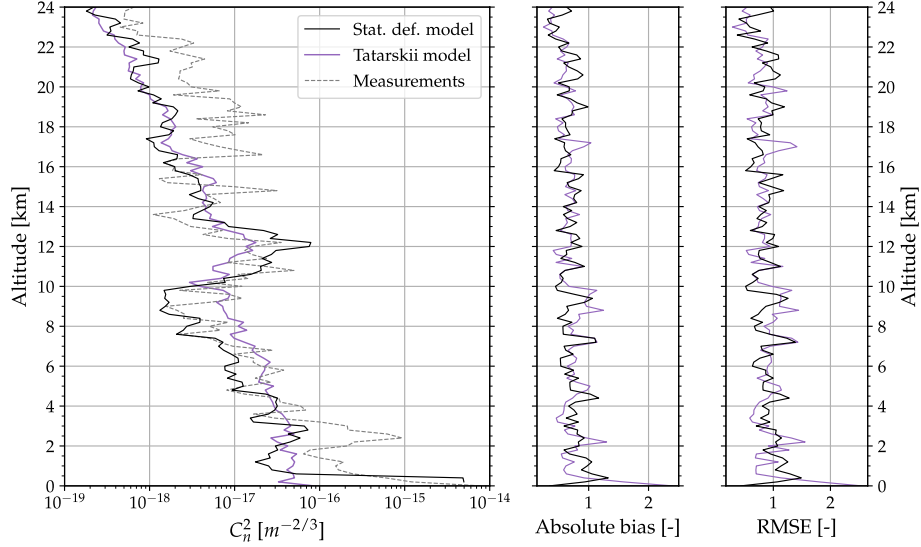


Figure 4.10 – Average (March 20 - April 26, 2006) profiles for T-REX campaign.

Table 4.8 – Altitude-averaged absolute bias and RMSE for T-REX campaign.

Parameters	C_n^2 stat. def. model	Tatarskii-based model
Absolute bias	0.690	0.682
RMSE	0.861	0.851

Chapter 3:

$$\text{ABSOLUTE BIAS}(z) = \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} |Y_i(z) - X_i(z)| \quad (3.58)$$

$$\text{RMSE}(z) = \sqrt{\frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} (Y_i(z) - X_i(z))^2}, \quad (3.59)$$

where N_{obs} is the number of measurements, X_i represents the i -th measurement of $\log_{10}(C_n^2)$ and Y_i is the logarithm in base 10 of the i -th modelled C_n^2 profile (either using the Tatarskii model or the statistical definition model).

The vertical profiles of absolute bias and RMSE obtained for both models are remarkably similar, indicating that the statistical definition C_n^2 model provides C_n^2 estimates comparable to those of the Tatarskii-based model. When averaged across all altitudes, the absolute bias and RMSE given in Table 4.8 demonstrate that the models exhibit comparable performance according to these metrics.

Moreover, in Figure 4.10, both modelled and measured C_n^2 profiles illustrate the impact of the tropopause layer (near 12 km altitude), where they exhibit good agreement. The decrease of C_n^2 in the troposphere is also evident. Measurements reveal the

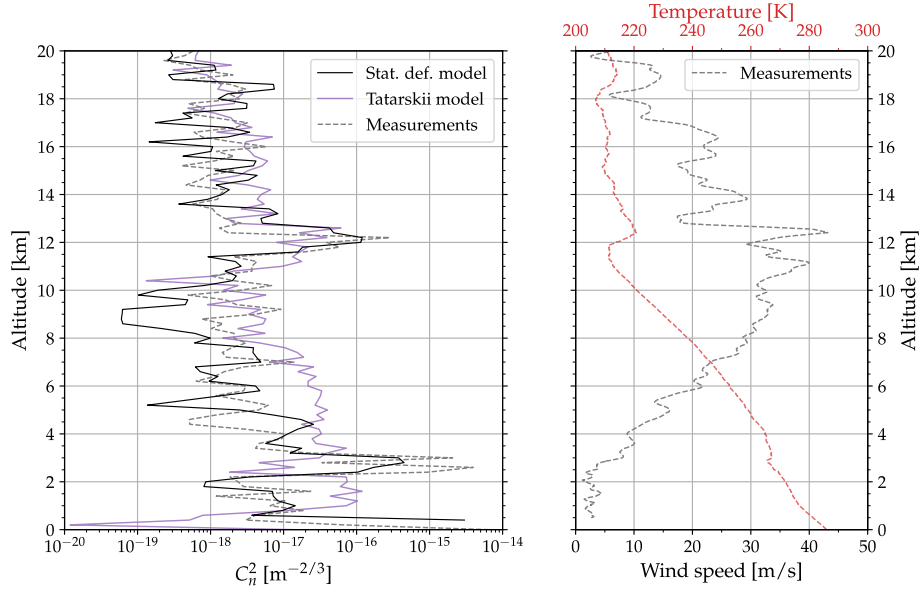


Figure 4.11 – C_n^2 profiles on March 31, 2006 at 00h48 UTC (T-REX019).

presence of a turbulent layer near the ground (2-3 km altitude) that is not accurately represented in the C_n^2 models. This discrepancy arises on days when the magnitude of the turbulent layer is underestimated. For instance, the C_n^2 profile recorded on March 31, 2006 is presented in Figure 4.11, where the C_n^2 statistical definition model uses $\Delta z = 200$ m. Isolated optical turbulent layers corresponding to temperature inversions are clearly visible, both in thermosonde measurements and for the radiosonde models. This is similar to what has been observed in Figure 4.8 for Hilo, HI, and in Svalbard's average profile (Figure 4.9).

4.5 Adaptation of the model for scintillation at radiofrequencies

The C_n^2 statistical definition model presented in Section 4.1 can be adapted to obtain C_n^2 profiles for radiofrequencies. This is presented in Section 4.5.1, highlighting the influence of humidity in C_n^2 . Its application to the radiosonde measurements presented in Section 4.3 is conducted in Section 4.5.2, while its validation with scintillation measurements in Norway at 19.68 GHz is detailed in Section 4.5.3.

4.5.1 Direct and indirect models

The adaptation of the optical C_n^2 statistical definition model to radiofrequencies is rather straightforward. It only requires modifying the refractive index definition used, substituting the optical refractive index expression by the one of the radiofre-

4 | Computation of C_n^2 based on its statistical definition

quency refractive index given in (2.5):

$$n = 1 + 10^{-6} \frac{77.6}{T} \left(p + \frac{4810 e}{T} \right). \quad (2.5)$$

This is referred to as the direct approach, detailed below. On the contrary, the indirect approach relies on C_θ^2 , C_q^2 , and $C_{\theta q}$, illustrating their respective contributions to obtain C_n^2 .

4.5.1.1 Direct model - computation of C_n^2

In Figure 4.2, the application of the refractive index expression to meteorological quantities (i.e., the third step in the figure) is modified, using (2.5) instead.

Hence, the additional knowledge of the humidity profile is now required, and can be obtained from several quantities: the specific humidity q [kg/kg], the partial pressure of water vapor e , or the relative humidity RH [%]. The following equations relate those quantities [FN08, p. 41], [Stu11, p. 90]:

$$q = 0.622 \frac{e}{p - 0.378e}, \quad (4.10)$$

$$\text{RH} = \frac{e}{e_s} \times 100\%, \quad (4.11)$$

$$e_s = 6.11 e^{\frac{17.2694(T-273.15)}{T-35.86}} \quad (\text{Tetens' formula}), \quad (4.12)$$

with p the atmospheric pressure [hPa], T the temperature [K], and e_s the saturation water vapor pressure [hPa]. Alternative expressions for e_s can be found in [FN08, p. 41] and [ITU19]. In practice, from the relative humidity profiles measured by radiosondes, the specific humidity q is estimated from (4.10) to (4.12) and the observed temperature and pressure profiles.

After the computation of the radiofrequency refractive index from the meteorological quantities, its fluctuations are extracted following Figure 4.2 and C_n^2 is estimated using (4.6). As a result, the C_n^2 profile is directly obtained from the refractive index, hence the name of *direct model*.

4.5.1.2 Indirect model - computations of C_θ^2 , C_q^2 , and $C_{\theta q}$

As an alternative to the direct model, C_n^2 can be estimated from (2.68) in Section 2.5.4, linking it to C_θ^2 , C_q^2 , and $C_{\theta q}$:

$$C_n^2 = A^2 C_\theta^2 + B^2 C_q^2 + 2AB C_{\theta q}. \quad (2.68)$$

For radiofrequency C_n^2 , the coefficients A and B are given by

$$A = -10^{-6} \frac{77.6}{T^2} \left(p + \frac{9620e}{T} \right) \left(\frac{p}{p_0} \right)^{R/c_p},$$

$$B = 1.62 p \frac{0.373}{T^2},$$

using (2.59), (2.60), (2.66) and (2.67).

Then, C_θ^2 , C_q^2 , and $C_{\theta q}$ can be estimated based on their statistical definitions, as done previously for the refractive index in (4.5). Assuming that the inertial range hypothesis is fulfilled, one has

$$\begin{aligned} C_\theta^2 &= \frac{\langle (\theta'(\mathbf{r} + \boldsymbol{\rho}) - \theta'(\mathbf{r}))^2 \rangle}{\rho^{2/3}}, \\ C_q^2 &= \frac{\langle (q'(\mathbf{r} + \boldsymbol{\rho}) - q'(\mathbf{r}))^2 \rangle}{\rho^{2/3}}, \\ C_{\theta q} &= \frac{\langle (\theta'(\mathbf{r} + \boldsymbol{\rho}) - \theta'(\mathbf{r})) (q'(\mathbf{r} + \boldsymbol{\rho}) - q'(\mathbf{r})) \rangle}{\rho^{2/3}}, \end{aligned}$$

where θ' and q' denote the fluctuations of potential temperature and specific humidity. These fluctuations are computed from the potential temperature and specific humidity profiles using a window average, as for the refractive index in Figure 4.2. Finally, C_θ^2 , C_q^2 , and $C_{\theta q}$ are obtained using equations similar to (4.6), with the same scaling factor c , and are introduced in (2.68).

Since C_n^2 comes from C_θ^2 , C_q^2 , and $C_{\theta q}$, and not directly from fluctuations of n , this model is named *indirect model*. It can also be applied to optical C_n^2 , using (2.71) and C_θ^2 (or (2.70) with C_T^2).

4.5.2 Application to radiosonde observations

The application of the radiofrequency C_n^2 statistical definition models, both direct and indirect, to the radiosonde measurements of Section 4.3 is presented in this section. The default configuration is used ($\omega = 2$, $\delta = \Delta z$) with $\Delta z = 200$ m and with the same scaling factor as identified for optical C_n^2 , i.e., $c = 0.5$.

The average C_n^2 profiles for the year 2020 at Trappes, Bordeaux, Hilo, and Svalbard, are given in Figures 4.12 to 4.15, in subfigures (a). Both optical (Opt.) and radiofrequency (RF) average C_n^2 profiles are depicted, coming from the direct and indirect approaches. As a reference, the HV-5/7 profile is also shown.

As a first observation from Figures 4.12 to 4.15, both direct and indirect models yield similar results, showing the consistency of mathematical developments in Section 2.5.4. Additionally, the influence of humidity fluctuations is clearly identified when comparing RF C_n^2 and optical C_n^2 . This observation aligns well with the average relative humidity profiles presented in subfigures (b) in all figures. Indeed, humidity is mostly located in lower atmospheric layers, impacting C_n^2 within those layers. Above the tropopause, RF C_n^2 and optical C_n^2 coincide. This is further illustrated in subfigures (c), showing the contribution of each term in (2.68). The dominance of $B^2 C_q^2$ in the troposphere and then of $A^2 C_\theta^2$ above the troposphere is clearly visible, and explain the equality between RF and optical C_n^2 at high altitudes. The term $ABC_{\theta q}$ is usually negligible compared to C_θ^2 and C_q^2 , and can take negative values. This is mostly the case at Trappes and Bordeaux, for altitudes higher than 10 km.

The difference between RF C_n^2 and optical C_n^2 profiles has already been presented

4 | Computation of C_n^2 based on its statistical definition

in the literature. In [Eat+88], measurements of RF C_n^2 profiles have been acquired with a VHF radar, while corrections for humidity effects leveraged optical C_n^2 profiles. In [Hil+88], ground measurements of optical and millimetre wave C_n^2 showed a difference of two orders of magnitude between optical and RF C_n^2 . This is consistent with the observations in Figures 4.12 to 4.15, for all four locations.

RF C_n^2 profiles acquired at different locations mostly differ in tropopause altitude and in the presence of isolated turbulent layers, as previously identified for optical C_n^2 profiles in Section 4.3. The isolated turbulent layer close to the ground in the average RF C_n^2 profiles is not only associated with temperature inversions as for optical C_n^2 , but also with humidity inversions (or gradients, as stated in [CVF95]). This is illustrated in Figure 4.16, showing the RF C_n^2 profile at Hilo for August 20, 2020, exactly as done previously in Figure 4.8 for optical C_n^2 . On the right part of the figure, the temperature and relative humidity profiles are depicted. At 2 km altitude, there is a really strong gradient in the relative humidity profile and a temperature inversion, that correlates very well with an important peak in C_q^2 and C_θ^2 , and hence in C_n^2 .

4.5.3 Validation with scintillation measurements in Norway

As introduced in Section 1.2.3.1, scintillation refers to amplitude fluctuations of the received EM waves due to atmospheric turbulence. It increases with the frequency and the propagation path through the atmosphere. Hence, communication links using radiofrequencies exceeding 10 GHz, as well as with low elevation angles, suffer from increased scintillation that necessitates mitigation. This motivates the collection of measurements at such frequencies, that have been used in this section.

For radiofrequencies, the modelling of short-term signal fades or enhancements caused by scintillation mostly relies on the long-term standard deviation (resp., variance) of the log-amplitude σ_χ (resp., σ_χ^2), also named scintillation standard deviation (resp., variance). Indeed, from σ_χ , probability distributions of amplitude scintillation can be obtained using the Karasawa [KYA88], van de Kamp [Kam+99], or the ITU-R [ITU17] models for example. These models assume that the distribution of log-amplitude scintillation χ in dB, with a PDF denoted as $p(\chi)$, can be obtained from [Kam+99]

$$p(\chi) = \int_0^\infty p(\chi|\sigma_\chi) p(\sigma_\chi) d\sigma_\chi,$$

where $p(\sigma_\chi)$ is the PDF of the long-term scintillation standard deviation, and $p(\chi|\sigma_\chi)$ is the PDF of χ knowing σ_χ , that is generally assumed to be Gaussian and for which expressions can be found in [KYA88; Kam+99].

Modelling of the scintillation standard deviation σ_χ is typically achieved using empirical models involving the surface wet refractivity N_{wet} [KYA88; Kam+99; ITU17; Otu96], or other surface meteorological quantities [Mar+99]. One of these models is presented in Section 4.5.3.2. Alternatively, models based on the vertical profile of the refractive index structure parameter C_n^2 have also been proposed in the literature [Vas99; Per+14].

This is the approach followed in this work to validate the RF C_n^2 statistical defini-

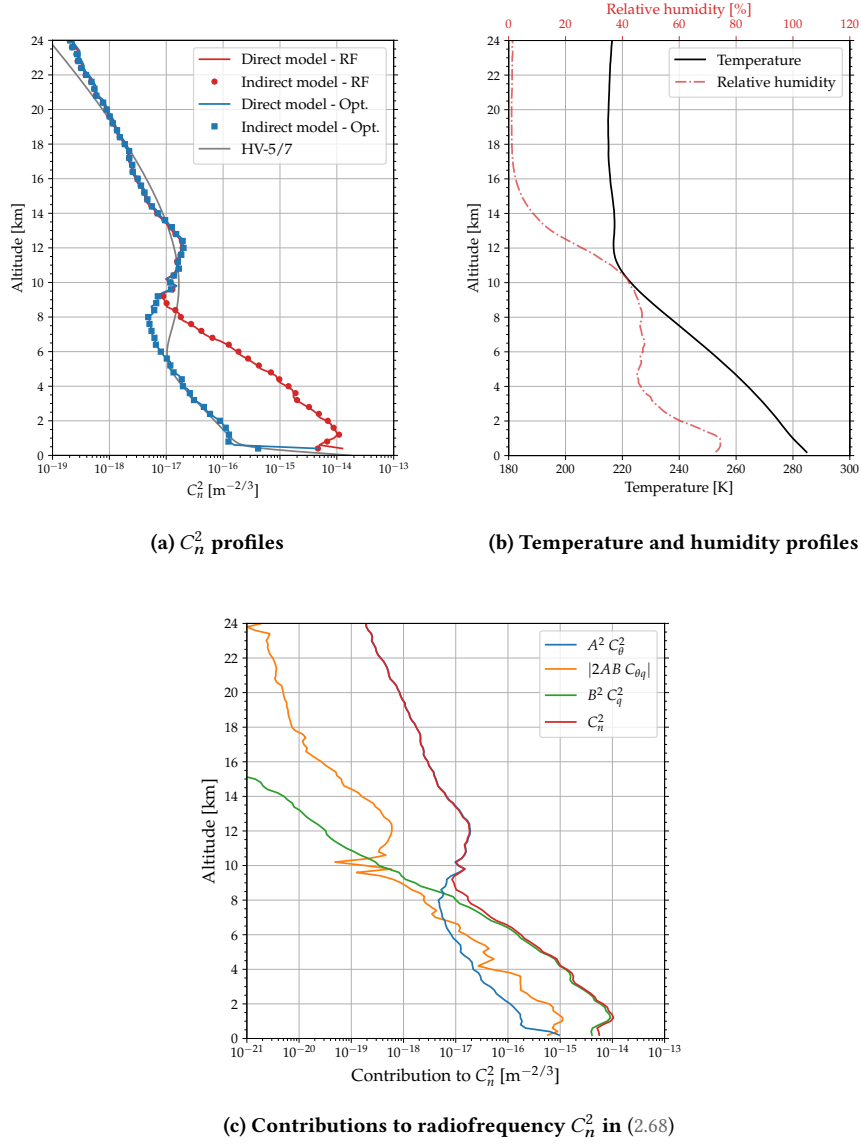


Figure 4.12 – Average profiles at Trappes for the year 2020.

4 | Computation of C_n^2 based on its statistical definition

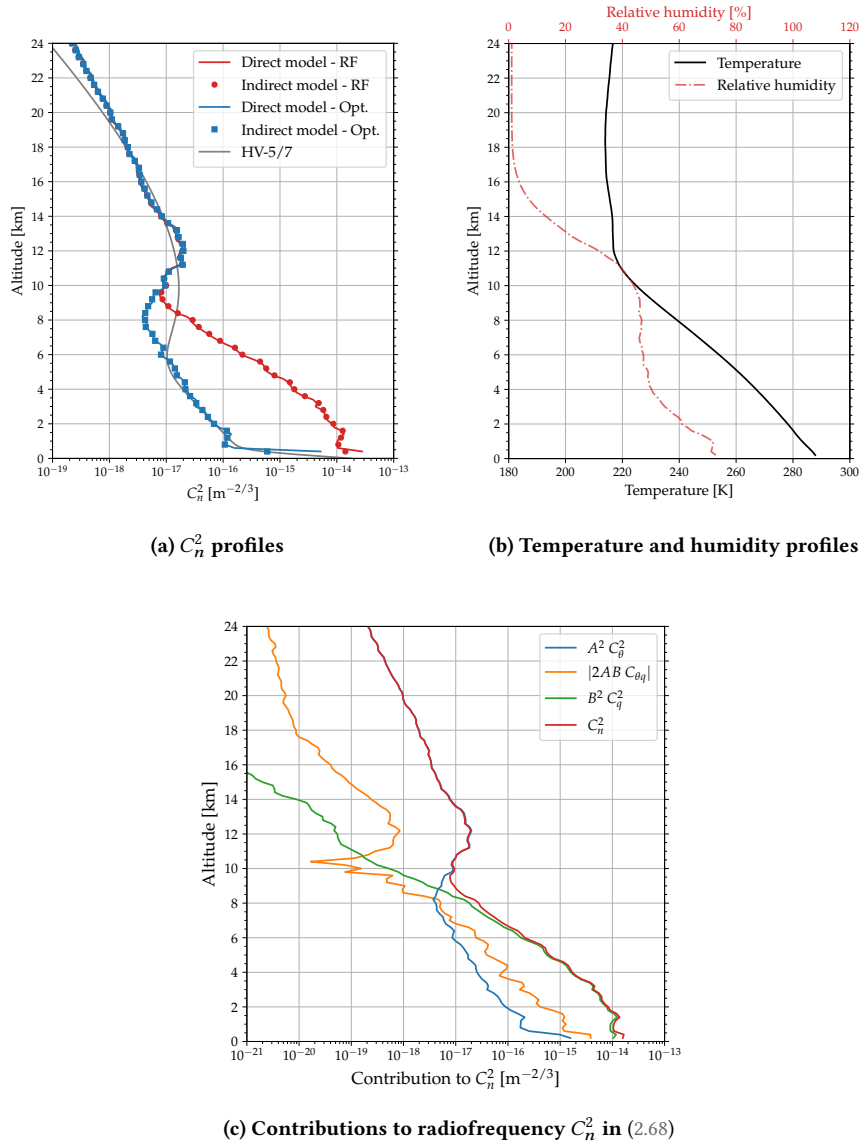


Figure 4.13 – Average profiles at Bordeaux for the year 2020.

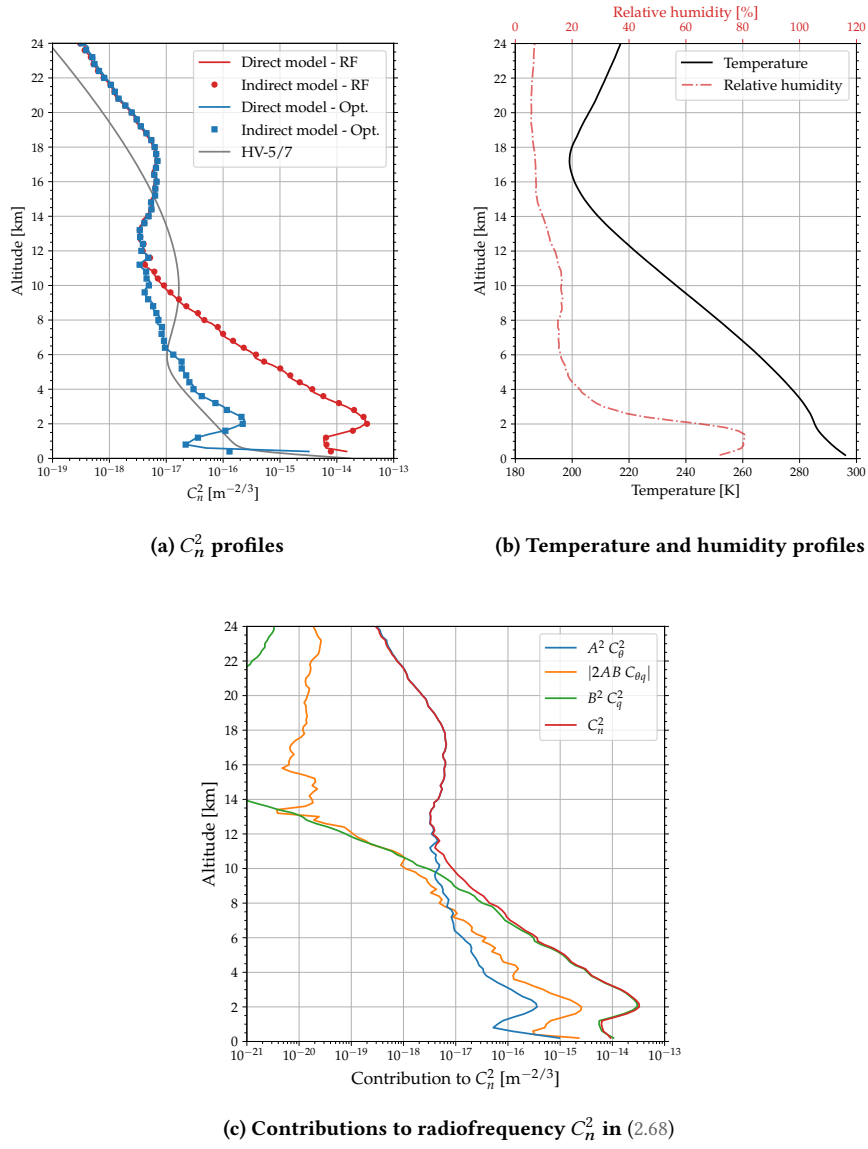


Figure 4.14 – Average profiles at Hilo, HI, for the year 2020.

4 | Computation of C_n^2 based on its statistical definition

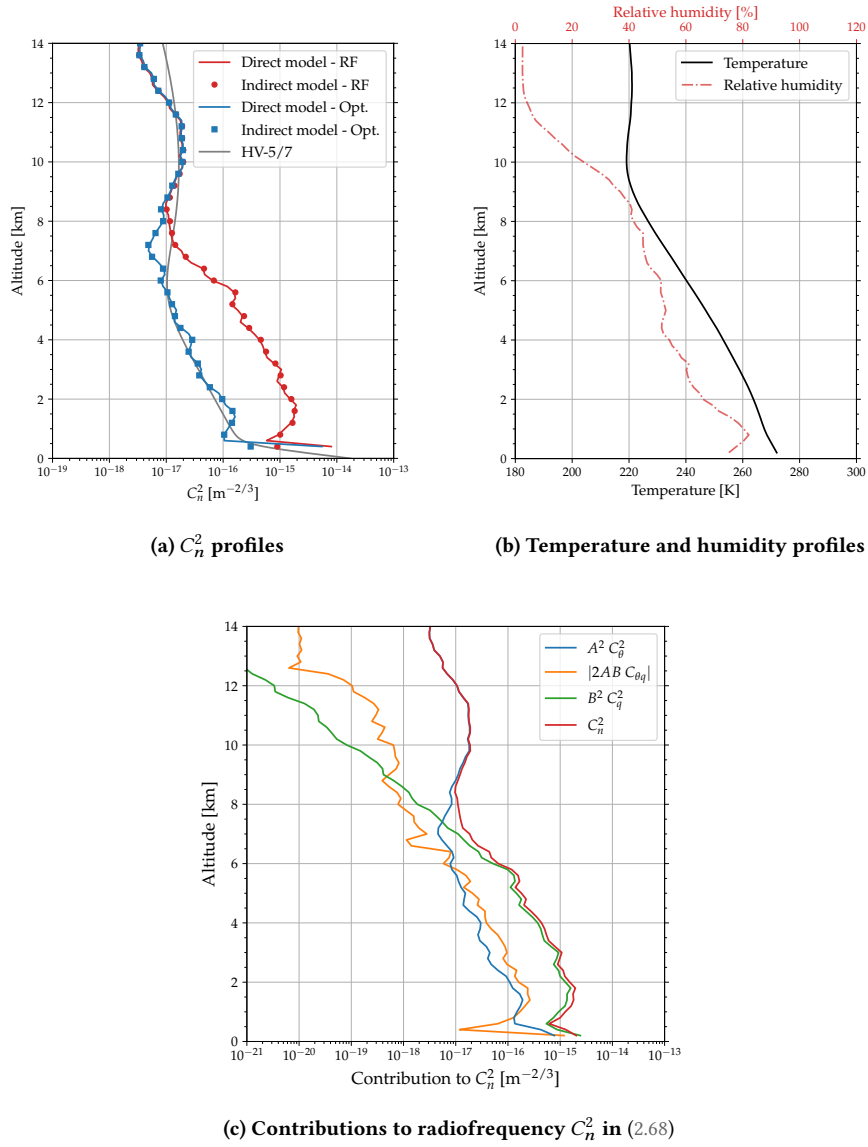


Figure 4.15 – Average profiles at Svalbard for the year 2020.

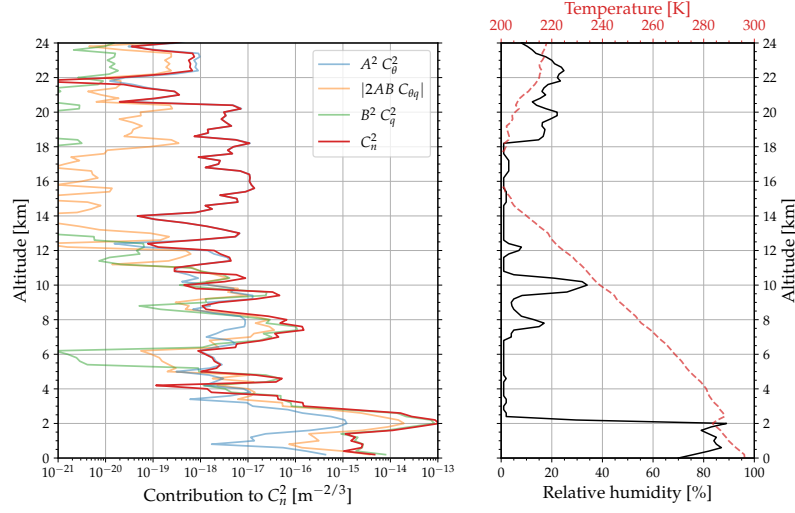


Figure 4.16 – Modelled radiofrequency C_n^2 profile at Hilo for August 20, 2020 at 12:00 UTC, and measured relative humidity and temperature profiles.

tion model using scintillation measurements in Norway, detailed in Section 4.5.3.1. The relationship between C_n^2 profiles and the scintillation standard deviation is presented in Section 4.5.3.2, while the validation results are discussed in Section 4.5.3.3.

4.5.3.1 Available scintillation measurements and radiosonde observations

One-year of scintillation measurements in the Arctic region from a satellite beacon at 19.68 GHz have been collected by the Norwegian Defence Research Establishment (FFI) [Ryt18], with simultaneous radiosonde observations.

Meteorological observations and attenuation measurements were gathered at Bjørnøya from May 2016 to April 2017. Situated in the Barents Sea, north of continental Norway, Bjørnøya is a small island measuring 14 km × 17 km. The island hosts a standard radiosounding station at latitude 74.5°N, longitude 19.0°E, and an elevation of 10 meters. During the attenuation measurement campaign, a receiver antenna for Ka-band was located close to the radiosounding station [Ryt18].

Scintillation measurements Attenuation time series have been obtained at Bjørnøya from a 19.68 GHz beacon transmitted by the Eutelsat Ka-Sat satellite. The elevation angle was 6.6°, with the path towards the satellite going from the North of the island to its South. The receiver, equipped with a 1.2 m diameter antenna, recorded data with a sampling rate of 10 samples/s.

From the attenuation data, scintillation time series have been extracted by isolating the high-frequency content using a 6th order Butterworth filter [RB14]. The chosen cut-off frequency was 2 mHz during clear-sky conditions, and 7 mHz during rain events. The availability of scintillation data over the studied 1-year period

4 | Computation of C_n^2 based on its statistical definition

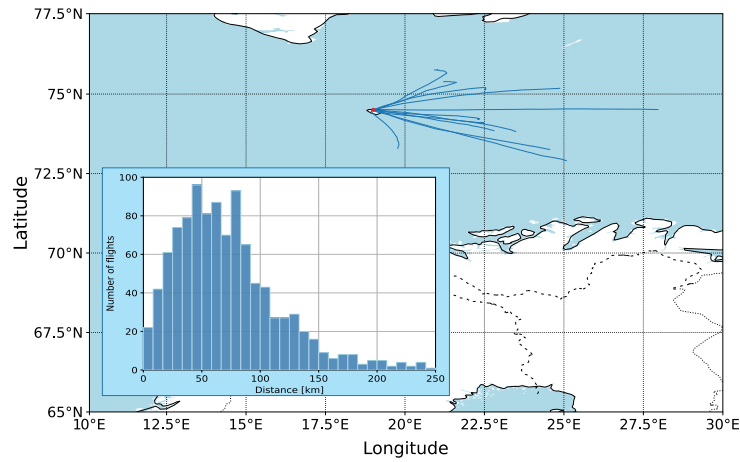


Figure 4.17 – Bjørnøya location and illustration of radiosonde drifting - examples and histogram of distances travelled. [Qua+24]

is 95.1%. Additional information about the hardware and experiments can be found in [Ryt18].

Radiosonde observations Radiosonde observations from Bjørnøya are publicly accessible on the Norwegian Meteorological Institute website⁴. The station (ID 01028) is equipped with high-density radiosondes, measuring vertical profiles of pressure, temperature, wind speed and direction, and relative humidity. The average vertical resolution of the measurements is 10 meters, from the ground up to 25 to 30 km of altitude. In summer (approximately from April to September), radiosondes are launched twice a day, around 00:00 UTC and 12:00 UTC. In winter (from October to March), observations are collected every 6 hours, at 00:00 UTC, 06:00 UTC, 12:00 UTC, and 18:00 UTC. A total of 1036 flights are available between May 2016 and April 2017. Following the exclusion of 9 flights associated with rainy conditions (measured rain rate larger than 2 mm/hr at launch time) and flights when no scintillation data have been collected, 1000 flights remain for further processing.

Figure 4.17 shows the location of Bjørnøya, depicted by the red dot. The continuous blue lines illustrate some radiosonde trajectories, influenced by the wind. For illustrative purposes, only the furthest radiosonde travels for each month between May 2016 and April 2017 are represented. The inset shows the histogram of all distances travelled by the radiosondes during the year of data collection, i.e., radiosonde drifting.

⁴<https://thredds.met.no/thredds/catalog/remotesensingradiosonde/catalog.html>, last accessed on 30 August, 2024.

4.5.3.2 Scintillation standard deviation models

Starting from (2.5), the refractivity $N = 10^6(n - 1)$ at radiofrequency can be obtained from

$$N = \frac{77.6 p}{T} + \underbrace{3.75 \times 10^5 \frac{e}{T^2}}_{N_{\text{wet}}}, \quad (4.13)$$

where p is the pressure [hPa], e is the water vapor pressure [hPa], and T is the temperature [K]. The second term in (4.13) is associated with the wet refractivity, denoted N_{wet} . The empirical σ_χ model described below makes use of N_{wet} to predict the scintillation standard deviation. It serves as a literature-based model against which the performance of the statistical definition model can be compared. Alternatively, an adaptation of the C_n^2 statistical definition model using N_{wet} in its calibration coefficient is also presented.

Empirical model based on N_{wet} Empirical measurements showed that the scintillation standard deviation is well correlated to the wet refractivity at the surface, denoted $N_{\text{wet,s}}$ [Ryt18; ITU17]. This led to several models of the form

$$\sigma_\chi = (a + b N_{\text{wet,s}}) f^c g(D_e) / \sin^d(\vartheta) \quad [\text{dB}], \quad (4.14)$$

with f the frequency in GHz and ϑ the elevation angle in degrees. The coefficients a , b , c , and d , have theoretical or empirical origins and depend on the model, e.g., Karasawa [KYA88], van de Kamp [Kam+99], or ITU-R [ITU17]. In this work, the coefficients computed from Bjørnøya data and other measurements in Arctic conditions are used, i.e., $a = 0.8 \times 10^{-3}$ and $b = 0.91 \times 10^{-4}$ [Ryt18], with the ITU-R values for $c = 7/12$ and $d = 1.2$ [ITU17]. The antenna aperture averaging function $g(D_e)$ is computed from [ITU17], with D_e the effective antenna diameter.

The surface wet refractivity $N_{\text{wet,s}}$ is computed from the definition of N_{wet} in (4.13), using the meteorological quantities measured by the radiosondes at their first level (less than 10 m above the ground). The conversion between the relative humidity, measured by the radiosondes, and the water vapor pressure e , needed in (4.13), is done according to [ITU19], similarly to (4.10) to (4.12).

In the following, the model in (4.14) is referred to as the N_{wet} model. It has initially been developed to provide long-term statistics [ITU17; Ryt18], even though its performance for instantaneous σ_χ is also presented in Section 4.5.3.3.

C_n^2 statistical definition model The link between the scintillation variance [dB^2] and the vertical profile of the refractive index structure parameter $C_n^2(z)$ is detailed in Section 6.1.2.1, and is given by [Ish78; Vas99]

$$\sigma_\chi^2 = 42.48 g^2(D_e) \frac{k^{7/6}}{(\sin(\vartheta))^{11/6}} \int_{\text{height}} C_n^2(z) z^{5/6} dz, \quad (4.15)$$

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where $k = \frac{2\pi}{\lambda}$ with λ the wavelength and z is the altitude above ground. The same antenna aperture averaging function as in (4.14) has been included in (4.15).

In (4.15), the $C_n^2(z)$ profile comes from the RF C_n^2 statistical definition model, in the default configuration ($\omega = 2$, $\delta = \Delta z$) with $\Delta z = 200$ m. Compared to the profiles presented in Section 4.5.2, the calibration coefficient (i.e., scaling factor) is obtained by ensuring a good fit between measured scintillation variances and modelled values from (4.15). In practice, two different calibration approaches have been studied for Bjørnøya data, leading to two distinct C_n^2 -based models for σ_χ :

- A constant factor γ , determined empirically as the average of the ratios between the measured and modelled σ_χ . This leads to the so-called C_n^2 *model* for σ_χ , where σ_χ is estimated from the square-root of (4.15), multiplied by γ .
- A $N_{\text{wet},s}$ -dependent calibration factor, i.e., $\alpha + \beta N_{\text{wet},s}$, inspired from literature-based N_{wet} models. The coefficients α and β are obtained by computing the best polynomial fit (according to the least squares criterion) between the monthly averages of $N_{\text{wet},s}$ and the monthly averages of the ratios between the measured and modelled σ_χ . This version of the C_n^2 model is referred to as the $C_n^2 + N_{\text{wet}}$ *model*, where the square-root of (4.15) is multiplied by $\alpha + \beta N_{\text{wet},s}$ to estimate σ_χ .

The coefficients α , β , γ ensuring the best fits between the measured and modelled σ_χ for Bjørnøya data are: $\alpha = 4.75$, $\beta = -0.05$, $\gamma = 3.16$.

4.5.3.3 Application to Bjørnøya and validation

The application of the three σ_χ models described in Section 4.5.3.2 to Bjørnøya data is presented in Figure 4.18, illustrating the monthly-averaged σ_χ and surface $N_{\text{wet},s}$. Given that modelled scintillation standard deviations are only available at the times of radiosonde launches, only the measurements collected in the one-hour window centred on each radiosonde launch have been considered, as depicted in Figure 4.19. This explains the slight variation observed in the blue curve presented in Figure 4.18 compared to the results outlined in [Ryt18], where all Bjørnøya measurements are considered, and not only the ones close to the radiosonde launches.

Observations In Figure 4.18, all three models exhibit comparable performance, capturing the seasonal variations in the monthly-averaged σ_χ . This is particularly the case for the N_{wet} and $C_n^2 + N_{\text{wet}}$ models, and is justified by the linear relationship of these models with $N_{\text{wet},s}$, which shares a similar seasonal variation as σ_χ (see brown dashed curve in Figure 4.18, associated with the y-axis on the right).

Figure 4.20 provides additional insights into the seasonal behaviour of the C_n^2 and $C_n^2 + N_{\text{wet}}$ models. It illustrates the average C_n^2 profiles derived from radiosonde observations during winter months (December, January, February, March) and summer months (June, July, August, September). The corresponding average temperature and N_{wet} profiles are also presented. The summer C_n^2 profile exhibits a peak at

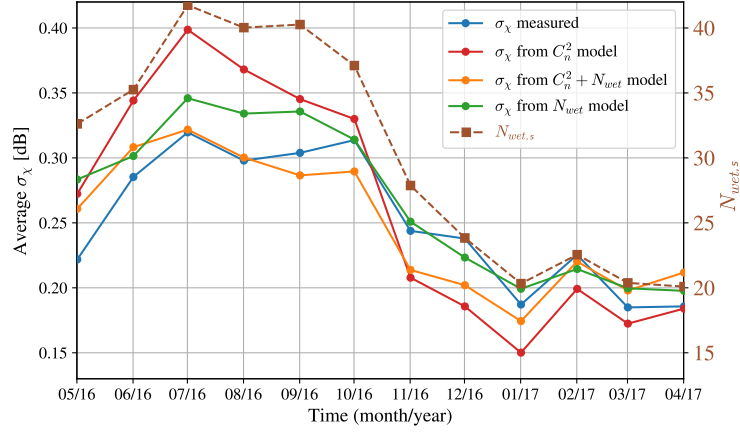


Figure 4.18 – Monthly averaged σ_χ and surface $N_{wet,s}$. [Qua+24]

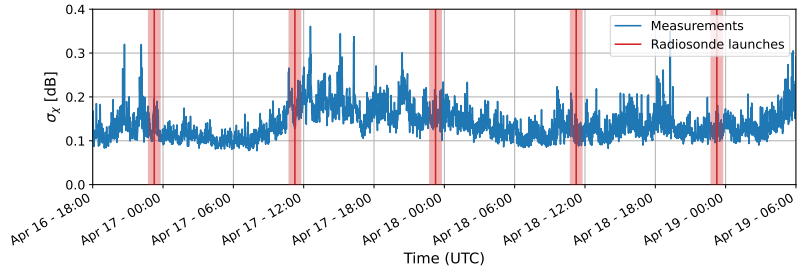


Figure 4.19 – Illustration of temporal series of scintillation measurements with simultaneous radiosonde launches, and one-hour windows centred on the launches.

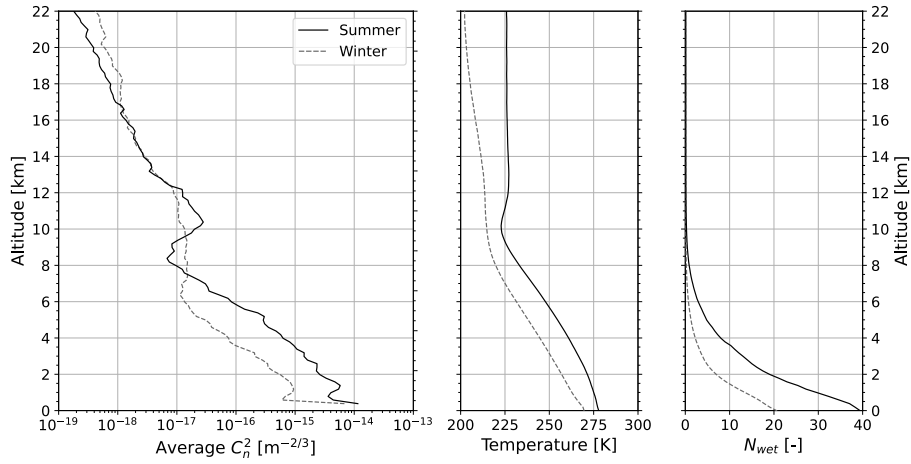


Figure 4.20 – Average C_n^2 , temperature, and wet refractivity profiles at Bjørnøya for winter and summer months. [Qua+24]

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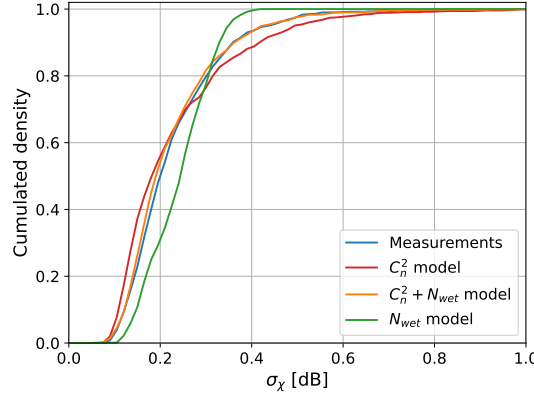


Figure 4.21 – Cumulative distributions of σ_χ . [Qua+24]

the tropopause altitude (approximately 10 km), indicative of a temperature inversion. Most variations between winter and summer C_n^2 profiles are observed in the troposphere (below 10 km) and are associated with seasonal changes in the N_{wet} profile. Specifically, higher N_{wet} values in summer contribute to elevated C_n^2 , consequently amplifying σ_χ .

Cumulative distributions of both measured and modelled σ_χ values, for the whole year, are reported in Figure 4.21. They show that the models successfully predict realistic σ_χ values compared to the measured data. The agreement is excellent for the $C_n^2 + N_{\text{wet}}$ model, while the N_{wet} model tends to give σ_χ values within a limited range, overestimating small measured σ_χ and underestimating large measured σ_χ .

This last observation is further illustrated in Figure 4.22, giving the correlation plots between modelled and measured σ_χ . The information presented in this plot goes further than the distributions, as it considers the temporal accuracy of the models, i.e., their capability to give σ_χ values in agreement with the measurements at a particular instant. It shows that σ_χ from the C_n^2 and $C_n^2 + N_{\text{wet}}$ models follow the correlation line, with a larger spread for large σ_χ . On the contrary, the N_{wet} model deviates more from the correlation line. This behaviour can be expected given that this model is designed to predict monthly and long-term statistics rather than instantaneous hourly averages [ITU17; Ryt18].

Finally, Table 4.9 presents the correlation coefficients and RMSE between each of the three models and the measurements. The C_n^2 model exhibits the highest correlation with the measurements, despite having the largest RMSE, primarily attributed to a few significant outlier values.

Validation The agreement between measured σ_χ and modelled values from the C_n^2 and $C_n^2 + N_{\text{wet}}$ models, as well as their comparable performance with literature-based models, tends to validate the C_n^2 statistical definition approach for RF scintillation.

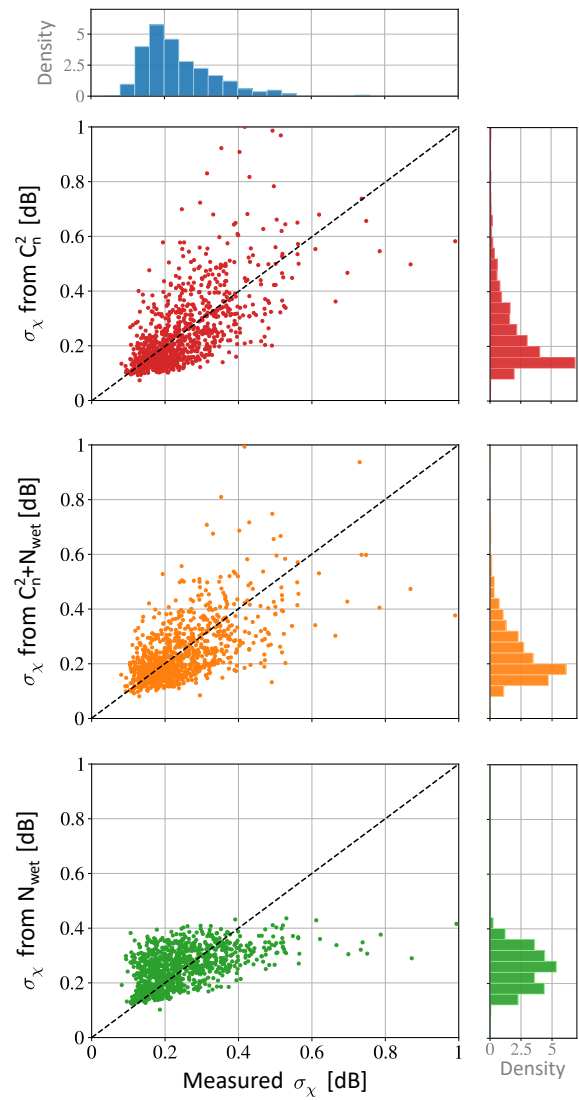


Figure 4.22 – Correlation plots of σ_{χ} . [Qua+24]

Table 4.9 – Correlation coefficients and RMSE for all models.

	N_{wet} model	C_n^2 model	$C_n^2 + N_{\text{wet}}$ model
Correlation coefficient	0.50	0.67	0.61
RMSE [dB]	0.095	0.104	0.096

4 | Computation of C_n^2 based on its statistical definition

However, certain aspects of the validation approach require additional discussion:

- Radiosonde drifting: C_n^2 profiles are derived from radiosonde measurements, and it is rather unlikely that they precisely align with the propagation path between the satellite and the ground station. This is especially evident due to the radiosonde drifting, illustrated in Figure 4.17. Consequently, the models outlined in Section 4.5.3.2 assume that C_n^2 vertical profiles exhibit relative consistency in the broader geographical region surrounding the location of interest. While this assumption is probably verified at high altitudes, where atmospheric features span over large areas, it is questionable close to the ground. Fortunately, radiosonde drifting is relatively small in the ground layer, and the turbulent features identified by the radiosonde are probably similar to those encountered by the communication signal. Nonetheless, the extent of these differences remains to be quantified.
- Temporal dependency: the temporal validity of the predicted C_n^2 profile must also be studied, as turbulence statistics vary in time. The literature suggests timescales of around 10 minutes for the stationarity of turbulence [VV95], which is smaller than the one-hour window employed for scintillation measurements and the average 2-hour flight time of radiosondes. As a result, the variability of C_n^2 profiles in both space and time could diminish the correlations between the models and measurements, although it might have minimal or no impact on the distributions.
- Profile validation: the validation of C_n^2 profiles is also made difficult by the comparison with measured integrated quantities, such as the scintillation standard deviation σ_χ . Ideally, the assessment of the C_n^2 profile model's performance should rely on measured vertical profiles of C_n^2 , but such data were not available.
- Calibration factors: the calibration coefficients used for the validation have been determined empirically fitting the measured σ_χ . Hence, they are different from the scaling factors introduced previously in Section 4.3, and their validity for other locations remains to be validated.

4.6 Discussion and generalization

The limitations of the C_n^2 statistical definition model are discussed in this section, highlighting possible alternatives.

4.6.1 Model limitations

This C_n^2 model does not claim to be an accurate description of the physics of optical turbulence because (i) the assumptions behind its equations may not be fully satisfied, (ii) it includes a fitted parameter without physical justification (scaling factor).

Nevertheless, it provides a simplified description of C_n^2 , easy to apply for practical scenarios. Moreover, despite its simplicity, the accuracy of the model is comparable to other literature-based models, as long as it is calibrated accordingly.

Inertial range hypothesis As briefly discussed in Section 4.1.2, the assumption of the inertial range, which is necessary for the validity of (4.5), might not hold given the available resolution of radiosonde measurements, potentially larger than the outer scale L_0 .

There is potential to extend this approach by incorporating physical scaling laws from the buoyancy range, i.e., the range made of scales larger than the outer scale L_0 . This could lead to a model that eliminates the need for fitting parameters. In the buoyancy range, several scaling laws for the temperature structure function have been proposed [Bol62; CB89; NVW97]. The underlying assumptions for these scaling laws are discussed in [BH22], suggesting temperature structure functions proportional to $\rho^{2/5}$ or ρ^2 in the buoyancy range.

Scaling factor The model includes a scaling factor that lacks a clear physical justification and is instead a fitted parameter. However, it is not uncommon to include empirically-based calibration parameters in C_n^2 models.

Applying the C_n^2 model to radiosonde measurements across different locations suggests that this coefficient exhibits minimal variation with location. Nevertheless, further validation is needed, and potential seasonal fluctuations could be explored.

Availability of radiosonde observations The C_n^2 statistical definition model has been applied to high-density radiosonde observations, which are primarily conducted for meteorological purposes and data assimilation in NWP. These measurements are relatively costly as they require balloon launches, and provide spatially and temporally localized data (restricted to the launch site and time). Consequently, they do not allow for continuous monitoring of C_n^2 variations.

Additionally, the analysis in this chapter is based on radiosonde observations with a high vertical resolution of 10 meters or less. However, not all radiosonde data offer such fine resolution, as some have vertical resolutions ranging from hundreds of meters to even kilometres.

Given these limitations, alternative approaches using NWP or UAVs as sources of meteorological data for the C_n^2 model are discussed below.

4.6.2 Use of NWP simulations

NWP simulations, detailed in Chapter 5, can provide vertical profiles of simulated meteorological quantities, to use as an alternative to radiosonde measurements. However, compared to radiosonde observations, they usually offer a lower vertical resolution while maintaining a reasonable computational cost, with vertical spacings ranging from tens to hundreds of meters in the troposphere, and reaching several kilometres in the stratosphere.

4 | Computation of C_n^2 based on its statistical definition

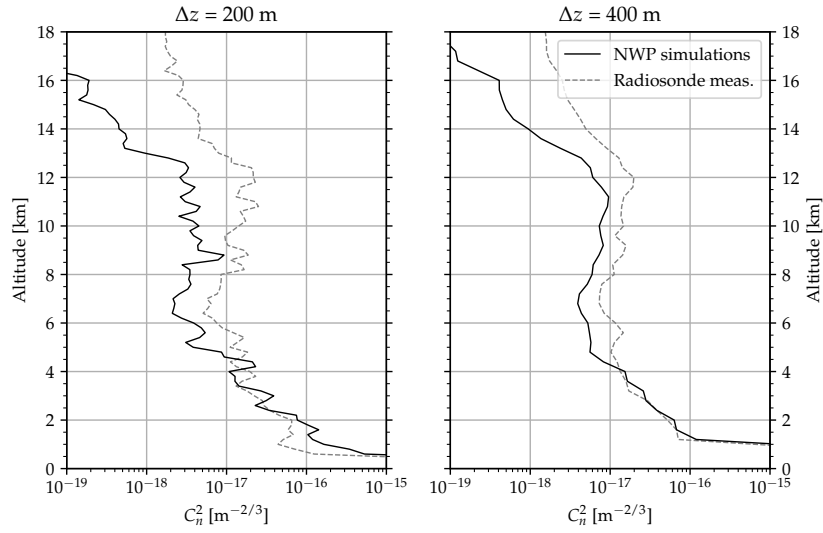


Figure 4.23 – Average C_n^2 profiles at Trappes in December 2020, with the C_n^2 statistical definition model applied to NWP data or radiosonde measurements.

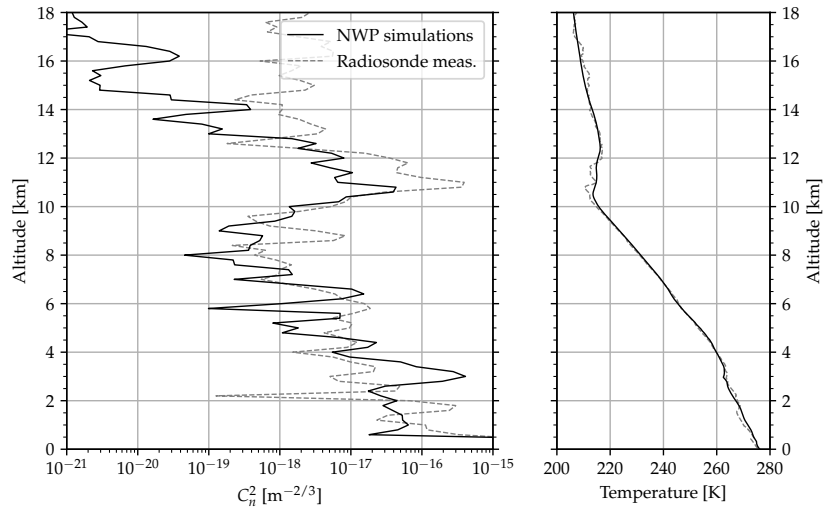


Figure 4.24 – C_n^2 and temperature profiles at Trappes on December 10, 2020, at 00h00 UTC.

The application of the C_n^2 statistical definition model to NWP data is illustrated in Figures 4.23 and 4.24 to radiosonde measurements obtained in December 2020 at Trappes. The NWP simulations are conducted using the WRF software, with three nested domains (9-3-1 km grids), 250 vertical levels, and 6-hour lead time (see Section 5.2 for details about these parameters). Vertical profiles of pressure and temperature are extracted above Trappes, and fed into the C_n^2 statistical definition model using $\omega = 2$ and $\delta = \Delta z$. Vertical spacings of 200 m and 400 m are chosen, and the average of C_n^2 profiles for December 2020 is depicted in Figure 4.23. The comparison with C_n^2 profiles coming from the C_n^2 statistical definition model applied to the radiosonde observations show that the profile trends are similar in the troposphere and can be used for the determination of the scaling factor. For radiosonde data, a scaling factor of 0.5 is found for both resolutions, while a value of 0.1 is obtained for the NWP model at 200 m resolution and, and 0.2 for the NWP model at 400 m resolution.

Regarding localized turbulent layers, they also appear in NWP C_n^2 profiles due to temperature inversion, as seen in Figure 4.24 for December 10, 2020, at 00h00 UTC in Trappes. However, temperature inversions may not arise at the exact altitude in the simulations as in the measured profiles, as it is clearly seen for the layers at 2 km in the measurements and 3 km in the simulations.

Finally, in both Figures 4.23 and 4.24, C_n^2 from NWP simulations is underestimated at high altitudes compared to the measured values. This can be explained by the lower vertical resolution in the simulations at these altitudes, the inherent smoothing in NWP simulations that do not resolve all turbulent scales, as well as the physics parametrization different in the troposphere and the stratosphere.

These different observations tend to limit the applicability of the C_n^2 statistical definition model with NWP data, justifying the use of other models presented in Chapter 5. As an alternative, the use of horizontal fluctuations with their associated structure functions is possible, as in [Fre+10],

4.6.3 Application to UAVs

The use of UAVs, i.e., drones, for atmospheric turbulence measurements has been presented in the recent literature. For meteorological parameter measurements, they offer improved control on the trajectory compared to radiosondes, increased spatial resolution, as well as reduced costs of operation. However, they are limited to the atmospheric BL, namely due to their autonomy [Gio+23]. Despite this limitation, they can be of interest to obtain high-quality C_n^2 profiles in the BL, and help for the parametrization of the BL in NWP software.

The estimation of C_n^2 based on the structure function and UAV measurements of refractive index or temperature fluctuations has been applied in [Van+12; Bey+12; Bra+16; Bon+15; Wai+15] for example.

4.7 Chapter summary

This chapter presented a simple approach to estimate vertical profiles of C_n^2 based on its statistical definition. This method was applied to high-density radiosonde profiles to derive optical and RF C_n^2 profiles, validated against thermosonde measurements of optical C_n^2 as well as scintillation measurements at radiofrequencies. While the approach is based on the physics, it includes empirically fitted parameters and may not fulfill all required hypotheses.

4.7.1 Main results and contributions

Using the statistical definition of C_n^2 to estimate vertical profiles of optical turbulence, as described in (4.5), is not a novel approach. However, with recent improvements in radiosonde vertical resolutions, this work explored the application of this model to radiosonde meteorological data source.

The primary challenge is associated to the inertial range hypothesis which may not be fully met even with improved radiosonde vertical resolution. It is partially addressed by introducing an empirical scaling factor c , ensuring consistency between C_n^2 profiles derived from the statistical definition model and those from literature-based profiles (e.g., HV model, Tatarskii-based models).

Despite this physical limitations, the model's simplicity makes it valuable for practical applications, such as site characterization for FSO. Unlike other C_n^2 models based on measurements, this approach can be applied to any location where radiosonde data is available, without being restricted to astronomical sites. Changing the location may only require a new model calibration, even though little variations in the scaling factors have been observed.

This C_n^2 model also enables to obtain optical and RF C_n^2 profiles using a similar approach, highlighting the relationship between both C_n^2 profiles and the dominant influence of temperature fluctuations on optical C_n^2 , and of temperature and humidity variations on RF C_n^2 . Moreover, it is capable of identifying localized turbulent layers associated to strong gradients in temperature and humidity profiles, such as temperature inversions. It offers performance comparable to literature models.

4.7.2 Future work

Extensions to this chapter model include further generalisation and comparison with existing literature models. For example, modifications to the HAP model are introduced in [SA23] to better fit the average C_n^2 profiles at Trappes obtained by this statistical definition model.

Additionally, a more in-depth study of the vertical resolution impact with respect to the outer scale could provide valuable insights into the scaling factor and the limitations of the model.

Exploring alternative data sources, such as UAVs or NWP data, is also a promising direction as radiosonde measurements, while valuable, are not always available.

NWP models can offer a broader spatial and temporal coverage, but their limited vertical resolution and inability to resolve small scales appear to prevent the use of the statistical definition model. Instead, alternative models are presented in the next chapter.

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Prediction of C_n^2 profiles from meteorological simulations | 5

The fifth chapter of this thesis delves into the modelling of C_n^2 profiles using Numerical Weather Prediction simulations. Utilizing NWP simulations for C_n^2 profile modelling provides the capability to obtain profiles over any area on Earth where simulations are feasible, eliminating the need for costly instruments. Additionally, it allows for the forecasting of C_n^2 profiles (up to several days in advance), much like weather forecasts. This capability is of significant interest for satellite-to-ground optical communications, particularly for site characterization, scheduling of optical links, and related applications, as introduced in Section 1.3.

The C_n^2 models discussed in this chapter are illustrated in Figure 5.1. One model is based on the literature review conducted in Chapter 3 and is a hybrid approach that combines MOST to determine the ground-level C_n^2 value and the BL C_n^2 profile, with the Tatarskii equation applied to the FA region. The second model directly uses the TKE as resolved by the NWP software (specifically, the WRF software in this work) to obtain C_n^2 . The following publications document the development of these models, applying them to various locations:

- [Qua+23a]: F. Quatresooz, R. Griffiths, L. Bardou, R. Wilson, J. Osborn, D. Vanhoenacker-Janvier, and C. Oestges. “Continuous daytime and nighttime forecast of atmospheric optical turbulence from numerical weather prediction models”. In: *Optics Express* 31.21 (2023), pp. 33850–33872. DOI: 10.1364/oe.500090.

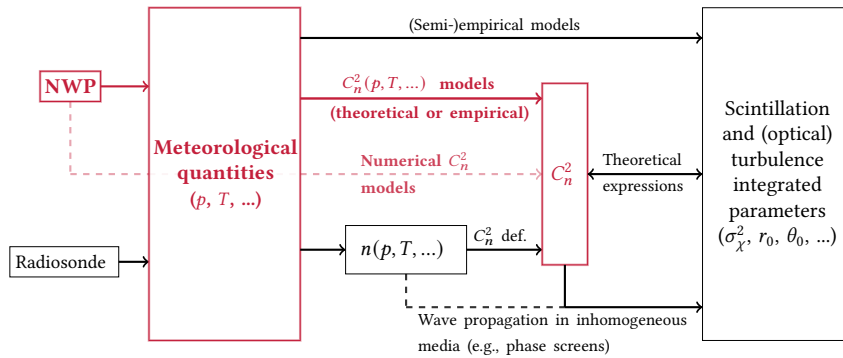


Figure 5.1 – Schematic of the thesis structure, highlighting the C_n^2 models presented in Chapter 5.

5 | Prediction of C_n^2 profiles from meteorological simulations

- [Qua+23b]: F. Quatresooz, G. Orban de Xivry, O. Absil, D. Vanhoenacker-Janvier, and C. Oestges. “Challenges for optical turbulence characterization and prediction at optical communication sites”. In: *International Conference on Space Optics—ICSO 2022*. Vol. 12777. SPIE. 2023, pp. 2410–2424. doi: 10.1117/12.2691034.
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The choice of these two models is motivated by their applicability to optical communication sites. Most C_n^2 models previously developed using NWP simulations have been designed for astronomical sites under nighttime conditions. However, OGS sites require both nighttime and daytime operations, where stronger turbulence is expected in the BL. Accurately modelling the BL motivates the use of models that solve for the TKE, or hybrid C_n^2 models that separately consider the BL and the FA.

This work makes the following contributions:

- A review of existing C_n^2 models derived from NWP simulations, mostly developed for astronomical sites;
- The development and application of two NWP-based C_n^2 models at astronomical sites (Paranal, Tenerife) and non-astronomical sites (T-REX campaign location, Redu, Louvain-la-neuve), and the evaluation of their performance for continuous OT forecast (i.e., daytime and nighttime);
- An overview of the relevant features of NWP software for C_n^2 profile prediction, offering insights into their strengths and limitations.

The validation of the same C_n^2 profiles across multiple locations and during both day and night is rarely addressed in the literature. This analysis highlights the limitations of OT predictions in regions where the BL is dominant, and suggests potential directions for future research.

Section 5.1 presents the general approach of OT forecast with NWP simulations, as well as its challenges and the current state of the art. Then, Section 5.2 details the relevant features of the WRF software for C_n^2 profile prediction, while Section 5.3 describes the two C_n^2 models considered. Finally, the application and validation of the C_n^2 models for several sites is given in Section 5.4.

5.1 General approach and main challenges for optical turbulence prediction

This section details the general approach used in this work that enables to forecast OT-related quantities. The challenges associated with the approach are also presented, and its application in the literature is explored.

General approach and main challenges for optical turbulence prediction | 5.1

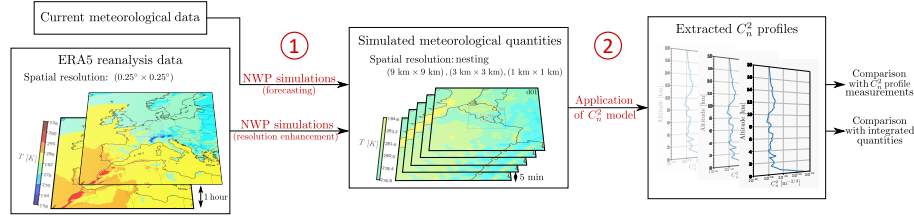


Figure 5.2 – General approach to conduct OT forecast from NWP simulations. [Qua+23a]

5.1.1 General approach

Inspired from the literature on OT forecasting for astronomical applications [MVB99; MMT20; CBL08; OS18; Bas+20], Figure 5.2 illustrates the general approach used to derive C_n^2 profiles above a specific site at times of interest. It involves two main steps: the NWP simulations and the application of a C_n^2 model [Qua+23a].

5.1.1.1 NWP simulations

The first step presented in Figure 5.2 uses meteorological data as input and predicts their evolution across a 3D domain using the NWP software. This stage serves two primary purposes:

1. Forecasting the future evolution of meteorological conditions above a desired location over subsequent hours or days. In this case, current meteorological data are used as inputs to the NWP software, and their temporal evolution is predicted (forecasting).
2. Enhancing the spatial and temporal resolutions of available meteorological data. This is of interest for OT characterization above a given site, i.e., to develop OT models and compare them with available measurements (collected previously). As inputs to the NWP software, reanalysis data are used. They correspond to meteorological quantities coming from large NWP simulations that describe past atmospheric states. They are stored in database but typically have coarse spatial and temporal resolutions, hence the use of NWP simulations to improve them (resolution enhancement).

In this work, for developing and assessing C_n^2 models adapted to OT forecast, input meteorological quantities come from the ERA5 reanalysis data of the European Centre for Medium-Range Weather Forecasts (ECMWF) [Cop23a; Cop23b], as illustrated in the left part of Figure 5.2. It is the fifth generation of ECMWF reanalysis atmospheric data, hence its name ERA5 (ECMWF Reanalysis v5). Being reanalysis data, they combine NWP models with observations across the world, using data assimilation. They cover the entire Earth with a horizontal resolution of $(0.25^\circ \times 0.25^\circ)$ in latitude and longitude, with 37 vertical (pressure) levels. Their temporal resolution

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is one hour, and they are easily accessible online thanks to the Copernicus climate data store [Cop23a; Cop23b].

As an output of NWP simulations (middle part of Figure 5.2), a 3D grid of meteorological quantities (e.g., temperature, pressure, wind speed, etc.) at the desired location is obtained, with its temporal evolution monitored. The spatial resolution of the grid can be chosen and be as small as 1 km horizontally by using nested domains (see Section 5.2.1.1). The temporal resolution at which NWP simulation outputs are recorded is typically 5 minutes in this work, and is larger than the time step used in the NWP software (see Section 5.2.1.2).

5.1.1.2 Application of C_n^2 models

The second step starts from the high-resolution simulated meteorological data and applies a C_n^2 model to translate meteorological quantities to C_n^2 . As a result, a 3D grid of C_n^2 values is obtained, and C_n^2 profiles above the location of interest can be extracted. The temporal evolution of these profiles is also monitored, as depicted in the right part of Figure 5.2. The C_n^2 models chosen in this work are presented in Section 5.3.

Finally, for model validation, predicted C_n^2 profiles are compared with measurements. In cases where C_n^2 profile measurements are available, the comparison can be conducted vertically, for all altitudes. However, in most cases, only integrated quantities are measured, such that the comparison is only conducted for vertical integrals of the predicted C_n^2 profiles.

5.1.2 Associated challenges

The methodology presented in Section 5.1.1 raises numerous challenges to achieve accurate C_n^2 predictions. These challenges are introduced below, based on [Qua+23b].

5.1.2.1 Spatial and temporal resolutions

Various types of NWP software are accessible for atmospheric modelling [MLF13; MTM19], including General Circulation Model (GCM), mesoscale models, LES, and DNS. Each of them resolves atmospheric phenomena at different scales, offering their own spatial and temporal resolutions:

- **General circulation model:** In such climate models, Navier-Stokes equations applied to the whole Earth are solved, offering a horizontal resolution of 10 to 20 km. These models could be of interest to obtain C_n^2 profiles in the FA and without needing site-specific calibration [OS18] but they hardly resolve optical turbulence [MLF13]. Indeed, the order of magnitude of OT spatial fluctuations is much smaller than the achievable resolution of GCMs.
- **Mesoscale models:** Mesoscale models are applied to a limited spatial area of the Earth. They are often initialized with data coming from GCMs and can then

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simulate mesoscale atmospheric phenomena, ranging from kilometres to hundreds of kilometres. In order to improve the spatial resolution near the location of interest, they use grid nesting (i.e., nested domains having smaller spatial extent but higher spatial resolution). In the centremost model, the achieved spatial resolution is typically of 1 km, or even lower, sometimes reaching 100 meters [MTM19]. This is the type of model most often used for OT forecast (see [MLF13; Gio+14; Bas+20] for example).

- **Large eddy simulation:** as detailed in Section 2.3.3, LES is a type of model used in computational fluid dynamics, enabling to simulate turbulent flows. They solve Navier-Stokes equations ignoring the smallest length scales thanks to low-pass filtering. Grid spacing is on the order of 10 meters [WF12]. LES simulations are often performed in the atmospheric BL in order to identify small-scale turbulence coming from convective plumes or gravity waves for example [Che+11; WF12]. Furthermore, they can also be used to improve locally the resolution of output quantities of mesoscale simulations [MLF13; MM10].
- **Direct numerical simulation:** in DNS, Navier-Stokes equations are solved directly without any turbulence model. Hence, all spatial and temporal scales of the turbulence are resolved, drastically increasing the computational cost of such simulations. They are therefore not practical for real atmospheric simulations of optical turbulence.

For this work, mesoscale models are of interest as they are applied to a limited area of Earth and simulate atmospheric phenomena with spatial scales ranging from kilometres to hundreds of kilometres. These models solve the conservation equations of mass, momentum and internal energy in the atmosphere over a 3D grid. For small-scale physical processes that are not resolved by the grid (i.e., by the chosen temporal and spatial discretization), physical schemes and approximations are used to describe them with grid-resolved parameters.

The limitation on the spatial resolution is mostly computational, as a higher resolution leads to a denser grid for the computations. As an alternative, grid nesting can be used to enhance the resolution only for a particular area (see Section 5.2.1.1). Moreover, the resolution of the topographic data must also be considered when reducing further the grid spacing.

Vertically, the grid spacing is usually non-linear, with small distances between levels close to the ground and larger distances between levels at high altitudes. As for the horizontal resolution, improving the vertical resolution by adding more levels increases the computation time.

Regarding the temporal resolution, it is related to the time step necessary for preventing numerical instabilities in NWP simulations, typically ranging from 1 to 5 seconds [Bas+20].

The application of those considerations for the chosen NWP software is described in details in Section 5.2.1.1. The associated challenge of parametrizing the NWP software physics is discussed in Section 5.2.1.3.

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5.1.2.2 C_n^2 models

As presented in Chapter 3, there exist numerous C_n^2 models in the literature and choosing a model adapted to NWP simulations is a challenge.

In this context, a parametric C_n^2 model depending on meteorological quantities is required. Ideally, the model should be free of any empirical parameterization, unless those parameters are not location-dependent or can be recomputed for the location of interest. It should also describe the whole atmosphere, even though a separate model for the BL can be used. Moreover, it is required to be applicable in any season and at any time of the day (daytime or nighttime) for the practical application of optical communication.

The C_n^2 models chosen in this work are described in Section 5.3. They are shown to meet all requirements stated above.

5.1.2.3 Boundary layer effect

The BL refers to the lower part of the Earth's atmosphere in proximity to the ground, where the influence from the Earth's surface is significant. Turbulence is typically pronounced in this region, posing challenges for accurate modelling. Furthermore, this presents a particular difficulty for optical communication sites situated at lower altitudes, as they experience intensified turbulence within the BL. A possible solution to ease this challenge is the use of hybrid C_n^2 models that differentiate between the FA and the BL [Ved+21]. Alternatively, a C_n^2 model relying on the TKE can also provide accurate predictions within the BL. This challenge is further discussed in Section 5.3.

5.1.2.4 Availability of measurements

To validate the C_n^2 prediction capabilities of the model, direct measurements of C_n^2 profiles are not always available. Instead, integrated quantities (e.g., seeing, isoplanatic angle, scintillation index, etc.) are often much simpler to obtain, and are related to the C_n^2 profile using the expressions given in Section 6.3 of Chapter 6. As a result, it is difficult to assess the model accuracy for all altitudes, especially since integrated quantities depend on specific parts of the atmosphere.

In Section 5.4, the validation of the presented models is conducted with C_n^2 profile measurements when available (namely the thermosondes from the T-REX campaign, introduced in Section 4.4), and with integrated quantities otherwise.

5.1.3 State of the art of optical turbulence forecast

The idea of OT forecast has initially been suggested for astronomy in 1986: seeing forecasts, based on meteorological quantities, can be used to assist and optimize observation scheduling of telescopes [Cou+86]. Following this idea, the first numerical simulation of atmospheric turbulence for astronomical site selection has been achieved in 1995 [Bou+95]. Since then, many ground-based optical telescopes are equipped with tools monitoring OT quantities and with software performing real-time OT predictions based on NWP simulations.

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This section presents some models commonly found in the literature about OT forecast for astronomical and optical communication applications. They are mainly grouped based on the chosen C_n^2 models.

5.1.3.1 Tatarskii-based C_n^2 models

The use of Tatarskii equation given in (3.13) and reminded below,

$$C_n^2 = a^2 L_0^{4/3} M^2, \quad (3.13)$$

for C_n^2 profile prediction is popular due to its simplicity of application. Indeed, various parametrizations for M and L_0 are available, as detailed in Section 3.2.2 of Chapter 3, that involve mostly macroscale meteorological quantities. These quantities can easily be obtained from NWP simulations as introduced in this chapter. For the parametrizations relying on the TKE, namely for L_0 , there are presented in the section associated to TKE-based models (Section 5.1.3.3) even though they rely on the Tatarskii equation.

As an example, [Jum+07] relies on the Dewan model (introduced in Section 3.2.2.1) applied to meteorological quantities predicted by the Fifth-Generation Penn State/National Center for Atmospheric Research Mesoscale Model (MM5) software in South of France. The predictions are compared with thermosonde observations. In [MM10], a Tatarskii-based model with WRF simulations is compared to the T-REX thermosonde measurements, and to balloon measurements in Hawaii.

Usually, Tatarskii-based C_n^2 models with NWP simulations serve as benchmark models, against which new or alternative C_n^2 models can be evaluated and compared. This is namely the case in [Raf+20; Raf+21; Raf+22], comparing the performance of the TV model with the ones of Tatarskii-based models. These latter relies on various parametrizations of L_0 , but also on L_0 -values directly deduced from thermosonde measurements (i.e., by isolating L_0 in the Tatarskii equation (3.13) and substituting C_n^2 by its measurements from thermosonde). The models are applied to WRF simulations at the Cerro Pachon Observatory (Chile), and compared with thermosonde measurements. Similarly, in [Fre+10], the Dewan model is used for comparison with a new C_n^2 approach based on estimations of the structure functions in the horizontal directions from the NWP simulation outputs, above New Mexico (USA).

Tatarskii-based models are mostly used for describing the FA OT. For example, in [Yan+21], the FA seeing is estimated from the Tatarskii model applied to Polar WRF outputs in Antarctic, while the BL seeing comes from an empirical expression. Hybrid models are therefore possible, as in [CCE18; Cue+24] where the FA C_n^2 profile comes from a Tatarskii model with the Jackson L_0 parametrization, and the BL C_n^2 profile is deduced from a TKE-based model. Results are compared with measured C_n^2 profiles at Cerro Armazones and Cerro Paranal (Chile).

Moreover, since Tatarskii-based models only require macroscale meteorological quantities, they can use NWP simulations as well as radiosonde measurements as inputs. In this case, it is possible to compare the predictions from the same models,

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but using different inputs (i.e., simulations or measurements). This is the approach followed in [Qin+20] and [Spr+21], with WRF simulations above the Tibetan Plateau and in Germany.

The various works cited in this section motivate the use of a hybrid C_n^2 profile relying on a Tatarskii-based model for the free atmosphere in this work, as described in Section 5.3.2.

5.1.3.2 Trinquet-Vernin C_n^2 models

The TV model, presented in Section 3.2.3, is applicable to NWP simulation outputs. The model only requires the vertical profile of potential temperature from which its vertical gradient is computed, and the horizontal wind velocities to get the wind shear. The C_n^2 profile is then obtained from (3.29), that is,

$$C_T^2(z) = \phi(z) \chi(z) S(z)^{1/2}, \quad (3.29)$$

where $\chi = \frac{d\theta}{dz}$, $S = \sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2}$, and $\phi(z)$ given in Table 3.2 for example [Gio+14].

The TV model is applied to WRF simulations at La Palma (Canary Islands) in [Gio+13; Gio+14], at Gaomeigu (China) in [Hon+13], and in Tibet in [Qia+21]. However, as stated in Section 3.2.3, the TV model can only be applied in stable layers ($\frac{d\theta}{dz} > 0$). If this is not the case ($\frac{d\theta}{dz} < 0$), C_n^2 can be obtained from a Tatarskii-based model instead. This approach is presented in [Gio+21] with WRF simulations at the Calern Observatory (France).

A site learning approach for $\phi(z)$, i.e., the evaluation of this function based on C_n^2 profile measurements collected locally, is also presented in [Gio+21] and applied to the Cerro Pachon Observatory (Chile) in [Raf+22].

5.1.3.3 TKE-based C_n^2 models

Thanks to the use of NWP simulations, numerical C_n^2 models relying on meteorological quantities that could not easily be measured in practice are made possible. Indeed, instead of using the measured values of these meteorological quantities, the numerical models use their simulated values. This is namely the case of C_n^2 models relying on the TKE.

Some models described in this section depends on Corrsin's expression, giving C_T^2 as [Cor51; HB15]

$$C_T^2 = 3.2 \tilde{\chi} \epsilon^{-1/3}, \quad (5.1)$$

where ϵ is the energy dissipation rate and $\tilde{\chi}$ is the temperature variance dissipation rate. Both parameters can be expressed as functions of the TKE, the exact relations depending on the models and their associated hypotheses. From C_T^2 , C_n^2 can be computed using (2.70) for example.

Astro-Meso-NH (Masciadri model) A well-known model used for astronomical application is the Astro-Meso-NH model, presented in [MVB99; MLF13]. It relies

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on the Mesoscale Non-Hydrostatic (Meso-NH) software to conduct the NWP simulations, and converts the meteorological outputs to C_n^2 values above locations of interest. From Corrsin's expression (5.1), an expression of C_T^2 involving the vertical gradient of potential temperature and a mixing length L , function of the TKE, is obtained. Hence, the predicted TKE in Meso-NH simulations is used to compute L , then obtaining C_T^2 and C_n^2 . More details about this model are given in Section 5.3.1 and in [MVB99].

In the last decades, the Astro-Meso-NH model has been applied to various astronomical locations and compared with several monitoring instruments. In [MVB99], it has been applied to Paranal and compared with SCIDAR profiles. Simulations conducted for the San Pedro Mártir Observatory (Mexico) are presented in [ME06], and simulations in Antarctica are given in [LMH11]. As part of the MOSE (Modelling ESO Sites) feasibility campaign, the model has also been extensively applied to Cerro Paranal and Cerro Armazones (Chile), and validated with Differential Image Motion Monitor (DIMM) and Generalized SCIDAR measurements [MLF13; Mas+17]. Its calibration is discussed in [HML11; Mas+17], and involves specifying a vertical profile of the minimal TKE in the Meso-NH simulations.

Recent extensions of the Astro-Meso-NH model for short-term forecasting can be found in [MMT20; MTF23]. They present the use of AR models coupling past on-site measurements with forecasts. Such AR models are applied for operational forecasts at the Large Binocular Telescope (LBT), Arizona [MMT20]; and have also been applied to the Very Large Telescope (VLT) at Cerro Paranal [MTF23]. AR models are further studied in Chapter 8 of this work.

The use of the WRF software with a similar TKE-based length-scale parametrization as in Astro-Meso-NH is presented in [CB13] and [LCB20]. A similar approach with a GCM is used in [OS18], where the TKE is estimated by the vertical wind shear. It is extended in [OCJ23], including a more accurate BL modelling.

Cherubini model In [Che+08; CBL08], the MM5 software is used. Its parameterization gives direct access to the energy dissipation rate ε , and to the temperature variance dissipation rate $\tilde{\chi}$ through the temperature turbulent fluxes $\overline{w'\theta'}$ and the TKE.

The Cherubini model has been applied to the Mauna Kea observatory, Hawaii, and validated with Multi-Aperture Scintillation Sensor (MASS) and G-SCIDAR measurements [Che+08; CBL08].

He-Basu model Another approach is presented in [He+14; HB15; Bas+20]. It relies on higher-order turbulence closure, solving prognostic equations for ε and $\tilde{\chi}$ with the WRF software and the MYNN2 scheme [NN04]. These quantities are derived from the TKE and the variance of potential temperature.

This model has been validated using thermosonde measurements collected over the Hawaiian islands [HB15; Bas+20]. In [Bas+20], a sensitivity analysis of OT predictions based on the number of vertical levels in WRF concludes that vertical grid

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spacing of 100 m (or smaller) is needed.

Alliss-Felton model The Alliss-Felton model is similar to the Astro-Meso-NH C_T^2 model (to a multiplicative factor), as the outer scale length is derived from the TKE obtained. It has been applied to WRF simulations conducted above Hawaii [AF09; FA11; AF14], above desert locations [All23], and in California [FHA12] for optical communication applications.

5.1.3.4 Surface C_n^2 models

Using only the NWP outputs close to the surface, the application of a bulk C_n^2 model relying on MOST (see Section 3.3.1.2) enables to obtain the surface value of C_n^2 [And88]. This methodology has been applied with the WRF software in Antarctic [Qin+16a; Qin+17], over the ocean [Qin+16b], above grassland in China [Qin+19], or over the Tibetan plateau [Yan+22]. The modelled C_n^2 values were then compared with C_n^2 measurements from micro-thermometers or scintillometers.

As an alternative to bulk models, already described in Section 3.3.1.2, [Che+11] extracts the surface fluxes and roughness lengths from the NWP simulations conducted with the ECMWF Integrated Forecasting System (IFS). Those quantities are then used in MOST equations to obtain surface C_n^2 for several sites in Germany and France. The predicted values are compared with scintillometer measurements. [Ull+19] uses a similar approach with the WRF software, for six different sites equipped with scintillometers or ultrasonic anemometers.

5.2 Parameterization of NWP simulations

The first step of the general approach depicted in Figure 5.2 consists in performing NWP simulations for forecasting and resolution enhancement. The NWP software used is the Weather Research and Forecasting software, presented in Section 5.2.1. An example of its parameterization for simulations conducted at Redu, Belgium, is detailed in Section 5.2.2.

5.2.1 Weather Research and Forecasting software

The WRF software is a fully compressible and nonhydrostatic mesoscale model that can simulate atmospheric phenomena ranging from kilometres to hundreds of kilometres above a limited area of Earth, for research and NWP applications. It has been developed conjointly by the National Center for Atmospheric Research (NCAR), the NOAA, the Air Force Weather Agency (AFWA), the Naval Research Laboratory (NRL), the University of Oklahoma, and the Federal Aviation Administration (FAA). Since its initial public release in 2000, it has become a true community model, and supports various prediction applications, such as air chemistry, wildland fires, hurricanes, regional climate, etc. More details about the history and development of the

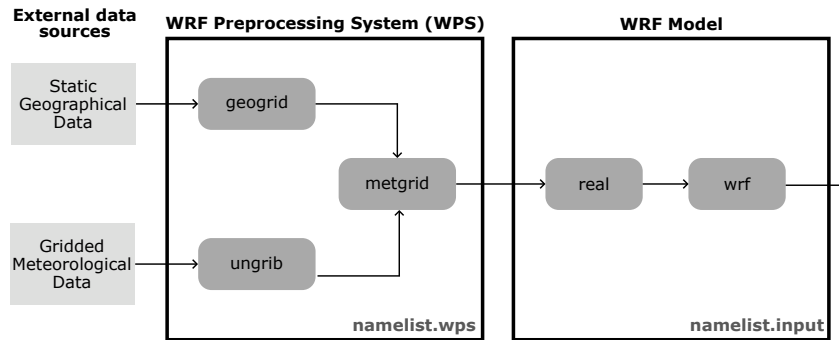


Figure 5.3 – Overview of WRF main components to conduct simulations for real scenarios.

WRF model can be found in [Pow+17], while advanced information are given the user guide [WRF] and in [Ska+19]. An online tutorial is also available¹.

The WRF version used in this work is version 4.2, with the Advanced Research WRF (ARW) solver. Figure 5.3, inspired from WRF online tutorial, gives an overview of WRF main components used to conduct simulations for real scenarios. It consists in the WRF Preprocessing System (WPS) and the WRF model.

WRF Preprocessing System The WPS prepares geographical and meteorological inputs for real-data simulation. It comprises three distinct programs: *geogrid* defines the model domains and interpolates static geographical data onto the grids; *ungrib* retrieves meteorological fields from GRIB-formatted files; and *metgrid* horizontally interpolates the extracted meteorological fields [WRF]. WPS is parameterized by the file “namelist.wps”.

WRF Model The WRF model itself is made of two programs: *real* interpolates vertically the meteorological fields to WRF vertical levels, and *wrf* does the numerical integration. It is parameterized by the file “namelist.input”.

Since WRF is a complex software, only the relevant features for this work are detailed below.

5.2.1.1 Grid decomposition and vertical levels

The WRF software uses a 3D grid with horizontal coordinates x and y , and vertical coordinates η corresponding to terrain-following pressure levels. The horizontal resolution can be enhanced thanks to grid nesting, as explained in this section.

¹Accessible at the following link: <https://www2.mmm.ucar.edu/wrf/OnLineTutorial/index.php>. Last accessed on 30 August 2024.

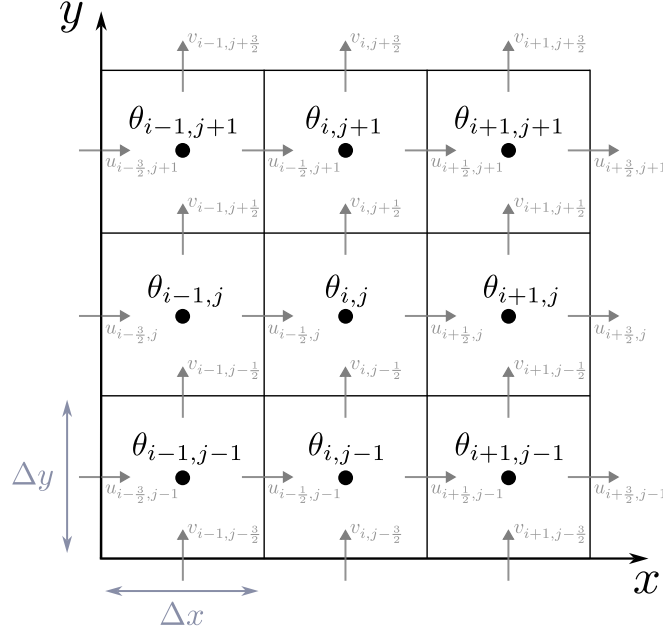


Figure 5.4 – Horizontal grid with mass points and “velocity” points (adapted from [Ska+19]).

Horizontal grid The horizontal simulation domain is centred on the specified latitude and longitude of the location of interest for the simulation. The domain is defined by the number of grid points in x and y directions (denoted by n_x and n_y) and by the horizontal grid spacings Δx and Δy . The grid spacings Δx and Δy are constant over the entire domain, and are usually equal. To account for varying physical grid distances due to the orthogonal projection between latitude-longitude coordinates to the x - y coordinates, map scale factors are included in WRF governing equations [Ska+19, p. 11]. Several projections are supported in WPS [WRF, p. 3-9], and the Lambert Conformal Conic projection is used for the simulations conducted in this work.

In the ARW solver, relevant physical quantities are specified on the simulation grid. Thermodynamic variables (e.g., potential temperature θ , moisture variables, etc.) are specified on the grid mass points, while velocities are defined on the edges of the grid, i.e., staggered one-half grid length from the mass points as shown in Figure 5.4 [Ska+19, p. 21].

A similar grid with mass and “velocity” points can be designed for the vertical direction. However, the spacing between vertical levels is not constant, as explained below.

Grid nesting It is possible to improve the horizontal spatial resolution over a specific region by defining several grids, called domains. The different domains are

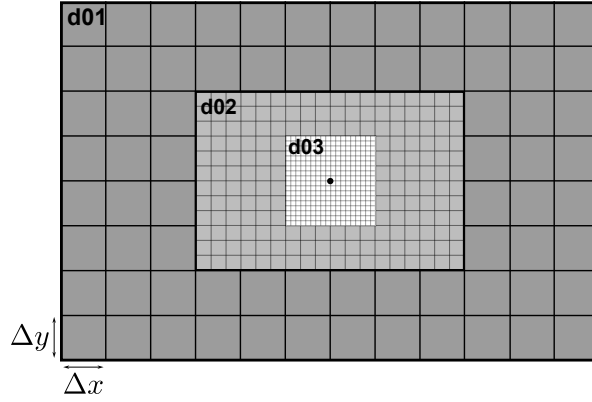


Figure 5.5 – Example of nested domains for improving the spatial resolution close to the location of interest, depicted by the dot at the centre (adapted from [Ska+19]).

nested together, as depicted in Figure 5.5. In this illustration, the larger domain with the coarser resolution is named $d01$, and is referred to as the parent domain for $d02$. The domain $d02$ is the child of $d01$, and is also the parent of $d03$. The inner domain $d03$, i.e., the child of $d02$, has the finest resolution but the smallest size. This practice is called nesting.

Nesting is of practical interest to obtain simulation results with a high spatial resolution around the location of interest, while limiting the computational cost of the simulations. Indeed, with nesting, WRF equations (described in Section 5.2.1.2) are solved in each domain, with the spatial resolution set by the domain. Feedback between parent and child domains during the simulation is then used. All nesting options are described in [Ska+19, p. 57] and [WRF], but the ones relevant in this work are listed below:

- The location of the nested domains inside their respective parents can be specified by the user. Here, nested domains are chosen to be exactly in the middle of their parent domains, all centred on the location of interest. Their location is fixed during the whole simulation (i.e., no moving domain).
- The variation of resolution between nested domains is set by an integer ratio (e.g., $\Delta x_{d01}/\Delta x_{d02} = m$, with $m \in \mathbb{N}_0$). Good practice is to use small odd integer numbers, such as three (e.g., in Figure 5.5). An odd number ensures that grid points in the parent domain are exactly aligned with some points in the child domain.
- One-way feedback is used, meaning that parent domains set the boundary conditions of their inner domains at each iteration in the resolution. However, there is no feedback from the inner domains to their parent (two-way nesting).

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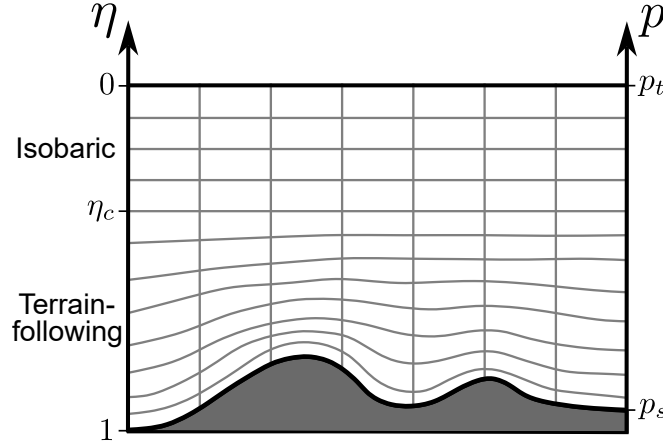


Figure 5.6 – WRF vertical levels - η and pressure coordinates (adapted from [Ska+19]).

Vertical levels η The vertical levels used in WRF are terrain-following pressure levels [Ska+19, p. 7]. Close to the ground, they tend to follow the terrain, while at high altitudes, they correspond to pure pressure coordinates. They are depicted by the variable η , varying between 0 (at the upper boundary of the domain) and 1 (at the surface). The transition between terrain-following levels and pure pressure coordinates arises at a value η_c . An illustration of the vertical η levels is given in Figure 5.6, where p_s (resp. p_t) is the hydrostatic component of pressure of dry air at the surface (resp. at the upper boundary).

Inspired by [Lap92; Par+13], the η coordinate is mathematically defined by

$$p = B(\eta) (p_s - p_t) + (\eta - B(\eta)) (p_0 - p_t) + p_t, \quad (5.2)$$

with p_0 the reference sea-level pressure and p the hydrostatic component of pressure of dry air, varying with altitude. The function $B(\eta)$ is a third-order polynomial that ensures the smooth transition between terrain-following and pure pressure coordinates. It is given by [Ska+19, p. 8]

$$B(\eta) = c_1 + c_2\eta + c_3\eta^2 + c_4\eta^3,$$

with

$$c_1 = \frac{2\eta_c^2}{(1 - \eta_c)^3}, \quad c_2 = \frac{-\eta_c (4 + \eta_c + \eta_c^2)}{(1 - \eta_c)^3}, \quad c_3 = \frac{2(1 + \eta_c + \eta_c^2)}{(1 - \eta_c)^3}, \quad c_4 = \frac{-(1 + \eta_c)}{(1 - \eta_c)^3}.$$

The values of c_1 to c_4 are deduced from the boundary conditions applied to $B(\eta)$: $B(1) = 1$, $\frac{dB}{d\eta}(1) = 1$, $B(\eta_c) = 0$ and $\frac{dB}{d\eta}(\eta_c) = 0$.

For the WRF simulations conducted in this work, the default value of the upper boundary hydrostatic pressure p_t is used (5000 Pa). The automatic option for defining the η coordinate is activated, with a logarithmic stretching close to the surface [WRF,

p. 4-9]. This is illustrated in Figures 5.8 and 5.9 for Redu, Belgium. The number of vertical levels is denoted n_z . Vertical nesting is possible in recent version of WRF, but is not used in this work.

Surface static quantities The topographic description of the surface in the simulation domains is computed by *geogrid*. It relies on a database of static data associated to the Earth's surface, and interpolates relevant quantities to the model grid, such as the terrain height, the soil type, the land use category, etc. [WRF, p. 3-3]. The default geographical dataset for WPS v4 is used².

5.2.1.2 Primitive equations and resolution

The ARW solver is based on Euler equations for compressible flows. As for Navier-Stokes equations, Euler equations are associated with the conservation of mass, momentum and energy in a flow (i.e., the core equations of continuous mechanics), but assuming a null viscosity. Compared to Navier-Stokes equations given in (2.10) and (2.11), the conservation of mass and momentum in Euler equations are, for compressible flows [Stu17, p. 746], [Qui20, p. 48]

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{u}) = 0 \quad (5.3)$$

$$\frac{\partial(\varrho \mathbf{u})}{\partial t} + \nabla \cdot (\varrho \mathbf{u} \mathbf{u}) = -\nabla p + \varrho \mathbf{g} + \mathcal{F}_{\mathbf{u}} \quad (5.4)$$

where the notations introduced in Section 2.2.2 are used. The external gravitational force is explicit in (5.4), with the vector $\mathbf{g} = (0, 0, g)^T$ and g the Earth gravitational acceleration [m/s^2]. External forces $\mathcal{F}_{\mathbf{u}}$ include other forces than gravitation.

Equations (5.3) and (5.4), combined with the equation of energy conservation, provide a set of coupled partial differential equations, called prognostic equations. With the addition of a diagnostic equation linking the energy to the other quantities (ϱ , p , $\mathbf{u}$), such as the ideal gas law, the dynamic evolution of the flow can be solved. As they enable to forecast fundamental (primitive) variables (e.g., p , $\mathbf{u}$, etc.), they are referred to as primitive equations. [Stu17, p. 747]

The application of Euler equations in the ARW solver is detailed in [Ska+19]. They are expressed with the terrain-following pressure coordinate η defined in (5.2), and explicitly include the potential temperature and humidity quantities (mixing ratios of water vapor, cloud water, rain water, etc.). In practice, they are solved for variables defined as perturbations from a hydrostatically-balanced reference state [Ska+19, p. 12].

Spatial and temporal discretization As introduced in Section 5.2.1.1, solving the primitive equations is conducted on a discrete 3D grid with given spacings. Hence,

²Available at https://www2.mmm.ucar.edu/wrf/users/download/get_sources_wps_geog.html, last accessed on 30 August 2024.

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all spatial scales are not resolved and it is necessary to add forcing terms in the equations to accurately model the atmosphere evolution at small and large scales. These terms are added as external forces, similarly to $\mathcal{F}_u$ in (5.4) for example. They arise from model physics, turbulent mixing, spherical projections and the Earth's rotation [Ska+19, p. 10]. Therefore, they are closely tied to the physics parametrization chosen in WRF, described in Section 5.2.1.3.

On top of the spatial discretization, the primitive equations are also discretized in time and solved using a time-split integration scheme (typically, third-order Runge-Kutta scheme for low-frequency modes with acoustic integration for high-frequency modes) [Ska+19, p. 15]. The time step Δt chosen for the temporal discretization must not be too large to avoid numerical instabilities over complex terrains and in high wind conditions. Its maximal value is related to the Courant-Friedrichs-Lewy condition, i.e., $\Delta t < c_r \Delta x / u_{\max}$ with Δx the spatial spacing (in one direction), $u_{\max}$ the maximum velocity in the simulation, and c_r a constant related to the Courant number [Ska+19, p. 28].

In practice, the time step Δt is on the order of seconds to a few minutes. It can be made adaptive to ensure faster time-to-completion of the simulations, especially in stable conditions.

Initial and boundary conditions Numerically solving partial differential equations require initial conditions (at the start of the simulation) and lateral boundary conditions (at the edges of the 3D grid).

For real scenarios, initial conditions come from external analysis or forecasts supplying a grid of meteorological quantities for the region of interest, at the time of interest. The supplied meteorological data are first interpolated to match the model grid's horizontal staggering in WRF (*metgrid*), and then interpolated vertically from pressure levels to eta levels (*real*). For nested domains, initial conditions are computed for each domain. [Ska+19, p. 45]

Regarding the lateral boundary conditions, they are deduced (e.g., interpolated) from external meteorological data provided at the initial time and at later simulation times. They are specified on the edges of the simulation domain (larger domain *d01* only in case of nesting), allowing for small relaxation zones between the edges and inner grid points. Both initial values and tendency terms describing the temporal evolution of the boundary conditions are computed. [Ska+19, p. 54]

In this work, initial and lateral boundary conditions come from the ERA5 database, introduced in Section 5.1.1, as WRF supports integration of ECMWF data.

Spin-up time To allow for convergence in WRF outputs, the simulations are initialized 6 to 12 hours before the time when data is of scientific interest. This time window corresponds to the model spin-up time, also named lead time.

Table 5.1 – WRF physics parameterization (default configuration).

Physical process	Model type (scheme)	Reference
Microphysics	WRF single-moment 6-class (WSM6)	[HL06]
Cumulus physics	Tiedtke (only in domain d01)	[ZWH11]
Shortwave radiation	Dudhia	[Dud89]
Longwave radiation	Rapid radiative transfer model (RRTM)	[Mla+97]
Surface layer	Revised MM5	[Jim+12]
Land surface	Thermal diffusion	[Dud96]
Planetary boundary layer	Mellor-Yamada Nakanishi Niino 2.5 (MYNN 2.5)	[NN09]

5.2.1.3 Physics parameterization

In order to represent sub-grid processes not resolved by the simulation grid, numerous physical schemes are available in WRF [Ska+19, p. 69]. They belong to the different categories:

- Microphysics: explicitly resolves water species (water vapor, precipitation, etc.);
- Cumulus physics: models the formation of cloud at sub-grid scales;
- Shortwave and longwave radiations: model solar radiation (shortwave) and infrared absorption and emission by gases and the Earth's surface (longwave);
- Surface layer physics: describes heat and moisture fluxes from the land surface and surface stresses in the planetary BL;
- Land surface physics: gives soil temperature and moisture;
- Planetary BL physics: models vertical sub-grid fluxes due to eddy transport (e.g., TKE).

For each of the category described above, different schemes can be chosen, as detailed in [Ska+19, p. 69], [WRF], and on the WRF physics website ³. In this work, the schemes presented in Table 5.1 are used. Their choice is motivated based on previous works conducted at UCLouvain, e.g., [Qui20; ROV23] and from good-practice recommendations for WRF. Of particular interest is the MYNN 2.5 scheme used for planetary BL physics, as it enables recording of the TKE in WRF outputs, useful for the C_n^2 model presented in Section 5.3.1.

The parameterization of WRF, i.e., the choice of physical schemes to use, is important as it may lead to different outputs. A sensitivity analysis of WRF outputs for OT forecast depending on the chosen physical schemes can be found in [Raf+22].

³Available at https://www2.mmm.ucar.edu/wrf/users/physics/phys_references.html, last accessed on 30 August 2024.

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Table 5.2 – Relevant WRF 3D output variables.

WRF variable	Description	Units	Dimensions	Symbol
PH + PHB	Geopotential	m^2/s	$(n_z + 1, n_y, n_x)$	ϕ_g
P + PB	Pressure	Pa	(n_z, n_y, n_x)	p
T + 300	Potential temperature	K	(n_z, n_y, n_x)	θ
QVAPOR	Vapor mixing ratio	kg/kg	(n_z, n_y, n_x)	q_v
U	U wind	m/s	$(n_z, n_y, n_x + 1)$	U
V	V wind	m/s	$(n_z, n_y + 1, n_x)$	V
QKE/2	Turbulent kinetic energy	m^2/s^2	(n_z, n_y, n_x)	k_ϵ
CLDFRA	Cloud fraction	/	(n_z, n_y, n_x)	/

Table 5.3 – Relevant WRF 2D output variables.

WRF variable	Description	Units	Dimensions	Symbol
HGT	Surface height	m	(n_y, n_x)	/
T2	Two-meter temperature	K	(n_y, n_x)	T_{2m}
Q2	Two-meter q mixing ratio	kg/kg	(n_y, n_x)	q_{2m}
LH	Latent heat flux	W/m^2	(n_y, n_x)	E
HFX	Sensible heat flux	W/m^2	(n_y, n_x)	H
UST	Friction velocity	m/s	(n_y, n_x)	u_*
PBLH	Planetary boundary layer height	m	(n_y, n_x)	h_{BL}
XLAT	Latitude	$^\circ\text{N}$	(n_y, n_x)	/
XLONG	Longitude	$^\circ\text{E}$	(n_y, n_x)	/
COSALPHA	Map rotation cosine	/	(n_y, n_x)	$\cos(\alpha)$
SINALPHA	Map rotation sine	/	(n_y, n_x)	$\sin(\alpha)$
COSZEN	Cosine of solar zenith angle	/	(n_y, n_x)	/

5.2.1.4 WRF output variables

As outputs, WRF returns various physical quantities, defined on the simulation grid (3D), or on the surface level (2D). A list of all WRF output variables can be found in the Registry file⁴ or directly in the output file headers, as given in [WRF, p. 5-131]. It is possible to remove (or add) some quantities using the “iofields” namelist parameter, as described in [WRF, p. 5-29].

The relevant quantities for this work are presented in Tables 5.2 and 5.3, inspired from [Qui20]. Table 5.2 gives the 3D variables, while Table 5.3 presents the 2D variables, mostly defined at the surface. The staggering of velocities and geopotentials, as introduced in Section 5.2.1.1, explains the extra value in the dimensions of these quantities. Moreover, a time-dimension (n_t) can be also added to account for the temporal evolution of the quantities.

The variables presented in Tables 5.2 and 5.3 are direct (or slightly modified) outputs of WRF simulations. In practice, other useful quantities are derived from the ones listed in both tables, such as:

⁴Available at https://github.com/wrf-model/WRF/blob/master/Registry/Registry.EM_COMMON, last accessed on 30 August 2024.

- The geopotential height h , computed from $h = \frac{\phi_g}{g_0}$, where $g_0 = 9.80665 \text{ m/s}^2$ is the standard Earth gravity. It approximately corresponds to the altitude above mean sea level.
- The temperature T , obtained from the pressure p and the potential temperature θ by inverting (2.19).
- The relative humidity, computed from the pressure p , the temperature T , and the water vapor mixing ratio q_v , using (4.10) to (4.12). The specific humidity q in (4.10) can be approximated by the water vapor mixing ratio q_v [FN08, p. 41].
- The wind components in the north-east directions. Indeed, the U - and V -winds from WRF are relative to the grid, following the projection used. To obtain their north-east components u and v , one needs to compute [Qui20, p. 59]

$$\begin{aligned} u &= U \cos(\alpha) - V \sin(\alpha), \\ v &= U \sin(\alpha) + V \cos(\alpha), \end{aligned}$$

where α , as defined in WRF, is the local rotation angle of the map projection.

The use of the presented relevant variables for C_n^2 modelling is given in Section 5.3, where their influences on C_n^2 profiles are made clear. Regarding the cloud fraction, it is mostly useful to determine if cloudy conditions can be expected, and hence impact the quality of the measurements (see Section 5.4.3 for example). The cosine of solar zenith angle helps to identify daytime to nighttime transitions, and vice-versa.

5.2.2 Example of parameterization for Redu, Belgium

As an example of WRF parameterization, simulations are conducted at Redu (Belgium) for February 14, 2019. Redu is located in the South of Belgium (50°00'06"N, 5°08'46.5"E) at an altitude of about 350 meters, and hosts several ground stations of the ESA's network (at European Space Security and Education Centre - ESEC). In the future, optical communications may be conducted from Redu, hence its interest for optical turbulence modelling. Furthermore, in the framework the Smart Adaptive-optics and Laser guide-star for medium-size Telescopes and Optical communications (SALTO) project [Orb+18; Orb+22], an 80-cm telescope equipped with adaptive optics and a laser guide star has been installed at Redu in 2022. Optical turbulence measurements have been collected in order to characterize the site prior to the installation (in 2019), and are compared with C_n^2 model predictions in Section 5.4.4. In this section, only the parameterization and the simulation inputs-outputs at Redu are illustrated.

The WRF configuration used, referred to as the "default configuration", is given in Table 5.1 for the physics, and in Table 5.4 for the grid parameters. The grid is centred on the location of interest (Redu in this case).

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Table 5.4 – Grid parameters for WRF simulations (default configuration).

Domain	Grid resolution (km)	Number of grid points	Domain size (km)	Number of vertical levels
d01	9	112×112	1008×1008	100
d02	3	112×112	336×336	100
d03	1	112×112	112×112	100

5.2.2.1 Input meteorological data

The necessary meteorological data to initialize WRF simulations (see left part of Figure 5.2) come from two distinct databases of ERA5 data:

- the ERA5 hourly data on pressure levels, including the vertical profiles (over 37 levels) of geopotential, relative humidity, specific humidity, temperature, wind components, and pressure.
- the ERA5 hourly data on single levels, giving the 10-meter wind components, 2-meter temperature and dewpoint temperature, mean sea level pressure, land sea mask, sea ice cover, sea surface and skin temperatures, snow depth, soil temperature and volumetric soil water layers.

By default, meteorological data in a region delimited by the latitude of the site of interest plus and minus 15°, as well as its longitude plus and minus 15°, are downloaded. This is illustrated in Figure 5.7 showing the two-meter temperature at Redu as downloaded from the ERA5 single level database. The location of Redu is highlighted, and the area where data are downloaded corresponds to the coloured region. The initialization time is 18h00 UTC on February 13, 2019.

The input meteorological data are then processed by WPS to interpolate them on the simulation grid, as explained previously in Figure 5.3.

5.2.2.2 Topography and nested domains

Figure 5.8 presents the WRF grid configuration for the simulations conducted at Redu. On the left part of this figure, the three nested domains *d01*, *d02* and *d03*, having the parameters from Table 5.4, are shown. A zoom on the topography of the inner domain is presented in the middle of the figure. This topography has been obtained from the default static geographical data recommended for WRF. Finally, the right part of the figure gives the vertical pressure levels along a slice at the latitude of Redu. The red dot depicts the exact location of Redu.

As explained in Section 5.2.1.1, the vertical levels follow the terrain close to the surface, and become pure pressure levels at higher altitude. This can be seen in Figure 5.8, but is also depicted in Figure 5.9. This figure presents the 3D-grid in the inner domain for a few samples of vertical levels, Figure 5.9a giving the lower levels and Figure 5.9b some upper levels. The horizontal grid resolution Δx and Δy is shown to be constant over the whole domain.

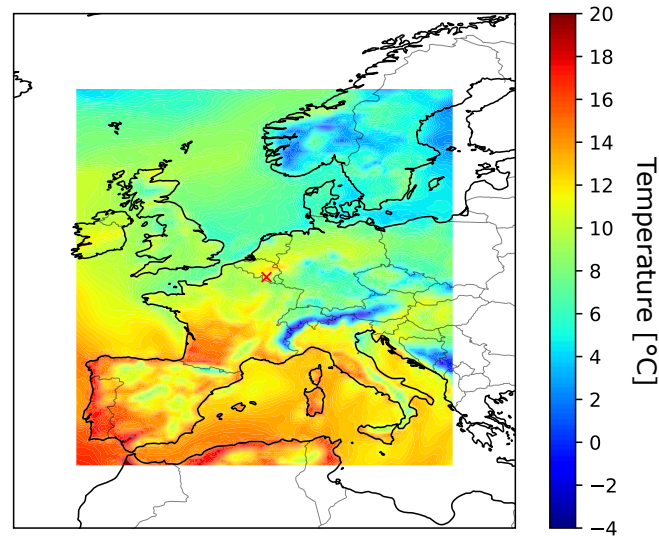


Figure 5.7 – Two-meter temperature on February 13, 2019, at 18h00 UTC, used as input for WRF simulations.

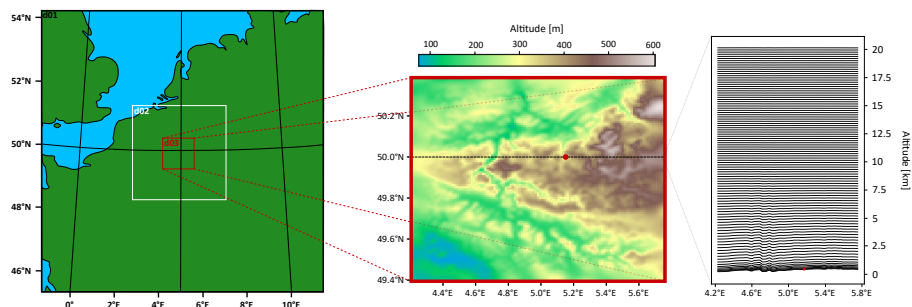


Figure 5.8 – WRF configurations for simulations conducted at Redu, Belgium.

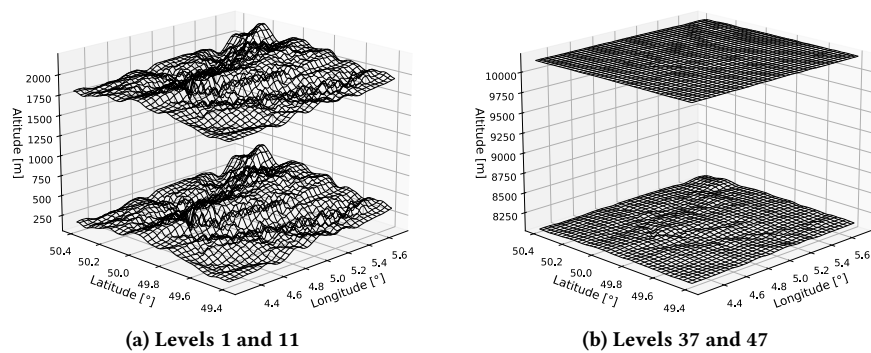


Figure 5.9 – Illustration of vertical levels used in the inner domain (d03) for WRF simulations conducted at Redu.

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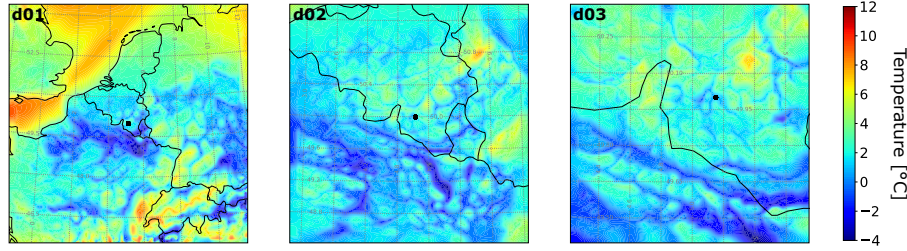


Figure 5.10 – Illustration of two-meter temperature in the three domains for Redu, on February 14, 2019, at 00h00 UTC.

5.2.2.3 Simulated meteorological quantities

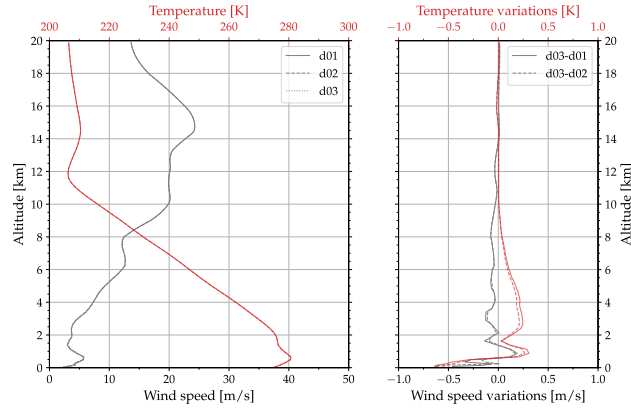
WRF simulation outputs are available starting from the initialization time until the end time of the simulation. In this case, the first outputs correspond to 18h00 UTC on February 13, 2019. However, assuming a 6-hour spin-up time, only the outputs starting from 00h00 UTC on February 14 should be considered for the practical application.

As an example of relevant 2D outputs, the two-meter temperature is given in Figure 5.10. The temperature field from the three nested domains (as shown in Figure 5.8) is represented, highlighting the gain in horizontal resolution between each domain. Indeed, in domain $d03$, smaller and more localized temperature variations are visible, compared to the coarser domains $d01$ and $d02$. Similar plots can be made for all the 2D quantities given in Table 5.3, and for all output times (here, saved every 5 minutes in the inner domain $d03$, and every 15 minutes in $d01$ and $d02$).

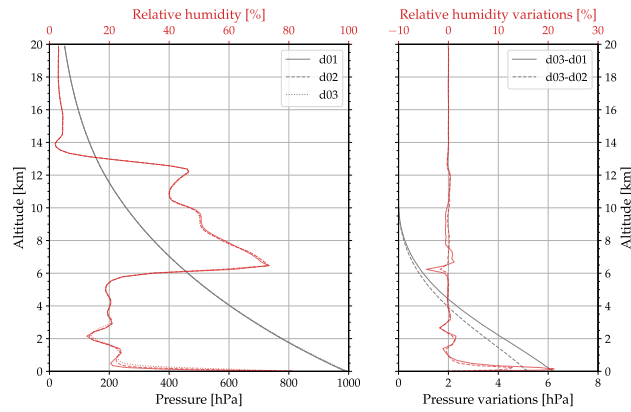
Figure 5.11 illustrates some 3D quantities from Table 5.2, sampled on the vertical levels directly above Redu. In each figure, the left part shows the quantity vertical profile as sampled from $d01$, $d02$ and $d03$ domains. The right part presents the variations between the profiles from the inner domain $d03$ and the other domains, defined as the difference between these profiles. For the TKE in Figure 5.11c, the absolute value of the difference is depicted on a logarithmic scale.

Regarding the temperature and wind speed profiles, the agreement between all domains is strong. More fluctuations are visible in the pressure, relative humidity and TKE profiles. This latter reaches its default minimal value ($5 \times 10^{-5} \text{ m}^2\text{s}^{-2}$) for high altitudes and for different altitudes in the troposphere.

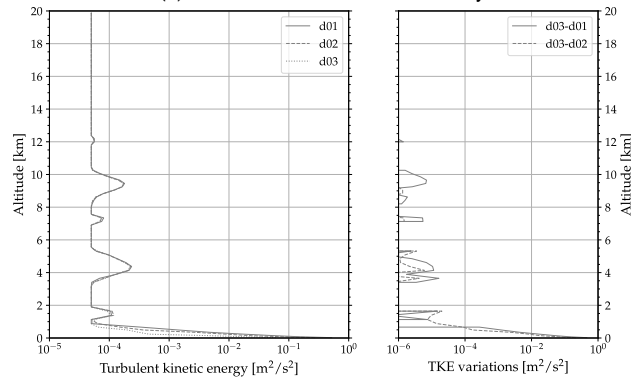
The good agreement between the profiles from different domains comes from the fact that the location of Redu is exactly on a grid point in each domain. Moreover, no vertical nesting is used, hence the same vertical resolution is available for all domains. Nevertheless, larger variations between the domains are visible in the atmospheric BL, i.e., for altitudes below one to two kilometres. This motivates the use of nesting for improving the modelling of the BL, which is particularly relevant to accurately predict the TKE. Indeed, as shown in Figure 5.11c, this quantity can vary from several orders of magnitude between the domain, which can have important consequences on predicted C_n^2 profiles using the model detailed in Section 5.3.1. For high altitude,



(a) Wind speed and temperature



(b) Pressure and relative humidity



(c) Turbulent kinetic energy

Figure 5.11 – Illustration of meteorological quantities vertical profiles in the three domains for Redu, on February 14, 2019, at 00h00 UTC.

5 | Prediction of C_n^2 profiles from meteorological simulations

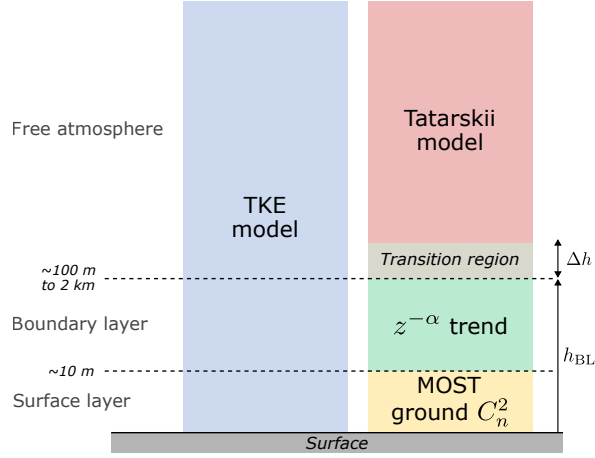


Figure 5.12 – Schematic of C_n^2 models chosen for NWP simulations.

since atmospheric phenomena of larger scales are expected, variations between the nested domains are negligible.

5.3 C_n^2 models suited for NWP simulations

For the second step in Figure 5.2, i.e., the application of a C_n^2 model to translate meteorological quantities to C_n^2 , two different C_n^2 models have been explored. Both models satisfy the requirements presented in Section 5.1.2.2 and describe the whole atmospheric profile, as shown in Figure 5.12.

The first considered C_n^2 model is a parametric and numerical TKE-based model, inspired from [MVB99]. It is presented in Section 5.3.1. The second model is a hybrid model based on the Tatarskii's equation for describing the FA, MOST for computing C_n^2 in the SL, and a decrease of C_n^2 in $z^{-\alpha}$ in the BL. It is further detailed in Section 5.3.2. The explanations of both models in Sections 5.3.1 and 5.3.2 come from [Qua+23a].

5.3.1 TKE-based C_n^2 model

The chosen TKE-based C_n^2 model has been extensively used for OT forecast in astronomy, namely in Astro-Meso-NH [MVB99; ME06; Mas+17]. This work aims at making it suitable for the optical communication application studied here. It also extends the model and results presented in [MVB99; ME06; Mas+17] by rigorously assessing the TKE model performance during daytime conditions, and comparing its performance with the hybrid Tatarskii model.

This model provides the refractive index structure parameter C_n^2 from the tem-

perature structure parameter C_T^2 thanks to (2.70), reminded below

$$C_n^2 = \left(\frac{80 \times 10^{-6} p}{T^2} \right)^2 C_T^2, \quad (2.70)$$

where T is the temperature in kelvin, and p is the pressure in hectopascal. A modified Tatarskii's expression is then used to obtain C_T^2 from the mixing length L and the square of the potential temperature vertical gradient, that is,

$$C_T^2 = 0.58 \phi_3 L^{4/3} \left(\frac{\partial \bar{\theta}}{\partial z} \right)^2, \quad (5.5)$$

with $\bar{\theta}$ representing the grid-averaged value of the potential temperature [Bou+95] and ϕ_3 equal to 0.78 [MVB99]. For stable layers, the mixing length L is associated to the Deardoff length L_D [Dea80], reminded below

$$L_D = \sqrt{\frac{2k_\epsilon}{\frac{g}{\theta_v} \frac{\partial \theta_v}{\partial z}}}. \quad (3.25)$$

As a result, meteorological quantities, such as the pressure p , the temperature T , the mixing ratio q_v , as well as the TKE k_ϵ , computed in WRF simulations can directly be fed into (2.70), (5.5) and (3.25) in order to obtain C_n^2 . As an example, Figure 5.13 illustrates the different quantities involved in the computation of C_n^2 with this model at Paranal, for March 03, 2023, at 16h00 UTC (see Section 5.4.2).

Turbulent layers, i.e., large values in the C_n^2 profile of Figure 5.13f, are associated with large potential temperature gradient values (Figure 5.13c) or large mixing length values (Figure 5.13d). As seen in Figure 5.13b, turbulent layers are associated with peak values of the TKE. For higher altitudes, the TKE tends to reach its minimal value set in the WRF simulations ($5 \times 10^{-5} \text{ m}^2 \text{ s}^{-2}$ by default). To avoid this phenomenon, a calibration of the minimum TKE value depending on the altitude has been presented in [HML11], and is a possible extension for the model given here. In the following, this TKE-based model is referred to as the *TKE model*.

5.3.2 Hybrid Tatarskii model

The hybrid Tatarskii model suggested in this work relies on Tatarskii's equation for the FA part, given by (3.13). It is rewritten below, highlighting that C_n^2 is given by the Tatarskii expression:

$$C_{n,\text{Tat}}^2(z) = a^2 L_0^{4/3} M^2, \quad (5.6)$$

with $a^2 = 2.8$ [BB88], and $M = \frac{dn}{dz}$ given by [CBL08]

$$M = - \frac{80 \times 10^{-6} p}{T \theta} \frac{d\theta}{dz}. \quad (3.14)$$

It involves the pressure p in hectopascal, the temperature T in kelvin, the po-

5 | Prediction of C_n^2 profiles from meteorological simulations

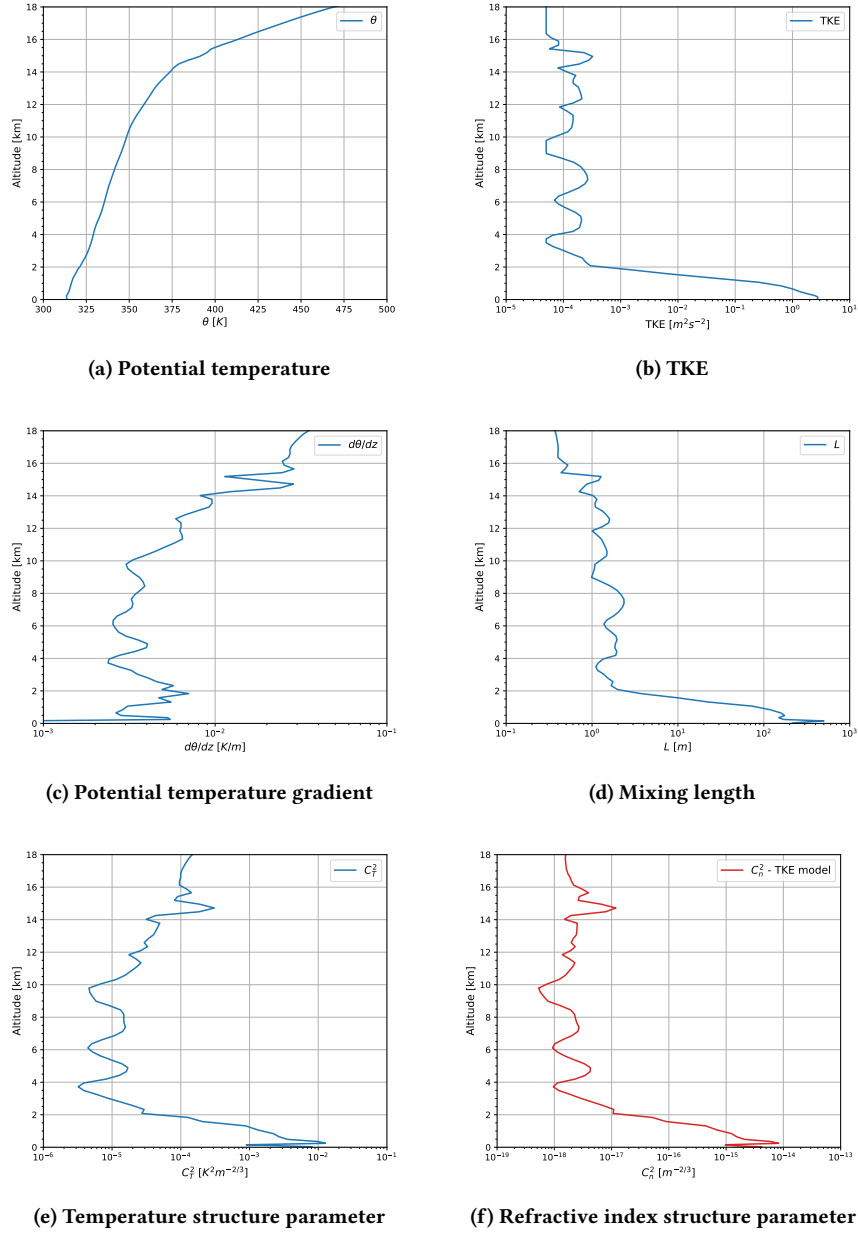


Figure 5.13 – Illustration of the different factors involved in the C_n^2 TKE model (Paranal, for March 03, 2023, at 16h00 UTC).

tential temperature θ in kelvin, the altitude z in meter, and the refractive index n . The chosen outer scale L_0 model is the HMNSP99 model [Dew93; RD02], detailed in Section 3.2.2.1 and given by (3.17) and (3.19).

In (3.19), S is the vertical shear of the horizontal wind velocity, while the tropopause height, that is, the limit between the troposphere and the stratosphere, depends on the location. With the presented models, M and L_0 are functions of the altitude z , such that (5.6) provides a vertical $C_n^2(z)$ profile. This is illustrated in Figure 5.14, giving C_n^2 at Paranal, for March 03, 2023, at 16h00 UTC (see Section 5.4.2). The major influence of the negative temperature gradient on the outer scale from the HMNSP99 model can be observed in Figures 5.14c and 5.14d. At high altitude, even if the wind shear reaches larger values (Figure 5.14b), a positive temperature gradient limits its impact on the outer scale. Peaks in the refractive index gradients (Figure 5.14e) are associated to peaks in the C_n^2 profile (Figure 5.14f).

This Tatarskii model has previously been compared with T-REX radiosonde measurements in [QVO23b], as described in Section 4.3 of Chapter 4. However, it has been found to underestimate C_n^2 in the BL, motivating the use of a hybrid C_n^2 model with a separated parametrization for the surface and boundary layers [QVO23a].

The BL C_n^2 profile is derived from MOST (Section 3.3.1.2), giving an estimation of the ground C_n^2 from

$$C_n^2 = A^2 C_T^2 + B^2 C_q^2 + 2ABC_{Tq}, \quad (3.46)$$

where A is equal to $79 \times 10^{-6} \frac{p}{T^2}$, and B to -56.4×10^{-6} [And88]. C_T^2 , C_q^2 , and C_{Tq} , are obtained from the temperature scaling parameter T_* and the humidity scaling parameter q_* , given in (3.42) to (3.44), where the similarity functions $g_T(\zeta)$, $g_q(\zeta)$, and $g_{Tq}(\zeta)$ are assumed to be equal and given by (3.45). The stability parameter ζ is chosen as [ULL+19]

$$\zeta = \frac{z}{L_{Ob}} = -\frac{0.4 g H z}{1.08 c_p u_*^3 T}.$$

The temperature scaling parameter T_* and the humidity scaling parameter q_* depend on the friction velocity u_* , the sensible heat flux H , the latent heat flux E , the density of air ρ , the specific heat capacity of air at constant pressure c_p , and the latent heat of vaporization of water l_v , through (3.54) and (3.55) [ULL+19].

The quantities u_* , H , and E , can be included in WRF simulation outputs, such that T_* and q_* can be computed.

As a result, Equation (3.46) evaluated at $z = 2$ meters provides the ground C_n^2 , denoted $C_{n,gr}^2$. From the C_n^2 ground value, a BL C_n^2 profile is computed up to the BL height h_{BL} using a vertical trend of $z^{-\alpha}$, as introduced in Section 3.3.2. During daytime, $\alpha = 4/3$ and this trend corresponds to the decrease of C_n^2 in the atmospheric BL dominated by convection. During nighttime, α is set to $2/3$. The implications of the α exponent are further discussed in Section 5.4.2.4. The BL height h_{BL} is a WRF simulation output. The continuity between this BL model and the Tatarskii model is ensured thanks to a linear interpolation, from h_{BL} up to $h_{BL} + \Delta h$. The final C_n^2 model,

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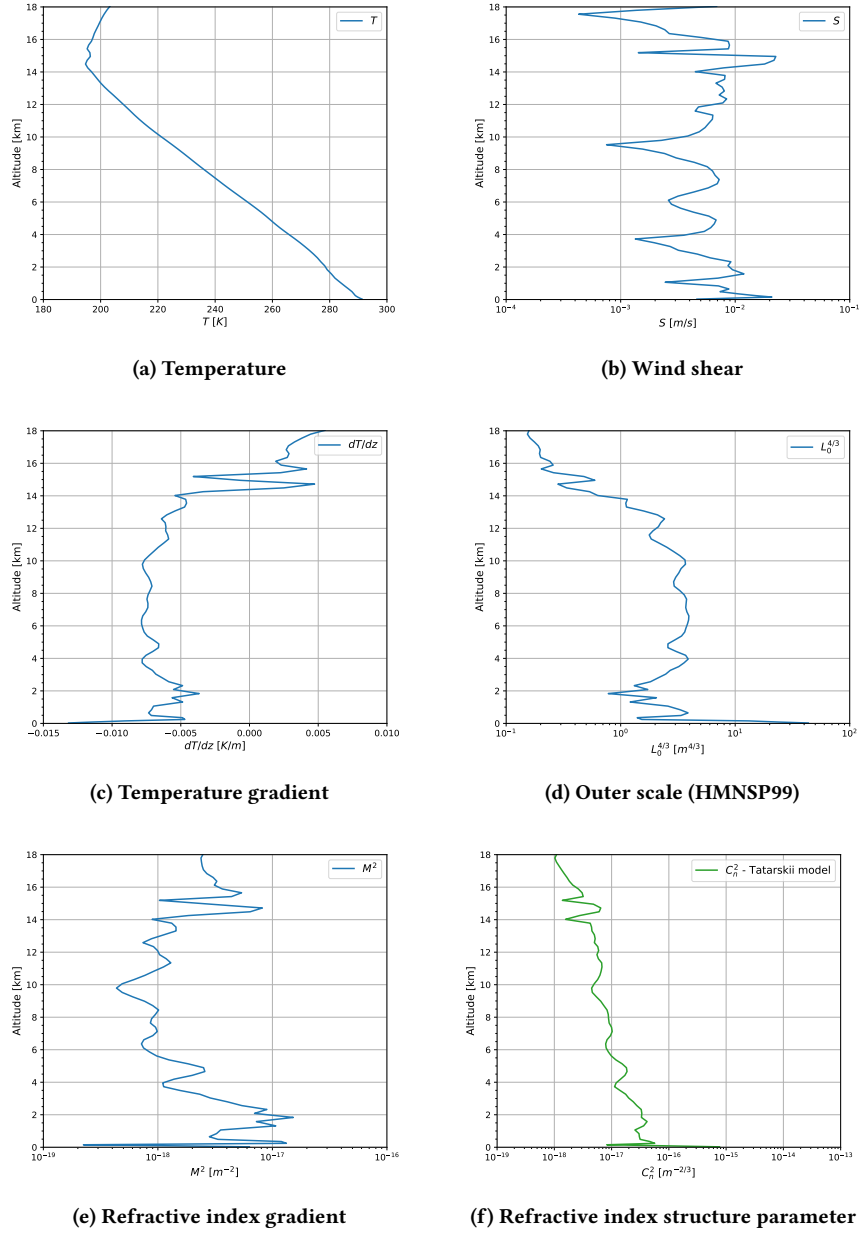


Figure 5.14 – Illustration of the different factors involved in the Tatarskii C_n^2 model (Paranal, for March 03, 2023, at 16h00 UTC).

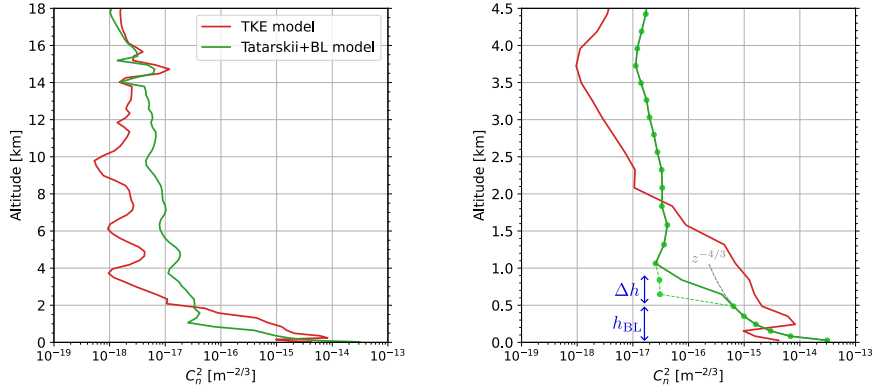


Figure 5.15 – Modelled C_n^2 profiles from TKE and Tatarskii+BL models (Paranal, for March 03, 2023, at 16h00 UTC). [Qua+23a]

referred to as *Tatarskii+BL* model, is given by [Qua+23a; QVO23a]

$$C_n^2(z) = \begin{cases} C_{n,gr}^2 \cdot \left(\frac{z}{2}\right)^{-\alpha} & \text{for } z \leq h_{BL}, \\ C_{n,gr}^2 \cdot \left(\frac{h_{BL}}{2}\right)^{-\alpha} \frac{h_{BL} + \Delta h - z}{\Delta h} + C_{n,Tat}^2(z) \frac{z - h_{BL}}{\Delta h} & \text{for } h_{BL} < z \leq h_{BL} + \Delta h, \\ C_{n,Tat}^2(z) & \text{for } z > h_{BL} + \Delta h. \end{cases} \quad (5.7)$$

In the following, a default value of $\Delta h = 0.2 h_{BL}$ is used. This choice is motivated by the little influence of h_{BL} and Δh that has been observed on the predicted coherence length at Paranal, in Section 5.4.2.

Figure 5.15 illustrates the modelled C_n^2 profiles for March 03, 2023, at 16h00 UTC, above the Paranal observatory (see Section 5.4.2). For comparison, C_n^2 from the TKE model, given in Figure 5.13f, is shown. The right part of the figure focuses on the BL, with $h_{BL} = 550$ m and $\Delta h = 300$ m (set for illustration purposes). The dots and dashed lines highlight the interpolation between the upper part coming from the Tatarskii model (Figure 5.14), and the lower part coming from the BL model. The smooth vertical trend in $z^{-4/3}$ in the BL can also be identified, further illustrated by the dashed gray line. The left part of the figure highlights a difference of one order of magnitude between the two C_n^2 models in the FA, that is also observed in Section 5.4.1 for T-REX thermosonde measurements.

5.4 Application and validation of C_n^2 profile prediction with WRF

The general approach described in Section 5.1, with the WRF parametrization given in Section 5.2 and both C_n^2 models detailed in Section 5.3, is applied to several locations where OT measurements are available. This enables to validate its OT prediction performance.

5 | Prediction of C_n^2 profiles from meteorological simulations

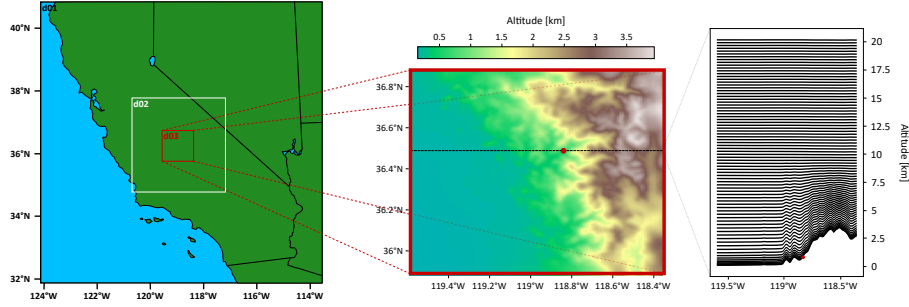


Figure 5.16 – WRF configurations for simulations conducted for T-REX campaign in Three Rivers, CA, USA. [Qua+23a]

First, the approach is applied to T-REX thermosonde measurements in Section 5.4.1, in order to compare predicted C_n^2 profiles with measured profiles. It is then applied to locations where integrated turbulence parameters have been obtained, such as Paranal (Section 5.4.2), Tenerife (Section 5.4.3), Redu (Section 5.4.4), and Louvain-la-Neuve (Section 5.4.5).

Most locations in this section have been chosen based on the availability of OT measurements, in relevant conditions for optical communication applications (e.g., daytime, non-astronomical sites, etc.). In locations where simultaneous measurements of meteorological quantities were obtained, these quantities are used for assessing the validity of WRF simulation outputs, prior to the application of C_n^2 models.

5.4.1 Validation with T-REX radiosonde and thermosonde measurements

Similarly to the validation conducted in Section 4.4 of Chapter 4, the T-REX thermosonde measurements, detailed in Appendix A, enable to compare predicted and measured C_n^2 profiles. All 15 flights have been exploited, with 8 being associated to nighttime conditions and 7 to daytime conditions (see Table A.1 of Appendix A for the exact launch times of T-REX thermosondes). The measured profiles have been binned using a vertical distance of 100 meters for the comparison with the modelled profiles. The results presented in this section come from [Qua+23a].

5.4.1.1 WRF configuration

The default configuration of WRF for this work is used, as given in Tables 5.1 and 5.4. The location of the nested domains, the ground topography, and an illustration of the vertical levels, can be seen in Figure 5.16.

5.4.1.2 Meteorological quantities

As a first step, a comparison between the measured meteorological parameters and those simulated using WRF is conducted. The simulations are initialized at least

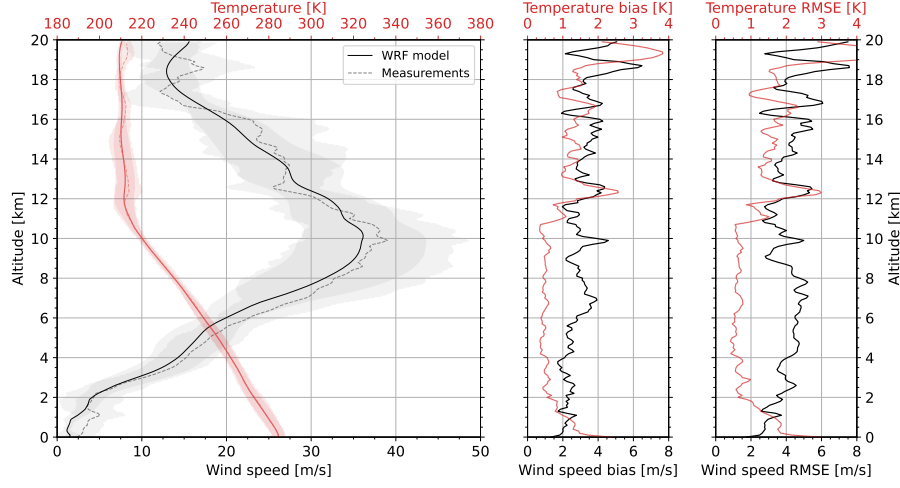


Figure 5.17 – Average temperature and wind speed profiles for T-REX campaign (left), and associated absolute bias and RMSE (right). [Qua+23a]

6 hours before the radiosonde launch time, and outputs are recorded at the same instants as the radiosonde launches.

The left part of Figure 5.17 illustrates the average temperature and wind speed profiles for the 15 T-REX flights, derived from either radiosonde measurements or WRF simulations. Individual profiles can be seen in Figures A.3 and A.4 of Appendix A. Shaded regions in Figure 5.17 represent the average profiles plus or minus their standard deviations, with the shaded areas delimited by dashed lines associated to the measurements. Good agreement is achieved, especially for the temperature in the troposphere (below ~ 10 km of altitude), which is important for further C_n^2 modelling. Wind speed fluctuations in the measurements are observed, and partially modelled in WRF simulations. Those agreements are quantified on the right part of Figure 5.17 using the absolute bias and the RMSE defined in Section 3.5 of Chapter 3:

$$\text{ABSOLUTE BIAS}(z) = \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} |Y_i(z) - X_i(z)| \quad (3.58)$$

$$\text{RMSE}(z) = \sqrt{\frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} (Y_i(z) - X_i(z))^2}, \quad (3.59)$$

with N being the number of measurements ($N = 15$), X_i the i -th measurement of the temperature (resp. wind speed) profile, and Y_i the modelled temperature (resp. wind speed) profile. They show that most temperature differences are located close to the ground and at high altitude. The error on the wind speed profile is roughly constant in the troposphere, and starts to increase for altitudes higher than 12 km.

Overall, the prediction of meteorological quantities from the WRF simulations is

5 | Prediction of C_n^2 profiles from meteorological simulations

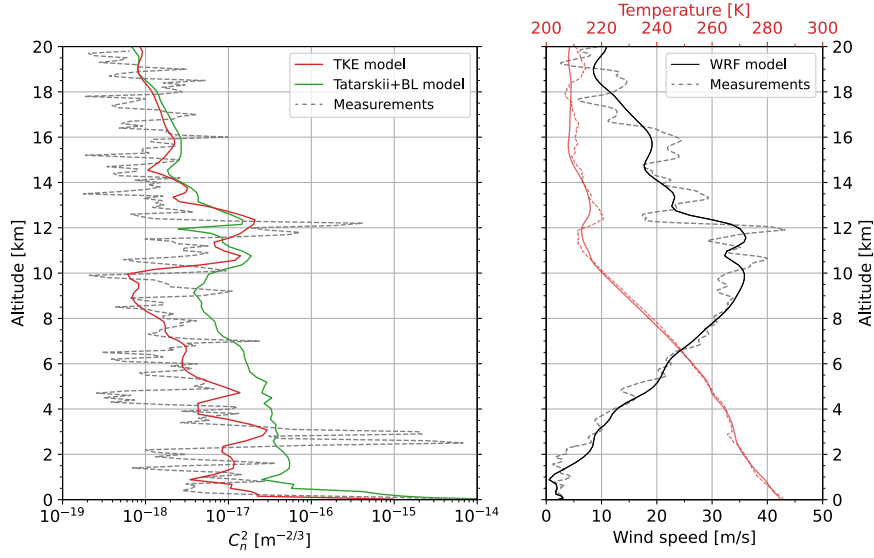


Figure 5.18 – C_n^2 , temperature, and wind speed profiles on March 31, 2006 at 00h48 UTC (T-REX019). [Qua+23a]

satisfactory when compared with the radiosonde measurements, allowing for further modelling of C_n^2 .

5.4.1.3 Validation with C_n^2 profiles

Both C_n^2 profiles presented in Section 5.3 are applied to the predicted meteorological quantities from the WRF simulations. In the Tatarskii+BL model, the tropopause height is set to 12 km as it approximately corresponds to the altitude where the average temperature profile ceases to decrease with altitude (Figure 5.17).

As an example of individual profile, Figure 5.18 depicts the temperature and wind speed measurements and forecasts for the T-REX019 flight (right part). On the left part of the figure, a comparison between the modelled C_n^2 profiles and the measured one is conducted. All individual profiles can be found in Appendix A.

In Figure 5.18, the TKE model accurately captures the overall shape and magnitude of the C_n^2 profile, while the Tatarskii+BL model tends to overestimate the turbulence in the troposphere. Additionally, two large turbulent layers corresponding to temperature inversion layers are observable in the measurements. They are also modelled by the TKE model, even though their magnitudes underestimate the measured values.

The left part of Figure 5.19 presents the median of all 15 profiles⁵. The shaded

⁵Compared to Section 4.4 in Chapter 4, the flight T-REX039 is not excluded from the analysis as it presents interesting features well-modelled by the TKE model (Figure A.4e in Appendix A). However, its large measured C_n^2 values have a dominant effect on the measured average C_n^2 profile, hence the choice of the median in Figure 5.19. Nevertheless, its impact is still present in the absolute bias and RMSE metrics depicted in the right part of the figure.

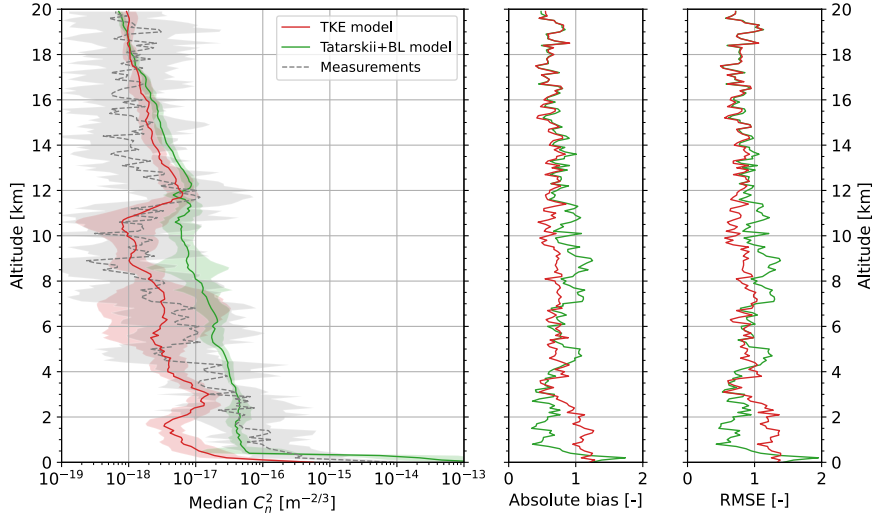


Figure 5.19 – Median C_n^2 profiles over the 15 T-REX flights (left), and associated absolute bias and RMSE (right).

areas represent the median plus or minus the standard deviation, computed from the base-10 logarithm of the C_n^2 profiles. Furthermore, the right part of the figure gives the absolute bias and RMSE using (3.58) and (3.59) applied to $\log_{10}(C_n^2)$.

Median trends are well described by the TKE model, although it appears to underestimate the turbulence below 5 km of altitude. On the contrary, the Tatarskii+BL model overestimates C_n^2 in the BL.

As depicted by the shaded areas in Figure 5.19, both models exhibit smaller standard deviations than the measurements, except for the TKE model around 7 km of altitude. It is associated to large C_n^2 values recorded during the T-REX039 flight and a peak in the C_n^2 profile (see Appendix A). While this peak is accurately captured by the TKE model, it is overlooked by the Tatarskii+BL model. The TKE model also identifies other localized turbulent layers, despite underestimating their magnitude, as previously observed in Figure 5.18.

Analysis of the absolute bias and RMSE metrics supports the observation that most discrepancies occur near the ground and at the tropopause altitude (approximately 10 km). Above 15 km, both models converge to similar average C_n^2 values. The average absolute bias (respectively RMSE) over all altitudes is 0.71 (resp. 0.86) for the TKE model, and 0.75 (resp. 0.90) for the Tatarskii+BL model.

5.4.1.4 Sensitivity analysis to the number of vertical levels used in WRF simulations

To conclude the comparison between T-REX thermosonde measurements and the modelled profiles, a study of the impact of the number of vertical levels used in the WRF simulations is conducted. In this case, the centered absolute bias and centered

5 | Prediction of C_n^2 profiles from meteorological simulations

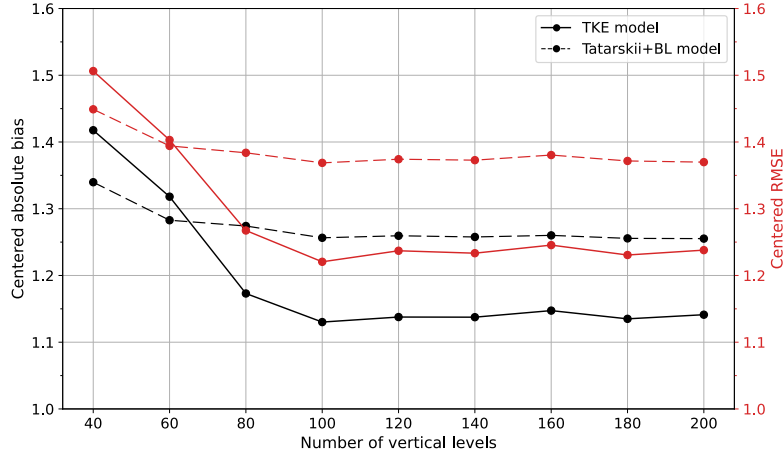


Figure 5.20 – Evolution of centered absolute bias and RMSE with the number of WRF vertical levels. [Qua+23a]

RMSE on $\log_{10}(C_n^2)$ are monitored. Here, centering involves removing the mean C_n^2 profile over the 15 flights before calculating the absolute bias and the RMSE. This eliminates the average bias between the measured and modelled profiles that does not vary with the number of vertical levels. Consequently, the centered metrics provide insights into how small C_n^2 fluctuations with altitude are better represented with the inclusion of more vertical levels, disregarding the differences in the average profiles.

The evolution of these two metrics, averaged across all altitudes from 0 to 20 km, is depicted in Figure 5.20, for varying numbers of levels (with a larger number of levels corresponding to longer computation times). The figure suggests that there is a limit to the achievable decrease of centered absolute bias and RMSE by increasing the number of vertical levels, observed to start around 80 to 100 levels. This is particularly true for the TKE model, while the decrease is less significant for the Tatarskii+BL model. This result motivates the choice of 100 vertical levels used in the default WRF configuration in this work. However, such findings are currently limited to the T-REX campaign case (15 flights), and should be subject to further investigation for other locations. Additional measurements and metrics need to be analysed to determine which atmospheric regions benefit most from an increase in the number of vertical levels.

5.4.2 Comparison with continuous measurements of Fried parameter at Paranal

In order to validate further the NWP simulations and the applied C_n^2 models, especially in the BL, comparisons with Fried parameter measurements collected at Paranal observatory (24.62615°S, 70.40387°W, altitude of 2625 m) have been conducted.

The measurements were carried out over a period of six days from 28th February

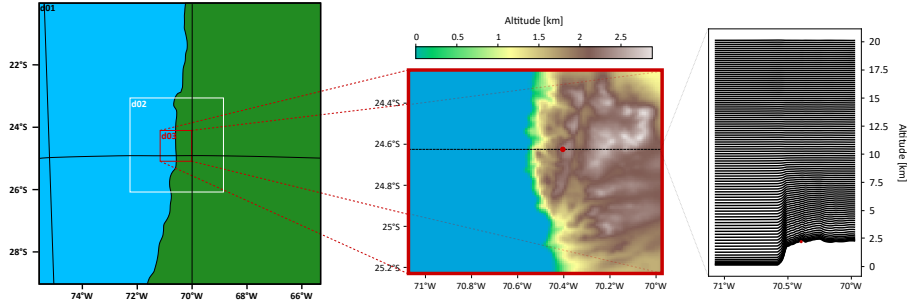


Figure 5.21 – WRF configurations for simulations conducted at Paranal, Chile. [Qua+23a]

to 5th March 2023, using a 24-hour SHIMM (24hSHIMM) [Gri+23]. The instrument was developed by Durham’s Centre for Advanced Instrumentation, and its working principle is presented in Chapter 7. The original aim of the measurement campaign was to compare several new generation OT monitoring instruments [Gri+24].

The instrument was mounted on a concrete pier with no dome, approximately 2 meters above the ground. Instead of the InGaAs camera used in the standard design, the system was operated with a ZWO ASI174MM CMOS camera. As a result, the effective wavelength was determined to be approximately 695 nm, but all measured parameters have been corrected to zenith and for a wavelength of 500 nm.

In this section, only measurements of the Fried parameter r_0 , derived from low resolution C_n^2 profiles estimated by the 24hSHIMM, are utilized. Such continuous measurements are of great interest to validate the daytime and nighttime applications of the presented models. Most of the results presented in this section comes from [Qua+23a].

5.4.2.1 WRF configuration

For the WRF simulations at Paranal, the default configuration of WRF for this work is used (given in Table 5.1 and 5.4). The location of the nested domains, the ground topography, and an illustration of the vertical levels, can be seen in Figure 5.21.

A six-hour lead time is ensured by initializing the simulations with the data at 18h00 UTC the day before, and the meteorological outputs are recorded every 5 minutes from midnight to midnight the next day. No validation of the predicted meteorological quantities have been conducted due to a lack of simultaneous weather measurements.

5.4.2.2 Time series of measured and predicted Fried parameters

From modelled C_n^2 profiles above the location of Paranal, the Fried parameter is computed using its analytical expression in the case of plane waves, detailed in Sec-

5 | Prediction of C_n^2 profiles from meteorological simulations

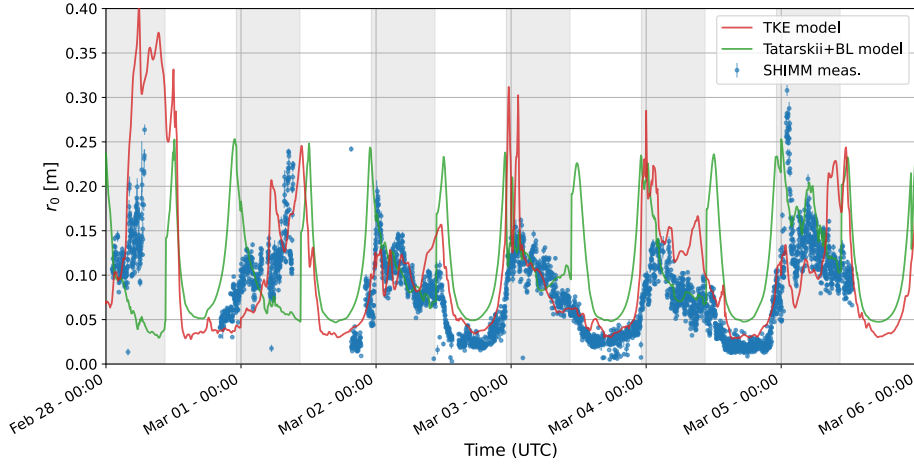


Figure 5.22 – Continuous measurements of Fried parameter at Paranal for 6 consecutive days, compared with the prediction from each C_n^2 model. [Qua+23a]

tion 6.3.1 [Rod81]

$$r_0 = \left(0.423 k^2 \sec(\tilde{\vartheta}) \int_{h_0}^{\infty} C_n^2(z) dz \right)^{-3/5}, \quad (5.8)$$

where $\tilde{\vartheta}$ is the elevation angle measured from zenith ($\tilde{\vartheta}=0^\circ$ in the following), and $k = 2\pi/\lambda$ is the wavenumber (λ is the wavelength, $\lambda=500$ nm in this case). Finally, h_0 is the starting altitude for the C_n^2 profile integration, chosen as the first level in WRF simulations, that is approximately located 20 meters above the ground. This leads to two time series of the predicted Fried parameter, simulated day by day, one for each C_n^2 model.

As seen from (5.8), the Fried parameter is dominated by large value of C_n^2 in its profile, arising mostly in the BL. As a result, Fried parameter measurements are useful to assess the validity of the BL profiles from both C_n^2 models. From observations of the average temperature profile at Paranal, the tropopause altitude is set to 14 km for the Tatarskii+BL model.

Figure 5.22 presents the temporal evolution of the measured Fried parameter r_0 using blue dots. Its associated uncertainty is depicted by the vertical blue bars. The red curve gives the forecast r_0 parameter from the TKE model, while the green curve corresponds to the forecast using the Tatarskii+BL model. Gray areas depict nighttime at Paranal, defined by civil twilight.

In Figure 5.22, the 24h variation of r_0 is clearly identified in the measurements. Indeed, during daytime, r_0 is low (as low as 2 cm), while it is larger in nighttime, when it can reach up to 30 cm. This nighttime-daytime variation is also modelled by both C_n^2 models, as explained below. Moreover, a small discontinuity in the r_0 predictions is sometimes observed at midnight UTC, as it corresponds to the transition between

two consecutive WRF simulations.

TKE model The TKE model is in good agreement with the measurements (nighttime and daytime), even though it suffers from oscillations not observed in the measurements.

Figure 5.23 presents some intermediate quantities, useful in the interpretation of r_0 predictions from the TKE model. It shows the C_n^2 profile integrated only in the BL (assumed from 0 to 2 km above the ground), i.e., $\int_0^{2000} C_n^2(z) dz$, as well as the TKE profile integrated in this same layer. It highlights the high values of the TKE in the BL during daytime, leading to large BL C_n^2 , and hence low r_0 values. The daytime-nighttime transitions in the predicted TKE is also noticeable in the middle figure.

The planetary BL height, useful for the Tatarskii+BL model, is shown in the bottom part of Figure 5.23.

Tatarskii+BL model The Tatarskii+BL model especially fits with measurements during nighttime. As seen from the distributions in Figure 5.26, it tends to predict realistic nighttime values, but not always at the right instants.

For daytime, the Tatarskii+BL model tends to overestimate r_0 (hence underestimating daytime turbulence in the C_n^2 profile). This overestimation can be mitigated by modifying the exponent α in (5.7). For example, a daytime exponent of 1, instead of 4/3 as detailed in Section 5.3.2, has been found to visually improve the agreement with the measurements during daytime. A quantitative study of the daytime exponent variation is conducted in Section 5.4.2.4.

Furthermore, the Tatarskii+BL model exhibits peaks in the predicted r_0 values during daytime to nighttime shifts. These peaks corresponds to abrupt declines in ground C_n^2 , coming from reductions in C_T^2 ground values during day to night transitions (refer to the first and second graphs in Figure 5.24). The decrease in C_T^2 originates from variations in the sensible heat flux H , transitioning between negative and positive values throughout the day and night (and vice-versa), ultimately reaching very low levels (in magnitude). As a result, the temperature scaling parameter T_* , expressed in (3.54), also diminishes significantly, thereby lowering C_T^2 through (3.42). This is depicted in the third graph in Figure 5.24. On the contrary, the impact of C_q^2 , and thus of q_* , on C_n^2 is rather limited. Similar day-to-night transitions in ground C_n^2 are also observed in [Ull+19].

5.4.2.3 Distributions and correlations

Binning r_0 measurements in 5-minute bins enables to compare forecast r_0 with the measurements at the same instants. This leads to the correlation plots presented in Figure 5.25, where dark dots are associated to daytime, while light squares correspond to nighttime. The distributions of both measurements and predictions are illustrated in Figure 5.26, segmented according to daytime and nighttime conditions.

5 | Prediction of C_n^2 profiles from meteorological simulations

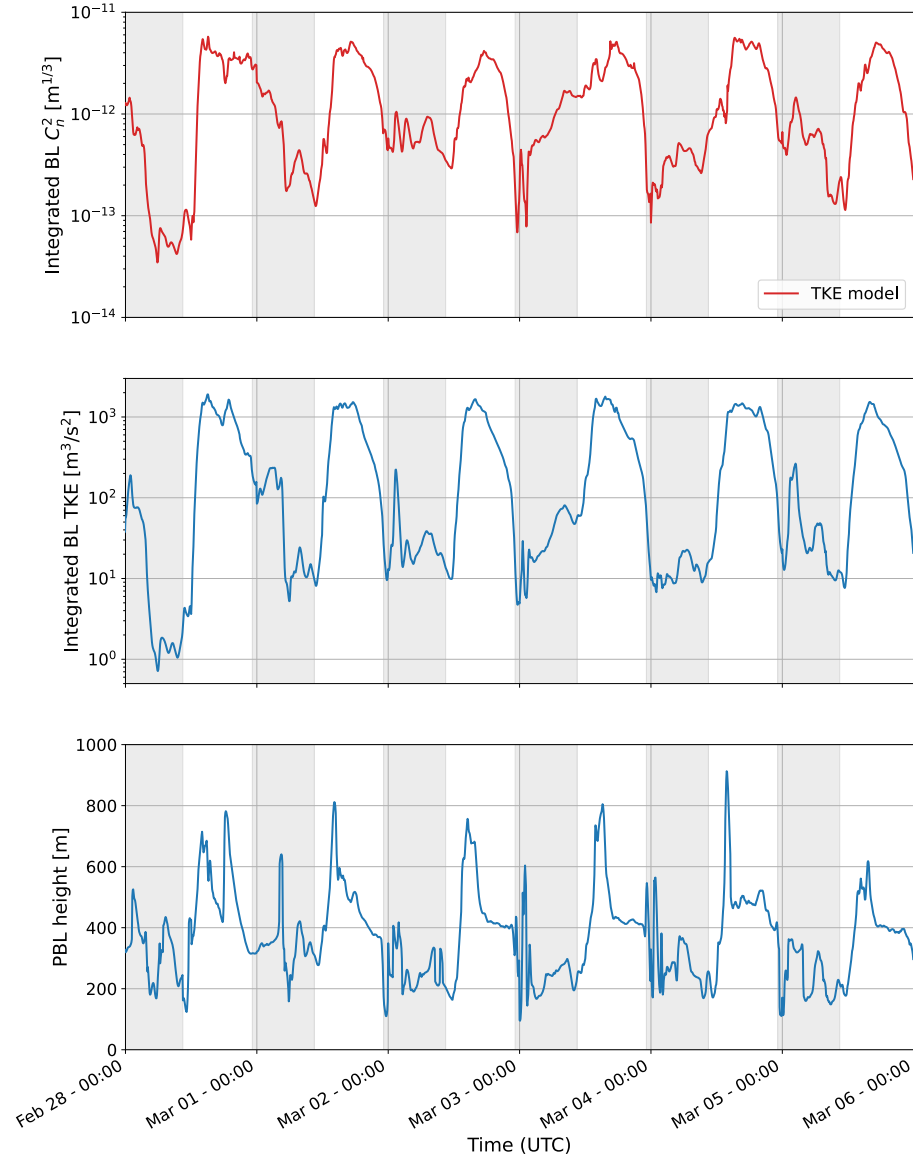


Figure 5.23 – Time series of C_n^2 from TKE model integrated in the boundary layer, TKE integrated in the boundary layer, and planetary boundary layer (PBL) height.

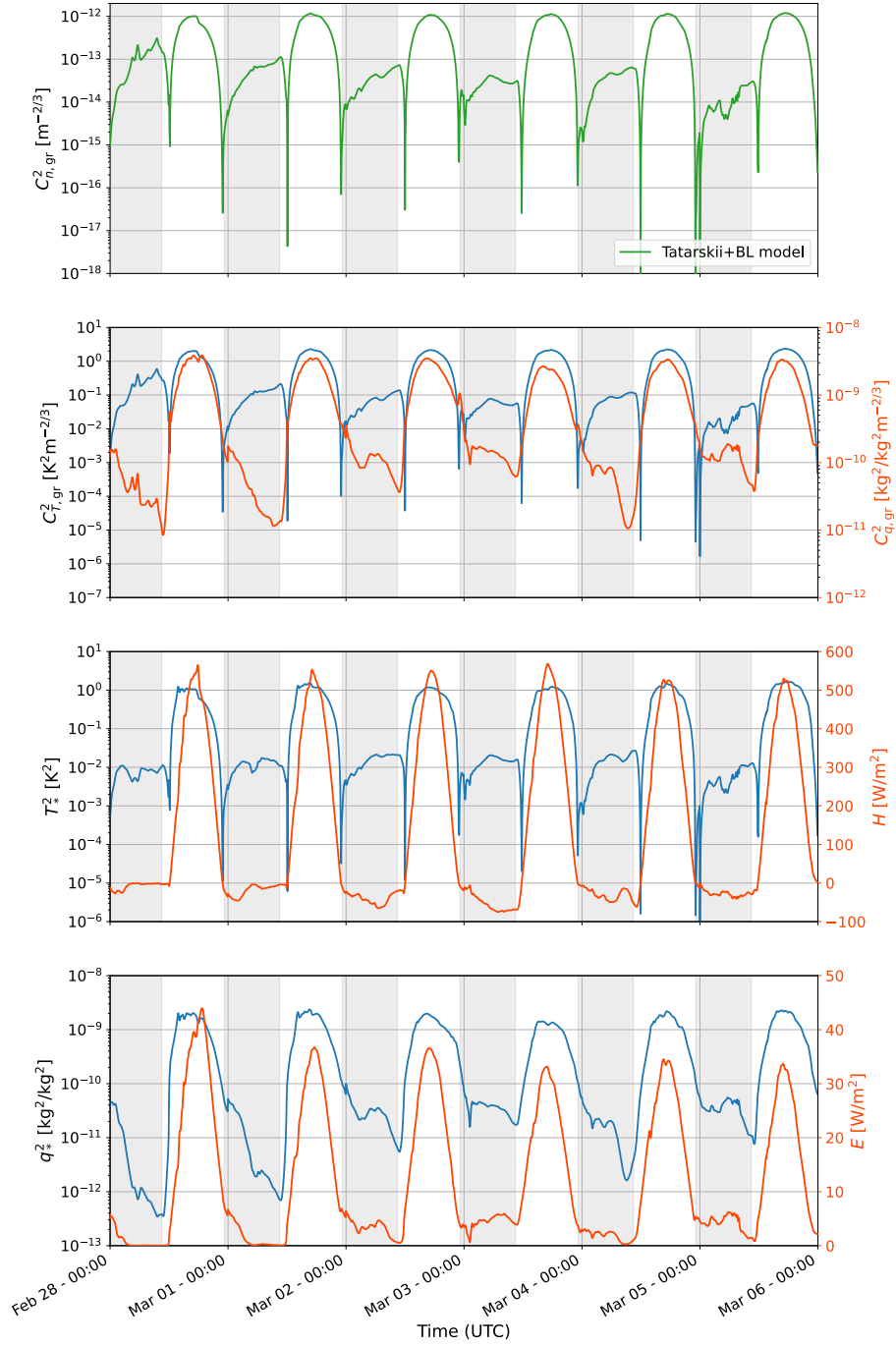


Figure 5.24 – Time series of ground C_n^2 from Tatarskii+BL model, ground C_n^2 and C_q^2 , temperature and humidity scaling parameters T_* and q_* , and sensible and latent heat fluxes H and E .

5 | Prediction of C_n^2 profiles from meteorological simulations

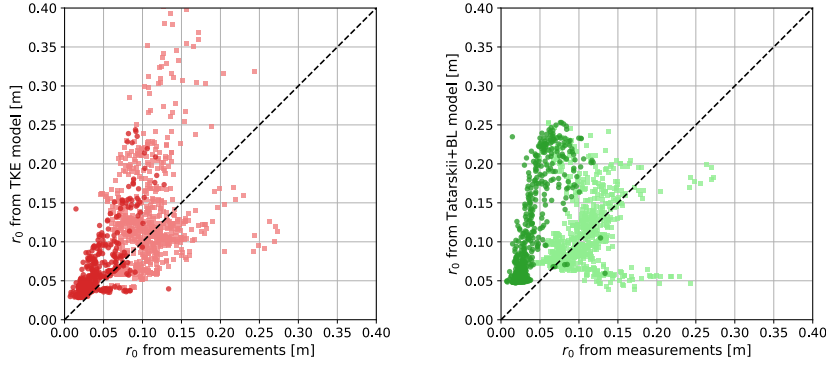


Figure 5.25 – Correlation between measured and predicted Fried parameters at Paranal (left: TKE model, right: Tatarskii+BL model; dark dots: daytime, light squares: nighttime). [Qua+23a]

TKE model Regarding the TKE model, it demonstrates strong prediction capabilities for low r_0 values, as shown by the correlation plot. A wider spread is observed for large r_0 values. A correlation coefficient of 0.59 is obtained between the 5-minute measurements and predictions, with a corresponding RMSE of 5.8 cm.

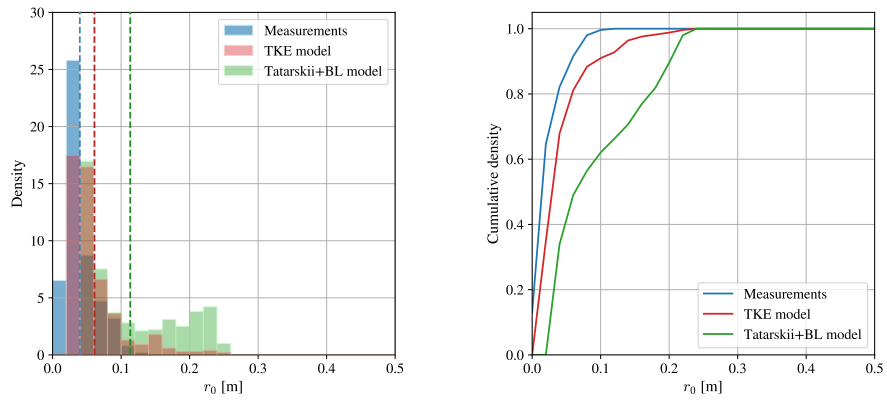
However, considering only daytime measurements and forecast, the correlation coefficient reaches 0.75, with a RMSE of 3.4 cm. The daytime distribution in Figure 5.26a shows good agreement with the distribution of the measurements. It also emphasizes the lack of small predicted r_0 values from the TKE model, hence shifting the distribution slightly to the right.

For nighttime conditions, the correlation coefficient is 0.22, and the RMSE is 7 cm. This large RMSE comes from the overestimation of r_0 during the night, especially for February 28. This also impacts the tail of the distribution in Figure 5.26b.

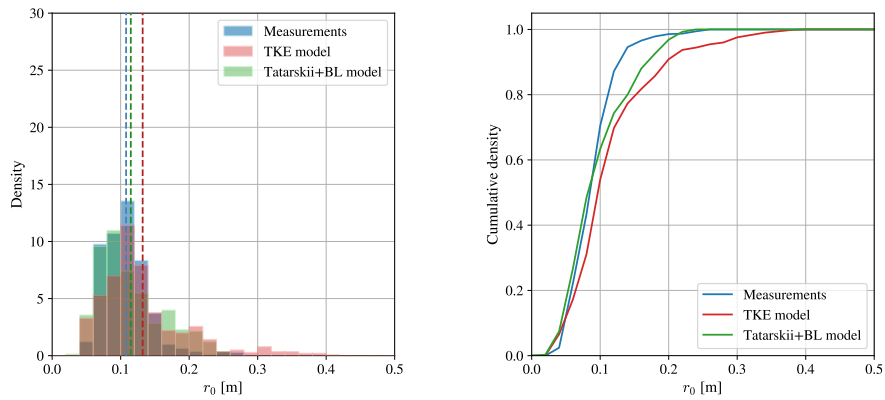
The overestimation of the Fried parameter during daytime (associated to the underestimation of the C_n^2 profile especially in the BL), as well as the large peaks in r_0 during nighttime in Figure 5.22, are related to an underestimation of the TKE in the BL. This is visible in Figure 5.23, for the night of February 28 for example, where low TKE leads to low C_n^2 , and thus high r_0 values. Hence, further investigations should focus on improving the parameterization of the BL in WRF, but require access to meteorological data not available for this study. Model improvements by modifying the SL parametrization in the NWP software have been demonstrated in the literature [LMH10].

Tatarskii+BL model In the correlation plot of the Tatarskii+BL model (Figure 5.25), two main clusters can be identified, related to daytime or nighttime predictions: the one following the equality line (nighttime), and a cluster of points overestimating r_0 , located to the left of the equality line (daytime). The global correlation coefficient is 0.24, while the RMSE is 7.2 cm.

In daytime only, the correlation coefficient is 0.76, showing that the daytime vari-



(a) Daytime distributions



(b) Nighttime distributions

Figure 5.26 – Distributions (density and cumulative density) of measured and predicted Fried parameters at Paranal. [Qua+23a]

5 | Prediction of C_n^2 profiles from meteorological simulations

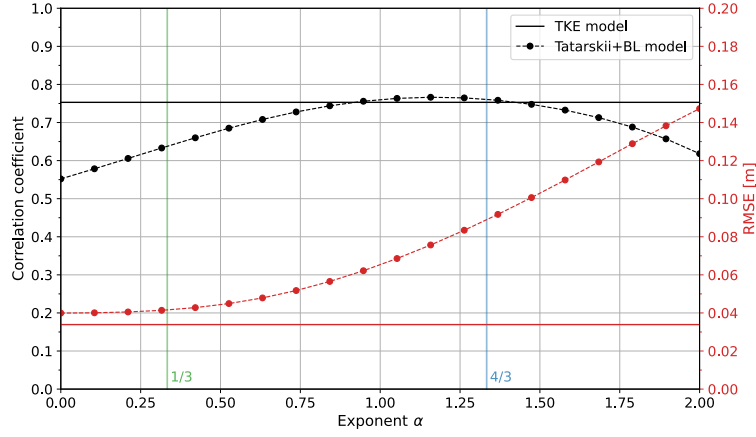


Figure 5.27 – Sensitivity analysis to daytime exponent α of Tatarskii+BL model. [Qua+23a]

ations are well depicted, up to a proportionality factor. Indeed, the daytime cluster in Figure 5.25 has actually an average slope close to 3 (based on empirical observation) rather than one. This suggests that daytime r_0 predictions from the Tatarskii+BL model overestimate the measurements in a consistent manner, explaining the large correlation coefficient. The RMSE is larger, being equal to 9.0 cm, and comes from the peaks at the beginning and the end of the day, as well as from the general overestimation of r_0 (see Figure 5.22). The latter also impacts the distribution presented in Figure 5.26b.

In nighttime, the correlation is 0.11, and the RMSE is 5.6 cm. However, the agreement between the predicted and measured distributions is good, as seen from Figure 5.26b. This shows that, even if the Tatarskii+BL model does not predict the correct r_0 at the right times, it provides realistic values of r_0 in agreement with the distribution of the measurements.

5.4.2.4 Sensitivity analysis to daytime exponent

The daytime exponent α in (5.7) plays a significant role in determining the overall performance of the Tatarskii+BL model. According to the literature, reviewed in Section 3.3.2, it is recommended to set α to 4/3. Nevertheless, previous studies have indicated that over complex terrains like mountainous areas, an exponent of 1/3 might be more appropriate [AP05]. To evaluate its impact on r_0 predictions, daytime correlation coefficients and RMSE were monitored for several values of the exponent, as shown in Figure 5.27. The TKE model is also represented for reference.

Interestingly, the correlation coefficient reaches its maximum for daytime exponents ranging from 1 and 4/3, indicating that these exponents offer the best fit for modelling daytime variations in the Fried parameter. This motivates the choice of the literature value of 4/3 set for α . However, larger daytime exponents tend to result in overestimations of r_0 , consequently increasing the RMSE, as illustrated in Figure 5.27.

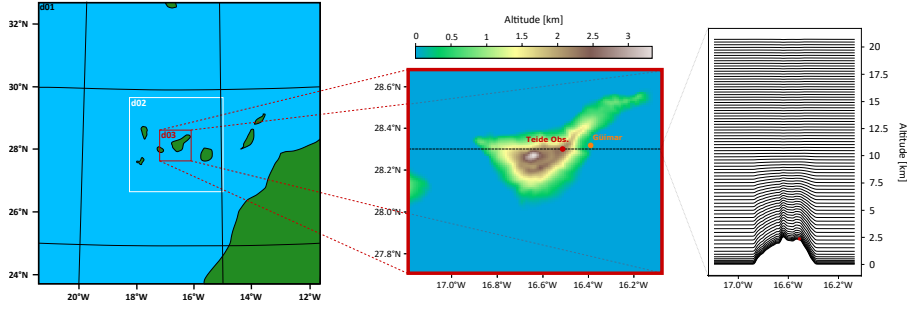


Figure 5.28 – WRF configurations for simulations conducted in Tenerife, Spain.

5.4.3 Comparison with daytime measurements in Tenerife

Evaluation of the daytime performance of the approach has also been conducted for the ESA optical ground station at Teide Observatory, Tenerife, Spain, using seeing measurements collected by a Miratlas' Sky Monitor⁶. The instrument continuously measures daytime seeing based on a Solar Scintillation Monitor (SSM), and for this work, measurements obtained in July 2021 have been exploited.

The results presented in this section are based on [QVO23a].

5.4.3.1 WRF configuration

The conducted NWP simulations are centered on the Teide Observatory (latitude $28^{\circ}18'03''\text{N}$, longitude $16^{\circ}30'43''\text{W}$, altitude 2400 m). The WRF default configuration is used (Tables 5.1 and 5.4), except that the number of vertical levels is set to 80 in all domains. The location of the nested domains, the ground topography, and an illustration of the vertical levels, is given in Figure 5.28.

For each day, simulations are started at 18h00 the day before, ensuring a spin-up time of 6 hours. Then, outputs are saved every 5 minutes above the location of interest, for the full day (i.e., from 00h00 to 00h00 the next day). Both C_n^2 models are exploited.

5.4.3.2 Time series of daytime seeing measurements and predictions

From the simulated meteorological quantities given by WRF, C_n^2 profiles are obtained every 5 minutes. The computation of the seeing $\mathcal{E}_0$ is conducted according to (see Section 6.3.2)

$$\mathcal{E}_0 = 5.285\lambda^{-1/5} \left(\sec(\tilde{\vartheta}) \int_{h_0}^{\infty} C_n^2(z) dz \right)^{3/5},$$

where h_0 is the altitude of the first WRF level, $\tilde{\vartheta} = 0^{\circ}$ is the elevation angle measured from zenith, and $\lambda = 500 \text{ nm}$ is the wavelength.

Figure 5.29 displays both measured and predicted seeing values for the month of July 2021. Seeing measurements are collected at one-minute intervals and subse-

⁶<https://www.miratlas.com/>, last accessed 30 August 2024.

5 | Prediction of C_n^2 profiles from meteorological simulations

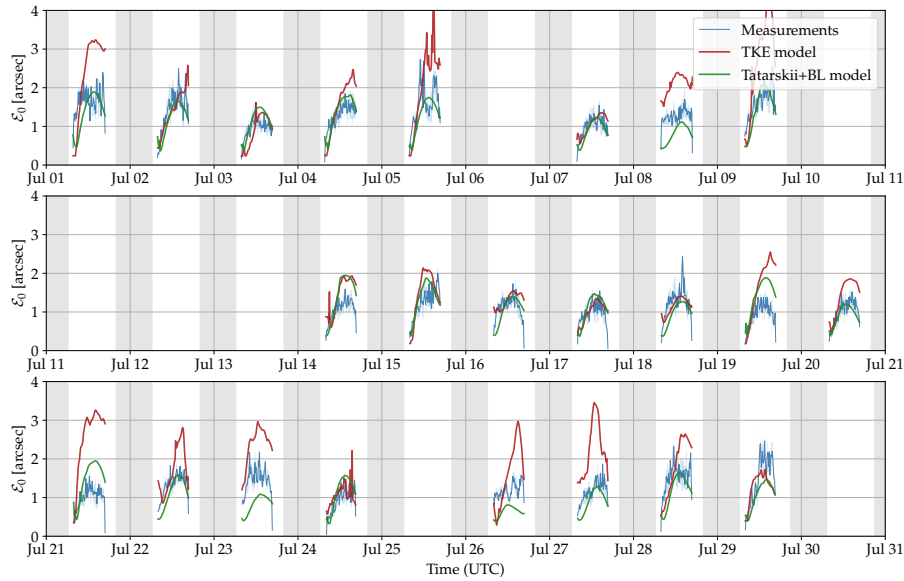


Figure 5.29 – Seeing measurements and predictions for the month of July 2021, at Tenerife. [QVO23a]

quently grouped into 5-minute bins. The mean value of each bin is represented by the blue line, while the shaded blue areas indicate the mean value plus or minus the standard deviation. Data is solely available during daytime, with nighttime periods denoted by gray areas. Day labels are positioned at 00:00 UTC for each respective day.

In Figure 5.29, days with seeing measurements larger than 3 arcsec have been discarded. They typically correspond to cloudy days, as identified by the cloud fraction in WRF simulation outputs⁷. As an example, Figure 5.30 presents the cloud fraction for low (from 300 m to 2 km), mid (from 2 km to 6 km), and high (above 6 km) altitudes in WRF domain *d01*, on July 12, 2021, at 06:00 UTC. For each pixel in the images, a cloud fraction of one means that it is fully covered with clouds, while zero relates to the absence of clouds. It is important to note that cloud predictions in WRF is challenging, and clouds may not be located at the exact locations and the right time.

In Figure 5.29, the estimated seeing magnitudes from both the TKE and Tatarskii+BL models align with the measurements for the majority of days. However, there are instances, particularly during the last ten days of July 2021, where predictions from the TKE model deviate from the measurements. Moreover, the models successfully capture the daytime variation, characterized by seeing increasing from sunrise to approximately noon and then decreasing until sunset.

⁷Due to the unavailability of cloud measurements directly above Tenerife at the time of the seeing observations, cloud predictions derived from WRF simulations are employed as a substitute for the missing cloud data.

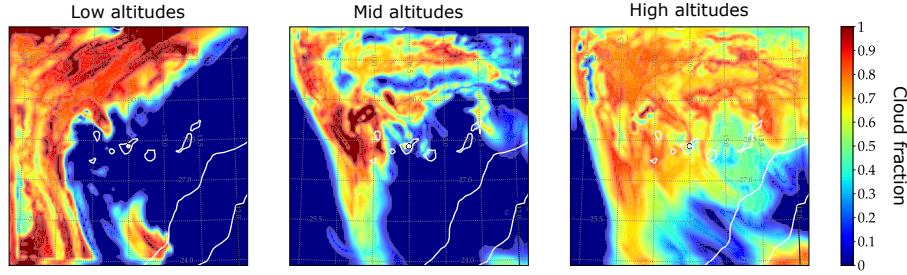


Figure 5.30 – WRF cloud fractions (low, mid, and high altitudes) above Tenerife on July 12, 2021, at 06:00 UTC.

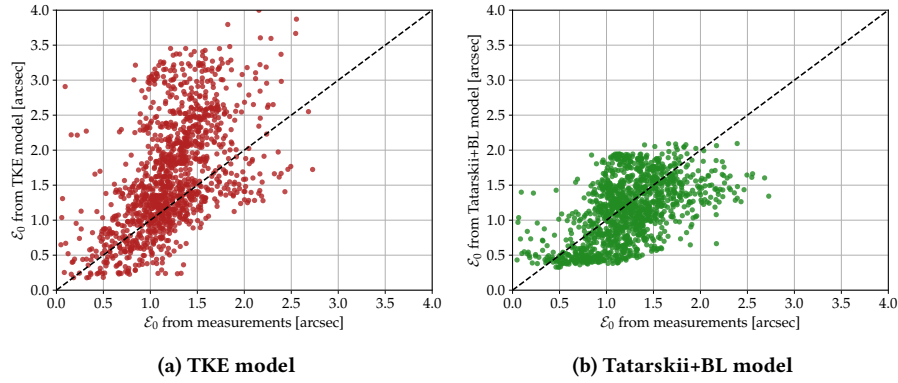


Figure 5.31 – Correlation plots between seeing measurements and their predictions in Tenerife. [QVO23a]

5.4.3.3 Correlations and distributions

The correlation plots between the measurements and the predicted values (from Figure 5.29, hence every 5 minutes) are given in Figure 5.31. In these plots, the over-estimation of the seeing by the TKE model is noticeable. The correlation coefficient for the TKE model stands at 0.56, while the Tatarskii+BL model achieves a slightly lower value of 0.53. Calculating the RMSE yields a value of 0.80 arcsec for the TKE model and 0.43 arcsec for the Tatarskii+BL model.

Finally, Figure 5.32 illustrates the distributions of measured and modelled seeing, with the CDFs given in Figure 5.32d. The mean μ and standard deviation σ of each distribution are provided in the legend. The mean is also depicted in black on the histograms, while the median is in cyan. Measurements tend to be symmetric around the mean of their distribution, while the predictions are skewed towards lower seeing values. Large seeing values are also obtained with the TKE model. Consequently, the CDFs of the predicted seeing deviate from the one of the measurements.

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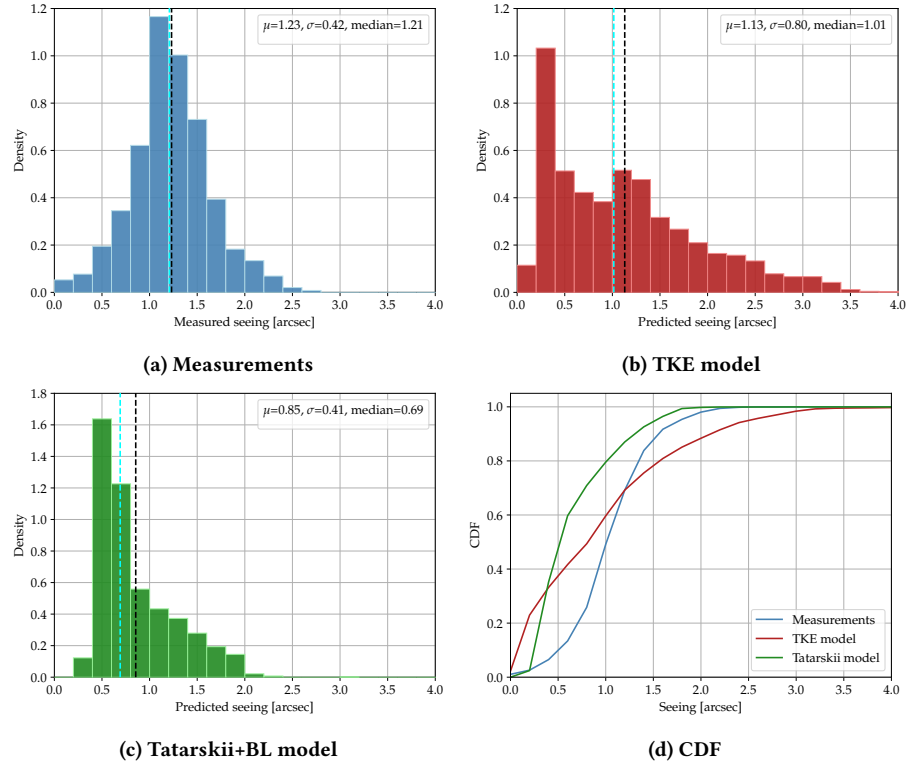


Figure 5.32 – Distributions of measured and modelled seeing in Tenerife. [QVO23a]

Table 5.5 – RMSE and correlation coefficients for seeing prediction in Redu.

Metrics	Dates	TKE model		Tatarskii+BL model	
		6h	12h	6h	12h
RMSE [arcsec]	05/02/2019	0.79	0.82	2.18	2.00
	14/02/2019	1.09	1.32	1.46	1.33
	04/09/2019	1.00	1.23	1.53	1.58
Corr. coeff.	05/02/2019	-0.49	-0.47	0.35	0.35
	14/02/2019	0.00	0.12	0.02	0.11
	04/09/2019	0.57	0.54	0.049	0.23

5.4.4 Seeing prediction in Redu

Seeing measurements have been collected in Redu for the SALTO project [Orb+18; Orb+22], and correspond to three different nights in 2019. They are compared with seeing predictions from the TKE and Tatarskii+BL models, complementing the work in [Qua+23b] with a new version of the models. Details about WRF simulations conducted in Redu are presented in Section 5.2.2.

The time series of the measurements with their uncertainty are depicted in Figures 5.33 and 5.34. Measurements are averaged over 5-minute bins, and compared to the TKE and Tatarskii+BL model predictions. The comparison also highlights the influence of the spin-up time, either set to 6 or 12 hours prior to the time of the first available measurements for the night. Table 5.5 presents the associated RMSE and correlation coefficients, for each night, model, and spin-up time.

As seen from the time series, seeing tends to be underestimated by both models, except for the night of February 05 and the TKE model. The discrepancies are the largest for the Tatarskii+BL model, leading to important RMSEs. Correlation coefficients have strong variations for the three nights, probably owing to the small number of available measurements. The agreement between measurements and predictions is far from perfect, and necessitates further study of the BL parametrisation in the NWP simulations, as seeing is mostly influenced by this atmospheric layer [Qua+23b].

In terms of spin-up time impact on seeing prediction, the time series in Figures 5.33 and 5.34 show that the TKE model has more variations with the spin-up time than the Tatarskii+BL model. Indeed, for the night of February 14 for example, the largest difference between the seeing predicted with 6 or 12-hour spin-up time is 0.48 arcsec for the TKE model, and 0.36 arcsec for the Tatarskii+BL model. Such variations are associated with changes in the meteorological quantities predicted during the night depending on the initial conditions used in the NWP simulations. A possible solution to reduce these differences can be the use of mathematical models combining recent seeing measurements with NWP predictions to provide accurate short-term seeing predictions, as detailed in Chapter 8.

5.4.5 Seeing prediction in Louvain-la-Neuve

A Miratlas' Sky Monitor, equipped with a SSM, a compact DIMM and a cloud monitoring instrument, has been installed at UCLouvain, in Louvain-la-Neuve (50.66863°N, 4.62390°E, 142 m). Measurements are collected continuously since spring 2024 and will enable to characterize the quality of Louvain-la-Neuve as a potential OGS site. Preliminary and qualitative results are depicted in Figure 5.35, corresponding to one week of seeing measurements under a clear sky (no cloud) in June 2024. They show seeing remains relatively large during the night, where it is underestimated by both models (using the default WRF configuration of this work). Daytime variations seem to be well modelled by the TKE model, while the Tatarskii+BL requires further work on the BL modelling in Louvain-la-Neuve to im-

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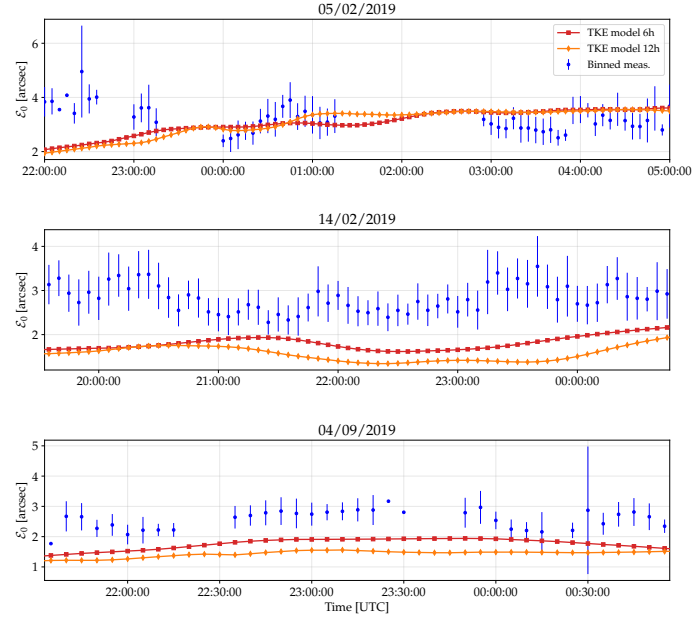


Figure 5.33 – Time series of predicted seeing in Redu from TKE model.

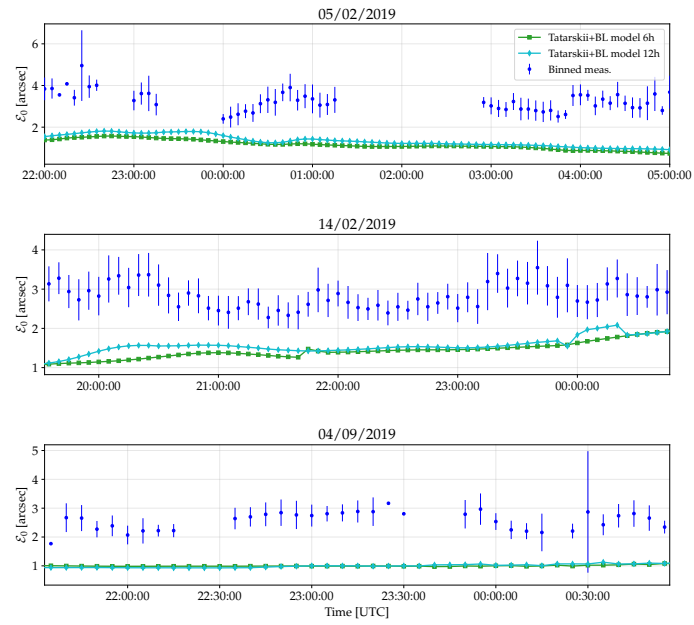


Figure 5.34 – Time series of predicted seeing in Redu from Tatarskii+BL model.

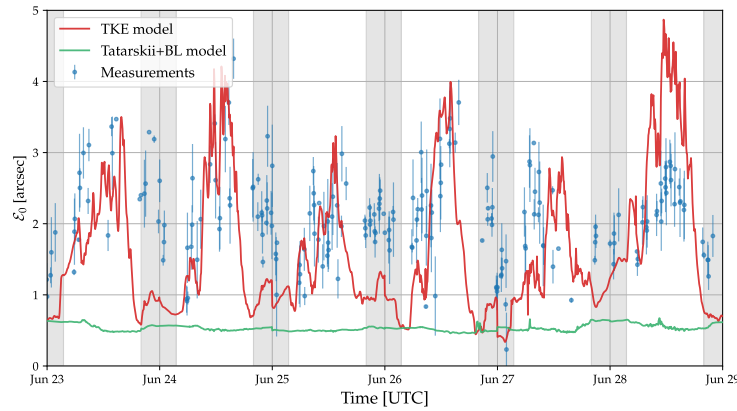


Figure 5.35 – Time series of measured and predicted seeing in Louvain-la-Neuve.

prove its seeing prediction accuracy. Discontinuities between daytime and nighttime measurements arise due to the use of distinct instruments for day and night operations. As more data are being collected, statistics will be possible and improvements of the prediction models are expected.

5.5 Chapter summary

Chapter 5 presented two different C_n^2 models based on NWP simulations, suited for optical communication sites. The NWP approach was detailed in Section 5.1 and the two C_n^2 models in Section 5.3. The features of the WRF software were given in Section 5.2, while the validation carried out for several sites in Section 5.4.

5.5.1 Main results and contributions

The main contribution of this chapter is the development and evaluation of NWP-based C_n^2 models for OGS sites. A literature model using the TKE has been applied for the first time to non-astronomical sites and during daytime, while a new hybrid model mixing MOST and the Tatarskii equation has been developed. They allowed to identify the challenges associated with BL modelling at FSOC sites, as seen in the validation with integrated parameters conducted in Section 5.4.

Moreover, the analysis conducted in this chapter enabled to highlight the advantages of the C_n^2 models studied, as well as their limitations, requiring further work.

Advantages of the models The NWP-based C_n^2 models can be applied to any location on Earth without the need for direct meteorological measurements from sources like radiosondes. However, on-site measurements may still be required for accurate model parameterization and validation. Unlike radiosondes, which provide a single profile at the time of launch, NWP-based models enable OT forecast with continuous predictions during both daytime and nighttime. Furthermore, they also offer a reasonable trade-off between the accuracy of the predictions, their spatial and temporal resolutions, and the computational time necessary for the NWP simulations.

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Challenges and limitations The C_n^2 models used in this chapter rely on the same WRF parameterization across all locations, informed by specific studies conducted at one site (e.g., the number of vertical levels studied using T-REX measurements). However, the WRF parameterization could be site-dependent, particularly regarding the physics of the BL. This aspect has not been explored here due to a lack of meteorological data within the BL at the sites of interest. Recent studies suggest that modifying the planetary BL schemes in WRF can significantly impact the predicted OT [Fre+10; PSB23].

Furthermore, the prediction inaccuracies in NWP models are greater for OT quantities than for standard meteorological quantities. This is due to the small scales involved in OT, which are not resolved in NWP simulations and require sub-grid models [Qin+20]. These inaccuracies are also influenced by the initialization data and the spin-up time for the simulations. However, while OT quantities may not be accurate at specific times, their overall statistics can still be effectively studied using NWP models [Qui20].

5.5.2 Future work

This work paved the way for applying NWP simulations at OGS sites to predict OT. It has also identified several avenues for future research:

- Improving OT modelling in the BL by relying on additional in-situ measurements and detailed site characterization, as in [Med+23]. Additionally, increasing the temporal and spatial resolutions of NWP simulations in the BL could help to better resolve small-scale turbulence. The potential to combine WRF with LES, as explored in [Udi+20], is also worth investigating.
- Enhancing WRF parameterization through the use of large datasets from on-site measurements and fine-tuning of WRF parameters.
- Leveraging the entire 3D grid within the simulation domain, rather than just a vertical profile at the location of interest from the inner domain. This could be valuable for modelling slant path propagation or using data from coarser domains if deemed relevant. Further study on the influence of vertical resolution, as explored in [Ull+23; Hu+23], is also recommended.
- Finally, adapting this model to predict RF C_n^2 could be beneficial, allowing exploration of the synergies between optical and RF turbulence modelling.

In conclusion, there is potential for using NWP simulations in planning and scheduling optical links, namely by predicting both cloud cover and OT at the same time. However, more OT measurements (especially C_n^2 profiles and BL meteorological quantities) at OGS sites are needed to design and validate these models in various conditions and refine the parametrization of NWP simulations. The next part of this thesis delves into OT monitoring instruments (Chapters 6 and 7) and explores models that combine real-time measurements with NWP simulations (Chapter 8).

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5 | Prediction of C_n^2 profiles from meteorological simulations

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Part II

Measurements and characterization of electromagnetic wave propagation in the turbulent atmosphere

Propagation of electromagnetic waves in the turbulent atmosphere and associated parameters

6

The previous chapters in the first part of the thesis focused on describing the propagation medium for satellite-to-ground optical communication, i.e., the turbulent atmosphere. This medium, treated as a random field, is statistically characterized by the knowledge of local C_n^2 .

This part of the thesis, and especially this chapter, examines the propagation of EM waves through the turbulent atmosphere. It provides:

- A review of theoretical and numerical approaches used in the literature to model EM propagation in random media;
- A detailed summary of key OT parameters that succinctly describe how optical waves are affected by atmospheric turbulence;
- An overview of available OT monitoring instruments that provide the necessary measurements for validating C_n^2 models.

This chapter enables to link the atmospheric description with EM propagation. It is associated with the part depicted in red in Figure 6.1.

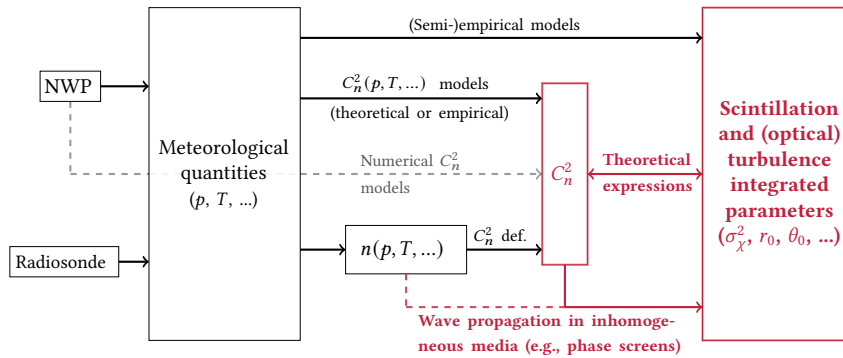


Figure 6.1 – Schematic of the thesis structure, highlighting the concepts detailed in Chapter 6.

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The parameters on the right part of Figure 6.1 are referred to as *integrated parameters* because they result from vertical integrations of C_n^2 profiles, weighted by different functions of the altitude.

Sections 6.1 and 6.2 present the theoretical and numerical approaches for EM wave propagation in random media. Then, Section 6.3 describes the OT integrated parameters, while Section 6.4 details the available OT monitoring instruments. Finally, Section 6.5 provides a summary of the main OT integrated parameters used in this thesis, and the associated monitoring instruments.

6.1 Propagation through random media

The propagation of EM waves through a random medium, such as atmospheric turbulence, leads to random fluctuations of the received amplitude and phase of the waves. Several analytical approaches have been developed to describe the amplitude and phase fluctuations, namely to relate them to the atmospheric C_n^2 profile. These different approaches can be found in [Tat61; Tat71; Ish78; AP05; Whe03].

For the purpose of this work, only the Rytov approximation is introduced in this section. It enables to obtain analytical expressions for the variance of the log-amplitude σ_χ^2 , used in Section 4.5.3.2 and Section 6.3.6. Details about this approach are given in [Ish78, p. 349] and [AP05, p. 143].

6.1.1 Maxwell's equations in a random medium

Assuming the permittivity is a random function of the position, thus neglecting temporal fluctuations (or assuming that they are sufficiently slow [AP05, p. 137]), i.e. $\epsilon_r(\mathbf{r})$, Maxwell's equations are given by [Ish78, p. 347]:

$$\begin{aligned}\nabla \times \mathbf{E}(\mathbf{r}) &= j\omega\mu_0\mathbf{H}(\mathbf{r}), \\ \nabla \times \mathbf{H}(\mathbf{r}) &= -j\omega\epsilon_0\epsilon_r(\mathbf{r})\mathbf{E}(\mathbf{r}),\end{aligned}$$

where a time dependency $e^{-j\omega t}$ for the electric field $\mathbf{E}$ and magnetic fields $\mathbf{H}$ is assumed (phasor quantities, ω being the angular frequency of the EM wave, and $j = \sqrt{-1}$ the imaginary unit), and $\nabla \times$ corresponds to the curl operator. Combining both equations, one has

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - \omega^2\mu_0\epsilon_0\epsilon_r(\mathbf{r})\mathbf{E}(\mathbf{r}) = 0.$$

Since $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ and $\nabla \cdot (\epsilon_r(\mathbf{r})\mathbf{E}(\mathbf{r})) = \nabla\epsilon_r(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) + \epsilon_r(\mathbf{r})\nabla \cdot \mathbf{E}(\mathbf{r})$ with $\nabla \cdot (\epsilon_r(\mathbf{r})\mathbf{E}(\mathbf{r})) = 0$, the following result holds

$$\nabla^2 \mathbf{E}(\mathbf{r}) + \omega^2\mu_0\epsilon_0\epsilon_r(\mathbf{r})\mathbf{E}(\mathbf{r}) - \nabla \left(\frac{\nabla\epsilon_r(\mathbf{r})}{\epsilon_r(\mathbf{r})} \cdot \mathbf{E}(\mathbf{r}) \right) = 0.$$

Using the refractive index field $n(\mathbf{r}) = \sqrt{\epsilon_r(\mathbf{r})}$ instead of the permittivity field, it

is expressed as

$$\nabla^2 \mathbf{E}(\mathbf{r}) + k_0^2 n^2(\mathbf{r}) \mathbf{E}(\mathbf{r}) - 2\nabla \left(\frac{\nabla n(\mathbf{r})}{n(\mathbf{r})} \cdot \mathbf{E}(\mathbf{r}) \right) = 0, \quad (6.1)$$

with $k_0^2 = \omega^2 \mu_0 \epsilon_0$.

Thanks to a study of the magnitude order of the different terms, (6.1) can be simplified [Ish78, p. 348], [AP05, p. 137]. Neglecting depolarization effects and assuming the wavelength λ is smaller than the smallest scale of turbulence l_0 , the last term is dropped:

$$\nabla^2 \mathbf{E}(\mathbf{r}) + k_0^2 n^2(\mathbf{r}) \mathbf{E}(\mathbf{r}) = 0.$$

Denoting by $U(\mathbf{r})$ one of the scalar components of the electric field that is transverse to the direction of propagation, the stochastic Helmholtz equation is obtained:

$$\nabla^2 U(\mathbf{r}) + k_0^2 n^2(\mathbf{r}) U(\mathbf{r}) = 0. \quad (6.2)$$

The index of refraction is fluctuating around its mean value $\bar{n}$, such that, defining the average wave number $k^2 = k_0^2 \bar{n}^2$, (6.2) can be expressed as

$$\nabla^2 U(\mathbf{r}) + k^2 (1 + n'(\mathbf{r}))^2 U(\mathbf{r}) = 0, \quad (6.3)$$

where n' corresponds to the refractive index fluctuations. Equation (6.3) is first considered in the weak fluctuation regime (Section 6.1.2), and then in the strong fluctuation regime (Section 6.1.3).

6.1.2 Weak fluctuations

Weak fluctuations typically occur when the intensity of turbulence (associated to C_n^2) is small, and for short propagation paths. It is also dependent on the wavelength: microwave propagation is in the weak fluctuation regime for distances of several tens of kilometres, while distances are limited to of a few kilometres for optical frequencies [Ish78, p. 347].

Condition The limit of validity of the weak fluctuation approximation is quantitatively expressed using the variance of the log-amplitude σ_χ^2 , defined below. As given in (6.4), σ_χ^2 is a function of the intensity of turbulence (C_n^2 , first assumed constant over the propagation path), the propagation distance (L), and the wavelength (λ , in $k = \frac{2\pi}{\lambda}$). A threshold value of 0.2 to 0.5 is typically used, such that the weak fluctuation hypothesis holds if [Ish78, p. 374]

$$\sigma_\chi^2 = 0.307 C_n^2 k^{7/6} L^{11/6} < 0.2 - 0.5 \quad (\text{weak fluctuations}). \quad (6.4)$$

A threshold value of one is suggested in [Whe03, p. 5]. [AP05, p. 140] expresses the condition in (6.4) in terms of Rytov variance σ_R^2 , computed using $\sigma_R^2 = 4\sigma_\chi^2$, with σ_χ^2 given by (6.4). In this case, weak fluctuations are considered for $\sigma_R^2 < 1$, in agreement

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with the threshold values suggested in (6.4). Conditions specific to Gaussian-beam waves are presented in [AP05, p. 140]. The origin of (6.4) is detailed in Section 6.1.2.1.

Rytov approximation It is assumed that $|n'(\mathbf{r})| \ll 1$, such that $(1 + n'(\mathbf{r}))^2$ is approximately equal to $1 + 2n'(\mathbf{r})$, neglecting the term in $n'^2(\mathbf{r})$. The Rytov approximation expresses the field U as

$$U(\mathbf{r}) = e^{\psi(\mathbf{r})},$$

and finds a series solution for $\psi(\mathbf{r})$. Substituting $U(\mathbf{r})$ in (6.3), and assuming weak fluctuations, one obtains the Ricatti equation [Ish78, p. 349]

$$\nabla^2 \psi(\mathbf{r}) + \nabla \psi(\mathbf{r}) \cdot \nabla \psi(\mathbf{r}) + k^2(1 + 2n'(\mathbf{r})) = 0. \quad (6.5)$$

Equation (6.5) is then solved for $\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \psi_1(\mathbf{r})$, with $\psi_0(\mathbf{r})$ denoting the solution in the absence of fluctuation, and $\psi_1(\mathbf{r})$ representing the complex phase-perturbation taking into account the fluctuations. This leads to the two equations [Ish78, p. 350]:

$$\nabla^2 \psi_0(\mathbf{r}) + \nabla \psi_0(\mathbf{r}) \cdot \nabla \psi_0(\mathbf{r}) + k^2 = 0, \quad (6.6)$$

$$\nabla^2 \psi_1(\mathbf{r}) + 2\nabla \psi_0(\mathbf{r}) \cdot \nabla \psi_1(\mathbf{r}) = -k^2 2n'(\mathbf{r}) - \nabla \psi_1(\mathbf{r}) \cdot \nabla \psi_1(\mathbf{r}). \quad (6.7)$$

Solving (6.6) gives the received field without perturbation, denoted as $U_0(\mathbf{r}) = e^{\psi_0(\mathbf{r})}$. It is then possible to obtain an integral form for $\psi_1(\mathbf{r})$ using (6.7), that can be solved iteratively [Ish78, p. 350],

$$\psi_1(\mathbf{r}) = \frac{1}{U_0(\mathbf{r})} \int_{V'} \frac{e^{jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} [\nabla \psi_1(\mathbf{r}') \cdot \nabla \psi_1(\mathbf{r}') + k^2 2n'(\mathbf{r}')] U_0(\mathbf{r}') dV'. \quad (6.8)$$

The iterative resolution of (6.8) generates a series expression for $\psi_1(\mathbf{r})$.

Log-amplitude and phase fluctuations The resolution of (6.8) gives access to the complex phase-perturbation $\psi_1(\mathbf{r})$. It is possible to relate it to the log-amplitude and the phase fluctuations of the field $U(\mathbf{r})$. Indeed, defining the amplitude of $U(\mathbf{r})$ by $A(\mathbf{r})$ and its phase by $\varphi(\mathbf{r})$, it now writes as

$$U(\mathbf{r}) = A(\mathbf{r})e^{j\varphi(\mathbf{r})} \quad \text{and} \quad \tilde{U}(\mathbf{r}) = \tilde{A}(\mathbf{r})e^{j\tilde{\varphi}(\mathbf{r})} = e^{\psi_0(\mathbf{r})},$$

where $\tilde{A}(\mathbf{r})$ and $\tilde{\varphi}(\mathbf{r})$ are the amplitude and phase of the field in the absence of perturbations. Since

$$U(\mathbf{r}) = e^{\psi(\mathbf{r})} = e^{\psi_0(\mathbf{r}) + \psi_1(\mathbf{r})} = \tilde{A}(\mathbf{r})e^{j\tilde{\varphi}(\mathbf{r})}e^{\psi_1(\mathbf{r})}$$

the complex phase perturbation $\psi_1(\mathbf{r})$ can be expressed as

$$\psi_1(\mathbf{r}) = \underbrace{\ln \left(\frac{A(\mathbf{r})}{\bar{A}(\mathbf{r})} \right)}_{\chi(\mathbf{r})} + \underbrace{J(\varphi(\mathbf{r}) - \tilde{\varphi}(\mathbf{r}))}_{\varphi'(\mathbf{r})}, \quad (6.9)$$

where $\ln(\cdot)$ is the natural logarithm. The real part of $\psi_1(\mathbf{r})$ is denoted $\chi(\mathbf{r})$ and corresponds to the fluctuations of the logarithm of the amplitude, also named *log-amplitude fluctuations*. The imaginary part of $\psi_1(\mathbf{r})$ is $\varphi'(\mathbf{r})$ and is called the *phase fluctuation* [Ish78, p. 351]. It will be simply denoted by $\varphi(\mathbf{r})$ in the following, with the understanding that it is defined relative to the unperturbed field.

6.1.2.1 Plane waves

For a plane wave in the z -direction, the solution in the absence of perturbation can be expressed as $\tilde{U}(\mathbf{r}) = e^{jkz}$. Its substitution in (6.8), for the first iteration (i.e., $\psi_1(\mathbf{r})$ set to zero in the integrand), leads to [Ish78, p. 352]

$$\psi_1(L, x, y) = \int_0^L dz' \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \frac{k^2}{2\pi} \frac{n'(x', y', z')}{L - z'} e^{jk \frac{(x-x')^2 + (y-y')^2}{2(L-z')}}, \quad (6.10)$$

when neglecting the backscattering (integration from $z = 0$ to $z = L$), and assuming $|x - x'|$ and $|y - y'|$ are much smaller than $|z - z'|$.

The direct resolution of (6.10) gives access to $\chi(\mathbf{r})$ and $\varphi(\mathbf{r})$, from which the covariance functions $B_\chi(\mathbf{r}_1, \mathbf{r}_2) = \langle \chi(\mathbf{r}_1) \chi(\mathbf{r}_2) \rangle$, $B_\varphi(\mathbf{r}_1, \mathbf{r}_2) = \langle \varphi(\mathbf{r}_1) \varphi(\mathbf{r}_2) \rangle$, $B_{\chi\varphi}(\mathbf{r}_1, \mathbf{r}_2)$, and structure functions $D_\chi(\mathbf{r}_1, \mathbf{r}_2)$, $D_\varphi(\mathbf{r}_1, \mathbf{r}_2)$, $D_{\chi\varphi}(\mathbf{r}_1, \mathbf{r}_2)$, can be derived.

Alternatively, spectral methods enable to directly obtain these functions, as presented in [Ish78, p. 353-358], [Tat61, p. 135]. Such approaches lead to expressions of $B(\mathbf{r}_1, \mathbf{r}_2)$ and $D(\mathbf{r}_1, \mathbf{r}_2)$ that involves an integration of the refractive index power spectrum $\Phi_n(\kappa)$, as given in Section 2.5.2, weighted by a filter function $f(\kappa)$. If $\Phi_n(\kappa)$ does not depend on the location, assuming local homogeneity and isotropy, the covariance and structure functions are [Ish78, p. 359]:

$$B_\iota(L, \rho) = 2\pi^2 k^2 L \int_0^{+\infty} \kappa J_0(\kappa \rho) f_\iota(\kappa) \Phi_n(\kappa) d\kappa, \quad (6.11)$$

$$D_\iota(L, \rho) = 4\pi^2 k^2 L \int_0^{+\infty} \kappa (1 - J_0(\kappa \rho)) f_\iota(\kappa) \Phi_n(\kappa) d\kappa, \quad (6.12)$$

where ι denotes either χ , φ , or $\chi\varphi$, and ρ is the spacing difference vector. J_0 is the Bessel function of order zero. The filter functions $f_\iota(\kappa)$ are

$$f_\chi(\kappa) = 1 - \frac{\sin(\kappa^2 L/k)}{\kappa^2 L/k},$$

$$f_\varphi(\kappa) = 1 + \frac{\sin(\kappa^2 L/k)}{\kappa^2 L/k},$$

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$$f_{\chi\varphi}(\kappa) = \frac{\sin^2(\kappa^2 L/2k)}{\kappa^2 L/2k}.$$

Looking at the shapes of the filter functions, analytical integrations of (6.11) and (6.12) can be made possible in different spatial regions, related to the Fresnel zone $\sqrt{\lambda L}$ [Ish78, p. 360]. In this work, the details are only presented for σ_χ^2 , obtained from $B_\chi(L, 0)$ or from $\frac{1}{2}D_\chi(L, \infty)$ [Tat71, p. 237].

Geometrical optics region ($L < l_0^2/\lambda$) In this region, the filter functions $f_\chi(\kappa)$ and $f_\varphi(\kappa)$ are approximated by [Ish78, p. 361]

$$\begin{aligned} f_\chi(\kappa) &\approx \frac{1}{6} (\kappa^2 L/k)^2, \\ f_\varphi(\kappa) &\approx 2. \end{aligned}$$

The integration of $B_\chi(L, \rho)$ using a modified von Kármán spectrum for $\Phi_n(\kappa)$ is detailed in [Ish78, p. 368], while [Tat71, p. 202-206] presents the integration with a Tatarskii spectrum. Instead, the use of a Kolmogorov spectrum, given by (2.47), integrated from 0 to $\kappa_m = 5.48/l_0$, gives [Tat61, p. 112]

$$\begin{aligned} \sigma_\chi^2 &= 2\pi^2 k^2 L \int_0^{\kappa_m} \kappa \frac{1}{6} \left(\frac{\kappa^2 L}{k} \right)^2 0.033 C_n^2 \kappa^{-11/3} d\kappa \\ &= 0.1086 C_n^2 L^3 \int_0^{\kappa_m} \kappa^{4/3} d\kappa \\ &= 2.46 C_n^2 L^3 l_0^{-7/3}, \end{aligned} \tag{6.13}$$

or $\sigma_\chi^2 = 2.95 C_n^2 L^3 l_0^{-7/3}$ if $\kappa_m = 5.92/l_0$ is used.

Inertial region ($l_0^2/\lambda < L < L_0^2/\lambda$) This is referred to as the diffraction region, where the filter functions are not simplified. Both l_0 and L_0 are important, but for κ small, $f_\chi(\kappa)$ is proportional to κ^2 , such that the effect of the outer scale L_0 is small [Ish78, p. 370]. As a result, a Kolmogorov spectrum can be used in the integration, giving [Ish78, p. 370], [Tat61, p. 153], [Tat71, p. 237]

$$\begin{aligned} \sigma_\chi^2 &= 2\pi^2 k^2 L \int_0^{+\infty} \kappa \left(1 - \frac{\sin(\kappa^2 L/k)}{\kappa^2 L/k} \right) 0.033 C_n^2 \kappa^{-11/3} d\kappa \\ &= 0.6514 k^2 L C_n^2 \left(\frac{k}{2L} \frac{k^{-11/6}}{L^{-11/6}} \underbrace{\int_0^{+\infty} \left(1 - \frac{\sin(\kappa')}{\kappa'} \right) (\kappa')^{-11/6} d\kappa'}_{=0.943} \right) \\ &= 0.307 C_n^2 k^{7/6} L^{11/6}. \end{aligned} \tag{6.4}$$

The change of variable $\kappa' = \kappa^2 L / k$ has been used in the integration. This result is also presented in [Whe03, p. 46], and is associated to Fresnel scattering.

Region for $L > L_0^2 / \lambda$ This region is not homogeneous. An expression of the variance of the log-amplitude is given by [Tat61, p. 144]

$$\sigma_\chi^2 = \langle n'^2 \rangle k^2 L L_n.$$

It depends on the size of the mean square refractive index fluctuations $\langle n'^2 \rangle$ and the integral scale of the turbulence L_n :

$$L_n = \frac{1}{\langle n'^2 \rangle} \int_0^{+\infty} B_n(\rho) d\rho.$$

Alternative expressions for this region are also presented in [Ish78, p. 373] and [Whe03, p. 51].

Conversion to dB² The expressions of σ_χ^2 , computed in the various regions, are expressed in squared neper (Np²). They can be converted in dB² by changing the basis of the logarithm used for defining the log-amplitude (now given in dB). This is equivalent to multiplying the expressions of σ_χ^2 in Np² by $(20 \log_{10}(e))^2 \approx 75.45$. With this factor, the expressions presented in [Van12] are obtained.

Varying C_n^2 on the propagation path For cases where C_n^2 is varying on the propagation path, such as for vertical or slant paths described by $C_n^2(z)$ profiles, the expressions of σ_χ^2 given by (6.13) and (6.4) need to be adapted.

Equations (6.11) and (6.12) cannot be used, as the power spectrum $\Phi_n(\kappa, z)$ now depends on the position through $C_n^2(z)$, considering a vertically-varying C_n^2 profile for example. The developments are conducted in [Ish78, p. 373], [Tat71, p. 246], [Tat61, p. 169], for the inertial region, and leads to

$$\sigma_\chi^2 = 0.563 k^{7/6} \int_0^L C_n^2(\eta) (L - \eta)^{5/6} d\eta,$$

where η can correspond to the altitude z for vertical propagations through C_n^2 profiles, or to a horizontal distance x for horizontal propagations through a varying C_n^2 . For satellite-to-ground (downlink) propagation along vertical and slant paths, the expression of the log-amplitude variance reduces to

$$\sigma_\chi^2 = 0.563 \frac{k^{7/6}}{(\sin(\vartheta))^{11/6}} \int_0^{H_0} C_n^2(z) z^{5/6} dz, \quad (6.14)$$

with H_0 the upper altitude of the atmosphere/the C_n^2 profile, and ϑ the elevation angle. The conversion of (6.14) in dB² and the addition of the aperture averaging function leads to (4.15), used in Section 4.5.3.2.

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In the geometrical optics region, σ_χ^2 is given by [Tat61, p. 169]

$$\sigma_\chi^2 = 7.37 l_0^{-7/3} \int_0^L C_n^2(\eta) (L - \eta)^2 d\eta.$$

Alternative expressions can be found in [Ish78, p. 374].

6.1.2.2 Spherical waves

Similar developments as conducted in Section 6.1.2.1 can be made for spherical waves, instead of plane waves. Only the log-amplitude variance in the inertial range is presented here, based on [Ish78, p. 376-379]. It is given by

$$\sigma_\chi^2 = 0.563 k^{7/6} \int_0^L C_n^2(\eta) \left(\frac{\eta(L - \eta)}{L} \right)^{5/6} d\eta \quad (6.15)$$

if C_n^2 is varying along the propagation path, and by

$$\sigma_\chi^2 = 0.124 C_n^2 k^{7/6} L^{11/6} \quad (6.16)$$

otherwise. This result is also presented in [Tat71, p. 253] and in [Tat61, p. 185-188].

6.1.2.3 Beam waves

For completeness, the mathematical developments to obtain σ_χ^2 for beam waves are presented in [Ish78, p. 380-384], or in [AP05] for the scintillation index, defined in Section 6.3.6.

6.1.3 Strong fluctuations

Propagation in the strong fluctuation regime, where (6.4) does not hold, is presented in [Ish78, p. 407] and [AP05, p. 229]. Several methods can be applied, and are equivalent under certain assumptions. In [Ish78, p. 407] and [AP05, p. 231], the parabolic equation method is detailed. Starting from the stochastic Helmholtz equation in (6.3) and assuming a wave propagating in the z -direction defined as $U(\mathbf{r})e^{jkz}$, with $U(\mathbf{r})$ a slowly varying function of z , one obtains¹

$$2jk \frac{\partial U(\mathbf{r})}{\partial z} + \nabla^2 U(\mathbf{r}) + 2k^2 n'(\mathbf{r}) U(\mathbf{r}) = 0.$$

As long as $U(\mathbf{r})$ is slowly varying in z and only varies over the scale size l of the random medium ($l \gg \lambda$), the derivative in z in the Laplacian can be neglected compared to the first order derivative in z (paraxial approximation). As a result,

$$2jk \frac{\partial U(\mathbf{r})}{\partial z} + \nabla_T^2 U(\mathbf{r}) + 2k^2 n'(\mathbf{r}) U(\mathbf{r}) = 0 \quad (6.17)$$

¹ $U(\mathbf{r})$ in this equation is thus different from $U(\mathbf{r})$ in Helmholtz equation (6.3), as the propagation in the z -direction has been made explicit.

that involves the transverse Laplacian $\nabla_T^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Equation (6.17) is named *parabolic equation*, and gives access to analytical expressions of the mean field solution and the mutual coherence function of the field (second-order moment) [Ish78, p. 407], [AP05, p. 229]. They enable to obtain the Gaussian beam spread and beam wander in strong fluctuations [AP05, p. 238, 246]. The phase screen propagation approach presented in Section 6.2.2 is also based on the parabolic equation.

6.2 Numerical propagation using phase screens

This section complements the analytical expressions provided in Section 6.1 by presenting numerical solutions to Maxwell's equations. The numerical approach of interest for modelling propagation through OT is the phase screen propagation method, detailed in Section 6.2.2.

6.2.1 Free space propagation

The Helmholtz equation, similar to (6.2) but without the stochastic refractive index field $n^2(\mathbf{r})$, is given by [Goo96, p. 39]

$$\nabla^2 U(\mathbf{r}) + k^2 U(\mathbf{r}) = 0, \quad (6.18)$$

with $U(\mathbf{r})$ a scalar wave field. As shown in the following of this section, the propagation of $U(\mathbf{r})$ can be computed using 2D Fourier transforms, either based on the Rayleigh-Sommerfeld integral, or on the angular spectrum approach.

6.2.1.1 Fresnel and Fraunhofer propagations

Starting from (6.18), it is possible to obtain the Rayleigh-Sommerfeld integral that describes the propagation of $U(\mathbf{r})$ in the scalar diffraction theory. Using rectangular coordinates, the Rayleigh-Sommerfeld integral is given by [Goo96, p. 66]

$$\begin{aligned} U(x, y, z) &= \frac{1}{j\lambda} \iint U_0(x', y') \frac{ze^{jk\sqrt{(x-x')^2 + (y-y')^2 + z^2}}}{(x-x')^2 + (y-y')^2 + z^2} dx' dy' \\ &= \frac{1}{j\lambda} \iint U_0(x', y') \frac{z}{r} \frac{e^{jkr}}{r} dx' dy', \end{aligned} \quad (6.19)$$

where the quantities x, y, z, x' and y' are depicted in Figure 6.2, and the integration is conducted over the whole domain of x' and y' . The field in $z = 0$ is denoted by $U_0(x', y')$, with the field in a position z is $U(x, y, z)$, and $r = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$. Equation (6.19) requires the propagation distance r to be larger than several wavelengths, i.e., $r \gg \lambda$ [Goo96, p. 66].

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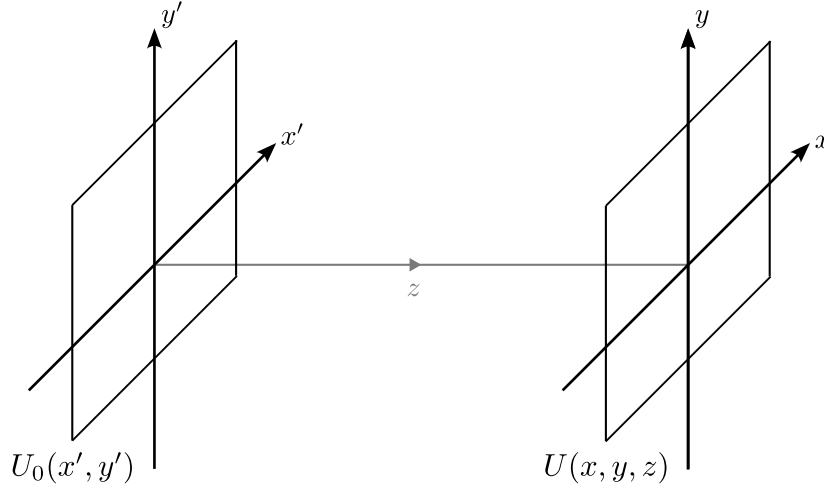


Figure 6.2 – Diffraction geometry, used to describe the propagation of $U_0(x', y')$ to $U(x, y, z)$.

Fresnel diffraction The propagation distance r can be rewritten as [Goo96, p. 66]

$$r = \sqrt{(x - x')^2 + (y - y')^2 + z^2} = z \sqrt{1 + \left(\frac{x - x'}{z}\right)^2 + \left(\frac{y - y'}{z}\right)^2}.$$

Using the Taylor series at $a = 0$ of $\sqrt{1 + a} = 1 + \frac{1}{2}a - \frac{1}{8}a^2 + \dots$, for a number a less than unity, r is approximated by

$$r \approx z \left(1 + \frac{1}{2} \left(\frac{x - x'}{z} \right)^2 + \frac{1}{2} \left(\frac{y - y'}{z} \right)^2 \right), \quad (6.20)$$

at the first order. Substituting (6.20) in (6.19) leads to the Fresnel approximation

$$U(x, y, z) = \frac{e^{jkz}}{j\lambda z} \iint U_0(x', y') e^{j\frac{k}{2z}[(x-x')^2 + (y-y')^2]} dx' dy', \quad (6.21)$$

where r at the denominator in (6.19) as been limited to the zero-th order ($r \approx z$), while r in the exponential is limited to the first order.

Equation (6.21) can be seen as a convolution between $U_0(x', y')$ and a convolution kernel $h(x', y') = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x'^2 + y'^2)}$. Alternatively, it can be expressed as, since $k = 2\pi/\lambda$,

$$U(x, y, z) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2 + y^2)} \iint \left[U_0(x', y') e^{j\frac{k}{2z}(x'^2 + y'^2)} \right] e^{-j\frac{2\pi}{\lambda z}(xx' + yy')} dx' dy', \quad (6.22)$$

where one recognises the 2D Fourier transform of the product of $U_0(x', y')$ and

a quadratic phase exponential, aside from multiplicative factors [Goo96, p. 67]. Both (6.21) and (6.22) are named Fresnel diffraction integrals, and describe the propagation in the near-field. The factor $e^{j\frac{k}{2z}(x'^2+y'^2)}$ is called *Fresnel propagator*.

Defining $\mathcal{F}\{A(x, y)\}(f_x, f_y)$ as the 2D Fourier transform operator applied to a 2D-field $A(x, y)$ and expressed in the domain of the spatial frequencies f_x, f_y , (6.22) becomes [KAU21, p. 146]

$$U(x, y, z) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \mathcal{F}\left\{U_0(x', y') e^{j\frac{k}{2z}(x'^2+y'^2)}\right\}\left(\frac{x}{\lambda z}, \frac{y}{\lambda z}\right). \quad (6.23)$$

Fraunhofer diffraction The Fraunhofer approximation assumes that the propagation distance z is larger than the maximal dimensions of $U_0(x', y')$, that is, [Goo96, p. 74]

$$z \gg \frac{k(x'^2 + y'^2)_{\max}}{2}.$$

If this condition is satisfied, then the quadratic exponential in (6.22) is approximated by one, and the diffracted field is now obtained from

$$U(x, y, z) = \frac{e^{jkz}}{i\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \iint U_0(x', y') e^{-j\frac{2\pi}{\lambda z}(xx' + yy')} dx' dy',$$

which exactly corresponds to the computation of the 2D-Fourier transform of $U_0(x', y')$, up to some multiplicative factors,

$$U(x, y, z) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \mathcal{F}\{U_0(x', y')\}\left(\frac{x}{\lambda z}, \frac{y}{\lambda z}\right). \quad (6.24)$$

6.2.1.2 Angular spectrum

As an alternative to the Rayleigh-Sommerfeld integral, the angular spectrum approach also expresses the field $U(x, y, z)$ after the propagation of $U_0(x', y')$ over a distance z . It is shown to give the same result as the Rayleigh-Sommerfeld solution [Goo96, p. 61].

Details about the angular spectrum approach can be found in [Goo96, p. 55]. It relies on the decomposition of the input field in plane waves by taking its Fourier transform, and then propagating each plane wave over a distance z . They are then added together with an inverse Fourier transform. Mathematically, the propagated field is given by [KAU21, p. 138]

$$U(x, y, z) = \mathcal{F}^{-1}\left\{\mathcal{F}\{U_0(x', y')\}(f_x, f_y) e^{jk_z z}\right\}(x, y),$$

with $k_z = 2\pi\sqrt{\frac{1}{\lambda^2} - f_x^2 - f_y^2}$.

For Fresnel diffraction, the angular spectrum form corresponds to the convolution seen in (6.21), computed in the frequency domain. Hence, it leads to the multiplication between the Fourier transforms of $U_0(x', y')$ and the convolution kernel

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$h(x', y')$. This latter is given by [Sch10, p. 95], using the result in [KAU21, p. 208]

$$\begin{aligned} H(f_x, f_y) &= \mathcal{F} \left\{ \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x'^2+y'^2)} \right\} (f_x, f_y) \\ &= e^{jkz} e^{-j\pi\lambda z(f_x^2+f_y^2)}. \end{aligned}$$

As a result, the propagated field, with the Fresnel approximation and computed from the angular spectrum approach, is [Sch10, p. 95-96]

$$U(x, y, z) = \mathcal{F}^{-1} \{ H(f_x, f_y) \mathcal{F} \{ U_0(x', y') \} (f_x, f_y) \} (x, y). \quad (6.25)$$

6.2.1.3 Numerical implementation

The numerical implementation of Fresnel and Fraunhofer diffractions, as given by (6.23) and (6.24), is detailed in [Sch10]. It requires the discretisation (i.e., sampling) of the source field over a 2D grid, with a constant grid spacing. Then, the use of 2D discrete Fourier transforms enables to compute the propagation of the field. The grid spacing in the propagated field plane is fixed by the grid spacing in the source plane, the propagation distance, and the wavelength [Sch10, p. 92].

In order to choose the resolution in the propagated field plane, the angular spectrum approach is used, as described in [Sch10, p. 96-102]. It introduces a tuneable scaling factor m , linking the grid spacings in the source and propagated field planes. The sampling constraints and requirement to fulfil to ensure an accurate propagation, e.g., without aliasing, are detailed in [Sch10, chap. 7-8]. In this work, the Python implementation of the angular spectrum approach given in the AOTools package is used [Tow+19].

6.2.2 Phase screen propagation

For propagation in non-vacuum media, such as the atmosphere, the split-step beam propagation method (i.e., a split-step Fourier method) is often used. It is a solution to the parabolic equation given in (6.17), as explained in [LF02, p. 3], [MF90] and the references therein. In this approach, the propagation path through a 3D random medium is split in an alternating sequence of vacuum propagations and phase screen perturbations. The phase screens are constructed based on the accumulated phase in the propagation volume, while the free space propagation makes use of 2D Fourier transforms, as introduced in Section 6.2.1.

Mathematically, the propagation, assumed on the z -axis, is split in several steps (in total, N_I steps). Each step describes the propagation between z_i and z_{i+1} ($i \in \{0, \dots, N_I - 1\}$) and involves:

1. A free-space propagation between z_i and z_{i+1}^- , denoted by the operator $\mathcal{P}$, such that

$$U(x, y, z_{i+1}^-) = \mathcal{P} \{ U(x', y', z_i) \} (x, y).$$

Here, the angular spectrum approach of the Fresnel approximation is used for $\mathcal{P}$, as given by (6.25).

2. The application of a phase screen as given by the operator $\mathcal{T}$, defined by

$$U(x, y, z_{i+1}^+) = \mathcal{T} \{U(x, y, z_{i+1}^-)\} = e^{-j\varphi_{i+1}(x, y)} U(x, y, z_{i+1}^-).$$

The notations z_{i+1}^- and z_{i+1}^+ corresponds to the distance z_{i+1} , respectively before the application of the phase screen, and after its application. The phase screen is described by the phase $\varphi_{i+1}(x, y)$, computed from [Sch10, p. 150]

$$\varphi_{i+1}(x, y) = k \int_{z_i}^{z_{i+1}} n'(x, y, z) dz. \quad (6.26)$$

In total, N_I steps are applied, such that $z_0 = 0$ is the source plane, and z_{N_I-1} corresponds to the total propagation distance. As the phase screen are computed from the refractive index fluctuations, they are random and depend on the atmospheric turbulence intensity through $C_n^2(z)$. Therefore, the received field in the observation plane (z_{N_I-1}) varies with the phase screen realisations, and simulations are repeated several times to obtain meaningful statistics (i.e., Monte Carlo simulations).

The generation of phase screens is desribed in Section 6.2.2.1, while the link with the $C_n^2(z)$ profile is presented in Section 6.2.2.2. A numerical illustration is given in Section 6.2.2.3, see Figure 6.3.

6.2.2.1 Phase screen generation

The random generation of phase screens is similar to the approach described in Section 2.5.3.1 to generate refractive index fluctuations, known as white noise coloring. However, instead of a 1D white noise sequence, a 2D image of white Gaussian noise is generated. Then, this image is filtered by a filter designed to ensure the desired power spectrum of the phase screen [Har+17]. Hence, the phase power spectrum $\Phi_\varphi(\boldsymbol{\kappa})$ is required.

Based on the definition of φ given in (6.26), its power spectrum is related to the refractive index power spectrum $\Phi_n(\boldsymbol{\kappa})$ by [Sch10, p. 160], [Rod81, p. 293 & 330]

$$\Phi_\varphi(\boldsymbol{\kappa}) = 2\pi k^2 \Delta z \Phi_n(\boldsymbol{\kappa}),$$

for a turbulent layer of width $z_{i+1} - z_i = \Delta z$ and constant C_n^2 . In cases where C_n^2 varies with the altitude z but is such that the refractive index power spectrum can be separated in the z and $\boldsymbol{\kappa}$ domains, i.e., $\Phi_n(\boldsymbol{\kappa}, z) = C_n^2(z) \hat{\Phi}_n(\boldsymbol{\kappa})$, the phase power spectrum is given by [Cha+20b]

$$\Phi_\varphi(\boldsymbol{\kappa}) = 2\pi k^2 \hat{\Phi}_n(\boldsymbol{\kappa}) \int_{z_i}^{z_i + \Delta z} C_n^2(z) dz.$$

Following the definition of the Fried parameter r_0 given in (6.32) described in

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Section 6.3.1, expressed over a layer of width Δz and at zenith, that is,

$$r_{0,i} = \left(0.423 k^2 \int_{z_i}^{z_i + \Delta z} C_n^2(z) dz \right)^{-3/5}, \quad (6.27)$$

the phase power spectrum is summarized by

$$\Phi_\varphi(\boldsymbol{\kappa}) = \frac{2\pi}{0.423} r_{0,i}^{-5/3} \hat{\Phi}_n(\boldsymbol{\kappa}).$$

Using a Kolmogorov power spectrum for $\Phi_n(\boldsymbol{\kappa})$, as given in (2.47), one obtains, with $\kappa = |\boldsymbol{\kappa}|$,

$$\Phi_\varphi(\kappa) = 0.49 r_{0,i}^{-5/3} \kappa^{-11/3}.$$

Instead, with a modified von Kármán spectrum from (2.50), the phase power spectrum is given by

$$\Phi_\varphi(\kappa) = 0.49 r_{0,i}^{-5/3} (\kappa^2 + \kappa_0^2)^{-11/6} e^{-\frac{\kappa^2}{\kappa_m^2}},$$

with $\kappa_m = 5.92/l_0$ and $\kappa_0 = 2\pi/L_0$.

In practice, from the phase power spectrum, there exist numerous methods to generate phase screens efficiently, with high accuracy, or with low memory requirement [Sch10, p. 166]. The subharmonic method, described in [LGD92], is chosen in this work to accurately represent the low-frequencies in the phase screens. Its numerical implementation is given in [Sch10, p. 168] and AOtools [Tow+19].

6.2.2.2 Sampling of C_n^2 profiles

In order to conduct phase screen propagation from a continuous $C_n^2(z)$ profile, it is necessary to choose the number of phase screen, their locations at altitude $\tilde{z}_i$, and their magnitudes. Their magnitudes are usually expressed through the Fried parameters $r_{0,i}$ for the i -th layer, or with equivalent $C_{n,i}^2 \Delta z_i$. The link between both quantities is [Sch10, p. 165], [RWH96, p. 70]

$$r_{0,i} = (0.423 k^2 C_{n,i}^2 \Delta z_i)^{-3/5},$$

as derived from (6.27).

The number of phase screens N_I , i.e., the number of propagation steps, is usually between 5 to 10 [Sch10, p. 165], and there exist several algorithms to compute their locations and magnitudes, such as

- **Matching of turbulence moments:** the values of $\tilde{z}_i$ and $C_{n,i}^2 \Delta z_i$ are chosen such that a set of discrete moments matches the continuous moments, i.e., [Sch10, p. 164], [RWH96, p. 67]

$$\sum_{i=0}^{N-1} \tilde{z}_i^m C_{n,i}^2 \Delta z_i = \int_0^L z^m C_n^2(z) dz,$$

with m the order of the moment ($0 \leq m \leq 7$ in [RWH96, p. 67]) and L the total propagation distance. The moments typically considered are the ones associated to the integrated quantities r_0 , θ_0 , etc. (see Section 6.3) for plane or spherical waves. They enable to construct a linear system, from which the phase screen altitudes $\tilde{z}_i$ and the magnitudes $C_{n,i}^2 \Delta z_i$ can be computed. Alternatively, the altitudes can be set beforehand and only the magnitudes are computed [TWR94].

- **Equivalent layers:** the atmosphere is divided in N_l slabs of same width, and for each slab, the equivalent $C_{n,i}^2 \Delta z_i$ is computed from [Fus+99; Tow+19]

$$C_{n,i}^2 \Delta z_i = \int_{z_{i,\min}}^{z_{i,\max}} C_n^2(z) dz, \quad (6.28)$$

where $z_{i,\min}$ and $z_{i,\max}$ are the minimal and maximal altitudes of the i -th slab. The equivalent altitude at which the phase screen is located is then computed from the C_n^2 -weighted integral

$$\tilde{z}_i = \left(\frac{\int_{z_{i,\min}}^{z_{i,\max}} C_n^2(z) z^{5/3} dz}{\int_{z_{i,\min}}^{z_{i,\max}} C_n^2(z) dz} \right)^{3/5}. \quad (6.29)$$

This approach, with the $5/3$ and $3/5$ exponents, is used in AOtools and conserves the isoplanatic angle.

Other C_n^2 -discretization algorithms are available in the literature, with several of them being compared in [Cha+20b]. An analysis of the required number of layers can be found in [Li+20; SX16].

In this work, the equivalent layer approach is used, as implemented in AOtools.

6.2.2.3 Numerical example

As an example, Figure 6.3 illustrates the phase screen propagation of a point source (i.e., planar wavefront) through the atmosphere. The atmospheric OT is assumed to be described by the HV-5/7 C_n^2 profile, from 0 to 25 km of altitude. This profile is sampled in five equivalent layers, as seen on the left part of Figure 6.3. The “atmosphere slabs” used for computing the equivalent layers according to (6.28) and (6.29) are equispaced and separated by black dashed lines. The altitudes $\tilde{z}_i$ and magnitudes $C_{n,i}^2 \Delta z_i$ of the equivalent layers are depicted by the red arrows (delta functions).

Based on the equivalent layer magnitudes and altitudes, phase screens are generated, as illustrated in the middle part of the figure. For illustration purposes, only the normalized magnitudes of the phase screens are shown. However, in practice, the magnitude of the phase screens depends on the equivalent layer magnitudes, i.e., $C_{n,i}^2 \Delta z_i$. The free-space propagation and phase screen application operators $\mathcal{P}$ and $\mathcal{T}$ are also depicted. Finally, from the output wavefront, the intensity of the received wave as imaged by a 11-inch telescope (27.94 cm, diameter of the SHIMM telescope) is

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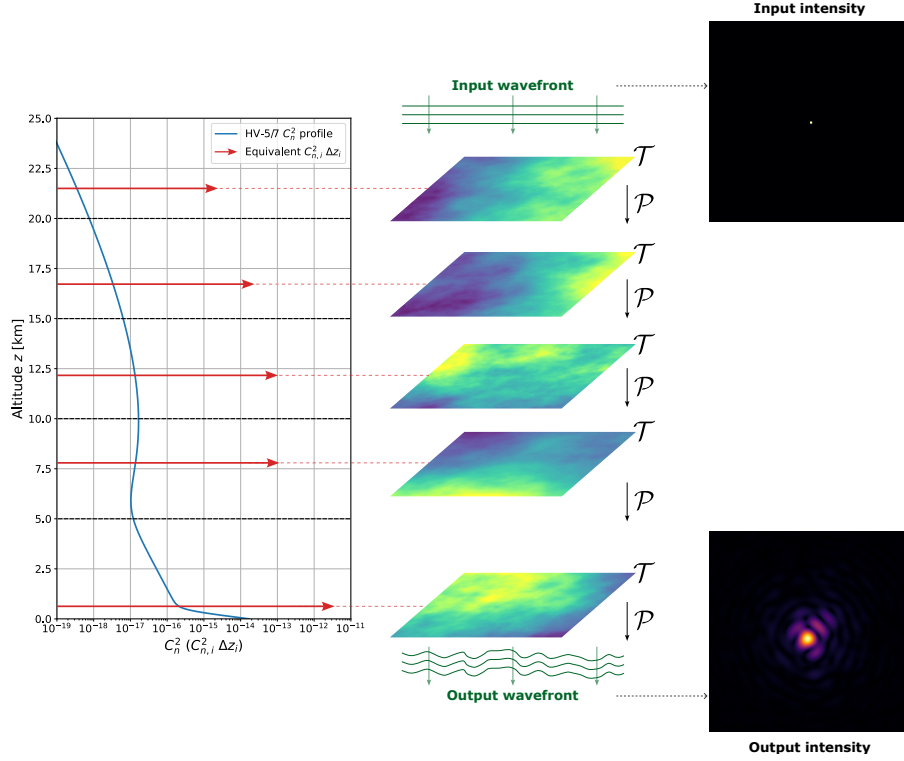


Figure 6.3 – Illustration of phase screen propagation through atmospheric turbulence for a plane wave (simulation parameters in Table 6.1).

computed and depicted on the bottom right part of the figure. Clearly, the received image suffers from OT distortions, and is far from the ideal Airy pattern expected in the case of free-space propagation through an annular pupil. All simulation parameters are summarized in Table 6.1. Kolmogorov turbulence is assumed for the generation of the phase, i.e., $l_0 \rightarrow 0$ and $L_0 \rightarrow \infty$.

The application of phase screen propagation is further presented and validated in Chapter 7, Section 7.2.2, in the particular case of the SHIMM instrument.

6.3 Important optical turbulence parameters

Refractive index fluctuations caused by atmospheric turbulence are described by vertical C_n^2 profiles, as detailed in the first part of this thesis. When considering the propagation of EM waves through such C_n^2 profiles, important OT parameters are derived. These parameters are presented in this section, highlighting their mathematical derivation and physical interpretation. A summary of most parameters introduced here can be found in Table 6.2, at the end of this chapter.

Table 6.1 – Parameters used for the phase screen propagation example.

Number of layers = 5			Parameters	Values
$\tilde{z}_i$ [m]	$C_{n,i}^2 \Delta z_i$ [m ^{1/3}]	$r_{0,i}$ [m]		
602	2.18×10^{-12}	0.16	Wavelength λ	1280 nm
7828	6.65×10^{-14}	1.26	Telescope diameter D	27.94 cm
12164	6.11×10^{-14}	1.33	Number of pixels in D	60
16720	1.41×10^{-14}	3.20	Oversampling factor	10
21501	1.34×10^{-15}	13.16	Phase screen size (in pixels)	600×600

6.3.1 Fried parameter

The *Fried parameter* r_0 , or *coherence length*, has been suggested by Fried in 1966 [Fri66]. Since then, it has become one of the key parameters to characterize atmospheric turbulence at astronomical sites and at optical communication sites [CCS22]. The success of the Fried parameter is probably related to its simplicity and its useful interpretation: it represents the diameter of a diffraction-limited telescope that would reach the same resolution as the one observed through atmospheric turbulence. In other words, under perfect conditions of observation (no atmosphere), the same resolution would be achieved with an ideal telescope of diameter r_0 , that is only limited by diffraction. Hence, “*atmospheric turbulence places an absolute upper limit on the resolution that can be obtained with a long exposure through the atmosphere*” [Fri66].

The term *resolution* used by Fried corresponds to the *resolving power*, whose definition is [Rod81, p. 300]:

$$\mathcal{R} = \int S(\mathbf{f}) \, d\mathbf{f} = \int A(\mathbf{f})T(\mathbf{f}) \, d\mathbf{f},$$

where $\mathbf{f} = (f_x, f_y)^T$ is the 2D spatial frequency in [m⁻¹], $S(\mathbf{f})$ is the Optical Transfer Function (OTF) of the whole system (telescope and atmosphere), and $A(\mathbf{f})$ and $T(\mathbf{f})$ are respectively the atmosphere and telescope OTFs.

The optical transfer function $O(\mathbf{f})$ is defined as the Fourier transform of the PSF. Based on [Hic08, p. 27], it can also be expressed as the autocorrelation of the complex field $U(\mathbf{x})$ received at the pupil, i.e.,

$$O(\boldsymbol{\rho}) = \int U(\mathbf{x})U^*(\mathbf{x} - \tilde{\boldsymbol{\rho}}) \, d\mathbf{x}, \quad (6.30)$$

with $\tilde{\boldsymbol{\rho}} = (\tilde{\rho}_x, \tilde{\rho}_y)^T$ in units of wavelength.

Diffraction-limited telescope In the case of a diffraction-limited telescope, the OTF is noted $T(\mathbf{f})$ and can be computed for a perfect plane wave incident to a circular

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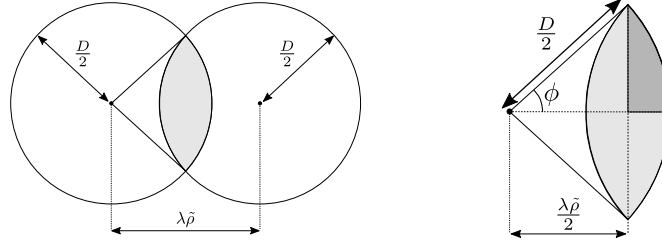


Figure 6.4 – Overlap area of two circles of diameter D separated by a distance $\lambda\tilde{\rho}$ (left). Zoom on the overlap area (right).

aperture of diameter D . In this case, the complex field is given by

$$U(\tilde{\rho}) = \begin{cases} 1 & \text{if } \lambda\tilde{\rho} \leq \frac{D}{2} \\ 0 & \text{otherwise} \end{cases},$$

with $\tilde{\rho} = |\tilde{\rho}|$. Hence, with such a function $U(\tilde{\rho})$, (6.30) reduces to computing the area of overlap of two circles separated by a distance $\lambda\tilde{\rho}$ [Hic08, p. 29].

Using the symmetry, the overlap area (shaded in light gray in Figure 6.4) as a function of $\tilde{\rho}$ can be computed by taking four times the area of the dark shaded region in Figure 6.4. Its area is given by the difference between the area of the circular sector of angle ϕ and the area of the right triangle of hypotenuse $\frac{D}{2}$, i.e.,

$$\frac{\phi}{2\pi} \frac{\pi D^2}{4} - \frac{1}{2} \frac{\lambda\tilde{\rho}}{2} \sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{\lambda\tilde{\rho}}{2}\right)^2}.$$

Moreover, the angle ϕ is given by $\phi = \arccos\left(\frac{\lambda\tilde{\rho}}{D}\right)$. Finally, the OTF as a function of $\tilde{\rho}$ is:

$$\begin{aligned} T(\tilde{\rho}) &= \frac{1}{A} 4 \left[\frac{1}{2\pi} \frac{\pi D^2}{4} \arccos\left(\frac{\lambda\tilde{\rho}}{D}\right) - \frac{1}{4} \frac{\lambda\tilde{\rho} D}{2} \sqrt{1 + \left(\frac{\lambda\tilde{\rho}}{D}\right)^2} \right] \\ &= \frac{2}{\pi} \left[\arccos\left(\frac{\lambda\tilde{\rho}}{D}\right) - \frac{\lambda\tilde{\rho}}{D} \sqrt{1 + \left(\frac{\lambda\tilde{\rho}}{D}\right)^2} \right], \end{aligned}$$

with A being a normalisation factor chosen to be equal to the area of the circular aperture ($A = \frac{\pi D^2}{4}$) [Fri66; Hic08]. With this choice, $T(0) = 1$.

Based on the OTF $T(\tilde{\rho})$, the resolving power $\mathcal{R}_D$ of a diffraction-limited telescope is obtained:

$$\mathcal{R}_D = \int T(\mathbf{f}) \, d\mathbf{f} = \int_0^{2\pi} \int_0^{D/\lambda} T(\tilde{\rho}) \, \tilde{\rho} \, d\tilde{\rho} \, d\phi$$

$$\begin{aligned}
 &= 4 \left(\frac{D}{\lambda} \right)^2 \int_0^1 \left[u \arccos(u) - u \sqrt{1+u^2} \right] du \\
 &= \frac{\pi}{4} \left(\frac{D}{\lambda} \right)^2,
 \end{aligned}$$

using the change of variable $u = \frac{\tilde{\rho}\lambda}{D}$ and the tabulated integrals $\int_0^1 u \arccos(u) du = \frac{\pi}{8}$ and $\int_0^1 u \sqrt{1+u^2} du = \frac{\pi}{16}$. This is indeed the result presented in [Rod81, p. 300].

Turbulence-limited telescope The OTF of atmospheric turbulence $A(f)$ is related to the coherence function $B(\tilde{\rho})$ thanks to the stationarity hypothesis that enables to substitute the integral in (6.30) by an ensemble average [Hic08]. Using the isotropy hypothesis and monochromatic plane waves, the coherence function is shown to be given by [Hic08, p. 30]:

$$B(\tilde{\rho}) = \exp \left(-4\pi^2 k^2 \int_0^z dz' \int_0^\infty \Phi(\kappa) [1 - J_0(\kappa \lambda \tilde{\rho})] \kappa d\kappa \right)$$

with $\Phi(\kappa)$ the power spectrum of the refractive index fluctuations. This is equation (5.18) in [Hic08, p. 30]. Writing the complex amplitude in terms of phase and log-amplitude fluctuations ($U = \exp(\chi + j\varphi)$) and reexpressing the coherence function, the following result is obtained [Hic08, p. 23]

$$B(\rho) = \exp \left(-\frac{1}{2} \mathcal{D}(\rho) \right),$$

with $\mathcal{D}(\rho) = \mathcal{D}_\chi(\rho) + \mathcal{D}_\varphi(\rho)$ called the *wave structure function*. Hence,

$$\mathcal{D}(\rho) = 8\pi^2 k^2 \int_0^z dz' \int_0^\infty \Phi(\kappa) [1 - J_0(\kappa \rho)] \kappa d\kappa.$$

Substituting $\Phi(\kappa)$ by the Kolmogorov power spectrum and performing the integration over κ , the wave structure function becomes:

$$\begin{aligned}
 \mathcal{D}(\rho) &= 0.033 (8\pi^2 k^2) \frac{3\Gamma(1/6)}{10\Gamma(11/6) 2^{2/3}} \rho^{5/3} \int_0^\infty C_n^2(l) dl \\
 &\approx 2.913 k^2 \sec(\tilde{\theta}) \rho^{5/3} \int_0^\infty C_n^2(z) dz \\
 &\approx 2K \rho^{5/3}
 \end{aligned} \tag{6.31}$$

relating a distance l along the line of sight by $l = z \sec(\tilde{\theta})$ with z the altitude and $\tilde{\theta}$ the elevation angle measured from zenith. This is equation (5.19) in [Hic08, p. 30], also expressed in equation (3.22) in [Rod81, p. 296]. The constant K corresponds to $\frac{2.913}{2} k^2 \sec(\tilde{\theta}) \int_0^\infty C_n^2(z) dz$. With these results and hypotheses, the atmosphere

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OTF $A(\mathbf{f})$ given by $B(\tilde{\rho})$, with $\tilde{\rho} = |\mathbf{f}|$ in this case, can be integrated to obtain the resolving power of the atmosphere (i.e., for a telescope with an infinitely large aperture):

$$\mathcal{R}_\infty = \int A(\mathbf{f}) d\mathbf{f} = \int_0^{2\pi} \int_0^\infty \exp\left(-K(\lambda\tilde{\rho})^{5/3}\right) \tilde{\rho} d\tilde{\rho} d\phi = \frac{1}{\lambda^2} \frac{6\pi}{5} K^{-\frac{6}{5}} \Gamma(6/5),$$

that is the result in equation (4.20) in [Rod81, p. 301] (with a different definition for K). It uses the mathematical result $\int_0^\infty u e^{-Ku^{5/3}} du = \frac{3\Gamma(6/5)}{5K^{6/5}}$ if the real part of K is positive.

Comparison and introduction of r_0 From $\mathcal{R}_D$ and $\mathcal{R}_\infty$, the Fried parameter can be determined based on its definition. For a telescope of diameter r_0 , one has:

$$\mathcal{R}_D = \mathcal{R}_\infty \quad \Leftrightarrow \quad \frac{\pi}{4} \left(\frac{r_0}{\lambda}\right)^2 = \frac{1}{\lambda^2} \frac{6\pi}{5} K^{-\frac{6}{5}} \Gamma(6/5)$$

such that

$$\begin{aligned} r_0 &= \sqrt{\frac{24}{5} \Gamma(6/5)} \left(\frac{2.913}{2} k^2 \sec(\tilde{\vartheta}) \int_0^\infty C_n^2(z) dz \right)^{-3/5} \\ &= \left(0.423 k^2 \sec(\tilde{\vartheta}) \int_0^\infty C_n^2(z) dz \right)^{-3/5}. \end{aligned} \quad (6.32)$$

This is the mathematical expression giving the Fried parameter r_0 as a function of the wavelength (dependency in $\lambda^{6/5}$), C_n^2 and the elevation angle (corresponding to (4.24) in [Rod81] and (4.31) in [Hic08]). The expression of r_0 for spherical waves is given in [Gre77].

With the Fried parameter defined as such, one can conveniently rewrite the wave structure function:

$$\mathcal{D}(\rho) = 6.8839 \left(\frac{\rho}{r_0} \right)^{5/3}, \quad (6.33)$$

and the coherence function:

$$B(\rho) = \exp \left(-3.44 \left(\frac{\rho}{r_0} \right)^{5/3} \right).$$

These expressions are very often used in the literature.

In summary, the Fried parameter r_0 aggregates in a single quantity the wavelength, the elevation angle, and the C_n^2 profile dependency. It typically varies between 2 and 20 cm in the visible, such that most astronomical observations are limited by the atmosphere. However, observations can be diffraction limited in the infrared under favourable circumstances [Rod81, p. 302].

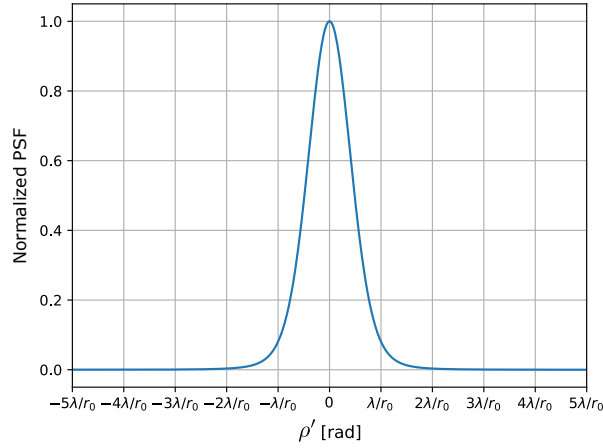


Figure 6.5 – Normalized PSF with atmospheric turbulence.

6.3.2 Seeing

The *seeing* $\mathcal{E}_0$ is defined as the FWHM of the PSF of a star for long exposures. Mathematically, the PSF can be obtained by taking the inverse Fourier transform of the OTF, which reduces to the Hankel transform of order 0 in case of circular symmetry. For a function $f(x, \phi)$ having a circular symmetry (i.e., $f(x, \phi) = f(x)$),

$$F(\rho) = \int_{x=0}^{+\infty} f(x) \underbrace{\int_{\phi=0}^{2\pi} e^{j\rho x \cos(\phi)} d\phi}_{2\pi J_0(\rho x)} x dx = 2\pi \int_0^{+\infty} f(x) J_0(\rho x) x dx,$$

with $J_0(\cdot)$ the Bessel function of first kind, order 0.

Application of this result to the OTF of turbulence $B(x)$ gives

$$\begin{aligned} I(\rho) &= 2\pi \int_0^{+\infty} e^{-3.44 \left(\frac{\lambda}{r_0} \frac{x}{2\pi} \right)^{5/3}} J_0(\rho x) x dx \\ &= 2\pi \left(\frac{r_0}{\lambda^2} \right)^2 \int_0^{+\infty} e^{-3.44 \left(\frac{x'}{2\pi} \right)^{5/3}} J_0 \left(\rho \frac{r_0}{\lambda} x' \right) x' dx', \end{aligned}$$

with $x' = x \frac{\lambda}{r_0}$. $I(\rho)$ can then be made dimensionless by stating $\rho' = \rho \frac{r_0}{\lambda}$.

Such integral cannot be solved analytically and hence numerical integration is used as stated in [Die88] and [Rac96]. The resulting PSF is depicted in Figure 6.5.

Based on Figure 6.5, the FWHM (i.e., where PSF norm.=0.5) is given by

$$\mathcal{E}_0 := \text{FWHM} = 2 \left(0.488 \frac{\lambda}{r_0} \right) = 0.976 \frac{\lambda}{r_0}, \quad (6.34)$$

which is the result often used and also presented in [Rac96]. $\mathcal{E}_0$ is commonly given in arcsec, the link between radian and arcsec being a multiplication factor $\frac{180}{\pi} \times 60 \times 60$.

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Equation (6.34) gives the link between the seeing $\mathcal{E}_0$ in arcsec and the Fried parameter r_0 . With this simple definition, both quantities can be used interchangeably. Using (6.32), the definition of $\mathcal{E}_0$ based on the C_n^2 profile is

$$\mathcal{E}_0 = 5.285\lambda^{-1/5} \left(\sec(\tilde{\vartheta}) \int_0^\infty C_n^2(z) dz \right)^{3/5}. \quad (6.35)$$

Hence, as seen from the C_n^2 integral, the seeing is mostly impacted by layers having large values of C_n^2 . Since it is often the BL that suffers from the largest turbulence fluctuations, this layer tends to contribute mostly to the seeing [Osby+15; Osby+10].

Typical values of $\mathcal{E}_0$ at astronomical sites are between 0.2 and 1 arcsec [Law+04].

6.3.3 Isoplanatic angle

The *isoplanatic angle* θ_0 defines the angular region where the turbulence is considered to be similar, or correlated. Mathematically, it is the angle at which the RMSE on the wavefront phase has increased by 1 radian. It is defined by considering two beams coincident at the telescope pupil and then separated by an angle Θ [Har98] (e.g., at a distance l from the telescope, they are separated by a distance Θl). The RMS phase difference between these two beams, averaged over the pupil, is given by [Hic08, p. 43]:

$$\sigma_\Theta^2 = \frac{1}{A} \int_A (\varphi(\mathbf{x}) - \varphi(\mathbf{x} + \Theta \mathbf{z}))^2 d\mathbf{x}.$$

This expression relates to the definition of the structure function $\mathcal{D}_\varphi$, that, in the case of astronomical applications, is often assumed to dominate compared to the log-amplitude fluctuations and thus $\mathcal{D}_\varphi(\rho) \approx \mathcal{D}(\rho)$. With these assumptions and $z_l = z \sec(\tilde{\vartheta})$,

$$\begin{aligned} \sigma_\Theta^2(\Theta) &\approx \mathcal{D}(\Theta z) = 2.913k^2 \sec(\tilde{\vartheta}) \int_0^\infty C_n^2(z) (\Theta z \sec(\tilde{\vartheta}))^{5/3} dz \\ &= 2.913k^2 (\sec(\tilde{\vartheta}))^{8/3} \Theta^{5/3} \int_0^\infty C_n^2(z) z^{5/3} dz \\ &= \left(\frac{\Theta}{\theta_0} \right)^{5/3}, \end{aligned}$$

with the *isoplanatic angle* θ_0 defined by:

$$\theta_0 = \left(2.913 k^2 (\sec(\tilde{\vartheta}))^{8/3} \int_0^\infty C_n^2(z) z^{5/3} dz \right)^{-3/5}. \quad (6.36)$$

This is the *modern* definition of the isoplanatic angle, that can be compared to the first definition proposed by Fried in 1982, where the numerical coefficient is slightly different [Fri82]. Equation (6.36) implies that the isoplanatic angle is mostly determined by high-level turbulence, due to the factor $z^{5/3}$ in the C_n^2 integral [Hic08]. It is therefore very sensitive to this high-level turbulence, that remains a challenge to

model with NWP simulations [HML11].

The isoplanatic angle is particularly useful in the context of AO with a reference natural star (or an artificial laser guide star), as it sets some constraints on the acceptable angular separation between this reference star and the science object of interest. Indeed, in order to assume their respective lights are passing through the same turbulence, they must have an angular separation smaller than the isoplanatic angle. The isoplanatic angle enables thus to specify the required field for AO observations [MLF13].

The isoplanatic angle is also of importance for optical communications and especially for uplink precompensation with a point-ahead angle (introduced in Section 1.2.3). Indeed, if this point-ahead angle is larger than the isoplanatic angle, anisoplanatism occurs and the turbulence observed on the downlink is different than the one on the uplink [Mar23].

Typical values of the isoplanatic angles are 2 arcsec for a wavelength of 0.5 μm , or 5 arcsec for 1 μm at astronomical sites [Hic08, p. 9].

6.3.4 Atmospheric time constant

The *atmospheric time constant* τ_0 (also named *coherence time*) defines the time after which the wavefront phase structure has changed [Hic08; Gio14]. It is expressed from the phase structure function $\mathcal{D}_\varphi(r)$ making use of the Taylor hypothesis, i.e., $r = vt$, with v the transverse wind speed. In this case, following (6.31),

$$\mathcal{D}_\varphi(t) = 2.913k^2 \sec(\tilde{\vartheta}) t^{5/3} \int_0^\infty C_n^2(z) v^{5/3}(z) dz = \left(\frac{t}{\tau_0} \right)^{5/3},$$

with τ_0 the atmospheric time constant given by

$$\tau_0 = \left(2.913k^2 \sec(\tilde{\vartheta}) \int_0^\infty C_n^2(z) v^{5/3}(z) dz \right)^{-3/5}. \quad (6.37)$$

6.3.5 Greenwood frequency

The Greenwood frequency [Gre77] is used in the context of AO to determine the required rate of the AO system for compensating atmospheric turbulence. It is given by [TF22; Hic08]:

$$f_G = \left(0.102k^2 \sec(\tilde{\vartheta}) \int_0^\infty C_n^2(z) v^{5/3}(z) dz \right)^{3/5}$$

with $v(z)$ the transverse wind speed profile. It sets constraints on the AO closed-loop bandwidth to be about ten times the Greenwood frequency [Hic08; GF76]. The Greenwood frequency is related to the atmospheric time constant by $f_G = \frac{0.1338}{\tau_0}$ [Hic08; Fri90].

Typically, the Greenwood frequency is in the range of tens to hundreds of

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Hertz [TF22]. For FSOC, the Greenwood frequency also depends on the satellite slew rate, that should be included in $v(z)$. In communication scenarios with low-Earth-orbit satellites, the satellite motion can lead to Greenwood frequencies of thousands of Hertz [CM20].

6.3.6 Scintillation index

The scintillation index σ_I^2 is associated to the normalized variance of intensity fluctuations I . It is defined by [Ish78, p. 445], [AP05, p. 157]

$$\sigma_I^2 = \frac{\langle (I - \langle I \rangle)^2 \rangle}{\langle I \rangle^2} = \frac{\langle I^2 \rangle}{\langle I \rangle^2} - 1. \quad (6.38)$$

In the weak fluctuation regime, the scintillation index is related through the log-amplitude variance σ_χ^2 according to [AP05, p. 157]

$$\sigma_I^2 = e^{4\sigma_\chi^2} - 1 \approx 4\sigma_\chi^2.$$

Hence, using (6.4), the scintillation index is given by

$$\sigma_I^2 = 1.23 C_n^2 k^{7/6} L^{11/6},$$

for plane waves with a constant C_n^2 over a propagation distance L [Ish78, p. 446], [AP05, p. 264]. For downlinks with a vertically-varying C_n^2 profile, the scintillation index, based on (6.14), is [AP05, p. 495]

$$\sigma_I^2 = 2.25 k^{7/6} \sec^{11/6}(\tilde{\theta}) \int_{h_0}^H C_n^2(z) (z - h_0)^{5/6} dz \quad (6.39)$$

This also corresponds to the result in [Rod81, p. 342] for the zenith. Using (6.15) and (6.16) instead, σ_I^2 for spherical waves is obtained [AP05, p. 264].

For weak fluctuations, the scintillation index is equal to the Rytov variance $\sigma_R^2 = 4\sigma_\chi^2$. For strong fluctuations, the scintillation index tends to saturate such that it is not depending linearly on σ_R^2 anymore. It is given by [AP05, p. 497]

$$\sigma_I^2 = e^{\left[\frac{0.49\sigma_R^2}{(1+1.11\sigma_R^{12/5})^{7/6}} + \frac{0.51\sigma_R^2}{(1+0.69\sigma_R^{12/5})^{5/6}} \right]} - 1,$$

for $0 \leq \sigma_R^2 < \infty$.

6.3.7 Strehl ratio

The *Strehl ratio* $\mathcal{S}$ is defined as the ratio between the received on-axis intensity I and that of an unaberrated beam $\tilde{I}$ (i.e., diffraction limited) [TF22]:

$$\mathcal{S} = \frac{I(0)}{\tilde{I}(0)},$$

where the argument 0 denotes the intensity as measured in the centre (on-axis, reference position). This central intensity is obtained by taking the integral of the OTF [Hic08]:

$$I(0) = \int \mathcal{S}(f) \, df,$$

which is exactly the resolving power as defined previously $\mathcal{R}$. Hence, the Strehl ratio can be expressed as $\mathcal{S} = \mathcal{R}/\mathcal{R}_D$.

It measures the optical quality of imaging and beam propagation, having a value between 0 and 1 (perfect imaging). It is of interest for taking into account lens aberrations or imaging through turbulence.

The Maréchal approximation is often used to estimate the Strehl ratio based on the phase variance: $\mathcal{S} \approx e^{-\sigma_\phi^2}$. A demonstration of this relation can be found in [Hic08].

6.3.8 Altitude dependency of optical turbulence integrated parameters

As a summary for this section, the altitude dependency of some OT integrated quantities is illustrated in Figures 6.6 and 6.7. Inspired from [Har+17], these figures depict the vertical profile of the integrand in the mathematical definitions of the Fried parameter, isoplanatic angle, coherence time, and the scintillation index. The reference C_n^2 profile used in the integration is the HV-5/7, shown on the left part of each row. The wind profile, necessary to compute the coherence time, is a Bufton profile, as given in (3.6), with $v_G = 10$ m/s, $v_T = 30$ m/s, $h_T = 9400$ m, and $L_T = 4800$ m. The considered wavelength is $\lambda = 500$ nm.

Graphs in the middle column present the weighting functions $f(z)$ that multiply the C_n^2 profile in the integrands of each OT integrated quantity. For example, a function in $z^{5/3}$ is seen in the isoplanatic angle expression given in (6.36). The last column shows the integrand (i.e., the product between the C_n^2 profiles in the first column and the weighting functions in the middle column), with the result from the integral (i.e., the OT integrated quantity) given in the legend. The gray-shaded areas depict the regions over which the integration can be performed to still obtain the associated OT quantity with an error smaller than 5% (compared to the integration when considering the whole C_n^2 profile). Hence, those areas highlight the most important altitude range sensed by each OT integrated quantity. It further emphasizes their complementarity, e.g., with the Fried parameter mostly depending on the BL, while high-altitude layers have increased influence on the isoplanatic angle. These

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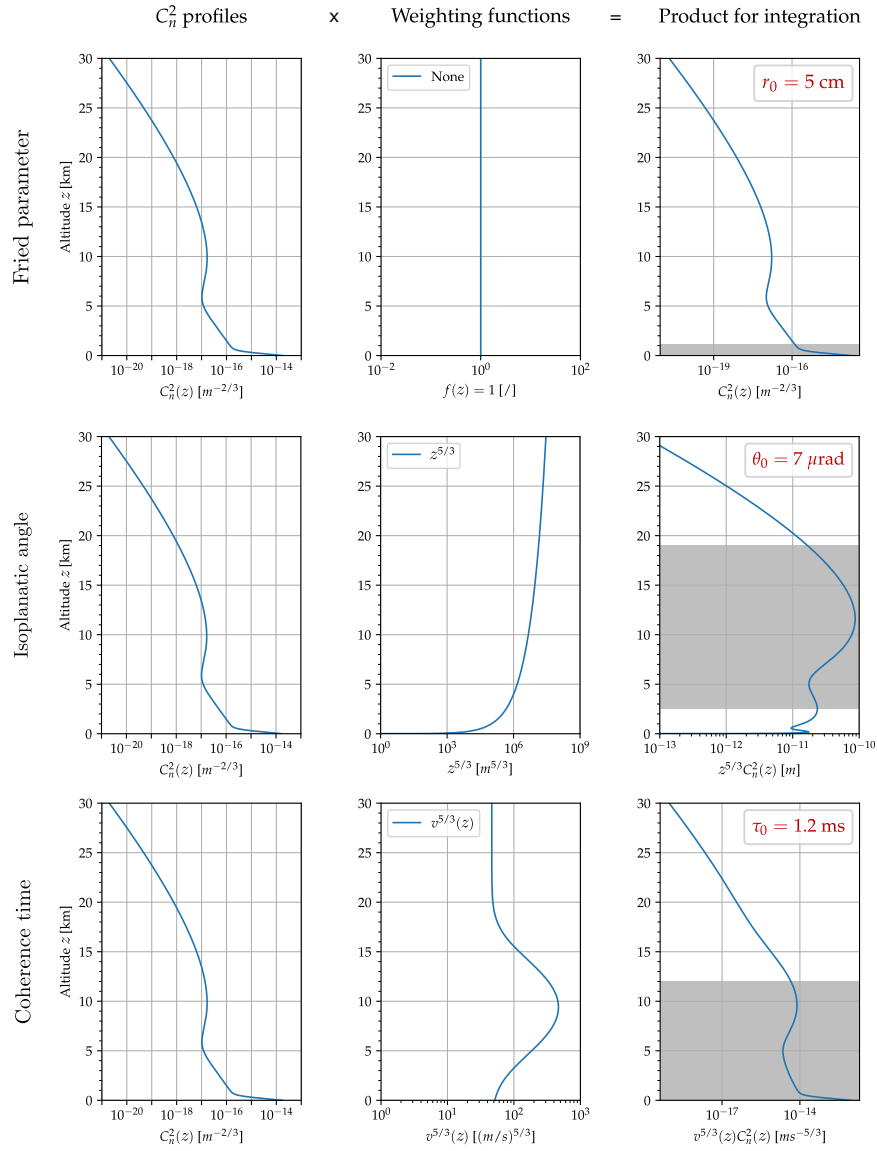


Figure 6.6 – Illustration of weighting functions used to compute C_n^2 -related quantities (part I).

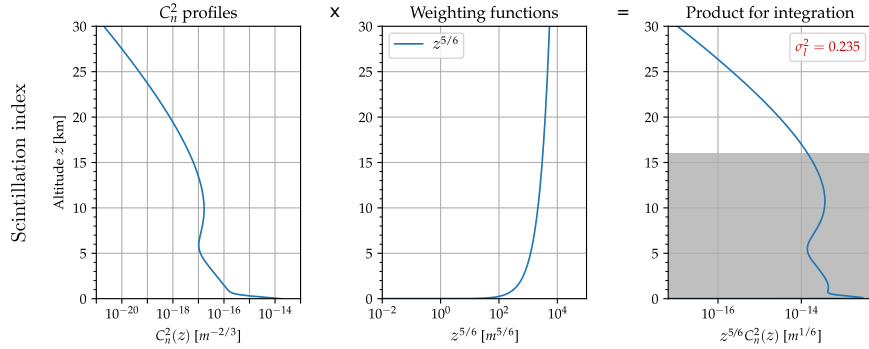


Figure 6.7 – Illustration of weighting functions used to compute C_n^2 -related quantities (part II).

results assume a HV-5/7 profile, which is an average profile. Depending on local conditions, instantaneous C_n^2 profiles could show features for which the conclusions stated above should be reconsidered.

6.4 Optical turbulence monitoring instruments

Various optical turbulence monitoring instruments have been developed in recent decades, namely to monitor OT at astronomical sites, during nighttime.

This section presents some of these instruments, serving as an introduction to the SHIMM instrument detailed in Chapter 7. Only the working principle of the DIMM instrument is presented thoroughly in Section 6.4.1, as it motivates the physics applied and some design choices used in other instruments. Other instruments are briefly introduced, with references provided.

The following references are of particular interest to get an overview of available OT monitoring instruments: [Osb10], [Per18], [Hic14], [BL22] and [Zia23]. A focus on instruments for daytime OT monitoring is presented in [JTY23].

6.4.1 Differential Image Motion Monitor (DIMM)

The Differential Image Motion Monitor is commonly used to measure the seeing (or equivalently, the Fried parameter) at astronomical sites. It is a compact instrument that measures the differential image motion of a star seen through two apertures. Thanks to this differential approach, the obtained seeing only depends on the image motions created by the turbulence, and rejects biases associated to vibrations, tracking errors and wind-shake [Osb10; Hic14]. The DIMM provides an estimate of the total atmospheric seeing (or Fried parameter), and does not give access to a C_n^2 profile.

The original idea of the DIMM is attributed to [SK61], while the instrument has first been developed by ESO for the site-selection of the VLT [SR90]. It is studied

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deeper in [VM95] and [Tok02]. Based on these references, the mathematical equations describing the DIMM instrument are presented in Section 6.4.1.1. Then, more recent extensions of the DIMM are given in Section 6.4.1.2, as well as an illustration of the instrument used at Redu for the SALTO project (Section 6.4.1.3). An illustration of the DIMM instrument can be found in [Tok02].

6.4.1.1 Mathematical derivation

In the following, the mathematics enabling to link the Fried parameter r_0 to the variances of the two image centroids are detailed. Firstly, an approximation valid when the distance d between the two subapertures is larger than the subaperture diameter D is presented, based on the work from Sarazin and Roddier [SR90]. Then, more accurate equations from Tokovinin are provided [Tok02].

According to [SR90], obtaining the variance of the differential image motion is achieved by first expressing the angle-of-arrival fluctuations. In the x -direction, i.e. the direction assumed to be the baseline between the two apertures, the angle-of-arrival α is given by

$$\alpha(x, y) = -\frac{\partial}{\partial x} z_c(x, y),$$

where x, y are the coordinates in the image plane and z_c is defined as the wavefront corrugation. This latter describes the geometric deformation of the wavefront coming from the wavefront phase variations φ , that is,

$$z_c(x, y) = \frac{\lambda}{2\pi} \varphi(x, y).$$

Tilted wavefront arriving on the DIMM apertures will induce a longitudinal motion of the image centroids related to the angle-of-arrival fluctuation. Its covariance function is

$$B_\alpha(\xi, \eta) = \langle \alpha(x, y) \alpha(x + \xi, y + \eta) \rangle,$$

where ξ and η are the separation distances along the x - and y -directions.

Since $\alpha(x, y)$ is related to $z_c(x, y)$ that itself depends on $\varphi(x, y)$, the covariance of the angle-of-arrival fluctuation $B_\alpha(\xi, \eta)$ is related to the covariance of the phase fluctuation $B_\varphi(\xi, \eta)$ by [SR90]

$$B_\alpha(\xi, \eta) = -\frac{\lambda^2}{4\pi^2} \frac{\partial^2}{\partial^2 \xi} B_\varphi(\xi, \eta).$$

Assuming homogeneity, the phase structure function $D_\varphi(\xi, \eta)$ is related to the covariance function of the phase by

$$D_\varphi(\xi, \eta) = 2 [B_\varphi(0, 0) - B_\varphi(\xi, \eta)],$$

meaning that the covariance of the angle-of-arrival fluctuation can be expressed as

$$B_\alpha(\xi, \eta) = \frac{\lambda^2}{8\pi^2} \frac{\partial^2}{\partial \xi^2} D_\varphi(\xi, \eta).$$

The phase structure function in the case of Kolmogorov turbulence has been expressed previously with the Fried parameter r_0 , as in (6.33). With this result, stating $r = \sqrt{\xi^2 + \eta^2}$, the covariance $B_\alpha(\xi, \eta)$ is finally given by

$$\begin{aligned} B_\alpha(\xi, \eta) &= 0.087\lambda^2 r_0^{-5/3} \frac{\partial^2}{\partial \xi^2} (\xi^2 + \eta^2)^{5/6} \\ &= 0.145\lambda^2 r_0^{-5/3} \left[(\xi^2 + \eta^2)^{-1/6} - \frac{1}{3}\xi^2 (\xi^2 + \eta^2)^{-7/6} \right]. \end{aligned} \quad (6.40)$$

Choosing $\eta = 0$ provides the longitudinal covariance $B_l(\xi) := B_\alpha(\xi, 0)$, i.e., the covariance in the direction of the tilt, whereas choosing $\xi = 0$ gives the transverse covariance $B_t(\eta) := B_\alpha(0, \eta)$. However, (6.40) is not valid at the origin ($\xi = 0, \eta = 0$) since its divergence is not physical. In fact, its value should be limited by the aperture averaging effect and is given by [SR90; Fri75]

$$B_\alpha(0, 0) = 0.179 \left(\frac{\lambda}{r_0} \right)^{5/3} \left(\frac{\lambda}{D} \right)^{1/3} = B_l(0) = B_t(0).$$

Except for the origin and points nearby, this aperture averaging effect becomes negligible as long as the distance d between the aperture is larger than $2D$ [SR90].

Finally, the variances of the longitudinal and lateral differential image motions observed over a distance d are obtained by

$$\begin{aligned} \sigma_l^2(d) &= 2 [B_l(0) - B_l(d)] = 2\lambda^2 r_0^{-5/3} \left[0.179D^{-1/3} - 0.0968d^{-1/3} \right], \\ \sigma_t^2(d) &= 2 [B_t(0) - B_t(d)] = 2\lambda^2 r_0^{-5/3} \left[0.179D^{-1/3} - 0.145d^{-1/3} \right]. \end{aligned}$$

They can be rewritten, defining $b = d/D$ as in [Tok02]:

$$\begin{aligned} \sigma_l^2(d) &= \lambda^2 r_0^{-5/3} D^{-1/3} 0.358 \left[1 - 0.541b^{-1/3} \right], \\ \sigma_t^2(d) &= \lambda^2 r_0^{-5/3} D^{-1/3} 0.358 \left[1 - 0.810b^{-1/3} \right]. \end{aligned}$$

These expressions are approximations, valid only if $b \gg 1$. In [Tok02], more general expressions are presented, in the cases of G-tilt and Z-tilt².

G-tilt The G-tilt (gradient tilt) is defined as the averaged tilt over a given aperture [TF22]. Mathematically, it is obtained thanks to [Tyl94], for a circular aperture of

²The wavefront tilt leads to a deviation in the direction of propagation of a beam. Instead of having a planar wavefront perpendicular to the optical axis, the wavefront can be tilted.

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radius R :

$$\theta_G = \frac{1}{\pi R^2} \frac{1}{k} \int W(\mathbf{r}/R) \nabla \varphi(\mathbf{r}) d\mathbf{r},$$

with $\varphi(\mathbf{r})$ the wavefront distortion at coordinate $\mathbf{r}$. The function $W(\mathbf{r})$ is such that $W(\mathbf{r}) = 1$ if $|\mathbf{r}| < 1$ and 0 otherwise. Equivalently, the G-tilt is obtained when averaging tilt measurements over multiple subapertures [TF22]. Hence, it can be interpreted as the average gradient over a given area.

Using G-tilts, the variances of the differential image motion are obtained thanks to [Tok02; Con+00]

$$\begin{aligned} \sigma_l^2(d) &= \lambda^2 r_0^{-5/3} D^{-1/3} 0.340 \left[1 - 0.570b^{-1/3} - 0.040b^{-7/3} \right], \\ \sigma_t^2(d) &= \lambda^2 r_0^{-5/3} D^{-1/3} 0.340 \left[1 - 0.855b^{-1/3} + 0.030b^{-7/3} \right]. \end{aligned}$$

Z-tilt Z-tilts (Zernike tilt) correspond directly to the Zernike polynomials describing the tilt. They can be obtained by determining the plane that best fits the wavefront distortion. The direction of the normal to this plane gives the Z-tilt [Ty194]. Mathematically, it is given by

$$\theta_Z = \frac{4}{\pi R^4} \frac{1}{k} \int W(\mathbf{r}/R) \varphi(\mathbf{r}) \mathbf{r} d\mathbf{r}.$$

[Sas12] provides a series-expansion for differential Z-tilts. Limiting this series to the first two terms, one has [Tok02]

$$\begin{aligned} \sigma_l^2(d) &= \lambda^2 r_0^{-5/3} D^{-1/3} 0.364 \left[1 - 0.532b^{-1/3} - 0.024b^{-7/3} \right], \\ \sigma_t^2(d) &= \lambda^2 r_0^{-5/3} D^{-1/3} 0.364 \left[1 - 0.798b^{-1/3} + 0.018b^{-7/3} \right]. \end{aligned}$$

According to [Tok02], Z-tilts are the tilts measured by a DIMM. Hence, this latter result should be used. Interestingly, it is quite comparable to the approximation presented by Sarazier and Roddier.

6.4.1.2 Extensions to the DIMM instrument

Several extensions or variations of the DIMM instrument are presented in the literature [Per18]. Some are listed below:

- **Generalized DIMM (GDIMM)**: making use of three subapertures [Ari+14]. Approaches to estimate the isoplanatic angle, the outer scale, and the coherence time are also presented in this reference.
- **Hartmann DIMM (H-DIMM)**: using a Hartmann mask to get more subapertures [Bal+96].
- **Solar DIMM (S-DIMM)**: relying on the solar limb observation through two apertures [ZB01].



Figure 6.8 – Illustration of the DIMM used at Redu in the framework of SALTO. [Qua+23]

- **S-DIMM+**: using a wavefront sensor and looking at solar granulation. It can estimate a rough profile of C_n^2 during daytime [SV10].

6.4.1.3 Application: DIMM instrument at Redu

In the framework of SALTO [Orb+18; Orb+22], already introduced in Section 5.2.2, a DIMM instrument has been set up by University of Liège at Redu, Belgium. It uses a Celestron C14 ($D = 35.6$ cm) with a mask of 12-cm aperture size and a baseline of 25 cm. The DIMM software has been implemented following the prescriptions of [Tok02]. An illustration of the setup at Redu is given in Figure 6.8. In this work, the measurements from the DIMM have been used (Section 5.4.4).

6.4.2 Scintillation detection and ranging (SCIDAR)

SCIntillation Detection And Ranging instruments are based on an optical triangulation technique. By observing two stars (e.g., binary stars) with an angular separation Θ , the instrument measures the correlation of scintillation patterns (intensity fluctuations) produced on the detector. Indeed, for a turbulent layer at altitude h , the scintillation patterns are expected to be correlated at a distance $h\Theta$ on the detector, the amplitude of the correlation peaks being related to the turbulence strength at this altitude h . Hence, SCIDAR instruments give access to vertical profile of the turbulence (C_n^2 profiles). Moreover, from the displacement of the correlation peaks, a wind-profile can be estimated.

The vertical resolution of the measured profiles depend on the width of the correlation peaks, that varies with the size of the telescope. This constraint requires

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the use of telescope larger than one metre for SCIDAR instruments. Furthermore, as scintillation mostly arises due to high-altitude OT, low atmospheric layers cannot be sensed by the instrument [Hic14; Osb10].

A non-exhaustive list of SCIDAR-based instruments is given below:

- **SCIDAR**: original design, presented in [VR73; RRV74].
- **Generalized SCIDAR (G-SCIDAR)**: design modification of SCIDAR enabling to measure low-altitude layers [AVM97].
- **High vertical resolution G-SCIDAR (HVR-GS)**: using the spatiotemporal cross-correlation images to improve the vertical resolution [EM07].
- **Low Layer SCIDAR (LOLAS)**: G-SCIDAR on a portable telescope but using widely separated target to reach high altitude resolution [Avi+08].
- **Single star SCIDAR (SSS)**: SCIDAR using a single star and not a double star [Hab+06].
- **Stereo-SCIDAR**: SCIDAR with a separate detector for each star [She+14].

6.4.3 Slope Detection and Ranging (SLODAR)

Slope Detection and Ranging (SLODAR) instruments rely on a similar principle as SCIDARs except that they measure the phase aberrations (i.e., wavefront distortions) instead of the scintillation (i.e., intensity fluctuations). They also require the use of a double star, and have a vertical resolution depending on the double star angular separation and the telescope diameter. Wavefront measurements are conducted using a Shack-Hartmann wavefront sensor, from which the slope correlations are computed. The altitude, strength and velocity of the atmospheric turbulent layers can then be estimated [Osb10; Per18].

- **SLODAR**: working principle, detailed in [Wil02; BWS06].
- **Generalized SLODAR (G-SLODAR)**: increasing the vertical resolution of SLODAR by using several datasets obtained for different wavefront conjugation heights [GJL07].
- **Surface-layer SLODAR (SL-SLODAR)**: using target stars with very wide separation to improve the resolution in the surface layer [Osb+10].
- **Solar SLODAR (So-SLODAR)**: daytime SLODAR making use of solar granulation [Tow+16].

6.4.4 Other instruments

Inspired by the working principles of the DIMM, SCIDAR, and SLODAR, numerous other instruments have been developed. A non-exhaustive list is given below:

- **Coupled SLODAR SCIDAR (CO-SLIDAR)**: estimates a C_n^2 profile based on the wavefront slope and scintillation correlations on two separated stars, recorded with a Shack-Hartmann wavefront sensor [Véd+07; Sau+21].
- **Single CO-SLIDAR (SCO-SLIDAR)**: CO-SLIDAR using a single star instead of two stars [Véd+07; Véd+10].
- **Shack-Hartmann Image Motion Monitor (SHIMM)**: using the same working principle as a SCO-SLIDAR [Per18; Per+23], with possible continuous 24-hour operation [Gri+23]. The SHIMM instrument is further detailed in Chapter 7.
- **Multi-Aperture Scintillation Sensor (MASS)**: by measuring the scintillation in different physical concentric apertures, it can estimate a few-layer C_n^2 profile. However, it is not sensitive to the ground-layer turbulence [Kor+03; TK07; Kor11].
- **MASS-DIMM**: combination of MASS and DIMM in a single instrument [Kor+07].
- **Ring-Image Next Generation Scintillation Sensor (RINGSS)**: estimates a C_n^2 profile using defocused ring-like images of a single star [Tok21]
- **Full Aperture Scintillation Sensor (FASS)**: processes scintillation images in annular pupils (from full aperture images) to estimate a few-layer C_n^2 profile [Gue+21].
- **Shadow Band Ranger (SHABAR)**: estimates C_n^2 in the BL from an array of solar scintillometers [Bec01].
- **Profiler of Moon Limb (PML)**: recovers the C_n^2 profile by observing the moon limb or the sun edge [Cha+20a].
- **Lunar scintillometer (LuSci)**: measures weak moonlight scintillation to estimate C_n^2 in the few hundred metres above the ground [TBB10].
- **Advanced Multiple Aperture Seeing Profiler (A-MASP)**: relies on solar granulation to measure the daytime optical turbulence profile [Ren+22].
- Etc.

6.5 Chapter summary

This chapter introduced the theory of wave propagation in random media, either using an analytical formalism (Section 6.1) or relying on numerical approaches (Section 6.2). It also highlighted important OT integrated quantities (Section 6.3), and numerous OT monitoring instruments (Section 6.4). It set the foundations for the

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theoretical concepts used in the previous and following chapters, regarding OT parameters and EM propagation.

A summary of C_n^2 -related quantities, including C_n^2 profiles and most important OT integrated quantities, is given in Table 6.2. It emphasizes the interpretation of each quantity, their limitations, and common tools used to monitor them.

As shown in Section 6.3.8, most of the OT integrated quantities are complementary, since they do not sense OT at the same altitudes. This motivates the need to evaluate C_n^2 profile predictions not only for one quantity (e.g., the seeing), but for several of them. However, this can be made difficult when different instruments are needed to obtain measurements of several integrated parameters. This further highlights the difficulties of validating C_n^2 models, as conducted in Chapters 4 and 5.

Regarding the numerous tools introduced in Section 6.4, comparisons of their working principles, advantages, and limitations, can be found in [Per18, p. 45], [BL22], [JTY23]. Analyses of the measurements collected simultaneously by several instruments at Paranal (SHIMM, RINGSS, MASS-DIMM, Stereo-SCIDAR) showed good agreement in terms of seeing and coherence time, but less correlation for the measured isoplanatic angles between the instruments [Gri+24]. Hence, discrepancies between the instruments are expected, especially for instruments using different approaches to sense OT or located at slightly different locations [Tok23].

The next chapter (Chapter 7) focuses on the SHIMM instrument, making use of phase screen propagation for its modelling. Chapter 8 uses multiple integrated quantities for assessing short-term OT predictions.

Table 6.2 – Summary of main C_n^2 -related quantities (for plane waves, downlink propagation)

Quantity	Interpretation	Limitations	Tools
C_n^2 profile	Complete description of atmospheric turbulence impacts on wave propagation.	/	Thermosondes, SCIDAR, SLODAR, etc.
Fried parameter r_0 $r_0 = \left(0.423k^2 \sec(\tilde{\theta}) \int_0^\infty C_n^2(z) dz\right)^{-3/5}$	Coherence length, representing the diameter of a diffraction-limited telescope offering the same resolution as observing through atmospheric turbulence.	Dominated by layers having a large C_n^2 value.	DIMM, MASS, etc.
Seeing $\mathcal{E}_0$ $\mathcal{E}_0 = 5.285\lambda^{-1/5} \left(\sec(\tilde{\theta}) \int_0^\infty C_n^2(z) dz\right)^{3/5}$	Full width at half maximum of the point spread function of a star for long exposure through atmospheric turbulence. The seeing is related to the Fried parameter by $\mathcal{E}_0 = 0.976 \frac{\lambda}{r_0}$.	Dominated by layers having a large C_n^2 value.	
Isoplanatic angle θ_0 $\theta_0 = \left(2.913k^2 (\sec(\tilde{\theta}))^{8/3} \int_0^\infty C_n^2(z) z^{5/3} dz\right)^{-3/5}$	Angular region where turbulence is assumed to be correlated. It corresponds to the angle at which the RMS error of the wavefront phase has increased by 1 rad.	Dominated by high-altitude turbulent layers.	Usually derived from measured C_n^2 profiles (or from scintillometers for σ_I^2).
Coherence time τ_0 $\tau_0 = \left(2.913k^2 \sec(\tilde{\theta}) \int_0^\infty C_n^2(z) v^{5/3}(z) dz\right)^{-3/5}$	It corresponds to the time after which the wavefront phase structure has changed. It is linked to the Greenwood frequency by $f_G = \frac{0.1338}{\tau_0}$.	Require knowledge of the wind velocity vertical profile $v(z)$.	
Scintillation index σ_I^2 $\sigma_I^2 = 2.25k^{7/6} \sec^{11/6}(\tilde{\theta}) \int_{h_0}^H C_n^2(z) (z - h_0)^{5/6} dz$	Normalized variance of the received intensity.	Dominated by high-altitude turbulence.	

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Measurements of optical turbulence using Shack-Hartmann wavefront sensors

7

This chapter presents the Shack-Hartmann Image Motion Monitor instrument, developed by Durham’s Centre for Advanced Instrumentation [Gri+23]. This instrument enables to continuously measure OT integrated parameters and low-resolution C_n^2 profiles, as long as clear sky conditions are ensured. This chapter thus complements the work in this thesis by detailing the working principle of a state-of-the-art OT monitoring instrument that provides OT measurements at OGS sites, enabling to validate the C_n^2 models developed in Chapters 4 and 5.

The key contributions of Chapter 7 are:

- A comprehensive description of the SHIMM instrument, highlighting the hypotheses behind its working principle and examining how its design influences the vertical sensing of C_n^2 profiles.
- The introduction of three distinct mathematical frameworks for reconstructing C_n^2 profiles from SHIMM measurements. The first approach is inspired from the literature, while the second and third approaches are novel methods that provide original descriptions of C_n^2 profiles, aiming to improve its vertical resolution. A performance comparison of these approaches on simulated data is also presented.
- In-depth sensitivity studies of various parameters affecting the reconstruction of C_n^2 profiles, offering insights about the instrument’s physics and the mathematical algorithms employed.

This chapter offers a thorough understanding of the instrument, as well as future research directions. Figure 7.1 depicts the methodology followed, and consists in two main parts:

- the direct model (forward model): it explains how a C_n^2 profile is “sensed” by the SHIMM instrument, giving access to intensity and angle-of-arrival fluctuation measurements. Two approaches are used: a theoretical (analytical) approach relying on the mathematical expressions of the SHIMM response, and a simulated (numerical) approach, based on phase screen simulations.

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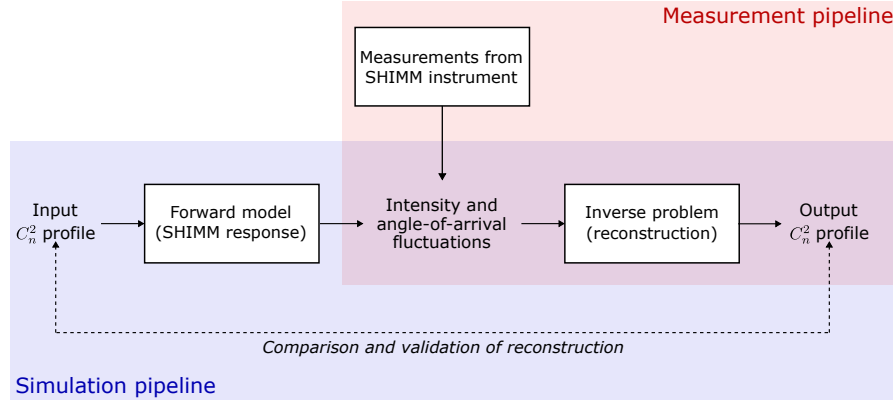


Figure 7.1 – Modelling (forward model) and inversion (reconstruction) approaches for the SHIMM instrument, as developed in this chapter.

- the inversion problem, or reconstruction: it corresponds to the algorithm used to recover C_n^2 profiles from the SHIMM measurements, i.e., from intensity and angle-of-arrival fluctuation measurements.

In simulations, or with theoretical developments, an input C_n^2 profile can be fed to the direct model to get modelled SHIMM measurements, used for reconstructing the profile. Then, the input and reconstructed profiles can be compared to assess the performance of the reconstruction algorithm, as depicted by the dashed arrow in Figure 7.1. On the contrary, with real measurements, only the inversion procedure is applied to estimate the C_n^2 profile, and validation of this profile can only be conducted using simultaneous measurements coming from other instruments (e.g, DIMM, SCIDAR, etc.) situated at the same location, as in [Gri+24a] for example.

First, the SHIMM instrument is described in Section 7.1. Then, the modelling of the instrument response is described in Section 7.2, for both the theoretical and numerical approaches. The C_n^2 profile reconstruction in the case of discrete turbulent layers is presented in Section 7.3, while Section 7.4 gives the reconstruction for other basis functions. Finally, Section 7.5 explores reconstruction approaches with sparse representation of C_n^2 profiles, and Section 7.6 presents a noise analysis and the application of the approaches to real SHIMM measurements.

7.1 Description of the SHIMM instrument

The SHIMM, as designed by Durham’s Centre for Advanced Instrumentation, is an off-the-shelf 11-inch (= 27.94 cm) telescope equipped with a Shack-Hartmann Wavefront Sensor (SHWFS). It is shown in the left part of Figure 7.2. The SHWFS is an array of micro-lenses separating the received light in multiple apertures, named sub-apertures. It enables to sense the received wavefront over the full aperture of the telescope, namely by monitoring the received intensity and angle of arrival in each

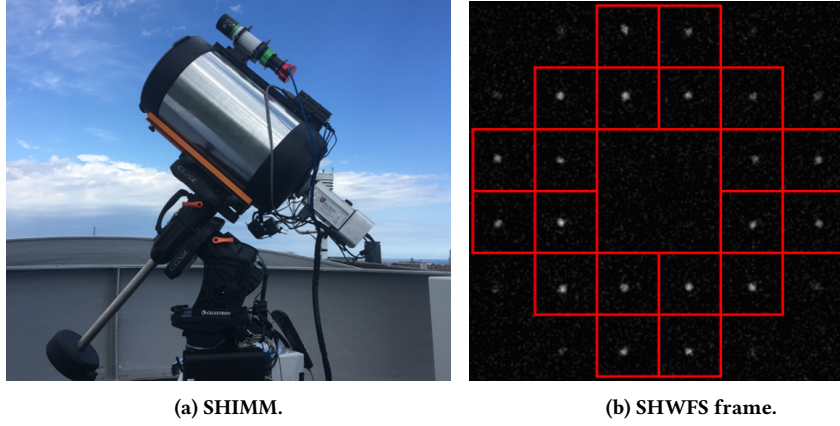


Figure 7.2 – Picture of the SHIMM instrument installed in Barcelona (left) and example of one frame as observed in the wavefront sensor (right).

subaperture. This device is often used in AO systems to measure the received wavefront, so that it can be compensated by the deformable mirror [Hic14]. An example of the received wavefront through the SHWFS is depicted in the right part of Figure 7.2.

7.1.1 Motivations

The main motivations and advantages of the SHIMM instrument are:

1. Portability and compactness: using a telescope with a diameter smaller than 30 cm makes it portable and easy to deploy, contrariwise to other instruments that require larger telescope diameters (e.g., SCIDAR or SLODAR). However, a smaller telescope diameter also limits the amount of light collected, hence reducing the number of available stars of sufficient brightness that can be used to collect measurements with the SHIMM. In practice, only stars with magnitude below 0 in the J-band enable to obtain a sufficiently high signal-to-noise ratio [Gri+23].
2. Low-cost: by being a small instrument and making use of off-the-self components, the cost of the instrument is limited.
3. Continuous operation: the SHIMM instrument observes stars in the infrared (J band, 1.1 to 1.4 μm), with an operating wavelength of 1280 nm. Doing so, it is possible to observe stars during daytime, which enables continuous monitoring of OT with a single instrument. Indeed, the daytime sky brightness is lower in the near-infrared than at optical wavelengths¹, reducing the amount of undesired photons collected by the camera. Moreover, the impacts of OT are

¹A major contribution to the sky brightness during daytime is the Rayleigh scattering of the solar light, that is inversely proportional to λ^4 , with λ being the wavelength. This explains the blue colour of the sky, as the wavelength of the blue colour is shorter and is therefore more scattered than other colours.

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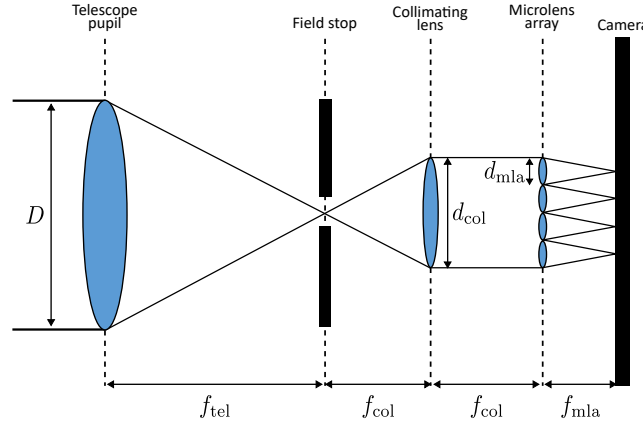


Figure 7.3 – Optical design of the SHIMM instrument (reproduced from [Gri+23]).

smaller in the near-infrared than in the visible, as seen by the wavelength dependency of most OT integrated quantities for example. As a result, the necessary hypothesis of weak perturbations on which relies the SHIMM instrument (see Section 7.2.1) is more valid. Nevertheless, observing in the near-infrared comes with the drawbacks of having a stronger detector noise and requiring more expensive cameras.

Overall, these advantages make the SHIMM instrument particularly suitable for OT monitoring and comparison at several sites, either for astronomical applications [Per+23], or for optical communication applications (e.g., site-selection of future OGS locations) [Gri+24b]. The continuous stream of data generated by the SHIMM is also beneficial for data assimilation into turbulence forecasting software, improving their prediction performance [Gri+23].

7.1.2 Optical design

The optical design of the SHIMM instrument is depicted in Figure 7.3. It includes a telescope having a diameter D of 27.94 cm, a collimating lens, and a microlens array (i.e., a Shack-Hartmann wavefront sensor) of $500\ \mu\text{m}$ pitch. Field stop is ensured by an iris with programmable aperture, and the camera used is an Indium Gallium Arsenide (InGaAs) camera. More details can be found in [Gri+23].

7.1.3 Available measurements and notations

The SHIMM instrument monitors the intensity and angle-of-arrival fluctuations of the received wavefront by sensing it through the SHWFS. In total, 20 subapertures are illuminated, as depicted in Figure 7.2.

The general pipeline for the SHIMM measurement acquisition is depicted in Figure 7.4. The received wavefront is first sensed by the SHWFS. Then, the intensities and angles of arrival in each subapertures are recorded. This corresponds to a single

frame, i.e., a single observation. By repeating the observations for a sufficient number of times, the covariance matrices of the intensity and angle-of-arrival fluctuations can be estimated.

Notations In the following, the variable m denotes the subaperture index, such that $m \in \{1, 2, \dots, 20\}$. The numbering of the SHIMM subapertures is done from left to right and top to bottom.

The variable p is used for the frame index, that refers to the temporal ordering of the wavefront images (called frames) that have been acquired. As a result, $p \in \{1, 2, \dots, N_{\text{frames}}\}$, with N_{frames} being on the order of a thousand, sufficiently large to obtain relevant statistics as described below.

Recorded quantities In each subaperture, and for each frame, the recorded quantities are:

- the total received intensity, computed by summing the intensity from all the subaperture pixels. It is denoted by $i_m^{(p)}$ for the m -th subaperture, in frame p . From the total intensity in each subaperture and each frame, the intensity fluctuations $\delta i_m^{(p)}$ are computed by removing the average intensity in this subaperture for all frames and normalizing, i.e., [Véd+10]

$$\delta i_m^{(p)} = \frac{i_m^{(p)} - \langle i_m \rangle}{\langle i_m \rangle}. \quad (7.1)$$

The average subaperture intensity is estimated from a temporal average over the frames, $\langle i_m \rangle = \frac{1}{N_{\text{frames}}} \sum_{p=1}^{N_{\text{frames}}} i_m^{(p)}$.

- the angle-of-arrival fluctuations, associated to the wavefront slopes in the x - and y -directions (i.e., both directions perpendicular to the direction of propagation, assumed to be along the z -axis). They are noted $s_{x,m}^{(p)}$ and $s_{y,m}^{(p)}$ for the m -th subaperture and the p -th frame, and computed using the optical axis (i.e., z -axis) as reference. They are expressed in radians or arcseconds. Their estimation based on centroid motions is not considered in this work and has been conducted by Durham University.

Matrices of covariances From the intensity fluctuations $\delta i_m^{(p)}$ and the angle-of-arrival fluctuations $s_{x,m}^{(p)}$ and $s_{y,m}^{(p)}$, their covariances between two subapertures are exploited. The ensemble average is substituted by a time-average such that, as an example, the covariance of intensity fluctuations between subapertures² m and n is computed from

$$\langle \delta i_m \delta i_n \rangle = \frac{1}{N_{\text{frames}}} \sum_{p=1}^{N_{\text{frames}}} \delta i_m^{(p)} \delta i_n^{(p)}.$$

²In this chapter, the variable n is used as an index for the SHWFS subapertures, except when it explicitly refers to the refractive index, as in the refractive index structure parameter C_n^2 , the refractive index power spectrum $\Phi_n(\kappa)$, or the refractive index covariance function $B_n(\rho)$.

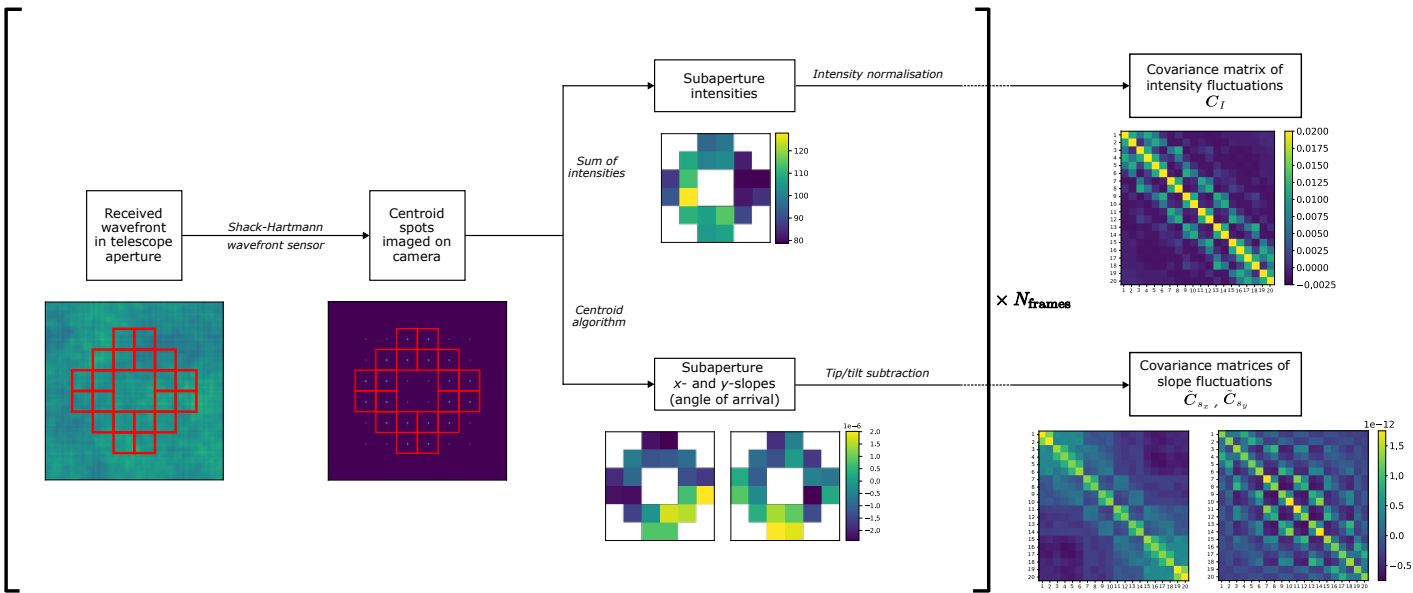


Figure 7.4 – SHIMM measurement pipeline.

The computation of the covariances of the angle-of-arrival fluctuations is similar. Both intensity and angle-of-arrival covariances can be related to the C_n^2 profile, as shown in Section 7.2.1.

In practice, covariance matrices corresponding to the complete set of possible differences between two subapertures are directly computed, instead of individual covariances one-by-one. For example, defining $\delta \mathbf{i}^{(p)}$ as a vector of length 20 containing the normalized intensities in each subaperture for one frame p , and $\Delta \mathbf{I}$ as the matrix constructed from appending all $\delta \mathbf{i}^{(p)}$ for $p = 1, \dots, N_{\text{frames}}$, i.e., $\Delta \mathbf{I} = \begin{pmatrix} \delta \mathbf{i}^{(1)} & \delta \mathbf{i}^{(2)} & \dots & \delta \mathbf{i}^{(N_{\text{frames}})} \end{pmatrix}^T$, then the intensity covariance matrix C_I is given by

$$C_I = \frac{1}{N_{\text{frames}}} \Delta \mathbf{I}^T \Delta \mathbf{I}. \quad (7.2)$$

The element m, n from C_I is indeed $(C_I)_{m,n} = \langle \delta i_m \delta i_n \rangle$. Matrices for the angle-of-arrival fluctuations C_{s_x} and C_{s_y} can be obtained similarly.

Number of measurements The number of entries in the covariance matrices is related to the number of subapertures. Let Ω denote the set containing the origins of all subapertures depicted in Figure 7.2. Its cardinality is $|\Omega| = 20$, i.e., they are 20 different subapertures. The cardinality of the set $\Omega - \Omega$, i.e., the set formed by the differences between 2 subapertures selected in Ω is computed based on Figure 7.5, and is equal to 81. The multiplicity of each difference between 2 subapertures is also presented in this figure. Due to the regularity of the grid, it is not possible to reach the largest possible cardinality of the set $\Omega - \Omega$, that is $|\Omega|^2 = 400$.

As a result, the covariance matrices C_I , C_{s_x} and C_{s_y} have $20 \times 20 = 400$ entries, with only 81 distinct entries. However, in practice, other symmetries described below further reduce the number of different measurements that are available.

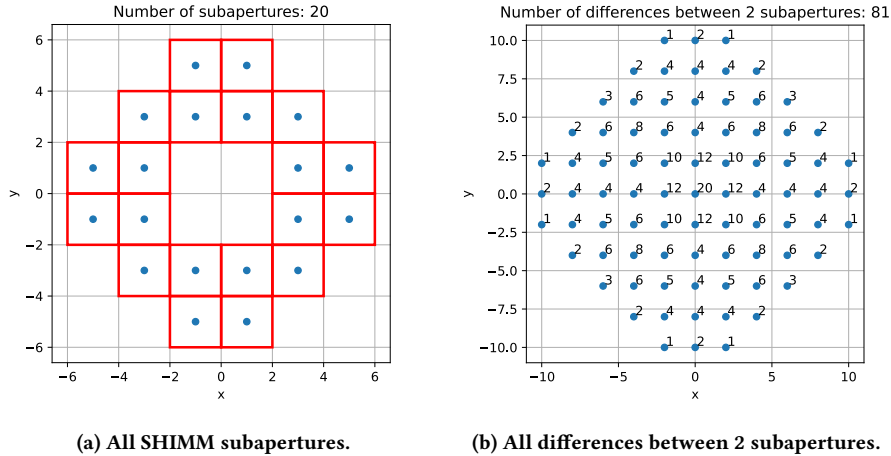


Figure 7.5 – SHIMM geometry: location of subapertures and differences (distances) between 2 subapertures, with their multiplicity.

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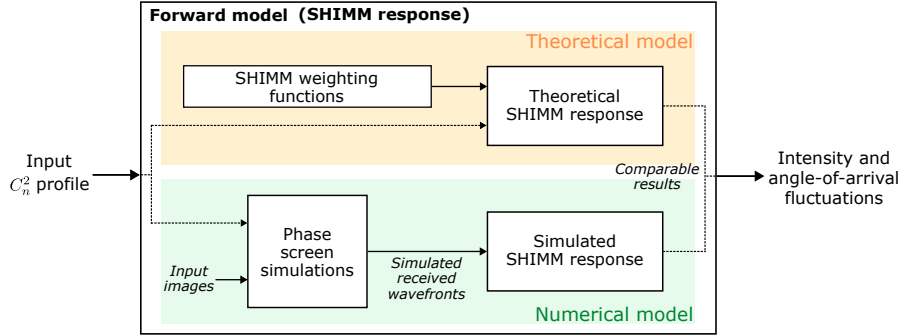


Figure 7.6 – SHIMM instrument response from theoretical and numerical models.

7.2 Modelling of the instrument response

The modelling of the SHIMM instrument corresponds to the first step shown in Figure 7.1 and relates the observed intensity and angle-of-arrival fluctuations to the C_n^2 profile. It is useful to obtain synthetic SHIMM measurements, but also to determine the relevant equations for the inversion, i.e., to recover C_n^2 profiles from the measurements.

The modelling of the instrument is conducted theoretically and numerically, as summarized in Figure 7.6. The theoretical model relies on the SHIMM weighting functions, defined in Section 7.2.1, while the numerical approach is based on phase screen propagation of plane waves. It is shown in Section 7.2.2 that the numerical approach gives results in agreement with the theoretical SHIMM model.

7.2.1 Theoretical formulation

Assuming monochromatic plane waves, the scalar components of the electric field that are perpendicular to the direction of propagation are given by their complex amplitudes $U(\mathbf{r}) = |U(\mathbf{r})|e^{j\varphi(\mathbf{r})}$, with $|U(\mathbf{r})|$ the amplitude of the field, $\varphi(\mathbf{r}) = \arg(U(\mathbf{r}))$ its phase, and $\mathbf{r}$ the position vector. It can also be expressed using the log-amplitude $\chi(\mathbf{r})$, that is,

$$U(\mathbf{r}) = e^{\chi(\mathbf{r}) + j\varphi(\mathbf{r})} \quad \text{with} \quad \chi(\mathbf{r}) = \ln(|U(\mathbf{r})|).$$

In the context of wave propagation through atmospheric turbulence, the log-amplitude and the phase are often expressed as fluctuations with respect to the unperturbed field $\tilde{U}(\mathbf{r})$, such that $\chi(\mathbf{r}) = \ln\left(\frac{|U(\mathbf{r})|}{|\tilde{U}(\mathbf{r})|}\right)$ and $\varphi(\mathbf{r}) = \arg(U(\mathbf{r})) - \arg(\tilde{U}(\mathbf{r}))$, as presented in (6.9).

Since plane waves are considered, the propagation distance z and the coordinates $\mathbf{x} = (x, y)^T$ in the plane perpendicular to the direction of propagation can be separated, as in [Rod81, p. 292]. This leads to express $U(\mathbf{r})$ after a propagation distance

z as

$$U_z(\mathbf{x}) = |U_z(\mathbf{x})|e^{j\varphi_z(\mathbf{x})}.$$

In the following, we consider a vertical propagation from a star/a satellite to the ground, such that $\mathbf{x}$ corresponds to the coordinates in a horizontal plane, and z to the altitude.

7.2.1.1 Thin turbulent layer

Considering a thin layer of width δz where the refractive index $n(\mathbf{x}, z)$ fluctuates, the phase shift introduced by the layer is [Rod81, p. 292]

$$\varphi(\mathbf{x}) = k \int_z^{z+\delta z} n(\mathbf{x}, h) dh, \quad (7.3)$$

where $k = \frac{2\pi}{\lambda}$ is the wavenumber, with λ the wavelength. This result relies on the thin screen approximation, i.e., the layer is considered sufficiently thin for neglecting diffraction effects over the distance δz . Assuming that the complex field is uniform and unitary at the layer input ($U_{z+\delta z}(\mathbf{x}) = 1$), the field at the output is given by

$$U_z(\mathbf{x}) = e^{j\varphi(\mathbf{x})}.$$

The goal is to derive the effects of this thin turbulent layer on the statistical properties of the complex field received at the ground. Of particular interest are the fluctuations of the squared modulus $|U_0(\mathbf{x})|^2$ describing the received intensity at the telescope aperture, and the fluctuations of the phase $\varphi_0(\mathbf{x})$ giving the shape of the received wavefront [Rod81, p. 328].

Relying on the Fresnel approximation as given by (6.21), the received complex field at the ground is obtained by the 2D convolution with respect to $\mathbf{x}$ [Goo96, p. 67]

$$U_0(\mathbf{x}) = U_z(\mathbf{x}) * \frac{1}{j\lambda z} e^{j\frac{\pi}{\lambda z} |\mathbf{x}|^2},$$

where the factor e^{jkz} is neglected as it introduces a constant phase shift over the whole wavefront, and $|\mathbf{x}|^2 = x^2 + y^2$.

The computation of the ground level coherence function $B_0(\boldsymbol{\rho}) = \langle U_0(\mathbf{x})U_0^*(\mathbf{x} + \boldsymbol{\rho}) \rangle$ is conducted in the general case in [Rod81, p.294]. In the following, only the computations in the *small perturbation approximation* are presented [Rod81, p. 329]. This is a strong assumption, not valid for horizontal propagation and not accounting for the saturation of scintillation.

This approximation leads to the assumption that the phase fluctuations introduced by the thin turbulent layer are small, i.e., $\varphi(\mathbf{x}) \ll 1$ for all $\mathbf{x}$. In this case, the complex field at the output of the layer can be approximated by its Taylor series of order 1:

$$U_z(\mathbf{x}) = e^{j\varphi(\mathbf{x})} \approx 1 + j\varphi(\mathbf{x}).$$

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The Fresnel propagation gives

$$U_0(\mathbf{x}) \approx (1 + j\varphi(\mathbf{x})) * \frac{1}{j\lambda z} e^{\frac{j\pi}{\lambda z} |\mathbf{x}|^2}.$$

The convolution between 1 and $\frac{1}{j\lambda z} e^{\frac{j\pi}{\lambda z} |\mathbf{x}|^2}$ is 1, while the other part of the convolution is denoted $\varsigma(\mathbf{x})$,

$$\varsigma(\mathbf{x}) = \varphi(\mathbf{x}) * \frac{1}{\lambda z} e^{\frac{j\pi}{\lambda z} |\mathbf{x}|^2}.$$

It provides the relative fluctuations of the complex amplitude at the ground, and, from the small perturbation approximation, $U_0(\mathbf{x}) \approx 1 + \varsigma(\mathbf{x}) \approx e^{\varsigma(\mathbf{x})}$, such that the real part of $\varsigma(\mathbf{x})$ describes the relative fluctuations of the log-amplitude $\chi(\mathbf{x})$, while the imaginary part gives the fluctuations of the phase $\varphi_0(\mathbf{x})$:

$$\chi(\mathbf{x}) = \varphi(\mathbf{x}) * \frac{1}{\lambda z} \cos\left(\frac{\pi}{\lambda z} |\mathbf{x}|^2\right), \quad (7.4)$$

$$\varphi_0(\mathbf{x}) = \varphi(\mathbf{x}) * \frac{1}{\lambda z} \sin\left(\frac{\pi}{\lambda z} |\mathbf{x}|^2\right). \quad (7.5)$$

Defining the two-dimensional power spectrum $W_\varphi(\mathbf{f})$ of $\varphi(\mathbf{x})$ as

$$W_\varphi(\mathbf{f}) = \int_{-\infty}^{+\infty} B_\varphi(\boldsymbol{\rho}) e^{-j2\pi \mathbf{f} \cdot \boldsymbol{\rho}} d\boldsymbol{\rho},$$

with $\mathbf{f} = (f_x, f_y)$, $\boldsymbol{\rho} = (\rho_x, \rho_y)$ and $B_\varphi(\boldsymbol{\rho}) = \langle \varphi(\mathbf{x} + \boldsymbol{\rho}) \varphi^*(\mathbf{x}) \rangle$, the power spectra for $\chi(\mathbf{x})$ and $\varphi_0(\mathbf{x})$ are [Rod81, p. 330]

$$\begin{aligned} W_\chi(\mathbf{f}) &= W_\varphi(\mathbf{f}) \left(\sin(\pi \lambda z |\mathbf{f}|^2) \right)^2, \\ W_{\varphi_0}(\mathbf{f}) &= W_\varphi(\mathbf{f}) \left(\cos(\pi \lambda z |\mathbf{f}|^2) \right)^2. \end{aligned}$$

This result is a direct application of the Wiener-Khinchin theorem, with the square modulus of the Fourier transforms of $\frac{1}{\lambda z} \cos\left(\frac{\pi}{\lambda z} |\mathbf{x}|^2\right)$ and $\frac{1}{\lambda z} \sin\left(\frac{\pi}{\lambda z} |\mathbf{x}|^2\right)$, given in Appendix B.

Since $B_\varphi(\boldsymbol{\rho})$ is related to the covariance of the refractive index fluctuations over the thin layer by [Rod81, p. 293]

$$B_\varphi(\boldsymbol{\rho}) = k^2 \delta z \int_{-\infty}^{+\infty} B_n(\boldsymbol{\rho}, h) dh,$$

from (7.3), and assuming that δz is much larger than the correlation scale of the refractive index fluctuations, such that the integration is from $-\infty$ to $+\infty$. Taking the Fourier transform of this result leads to

$$W_\varphi(\mathbf{f}) = k^2 \delta z W_n(\mathbf{f}, 0) \quad \text{with} \quad W_n(f_x, f_y, f_z) = (2\pi)^3 \Phi_n(2\pi f_x, 2\pi f_y, 2\pi f_z). \quad (7.6)$$

Equation (7.6) thus provides the link between the phase power spectrum, and the

power spectrum of the refractive index. Choosing the Kolmogorov power spectrum with $f = |\mathbf{f}|$,

$$W_n(\mathbf{f}, 0) = (2\pi)^3 \times 0.033 C_n^2(z) (2\pi f)^{-11/3},$$

that can be substituted in the results above, and the dependency of the power spectra in $C_n^2(z)\delta z$ can be isolated [Rod81, p. 331]:

$$W_\chi(\mathbf{f}) = 0.38\lambda^{-2}f^{-11/3}C_n^2(z)\delta z (\sin(\pi\lambda z f^2))^2, \quad (7.7)$$

$$W_{\varphi_0}(\mathbf{f}) = 0.38\lambda^{-2}f^{-11/3}C_n^2(z)\delta z (\cos(\pi\lambda z f^2))^2. \quad (7.8)$$

7.2.1.2 Multiple layers

Under the approximation of small perturbations, fluctuations produced at ground level by several turbulent layers add linearly. This is demonstrated in [Rod81, p. 331]. As a result, (7.7) and (7.8) can be generalized for any distribution of turbulence $C_n^2(z)$:

$$W_\chi(\mathbf{f}) = 0.38\lambda^{-2}f^{-11/3} \int_0^{+\infty} C_n^2(z) (\sin(\pi\lambda z f^2))^2 dz,$$

$$W_{\varphi_0}(\mathbf{f}) = 0.38\lambda^{-2}f^{-11/3} \int_0^{+\infty} C_n^2(z) (\cos(\pi\lambda z f^2))^2 dz.$$

Furthermore, χ and φ_0 are Gaussian variables, statistically independent [Rod81, p. 332].

7.2.1.3 Intensity fluctuations

The intensity $I(\mathbf{x})$ is given by $|U_0(\mathbf{x})|^2 = e^{2\chi(\mathbf{x})}$, such that the intensity fluctuations $\delta i(\mathbf{x})$ are equal to $2\chi(\mathbf{x})$ under the small perturbation hypothesis. Using (7.4),

$$\delta i(\mathbf{x}) = 2\varphi(\mathbf{x}) * \frac{1}{\lambda z} \cos\left(\frac{\pi}{\lambda z}|\mathbf{x}|^2\right).$$

In practice, the intensity is measured over a subaperture, such that, for an aperture m , the received intensity fluctuations are

$$\delta i_m = 2\varphi(\mathbf{x}) * \frac{1}{\lambda z} \cos\left(\frac{\pi}{\lambda z}|\mathbf{x}|^2\right) * P(\mathbf{x}),$$

where $P(\mathbf{x})$ is the subaperture averaging function, taking into account the shape and dimensions of the subaperture. This is the result presented in [Rob+06], particularized here for point sources.

Then, by computing the cross-correlations of the intensity fluctuations between two subapertures m and n , one gets access to one measurement of the intensity covariance $B_i(\boldsymbol{\rho})$ for a distance $\boldsymbol{\rho}$ equal to the subaperture spacing $\mathbf{d}_{mn}$. The link with the log-amplitude power spectrum is

$$W_i(\mathbf{f}) = 4 W_\chi(\mathbf{f}) \hat{P}^2(\mathbf{f}) = 4 W_\varphi(\mathbf{f}) (\sin(\pi\lambda z f^2))^2 \hat{P}^2(\mathbf{f}),$$

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where $\hat{P}(\mathbf{f})$ is the Fourier transform of $P(\mathbf{x})$. If the Kolmogorov power spectrum is used,

$$W_i(\mathbf{f}) = 4 \hat{P}^2(\mathbf{f}) 0.38\lambda^{-2} f^{-11/3} \int_0^{+\infty} C_n^2(z) (\sin(\pi\lambda z f^2))^2 dz.$$

As a result, the quantity $\langle \delta i_m \delta i_n \rangle$ comes from a weighted integral of the $C_n^2(z)$ profile [Véd+10]:

$$\begin{aligned} \langle \delta i_m \delta i_n \rangle &= B_i(\mathbf{d}_{mn}) \\ &= \mathcal{F}^{-1} \left\{ 4 \hat{P}^2(\mathbf{f}) 0.38\lambda^{-2} f^{-11/3} \int_0^{+\infty} C_n^2(z) (\sin(\pi\lambda z f^2))^2 dz \right\}(\mathbf{d}_{mn}) \\ &= \int_0^{+\infty} C_n^2(z) \underbrace{\mathcal{F}^{-1} \left\{ 4 \hat{P}^2(\mathbf{f}) 0.38\lambda^{-2} f^{-11/3} (\sin(\pi\lambda z f^2))^2 \right\}}_{w_i(z, \mathbf{d}_{mn})}(\mathbf{d}_{mn}) dz \\ &= \int_0^{+\infty} C_n^2(z) w_i(z, \mathbf{d}_{mn}) dz, \end{aligned} \quad (7.9)$$

with the definition of the intensity weighting functions $w_i(z, \mathbf{d}_{mn})$ highlighted. More complex models of $W_\varphi(\mathbf{f})$ are often used, depending on the chosen power spectrum of the refractive index fluctuations, see [Véd+10] for example. Nevertheless, they involve a multiplication with the $C_n^2(z)$ profile that can be isolated to express it as in (7.9), with a different definition for $w_i(z, \mathbf{d}_{mn})$. Indeed, in a general case, the weights $w_i(z, \mathbf{d}_{mn})$ come from [Véd+10]

$$w_i(z, \mathbf{d}_{mn}) = 4 \mathcal{F}^{-1} \left\{ \hat{P}^2(\mathbf{f}) \tilde{W}_\varphi(\mathbf{f}) (\sin(\pi\lambda z f^2))^2 \right\}(\mathbf{d}_{mn}), \quad (7.10)$$

where $\tilde{W}_\varphi(\mathbf{f})$ is the power spectrum of the phase without the $C_n^2(z)$ dependency, i.e., $W_\varphi(\mathbf{f}) = C_n^2(z) \tilde{W}_\varphi(\mathbf{f})$. As an alternative to the Kolmogorov power spectrum presented above, a modified von Kármán spectrum for the refractive index fluctuations can be considered, that is,

$$\Phi_n(\kappa) = 0.033 C_n^2 (\kappa^2 + \kappa_0^2)^{-11/6} e^{-\frac{\kappa^2}{\kappa_m^2}} \quad \text{for} \quad 0 \leq \kappa < \infty, \quad (2.50)$$

with κ the spatial pulsation [rad/m], $\kappa_m = 5.92/l_0$ and $\kappa_0 = 2\pi/L_0$. With this choice,

$$\begin{aligned} \tilde{W}_\varphi(\mathbf{f}) &= k^2 (2\pi)^3 0.033 ((2\pi f)^2 + \kappa_0^2)^{-11/6} e^{-\frac{(2\pi f)^2}{\kappa_m^2}} \\ &= k^2 0.033 (2\pi)^{-2/3} \left(f^2 + \frac{1}{L_0^2} \right)^{-11/6} e^{-\left(\frac{2\pi l_0}{5.92} f \right)^2} \\ &= 0.033 \frac{(2\pi)^{4/3}}{\lambda^2} \left(f^2 + \frac{1}{L_0^2} \right)^{-11/6} e^{-\left(\frac{2\pi l_0}{5.92} f \right)^2}, \end{aligned}$$

which corresponds to the phase power spectrum used in [Véd+10].

The correlations of intensity between any pairs of subaperture can be computed, giving $N_{\text{mes}} = 400$ measurements of $\langle \delta i_m \delta i_n \rangle$, separated by different distances $\mathbf{d}_{mn}$. From those measurements, the only unknown that remains is the $C_n^2(z)$ profile, as the SHWFS geometry is known and the $w_i(z, \mathbf{d}_{mn})$'s can be computed once for all.

7.2.1.4 Angle-of-arrival fluctuations (slopes)

As an alternative to intensity fluctuations measured by the SHIMM in each subaperture, fluctuations of the angles of arrival in each subaperture, i.e., the local slope, can be measured. In practice, this requires to use a centroid algorithm, measuring the centroid motions in each subaperture.

Using the small angle approximation, the angles of arrival in the x - and y -directions, denoted s_x and s_y (i.e., slopes in x and y), is given by the received phase at the ground φ_0 (similar to Section 6.4.1.1), according to [Rod81],[Véd+10]

$$s_x = -\frac{1}{k} \frac{d\varphi_0(\mathbf{x})}{dx} \quad \text{and} \quad s_y = -\frac{1}{k} \frac{d\varphi_0(\mathbf{x})}{dy}. \quad (7.11)$$

For the SHIMM instrument, $\varphi_0(\mathbf{x})$ is given by (7.5), with the addition of the subaperture averaging function $P(\mathbf{x})$, that is,

$$\varphi_0(\mathbf{x}) = \varphi(\mathbf{x}) * \frac{1}{\lambda z} \sin\left(\frac{\pi}{\lambda z} |\mathbf{x}|^2\right) * P(\mathbf{x}).$$

Using the Wiener-Kintchine theorem, the power spectrum of the slopes in the x -direction is given by

$$\begin{aligned} W_{s_x}(\mathbf{f}) &= \frac{1}{k^2} (2\pi f_x)^2 W_\varphi(\mathbf{f}) (\cos(\pi \lambda z f^2))^2 \hat{P}^2(\mathbf{f}) \\ &= \lambda^2 f_x^2 W_\varphi(\mathbf{f}) (\cos(\pi \lambda z f^2))^2 \hat{P}^2(\mathbf{f}), \end{aligned} \quad (7.12)$$

and for the y -direction

$$W_{s_y}(\mathbf{f}) = \lambda^2 f_y^2 W_\varphi(\mathbf{f}) (\cos(\pi \lambda z f^2))^2 \hat{P}^2(\mathbf{f}). \quad (7.13)$$

Mixing slopes in x and y is also possible, leading to

$$W_{s_{xy}}(\mathbf{f}) = \lambda^2 f_x f_y W_\varphi(\mathbf{f}) (\cos(\pi \lambda z f^2))^2 \hat{P}^2(\mathbf{f}). \quad (7.14)$$

Substituting $W_\varphi(\mathbf{f})$ by its expression assuming a Kolmogorov spectrum for example, a result similar to (7.9) is obtained:

$$\begin{aligned} \langle s_{x,m} s_{x,n} \rangle &= B_{s_x}(\mathbf{d}_{mn}) \\ &= \mathcal{F}^{-1} \left\{ \lambda^2 f_x^2 \hat{P}^2(\mathbf{f}) 0.38 \lambda^{-2} f^{-11/3} \int_0^{+\infty} C_n^2(z) (\cos(\pi \lambda z f^2))^2 dz \right\} (\mathbf{d}_{mn}) \end{aligned}$$

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$$\begin{aligned}
&= \int_0^{+\infty} C_n^2(z) \underbrace{\mathcal{F}^{-1} \left\{ f_x^2 \hat{P}^2(\mathbf{f}) 0.38 f^{-11/3} (\cos(\pi \lambda z f^2))^2 \right\}}_{w_{s_x}(z, \mathbf{d}_{mn})}(\mathbf{d}_{mn}) dz \\
&= \int_0^{+\infty} C_n^2(z) w_{s_x}(z, \mathbf{d}_{mn}) dz,
\end{aligned} \tag{7.15}$$

with the definition of the slope weighting functions $w_{s_x}(z, \mathbf{d}_{mn})$. Functions $w_{s_y}(z, \mathbf{d}_{mn})$ and $w_{s_{xy}}(z, \mathbf{d}_{mn})$ are defined similarly, and other spectra than the Kolmogorov power spectrum can be used. The units of such functions are $[\text{rad}^2]$.

Tip/tilt subtraction In practice, the slope functions cannot directly be used, as common telescope motion induced by wind shake and tracking errors should be removed [BWS06]. Hence, the average slope over all subapertures is subtracted (called tip/tilt subtraction) prior to the computation of the slope covariance matrices. This gives, for the tip/tilt corrected slope covariance matrix in the x -direction for example:

$$(\tilde{C}_{s_x})_{m,n} = \langle (s_{x,m} - \bar{s}_x)(s_{x,n} - \bar{s}_x) \rangle = \underbrace{(C_{s_x})_{m,n}}_{\langle s_{x,m} s_{x,n} \rangle} - \langle s_{x,m} \bar{s}_x \rangle - \langle \bar{s}_x s_{x,n} \rangle + \bar{s}_x^2, \tag{7.16}$$

where $\bar{s}_x = \frac{1}{20} \sum_{m=1}^{20} s_{x,m}$. The application of this correction to the theoretical slope functions computed from (7.15) leads to the definitions of functions $\tilde{w}_{s_x}(z, \mathbf{d}_{mn})$ and $\tilde{w}_{s_y}(z, \mathbf{d}_{mn})$.

In practice, the ensemble averages in (7.16) are computed temporally, over the different frames, that is,

$$(\tilde{C}_{s_x})_{m,n} = \langle (s_{x,m} - \bar{s}_x)(s_{x,n} - \bar{s}_x) \rangle = \frac{1}{N_{\text{frames}}} \sum_{p=1}^{N_{\text{frames}}} \left(s_{x,m}^{(p)} - \bar{s}_x^{(p)} \right) \left(s_{x,n}^{(p)} - \bar{s}_x^{(p)} \right),$$

with $\bar{s}_x^{(p)} = \frac{1}{20} \sum_{m=1}^{20} s_{x,m}^{(p)}$.

Denoting the tip/tilt corrected slopes of the m -th subaperture by $\tilde{s}_{x,m}$ and $\tilde{s}_{y,m}$, e.g., $\tilde{s}_{x,m} = s_{x,m} - \bar{s}_x$, (7.15) is now written

$$\langle \tilde{s}_{x,m} \tilde{s}_{x,n} \rangle = \int_0^{+\infty} C_n^2(z) \tilde{w}_{s_x}(z, \mathbf{d}_{mn}) dz. \tag{7.17}$$

7.2.1.5 Representations of the weighting functions

The intensity and slope theoretical weighting functions are computed from (7.10) and from the inverse Fourier transforms of (7.12) to (7.14), assuming a Kolmogorov power spectrum and a wavelength $\lambda = 1280$ nm and $d = 4.7$ cm. The parameter d is the pitch of the array *projected on the telescope aperture*³, involved in $\mathbf{d}_{mn}$ as well as

³The pitch is $500 \mu\text{m}$ in the lenslet array, with 6 subapertures across. Those six subapertures, following the inverse optical path, project on the telescope aperture measuring 279.4 mm. Hence, dividing this number by six gives 4.65667 cm.

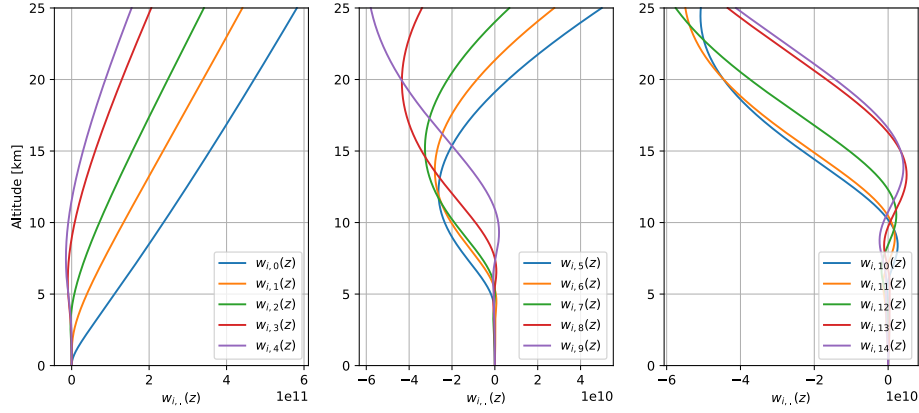


Figure 7.7 – All intensity weighting functions $w_{i,\cdot}(z)$.

in the subaperture function $P(\mathbf{x})$. Here, the aperture is a square, such that its Fourier transform is

$$\hat{P}(\mathbf{f}) = \text{sinc}(f_x d) \text{sinc}(f_y d)$$

with $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ [Tok02]. A discrete 2D inverse FFT algorithm is used for the computation of the weighting functions, with 128 samples and a sampling frequency of $2/d$.

Intensity weighting functions Figures 7.7 shows the altitude dependency (z) of the computed intensity weighting functions. In total, 15 different intensity weighting functions are available, due to the symmetries involved in the computation and the multiplicity of the SHIMM subaperture spacings. The mapping between their index (from 0 to 14) and the spacings is given in Table B.1 in Appendix B. It shows that the intensity weighting functions only depends on the magnitude of the vector $\mathbf{d}_{mn}$, such that the symmetry is $w_i(z, \mathbf{d}_{mn}) = w_i(z, |\mathbf{d}_{mn}|)$.

Figure 7.8 shows a 1D-slice of the intensity weighting functions taken along the x -axis spacings (i.e., a spacing of 0 is $w_{i,0}(z)$, while a spacing of 1 is $w_{i,1}(z)$, a spacing of 2 is $w_{i,3}(z)$, etc., as seen from the mapping in Table B.1). Five different altitudes are presented, and one can already notice that the ground layer cannot be sensed by the intensity fluctuations. From Figure 7.7, it is clear that low altitude layers lead to fewer intensity fluctuations, and are therefore difficult to reconstruct from the intensities.

Angle-of-arrival (slope) weighting functions Slope weighting functions in the x -direction (same as for the y -direction) are given in Figures 7.9 and 7.10, following the mapping given in Table B.2 of Appendix B. Here, the low-altitude layers have a similar or stronger influence than the high-altitude layers. Hence, the slope weighting functions enable to reconstruct low altitudes. In total, 26 different slope weighting functions are found. Considering a vector $\mathbf{d}_{mn} = [\mu, \nu]^T$, with μ and ν being multiples of the pitch spacing, the symmetries on the slope weighting functions can

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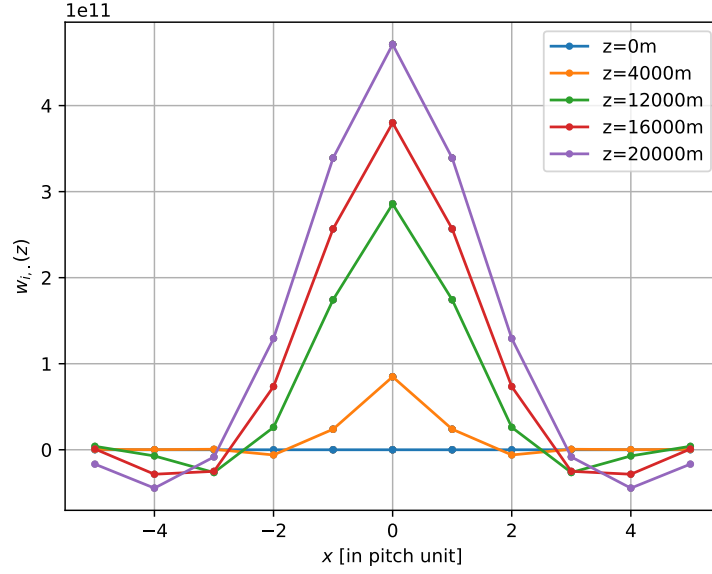


Figure 7.8 – 1D-slice of intensity weighting functions $w_{i,}(z)$.

be expressed as

$$w_{s_x}(z, [\mu, \nu]^T) = w_{s_x}(z, [-\mu, \nu]^T) = w_{s_x}(z, [\mu, -\nu]^T) = w_{s_x}(z, [-\mu, -\nu]^T)$$

with $w_{s_x}(z, [\mu, \nu]^T) = w_{s_y}(z, [\nu, \mu]^T)$.

The functions depicted in Figure 7.9 do not consider the tip/tilt subtraction. They are also quite sensitive to the number of samples used in the FFT. Indeed, close to the zero frequencies, the Kolmogorov power spectrum tends to infinity. This problem is avoided for the intensity weighting functions, as the power spectrum is multiplied by a sine function, equal to zero for zero frequencies. However, for the slope weighting functions, it is multiplied by a cosine, hence the zero frequency leads to a relatively large constant component on the computed slope weighting functions. Nevertheless, with tip/tilt subtraction, this effect disappears as the constant component is removed. This is illustrated in Figure 7.10, where the functions obtained after tip/tilt subtraction are given in dashed lines. They actually correspond to the average of all slope weighting functions computed with such spacing after the tip/tilt subtraction, their individual values being given by the crosses. After tip/tilt subtraction, the number of different weighting functions is increased to 60, as computed numerically.

Finally, the cross-slope weighting functions are presented in Figure 7.11, using the mapping given in Table B.3. A total of 19 different functions are available, with an odd symmetry. Again, expressing $\mathbf{d}_{mn}$ as $[\mu, \nu]^T$, the symmetries are

$$\begin{aligned} w_{s_{xy}}(z, [0, \nu]^T) &= w_{s_{xy}}(z, [\mu, 0]^T) = 0, \\ w_{s_{xy}}(z, [\mu, \nu]^T) &= w_{s_{xy}}(z, [-\mu, -\nu]^T) = w_{s_{xy}}(z, [\nu, \mu]^T) = w_{s_{xy}}(z, [-\nu, -\mu]^T), \end{aligned}$$

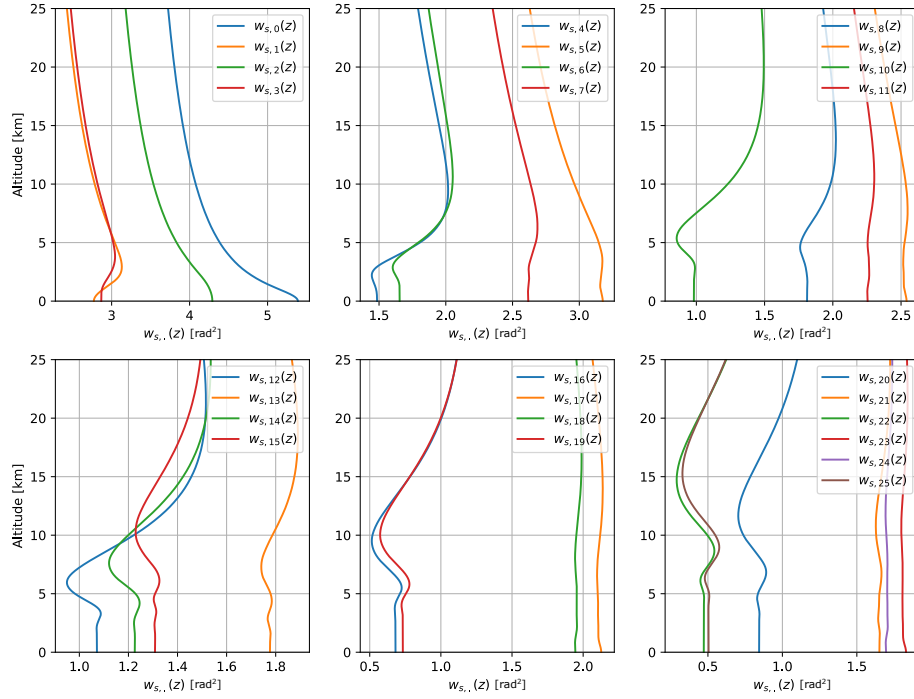


Figure 7.9 – All x -slope and y -slope weighting functions $w_{s_{x..}}(z)$ and $w_{s_{y..}}(z)$.

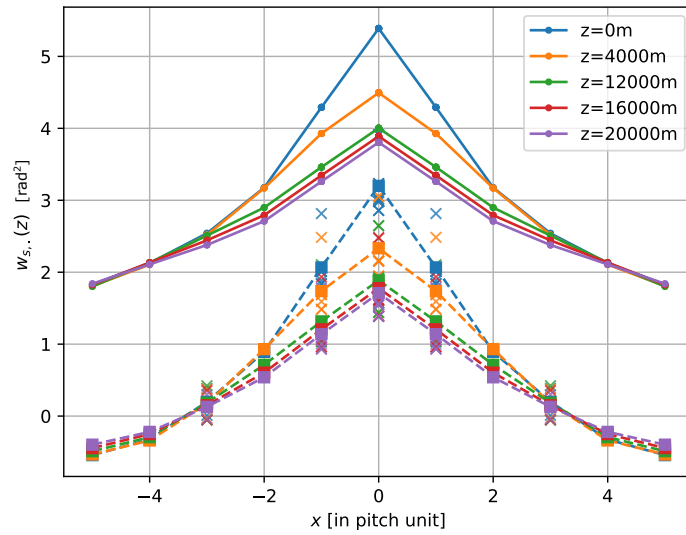


Figure 7.10 – 1D-slice of x -slope weighting functions $w_{s_{x..}}(z)$ and $\tilde{w}_{s_{x..}}(z)$.

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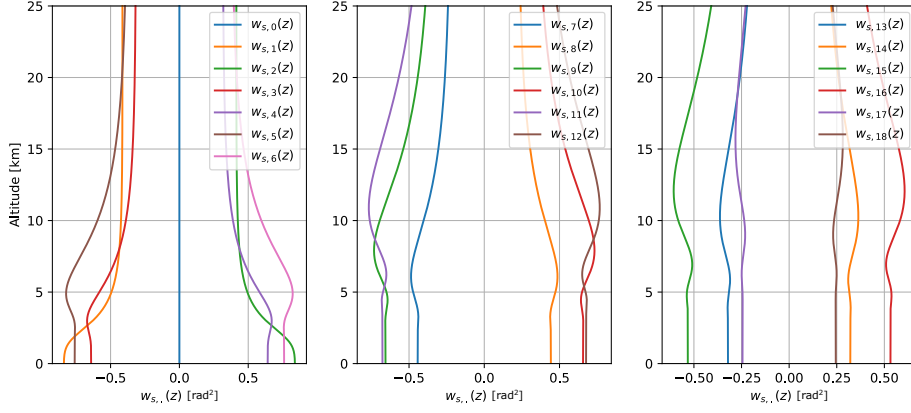


Figure 7.11 – All cross-slope weighting functions $w_{s_{xy}}(z)$.

$$w_{s_{xy}}(z, [\mu, \nu]^T) = -w_{s_{xy}}(z, [\mu, -\nu]^T) = w_{s_{xy}}(z, [-\mu, \nu]^T).$$

In practice, these functions are rarely used [Sau+21].

7.2.1.6 Study of weighting functions

In order to quantify the similarity between the weighting functions, the normalized scalar product (correlation) is used, having for definition

$$\mu_{mn} = \frac{1}{\sigma_m \sigma_n} \int_0^{+\infty} w_{.,m}(z) w_{.,n}(z) dz,$$

where $w_{.,m}(z)$ is any of the intensity or slope weighting functions. σ_m is defined as

$$\sigma_m^2 = \int_0^{+\infty} w_{.,m}^2(z) dz.$$

Ideally, μ_{mn} should be close to zero for all pairs of weighting functions, such that all functions are uncorrelated and provide useful information about C_n^2 profiles sampled by the instrument.

The absolute values of μ_{mn} for the intensity and x -slopes (y -slopes) weighting functions are given in Figure 7.12. As mentioned above, there are 15 different intensity weighting functions, complemented by 26 different slope weighting functions (not including tip/tilt subtraction), giving a total of 41 weighting functions. The normalized scalar product is close to one for the slope weighting functions, as well as for several intensity weighting functions, which limit the amount of uncorrelated observations obtained by the SHIMM instrument.

When considering tip/tilt subtraction, a total of 75 different weighting functions are obtained (intensity and slopes), that show similar scalar products as in Figure 7.12.

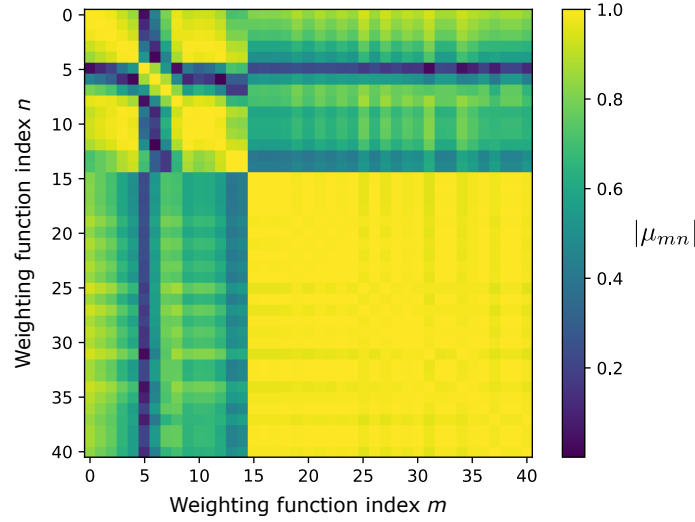


Figure 7.12 – Normalized scalar product for all intensity and slope weighting functions.

7.2.2 Numerical simulations (phase screen propagation)

The numerical modelisation of the SHIMM instrument response is based on phase screen propagation (Section 6.2): for a given C_n^2 profile in input, a plane wave is propagated through the atmosphere that has been discretized with several phase screens, and the output wavefront at the entrance of the telescope pupil is recorded. Then, the response of the SHIMM instrument to this wavefront is modelled, giving access to the intensity and slope fluctuations for a single frame. The simulation is repeated several times to obtain a relevant number of frames, from which the covariance matrices are estimated.

The parameters used for the phase screen propagation are given in Table 7.1. They are similar to the ones presented in Table 6.1, except that the number of vertical equivalent layers has been increased to 10, as well as the oversampling factor to 40, to avoid aliasing arising from high-altitude turbulent layers. The telescope diameter is associated to six subapertures in the SHIMM instrument, each having 10 pixels.

The illustrations given in Figure 7.4 are examples of the received field (i.e., wavefront) and the computed intensities and slopes for a phase screen simulation. In comparison with the real SHIMM instrument, the subaperture lenses are not modelled, such that the intensity and angle-of-arrival fluctuations are not computed from the centroid spots in the camera. Instead, the following computations are conducted:

- the intensity in a subaperture is obtained by summing the intensity from the received field of each pixel in the subaperture. Then, the normalization given in (7.1) is used, and the intensity fluctuations in each subaperture are obtained (i.e., one scalar value in each subaperture).

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Table 7.1 – Parameters used for the phase screen propagation for the numerical modelling of the SHIMM instrument.

Parameters	Values
Wavelength λ	1280 nm
Telescope diameter D	27.94 cm
Number of pixels in D	60
Subaperture size d	$D/6=4.657$ cm
Number of pixels per subaperture	10
Oversampling factor	40
Phase screen size (in pixels)	600×600
Number of equivalent layers	10

- the angles of arrival, i.e., the slopes, are obtained by computing the average gradient of the received wavefront phase in each subaperture, following (7.11). This is a simplification of the SHIMM model since, in practice, the slopes are estimated from the centroid motions observed on the camera.

The validation of the SHIMM numerical model is conducted by comparing the theoretical and numerical responses for different input C_n^2 profiles, as shown in Figure 7.6. The response to a single turbulent layer is presented in Section 7.2.2.1, while the response to a complete C_n^2 profile is shown in Section 7.2.2.2. Examples of both cases are shown in Figure 7.13, where the SCIDAR C_n^2 profile has been acquired at Paranal on March 1, 2023, at 00h50 UTC, for the measurement campaign in [Gri+24a]. A sensitivity to the number of frames is detailed in Section 7.2.2.3.

7.2.2.1 Validation for single turbulent layers

The case of a single turbulent layer corresponds to the theoretical development given in Section 7.2.1.1. The C_n^2 profile is written as $C_n^2(z) = c \delta(z - z_e)$, where c is a constant, $\delta(\cdot)$ is the Dirac function, and z_e is the altitude at which is located the layer. Substituting this expression of $C_n^2(z)$ in (7.9) and (7.17) leads to

$$\begin{aligned} \langle \delta i_m \delta i_n \rangle &= \int_0^{+\infty} C_n^2(z) w_i(z, \mathbf{d}_{mn}) dz, \\ &= c w_i(z_e, \mathbf{d}_{mn}), \end{aligned} \quad (7.9)$$

and

$$\begin{aligned} \langle \tilde{s}_{x,m} \tilde{s}_{x,n} \rangle &= \int_0^{+\infty} C_n^2(z) \tilde{w}_{s_x}(z, \mathbf{d}_{mn}) dz, \\ &= c \tilde{w}_{s_x}(z_e, \mathbf{d}_{mn}). \end{aligned} \quad (7.17)$$

Hence, the obtained intensity and angle-of-arrival covariances correspond to the intensity and slope weighting functions sampled at the altitude z_e of the layer, with a

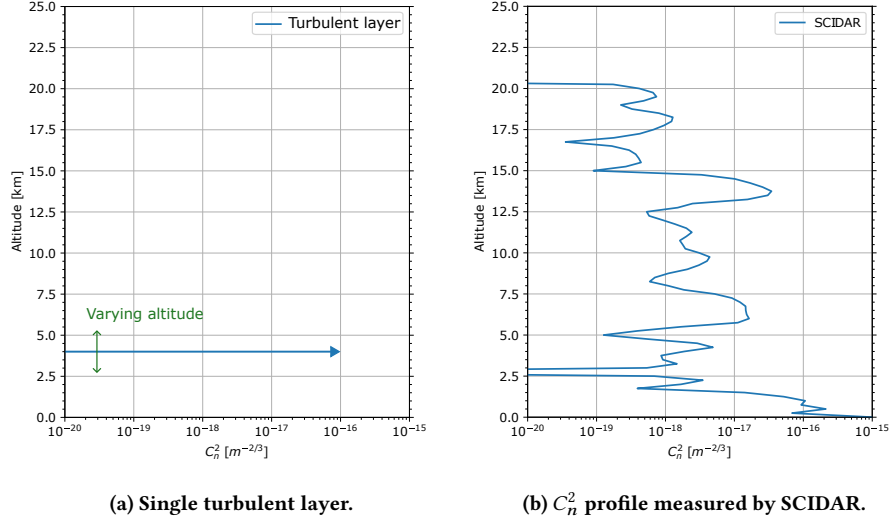


Figure 7.13 – Considered C_n^2 profiles for the validation of the numerical simulations of the SHIMM instrument (SCIDAR profile at Paranal, March 1, 2023, at 00h50 UTC).

constant factor c . By varying the altitude z_e and repeating the simulation, it is therefore possible to obtain numerically the weighting functions. As an illustration, the single-layer C_n^2 profile with $z_e = 4$ km and $c = 10^{-16}$ is shown in Figure 7.13a. In order to obtain accurate covariance estimations (i.e., $\langle \delta i_m \delta i_n \rangle$ and $\langle \tilde{s}_{x,m} \tilde{s}_{x,n} \rangle$), 10000 independent frames have been simulated relying on phase screen propagation. In the case of a single turbulent layer, there is only a single phase screen used in the simulation, located at the altitude z_e .

Figure 7.14 presents the simulations conducted for single turbulent layers located at 0, 4, 12, 16, or 20 km. It shows the values of several weighting functions sampled at those altitudes, similarly to Figures 7.8 and 7.10 (with tip/tilt subtraction). In Figure 7.14, plain lines with squares correspond to the averages of all theoretical weighting functions for subaperture pairs separated by the distance x (in pitch unit). Crosses correspond to the individual samples of the theoretical weighting functions, averaged to obtain the plain lines with squares. Regarding the simulated (numerical) weighting functions, stars represent the individual samples, and their averages is depicted by the dashed lines. As seen in both subfigures, theoretical and numerical weighting functions are superimposed, hence validating the phase screen simulation model for the propagation of a single layer.

7.2.2.2 Validation for complete C_n^2 profiles

From (7.9) and (7.15), the theoretical covariances associated to a given C_n^2 profile can be computed. The numerical covariances are obtained from the phase screen

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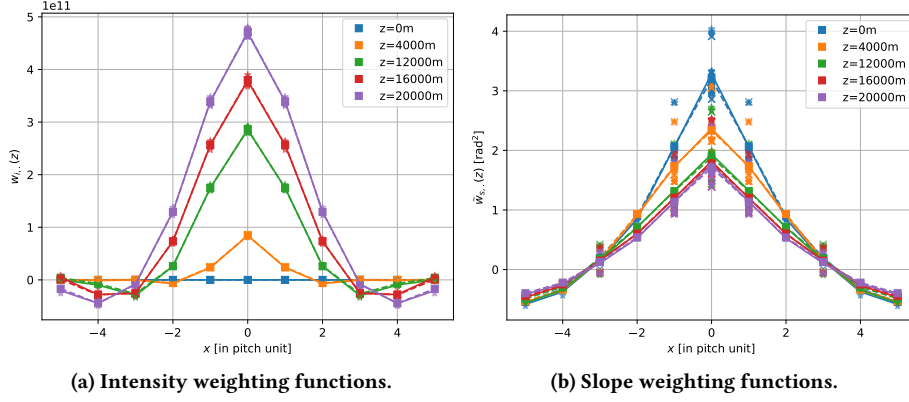


Figure 7.14 – Validation of numerical models of the SHIMM instrument, by sampling the theoretical and numerical weighting functions at 0, 4, 12, 16, and 20 km of altitude.

simulations, with the parameters from Table 7.1.

As an example, the C_n^2 profile measured by a SCIDAR instrument at Paranal, on March 1, 2023, at 00h50 UTC, is used, given in Figure 7.13b. The theoretical and numerical covariance matrices C_I and $\tilde{C}_{s_x}$ (i.e., the values of all $\langle \delta i_m \delta i_n \rangle$ and $\langle \tilde{s}_{x,m} \tilde{s}_{x,n} \rangle$ for any pair m, n) are shown in Figure 7.15. The numerical covariances have been computed with 12000 independent frames from the phase screen simulations.

Both the intensity and slope covariance matrices have maximal values on their diagonal, corresponding to auto-covariances (i.e., covariance of a given subaperture with itself). Other large values are found for small spacings between the two considered subaperture pairs. The agreement between the theoretical and numerical covariances appears excellent, and is quantified in Figures 7.15e and 7.15f.

Figure 7.15e shows the relative absolute differences (i.e., the error) between the theoretical and simulated C_I , in percentage of the maximal value in the theoretical C_I . For all subaperture pairs, the relative error is lower than 3%. Similarly, figure 7.15f shows the absolute error between the logarithms in base 10 of the theoretical and simulated $\tilde{C}_{s_x}$. The agreement is excellent, except for two pairs of subapertures where the error reaches two orders of magnitude. However, it appears that covariance values for those apertures is exactly 0, which is impossible to obtain in practice without requiring an infinite number of frames, and due to the numerical precision limitation. Nevertheless, no influence on the reconstruction performance is expected, since those values are at least four order of magnitudes lower than the maximal slope covariances (10^{-12} vs 10^{-16}).

7.2.2.3 Sensitivity to the number of frames and evaluation dataset

The validation presented in Section 7.2.2.2 is only conducted for a single C_n^2 profile, and for a fixed number of frames. It is therefore relevant to consider more profiles, and a varying number of frames to assess the sensitivity of the performance to this

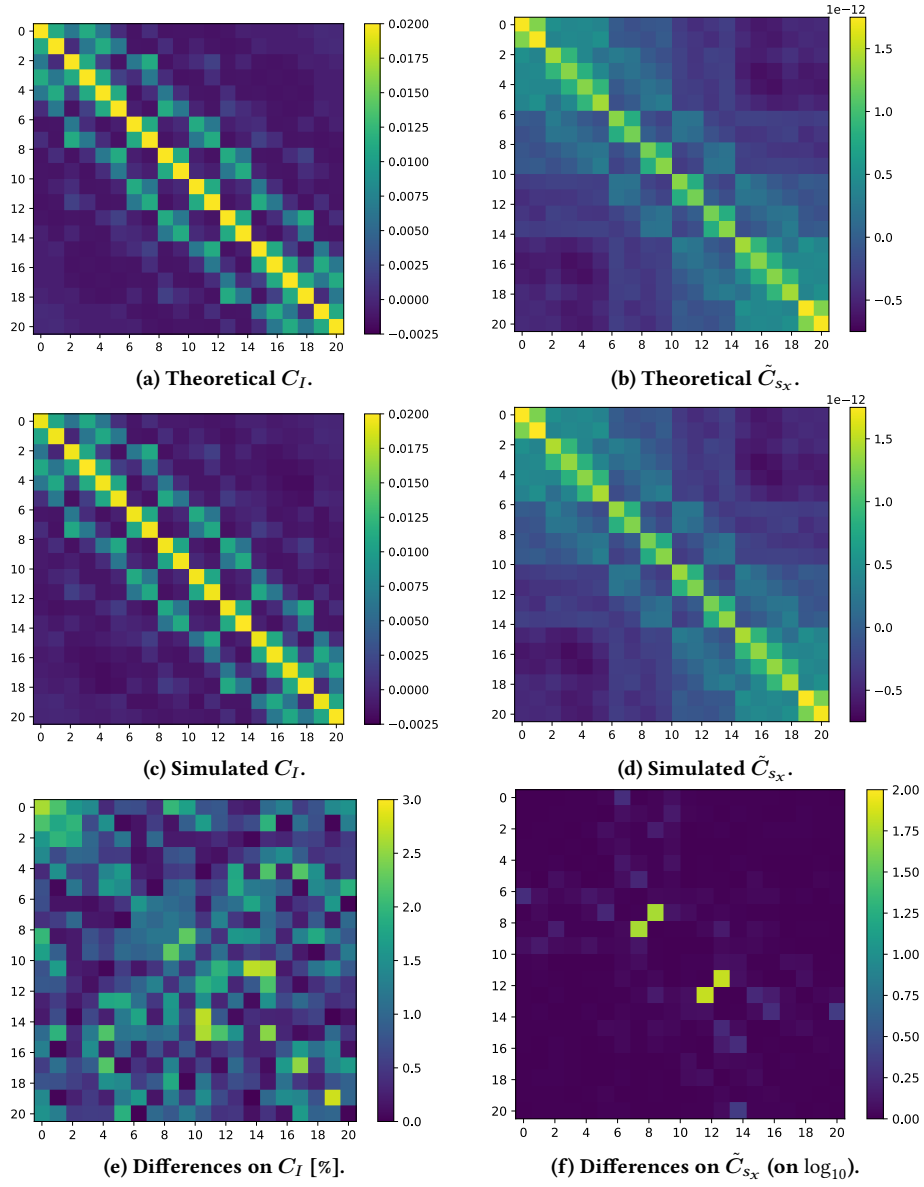


Figure 7.15 – Theoretical and simulated covariance matrices associated to the C_n^2 profile given in Figure 7.13b.

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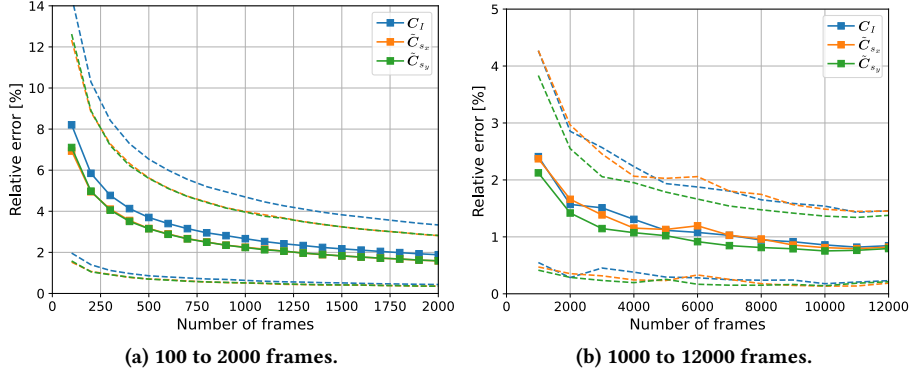


Figure 7.16 – Evolution of the mean relative errors on the intensity and slope covariances with the number of frames used in the estimation, for the 100 SCIDAR profiles in the dataset (left), and for the single SCIDAR profile in Figure 7.13b (right).

parameter.

From the measurement campaign at Paranal, described in [Gri+24a], a dataset of SCIDAR profiles measured between March 1 and March 5, 2023, have been acquired. From this dataset, 100 independent profiles have been chosen, separated by approximately 20 minutes, with roughly 20 profiles per night. They enabled to build a dataset of profiles against which the simulation performance could be compared. This same dataset is used to assess the reconstruction performance in Sections 7.3 to 7.5. As the SCIDAR solely operates during nighttime, the evaluation dataset only contains nighttime profiles.

For each profile, 2000 frames have been simulated with phase screen propagation to estimate the covariance matrices. The estimation of the covariance matrices from a subset of those 2000 frames is studied in Figure 7.16a, that depicts the average relative error between the theoretical and simulated covariances, in percentage of their maximal values. The average have been taken over all subapertures, and over the 100 profiles in the dataset. It is depicted by the plain lines with squares, while the dashed lines show the average plus or minus the standard deviation. Multiples of 100 frames have been used. It shows that mean relative error lower than 2% on the intensity and slope covariances can be achieved, provided that nearly 2000 frames are used. This motivates the choice of 2000 frames, and not a lower number. However, using many frames implies a larger computational cost, since each frame requires an independent phase screen propagation realisation.

This computational cost also motivates the use of a single SCIDAR profile to further assess the evolution of the error metrics for numbers of frames larger than 2000. The result is shown in Figure 7.16b, highlighting that little gain is obtained in this case. Hence, using 2000 frames appears as a relevant trade-off between the accuracy of the covariance estimations and the computational burden.

7.3 Layer-based inversion approach

As a first approach, a C_n^2 profile made of a few discrete layers is reconstructed from the SHIMM observations. It corresponds to the method presented in [Gri+23]. Such approach naturally emerges from the description of OT with discrete layers, as presented in Section 7.2.1.1, and from the phase screen propagation method (Section 6.2).

The formulation of the inverse problem for a layer-based C_n^2 profile is detailed in Section 7.3.1. Then, the approach is numerically validated for single turbulent layers in Section 7.3.2, before being applied to C_n^2 profiles from the SCIDAR dataset in Section 7.3.3.

7.3.1 Formulation of the inverse problem

The inversion approach is based on (7.9) and (7.17) for the x - and y -slopes, reminded below,

$$\langle \delta i_m \delta i_n \rangle = \int_0^{+\infty} C_n^2(z) w_i(z, \mathbf{d}_{mn}) dz, \quad (7.9)$$

$$\langle \tilde{s}_{x,m} \tilde{s}_{x,n} \rangle = \int_0^{+\infty} C_n^2(z) \tilde{w}_{s_x}(z, \mathbf{d}_{mn}) dz, \quad (7.17)$$

$$\langle \tilde{s}_{y,m} \tilde{s}_{y,n} \rangle = \int_0^{+\infty} C_n^2(z) \tilde{w}_{s_y}(z, \mathbf{d}_{mn}) dz. \quad (7.18)$$

The cross-slopes, e.g., $\langle \tilde{s}_{x,m} \tilde{s}_{y,n} \rangle$, are not used in this work.

The expression of the C_n^2 profile made by a number of distinct layers N_{layers} , located at altitudes z_q for $q \in \{1, \dots, N_{\text{layers}}\}$, and having an equivalent magnitude c_q at those altitudes, is given by

$$C_n^2(z) = \sum_{q=1}^{N_{\text{layers}}} c_q \delta(z - z_q). \quad (7.19)$$

The units of c_q are $[\text{m}^{1/3}]$, as it is an equivalent C_n^2 corresponding to $C_n^2 dz$ at an altitude z_q . The number of layers N_{layers} is the number of C_n^2 layers used in the reconstruction and should not be confused with the number of equivalent layers used in the phase screen propagation approach.

The insertion of (7.19) in (7.9), (7.17) and (7.18) leads to linear expressions, giving the SHIMM observations as weighted sums of the weighting functions sampled at altitudes z_q :

$$(\mathbf{C}_I)_{m,n} = \langle \delta i_m \delta i_n \rangle = \sum_{q=1}^{N_{\text{layers}}} c_q w_i(z_q, \mathbf{d}_{mn}), \quad (7.20)$$

$$(\tilde{\mathbf{C}}_{s_x})_{m,n} = \langle \tilde{s}_{x,m} \tilde{s}_{x,n} \rangle = \sum_{q=1}^{N_{\text{layers}}} c_q \tilde{w}_{s_x}(z_q, \mathbf{d}_{mn}), \quad (7.21)$$

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$$(\tilde{C}_{s_y})_{m,n} = \langle \tilde{s}_{y,m} \tilde{s}_{y,n} \rangle = \sum_{q=1}^{N_{\text{layers}}} c_q \tilde{w}_{s_y}(z_q, \mathbf{d}_{mn}). \quad (7.22)$$

The link with the intensity and slope covariance matrices has been made explicit in (7.20) to (7.22).

By aggregating all c_q in a vector $\mathbf{c}$ of dimensions $N_{\text{layers}} \times 1$ and vectorizing the covariance matrices $\mathbf{C}_I$, $\tilde{\mathbf{C}}_{s_x}$ and $\tilde{\mathbf{C}}_{s_y}$, (7.20) to (7.22) can be written with the matrix form

$$\underbrace{\begin{bmatrix} \text{vec}(\mathbf{C}_I) \\ \alpha \text{vec}(\tilde{\mathbf{C}}_{s_x}) \\ \alpha \text{vec}(\tilde{\mathbf{C}}_{s_y}) \end{bmatrix}}_{:=\mathbf{y}} = \underbrace{\begin{bmatrix} \mathbf{A}_I \\ \alpha \mathbf{A}_{s_x} \\ \alpha \mathbf{A}_{s_y} \end{bmatrix}}_{:=\mathbf{A}} \mathbf{c} \quad \Leftrightarrow \quad \mathbf{y} = \mathbf{A} \mathbf{c}, \quad (7.23)$$

where the matrices $\mathbf{A}_I$, $\mathbf{A}_{s_x}$ and $\mathbf{A}_{s_y}$ have dimensions $N_{\text{mes}} \times N_{\text{layers}}$, with the number of measurements N_{mes} equal to 400. They are concatenated vertically in (7.23), and are given by

$$(\mathbf{A}_I)_{v,q} = w_i(z_q, \mathbf{d}_{mn}), \quad (7.24)$$

$$(\mathbf{A}_{s_x})_{v,q} = \tilde{w}_{s_x}(z_q, \mathbf{d}_{mn}), \quad (7.25)$$

$$(\mathbf{A}_{s_y})_{v,q} = \tilde{w}_{s_y}(z_q, \mathbf{d}_{mn}), \quad (7.26)$$

where the index v corresponds to the (m, n) -index pair, i.e., $v \in \{1, \dots, N_{\text{mes}}\}$, conserving the same ordering as the $\text{vec}(\cdot)$ operation.

The coefficient α in (7.23) is a scalar value chosen such that the products between the vector $\mathbf{c}$ and the slope functions (in $\mathbf{A}_{s_x}$ and $\mathbf{A}_{s_y}$) have the same order of magnitude in the reconstruction as the product between $\mathbf{c}$ and the intensity functions (in $\mathbf{A}_I$). A value of 10^{11} is suggested for α and is motivated in Section 7.3.3.1. It is named *intensity-slope coefficient*, or equivalently, *intensity-slope ratio*.

Finally, the vector $\mathbf{y}$ and the matrix $\mathbf{A}$ defined in (7.23) have for dimensions $3N_{\text{mes}} \times 1$ and $3N_{\text{mes}} \times N_{\text{layers}}$, respectively. The vector $\mathbf{y}$ is denoted as the *vector of observations* (or *vector of measurements*), and $\mathbf{A}$ is the *observing matrix*, or *sensing matrix*. Hence, in the case of a layer-based C_n^2 profile, the inverse problem is stated as:

How to obtain the C_n^2 vector $\mathbf{c}$ from the vector of observations $\mathbf{y}$, given the theoretical model of the instrument in (7.23), that is, $\mathbf{y} = \mathbf{A} \mathbf{c}$ with $\mathbf{A}$ the sensing matrix?

This simple formulation hides undesired effects, such as noise, and constraints on $\mathbf{c}$. It is expressed mathematically in the next paragraphs.

Least squares reconstruction The inversion of a linear matrix equation $\mathbf{y} = \mathbf{A} \mathbf{c}$ where the number of observations is larger than the number of unknowns, i.e., the number of layers in this case, can be formulated as a Least Squares (LS) prob-

lem [LH95]. In this case, the estimated C_n^2 vector $\hat{\mathbf{c}}$ is given as the solution of

$$\hat{\mathbf{c}} = \underset{\mathbf{c}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{A}\mathbf{c} - \mathbf{y}\|_2^2, \quad (7.27)$$

where $\|\cdot\|_2$ is the L2-norm. Making the influence of the intensity and slope observations explicit, (7.27) can be written as

$$\hat{\mathbf{c}} = \underset{\mathbf{c}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{A}_I \mathbf{c} - \operatorname{vec}(\mathbf{C}_I)\|_2^2 + \frac{\alpha}{2} \left(\left\| \mathbf{A}_{s_x} \mathbf{c} - \operatorname{vec}(\tilde{\mathbf{C}}_{s_x}) \right\|_2^2 + \left\| \mathbf{A}_{s_y} \mathbf{c} - \operatorname{vec}(\tilde{\mathbf{C}}_{s_y}) \right\|_2^2 \right).$$

For ordinary LS, there exists a closed-form expression and $\hat{\mathbf{c}}$ is obtained by the product between the pseudo-inverse matrix $\mathbf{A}^+$ and $\mathbf{y}$, that is,

$$\hat{\mathbf{c}} = \mathbf{A}^+ \mathbf{y} = \left(\mathbf{A}^T \mathbf{A} \right)^{-1} \mathbf{A}^T \mathbf{y}. \quad (7.28)$$

In (7.28), the pseudo-inverse $\mathbf{A}^+$ is equal to $(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ [LH95, p. 39]. This is valid when the rank of $\mathbf{A}$ is N_{layers} , for a matrix $\mathbf{A}$ that is $3N_{\text{mes}} \times N_{\text{layers}}$. This is always the case as long as the number of layers N_{layers} is smaller than three times the number of measurements, minus the repeated observations from the symmetries in the weighting functions.

For the practical implementation of (7.27), the ordinary least squares solver from the Python package `scipy.optimize` is used [Vir+20].

Non-negative least squares reconstruction The LS problem in (7.27) does not ensure that the estimated C_n^2 vector contains only positive coefficients. Since, in practice, C_n^2 is a positive quantity, a non-negativity constraint should be added to the LS problem. This leads to the Non-Negative Least Squares (NNLS) problem given by

$$\hat{\mathbf{c}} = \underset{\mathbf{c}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{A}\mathbf{c} - \mathbf{y}\|_2^2, \quad \text{such that} \quad \mathbf{c} \geq \mathbf{0}. \quad (7.29)$$

Due to the non-negativity constraint, there is no closed-form solution, and iterative algorithms are used to solve (7.29) [LH95, p. 161]. The implementation from `scipy.optimize` is used [Vir+20].

7.3.2 Validation for single turbulent layers

First, the reconstruction by the SHIMM instrument of a single turbulent layer is studied. Without loss of generality, five layers are considered for the reconstruction, i.e., $N_{\text{layers}} = 5$, as depicted in Figure 7.17. These five layers are located at altitudes 0, 4, 12, 16, and 20 km; and are associated to the C_n^2 vector $\mathbf{c} = [c_1, c_2, c_3, c_4, c_5]^T$.

The primary objective is to determine how the SHIMM instrument, in its 5-layer version, reconstructs a single turbulent layer. Ideally, it is expected that if the turbulent layer is located exactly at the altitude of one of the five reconstructed layers, it should only “activate” this layer in the reconstruction. However, for a turbulent layer

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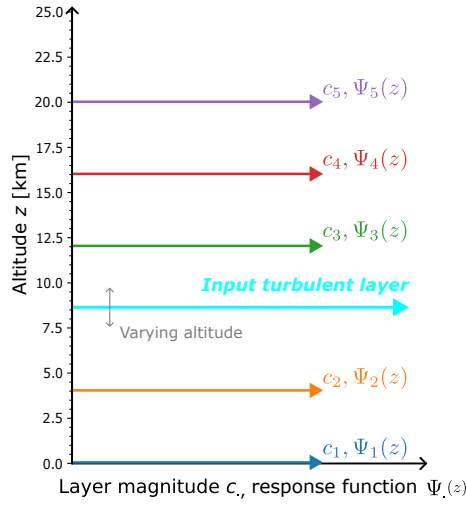


Figure 7.17 – Schematic of the five layers used for the reconstruction of a single turbulent layer (in cyan), with their associated magnitudes c_q and response functions $\Psi_q(z)$, defined below, for $q \in \{1, \dots, 5\}$.

that does not correspond to one of the possible five layers, it should be “distributed” among these layers.

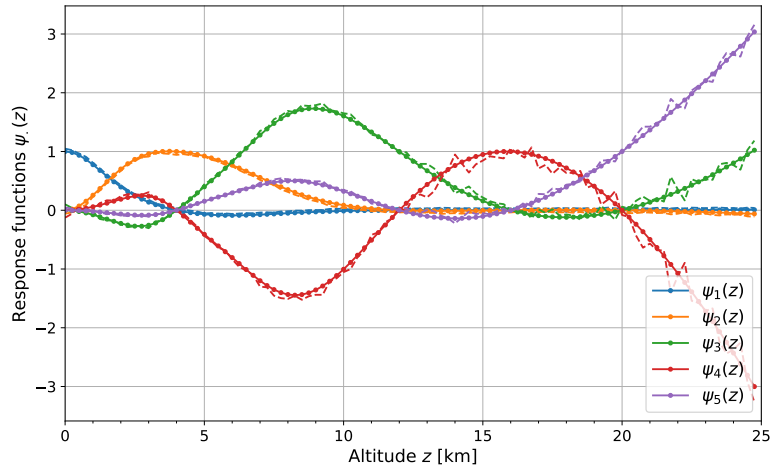
For this study, both the theoretical and numerical models are used to obtain the SHIMM instrument response for a single layer, as well as both LS and NNLS algorithms for the reconstruction. The altitude of the layer to reconstruct varies from 0 to 25 km, using a spatial resolution of 250 m.

Figure 7.18 presents the values for the reconstructed C_n^2 vector c , with the following colour code: blue for c_1 , orange for c_2 , green for c_3 , red for c_4 , and purple for c_5 . It should be read vertically to obtain the values of all c_q ($q \in \{1, \dots, 5\}$) for a given altitude of the turbulent layer. For example, a layer situated at an altitude of 4 km (4 km on the x -axis), with a normalized magnitude⁴, leads to a coefficient c_2 equal to one, with all others coefficients equal to zeros. On the contrary, a layer at 5 km is distributed between the coefficients c_2 and c_3 . This distribution depends on the reconstruction algorithm used, either LS or NNLS.

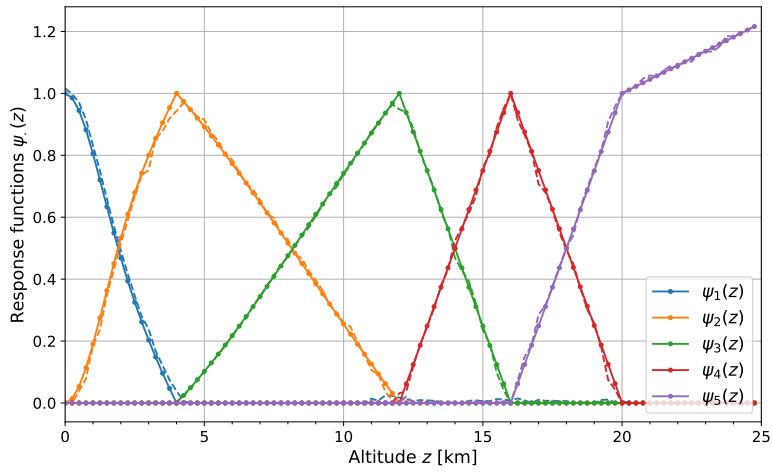
Several comments result from this figure:

- Plain lines with dotted points correspond to the reconstructed c based on the theoretical SHIMM response to a single layer, while the dashed lines are associated to the reconstruction with the numerical SHIMM response. This numerical response has been obtained by the phase screen simulations presented in Section 7.2.2.1. The agreement between both approaches is expected, as the theoretical and numerical SHIMM responses are similar. It is indeed the case, except for small fluctuations with the LS reconstruction that could be

⁴In practice, layers are given a magnitude of 10^{-16} for the simulations, and then the reconstructed coefficients c are normalized with respect to this value.



(a) LS response functions.



(b) NNLS response functions

Figure 7.18 – Response functions for a single-layer in input and five layers in the reconstruction, as given in Figure 7.17.

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prevented by further increasing the number of frames used in the numerical model (currently, 10000 frames).

- In both LS and NNLS cases, the C_n^2 vector $\mathbf{c}$ has a single component equal to 1 at one of the five layer altitudes (0, 4, 12, 16, and 20 km). This behaviour can be related to the behaviour of interpolating functions between known samples, here corresponding to the five altitudes. In this framework, the terminology of *response functions* is used, to emphasize that these curves present the SHIMM instrument response to an individual layer, presenting how it is distributed between its five layers. These response functions are denoted $\Psi_q(z)$, with $q \in \{1, \dots, 5\}$, as given in the figure legend.
- With the LS reconstruction, negative values are obtained for some altitudes. They are related to the weighting functions themselves, through the matrices $\mathbf{A}_I$, $\mathbf{A}_{s_x}$, and $\mathbf{A}_{s_y}$ in the matrix $\mathbf{A}$, as shown by (7.23) to (7.26). Since these weighting functions can be negative for some altitudes and $\mathbf{d}_{mn}$ spacings, as illustrated in Figures 7.7 and 7.10, it is not surprising to obtain negative values in the LS response functions. However, this challenges the interpretation of the C_n^2 vector as equivalent C_n^2 layers.
- In the NNLS case, at most two layers are “activated” for a single turbulent layer, corresponding to the closest layers to this turbulent layer. This is not valid for altitudes higher than 20 km. Instead, an overshoot is observed in $\Psi_5(z)$ to compensate for the turbulence generated by high altitudes (higher than 20 km).

The formalism of response functions provides an equivalent modelling of the SHIMM instrument response and reconstruction, in the case of LS reconstruction. This is illustrated in Figure 7.19, completing Figure 7.1 for a SHIMM instrument with five layers. Considering a C_n^2 profile discretized over 100 layers at altitudes z'_q , i.e.,

$$C_n^2(z) = \sum_{q=1}^{100} c'_q \delta(z - z'_q),$$

which corresponds to a spatial resolution of 250 m between 0 and 25 km of altitude, a vector $\mathbf{c}'$ of length 100 is defined. The SHIMM forward model relies on a matrix $\mathbf{A}'$, obtained from (7.23) and from (7.24) to (7.26), with z'_q instead. Hence, the intensity and angle-of-arrival fluctuations are given by $\mathbf{y} = \mathbf{A}'\mathbf{c}'$. This corresponds to the first part of Figure 7.19, where the vector and matrix dimensions are specified.

The reconstruction can then be performed using LS or NNLS. Both cases involve the matrix $\mathbf{A}$, defined with five layers, and provides the output C_n^2 profile vector $\hat{\mathbf{c}}$ of dimension 5.

Alternatively, the complete processing of the SHIMM instrument is substituted by the application of the response functions, as shown in the lower part of Figure 7.19. In this case, the output C_n^2 profile is obtained by the projection of the input C_n^2 profile

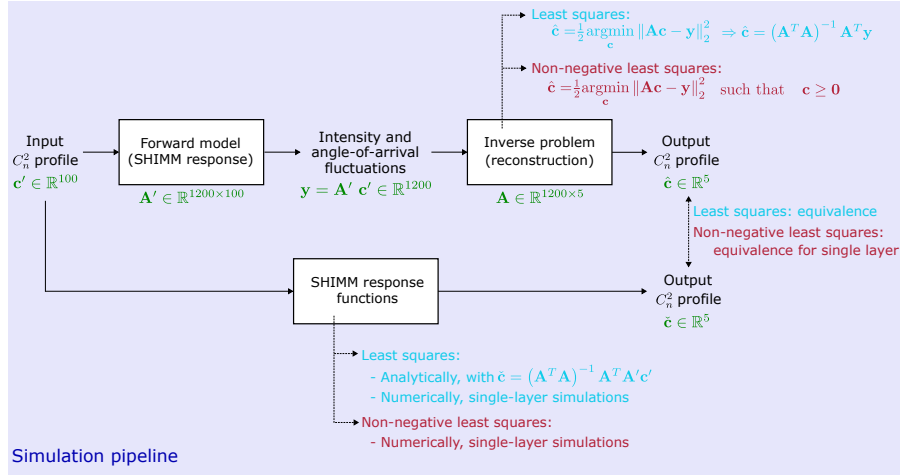


Figure 7.19 – Full modelling of the SHIMM instrument and response function approach. Both approaches are equivalent in the LS case, or with a single turbulent layer as input.

on those response functions, i.e.,

$$\check{c}_q = \int_0^{25 \text{ km}} C_n^2(z) \Psi_q(z) dz, \quad \text{for } q \in \{1, \dots, 5\}, \quad (7.30)$$

or with the equivalent discrete form for the integration. The notation $\check{c}$ is used to differentiate both approaches.

Regarding the response functions, they are computed numerically, as explained above, with a single turbulent layer as input (hence, only one non-zero value in c'). With the LS reconstruction, they can also be obtained analytically, and correspond to the rows of the matrix $(A^T A)^{-1} A^T A'$. The dimensions of matrices A and A' clearly highlight the change of resolution in the C_n^2 profile, from 100 layers to 5 layers in this scenario.

The equivalence between $\hat{c}$ and $\check{c}$ is ensured in scenarios with a single turbulent layer as input, since this corresponds to the approach followed to obtain the response functions. Moreover, in the LS case, the equivalence is also true and $\hat{c} = \check{c}$. Indeed, both approaches, i.e., either the application of the forward model and the inversion or the projection on the response functions, are linear operations.

However, in the NNLS scenario, there is no guarantee that the linear approach associated to the projection on the response functions will provide similar results as the complete SHIMM modelling, that involves a non-linear operation. This is further illustrated in Section 7.3.3 and Figure 7.20 for C_n^2 profiles in input.

Similar observations and results are obtained when the reconstruction is conducted with fewer or more layers, with response functions showing the same properties as the ones highlighted in Figure 7.18.

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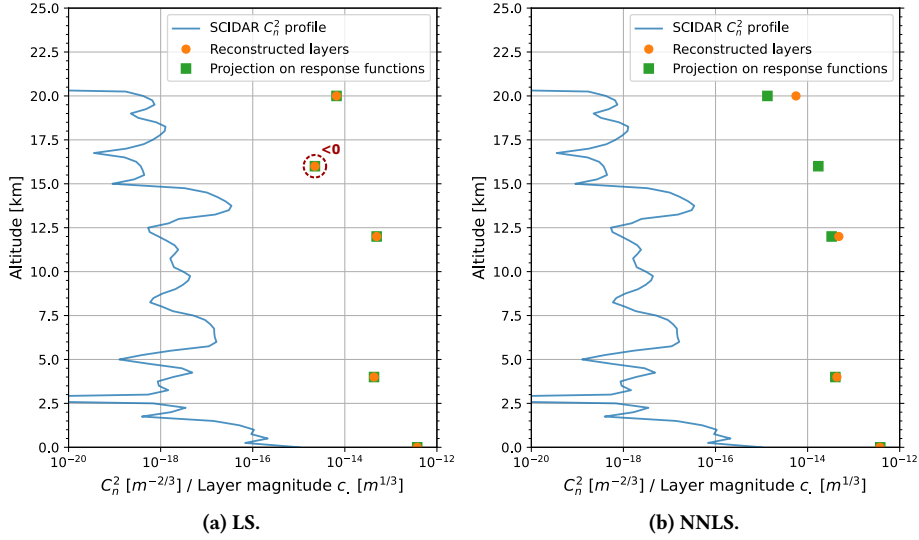


Figure 7.20 – Comparison between projection and reconstruction on 5 layers for the C_n^2 profile given in Figure 7.13b.

7.3.3 Validation with C_n^2 profiles

The inversion procedure described in Section 7.3.1 is further validated with C_n^2 profiles given as inputs and not single layers. The dataset of 100 SCIDAR profiles and the associated simulated SHIMM responses, presented in Section 7.2.2.2, is used. These profiles are considered as continuous profiles even though, in practice, they are sampled with a vertical resolution of 250 m.

Example for one C_n^2 profile Figure 7.20 presents the reconstructed 5-layer profile for the SCIDAR C_n^2 profile given in Figure 7.13b, and shown in blue in this figure. The layers, i.e., the vector $\hat{c}$, obtained with the LS or NNLS approaches is depicted with orange dots. In the LS case, the fourth layer contains a negative coefficient, which absolute magnitude is shown on the graph. For the NNLS approach, the fourth layer is set to zero, and is therefore not visible in the figure.

Green squares present the projection on the C_n^2 input profile onto the response functions given in Figure 7.18, using (7.30), i.e., the lower approach shown in Figure 7.19 giving $\check{c}$. As expected, a perfect agreement is observed between the LS reconstruction and the projection on the response functions. On the contrary, in the NNLS case, discrepancies between both approaches are observed, especially for the fourth layer. Table 7.2 gives the exact values of the coefficients for the LS and NNLS reconstructions and projections. As an observation, the zero coefficient in the NNLS reconstruction is associated to the negative coefficient in the LS reconstruction and projection (c_4).

In Table 7.2, the integrated parameters obtained based on the equivalent layers (projected or reconstructed) are presented. The Fried parameter r_0 , the isoplanatic

Table 7.2 – C_n^2 vector c and integrated parameters for the C_n^2 profile given in Figure 7.13b.

C_n^2 vector and other parameters	LS		NNLS	
	Projection ($\hat{c}$)	Reconstruction ($\hat{c}$)	Projection ($\hat{c}$)	Reconstruction ($\hat{c}$)
c_1 ($z = 0$ km)	3.73×10^{-13}	3.73×10^{-13}	3.78×10^{-13}	3.73×10^{-13}
c_2 ($z = 4$ km)	4.31×10^{-14}	4.31×10^{-14}	4.00×10^{-14}	4.33×10^{-14}
c_3 ($z = 12$ km)	4.91×10^{-14}	4.91×10^{-14}	3.32×10^{-14}	4.77×10^{-14}
c_4 ($z = 16$ km)	-2.24×10^{-15}	-2.24×10^{-15}	1.71×10^{-14}	0
c_5 ($z = 20$ km)	6.60×10^{-15}	6.60×10^{-15}	1.34×10^{-15}	5.59×10^{-15}
r_0 [cm]	12.65	12.65	12.65	12.64
θ_0 [μ rad]	10.58	10.58	10.34	10.58
σ_I^2	0.0795	0.0795	0.0789	0.0795

angle θ_0 and the scintillation index σ_I^2 are computed according to their definitions in (6.32), (6.36) and (6.39), assumed at zenith and for a wavelength $\lambda = 500$ nm. The continuous integrals are substituted by a summation over the five equivalent layers. Interestingly, the integrated parameters obtained by the LS or the NNLS reconstruction are equal, even though the C_n^2 vectors $\hat{c}$ are different, and even contain a negative value in the LS case. However, differences are observed with the integrated parameters coming from the projection on the NNLS response functions, as anticipated. The genuine values of these parameters, computed from the input SCIDAR profiles, are: $r_0 = 15.30$ cm, $\theta_0 = 10.60$ μ rad, and $\sigma_I^2 = 0.087$. The coherence time τ_0 is not evaluated, as the wind speed profile, necessary in (6.37), is not estimated with this version of SHIMM processing.

Evaluation over the dataset Similarly, the integrated parameters are computed for all SCIDAR profiles in the evaluation dataset, and the comparisons between the integrated parameters from the NNLS reconstructed-layers or from the initial SCIDAR profile are shown in Figure 7.21. Numerous metrics are also presented in Table 7.3, including the RMSE, MRE, and the correlation coefficient.

Based on these results, the isoplanatic angle seems to be the quantity that is the best reconstructed, i.e., with the highest correlation coefficient and the lowest MRE. This is confirmed in Figure 7.21b. The reconstructions of the Fried parameter and the scintillation index achieve good performance, with correlation coefficients of 0.970 and 0.936 respectively, and MRE of 9.46% and 16.10%. As seen in Figure 7.21, these latter two quantities tend to be underestimated in the SHIMM reconstruction.

As a result, even if negative equivalent C_n^2 layers are obtained with the LS reconstruction and are difficult to interpret, the integrated parameters are consistent.

7.3.3.1 Determination of the intensity-slope coefficient

The relative weight between the intensity fluctuations and the angle-of-arrival fluctuations in the reconstruction algorithm is set by the coefficient α in (7.23). Its impact

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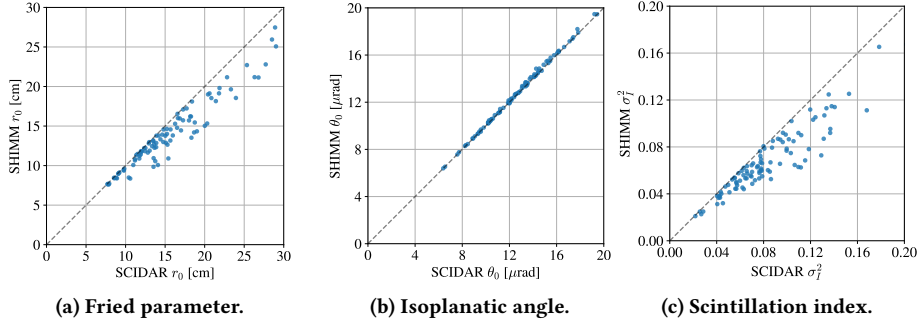


Figure 7.21 – Correlation plots between the integrated parameters computed from the input SCIDAR profiles and the ones obtained from the NNLS reconstructed SHIMM profiles.

Table 7.3 – Error metrics used to validate the C_n^2 layer reconstruction for the dataset SCIDAR profiles.

Integrated parameters	LS reconstruction			NNLS reconstruction		
	RMSE	MRE [%]	Corr. coeff.	RMSE	MRE [%]	Corr. coeff.
r_0 [cm]	2.51	9.45	0.970	2.51	9.46	0.970
θ_0 [μrad]	0.123	0.61	0.999	0.103	0.54	0.999
σ_I^2	0.02	16.13	0.936	0.02	16.10	0.936

on the reconstruction performance of integrated parameters for the SCIDAR dataset, as studied in the previous section, is presented in Figure 7.22, highlighting the RMSE, MRE, and correlation coefficient evolution with α , in the 5-layer scenario.

It shows that, in terms of integrated parameters, it has little influence over a whole range of values, except around 10^{11} where it leads to a transition in the metrics. Increasing α , thus increasing the weights of the slopes compared to the intensity, tends to improve the reconstruction of the scintillation index, while degrading the reconstruction of the isoplanatic angle. It has almost no influence on the reconstructed Fried parameter. These observations can be explained by:

1. The small values (close to zero, and even zero at the ground) of the intensity response functions at low altitudes, as shown in Figure 7.7. Hence, low-altitude layers can only be reconstructed from the angle-of-arrival fluctuations, that will have important influence at these altitudes independently on the value of α for the range considered in Figure 7.22. It explains why the Fried parameter is always well reconstructed, since it is mostly dependent on the strong turbulent layers at low altitudes.
2. The reduced influence of intensity weighting functions at high altitudes compared to the slope weighting functions for large values of α . This slightly degrades the reconstruction performance of high-altitude layers, hence impacting the isoplanatic angle. On the contrary, mid-altitude layers may be better

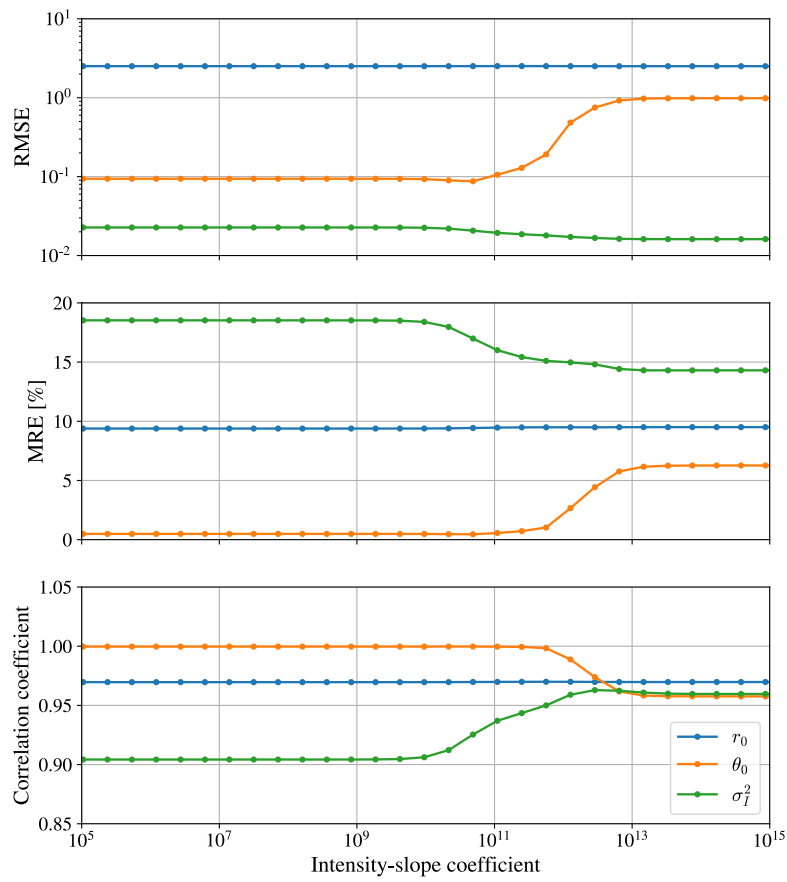


Figure 7.22 – Evolution of metrics on integrated parameters with the intensity-slope coefficient (NNLS reconstruction on 5 layers).

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reconstructed, improving the scintillation index.

There is therefore a trade-off between the focus on low altitudes or high altitudes in the reconstruction, and a value of 10^{11} for α seems suitable. It corresponds to the order of magnitude between the intensity and slope weighting functions, as seen in Figure 7.8 and 7.10, and therefore agrees with the intuition of giving a similar importance to both. It is also in agreement with the use of slope functions expressed in $[\text{arcsec}^2]$ instead of $[\text{rad}^2]$ in the SHIMM algorithm in [Gri+23], as one radian is equal to 206265 arcsec, and $206265^2 = 4.25 \times 10^{10}$, that is close to 10^{11} .

This conclusion is not influenced by the number of layers used in the reconstruction, as well as by the use of the LS approach. However, in this case, stronger degradations of the scintillation index and the isoplanatic angle are observed for values of α larger than 10^{12} .

7.3.3.2 Study of the number of reconstructed layers

For a value of α set to 10^{11} , the variations of the RMSE, MRE and correlation coefficient metrics on the integrated parameters is further studied, with the evolution of the number of layers used in the SHIMM reconstruction. This is depicted in Figure 7.23, in the NNLS case. For a given number of layers, layers are considered to be linearly equispaced between the ground and 25 km altitude.

According to the metrics on the integrated parameters, increasing the number of layers tends to improve the performance, leading to smaller RMSE and MRE, and larger correlation coefficients. However, it also increases the number of zeros obtained in the reconstructed C_n^2 vectors $\hat{c}$, as shown in the last graph in Figure 7.23. The black plain line with dots corresponds to the average number of zeros in the reconstructed vectors for the 100 profile in the dataset, while the dashed black lines correspond to the average plus and minus the standard deviation. Clearly, there seems to be a linear increase of the number of zeros with the number of layers.

The condition number of the inversion matrix is also shown in red. It is computed based on the matrix $\tilde{A}$, that is made of a subset of the different rows of the matrix A , hence removing rows that are duplicated. As this condition numbers increases with the number of layers, the inversion approach is degraded (ill-conditioned matrix).

Similar conclusions are obtained with the LS approach, with an additional degradations of the performance on θ_0 for more than 12 layers. In this LS case, no zeros are obtained in the C_n^2 vector.

7.4 Dictionary-based inversion approaches

As an alternative to the layer-based inversion approach, the reconstruction of C_n^2 based on a dictionary of continuous functions of the altitude is presented in this section. The concepts and the formulation of the inverse problem are detailed in Section 7.4.1, before being applied to rectangular functions in Section 7.4.2 and to principal component analysis in Section 7.4.3.

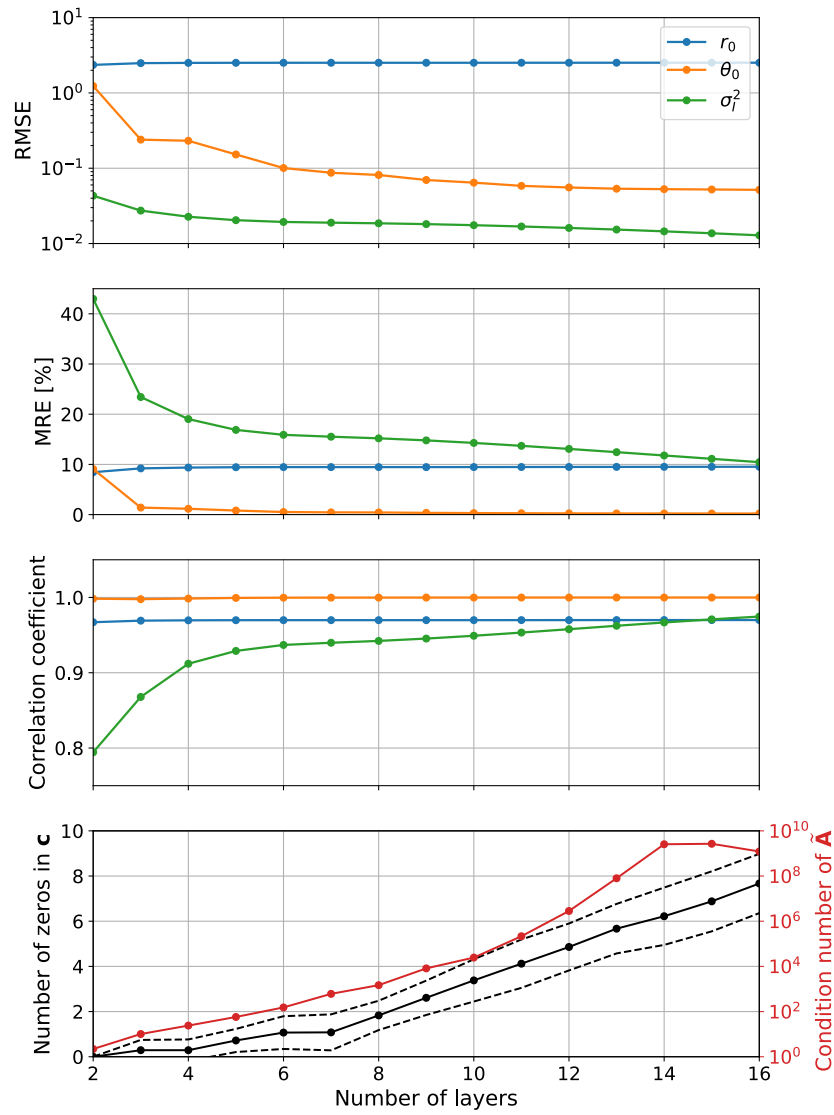


Figure 7.23 – Evolution of metrics on integrated parameters, number of zeros in $\mathbf{c}$ and $\mathbf{\hat{A}}$ matrix condition number with the number of layers used in the reconstruction.

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7.4.1 Formulation of the inverse problem

Instead of linearizing the SHIMM equations by assuming a discrete C_n^2 profile made of individual layers, the use of a finite set of *basis functions*, i.e., a dictionary, can be of interest. With this approach, the $C_n^2(z)$ profile is expressed as a linear combination of continuous functions $\Phi_\theta(z)$, weighted by coefficients c_θ :

$$C_n^2(z) = \sum_{\theta \in \Theta} c_\theta \Phi_\theta(z). \quad (7.31)$$

The dictionary, i.e., the set of all $\Phi_\theta(z)$, is written Θ and has a cardinality $|\Theta|$. It should be chosen to ensure accurate representations of real C_n^2 profiles when expressed with (7.31), such that numerous choices are possible.

The insertion of (7.31) in the SHIMM equations (7.9), (7.17) and (7.18) leads to:

$$(C_I)_{m,n} = \langle \delta i_m \delta i_n \rangle = \sum_{\theta \in \Theta} c_\theta \int_0^{+\infty} \Phi_\theta(z) w_i(z, \mathbf{d}_{mn}) dz, \quad (7.32)$$

$$(\tilde{C}_{s_x})_{m,n} = \langle \tilde{s}_{x,m} \tilde{s}_{x,n} \rangle = \sum_{\theta \in \Theta} c_\theta \int_0^{+\infty} \Phi_\theta(z) \tilde{w}_{s_x}(z, \mathbf{d}_{mn}) dz, \quad (7.33)$$

$$(\tilde{C}_{s_y})_{m,n} = \langle \tilde{s}_{y,m} \tilde{s}_{y,n} \rangle = \sum_{\theta \in \Theta} c_\theta \int_0^{+\infty} \Phi_\theta(z) \tilde{w}_{s_y}(z, \mathbf{d}_{mn}) dz. \quad (7.34)$$

Based on (7.32) to (7.34), a similar matrix expression as (7.23) can be obtained, i.e.,

$$\underbrace{\begin{bmatrix} \text{vec}(C_I) \\ \alpha \text{vec}(\tilde{C}_{s_x}) \\ \alpha \text{vec}(\tilde{C}_{s_y}) \end{bmatrix}}_{:=\mathbf{y}} = \underbrace{\begin{bmatrix} \mathbf{A}_I \\ \alpha \mathbf{A}_{s_x} \\ \alpha \mathbf{A}_{s_y} \end{bmatrix}}_{:=\mathbf{A}} \mathbf{c} \quad \Leftrightarrow \quad \mathbf{y} = \mathbf{A} \mathbf{c}, \quad (7.23)$$

provided that the matrices $\mathbf{A}_I$, $\mathbf{A}_{s_x}$ and $\mathbf{A}_{s_y}$ are now defined by

$$(\mathbf{A}_I)_{v,\theta} = \int_0^{+\infty} \Phi_\theta(z) w_i(z, \mathbf{d}_{mn}) dz, \quad (7.35)$$

$$(\mathbf{A}_{s_x})_{v,\theta} = \int_0^{+\infty} \Phi_\theta(z) \tilde{w}_{s_x}(z, \mathbf{d}_{mn}) dz, \quad (7.36)$$

$$(\mathbf{A}_{s_y})_{v,\theta} = \int_0^{+\infty} \Phi_\theta(z) \tilde{w}_{s_y}(z, \mathbf{d}_{mn}) dz. \quad (7.37)$$

The index v corresponds to the (m, n) -index pair, i.e., $v \in \{1, \dots, N_{\text{mes}}\}$, and conserves the same ordering as the $\text{vec}(\cdot)$ operation used in (7.23). The coefficient α is the *intensity-slope ratio*, for which the value of 10^{11} is still valid for this approach, as motivated in Section 7.4.2.2.

The matrices $\mathbf{A}_I$, $\mathbf{A}_{s_x}$ and $\mathbf{A}_{s_y}$ have dimensions $N_{\text{mes}} \times |\Theta|$, with the number of measurements N_{mes} equal to 400. Their concatenation in (7.23) leads to a matrix $\mathbf{A}$

of dimensions $3N_{\text{mes}} \times |\Theta|$, while the vector $\mathbf{y}$ keeps the same dimensions $3N_{\text{mes}} \times 1$ as in the layer-based inversion case, and $\mathbf{c}$ is of dimensions $|\Theta| \times 1$.

The computation of the matrices $\mathbf{A}_I$, $\mathbf{A}_{s_x}$ and $\mathbf{A}_{s_y}$, following (7.35) to (7.37), is part of a pre-processing step of the inversion approach. Indeed, the basis functions $\Phi_\theta(z)$ of the dictionary Θ are chosen a priori, such that their projection on the SHIMM weighting functions, as mathematically expressed by (7.35) to (7.37), is independent on the SHIMM measurements in $\mathbf{y}$. Hence, for a given choice of dictionary, the matrix $\mathbf{A}$ is only computed once, as in the layer-inversion case. It is also referred to as the *sensing matrix*.

Provided that the number of basis functions, i.e., the cardinality $|\Theta|$, is smaller than the number of independent observations (i.e., 75 with tip/tilt-subtracted slope functions, as shown in Section 7.2.1.5), it is possible to obtain the coefficients $\mathbf{c}$ using a LS reconstruction. Cases where the dictionary contains more basis functions than the number of independent observations are described in Section 7.5. In certain scenarios where several basis functions have the same projections on the SHIMM weighting functions (non-optimal choice of the dictionary), the number of independent observations can be reduced. When the coefficients $\mathbf{c}$ are computed, they can be reexpressed as a $C_n^2(z)$ profile using (7.31).

Least squares reconstruction The LS reconstruction is the same as in Section 7.3, with the new definition of the matrix $\mathbf{A}$ and the vector of coefficients $\mathbf{c}$:

$$\hat{\mathbf{c}} = \underset{\mathbf{c}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{Ac} - \mathbf{y}\|_2^2, \quad (7.27)$$

where $\|\cdot\|_2$ is the L2-norm. The closed-form expression (7.28) is also valid.

Non-negative least squares reconstruction The non-negativity of the coefficients in $\mathbf{c}$ could be required to ensure a positive $C_n^2(z)$ profile, depending on the choice of the basis functions. The NNLS problem is formulated as previously

$$\hat{\mathbf{c}} = \underset{\mathbf{c}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{Ac} - \mathbf{y}\|_2^2, \quad \text{such that} \quad \mathbf{c} \geq \mathbf{0}. \quad (7.29)$$

7.4.2 Reconstruction with rectangular basis functions

The use of rectangular functions, or window functions, as basis functions in the dictionary Θ is explored in this section. Two distinct cases are considered:

1. A number $N_{\text{functions}}$ of equispaced rectangular functions, all having the same widths, spanning the altitude range from 0 to 25 km. Such functions are mathematically defined by

$$\Phi_q(z) = \begin{cases} 1 & \text{if } z \in [z_q - \frac{\Delta z}{2}, z_q + \frac{\Delta z}{2}] \\ 0 & \text{otherwise,} \end{cases}$$

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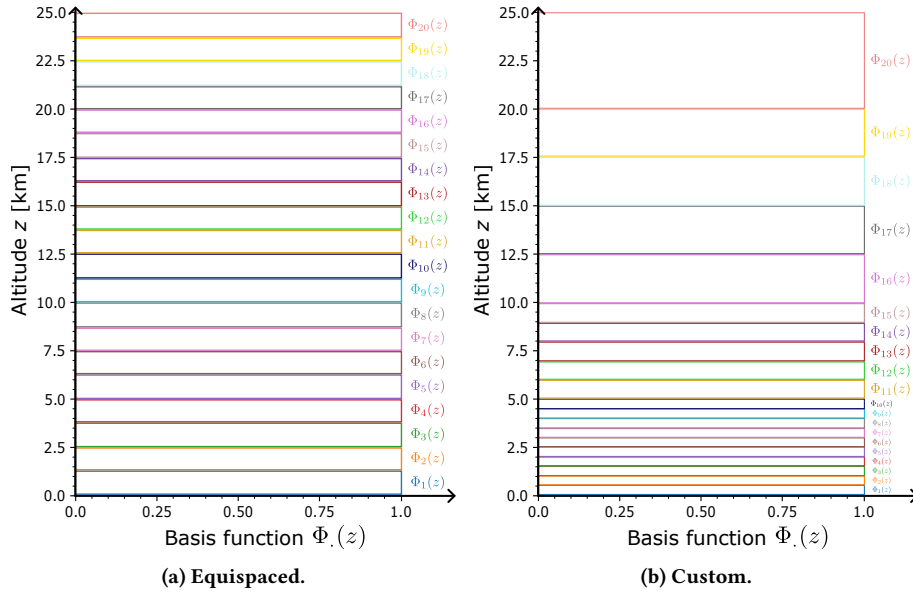


Figure 7.24 – Example of 20 rectangular basis functions, either of equal spacing or of customized width.

with q from 1 to $N_{\text{functions}}$, and $\Delta z = 25/N_{\text{functions}}$ km. The altitudes z_q correspond to the middle altitude of the rectangular functions, i.e., $z_q = \frac{\Delta z}{2} + q\Delta z$. Figure 7.24a presents these equispaced rectangular functions with $N_{\text{functions}} = 20$.

2. A set of custom rectangular basis functions having varying widths depending on the altitudes. As an example, Figure 7.24b shows 20 rectangular functions with a smaller width (i.e., finer vertical resolution) close to the ground (equal to 500 m), and a coarser resolution at high altitudes (progressively, width of 1 km then 2.5 km and finally 5 km). Such choice is motivated by the improved reconstruction performance achieved at low altitude thanks to the larger differences between the slope weighting functions close to the ground (Figures 7.9 and 7.10). It is also the region where C_n^2 is the largest and have strong altitude variations (boundary layer), hence requiring an improved vertical resolution.

For example, the reconstructions of the SCIDAR profile in Figure 7.13b, making use of the basis functions shown in Figure 7.24, are depicted in Figure 7.25. NNLS reconstructions are used, as the reconstructed C_n^2 profiles should be positive, which is only possible if the coefficients c are positive. Rectangular functions not visible in the figures have been set to zero by the NNLS algorithm. In practice, these zero values in the C_n^2 profile are set to 10^{-24} for the computation of the error metrics. For the same complexity (i.e., 20 coefficients to estimate), the custom basis provides a C_n^2 profile that is, at least visually, closely related to the input SCIDAR profile. This is particularly the case for the BL and other layers of strong turbulence, and justifies

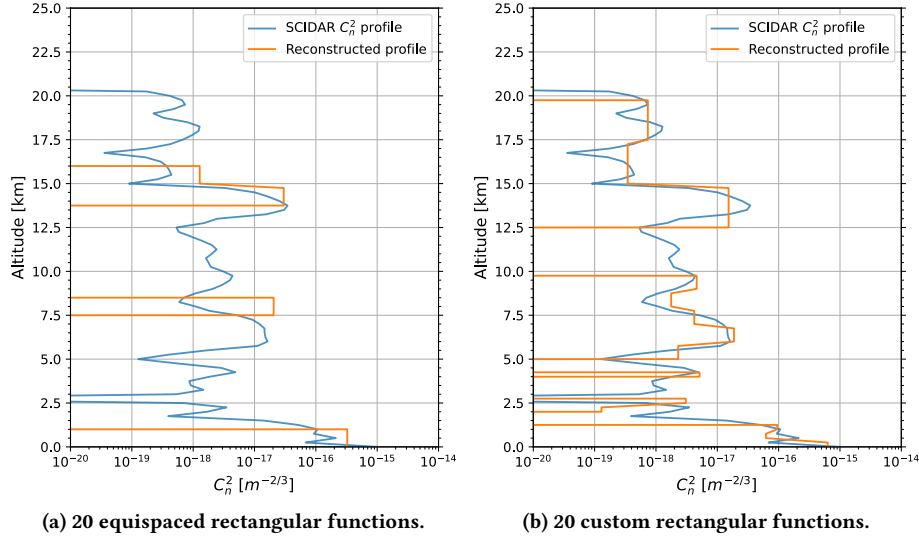


Figure 7.25 – Reconstruction (NNLS) on 20 rectangular functions for the C_n^2 profile given in Figure 7.13b.

Table 7.4 – Integrated parameters and C_n^2 metric for the C_n^2 profile given in Figure 7.13b obtained from its reconstruction on 20 rectangular functions (equispaced or custom distribution, see Figure 7.24).

Integrated parameters and C_n^2 metric	SCIDAR C_n^2 profile	Equispaced functions		Custom functions	
		LS	NNLS	LS	NNLS
r_0 [cm]	15.30	0.02	13.32	14.12	14.10
θ_0 [μ rad]	10.60	0.003	10.56	10.60	10.60
σ_I^2	0.0872	32000	0.0982	0.0900	0.0901
RMSE on $\log_{10}(C_n^2)$	/	/	4.87	/	2.83

the choice of this basis.

In terms of integrated parameters, both profiles in Figure 7.25 provide values in good agreement with the SCIDAR input profile, as given in Table 7.4. As a result, even if the use of equispaced rectangular functions leads to reconstructed profiles with numerous altitudes with a null C_n^2 , they are equivalent when considering the integrated parameters. However, the altitude-averaged RMSE on the logarithms in base 10 of the C_n^2 profiles, between the reconstructed profile and the initial SCIDAR profile, is larger in the equispaced case, as expected from Figure 7.25a.

LS reconstruction has also been considered to determine whether equivalent C_n^2 profiles, that may contain negative coefficients, can still provide relevant integrated parameters. On this particular example, it is the case with the custom rectangular functions, but not with the equispaced functions. This is explained by the observation of large coefficients for the high-altitude rectangular functions, with alternating

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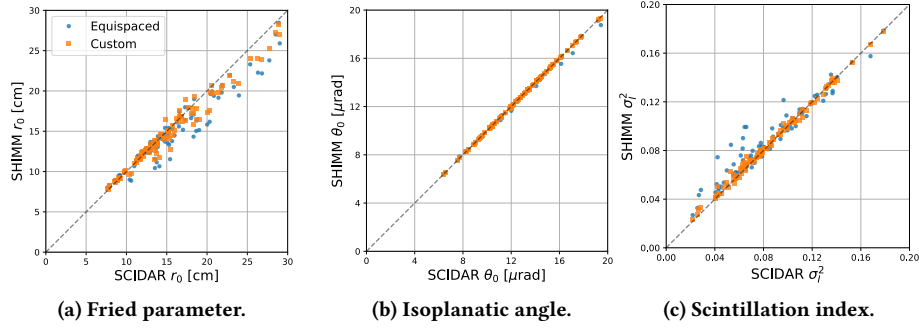


Figure 7.26 – Correlation plots between the integrated parameters computed from the input SCIDAR profiles and the ones obtained from the NNLS reconstructed SHIMM profiles using 20 rectangular windows.

Table 7.5 – Error metrics used to validate the C_n^2 reconstruction (NNLS) with 20 rectangular functions for the dataset SCIDAR profiles.

Integrated parameters	Equispaced functions			Custom functions		
	RMSE	MRE [%]	Corr. coeff.	RMSE	MRE [%]	Corr. coeff.
r_0 [cm]	1.97	7.05	0.972	1.31	4.82	0.988
θ_0 [μrad]	0.16	0.49	0.999	0.04	0.12	0.9999
σ_I^2	0.010	9.86	0.969	0.003	2.95	0.998

positive and negative values. This behaviour can be avoided by limiting the number of functions $N_{\text{functions}}$. Nevertheless, for practical scenarios, positive profiles are required, and only the NNLS reconstruction is further studied in this section.

7.4.2.1 Validation with C_n^2 profiles

The evaluation of the reconstruction in the rectangular bases made of 20 equispaced or custom functions for all SCIDAR profiles in the dataset is presented in Figure 7.26 and Table 7.5.

In Table 7.5, the same metrics as in Section 7.3 are used on the integrated parameters. The performance over the SCIDAR dataset are slightly improved compared to the layer-based inversion (Table 7.3). The isoplanatic angle is the integrated parameter that is best reconstructed, and the scintillation index is particularly well reconstructed with the 20 custom rectangular functions. This can also be observed in Figure 7.26c, where the integrated parameter derived from the 20 custom functions are depicted by orange squares, and blue dots correspond to the reconstruction with the 20 equispaced rectangular functions. The improvement is substantial on the Fried parameter.

The improved reconstruction of the Fried parameter and the scintillation index in the case of custom rectangular functions can be explained by the gain in vertical resolution at low altitudes, which have a strong influence on these parameters.

Nevertheless, it is remarkable to observe that the gain in resolution at low altitude that leads to a loss of resolution at higher altitude (to ensure a fixed number of basis functions) does not degrade the reconstruction of the isoplanatic angle, and even improves it. This further motivates to use the custom rectangular basis instead of the equispaced one.

Finally, the average RMSE on the logarithm in base 10 of C_n^2 , between the SCIDAR profile and the reconstructed profile, is 4.64 with the equispaced basis, and 3.51 with the custom basis. This metric is mostly influenced by the number of zero coefficients (and hence zero C_n^2 values in the profile), which is reduced with the custom rectangular functions, as already observed in Figure 7.25 for a particular example.

7.4.2.2 Determination of the intensity-slope coefficient

As similar study as in Section 7.3.3.1 for layer reconstruction shows that the value of $\alpha = 10^{11}$ is also suited with the reconstruction on rectangular functions. Indeed, as seen in Figure 7.27, there is a range of values between 10^8 and 10^{12} where the intensity-slope coefficient α has little influence on the integrated parameter performance metrics. This study is valid for 20 equispaced rectangular functions.

Moreover, the following observation in Figure 7.27 is of interest: for small α , the isoplanatic angle and the scintillation index are well reconstructed (low RMSE, low MRE, high correlation). However, the Fried parameter is poorly modelled. Increasing α above 10^8 improves the reconstruction of r_0 , while slightly degrading the one of θ_0 and σ_I^2 . For α larger than 10^{12} , θ_0 and σ_I^2 are strongly degraded, while the Fried parameter is unchanged. This is a direct consequence of the fact that the slope weighting functions give more importance to low-altitude layers, while the intensity weighting functions are more influenced by high-altitude turbulence. As a result, giving more importance to the slope weighting functions (i.e., increasing α) improves the reconstruction of the low layer and therefore of r_0 , while increasing the influence of the intensity weighting functions (i.e., reducing α) leads to an improved modelling of high altitudes, and therefore of θ_0 and σ_I^2 . There is thus a trade-off to ensure a good reconstruction on all altitudes.

7.4.2.3 Study of the number of basis functions

The influence of the number of rectangular functions $N_{\text{functions}}$, in the equispaced case, is further studied, to motivate the choice of 20 functions used previously.

The monitoring of the integrated parameter metrics, computed with C_n^2 reconstructed with a varying number of equispaced rectangular functions, is shown in Figure 7.28 (three first plots). For all metrics, increasing the number of functions improves the reconstruction performance. While the performance gains are substantial for low numbers of functions, increasing $N_{\text{functions}}$ above approximately 20 shows marginal gains. The stability of the inversion algorithm is also degraded, as the condition number of the matrix $\hat{\mathbf{A}}$, that is, the matrix $\mathbf{A}$ without duplicated rows, increases.

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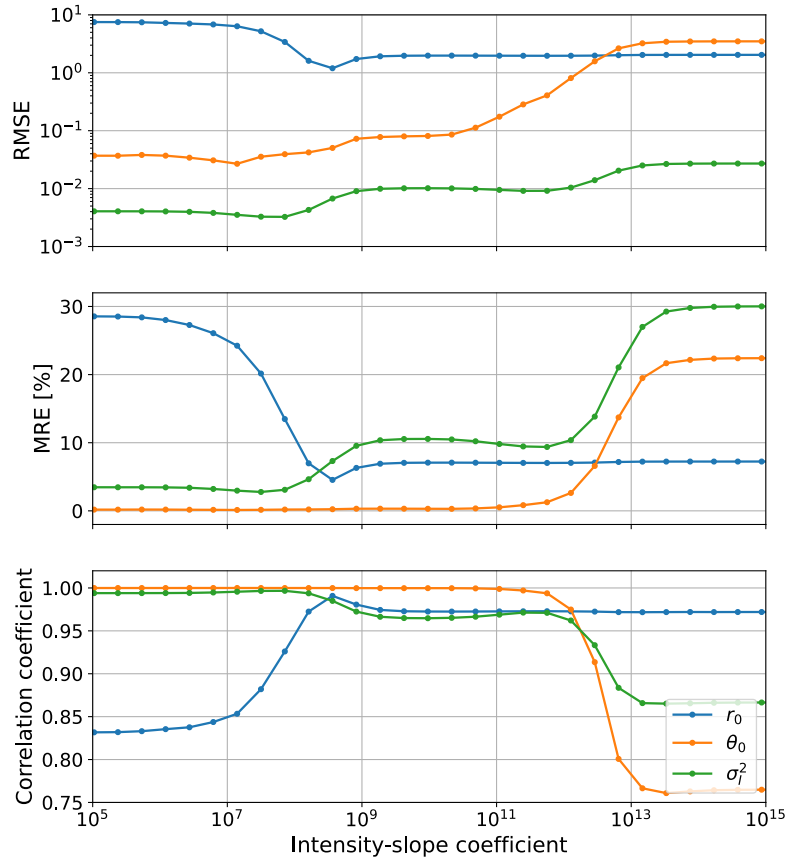


Figure 7.27 – Evolution of metrics on integrated parameters with the intensity-slope coefficient (NNLS reconstruction with 20 equispaced rectangular functions).

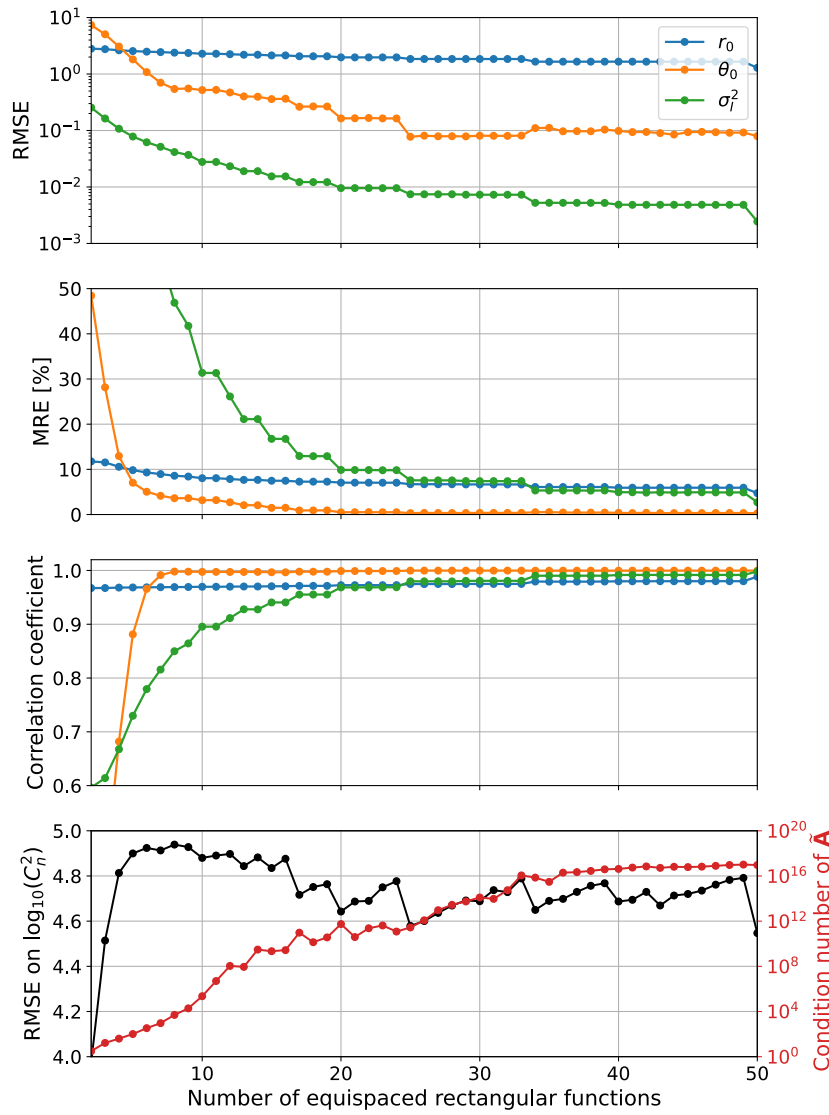


Figure 7.28 – Evolution of metrics on integrated parameters, RMSE on $\log_{10}(C_n^2)$ and $\tilde{\mathbf{A}}$ matrix condition number with the number of equispaced rectangular functions used in the reconstruction.

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In terms of RMSE on the logarithm in base 10 of C_n^2 , it tends to remain relatively constant between 20 and 50 functions. It is associated to the number of zero coefficients in $\mathbf{c}$, being close to 70% on average (standard deviation of 8.78%) for the region between 20 and 50 functions.

Hence, using 20 equispaced rectangular basis functions seems sufficient to achieve good performance. However, as shown in this section, a clever choice of these 20 functions can further improve the performance, for the same computational cost.

7.4.3 Reconstruction with basis functions from principal component analysis

A dictionary made of basis functions extracted from a Principal Component Analysis (PCA) [Jol02] of the SCIDAR dataset is explored in this section. The main drawback of this learning approach for the basis functions is its location dependency, as it is specific to the dataset used. However, it serves as an illustration of the multitude of basis functions (and of dictionaries) that is made possible with the formalism in (7.31).

PCA enables to reduce the dimensions of a dataset by linearly transforming the coordinate systems in which it is expressed, while capturing the largest variance. Considering the dataset of SCIDAR profiles of dimensions $100 \times N_{\text{profiles}}$, denoted $\mathbf{X}$, the application of the PCA matrix $\mathbf{V}$ is such that the transformed dataset $\mathbf{Y} = \mathbf{V}\mathbf{X}$ has a diagonal covariance matrix. The matrix $\mathbf{V}$ can be related to the Eigenvalue Decomposition (EVD) of the covariance matrix of $\mathbf{X}$ after centring, i.e., removing the mean, with its rows associated to the EVD eigenvectors. The EVD eigenvalues are then found, in decreasing order of magnitude, on the diagonal of $\mathbf{Y}$ covariance matrix. As a result, it is possible to select fewer components (i.e., less than 100) in $\mathbf{Y}$, while still considering most of the variations in the initial dataset. The scikit-learn Python package has been used for the PCA computation [Ped+11].

In this work, PCA is used to obtain relevant eigenvectors that can serve as basis functions $\Phi_i(z)$. Two datasets are considered: the dataset of SCIDAR profiles collected during the month of March 2023 where the 100 profiles used for validating the SHIMM reconstruction approach are excluded (in total, 511 profiles remain), or a dataset of 19356 SCIDAR profiles measured between 2016 and 2019 [Osb+18]. Both datasets have been acquired at Paranal, and contain 100 features that are the altitudes at which C_n^2 is measured (every 250 m, from 0 to 25 km altitude). PCA is applied on the logarithm in base 10 of C_n^2 , after removal of the mean log-profile of the dataset. As an example, Figure 7.29a shows the first five eigenvectors of the PCA decomposition of March 2023 dataset, i.e., the first five rows of the matrix $\mathbf{V}$. Their corresponding C_n^2 profiles, chosen for basis functions⁵, are depicted in Figure 7.29b, and are obtained by adding the dataset mean log-profile to the eigenvector (decentring), and then taking the power of 10. The obtained basis functions appear similar,

⁵As the SHIMM instrument responds to the C_n^2 profile and not to its logarithm profile, it is necessary to express the basis functions in the C_n^2 domain.

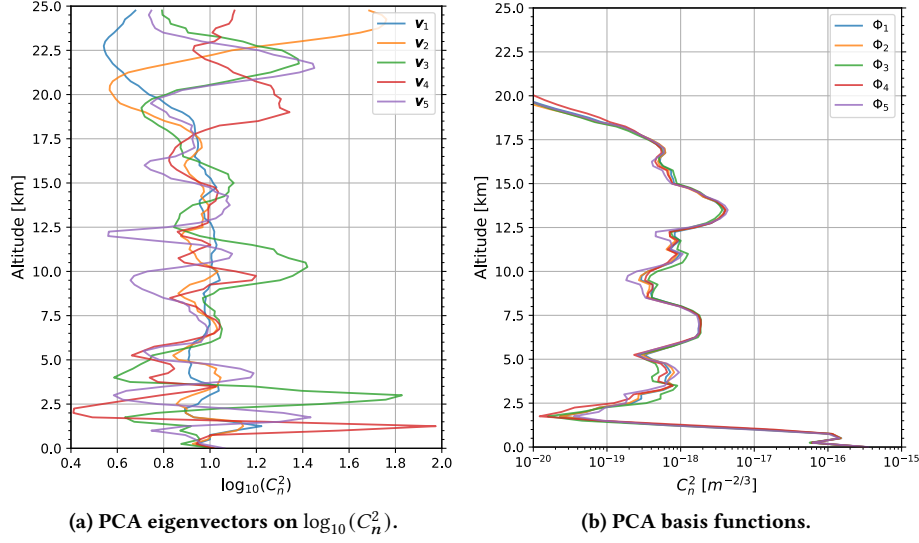


Figure 7.29 – Illustration of the first five eigenvectors and basis functions associated to the PCA of the March 2023 dataset.

due to the major influence of the mean log-profile. Nevertheless, their projections on the SHIMM weighting functions, as expressed in (7.35) to (7.37), are sufficiently different to enable the reconstruction of C_n^2 profiles.

Number of PCA components Since there are 100 features in the initial dataset, up to 100 PCA components are available. However, it is possible to use a smaller number of components to transform measured C_n^2 profiles in the PCA basis, while still ensuring good reconstruction of those profiles (i.e., good inversion of PCA transform) and explaining most of the dataset variation. This is depicted in Figure 7.30 that shows the percentage of explained variance between the initial C_n^2 basis and the PCA basis, depending on the number of selected components. It also presents the altitude-averaged RMSE over all profiles in the March 2023 dataset, after the inverse PCA transform has been applied and a smaller number of components have been kept. The average RMSE for the whole dataset is giving by the red dots, while the dashed lines give the average plus or minus the standard deviation. It shows that 20 components provide an explained variance ratio of 83%, while giving an average RMSE close to 0.75. In the following, 20 components, thus 20 basis functions, are used for the SHIMM reconstruction.

Reconstruction with PCA basis functions The suggested reconstruction with PCA basis functions makes use of the NNLS in (7.29). Since all basis functions are positive, the restriction of the coefficients c to positive values ensure the positivity of the reconstructed C_n^2 profiles.

This algorithm limits the range of possible solutions, and a simple LS reconstruc-

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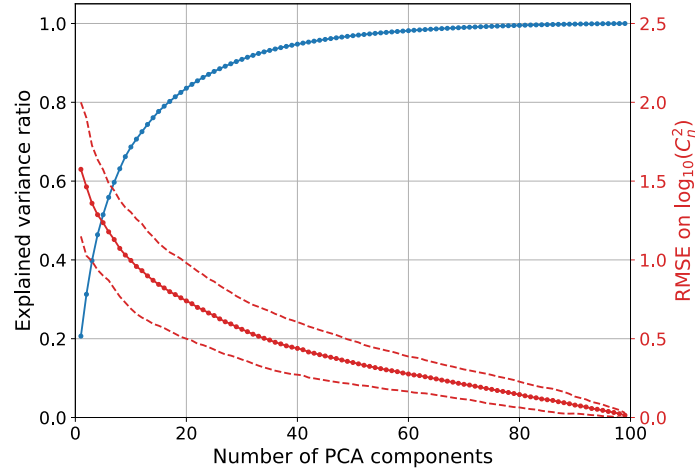


Figure 7.30 – Evolution of explained variance ratio and RMSE on $\log_{10}(C_n^2)$ with the number of PCA components, for March 2023 dataset.

tion is also possible. However, since in this case negative coefficients in c may be obtained, an additional constraint directly on the reconstructed C_n^2 profile computed by (7.31) should be added. This has not been explored in this work.

Example on one SCIDAR profile The reconstruction of the SCIDAR profile given in Figure 7.13b using 20 PCA basis functions is shown in Figure 7.31. The two distinct datasets for computing the basis functions are considered, with their mean C_n^2 profiles depicted in red. Two values have been considered for the intensity-slope factor α (10^9 and 10^{11}).

In Figure 7.31, it can be observed that the reconstruction of C_n^2 with PCA basis functions leads to profiles that appear more realistic, with fewer layers set to zero. However, the agreement with the input SCIDAR profile is far from perfect, and is deeply dependent on the dataset used. The March 2023 dataset, i.e., the dataset collected at the same time as the input SCIDAR C_n^2 profile, offers accurate reconstruction of the BL and high-turbulent layers, as they appear in the mean profile. On the contrary, the 2016-2019 dataset has an average profile relatively flat, which leads to flat reconstructed C_n^2 profiles. As a result, the PCA reconstruction seems very biased by the dataset used.

In terms of integrated parameters, the influence of the intensity-slope factor is important. Indeed, $\alpha = 10^9$ leads to an improved reconstruction of the turbulence at high altitudes, hence an improved reconstructed isoplanatic angle θ_0 . Contrariwise to the rectangular basis function case, this comes at the expense on the accuracy on the lower layers, and therefore of the Fried parameter r_0 . On the contrary, $\alpha = 10^{11}$ leads to a great reconstruction of r_0 , with a worse reconstructed θ_0 . Based on an analysis similar to Section 7.27, no value of α enabling to maximize the reconstruction on low- and high-altitude turbulence has been found with the PCA basis functions.

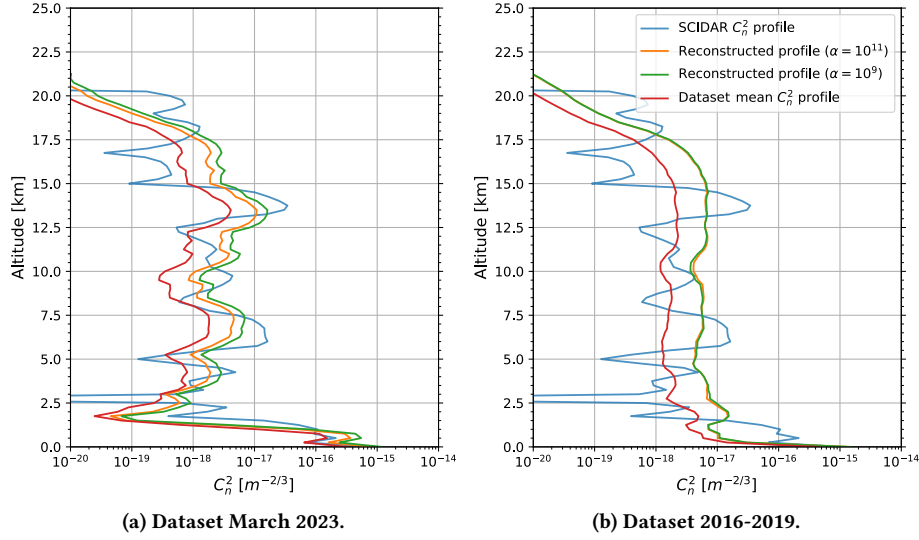


Figure 7.31 – Reconstruction (NNLS) on 20 PCA functions for the C_n^2 profile given in Figure 7.13b.

Table 7.6 – Integrated parameters and C_n^2 metric for the C_n^2 profile given in Figure 7.13b (SCIDAR profile at Paranal, March 1, 2023, at 00h50 UTC) obtained from its NNLS reconstruction using 20 PCA functions (using two different datasets, and $\alpha = 10^9$ or $\alpha = 10^{11}$).

Integrated parameters and C_n^2 metric	SCIDAR C_n^2 profile	Dataset March 2023		Dataset 2016-2019	
		$\alpha = 10^9$	$\alpha = 10^{11}$	$\alpha = 10^9$	$\alpha = 10^{11}$
r_0 [cm]	15.30	11.55	14.72	16.71	16.85
θ_0 [μ rad]	10.60	10.60	13.51	10.60	10.63
σ_I^2	0.0872	0.98	0.065	0.0802	0.80
RMSE on $\log_{10}(C_n^2)$	/	1.63	1.56	1.78	1.77

Regarding the RMSE on $\log_{10}(C_n^2)$, it is not influenced anymore by altitudes with C_n^2 of zero, and thus better represents the agreement between the SCIDAR profile and the reconstructed profiles, looking at all altitudes. Values in Table 7.6 show that the March 2023 dataset indeed leads to improved reconstructions of the profile.

7.4.3.1 Validation with C_n^2 profiles

The validation with all 100 profiles from the SCIDAR dataset is presented in Figure 7.32 (only for the March 2023 dataset) and Table 7.7. It confirms the conclusions previously obtained for a single profile:

- The intensity-slope parameter α has a strong influence on either the reconstruction of the Fried parameter, or the isoplanatic angle. This is visible in the correlation plots (Figure 7.32), as well as from the metrics in Table 7.7, where

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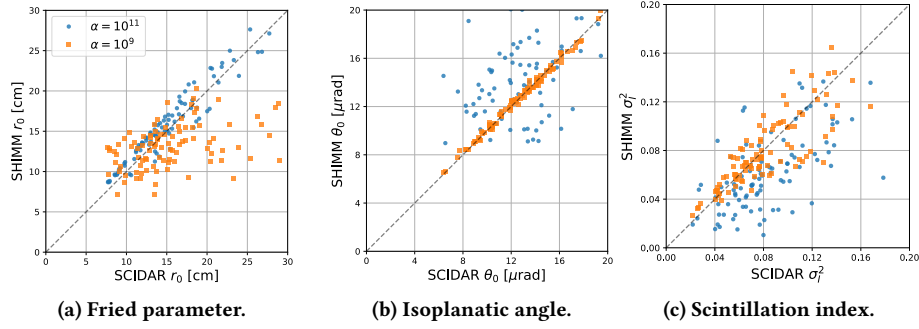


Figure 7.32 – Correlation plots between the integrated parameters computed from the input SCIDAR profiles and the ones obtained from the NNLS reconstructed SHIMM profiles using 20 principal component eigenvector functions from March 2023 dataset.

Table 7.7 – Error metrics used to validate the C_n^2 reconstruction (NNLS) with 20 principal component eigenvector functions for the dataset SCIDAR profiles, for $\alpha = 10^{11}$. Values for $\alpha = 10^9$ are in parentheses.

Integrated parameters	PCA dataset March 2023			PCA dataset 2016-2019		
	RMSE	MRE [%]	Corr. coeff.	RMSE	MRE [%]	Corr. coeff.
r_0 [cm]	1.93 (6.28)	9.80 (23.93)	0.962 (0.487)	3.92 (6.70)	21.79 (40.91)	0.964 (0.457)
θ_0 [μrad]	6.67 (0.27)	38.68 (1.68)	0.240 (0.997)	4.61 (0.23)	25.50 (1.37)	0.283 (0.998)
σ_I^2	0.035 (0.019)	34.02 (15.82)	0.640 (0.852)	0.030 (0.026)	29.55 (20.26)	0.673 (0.833)

the values from the reconstruction with $\alpha = 10^9$ are in parentheses. It is not possible to have accurate reconstruction of r_0 and θ_0 at the same time. The reconstruction performance on the scintillation index are improved with $\alpha = 10^9$ compared to $\alpha = 10^{11}$, as high-altitude layers are better reconstructed.

- The dataset influences the performance, and all error metrics are lower with PCA basis functions from the March 2023 dataset than with functions from the 2016-2019 dataset.

Regarding the evaluation of the performance with the number of PCA components, little variations have been observed. This highlights the major influence of the dataset mean C_n^2 profile in the reconstruction, as well as the one of α .

Even though the reconstruction of integrated parameters with the PCA is not as good as with rectangular functions, the RMSEs on $\log_{10} C_n^2$ are improved. For the March 2023 dataset, the RMSE is 1.81 (resp. 1.81) with $\alpha = 10^{11}$ (resp. $\alpha = 10^9$), while it is 1.93 (resp. 1.91) with $\alpha = 10^{11}$ (resp. $\alpha = 10^9$) for the 2016-2019 dataset.

7.5 Dictionary-based inversion approaches with sparse representations

Depending on the choice of the dictionary of basis functions, as introduced in Section 7.4, the resulting linear system in (7.23) may be undetermined. This is namely the case where the number of basis functions, i.e., the cardinality of Θ , is larger than the number of independent observations (75, as computed above). In such scenarios, it may be possible to obtain a solution to the inverse problem by relying on the framework of *compressive sensing*.

Compressive sensing is a signal processing technique that enables the reconstruction of sparse signals from a small number of measurements (typically, smaller than the number of measurements required by the Shannon-Nyquist sampling theorem). It leverages the sparsity of some signals in a well-chosen basis, i.e., their representation with a few non-zero coefficients in the basis. The sensing operation (i.e., the matrix $\mathbf{A}$), is linear and captures the information in a compressed form (i.e., in the vector $\mathbf{y}$). Non-linear reconstruction algorithms are then required to recover the original signal (i.e., the coefficients $\mathbf{c}$), expressed with sparse coefficients in the a-priori chosen basis (e.g., the basis functions $\Phi_\theta(z)$).

The formulation of the inverse problem using the framework of compressive sensing is presented in Section 7.5.1. Its application to a dictionary made of wavelet basis functions is then presented in Section 7.5.2. Finally, some perspectives for generalization are discussed in Section 7.5.3.

7.5.1 Formulation of the inverse problem

The formulation of the inverse problem is similar to Section 7.4.1, based on (7.31) to (7.37). However, in (7.31), it is expected that the coefficients c_θ are sparse, meaning that only a few of them are non-zero. The inverse problem described by (7.23) is then solved with LS combined with *regularization* approaches.

Regularized least squares reconstruction with L_1 -norm Exploiting the sparsity in $\mathbf{c}$, a possible formulation for the inverse problem is

$$\hat{\mathbf{c}} = \underset{\mathbf{c}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{A}\mathbf{c} - \mathbf{y}\|_2^2, \quad \text{such that} \quad \|\mathbf{c}\|_0 \leq K. \quad (7.38)$$

It involves the L_0 -norm of $\mathbf{c}$, i.e., the number of non-zero coefficients in $\mathbf{c}$, that should be lower than a natural number K . There exist several algorithms to approximately solve (7.38), such as iterative hard thresholding or matching pursuit. However, the knowledge of the number of non-zero coefficients K is not available, and the L_0 -norm makes differential approaches difficult. As a result, *relaxation* is often used and the problem is formulated with the L_1 -norm instead.

The formulation chosen here involves the regularization of the LS solution with

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the L_1 -norm, weighted by a parameter τ , i.e.,

$$\hat{\mathbf{c}} = \underset{\mathbf{c}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{Ac} - \mathbf{y}\|_2^2 + \tau \|\mathbf{c}\|_1. \quad (7.39)$$

For a given value of τ , it is solved using the forward-backward algorithm [BT09] implemented by the PyUNLocBoX Python package [DPP]. Larger τ values ensure sparser solutions, while smaller τ values gives stronger fidelity to the measurements. This is further illustrated in Section 7.5.2.

In (7.39), there is no constraint on the positivity of the solution. Ideally, the constraint should not be set on the computed coefficients $\mathbf{c}$, that may be negative, but rather on the C_n^2 profile itself, that is given by (7.31). As such constraint would complexify the resolution of (7.39), it has not been implemented in this work. Hence, the reconstructed C_n^2 profiles are mostly of mathematical interest, with limited physical interpretation.

7.5.2 Reconstruction with wavelet basis functions

In signal processing, the wavelet transform enables to decompose a signal into its components at different scales, i.e., at various levels of resolution. By applying the wavelet transform, the signal is successively broken down into approximation and detail coefficients, that are associated to its low-frequency and high-frequency components. In the case of Haar wavelets, the approximation coefficients are obtained by applying the scaling function given in Figure 7.33a to the signal at its current resolution, while the detail coefficients come from the application of the wavelet function. Recursively, scaling and wavelet functions are applied at decreasing resolutions (i.e., for increasing intervals over which the functions are defined, that is normalized to one in Figure 7.33a), therefore reducing the number of approximation and detail coefficients obtained.

For example, in the case of a discrete signal having 2^{N_c} coefficients, the first wavelet decomposition (first level) is made of 2^{N_c-1} approximation coefficients and 2^{N_c-1} detail coefficients, obtained by averaging and differencing neighbouring coefficients at the initial resolution taken by pairs (in the case of Haar wavelets). For the next level, the 2^{N_c-1} approximation coefficients are further split into 2^{N_c-2} approximation coefficients and 2^{N_c-2} detail coefficients at a coarser resolution. By recursive application, at the last level (N_c), the signal is separated into a single approximation coefficient and a set of $2^{N_c} - 1$ detail coefficients from the various resolutions. Using all coefficients, the inverse wavelet transform enables perfect reconstruction of the signal.

In data compression or noise reduction applications, it is expected that the signal decomposition in the wavelet basis, i.e., the coefficients from the wavelet transform, will be sparse. This is also of interest for compressive sensing approaches, as explored in this section.

Figure 7.33b presents some of the basis functions associated to the wavelet trans-

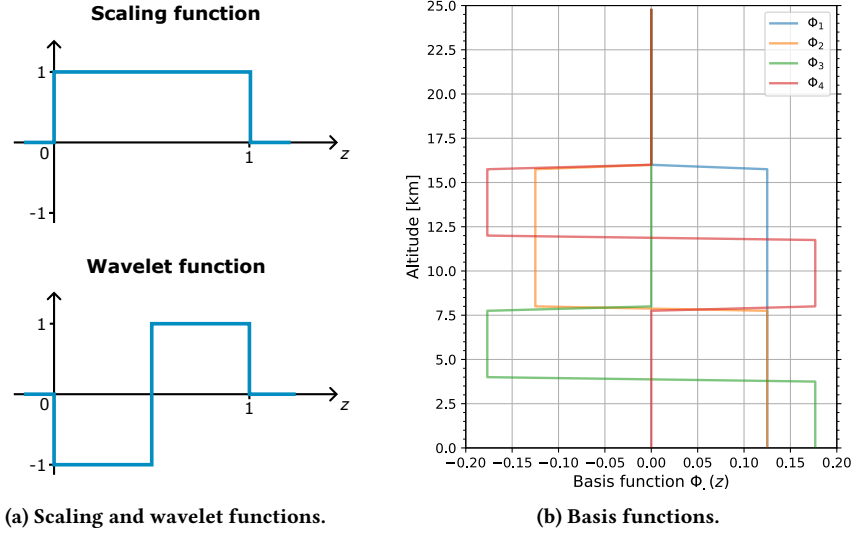


Figure 7.33 – Haar scaling and wavelet functions (left), and examples of associated basis functions (right).

form applied to the SCIDAR dataset C_n^2 profiles. Since these profiles have an initial resolution of 250 m from 0 to 25 km of altitude and therefore contain 100 values, their Haar wavelet transformation leads to 103 coefficients (level 6 wavelet). Indeed, the wavelet transform leads to more than 100 coefficients in this case since the initial number of values in the C_n^2 profile (100) is not a power of two and is separated in the sum of three powers of two, i.e., $64 + 32 + 4 = 100$. As a result, 103 basis functions are defined. In Figure 7.33b, $\Phi_1(z)$ is associated to the approximation coefficient over the 64 first values (altitudes from 0 to 16 km), $\Phi_2(z)$ corresponds to the wavelet function over the same altitude range and at the same resolution, while $\Phi_3(z)$ and $\Phi_4(z)$ are the wavelet functions at one higher level of resolution. Other basis functions are defined similarly, further completing the detail coefficients over the range 0 to 16 km, as well as giving the approximation and detail coefficients for other altitudes. The computation of the wavelet transform is conducted in Python using the PyWavelets package [Lee+19].

Example on one SCIDAR profile The reconstruction of the SCIDAR profile given in Figure 7.13b with the wavelet basis functions is depicted in Figure 7.34, for several regularization coefficients τ . A value of $\alpha = 10^{11}$ is used for the intensity-slope factor. The associated integrated parameters are presented in Table 7.8.

In Figure 7.34, the influence of the regularization parameter τ is striking. For large values (e.g., 10^{13}), very few coefficients are different from zero in c , such that few wavelet basis functions are activated. This leads to a coarse reconstructed C_n^2 profile. On the contrary, for smaller τ (e.g., 10^{11}), most of the coefficients in c are non-zero, and fine details are visible in the reconstructed profile due to the large number of

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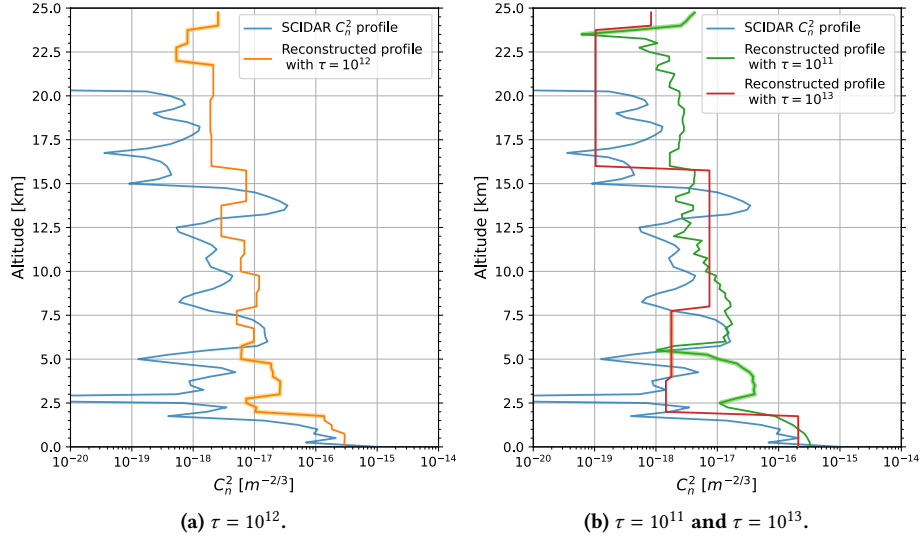


Figure 7.34 – Sparse reconstruction on wavelet functions for the C_n^2 profile given in Figure 7.13b.

Table 7.8 – Integrated parameters and C_n^2 metric for the C_n^2 profile given in Figure 7.13b obtained from its sparse reconstruction with the wavelet functions.

Integrated parameters	SCIDAR C_n^2 profile	Wavelet reconstruction with		
		$\tau = 10^{11}$	$\tau = 10^{12}$	$\tau = 10^{13}$
r_0 [cm]	15.30	13.32	13.24	13.01
θ_0 [μ rad]	10.60	10.39	10.46	10.48
σ_I^2	0.0872	0.097	0.099	0.112

activated wavelet functions. In-between, for example for $\tau = 10^{12}$, important details are visible in the profile, such as strong turbulent layers, while limiting the number of activated basis functions.

Nevertheless, the reconstructed profiles remain different from the input SCIDAR profile. The general trend is usually well depicted (large layer close to the ground and at the tropopause), but unrealistic C_n^2 values may be reconstructed. Indeed, the positivity constraint on C_n^2 has not been implemented in (7.39), such as negative values have been observed in the reconstructed profiles. They correspond to the highlighted regions in Figure 7.34, located between 2.5 and 5 km altitude, and above 20 km. Hence, Figure 7.34 actually shows the absolute values of the reconstructed C_n^2 profile.

In the profiles, it seems that overestimated C_n^2 values are often compensated with neighbouring negative C_n^2 values, thereby ensuring accurate reconstruction of the integrated parameters, as seen in Table 7.8. Even though the profiles have different shapes depending on τ , their integrated parameters are similar and closely related to the input SCIDAR profile.

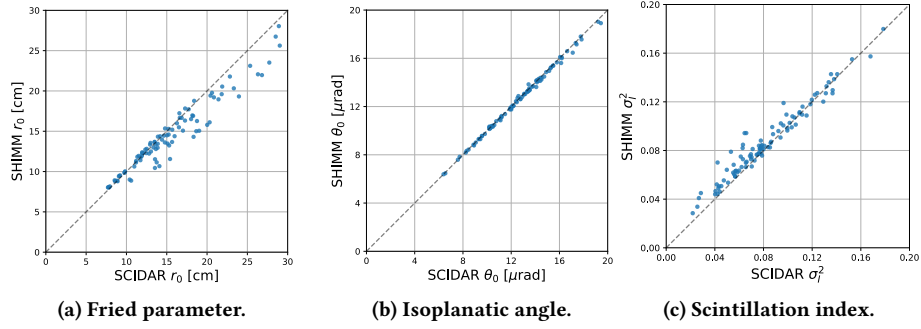


Figure 7.35 – Correlation plots between the integrated parameters computed from the input SCIDAR profiles and the ones obtained from the reconstructed SHIMM profiles with wavelet basis functions ($\tau = 10^{12}$).

Table 7.9 – Error metrics used to validate the C_n^2 reconstruction with wavelet functions for the dataset SCIDAR profiles.

Integrated parameters	Wavelet reconstruction with $\tau = 10^{12}$		
	RMSE	MRE [%]	Corr. coeff.
r_0 [cm]	2.06	7.24	0.973
θ_0 [μrad]	0.179	0.92	0.999
σ_I^2	0.009	10.30	0.977

7.5.2.1 Validation with C_n^2 profiles

The validation for all 100 C_n^2 profiles in the dataset is presented in Figure 7.35 and Table 7.9, for a regularization parameter $\tau = 10^{12}$ and an intensity-slope factor of $\alpha = 10^{11}$.

As seen from the correlation plots in Figure 7.35, the reconstructed isoplanatic angle is in excellent agreement with the isoplanatic angle from the SCIDAR profile. Very good agreements are also reached for the Fried parameter and the scintillation index. The achieved performance, as evaluated with the error metrics in Table 7.9, are comparable to the ones obtained with 20 equispaced rectangular basis functions (Table 7.5). A slight improvement on the Fried parameter and the scintillation index compared to the layer-based reconstruction (Table 7.3) is also observed.

7.5.2.2 Determination of the intensity-slope coefficient

A similar study to the ones conducted in Sections 7.3.3.1 and 7.4.2.2 for the intensity-slope coefficient led to the same conclusions as for the rectangular basis functions (Section 7.4.2.2). Indeed, an optimal range for α between 10^8 and 10^{12} has been identified, where all integrated parameters are accurately reconstructed. For α smaller than 10^8 , only the isoplanatic angle and scintillation index are properly reconstructed, while for α larger than 10^{12} , only the Fried parameter is well reconstructed.

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Hence, the value of $\alpha = 10^{11}$ is also suited with the sparse reconstruction using wavelet basis functions.

7.5.2.3 Study of the regularization parameter

The parameter with the most influence on the reconstructed C_n^2 profiles is the regularization parameter τ , as briefly presented in Figure 7.34 for one particular example. A complete study of its influence on the integrated parameter metrics and the C_n^2 profiles is presented in Figure 7.36.

It can be observed that τ has little impact on the integrated parameters for values between 10^9 and 10^{12} . However, above 10^{12} , the performance degrades for all metrics (larger RMSE and MRE, lower correlation), with the stronger impact on the scintillation index.

In terms of RMSE on the logarithm in base 10 of C_n^2 , there seems to be a minimal value close to $\tau = 10^{12}$. However, this metric only serves as a guideline as it does not consider the regions of the reconstructed C_n^2 profiles where C_n^2 is negative. Moreover, it is undefined for large τ where few coefficients are different from 0 in c and actually leads to zero values in the C_n^2 profiles themselves. This can be observed on the red curve, giving the evolution with τ of the average (over the profiles in the dataset) number of zeros in c (plain line with dots), and the average plus or minus the standard deviation (dashed lines). Clearly, increasing τ leads to sparser c vectors.

Consequently, the choice $\tau = 10^{12}$, as used above, seems to be a good trade-off between the sparsity of the profiles and the accuracy of their reconstructions.

7.5.3 Generalization perspectives

This section merely introduced the foundational concepts of compressive sensing and sparse representation. It showed that the achievable performance are similar to the other approaches, namely in terms of integrated parameters, and that C_n^2 profiles could be reconstructed with high-resolution details where desired. However, much work remains to ensure the positivity of the reconstructed profiles, and thus their physical interpretation.

The compressive sensing approach could be applied to any other dictionary, or even combination of the dictionaries presented here. Indeed, there is no limit to the number of basis functions in the dictionary. However, the representation of C_n^2 profiles with the chosen basis functions must be sparse. This sparsity prior is a strong assumption that is particularly challenging to fulfil due to:

- the shapes of C_n^2 profiles, that include strong and localized turbulent layers;
- the magnitudes of C_n^2 profiles, that can span several orders and motivates the use of logarithmic functions.

As a result, the work presented in this section is mostly preliminary and could be extended in many ways.

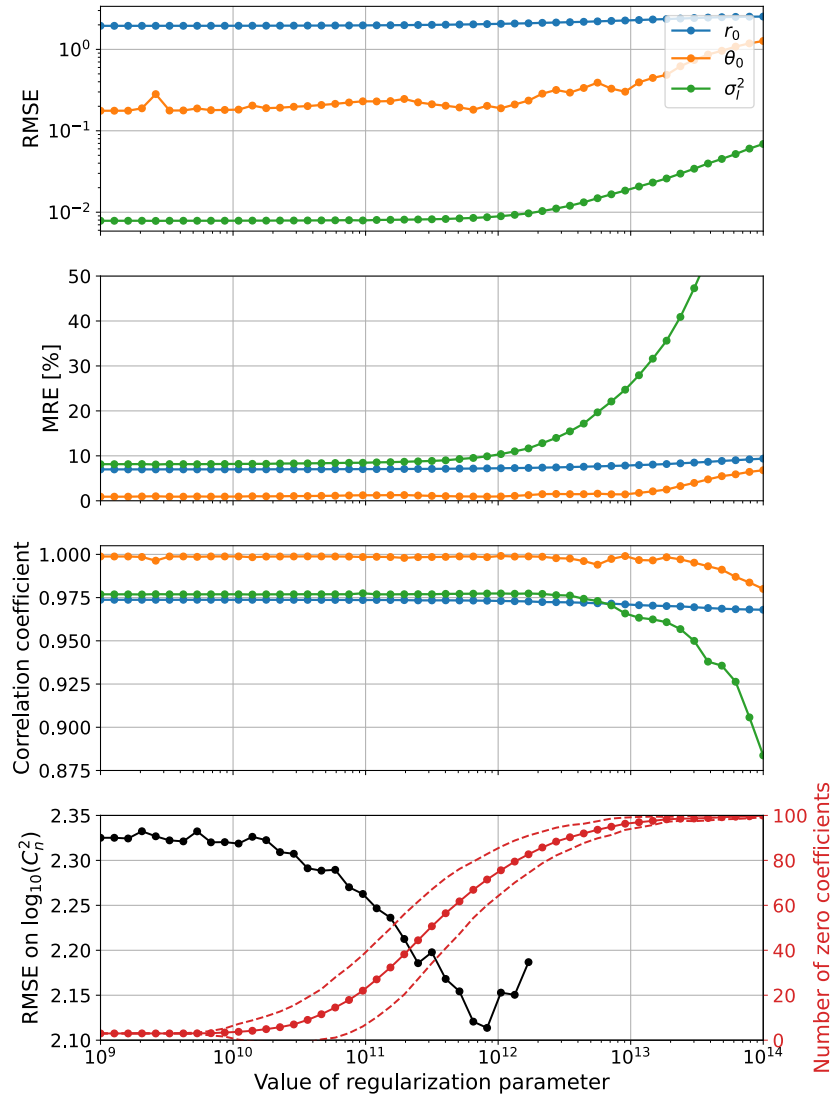


Figure 7.36 – Evolution of metrics on integrated parameters, RMSE on $\log_{10}(C_n^2)$ and number of zero coefficients in c with the regularization parameter τ .

7.6 Noise analysis and application to SHIMM measurements at Paranal

The study conducted in Sections 7.3 to 7.5 is mainly theoretical and does not include non-idealities existing in the SHIMM instrument. In this section, the impact of noise on the reconstruction algorithms is studied (Section 7.6.1), and their application to real measurements collected at Paranal is presented (Section 7.6.2).

7.6.1 Noise analysis

In this work, the considered sources of noise in the wavefront sensor images are dark noise, photon shot noise, and readout noise. They impact the intensity and slope measurements computed in each subaperture of the Shack-Hartmann.

7.6.1.1 Noise sources and distributions

Dark noise originates from the thermal generation of electrons in the camera, even when there is no light on the detector. In order to mitigate its influence, *dark frames*, i.e., images captured with the camera shutter closed, are regularly measured and then subtracted from the actual scientific images. This procedure is implemented in the SHIMM instrument, such that it is assumed that dark noise has little influence on the measurements, and thus does not require further modelling.

Photon shot noise is inherent to the discrete nature of light, that is made of photons. It is related to photon counting on the detector, and follows a Poisson distribution, especially in low-light conditions.

Readout noise arises in the detector during the conversion of electric charges (associated to the number of received photons) to digital signals. It is assumed to follow a Gaussian distribution, and has a dominant contribution at low-light levels [Tho+06].

The influence of photon and readout noises on a SHWFS is presented in [Tho+06], enabling to quantify the phase noise variance depending on the utilized centroid algorithm. It is further studied in [Voy+12] in the case of the CO-SLIDAR instrument, and in [BWS06] for the SLODAR instrument. Since no centroid algorithm has been implemented in this work, a simple Gaussian noise model that assumes to combine the photon shot noise and the readout noise is used in the following. Similarly, the noise on the measured intensity fluctuations is assumed to follow a Gaussian distribution.

7.6.1.2 Theoretical modelling of the influence of noise

Using the notations introduced in (7.1), the intensity fluctuations in the m -th subaperture and in frame p is corrupted by a noise $v_{\delta i, m}^{(p)}$ following a zero-mean Gaussian distribution of variance $\sigma_{v_{\delta i}}^2$, denoted

$$\delta i_m^{(p)} + v_{\delta i, m}^{(p)} \quad \text{with} \quad v_{\delta i, m}^{(p)} \sim \mathcal{N}(0, \sigma_{v_{\delta i}}^2). \quad (7.40)$$

Similarly, noise on the angle-of-arrival fluctuations, after tip/tilt subtraction, $\tilde{s}_{x,m}^{(p)}$ and $\tilde{s}_{y,m}^{(p)}$, is denoted $v_{s_x,m}^{(p)} \sim \mathcal{N}(0, \sigma_{v_{s_x}}^2)$ and $v_{s_y,m}^{(p)} \sim \mathcal{N}(0, \sigma_{v_{s_y}}^2)$. Both for intensity and slope fluctuations, noise realizations in different subapertures and different frames are assumed to be independent. The noise variance is the same for all subapertures.

The computation of the covariance matrices of intensity and angle-of-arrival fluctuations is impacted by the noise, leading to biases on their diagonal (auto-covariance) [BWS06]. Indeed, as an example for the intensity covariance matrix, one has

$$\begin{aligned} (C_I)_{m,n} &= \langle (\delta i_m + v_{\delta i,m})(\delta i_n + v_{\delta i,n}) \rangle, \\ &= \langle \delta i_m \delta i_n \rangle + \langle \delta i_m v_{\delta i,n} \rangle + \langle v_{\delta i,m} \delta i_n \rangle + \langle v_{\delta i,m} v_{\delta i,n} \rangle, \\ &= \langle \delta i_m \delta i_n \rangle + \sigma_{v_{\delta i}}^2 \delta_{mn}, \end{aligned}$$

since the noise is zero-mean and independent for subapertures m and n . The notation δ_{mn} is the Kronecker delta function, i.e., $\delta_{mn} = 1$ if $m = n$ and 0 otherwise.

As a result, for $m = n$, i.e., on the diagonal of the covariance matrix, a non-zero bias is added due to the presence of noise. It is also the case for the angle-of-arrival covariance matrices, such that the covariance matrices are

$$\begin{aligned} C_I &= C_I^{\text{noiseless}} + \Sigma_{v_{\delta i}}, \\ \tilde{C}_{s_x} &= \tilde{C}_{s_x}^{\text{noiseless}} + \Sigma_{v_{s_x}}, \\ \tilde{C}_{s_y} &= \tilde{C}_{s_y}^{\text{noiseless}} + \Sigma_{v_{s_y}}, \end{aligned}$$

with $\Sigma_{v_{\delta i}}$, $\Sigma_{v_{s_x}}$ and $\Sigma_{v_{s_y}}$ diagonal matrices containing the noise variances $\sigma_{v_{\delta i}}^2$, $\sigma_{v_{s_x}}^2$ or $\sigma_{v_{s_y}}^2$ on their diagonals. The noiseless expressions of the covariance matrices are given in (7.2) and (7.16).

Due to the noise bias, overestimation of reconstructed C_n^2 values has been found in [Voy+12]. In cases where the noise variances is known, it is possible to subtract this bias prior to the reconstruction. As an alternative, measurements associated to diagonal terms in the covariance matrices can be neglected in the reconstruction algorithm, thus decreasing the number of available information.

7.6.1.3 Simulations of SHIMM reconstruction including noise

The noise analysis is conducted in simulations over the dataset of 100 SCIDAR C_n^2 profiles, detailed in Section 7.2.2.3. In this evaluation dataset, 2000 frames generated by phase screen propagation are available for each C_n^2 profile. Noise is either added on the intensity fluctuations $\delta i_m^{(p)}$ or on the angle-of-arrival fluctuations $\tilde{s}_{x,m}^{(p)}$ and $\tilde{s}_{y,m}^{(p)}$, with independent realizations for each subaperture m , frame p , and C_n^2 profile. The simple Gaussian noise model given in (7.40) is considered, both for the intensities and the angles-of-arrival. After the addition of noise, the intensity and angle-of-arrival fluctuation covariance matrices are computed, and C_n^2 profiles are reconstructed.

The performance of the reconstruction with noise are monitored by the RMSE,

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MRE and correlation coefficient metrics on the integrated parameters. Four different reconstruction algorithms are considered, referenced by the names:

- **5 layers** - NNLS reconstruction over five layers located at altitudes 0, 4, 12, 16 and 20 km, as presented in Section 7.3.
- **20 rect.** (rectangular windows) - NNLS reconstruction using 20 rectangular windows with the custom spacing given in Figure 7.24b, in Section 7.4.2.
- **20 PCA** - NNLS reconstruction using 20 PCA-based basis functions from the 2016-2019 dataset, detailed in Section 7.4.3
- **Haar wavelets** - sparse reconstruction on Haar wavelet basis functions, following Section 7.5.2.

Furthermore, the reconstructions with the neglect of the diagonal terms in the covariance matrices, i.e., the terms biased by the noise, are also evaluated.

Figure 7.37 presents the evolution of the error metrics on Fried parameter with the variance of the noise on the intensity fluctuations $\sigma_{v_{SI}}^2$. Figure 7.38 shows those same metrics in the case of noise on the angle-of-arrival fluctuations. The metrics for the isoplanatic angle and the scintillation index, not depicted, follow a similar behaviour.

In both figures, the caption specifies the reconstruction algorithm used (plain lines). Dashed lines refer to the application of these algorithms with the neglect of the diagonal terms of the covariance matrices.

Regarding noise on the intensity fluctuations, it appears that the 5 layers and 20 rect. reconstruction approaches have similar behaviour with noise, since their error metrics are superimposed for noise variances larger than 10^{-1} . On the contrary, the 20 PCA reconstruction seems to be more robust to noise, while the wavelet approach is more sensitive. Neglecting the noise biased terms (dashed lines) leads to a slightly reduced noise sensitivity for all reconstruction approaches, except the Haar wavelet reconstruction. This is visible in Figure 7.37 for the RMSE and MRE, with the dashed lines showing a smaller error than the plain lines for a given noise variance. However, the correlation coefficients are more degraded when neglecting the biased terms. Overall, intensity noise variances smaller than 10^{-2} have no impact on the reconstruction performances.

Figure 7.38 shows a progressive degradations of the reconstructed Fried parameters for angle-of-arrival fluctuation noise variances approximately larger than 10^{-13} . In this case, the 5 layer reconstruction seems more robust to noise than the 20 rect. approach. The 20 PCA reconstruction is more robust, while the Haar wavelet approach experiences a stronger sensitivity to noise. The achieved correlation coefficients are identical for the 5 layers, 20 rect., and 20 PCA approaches. For these approaches, neglecting biases terms improves the performances.

Hence, the 20 PCA approach seems to be more robust to intensity and angle-of-arrival fluctuation noises. This can be explained by the limitations of the basis func-

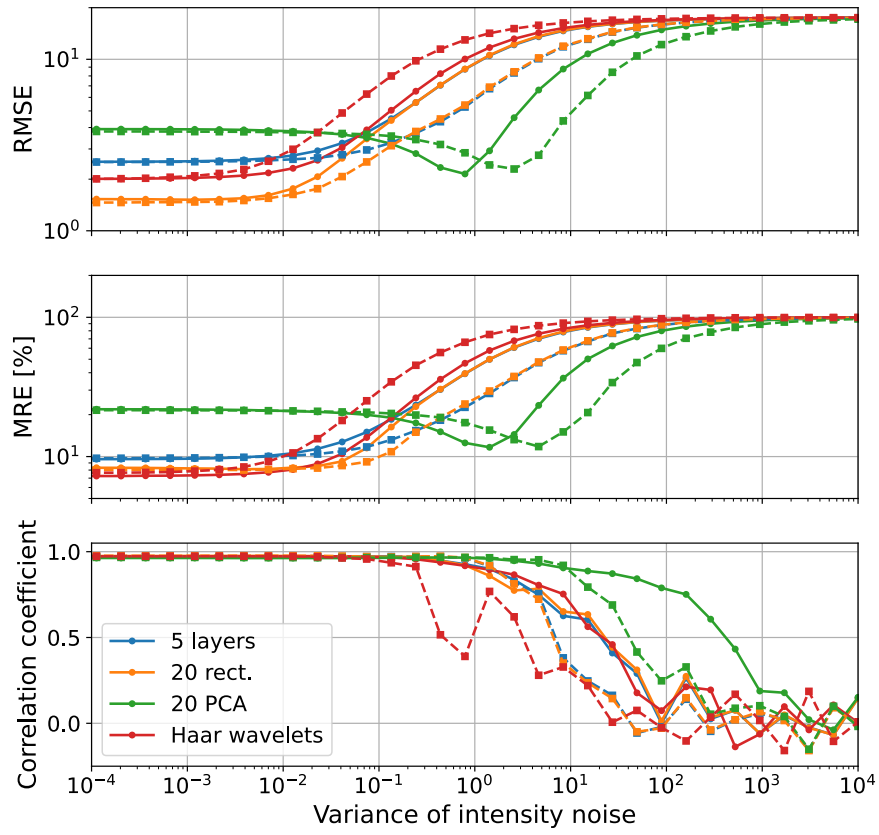


Figure 7.37 – Evolution of error metrics on Fried parameter with the intensity noise variance $\sigma_{\delta I}^2$.

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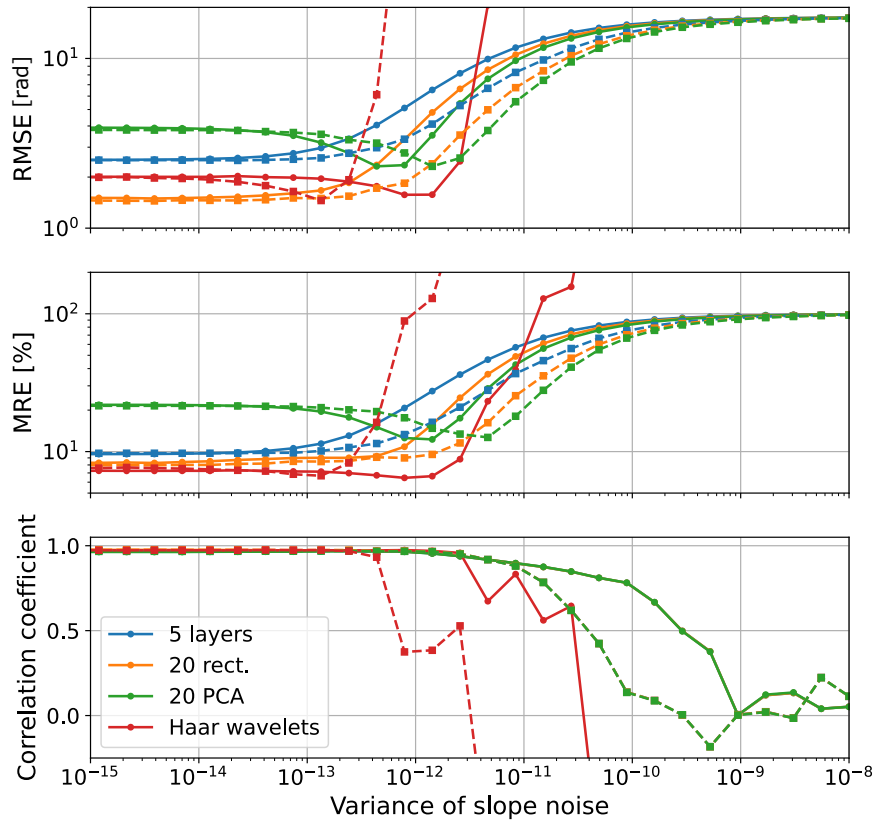


Figure 7.38 – Evolution of error metrics on Fried parameter with the angle-of-arrival (slope) noise variance $\sigma_{v_{sx}}^2 = \sigma_{v_{sy}}^2$.

tions to the PCA eigenvectors, that corresponds to realistic C_n^2 profiles only. Nevertheless, without noise, the PCA approach is too dependent on the learning set and has the worse error metrics. Conclusions regarding the Haar wavelet approach should be mitigated by the fact that the regularization parameter τ has not been optimized in the presence of noise. Moreover, the lack of non-negativity constraint explains the steep behaviour seen in Figure 7.38.

7.6.2 SHIMM measurements at Paranal

Measurements of intensity and angle-of-arrival fluctuations taken at Paranal have been provided by the Centre for Advanced Instrumentation of Durham University [Gri+24a; Qua+23]. They enable to evaluate the reconstruction algorithms presented in this chapter with real data.

In total, 4041 measurements acquired by the SHIMM instrument between February 28 and March 5, 2023, have been exploited. Each measurement consists in 3600 frames acquired with a frame rate of 60 Hz, containing the estimated intensity and angle-of-arrival fluctuations for all subapertures⁶, i.e., all $\delta i_m^{(p)}$, $\tilde{s}_{x,m}^{(p)}$ and $\tilde{s}_{y,m}^{(p)}$. Covariance matrices are then estimated based on the 3600 frames, and used for the reconstruction.

The performance of the four reconstruction approaches considered in Section 7.6.1.3 are evaluated thanks to simultaneous measurements of SCIDAR profiles. However, these measurements are only available during nighttime, such that the comparison is only conducted at night. With respect to the analysis conducted in Sections 7.3 to 7.5, the SCIDAR profiles are not used to generate expected SHIMM measurements since, instead, real measurements are available.

Example of reconstructed profiles Figure 7.39 presents the reconstructed C_n^2 profiles (or layers) at Paranal on March 1, 2023, at 00h50 UTC. The SCIDAR profile corresponds to the one given in Figure 7.13b, already used as an illustration for the reconstructions with simulated data (in Sections 7.3 to 7.5). However, in Figure 7.39, using real SHIMM measurements, the agreement between the SHIMM profiles and the SCIDAR profile is relatively poor. The layer and window-based profiles contain zero C_n^2 values, while the wavelet-based profile is negative for several altitudes (highlighted in Figure 7.39).

The integrated parameters associated to each C_n^2 profile given in Figure 7.39 are presented in Table 7.10. The PCA-based reconstruction provides the integrated parameters the closest to the ones measured by the SCIDAR profiles for this example, while other reconstruction approaches obtain similar values.

Application to all measurements Figure 7.40 gives the integrated parameters computed from the SCIDAR profiles and the SHIMM reconstructed profiles using

⁶The computation of intensity and angle-of-arrival fluctuations in each subapertures has been conducted by Durham University Centre for Advanced Instrumentation, based on the raw wavefront sensor images and a centroid algorithm. It is not part of this study.

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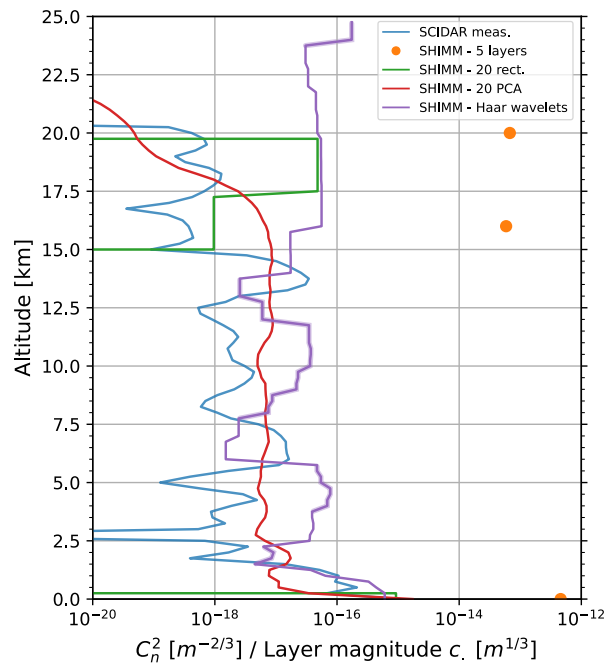


Figure 7.39 – Reconstructions of C_n^2 profiles from SHIMM measurements collected at Paranal on March 1, 2023, at 00h50 UTC, and comparison with simultaneous SCIDAR profile (similar as Figure 7.13b).

Table 7.10 – Integrated parameters for the example C_n^2 profile given in Figure 7.13b with different reconstruction approaches, based on real SHIMM measurements.

Integrated parameters	SCIDAR C_n^2 profile	Reconstruction approach			
		5 layers	20 rect.	20 PCA	Haar wavelets
r_0 [cm]	15.30	11.09	12.66	15.04	11.76
θ_0 [μ rad]	10.60	4.80	4.78	9.13	4.33
σ_I^2	0.0872	0.192	0.201	0.100	0.240

Table 7.11 – Error metrics on integrated parameters using SCIDAR profiles as reference.

Integrated parameters	Metrics	Reconstruction approach			
		5 layers	20 rect.	20 PCA	Haar wavelets
r_0 [cm]	RMSE	4.23	3.94	4.39	4.03
	MRE [%]	23.97	24.42	28.49	23.69
	Corr. coeff.	0.47	0.48	0.53	0.47
θ_0 [μ rad]	RMSE	5.86	5.71	3.55	7.08
	MRE [%]	41.93	40.66	21.64	51.96
	Corr. coeff.	0.37	0.37	0.35	0.33
σ_I^2	RMSE	0.063	0.067	0.031	0.085
	MRE [%]	56.14	62.06	30.87	81.00
	Corr. coeff.	-0.01	0.02	0.40	-0.02

the five layers and 20 rectangular windows approaches. The good agreement between the SCIDAR and SHIMM Fried parameters is noticeable in the upper part of Figure 7.40. However, strong differences are observed for the isoplanatic angle and for the scintillation index to a lesser extent. Such findings agree with [Gri+24a]. For readability purposes, the PCA and wavelet-based reconstructions are not depicted in this figure.

The integrated parameters obtained from the different SHIMM reconstruction approaches tend to have similar values. This is also shown in Figure 7.41, giving the distributions of the integrated parameters. Clearly, parameters coming from the SHIMM five layer, the 20 rectangular windows, and the Haar wavelet reconstructions lead to alike distributions. This is not the case of the PCA-based reconstruction, which integrated parameter distributions tend to be closer to the SCIDAR distributions.

It is worth highlighting that the SCIDAR distributions only includes nighttime measurements, while the SHIMM distributions include both daytime and nighttime observations. This observation explains the two peaks observed in the Fried parameter distribution (left graphs), with the one close to 5 cm associated to daytime and the one close to 10 cm related to nighttime.

Using the SCIDAR profiles as a reference (hence, only nighttime measurements), RMSE, MRE and correlation coefficients can be computed for all integrated parameters. They are presented in Table 7.11. While all approaches tend to have similar error metrics for the Fried parameter, the PCA approach achieves smaller discrepancies with the SCIDAR data for the isoplanatic angles and the scintillation index. This confirms the observations made previously.

Conclusions The conclusions from the analysis of the SHIMM reconstruction approaches with real measurements are twofold:

- It seems that, whatever the reconstruction approach used, the reconstructed

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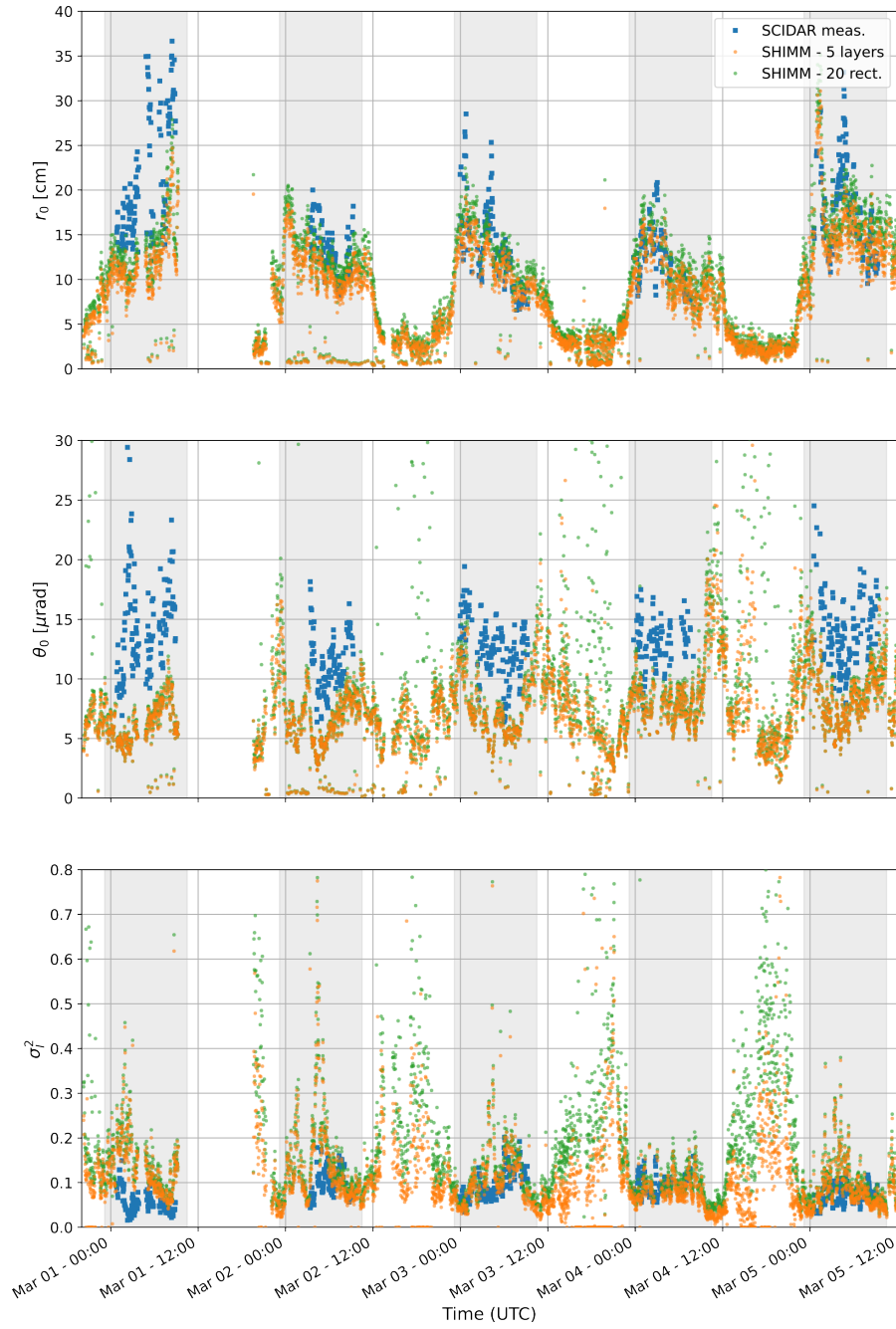


Figure 7.40 – Time series of reconstructed integrated parameters for SHIMM measurements at Paranal.

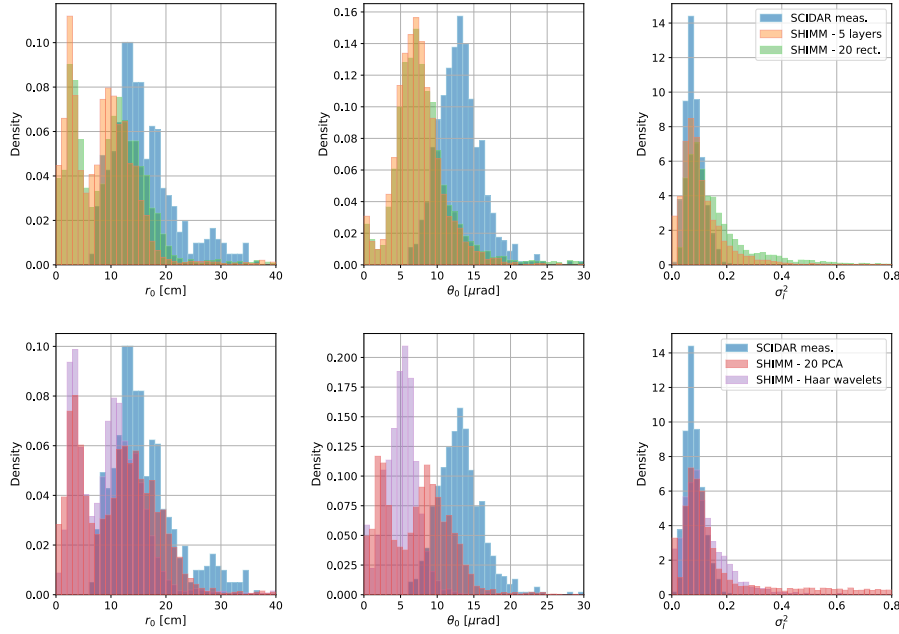


Figure 7.41 – Distributions of reconstructed integrated parameters for SHIMM measurements at Paranal (SCIDAR: nighttime only, SHIMM: daytime and nighttime).

integrated parameters are similar. As a result, any approach can be used if only integrated parameters are required. This is however not true for the reconstruction with PCA, which is biased towards C_n^2 profiles computed from the training dataset.

- The comparison with SCIDAR simultaneous measurements highlights that the reconstruction of Fried parameters from different instruments tend to agree, while this is not the case for isoplanatic angle and scintillation index.

This latter could be explained by differences in location of the instruments, in sensed atmospheric areas, as well as in instrument sensitivities to low and high-altitude turbulence. The SCIDAR profiles were used as a reference (ground truth), but they main contain some biases inherent to the SCIDAR instrument. Interestingly, the SHIMM instrument can provide a similar response as the SCIDAR when using the PCA reconstruction, i.e., when reconstructing over a set of basis functions usually associated to SCIDAR-reconstructed profiles.

7.7 Chapter summary

The SHIMM instrument has been discussed in this chapter. Section 7.1 provided a description of the instrument, while its theoretical and numerical modelling was presented in Section 7.2. Sections 7.3 to 7.5 explored three different approaches for

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C_n^2 profile reconstruction (layer-based, dictionary-based, and dictionary-based with sparse representation), whereas Section 7.6 briefly examined the influence of noise on reconstruction and the application to real measurements.

The main results and contributions of this chapter are summarized below, along with potential directions for future research.

7.7.1 Main results and contributions

The key contribution of this chapter is the introduction of new reconstruction approaches that can estimate C_n^2 profiles with high vertical resolution or as continuous functions of the altitude z . These approaches were compared to the classical layer-based method and demonstrated similar performance in reconstructing integrated parameters. However, differences in noise sensitivity were noted, as discussed in Section 7.6.1, and require further study.

The detailed modelling of the SHIMM instrument is also an important part, that revealed the inherent limitations of the instrument for C_n^2 profile sensing. The presentation of the instrument weighting functions as functions of the altitude, as well as the study of their correlations, is an original approach.

The main finding of this study is that, regardless of the reconstruction method or the shape of the reconstructed C_n^2 profiles, the integrated parameters tend to remain consistent. This suggests that simpler approaches may be sufficient when only integrated parameters are of interest. However, for profile comparison, exploring more advanced reconstruction techniques may provide reconstructed profiles constrained by predefined shapes (e.g., rectangular windows, wavelets, etc.), which could enhance the comparison between profiles sensed by different instruments or derived from NWP predictions.

7.7.2 Future work

The study presented in this chapter is preliminary and opens several research avenues in the areas of instrument modelling, reconstruction algorithms, and instrument design.

Regarding the SHIMM modelling, this chapter focused on an idealized instrument capturing intensity and angle-of-arrival fluctuations under Kolmogorov turbulence conditions. In practice, additional factors must be considered to achieve a more accurate representation of the actual instrument. These include the finite exposure time, the impact of wind on OT, and the influence of the outer scale, among others. Furthermore, a more realistic noise model could be developed, including other error sources, such as those arising from the centroid algorithm used to compute the slopes or from saturation effects.

In terms of reconstruction algorithms, the exploration of dictionary-based approaches has just started. A more in-depth study of potential basis functions could be undertaken, aiming to develop a dictionary where C_n^2 profiles have a sparse representation. However, this raises significant challenges due to the presence of localized

turbulent layers and the broad range of magnitudes spanned by C_n^2 . Additionally, the mathematical techniques employed (e.g., LS or NNLS) are relatively straightforward and could benefit from more advanced, data-driven optimization. For instance, weighting the instrument's observations based on the measured noise could improve reconstruction accuracy, i.e., by weighting the observations with the inverse of the measured noise standard deviations (whitening). This can be achieved by relying on the Bayesian framework presented in [LM08], namely to include knowledge on the priors in the inversion approach, e.g., using a Maximum a Posteriori resolution as in [Voy+14]. Moreover, ensuring physically consistent profiles, by preventing the occurrence of negative C_n^2 values, is also an essential aspect to address.

Finally, the instrument design could be further optimized based on the theoretical and simulation approaches presented in this study. For instance, achieving uncorrelated weighting functions or increasing their number would enhance the instrument sensing capabilities. This could be accomplished by modifying the SHIMM geometry or exploring the use of aperture coding. However, reconstructing the profile will remain challenging, as regions with large C_n^2 values will dominate the instrument sensitivity.

Nevertheless, the necessary measurements of OT parameters and C_n^2 profiles to obtain at FSOC sites remain to be determined and quantified (e.g., required accuracy, necessary vertical resolution on C_n^2 profiles). This is strongly dependent on the application, as illustrated in the next chapter where only OT integrated parameters are exploited for short-term forecast. After specifying the application requirements, rigorous comparisons between different OT monitoring instruments and/or inversion approaches will be possible and shall be explored.

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Application of autoregressive models for short-term forecast

8

For practical operations of satellite-to-ground optical communication systems, there is significant interest in short-term prediction of expected OT conditions on the optical link. Short-term predictions typically refer to predictions with timescales of seconds to minutes, as opposed to long-term forecasts, which span from hours to days. NWP-based models primarily offer long-term predictions, while measurements taken directly at optical sites provide highly accurate information about OT conditions at the time of observation but do not allow for predictions. Hence, the aim of this chapter is to explore approaches merging both long-term prediction models with real-time measurements to enhance short-term OT forecast.

This chapter builds on the work presented in Chapters 5 and 7 and is based on the publication:

- [Qua+24]: F. Quatresooz, R. Griffiths, J. Osborn, D. Vanhoenacker-Janvier, and C. Oestges. “Application of autoregressive models for optical turbulence prediction at optical communication sites”. In: *International Conference on Space Optics—ICSO 2024*. 2024.

It includes the following contributions:

- An overview of short-term OT prediction models for astronomical and optical communication applications, including ML-based approaches;
- The development of AR models that merge recent measurements with NWP forecasts, along with a study of the key parameters that offer insights into the design of short-term prediction models and the necessary measurements;
- A validation of the presented models in two distinct locations (Paranal and Barcelona), using OT integrated quantities time series collected by the SHIMM instrument.

In contrast to models developed for astronomy, this work also considers daytime operations and targets optical communication sites.

Section 8.1 presents the state of the art in OT short-term forecasting approaches, while Section 8.2 details the AR models used and the available measurements. The design of AR models is studied in Section 8.3 using SHIMM measurements from Paranal,

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and the model performance is evaluated and discussed in Section 8.4. Finally, Section 8.5 summarizes the key findings and model limitations, providing perspectives for future research.

8.1 State of the art of optical turbulence short-term forecast

The literature relevant to short-term OT forecast is explored in this section. It complements the state of the art conducted in Section 5.1.3 of Chapter 5, mostly associated to long-term OT predictions using NWP models.

The short-term forecast of OT usually relies on empirical approaches exploiting measurements of desired physical quantities (e.g., Fried parameter, coherence time, etc.) that have been recently acquired at a site of interest. From the time series of these measurements, future values of the physical quantities are predicted, using various of the models presented in Section 8.1.1. The addition of extra input parameters to the time series, such as weather quantities (e.g., surface temperature, relative humidity, etc.), is also possible and is detailed in Section 8.1.2.

8.1.1 Time series forecasting of optical turbulence

The models given in this section rely exclusively on the time series of a given quantity to predict its future evolution. They are classified in three categories, according to their complexity and relevance for this work.

8.1.1.1 Autoregressive models

Autoregressive models linearly describe the temporal evolution of a process, assuming that the future values exclusively depend on a linear combination of the previous values and on a stochastic term (e.g., noise or model imperfection). Different types of AR models are available in the literature, and are further detailed in Section 8.2.

It appears that the use of autoregressive models for seeing forecast was first referenced in 1994, during the construction of the VLT at Paranal [AMS94]. The primary goal was to predict temperature and seeing conditions, facilitating the flexible scheduling of astronomical observations. However, non-linear models were preferred in [AMS94] to achieve this goal.

The recent application of AR models for short-term seeing forecast is presented in [Kor16], for OT measurements above Mount Shatdzhatmaz. It shows improvements in the PDFs of the forecast quantities, using simple ARIMA models (see Section 8.2.1). In [Ni+22], ARIMA models are also used for seeing prediction at the Large sky Area Multi-Object fiber Spectroscopic Telescope (LAMOST), serving as a reference for the comparison with more complex ML models.

Since OT time series contain non-linear and non-stationary components, [Li+22] recommends to first conduct a mode decomposition of the series in stationary com-

ponents. Then, forecast can be conducted for each component, their summation giving the final forecast for the OT quantity. This approach is applied in [Li+22] using a variational mode decomposition (VMD) prior to the application of the AR models. It shows improvements in comparison with a seasonal ARIMA model or NWP predictions. As an alternative, Empirical Mode Decomposition (EMD) is presented in [Li+23], and successfully applied to AR models in [Klo23].

8.1.1.2 Autoregressive models combined with long-term NWP

Instead of applying AR models directly on the OT quantity time series, they can be used in combination with long-term forecast to remove possible biases. In this case, the AR models are applied on the modelling error, i.e., the difference between the long-term forecast and the past observations.

This hybrid approach is followed in [TMM18; MMT20; MTF23], where the long-term forecasts originate from the Astro-Meso-NH model (Section 5.1.3.3). Its application for weather parameters and seeing short-term prediction at the LBT is presented in [MMT20]. It shows performance gains from 2.7 to 4.9 on the RMSE between the long-term/short-term forecasts and the observations. Generalization of the approach to other OT quantity at the VLT can be found in [MTF23]. The addition of a Kalman filter on the predicted short-term series is presented in [TMM18].

8.1.1.3 Other machine learning-based approaches

As an alternative to linear AR models, non-linear ML-based approaches have been explored in AR literature. Similarly to the AR models presented above, all ML models presented in this section only uses the time series of the desired quantity as an input. This is for example the case in [AMS94], where a dynamic recurrent neural network enables to predict nightly-averaged seeing values at Paranal. The use of deep neural networks is also further explored in [Klo23], for Fried parameter predictions at Tenerife.

In [Li+23], a Long Short-Term Memory (LSTM) neural network is applied to Fried parameter predictions in the North of Beijing, China. It demonstrates significant improvements compared to WRF forecasts at this location, reducing the RMSE by a factor of 4.7. The model combines EMD and multi-step time series prior to the application of the LSTM network.

Regarding the forecast of C_n^2 profiles, [Hou+23a] uses a deep convolutional neural network to predict the temporal evolution of turbulent layers between 800 m and 3.5 km altitudes. The C_n^2 profile measurements are acquired by a BL wind profile radar located in Sichuan (China), and multi-step predictions over several hours are presented.

8.1.2 Optical turbulence forecast models with external inputs

OT forecasts can benefit from incorporating external inputs in the model, i.e., measurements of other parameters found relevant to predict the evolution of OT. In this

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approach, not only are the time series of the quantity to be forecast used, but also other parameters like weather data (e.g., surface temperature, wind speed, relative humidity, etc.). Two distinct methodologies are identified:

1. “One-step approach”: the OT parameter time series and the external inputs are all fed directly into the forecast model to obtain the OT parameter short-term temporal evolution.
2. “Two-step approach”: first, the forecast of weather parameters is conducted. Then, OT is inferred from the predicted weather data using regression models.

8.1.2.1 “One-step”: Approaches combining external inputs and OT measurements

In [Mil+19], several ML approaches are explored to predict OT parameters at Paranal Observatory for the next two hours. These models leverage DIMM and MASS-DIMM measurements, along with meteorological parameters collected one to two hours before the prediction. The outputs include predictions for seeing, coherence time, and the ground layer fraction for the next one to two hours. The three ML models studied (random forest, multilayer perceptron, and LSTM) show modest performance improvements compared to the persistence scenario, primarily due to the high-frequency behaviour of turbulence. This work is further extended in [TMF22], showing the sensitivity of the predictions with the learning dataset and the temporal horizon.

Furthermore, multiple data-driven approaches applied to weather data and seeing measurements at LAMOST can be found in [Ni+22].

8.1.2.2 “Two-step”: Forecast of weather parameters and regression models

In a “two-step” approach, weather parameters are forecast before being translated into OT parameters. For example, [Hou+23b] uses LSTM for the predictions of meteorological parameters at Dome A, Antarctica. Then, a Gaussian process regression (GPR) model is trained to derive the seeing from the weather parameters. Overall, the presented approach reduces the RMSE by 37% in comparison with a simple persistence model.

A similar framework is applied in [Ni+24], where the regression is based on Light-GBM (Light Gradient Boosting Machine). The input features consist in measured meteorological parameters, along with their spatial and temporal differences. The detailed model achieves the lowest mean square error compared to various ML approaches, while maintaining low training and forecasting computational times.

In [CLB22], global forecasting system (GFS) forecasts of meteorological parameters for the five following nights are used as inputs to classification and regressor algorithms. These enable the prediction of average OT parameters for each of the five nights, as well as classify of future OT conditions.

8.2 Autoregressive models and motivations

This section details and motivates the autoregressive models used in this work. They are applied to time series of integrated parameters collected by the SHIMM instrument, presented in Section 8.2.3.

8.2.1 Autoregressive models

A generic scalar quantity y is considered, that could correspond to any of the OT integrated parameter. The quantity is measured at instants t_k , such that a discrete series of measurements $y_k = y(t_k)$ is defined, with $k \in \mathbb{N}$ the temporal index of the series. The goal of autoregressive models is to describe the evolution of y based on its previous values.

AR models In its simplest form, an autoregressive model describes a random process by a linear regression of the random process variable against itself, hence the name “autoregression”. Considering the discrete time series of a variable y , it is said to follow an autoregressive model of order P , denoted $\text{AR}(P)$, if [Box+94, p. 52]

$$y_k = \sum_{i=1}^P \alpha_i y_{k-i} + \nu_k. \quad (8.1)$$

In (8.1), α_i for $i \in \{1, \dots, P\}$ are the AR coefficients, while ν_k is a realisation of a zero-mean Gaussian noise.

ARMA models A possible extension of the simple AR model form given in (8.1) is the Autoregressive Moving Average (ARMA) model, in which the noise contributions $\nu_{0:k-1}$, i.e., the noise realisations $\nu_0, \nu_1, \dots, \nu_{k-1}$, also linearly appear in the process. These contributions are affected by coefficients β_j with $j \in \{1, \dots, Q\}$, leading to the form $\text{ARMA}(P, Q)$ [Box+94, p. 53]

$$y_k = \sum_{i=1}^P \alpha_i y_{k-i} + \sum_{j=1}^Q \beta_j \nu_{k-j}.$$

ARIMA and SARIMA models AR and ARMA models are stationary random processes, following the stationarity conditions discussed in [Box+94, p. 54, 77]. In practical cases where observed time series are non-stationary, it is possible that the differenced series, i.e., $y_k - y_{k-1}$, may be stationary. Such series are described by Autoregressive Integrated Moving Average (ARIMA) models, and correspond for example to series where a non-stationary mean function (i.e., the trend) can be removed [Box+94, p. 89]. Seasonality may also appear in real-world observed time series, and could be handled by seasonal autoregressive integrated moving average (SARIMA) models.

8.2.2 Predictions using AR models

This work is restricted to the simplest form of AR model, i.e., AR(P). Using such models, one-step and multi-step predictions of an observed time series are possible, provided that the AR model coefficients α_i are known. These coefficients are usually estimated (i.e., fitted) based on the observed time series $y_{0:k-1} = \{y_0, y_1, \dots, y_{k-1}\}$, using LS regressions with a chosen order P . In this work, the estimation of the AR model based on observations is conducted with the statsmodels Python package [SP10].

One-step ahead predictions Given an observed time series $y_{0:k-1}$, the linear minimum mean square error (MMSE) estimator for the one-step ahead prediction $\hat{y}_k$ is given by [Box+94, p. 131]

$$\begin{aligned}\hat{y}_k &= \mathbb{E}[y_k | y_{0:k-1}] = \mathbb{E}\left[\sum_{i=1}^P \alpha_i y_{k-i} + v_k \mid y_{0:k-1}\right] = \sum_{i=1}^P \alpha_i y_{k-i} \\ &= \alpha_1 y_{k-1} + \alpha_2 y_{k-2} + \dots + \alpha_P y_{k-P},\end{aligned}\quad (8.2)$$

since the noise is zero-mean and the $y_{0:k-1}$ are assumed to be known. Hence, the forecast value for y_k is a linear combination of the past observations $y_{0:k-1}$, weighted by the AR model coefficients. In the determination of the AR model coefficients and the application of (8.2), the addition of a constant coefficient α_0 can be considered.

Multi-step predictions Multi-step predictions are made recursively, according to

$$\begin{aligned}\hat{y}_{k+n} &= \mathbb{E}[y_{k+n} | y_{0:k-1}] = \mathbb{E}\left[\sum_{i=1}^P \alpha_i y_{k+n-i} + v_{k+n} \mid y_{0:k-1}\right] \\ &= \sum_{i=1}^n \alpha_i \hat{y}_{k+n-i} + \sum_{i=1}^{P-n} \alpha_{i+n} y_{k-i} \\ &= \alpha_1 \hat{y}_{k+n-1} + \dots + \alpha_n \hat{y}_k + \alpha_{n+1} y_{k-1} + \dots + \alpha_P y_{k-P},\end{aligned}\quad (8.3)$$

for a $(n+1)$ -step prediction for example, knowing $y_{0:k-1}$. It shows that the available observations in $y_{0:k-1}$ directly appear, weighted by corresponding AR coefficients, while the other variables $y_{k:k+n-1}$ are substituted by their estimations since they have not been observed yet.

For large values of n and for coefficients α_i lower than 1, the prediction $\hat{y}_{k+n}$ goes towards zero as large powers and/or products of the α_i coefficients will appear in (8.3). In the case of the addition of a constant coefficient α_0 in the AR model, the prediction tends to this constant.

8.2.3 Available measurements and motivations

This chapter exploits measured time series of OT integrated parameters by the SHIMM instrument, at Paranal and in Barcelona.

Paranal measurements The SHIMM measurements obtained at Paranal are presented in Section 5.4.2 of Chapter 5 and in Section C.1 of Appendix C. They correspond to the data of the six-day campaign conducted in February and March 2023 at Paranal [Gri+23; Qua+23]. The time series of the measured Fried parameter, isoplanatic angle, and scintillation index are considered, depicted in Figure C.1 of Appendix C. The associated time series computed using the TKE and Tatarskii+BL models of Section 5.3 are also depicted.

Barcelona measurements Measurements in Barcelona (41.38922°N, 2.11205°E, altitude of 115 m) have been acquired during the TURBulence mOnitoring (TURBO) campaigns, held in November 2023 and May 2024 [Gri+24]. The time series are given in Figures C.2 and C.3, respectively corresponding to two and three days of almost continuous data. The associated predictions from WRF simulations with the C_n^2 models of Section 5.3 are also shown in the figures. For the second campaign (May 2024), surface weather parameters have been acquired by a weather station next to the instrument (Figure C.4).

AR model and motivations In order to design AR models based on this data, the measured time series have been binned using a five-minute period. The stationarity of the series has been verified using the Augmented Dickey Fuller and Kwiatkowski-Phillips-Schmidt-Shin tests, implemented in the statsmodel Python package [SP10].

The motivations for using AR models to improve short-term forecast come from the observed autocorrelations of the OT integrated parameter time series. They are depicted in Figure 8.1 for the Fried parameter, the isoplanatic angle, and the scintillation index measured at Paranal. They exploit the last three days and four nights of continuous measurements (from 02/03/2023 at 00:00 to 05/03/2023 at 12:30 UTC).

Regarding the measurements, a large autocorrelation is obtained for time lags under one to two hours for all three integrated parameters (normalized autocorrelation larger than or close to 0.7). It implies that the value of the parameter measured at one instant is strongly correlated with the previously measured values. This is of great interest for AR models, as they indeed make use of past observations to predict future values. It is also beneficial for persistence models, and a similar behaviour is observed on the autocorrelations from the NWP simulation time series.

Furthermore, a diurnal effect is noticeable, as peaks in the autocorrelation appear every multiple of 24 hours. It is associated to the daytime variations, and is also seen in the NWP time series. However, for the autocorrelation of Fried parameters given by the Tatarskii+BL model, strong peaks arise every 12 hours instead. They are related to the peaks in r_0 seen in the time series (Figure 5.22 or equivalently Figure C.1a), coming from the BL thermal flux that goes to zero at sunrise and sunset, drastically decreasing the modelled OT in this layer (Section 5.4.2).

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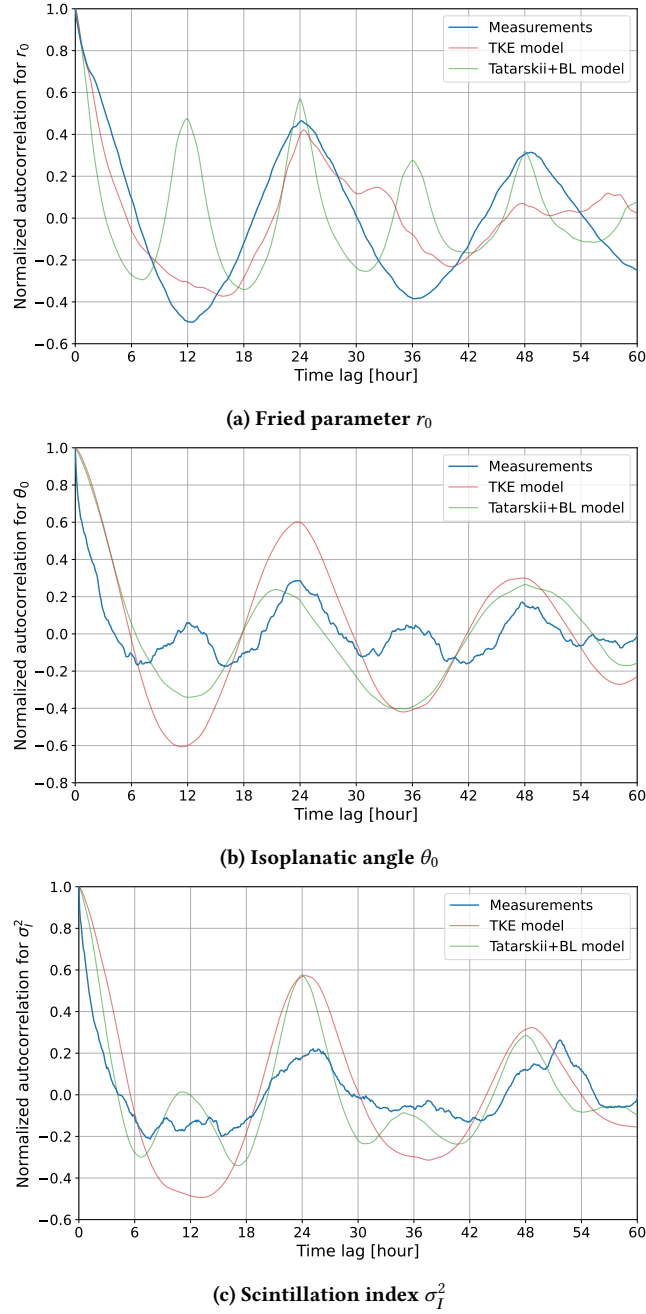


Figure 8.1 – Autocorrelation of time series for the measured integrated parameters by the SHIMM instrument at Paranal.

8.2.4 Autoregressive models chosen for this work

Following the analysis presented in Sections 8.2.1 and 8.2.2, three types of short-term forecast models are explored in this work. They are expressed below, for one-step ahead predictions:

1. **Persistence model:** in this case, the prediction of the next value in the sequence is equal to its most recent measurement, i.e.,

$$\hat{y}_k = y_{k-1}.$$

It is a very simple model, used as a benchmark for performance evaluation of the other models.

2. **Autoregression on measurements:** the predicted $\hat{y}_k$ comes from a linear combination of previous measurements, weighted by coefficients α_i . The time horizon of this combination is set to P , i.e.,

$$\hat{y}_k = \sum_{i=1}^P \alpha_i y_{k-i}. \quad (8.4)$$

The coefficients α_i for $i \in \{1, \dots, P\}$ must be determined and “learned” based on a set of measurements. Equation (8.4) behaves as a sliding window of width P on the time series, in which measured values of y are weighted and summed to give $\hat{y}_k$.

3. **Autoregression on the modelling error:** in cases where a forecast sequence $y^{(\text{mod.})}$ from another model is available, such as from NWP models, the autoregression can be performed on the modelling error x , defined as $x_k = y_k - y_k^{(\text{mod.})}$. It corresponds to the hybrid approaches presented in [MMT20; MTF23]. In this case, the predicted value $\hat{y}_k$ from the autoregression is given by

$$\hat{y}_k = y_k^{(\text{mod.})} + x_k \quad \text{with} \quad x_k = \sum_{i=1}^P \alpha_i \underbrace{(y_{k-i} - y_{k-i}^{(\text{mod.})})}_{=x_{k-i}}. \quad (8.5)$$

The coefficients α_i for $i \in \{1, \dots, P\}$ must be determined.

Any of the models presented above can be used for improving the short-term prediction of the Fried parameter, isoplanatic angle, and scintillation index time series. The modelled values $y^{(\text{mod.})}$ can either come from the TKE or the Tatarskii+BL models. A constant term α_0 could also be included in the AR models presented above.

As an example, Figure 8.2 illustrates the application of the three models for short-term forecasting of the Fried parameter at Paranal. Both the autoregressive models on the measurements (AR meas.) and on the modelling errors (AR TKE model) include ten coefficients ($P = 10$), fitted based on the measurements in the four-hour window (between 8:00 and 12:00 UTC on March 4). The predictions are made for the next

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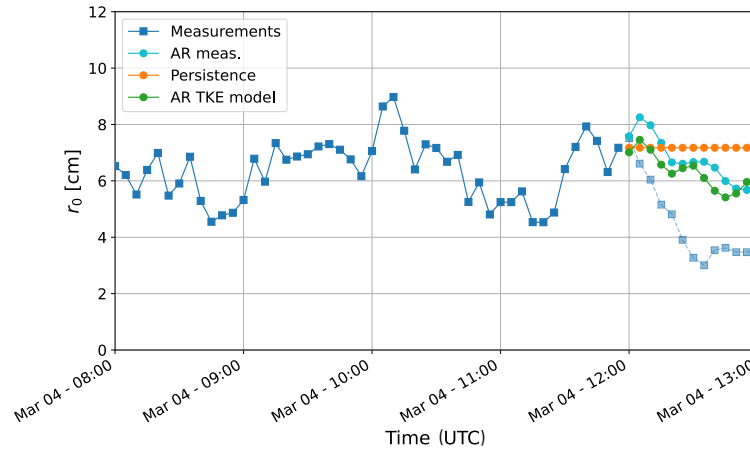


Figure 8.2 – Illustration of AR models applied to measurements at Paranal, predictions over a time horizon of one hour. [Qua+24]

following hour, relying on multi-step predictions. Only the TKE model is considered for obtaining $y^{(\text{mod.})}$.

As expected, the persistence model provides a constant estimation of the Fried parameter based on the last acquired measurements. On the contrary, the autoregressive models on measurements and on the modelling error both exploit past measurements to predict future values. They predict a slight decrease of the Fried parameter. For comparison, the measurements collected during the one-hour prediction window are shown (light blue squares with dashed line). It highlights that both AR models are closer to the measurements than the persistence model in this studied case.

The use of a training window (here, four hours) and a prediction window (here, one hour) is further presented in the next section, in order to study the parametrization of the different models and quantify their performance.

8.3 Design of autoregressive models

The application to practical scenarios of the AR models given in (8.4) and (8.5) requires the determination of:

- the number of coefficients P in the autoregression;
- the training window duration, used to fit the AR coefficients;
- the prediction window durations, i.e., the time horizon for the future forecast.

These three parameters are studied based on the data collected at Paranal (last three days and four nights of continuous measurements), as given in Figure 8.3 for the Fried parameter. The definitions of the training and prediction windows are shown. These windows are progressively slid with a time step of 5 minutes, and performance

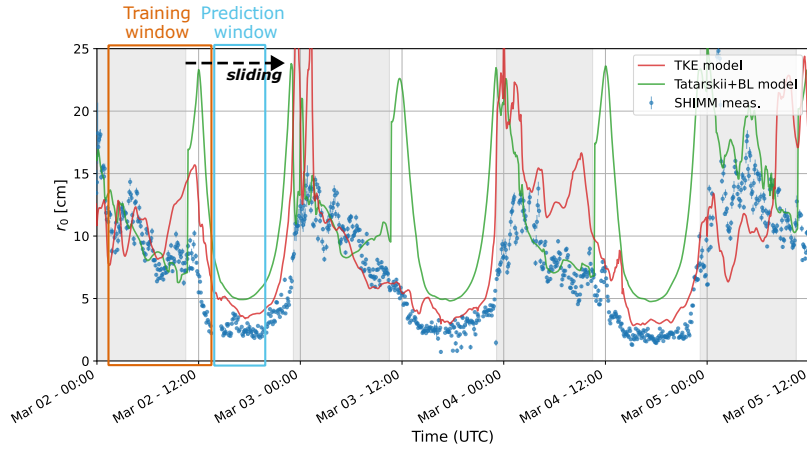


Figure 8.3 – Data used for designing the AR models, and definition of training and prediction windows (Paranal, February-March 2023). [Qua+24]

metrics are evaluated by comparing the forecasts with the measurements in the prediction windows.

In the case of punctual missing measurements, gaps in the time series are filled with the previous measured values. This has the downside of artificially increasing the performance of the persistence model, but facilitates the training of AR models.

8.3.1 Study of the training window duration

The first model parameter studied is the training window duration. Figure 8.4 depicts the evolution of the RMSE on the integrated parameters with the training window duration. The RMSE is computed for one-step ahead predictions using the AR model on the measurements, and is averaged over the whole duration of the sliding window process (84 hours, as shown in Figure 8.3). AR models with a number of coefficients P equal to 2, 5, 10, and 20 are studied. For this parameter study, only the AR model on the measurements in (8.4) is considered.

Figure 8.4 illustrates that an increase in the number of autoregression coefficients requires a longer training window to better fit these coefficients and reduce prediction errors. There is an asymptotic level below which the RMSE cannot be reduced, regardless of increasing the number of coefficients or extending the training window duration. The convergence to this level is faster for the isoplanatic angle and the scintillation index than for the Fried parameter.

8.3.2 Study of the number of coefficients in the autoregression

The evolution of the one-step ahead RMSE and correlation coefficients with the number of coefficients in the autoregression is presented in Figure 8.5. A training window of 24 hours is considered, while the performance metrics (RMSE and correlation coefficient) are computed for one-step ahead predictions. The metrics associated to the

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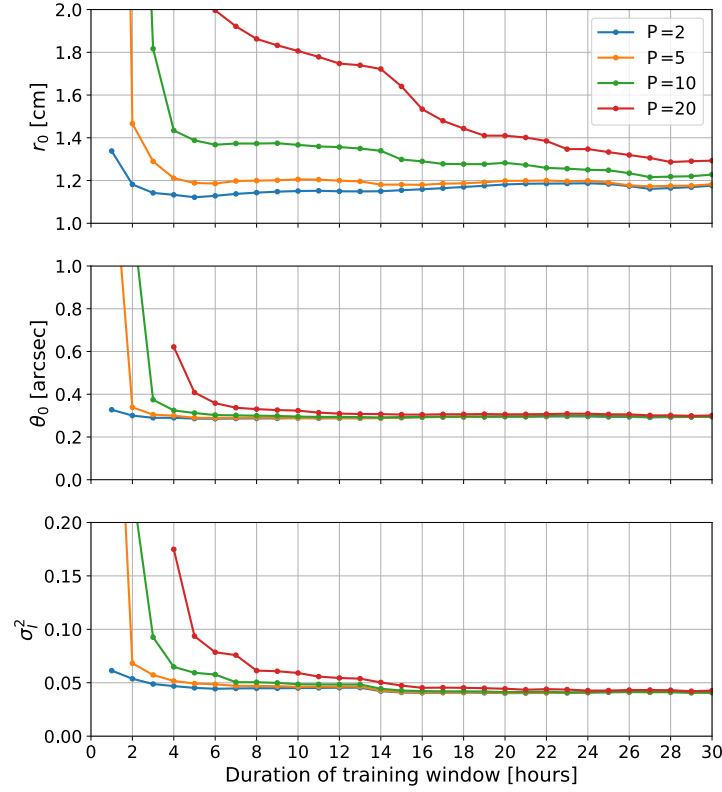


Figure 8.4 – Evolution of RMSE on integrated parameters with the training window duration, based on AR models using past measurements.

persistence model are also depicted, for reference.

Interestingly, the one-step-ahead predictions appear to degrade with an increasing number of coefficients in the AR models, as evidenced by the rise in RMSE and the drop in correlation coefficients. For large numbers of coefficients, this degradation is partially explained by the limited duration of the training window, as previously discussed (strong influence on the Fried parameter). Nevertheless, this effect remains relatively marginal, and AR models on measurements and on NWP modelling errors have similar trends.

Regarding the comparison with the persistence model, it appears that AR models with small numbers of coefficients offer better performance for one-step ahead prediction of isoplanatic angle and scintillation index, namely in terms of RMSE. However, this is not the case for the Fried parameter, where the persistence model has the smallest RMSE and the largest correlation coefficients. The autoregression on the measurements also provides improved short-term forecasts than the autoregressions on the TKE or the Tatarskii+BL modelling errors.

A possible explanation of the small influence of the number of coefficients on the performance metrics can be seen in Figure 8.6. It depicts the averages and standard

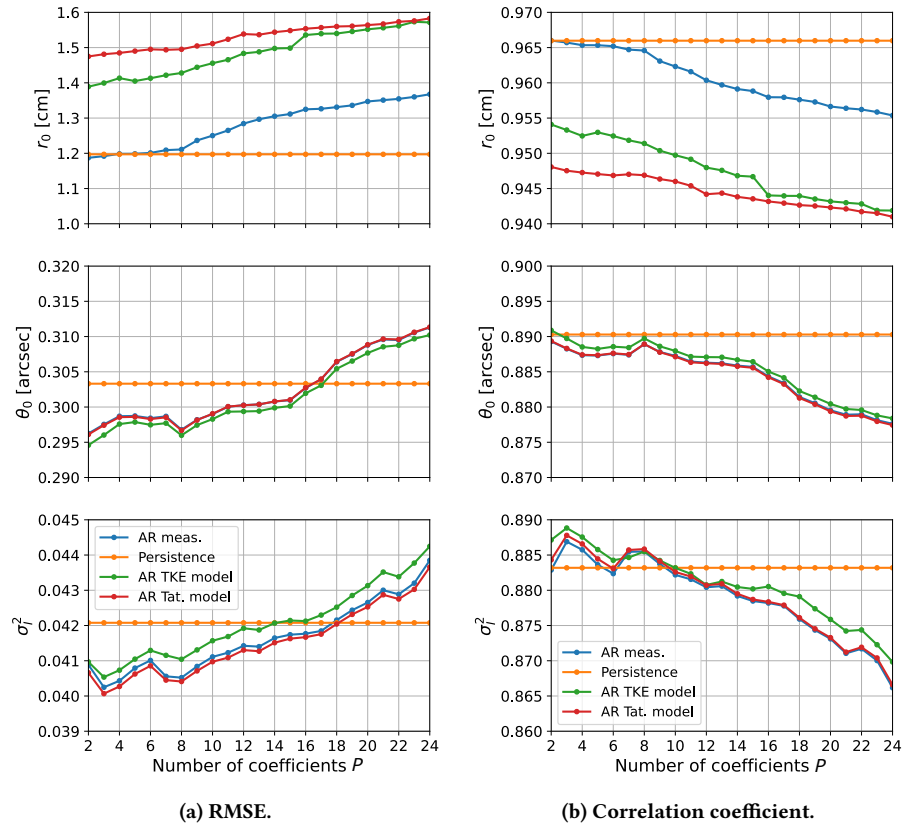


Figure 8.5 – Evolution of RMSE and correlation coefficient on the integrated parameters with the number of coefficients in the AR models.

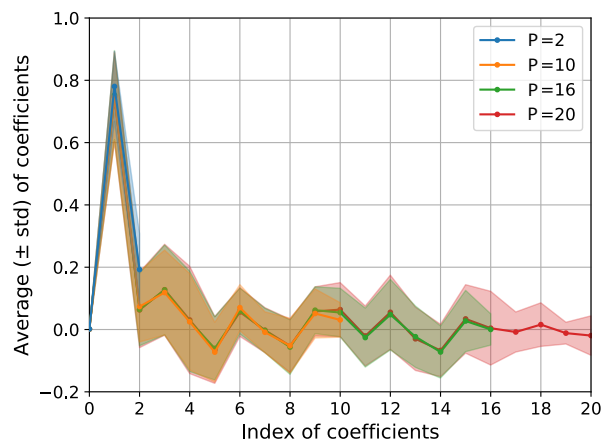


Figure 8.6 – Average ($\pm$ standard deviation) of the AR model coefficients (on Fried parameter).

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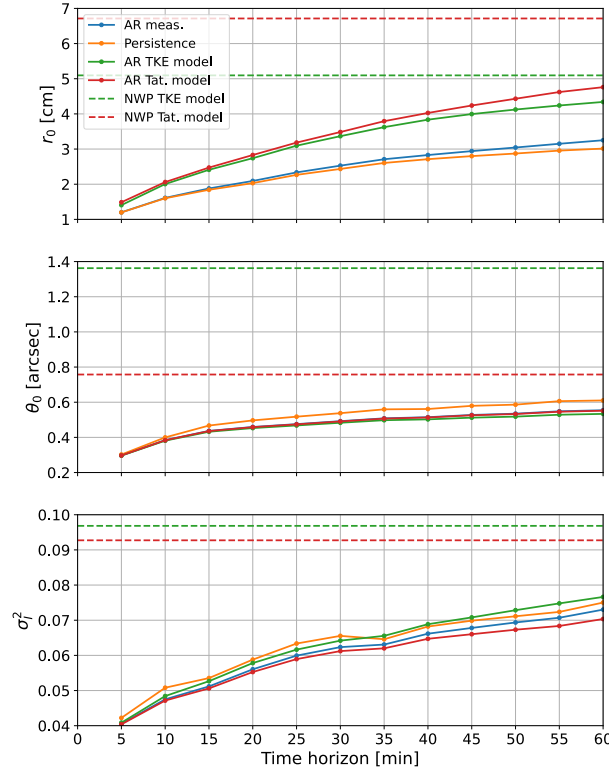


Figure 8.7 – Influence of time horizon on RMSE for AR short-term forecast. [Qua+24]

deviations of all AR coefficients computed with the sliding training window, for the AR model on the Fried parameter measurements. The coefficient index is given on the x -axis. Coefficient 0 is the constant α_0 added to the autoregression, fitted close to 0. The dominance of the first coefficient α_1 is clearly visible, and also explains the good performance of the persistence model (i.e., AR model with $\alpha_1 = 1$ and other coefficients equal to zero). The diminishing importance of large coefficients is visible, as their averages get closer to zero. They tend to alternate between opposite positive and negative values, thus partially cancelling each other in the autoregression. Similar behaviours are observed for autoregressions on the isoplanatic angle and the scintillation index.

8.3.3 Study of the time horizon for the prediction

The performance evaluation for multi-step predictions is presented in Figure 8.7. The number of coefficients in the AR models is set to 4, in line with the observations in Section 8.3.2. The training window is 8 hours, which is sufficient to reach convergence on the RMSE with only 4 coefficients in the autoregression (Figure 8.4). The prediction window duration varies from 5 to 60 minutes, and the RMSE for each time horizon is computed.

As the time horizon increases, the RMSE increases for all short-term prediction approaches. For comparison, the RMSE based on the NWP long-term forecasts are depicted, computed over the 84 hours of Paranal data studied here. It shows that all short-term prediction approaches provide lower errors on the predicted integrated parameters than the pure NWP forecasts. Hence, using autoregressions on the modelling error, i.e., merging NWP forecasts and recent measurements, is of interest to improve forecast capabilities. The autoregressions on the measurements only also show very good short-term forecast performance.

Finally, for the isoplanatic angle and the scintillation index, AR models lead to lower RMSE than the persistence model. However, the persistence model provides the best short-term forecast for all time horizons on the Fried parameter.

8.3.4 Conclusions for practical applications

The observations presented in this Section 8.3 come from the analysis of a small dataset. They therefore require further validation as more data become available. Nevertheless, they motivate the use of AR models with few coefficients, such that small training windows can be used. This is of interest for practical scenarios where few measurements may be collected prior to the forecasts.

8.4 Evaluation of autoregressive models

The evaluation of the short-term forecast approaches for Paranal and Barcelona measurements is presented in this section. For each dataset, the AR models chosen for evaluation include $P = 4$ coefficients (plus a constant), estimated with 4-hour training windows. The RMSE, MRE and correlation coefficients for the OT integrated parameters are monitored, averaged over the whole duration of the dataset. One-step predictions (time horizon of 5 minutes) and multi-step predictions (three steps, time horizon of 15 minutes) are evaluated.

Tables 8.1 to 8.3 present the error metrics for the short-term forecasts at Paranal (February-March 2023) and Barcelona (November 2023 and May 2024). Numbers in parentheses are associated to the multi-step predictions (15 minutes), while other numbers correspond to the one-step ahead predictions (5 minutes).

Based on the analysis of Tables 8.1 to 8.3, the following conclusions are drawn:

- **RMSE:** the achieved RMSE on the integrated parameters, and especially on the Fried parameter, are in agreement with literature values [Mil+19; Li+22]. However, the datasets used are different, and nighttime and daytime data are considered here.
- **RMSE gains on NWP models:** the use of recent measurements with an autoregression on the modelling error drastically improves the performance of the NWP models, both for the TKE and the Tatarskii+BL models. The gains on the RMSE, i.e., the ratios between the RMSE from the NWP only model (NWP

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TKE and NWP Tat. models) and the NWP combined with measurements (AR TKE and AR Tat. models), ranges from 3 to 5.7 for the Fried parameters, depending on the dataset. On the isoplanatic angles, the gains vary between 2.6 and 7.3, while they are between 1.5 and 4 on the scintillation index. Similar gains on the RMSE for the seeing can be found in [MMT20]. The values of the RMSE gains not only depend on the good parametrization of the AR models, but also on the initial accuracy of the NWP predictions.

- **MRE:** mean relative errors on the persistence and AR models are on the order of 10 to 20%, depending on the considered integrated quantity and dataset. It is considerably lower than the MRE for the NWP models, that can reach more than 100%.
- **Correlation coefficients:** maximal values of the correlation coefficients are achieved by the persistence model.
- **Best performance:** considering all metrics, integrated quantities and datasets, the persistence model offer the best performance, both for 5 and 15-minute predictions. This is slightly different from the results observed in Section 8.3 where, in some cases, the AR models lead to smaller error than the persistence model. This difference mainly originates in the training window duration considered here (4 hours). The AR model on the measurements has the second-best performance.
- **NWP models:** as presented in Section 5.4.2, the TKE model gives smaller RMSE for the Fried parameter than the Tatarskii+BL model. On the contrary, the Tatarskii+BL model provides better estimates of θ_0 . Such conclusions are still verified after the application of AR models, even though the differences between both NWP models tend to decrease (especially for the isoplanatic angle).
- **Time horizon:** as expected, increasing the time horizon reduces the accuracy of the predictions, for all quantities and datasets.

Hence, combining recent measurements with NWP models is required to significantly improve their short-term predictions. However, in scenarios where such measurements are available, using simple persistence models or autoregressions on the measurements may offer the best predictions, without requiring to use long-term forecasts.

These findings require further validation owing to the small size of the dataset. They also depend on the quality of the NWP forecasts, and NWP models specifically designed and calibrated at locations of interest may offer improved long-term forecasts, that could add extra values for the short-term forecasts.

Table 8.1 – Error metrics on integrated parameter short-term forecasts using autoregressive models: 5-minute and 15-minute (in parentheses) predictions (Paranal campaign).

Models	r_0 [cm]			θ_0 [arcsec]			σ_I^2		
	RMSE	MRE [%]	R_{XY}	RMSE	MRE [%]	R_{XY}	RMSE	MRE [%]	R_{XY}
Persistence	1.11 (1.69)	12.21 (17.04)	0.97 (0.93)	0.28 (0.43)	11.71 (17.70)	0.90 (0.76)	0.059 (0.066)	17.78 (27.27)	0.80 (0.75)
AR meas.	1.21 (2.00)	13.19 (18.76)	0.96 (0.91)	0.29 (0.45)	12.59 (18.88)	0.88 (0.72)	0.062 (0.073)	21.12 (31.11)	0.79 (0.74)
AR TKE model	1.47 (2.82)	14.45 (23.83)	0.94 (0.82)	0.29 (0.44)	12.69 (18.95)	0.89 (0.73)	0.061 (0.071)	20.19 (29.97)	0.79 (0.73)
AR Tat. model	1.60 (3.03)	16.61 (30.78)	0.93 (0.80)	0.29 (0.45)	12.60 (18.90)	0.88 (0.72)	0.061 (0.071)	21.08 (30.97)	0.79 (0.74)
NWP TKE model	5.10	61.03	0.60	1.36	77.38	0.38	0.097	49.80	0.56
NWP Tat. model	6.71	110.35	0.46	0.76	43.08	0.23	0.093	34.60	0.47

Table 8.2 – Error metrics on integrated parameter short-term forecasts using autoregressive models: 5-minute and 15-minute (in parentheses) predictions (Barcelona November 2023 campaign).

Models	r_0 [cm]			θ_0 [arcsec]			σ_I^2		
	RMSE	MRE [%]	R_{XY}	RMSE	MRE [%]	R_{XY}	RMSE	MRE [%]	R_{XY}
Persistence	0.64 (0.95)	7.47 (12.09)	0.90 (0.78)	0.16 (0.25)	10.75 (16.96)	0.96 (0.89)	0.144 (0.212)	16.43 (28.02)	0.91 (0.80)
AR meas.	0.68 (1.02)	8.50 (13.23)	0.89 (0.74)	0.18 (0.27)	11.51 (18.36)	0.95 (0.87)	0.156 (0.229)	22.13 (40.21)	0.89 (0.75)
AR TKE model	0.79 (1.49)	10.19 (19.68)	0.85 (0.55)	0.20 (0.30)	12.82 (23.51)	0.93 (0.85)	0.163 (0.266)	22.08 (40.86)	0.88 (0.70)
AR Tat. model	1.08 (2.01)	11.44 (20.38)	0.75 (0.44)	0.19 (0.29)	11.97 (18.99)	0.93 (0.85)	0.160 (0.239)	23.48 (42.20)	0.88 (0.73)
NWP TKE model	3.04	36.45	0.32	0.96	111.53	0.54	0.346	49.13	0.62
NWP Tat. model	6.14	96.77	-0.07	0.84	128.27	0.54	0.470	50.86	0.22

Table 8.3 – Error metrics on integrated parameter short-term forecasts using autoregressive models: 5-minute and 15-minute (in parentheses) predictions (Barcelona May 2024 campaign).

Models	r_0 [cm]			θ_0 [arcsec]			σ_I^2		
	RMSE	MRE [%]	R_{XY}	RMSE	MRE [%]	R_{XY}	RMSE	MRE [%]	R_{XY}
Persistence	0.58 (0.89)	6.39 (10.11)	0.95 (0.89)	0.18 (0.27)	9.95 (16.76)	0.88 (0.73)	0.112 (0.150)	15.67 (26.06)	0.83 (0.70)
AR meas.	0.62 (0.97)	7.27 (12.00)	0.95 (0.87)	0.19 (0.29)	11.84 (20.12)	0.86 (0.66)	0.117 (0.164)	21.74 (35.91)	0.80 (0.60)
AR TKE model	1.17 (3.67)	11.19 (27.17)	0.85 (0.47)	0.20 (0.31)	12.54 (22.04)	0.84 (0.63)	0.119 (0.189)	22.09 (37.12)	0.79 (0.52)
AR Tat. model	1.71 (2.03)	12.41 (22.82)	0.72 (0.56)	0.20 (0.30)	11.94 (20.44)	0.84 (0.64)	0.117 (0.169)	22.00 (36.97)	0.80 (0.60)
NWP TKE model	6.57	87.89	0.56	1.45	152.33	0.14	0.264	65.75	0.18
NWP Tat. model	5.12	88.04	-0.01	0.75	82.55	0.23	0.244	37.13	-0.01

8.5 Chapter summary

This chapter introduced several AR models designed for short-term OT forecasting by integrating recent measurements with NWP models. It reviewed the latest research in the field and connected the work conducted in several chapters of this thesis. The main results are outlined below, followed by a discussion of the current limitations and directions for future research.

8.5.1 Main results and contributions

The study in this chapter demonstrated that applying AR models for short-term OT forecasting significantly improves the accuracy of NWP model predictions by incorporating on-site measurements. Moreover, AR models that rely solely on these measurements or simple persistence models also demonstrate strong predictive capabilities. Hence, this work highlights the importance of collecting continuous on-site measurements at optical communication sites, for improving long-term predictions from NWP models and for refining short-term forecast approaches.

While the conclusions obtained depend on the AR model parametrization, the datasets used, and the specific NWP models applied, they offer insights into the design of AR models. In particular, they advocate for few coefficients in the autoregressions. This allows for shorter training sequences, but also limits the predictability of the process (short-term dependence only). These results apply for both daytime and nighttime OT conditions, across astronomical and non-astronomical sites.

8.5.2 Model limitations and future work

Several limitations of the presented AR models and possible research directions are presented below.

Availability of measurements The models developed in this work have been applied to a limited dataset of OT turbulence measurements from Paranal and Barcelona. Applications to other locations with more continuous measurement data will provide opportunities to further refine the models and validate the conclusions from this study.

NWP models The conclusions associated to AR models on the modelling errors may depend on the chosen NWP models and the accuracy of their long-term forecasts. Further improvements of these models, for example with refined calibrations at OGS sites using available measurements of ground meteorological parameters (e.g., Figure C.4), will benefit both long-term and short-term forecasts. Moreover, the corrections to NWP predictions made by AR models using recent measurements can facilitate the development of site-specific models, by “learning” the biases in NWP predictions using available measurements. Once these biases are corrected, the fore-

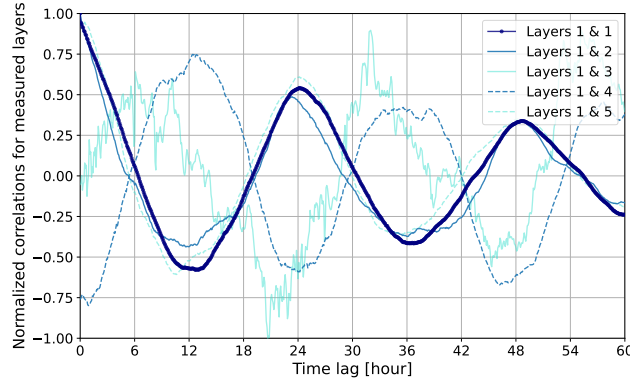


Figure 8.8 – Correlations between logarithms of $C_n^2 dz$ in different layers (SHIMM measurements).

casts will become more reliable, enabling improved estimation of OT statistics at different sites.

Scalar models The AR models on measurements and modelling errors have been developed for scalar quantities, i.e., independently for each integrated parameters. Other options worth exploring are vector models or state-space representations, especially in scenarios with measurements of C_n^2 profiles or layers. As an example, a vector of integrated quantities, such as r_0 and θ_0 , can be obtained linearly from a state vector made of C_n^2 layers c (e.g., measured by the SHIMM instrument) using

$$\begin{bmatrix} \frac{r_0^{-5/3}}{0.423k^2 \sec(\hat{\theta})} \\ \frac{\theta_0^{-5/3}}{2.913k^2 (\sec(\hat{\theta}))^{8/3}} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ h_1^{5/3} & h_2^{5/3} & h_3^{5/3} & h_4^{5/3} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}.$$

This example assumes a 4-layer description of the C_n^2 profile with layer coefficients c_1 to c_4 , at altitudes h_1 to h_4 . It used the equations for computing r_0 and θ_0 based on C_n^2 given in Table 6.2.

Moreover, Figure 8.8 shows that the correlations between the different layers measured by the SHIMM instrument (for Paranal campaign, as for Figure 8.1) can be important, especially between layers 1 and 2. It is thus of interest to explore models where influences between neighboring C_n^2 layers are allowed, as in [Hou+23a].

Seasonality and daytime variations The developed AR models use few coefficients, and therefore relies on the most recent measurements. As highlighted in

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Figure 8.1, measurements separated by 24 hours also show significant correlations, that could be exploited by adding seasonality in the AR models.

Number of AR coefficients The number of coefficients P to use in the AR models is determined from a parameter study conducted on Paranal data. The results can be different using other dataset, and should therefore be validated. The use of statistical tests for determining P can be considered [Li+22].

External inputs The addition of external inputs, such as meteorological surface parameters, can also improve the short-term forecasts, as discussed in Section 8.1.2. The use of gradients, namely temperature gradients, is suggested in [Hou+23b] or [Ni+24]. Inspiration from models developed for other fields, such as wind speed or air pollutant forecasts, is another possibility [THH18; Liu+21].

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The use of FSOC will be crucial in meeting the growing demand for Earth-space communications, supporting a wide range of applications from enhancing terrestrial connectivity to managing data from Earth observation satellites and deep space missions. However, transmitting optical signals through Earth's atmosphere raises significant challenges, requiring innovative solutions such as ground station diversity to mitigate cloud cover or adaptive optics to counteract atmospheric turbulence. The latter requires the precise characterization of OT at OGS sites, which is the core topic of this thesis, exploring both OT modelling and monitoring.

The first part of the thesis (Chapters 2 to 5) focused on OT modelling, reviewing and developing C_n^2 models for optical communication sites. The second part (Chapters 6 to 8) namely delved into OT monitoring, presenting the SHIMM instrument and short-term OT forecast. Together, these efforts led to the following contributions, as introduced in Section 1.3.2:

1. **Comprehensive review of C_n^2 modelling and optical turbulence:** Chapters 2 and 6 presented the necessary concepts for understanding and modelling OT. Chapter 2 namely introduced the statistical framework for describing OT, with a focus on the refractive index structure parameter C_n^2 . Chapter 6 explored EM wave propagation in random media, leading to the derivation of integrated OT parameters, as summarized in Table 6.2. In addition, Chapter 3 provided a comprehensive, up-to-date review of C_n^2 models from the literature, necessary for evaluating their applicability to optical communication sites.
2. **Development of C_n^2 models for satellite-to-ground optical communications:** Chapter 4 detailed a novel approach for estimating C_n^2 from high-density radiosonde profiles of meteorological parameters, making use of the refractive index structure parameter statistical definition. In Chapter 5, C_n^2 models based on NWP simulations were introduced and validated. The models from both chapters are applicable at FSOC locations and for daytime and nighttime conditions.
3. **Study of optical turbulence monitoring instruments and application to short-term forecast:** In Chapter 7, the SHIMM was presented and new mathematical approaches to estimate C_n^2 profiles from its observations were explored. Finally, Chapter 8 introduced AR models for enhancing short-term OT predictions at both astronomical and optical communication sites.

The key results and takeaways from this work are described in Section 9.1, while

Section 9.2 presents perspectives for future research to complement this study.

9.1 Key results and takeaways

Besides the contributions stated above, the work conducted in the various chapters also led to important conclusions and results regarding the application studied in this work, i.e., the modelling and monitoring of atmospheric turbulence for FSOC. The key takeaways from each of the contributions are:

1. Despite the numerous C_n^2 models available in the literature, further research and development of new models need to be conducted, as different locations and conditions are targeted.
2. NWP simulations alone are insufficient for achieving accurate OT predictions, and should be extended (e.g., using LES in the BL) and/or complemented with on-site measurements.
3. There is a need for OT monitoring instruments that provide continuous measurements at OGS, with flexibility in their design and operating principles.

Additionally, transversal takeaways associated to all contributions have been identified and are described in this section.

9.1.1 Optical turbulence modelling and monitoring for FSOC is multidisciplinary

The multidisciplinary nature of OT was highlighted through the various chapters of this work. It showed that the modelling of OT requires solid understanding of fluid mechanics and meteorology, particularly at small scales (Chapter 2). Knowledge of NWP software is also of interest for OT prediction (Chapter 5). Moreover, it involves advanced mathematical concepts from stochastic processes and random fields (Chapter 2) and may require skills in ML in the future for further improving the models (Chapter 3). Additionally, understanding the impact of OT on EM waves and optical communication necessitates expertise in EM wave propagation and telecommunications (Chapter 6), where synergies between radio frequencies (RF) and optical frequencies can be exploited (Chapter 4).

In terms of OT monitoring, skills in optical design and instrument operation are essential, as well as competence in signal processing (Chapter 7). Finally, for short-term forecasting, advanced mathematical models or ML approaches offer great potential (Chapter 8).

9.1.2 The statistical description of OT deeply relies on the refractive index structure parameter C_n^2

The statistical description of atmospheric turbulence is based on Kolmogorov's hypotheses, which assume isotropic and locally homogeneous random flows, as well

as energy cascade in the inertial range. Under these assumptions, OT is statistically characterized by the refractive index structure parameter C_n^2 , varying with altitude (Chapter 2). Obtaining this parameter is challenging because it depends on microscale meteorological quantities. Therefore, various models, both empirical and theoretical, have been developed to estimate C_n^2 (Chapter 3). While these models were originally designed for astronomical sites, their extension to FSOC locations is under study.

For FSOC applications, as studied in this work, it is crucial to have location- and season-specific C_n^2 profiles, tailored to local weather conditions. Such profiles are provided by C_n^2 models that rely on macroscale meteorological quantities, that can be easily measured or accurately predicted.

9.1.3 FSOC sites are not astronomical sites

C_n^2 models developed for astronomical applications and locations cannot be directly applied to optical communication sites. The main difference lies in the altitude of OGSs, which are often situated near cities rather than on mountaintops, where air density is lower and atmospheric turbulence effects are less pronounced. This results in a stronger contribution from the BL, particularly during daytime communications. Therefore, C_n^2 models specific to this layer, or hybrid models that consider it separately, are essential for accurate predictions (Chapters 3 and 5).

Furthermore, monitoring OT at OGS sites may require the development of new instruments capable of operating continuously both day and night, and of discriminating between different turbulent layers, such as the BL contribution. Such instruments must be relatively compact and easy to operate, ensuring practical deployment at a wide range of OGS sites while maintaining the capability to provide continuous, reliable data under varying conditions (Chapters 6 and 7).

Hence, *site characterization* for FSOC is crucial for making informed site selection and for designing optical terminals. This can be achieved by deploying monitoring instruments at various locations or by utilizing meteorological data (e.g., radiosondes, NWP) in conjunction with validated C_n^2 models.

9.1.4 OT prediction will ensure the reliability of optical communication networks

Effectively managing optical communication networks will require real-time hand-over decisions for establishing optical links between satellites and ground stations distributed across countries or continents. In this context, NWP models are of great importance, especially if they can accurately predict both cloud cover and atmospheric turbulence simultaneously (Chapter 5). Moreover, enhancing these models with on-site measurements (Chapter 7), combined with sophisticated mathematical or ML approaches (Chapter 8), could significantly improve short-term forecasting and decision-making. However, this remains a significant challenge due to the complex behaviour of the BL, the small-scale atmospheric phenomena involved, and the

availability of OT measurements.

9.2 Perspectives for future research

The full integration of space optical communication networks with ground telecommunication infrastructure is getting closer and is expected in the coming years. Significant progress is being made, with technology maturing, numerous demonstrations conducted, and standardization efforts underway. The needs are also present, given the increasing congestion of the RF spectrum for space communications and the growing volume of data from Earth observation satellites, space probes, and for humanity communications in general.

What remains to be developed is a reliable network capable of operating under all conditions, particularly with regard to atmospheric turbulence. Future studies will focus on enhancing this reliability by exploiting data from site characterization campaigns and appropriately designing optical terminals and signals. Short-term prediction will be another major step for achieving this reliability. As previously stated, this will require on-site and real-time data from OT monitoring instruments, as well as advanced modelling techniques.

9.2.1 Optical turbulence modelling

Regarding the modelling of OT at FSOC locations, there is potential for the development of new C_n^2 models applicable to a wider range of conditions (e.g., daytime and nighttime, different seasons) and various locations. These models can also incorporate data from additional sources, such as UAVs, improving their modelling of the BL. In addition, extensions to non-Kolmogorov turbulence can be considered.

The models presented in this work (Chapters 4 and 5) will benefit from further validation at other locations where OT measurements and meteorological data are available. For example, the C_n^2 statistical definition model could be integrated into a more comprehensive physical understanding of OT, extending beyond the inertial range. For NWP models, improving the BL parameterization based on in-situ measurements and integrating LES will further enhance the accuracy of the predictions.

9.2.2 Optical turbulence monitoring

The development and validation of OT monitoring instruments is an ongoing process. Currently, there remain open questions regarding the sufficient information to collect at FSOC sites for predicting link quality and communication metrics and the specific instruments to deploy at OGS sites. For instance, the required vertical resolution of C_n^2 profiles at FSOC sites is still under study, as integrated parameters might be adequate for most applications. This aspect was not explored in this work, even though Chapter 7 demonstrated that various C_n^2 profiles could be reconstructed from the same instrument observations while giving identical integrated parameters. Fu-

ture improvements in OT instrumentation will thus focus on enhancing the accuracy of the measured parameters in any meteorological conditions.

Moreover, OT monitoring enables short-term forecasting, and enhancements in short-term forecast models will likely arise from the analysis of extensive OT datasets being collected at OGS sites, potentially using ML approaches.

9.2.3 Short-term perspectives: site selection and operation of FSOC

In the coming years, an increasing number of sites will be selected for building OGSs as part of the future FSOC network. As the network expands, its operation, including the scheduling of optical communication links, will become more complex yet more reliable. The next steps to be taken in the near term for site selection and the operation of FSOC are presented below, detailing the available instruments and software of interest.

9.2.3.1 Site selection and FSOC terminal design

Accurate OT characterization across numerous sites is essential for site selection and FSOC terminal design. This primarily involves knowledge of C_n^2 profiles and long-term statistics under various conditions. Obtaining such characterization can be costly and may require extended observation campaigns. However, unlike the real-time requirements for FSOC network operations (see Section 9.2.3.2), there is no immediate time constraint for this phase.

Collecting long-term measurements of C_n^2 profiles at sites of interest is challenging and cannot be achieved with instruments commonly used in astronomical observatories, such as SCIDAR or SLODAR. Indeed, these instruments require a substantial investment and are limited to nighttime profiling. An alternative could be thermosonde measurements, which offer high-resolution vertical C_n^2 profiles, but they remain costly and only provide punctual observations. Instead, next-generation OT monitoring instruments like the SHIMM are well-suited to meet the goals of long-term campaigns, as they are portable and relatively affordable. Improvements in the robustness of these instruments to diverse meteorological conditions and enhancements in their ability to obtain accurate C_n^2 profiles should thus be priorities.

However, measuring C_n^2 profiles is time-intensive, and profiles can only be obtained along the line of sight to observed stars, laser guide stars, or satellites. NWP models offer a valuable solution by providing C_n^2 profiles over a given area (3D grid), but the challenge lies in parameterizing these simulations correctly and validating their performance (for which measurement campaigns will be needed). With these models, long-term statistics could potentially be generated through simulations, thus obtained significantly faster than collecting on-site data over extended periods. In this context, developing customized NWP simulations that yield the most accurate C_n^2 profiles for a given location is of major interest.

9 | Conclusions

9.2.3.2 Scheduling and operation of FSOC

For the scheduling and operation of FSOC links, real-time constraints are critical, as forecasts of OT conditions at various locations are needed shortly before establishing communication links. Therefore, scheduling radiosonde or thermosonde launches to acquire C_n^2 profiles would be impractical. NWP simulations can be useful, but as demonstrated in Chapter 8, they should be supplemented with on-site measurements, such as continuous OT monitoring from a SHIMM instrument.

The main research direction here involves developing models and algorithms (e.g., machine learning) that integrate all available information to improve forecasting, i.e., by combining real-time OT measurements from SHIMM-like instruments with NWP outputs. Ground meteorological data measured from local weather stations may also play a role in enhancing these forecasts. The benefits of running complex, customized NWP simulations should also be explored, as classical NWP simulations (i.e., intended for weather forecasting) that are already available might be sufficient, when combined with appropriate C_n^2 models.

9.2.4 Extensions to other fields beyond FSOC

The study conducted in this thesis has implications beyond satellite-to-ground optical communications. Astronomy, for instance, can benefit from enhanced OT models and monitoring techniques, namely for improving the quality and correction of ground-based optical observations. Additionally, strategic fields such as space surveillance and tracking (SST) and the monitoring of Near-Earth Objects (NEO) rely on accurate atmospheric turbulence models to correct the degradations in satellite and space debris images observed from the ground, thus benefiting from this research. Furthermore, high-energy laser applications through Earth's atmosphere, such as space solar power systems (i.e., geostationary satellites converting solar energy into laser light and beaming it to the ground) can also gain from this research.

Part III

Appendix

T-REX thermosonde measurements



The use of thermosonde measurements enabled to validate the C_n^2 statistical definition model presented in Chapter 4 as well as the NWP-based C_n^2 models in Chapter 5, by comparing modelled C_n^2 profiles with the measured profiles (from the thermosonde).

In contrast to radiosonde measurements, which involve the collection of macroscale meteorological parameters such as pressure and temperature, thermosonde measurements focus specifically on C_n^2 or C_T^2 and are less numerous, requiring specialized hardware. In this work, the dataset of thermosonde measurements used comes from the T-REX campaign, accessible online from the Earth Observing Laboratory data archive [Jum07].

The description of the T-REX campaign is given in Section A.1. The individual profiles coming from the C_n^2 statistical definition model are presented in Section A.2, while the results from the NWP-based models are shown in Section A.3.

A.1 Description

The T-REX campaign has been conducted by the AFRL in Sierra Nevada Mountains, near Three Rivers, CA, USA. The exact launch site was located at a latitude of 36.4872°N, a longitude of 118.84048°W, and an altitude of 503 meters above sea level. The campaign took place from 20 March 2006 to 6 April 2006 [Jum06].

The purpose of the experiments was to study high altitude mountain waves and the associated turbulence, thanks to simultaneous measurements from radiosonde and thermosonde (measurements of C_n^2 from C_T^2 , as detailed in Section 4.4 of Chapter 4). The radiosoundings offered a vertical resolution of 10 meters, approximately constant during the whole flight (standard deviation: 3.7 m). They recorded the altitude, pressure, temperature, dew point temperature, relative humidity, wind speed and wind direction.

The description of the launching conditions during the T-REX campaign can be found in [Jum06]. Table A.1 presents the launch times of all fifteen flights, emphasizing those linked to either daytime or nighttime conditions, based on the civil twilight definition. A total of seven profiles were measured under daytime conditions, with an additional eight profiles acquired during nighttime.

A | T-REX thermosonde measurements

Table A.1 – Launch times of T-REX flights.

Flight ID	Launch time (UTC)	Launch time (local time)	Daytime/nighttime
T-REX004	22/03/2006 02:07	21/03/2006 18:07	Daytime
T-REX005	23/03/2006 02:18	22/03/2006 18:18	Daytime
T-REX006	23/03/2006 02:19	22/03/2006 18:19	Daytime
T-REX007	25/03/2006 05:50	24/03/2006 21:50	Nighttime
T-REX009	25/03/2006 07:53	24/03/2006 23:53	Nighttime
T-REX011	25/03/2006 09:47	25/03/2006 01:47	Nighttime
T-REX012	25/03/2006 09:57	25/03/2006 01:57	Nighttime
T-REX017	28/03/2006 03:47	27/03/2006 19:47	Nighttime
T-REX019	31/03/2006 00:48	30/03/2006 16:48	Daytime
T-REX021	31/03/2006 02:52	30/03/2006 18:52	Nighttime
T-REX029	03/04/2006 00:02	02/04/2006 17:02	Daytime
T-REX031	03/04/2006 01:51	02/04/2006 18:51	Daytime
T-REX039	06/04/2006 01:40	05/04/2006 18:40	Daytime
T-REX040	06/04/2006 03:27	05/04/2006 20:27	Nighttime
T-REX041	06/04/2006 05:14	05/04/2006 22:14	Nighttime

A.2 C_n^2 statistical definition and Tatarskii models based on radiosonde measurements

The measured temperature, wind speed, and C_n^2 profiles for all T-REX flights are presented in Figures A.1 and A.2. The corresponding modelled C_n^2 profiles using the statistical definition model and the Tatarskii-based model from Chapter 4 are also depicted in these figures. Their average is presented in Figure 4.10.

A.3 C_n^2 numerical weather prediction models

The measured temperature, wind speed, and C_n^2 profiles for all T-REX flights are also presented in Figures A.3 and A.4. The modelled temperature, wind speed and C_n^2 profiles from the NWP simulations are depicted, using the models detailed in Chapter 5. The averages of the temperature and wind speed profiles are presented in Figure 5.17, while the median C_n^2 profile is given in Figure 5.19.

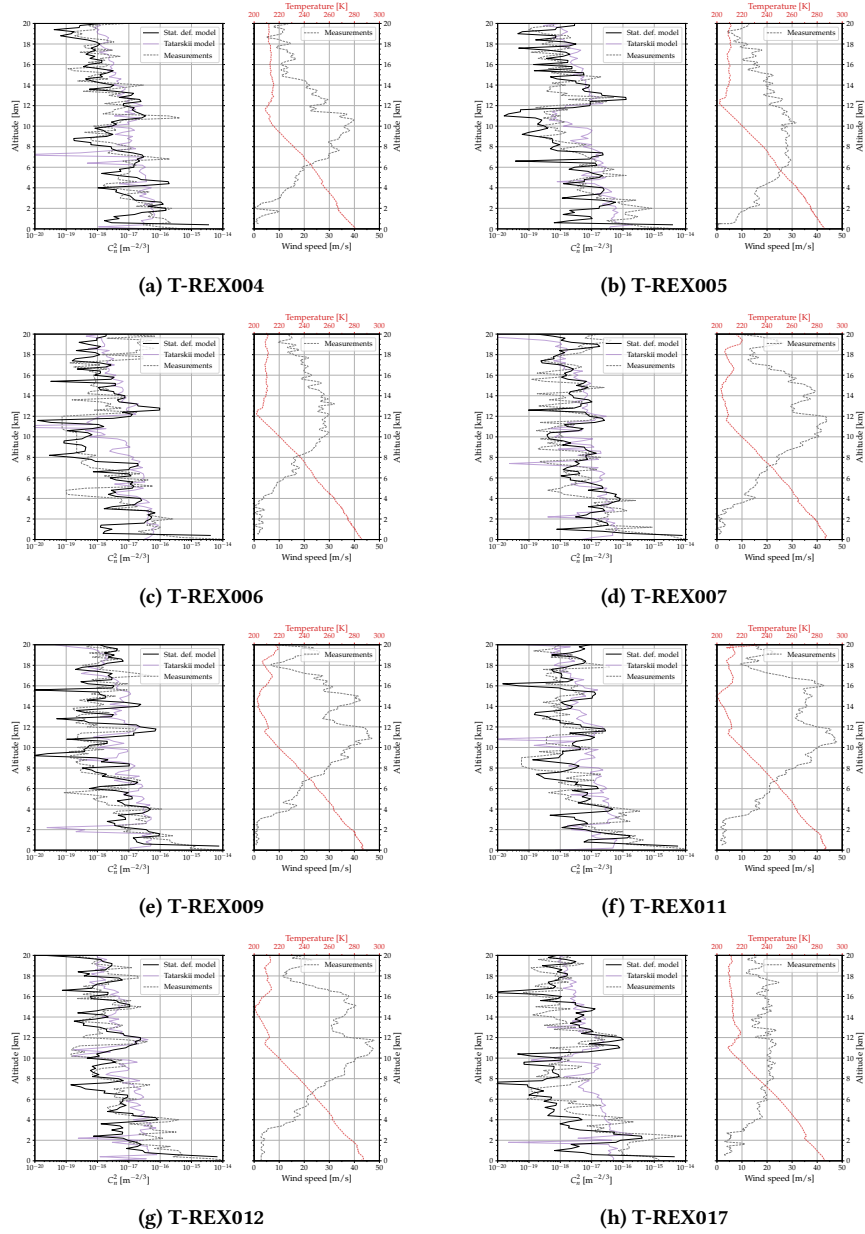


Figure A.1 – T-REX flights with radiosonde models - Part I.

A | T-REX thermosonde measurements

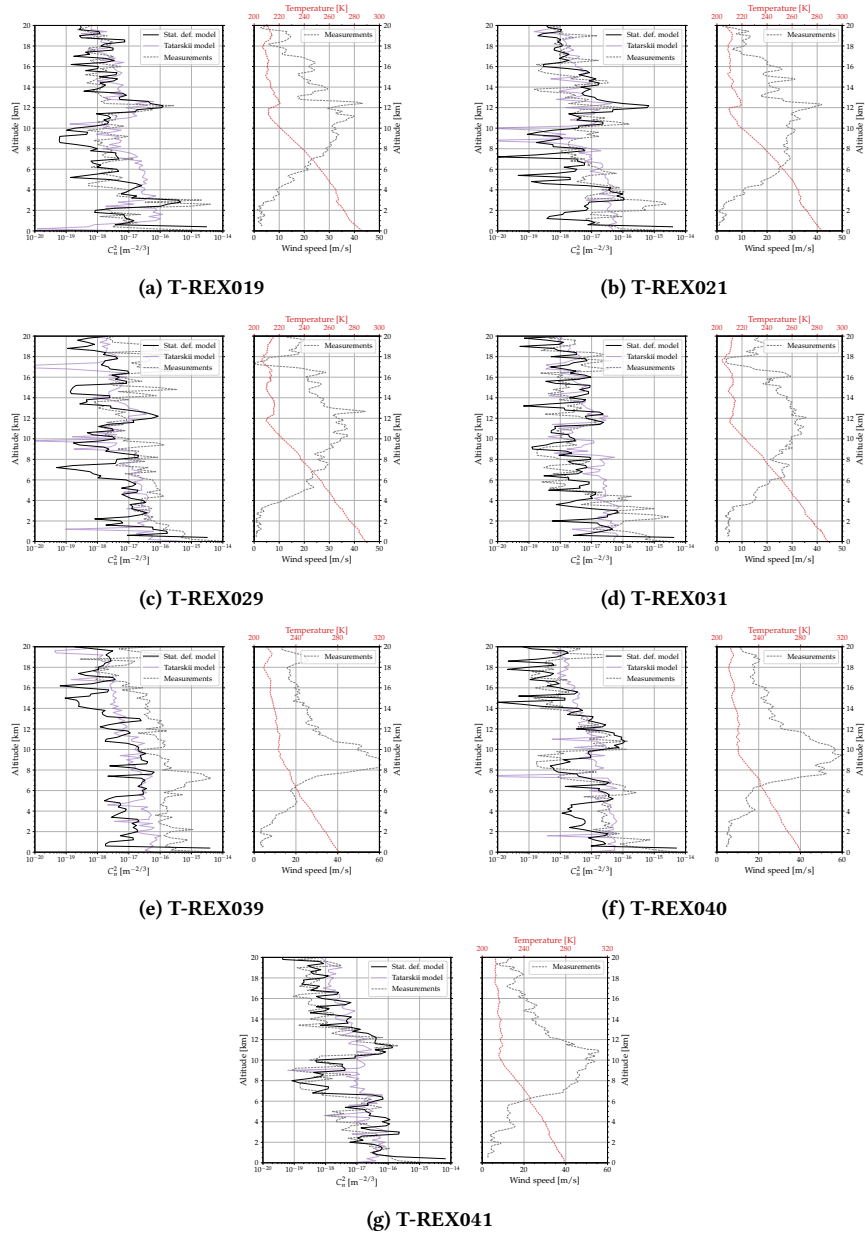


Figure A.2 – T-REX flights with radiosonde models - Part II.

C_n^2 numerical weather prediction models | A.3

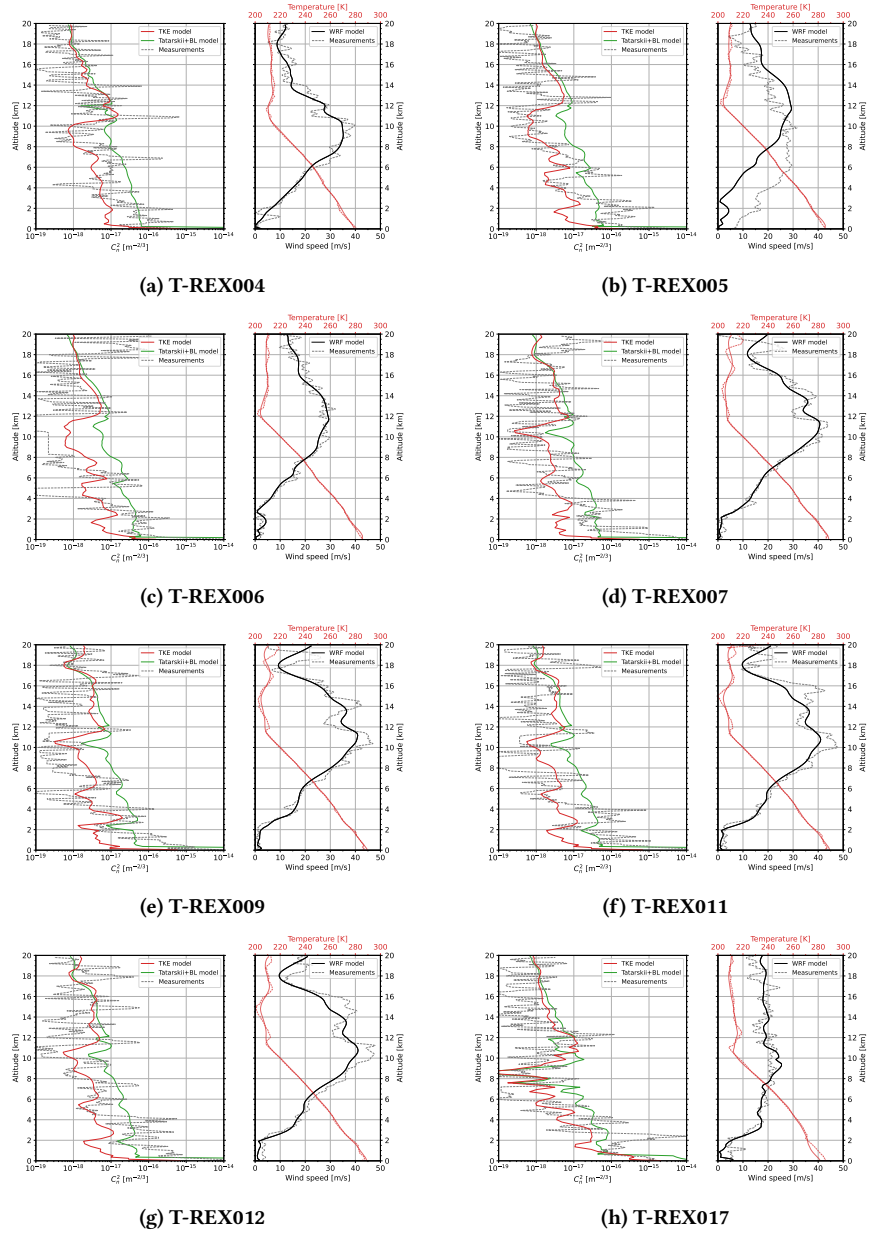


Figure A.3 – T-REX flights with NWP models - Part I. [Qua+23]

A | T-REX thermosonde measurements

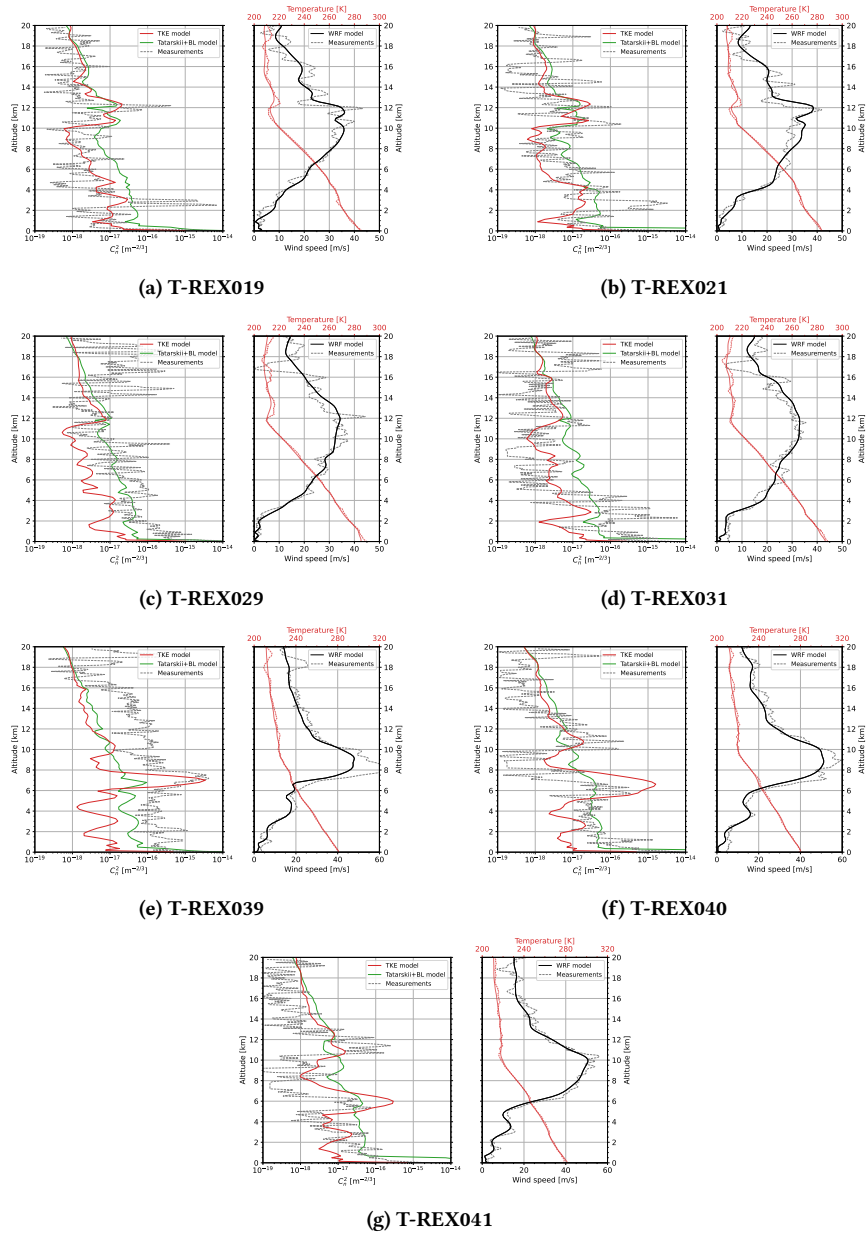


Figure A.4 – T-REX flights with NWP models - Part II. [Qua+23]

Fourier transforms and SHIMM weighting functions

B

This appendix complements the modelling of the SHIMM instrument presented in Section 7.2 of Chapter 7. Section B.1 details the computation of the Fourier transforms used in Section 7.2.1.1, while Section B.2 details the numbering convention of the SHIMM intensity and angle-of-arrival weighting functions, depicted in Section 7.2.1.5.

B.1 Computation of Fourier transforms

The Fourier transforms of the functions $\frac{1}{\lambda z} \cos\left(\frac{\pi}{\lambda z} |\mathbf{x}|^2\right)$ and $\frac{1}{\lambda z} \sin\left(\frac{\pi}{\lambda z} |\mathbf{x}|^2\right)$, used in (7.4) and (7.5), are detailed in this section. Both Fourier transforms can be computed from the Fourier transforms of $\frac{1}{\lambda z} e^{-\frac{j\pi}{\lambda z} |\mathbf{x}|^2}$ and $\frac{1}{\lambda z} e^{\frac{j\pi}{\lambda z} |\mathbf{x}|^2}$. Indeed, from Euler's formula,

$$\frac{1}{\lambda z} \cos\left(\frac{\pi}{\lambda z} |\mathbf{x}|^2\right) = \frac{1}{\lambda z} \frac{e^{\frac{j\pi}{\lambda z} |\mathbf{x}|^2} + e^{-\frac{j\pi}{\lambda z} |\mathbf{x}|^2}}{2} \quad (\text{B.1})$$

$$\frac{1}{\lambda z} \sin\left(\frac{\pi}{\lambda z} |\mathbf{x}|^2\right) = \frac{1}{\lambda z} \frac{e^{\frac{j\pi}{\lambda z} |\mathbf{x}|^2} - e^{-\frac{j\pi}{\lambda z} |\mathbf{x}|^2}}{2j} \quad (\text{B.2})$$

First, the details for the Fourier transform of $\frac{1}{\lambda z} e^{-\frac{j\pi}{\lambda z} |\mathbf{x}|^2}$ are presented. The definition of the Fourier transform is

$$\mathcal{F}\left\{\frac{1}{\lambda z} e^{-\frac{j\pi}{\lambda z} |\mathbf{x}|^2}\right\} = \int_{-\infty}^{+\infty} \frac{1}{\lambda z} e^{-\frac{j\pi}{\lambda z} |\mathbf{x}|^2} e^{-j2\pi \mathbf{f} \cdot \mathbf{x}} d\mathbf{x}, \quad (\text{B.3})$$

with $\mathbf{f} = (f_x, f_y)^T$ and $\mathbf{x} = (x, y)^T$.

In the following, Gaussian integrals will be useful:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}},$$

for any constant a and b . Of particular interest is their generalization for purely imaginary argument [Uni18]:

$$\int_0^{\infty} e^{-j a x^2} dx = e^{-j\pi/4} \sqrt{\frac{\pi}{4a}} \quad \text{for } a > 0, \quad (\text{B.4})$$

B | Fourier transforms and SHIMM weighting functions

$$\int_0^{\infty} e^{Jax^2} dx = e^{J\pi/4} \sqrt{\frac{\pi}{4a}} \quad \text{for } a > 0. \quad (\text{B.5})$$

Equation (B.3) can be rewritten as, where the 2D integration is explicit,

$$\begin{aligned} \mathcal{F} \left\{ \frac{1}{\lambda z} e^{-\frac{J\pi}{\lambda z} |\mathbf{x}|^2} \right\} &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\lambda z} e^{-\frac{J\pi}{\lambda z} (x^2+y^2)} e^{-J2\pi(f_x x + f_y y)} dx dy \\ &= \frac{1}{\lambda z} \underbrace{\left(\int_{-\infty}^{+\infty} e^{-\frac{J\pi}{\lambda z} x^2} e^{-J2\pi f_x x} dx \right)}_{I_1} \underbrace{\left(\int_{-\infty}^{+\infty} e^{-\frac{J\pi}{\lambda z} y^2} e^{-J2\pi f_y y} dy \right)}_{I_2}, \quad (\text{B.6}) \end{aligned}$$

since the dependency in x and y can be separated. Solving for integral I_1 first,

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} e^{-J\left(\frac{\pi}{\lambda z} x^2 + 2\pi f_x x\right)} dx \\ &= \int_{-\infty}^{+\infty} e^{-J\left(\left(\sqrt{\frac{\pi}{\lambda z}} x + \sqrt{\pi \lambda z} f_x\right)^2 - \pi \lambda z f_x^2\right)} dx \\ &= e^{J\pi \lambda z f_x^2} \int_{-\infty}^{+\infty} e^{-J\left(\sqrt{\frac{\pi}{\lambda z}} x + \sqrt{\pi \lambda z} f_x\right)^2} dx \end{aligned}$$

using the completion of the square. A Gaussian integral with an imaginary argument, see (B.4), is identified with $x' = \sqrt{\frac{\pi}{\lambda z}} x + \sqrt{\pi \lambda z} f_x$, leading to

$$\begin{aligned} I_1 &= e^{J\pi \lambda z f_x^2} \sqrt{\frac{\lambda z}{\pi}} \int_{-\infty}^{+\infty} e^{-Jx'^2} dx' \\ &= e^{J\pi \lambda z f_x^2} \sqrt{\frac{\lambda z}{\pi}} 2 \int_0^{+\infty} e^{-Jx'^2} dx' \\ &= e^{J\pi \lambda z f_x^2} \sqrt{\frac{\lambda z}{\pi}} 2 \sqrt{\frac{\pi}{4}} e^{-J\frac{\pi}{4}} \\ &= \sqrt{\lambda z} e^{-J\frac{\pi}{4}} e^{J\pi \lambda z f_x^2}. \end{aligned}$$

I_2 leads to a similar result, such that (B.6) becomes

$$\begin{aligned} \mathcal{F} \left\{ \frac{1}{\lambda z} e^{-\frac{J\pi}{\lambda z} |\mathbf{x}|^2} \right\} &= e^{-J\frac{\pi}{4}} e^{J\pi \lambda z f_x^2} e^{-J\frac{\pi}{4}} e^{J\pi \lambda z f_y^2} \\ &= -J e^{J\pi \lambda z (f_x^2 + f_y^2)} \\ &= \sin(\pi \lambda z |\mathbf{f}|^2) - J \cos(\pi \lambda z |\mathbf{f}|^2). \quad (\text{B.7}) \end{aligned}$$

Equivalently, but using (B.5), the Fourier transform of $\frac{1}{\lambda z} e^{\frac{J\pi}{\lambda z} |\mathbf{x}|^2}$ is computed:

$$\begin{aligned} \mathcal{F} \left\{ \frac{1}{\lambda z} e^{\frac{J\pi}{\lambda z} |\mathbf{x}|^2} \right\} &= e^{J\frac{\pi}{4}} e^{-J\pi \lambda z f_x^2} e^{J\frac{\pi}{4}} e^{-J\pi \lambda z f_y^2} \\ &= J e^{-J\pi \lambda z (f_x^2 + f_y^2)} \end{aligned}$$

$$= \sin(\pi\lambda z|\mathbf{f}|^2) + J \cos(\pi\lambda z|\mathbf{f}|^2). \quad (\text{B.8})$$

Finally, the desired Fourier transforms are obtained from (B.1), (B.2) and (B.7), (B.8):

$$\begin{aligned} \mathcal{F} \left\{ \frac{1}{\lambda z} \cos \left(\frac{\pi}{\lambda z} |x|^2 \right) \right\} &= \sin(\pi\lambda z|\mathbf{f}|^2) \\ \mathcal{F} \left\{ \frac{1}{\lambda z} \sin \left(\frac{\pi}{\lambda z} |x|^2 \right) \right\} &= \cos(\pi\lambda z|\mathbf{f}|^2). \end{aligned}$$

B.2 SHIMM weighting functions

This section presents the numbering convention for the SHIMM weighting functions represented in Section 7.2.1.5.

Intensity weighting functions The numbering (from 0 to 14) of the 15 distinct intensity weighting functions is shown in Table B.1. These functions are depicted in Figure 7.7. The functions in blue in Table B.1 correspond to the functions given in Figure 7.8.

Table B.1 – All intensity weighting functions associated to the SHIMM instrument.

Function index	Spacing differences $\mathbf{d}_{mn}$ (in pitch unit)
$w_{i,0}(z)$	$[(0, 0)]$
$w_{i,1}(z)$	$[(0, 1), (1, 0), (0, -1), (-1, 0)]$
$w_{i,2}(z)$	$[(1, 1), (-1, -1), (1, -1), (-1, 1)]$
$w_{i,3}(z)$	$[(0, 2), (2, 0), (0, -2), (-2, 0)]$
$w_{i,4}(z)$	$[(2, 1), (1, 2), (2, -1), (-2, 1), (1, -2), (-1, 2), (-2, -1), (-1, -2)]$
$w_{i,5}(z)$	$[(2, 2), (-2, -2), (2, -2), (-2, 2)]$
$w_{i,6}(z)$	$[(0, 3), (3, 0), (0, -3), (-3, 0)]$
$w_{i,7}(z)$	$[(3, 1), (1, 3), (3, -1), (-3, 1), (1, -3), (-1, 3), (-3, -1), (-1, -3)]$
$w_{i,8}(z)$	$[(3, 2), (2, 3), (3, -2), (-3, 2), (2, -3), (-2, 3), (-3, -2), (-2, -3)]$
$w_{i,9}(z)$	$[(3, 3), (-3, -3), (3, -3), (-3, 3)]$
$w_{i,10}(z)$	$[(0, 4), (4, 0), (0, -4), (-4, 0)]$
$w_{i,11}(z)$	$[(4, 1), (1, 4), (4, -1), (-4, 1), (1, -4), (-1, 4), (-4, -1), (-1, -4)]$
$w_{i,12}(z)$	$[(2, 4), (4, 2), (2, -4), (-2, 4), (4, -2), (-4, 2), (-2, -4), (-4, -2)]$
$w_{i,13}(z)$	$[(0, 5), (5, 0), (0, -5), (-5, 0)]$
$w_{i,14}(z)$	$[(5, 1), (1, 5), (5, -1), (-5, 1), (1, -5), (-1, 5), (-5, -1), (-1, -5)]$

Slope weighting functions The angle-of-arrival (slope) weighting functions are depicted in Figure 7.9 for the x -direction, and correspond to the mapping given in Table B.2. Functions in blue are presented in Figure 7.10. In total, 26 different functions are observed (without tip/tilt subtraction).

The y -slope weighting functions are such that $w_{s_x}(z, [\mu, \nu]^T) = w_{s_y}(z, [\nu, \mu]^T)$, for $\mathbf{d}_{mn} = [\mu, \nu]^T$ with μ and ν being integers of the pitch spacing. The cross-slope weighting functions $w_{s_{xy}}(z, \mathbf{d}_{mn})$ are shown in Figure 7.11, using the numbering in Table B.3.

B | Fourier transforms and SHIMM weighting functions

Table B.2 – All x -slope weighting functions associated to the SHIMM instrument.

Function index	Spacing differences d_{mn} (in pitch unit)
$w_{s,0}(z)$	$[(0, 0)]$
$w_{s,1}(z)$	$[(0, 1), (0, -1)]$
$w_{s,2}(z)$	$[(1, 0), (-1, 0)]$
$w_{s,3}(z)$	$[(1, 1), (-1, -1), (1, -1), (-1, 1)]$
$w_{s,4}(z)$	$[(0, 2), (0, -2)]$
$w_{s,5}(z)$	$[(2, 0), (-2, 0)]$
$w_{s,6}(z)$	$[(1, 2), (1, -2), (-1, 2), (-1, -2)]$
$w_{s,7}(z)$	$[(2, 1), (2, -1), (-2, 1), (-2, -1)]$
$w_{s,8}(z)$	$[(2, 2), (-2, -2), (2, -2), (-2, 2)]$
$w_{s,9}(z)$	$[(3, 0), (-3, 0)]$
$w_{s,10}(z)$	$[(0, 3), (0, -3)]$
$w_{s,11}(z)$	$[(3, 1), (3, -1), (-3, 1), (-3, -1)]$
$w_{s,12}(z)$	$[(1, 3), (1, -3), (-1, 3), (-1, -3)]$
$w_{s,13}(z)$	$[(3, 2), (3, -2), (-3, 2), (-3, -2)]$
$w_{s,14}(z)$	$[(2, 3), (2, -3), (-2, 3), (-2, -3)]$
$w_{s,15}(z)$	$[(3, 3), (-3, -3), (3, -3), (-3, 3)]$
$w_{s,16}(z)$	$[(0, 4), (0, -4)]$
$w_{s,17}(z)$	$[(4, 0), (-4, 0)]$
$w_{s,18}(z)$	$[(4, 1), (4, -1), (-4, 1), (-4, -1)]$
$w_{s,19}(z)$	$[(1, 4), (1, -4), (-1, 4), (-1, -4)]$
$w_{s,20}(z)$	$[(2, 4), (2, -4), (-2, 4), (-2, -4)]$
$w_{s,21}(z)$	$[(4, 2), (4, -2), (-4, 2), (-4, -2)]$
$w_{s,22}(z)$	$[(0, 5), (0, -5)]$
$w_{s,23}(z)$	$[(5, 0), (-5, 0)]$
$w_{s,24}(z)$	$[(5, 1), (5, -1), (-5, 1), (-5, -1)]$
$w_{s,25}(z)$	$[(1, 5), (1, -5), (-1, 5), (-1, -5)]$

Table B.3 – All xy -slope weighting functions associated to the SHIMM instrument.

Function index	Spacing differences d_{mn} (in pitch unit)
$w_{s,0}(z)$	$[(0, 0), (0, \pm q), (\pm q, 0)]$ for $q = \{1, 2, 3, 4, 5\}$
$w_{s,1}(z)$	$[(1, 1), (-1, -1)]$
$w_{s,2}(z)$	$[(1, -1), (-1, 1)]$
$w_{s,3}(z)$	$[(1, 2), (-1, -2), (2, 1), (-2, -1)]$
$w_{s,4}(z)$	$[(2, -1), (-2, 1), (1, -2), (-1, 2)]$
$w_{s,5}(z)$	$[(2, 2), (-2, -2)]$
$w_{s,6}(z)$	$[(2, -2), (-2, 2)]$
$w_{s,7}(z)$	$[(3, 1), (1, 3), (-1, -3), (-3, -1)]$
$w_{s,8}(z)$	$[(3, -1), (-3, 1), (1, -3), (-1, 3)]$
$w_{s,9}(z)$	$[(3, 2), (-3, -2), (2, 3), (-2, -3)]$
$w_{s,10}(z)$	$[(3, -2), (-3, 2), (2, -3), (-2, 3)]$
$w_{s,11}(z)$	$[(3, 3), (-3, -3)]$
$w_{s,12}(z)$	$[(3, -3), (-3, 3)]$
$w_{s,13}(z)$	$[(4, 1), (1, 4), (-4, -1), (-1, -4)]$
$w_{s,14}(z)$	$[(4, -1), (-4, 1), (1, -4), (-1, 4)]$
$w_{s,15}(z)$	$[(2, 4), (4, 2), (-2, -4), (-4, -2)]$
$w_{s,16}(z)$	$[(2, -4), (-2, 4), (4, -2), (-4, 2)]$
$w_{s,17}(z)$	$[(5, 1), (1, 5), (-5, -1), (-1, -5)]$
$w_{s,18}(z)$	$[(5, -1), (-5, 1), (1, -5), (-1, 5)]$

SHIMM measurements at Paranal and Barcelona



This appendix presents the time series of OT integrated parameters used in Chapter 8 for the development of short-term OT prediction models. Data from the SHIMM instrument at Paranal (Section C.1) and in Barcelona (Section C.2) are presented, as introduced in Section 8.2.3. The predictions from the NWP-based C_n^2 models given in Chapter 5 are also depicted.

C.1 Paranal

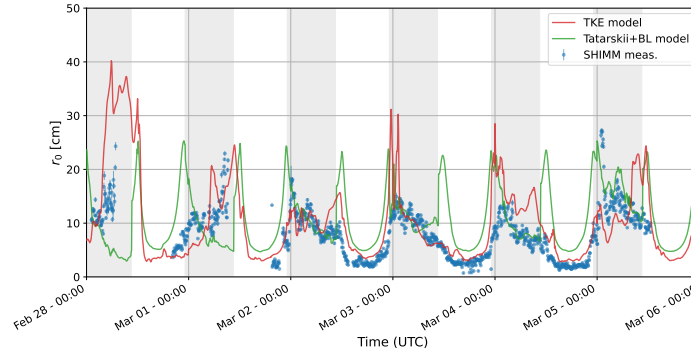
Figure C.1 presents the time series of the Fried parameter, isoplanatic angle, and scintillation index measured at Paranal, between February 28 and March 6, 2023. The times series of the Fried parameter has already been depicted in Figure 5.22 of Chapter 5.

C.2 Barcelona

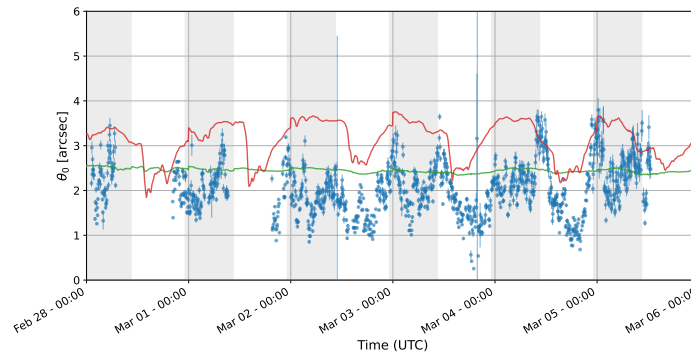
Figures C.2 and C.3 give the time series of the Fried parameter, isoplanatic angle and scintillation index measured in Barcelona in November 2023 and May 2024. Gaps in the measurements are associated to cloudy observations.

For the measurement campaign in May 2024, simultaneous meteorological data collected by an on-site weather station are shown in Figure C.4. A comparison with the surface meteorological quantities predicted by the NWP simulations is visible in the figure.

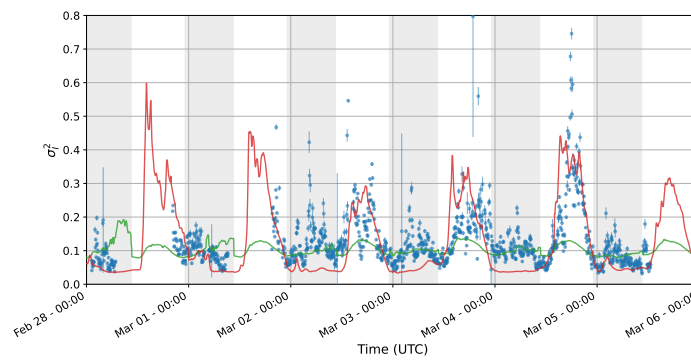
C | SHIMM measurements at Paranal and Barcelona



(a) Fried parameter r_0

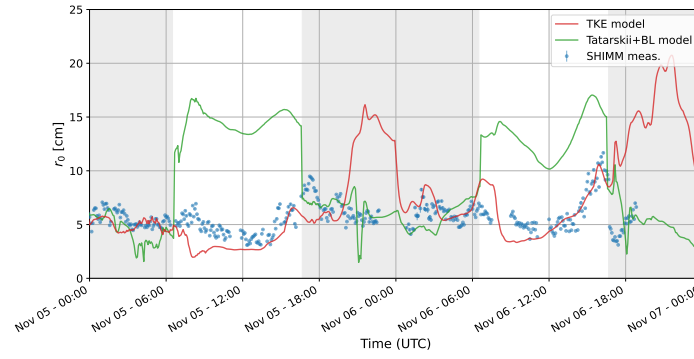


(b) Isoplanatic angle θ_0

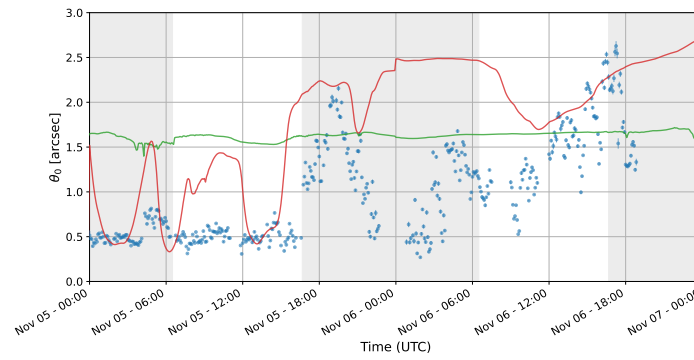


(c) Scintillation index σ_I^2

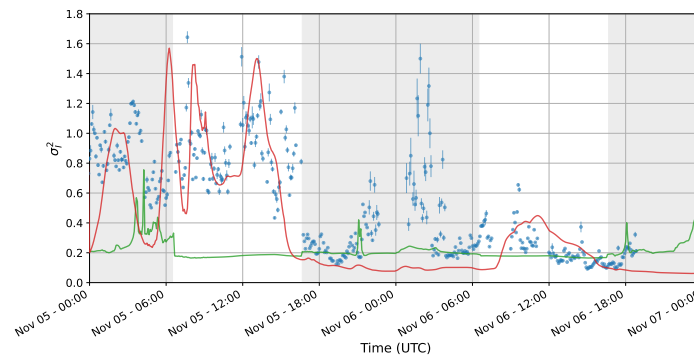
Figure C.1 – Time series of measured and modelled integrated parameters at Paranal.



(a) Fried parameter r_0



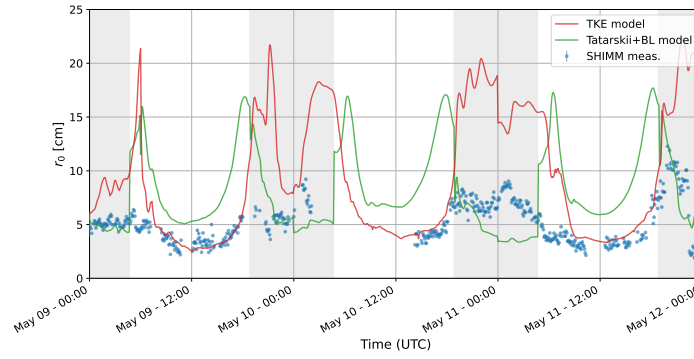
(b) Isoplanatic angle θ_0



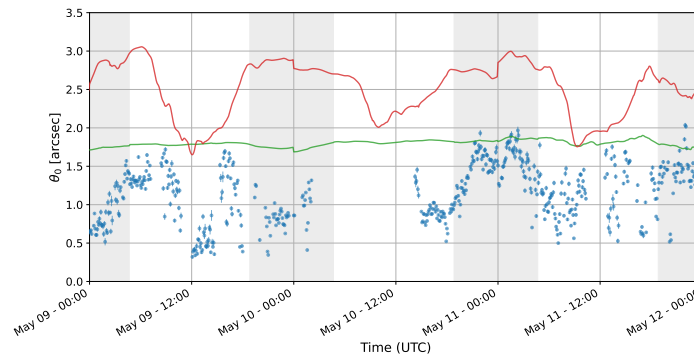
(c) Scintillation index σ_I^2

Figure C.2 – Time series of measured and modelled integrated parameters in Barcelona, in November 2023.

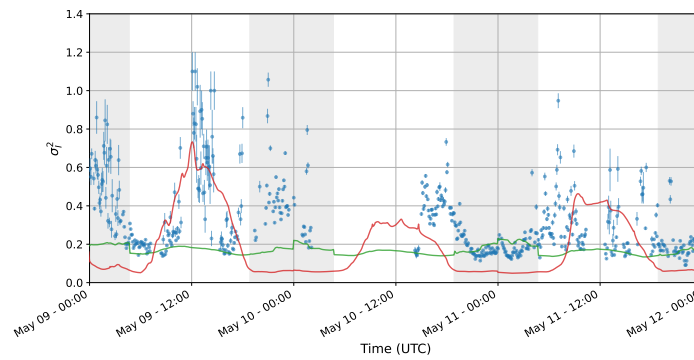
C | SHIMM measurements at Paranal and Barcelona



(a) Fried parameter r_0

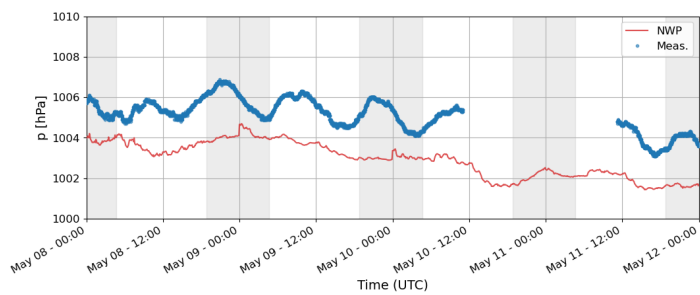


(b) Isoplanatic angle θ_0

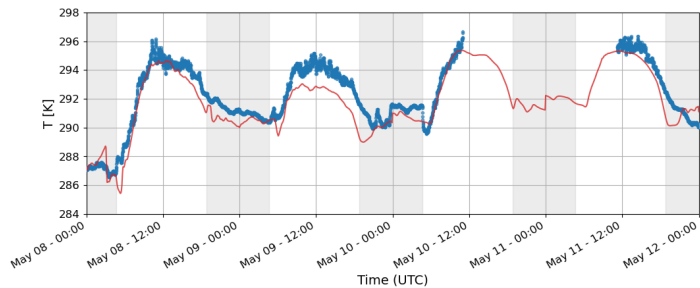


(c) Scintillation index σ_I^2

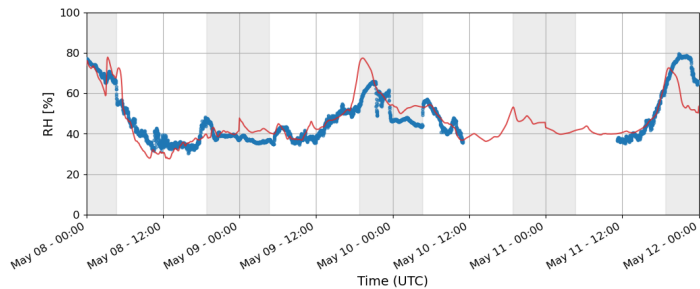
Figure C.3 – Time series of measured and modelled integrated parameters in Barcelona, in May 2024.



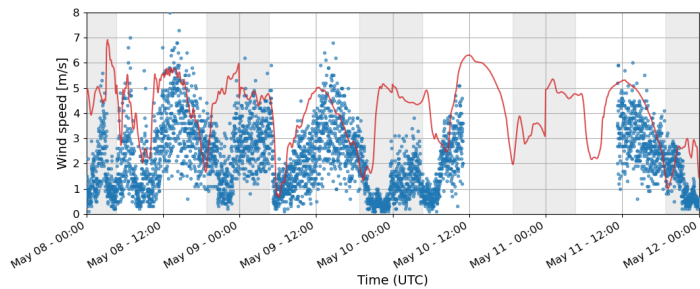
(a) Surface pressure.



(b) Ground temperature.



(c) Relative humidity.



(d) Wind speed.

Figure C.4 – Time series of ground meteorological quantities in Barcelona (May 2024).

References

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