

The $1 + 1$ Dimensional Abelian Higgs Model Revisited: Non-perturbative Dynamics in the Physical Sector¹

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In this paper the two dimensional Abelian Higgs model is revisited. We show that in the physical sector, this model describes the coupling of the electric field to the radial part, in field space, of the complex scalar field.

1 Introduction

Two dimensional scalar electrodynamics is the coupling of a $U(1)$ gauge field to a scalar complex field, where the potential can be chosen arbitrarily. Associated to the Higgs potential, the model corresponds to the well-known abelian Higgs model which has been used to illustrate spontaneous gauge symmetry breaking, or provides an effective description for superconductivity through the Landau–Ginzburg model in four dimensional Minkowski spacetime. In this contribution we show [1] that in its physical gauge invariant sector this model corresponds to the coupling of the electric field to the radial part—in field space—of the initial complex scalar field.

This paper is organised as follows. In Section 2, the two dimensional abelian Higgs model is introduced. In Section 3, the non-perturbative dynamics in the physical sector of the system is identified. Concluding remarks are provided in Section 4.

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2 The Two Dimensional Abelian Higgs Model

2.1 Choice of Lagrangian

Let us consider a two dimensional Minkowski spacetime, where the metric $\eta_{\mu\nu}$ ($\mu, \nu = 0, 1$) is of signature $\eta_{\mu\nu} = \text{diag}(+ -)$. The coupling of a gauge field A_μ to the complex scalar field ϕ is characterised by the gauge coupling constant e , while the scalar field ϕ possesses self-interactions associated to a U(1) invariant potential $V(|\phi|)$. By construction this U(1) symmetry defines the associated gauge group. The Lagrangian density of this model is,

$$\mathcal{L} = -\frac{1}{4} [\partial_\mu A_\nu - \partial_\nu A_\mu] [\partial^\mu A^\nu - \partial^\nu A^\mu] + |(\partial_\mu + ieA_\mu)\phi|^2 - V(|\phi|),$$

where it is assumed without loss of generality that $e > 0$. More explicitly, by expanding the implicit summations over the repeated spacetime indices, one has

$$\mathcal{L} = \frac{1}{2} (\partial_0 A_1 - \partial_1 A_0)^2 + [\partial_0 \phi^* - ieA_0 \phi^*] [\partial_0 \phi + ieA_0 \phi] - [\partial_1 \phi^* - ieA_1 \phi^*] [\partial_1 \phi + ieA_1 \phi] - V(|\phi|). \quad (1)$$

In order to conveniently factorise the gauge invariant and variant contributions to the dynamics, let us introduce a polar parametrisation of the complex scalar field as follows,

$$\phi(x^\mu) = \frac{1}{\sqrt{2}} \rho(x^\mu) e^{i\varphi(x^\mu)}, \quad (2)$$

where the factor $1/\sqrt{2}$ is a convenient choice of normalisation. It is understood that the functions $\rho(x^\mu)$ and $\varphi(x^\mu)$ are continuous on the two dimensional Minkowski spacetime, the sign of $\rho(x^\mu)$ being arbitrary. However the choice for the sign of $\rho(x^\mu)$ is correlated to the choice of the evaluation for $\varphi(x^\mu)$ modulo π as

$$[(-1)^k \rho, \varphi + k\pi] \longleftrightarrow [\rho, \varphi], \quad k \in \mathbb{Z}.$$

Furthermore, $\varphi(x^\mu)$ can be multi-valued at the zeros of $\rho(x^\mu)$. We thus also need to take into account for the gauge invariance of physical configurations their invariance properties under the associated $\mathbb{Z}_2$ symmetry, $\rho \leftrightarrow -\rho$.

Substituting (2) into (1), we have

$$\mathcal{L} = \frac{1}{2} [\partial_0 A_1 - \partial_1 A_0]^2 + \frac{1}{2} (\partial_0 \rho)^2 + \frac{1}{2} \rho^2 (\partial_0 \varphi + eA_0)^2 - \frac{1}{2} (\partial_1 \rho)^2 - \frac{1}{2} \rho^2 (\partial_1 \varphi + eA_1)^2 - V(\rho). \quad (3)$$

2.2 Hamiltonian formulation

The degrees of freedom of the model in its Lagrangian description are

$$A_0(t, x), A_1(t, x), \rho(t, x), \varphi(t, x). \quad (4)$$

Note that the electric field is given by

$$E = -\partial_1 A^0 - \partial_0 A^1 = \partial_0 A_1 - \partial_1 A_0 = F_{01}. \quad (5)$$

The conjugate momenta associated to these degrees of freedom are, respectively,

$$\pi^0 = 0, \quad \pi^1 = E, \quad \pi_\rho = \partial_0 \rho, \quad \pi_\varphi = \rho^2 (\partial_0 \varphi + eA_0). \quad (6)$$

The associated non-vanishing Poisson brackets are given by,

$$\begin{aligned} \{A_0(t, x), \pi^0(t, y)\} &= \delta(x - y) = -\{\pi^0(t, x), A_0(t, y)\}, \\ \{A_1(t, x), \pi^1(t, y)\} &= \delta(x - y) = -\{\pi^1(t, x), A_1(t, y)\}, \\ \{\varphi(t, x), \pi_\varphi(t, y)\} &= \delta(x - y) = -\{\pi_\varphi(t, x), \varphi(t, y)\}, \\ \{\rho(t, x), \pi_\rho(t, y)\} &= \delta(x - y) = -\{\pi_\rho(t, x), \rho(t, y)\}. \end{aligned} \quad (7)$$

Given these degrees of freedom, their conjugate momenta and the associated Poisson brackets, the dynamics of the system also derives from the following canonical Hamiltonian

$$H_0 = \int dx \left(\frac{1}{2} (\pi^1)^2 + \frac{1}{2} (\pi_\rho)^2 + \frac{1}{2} (\partial_1 \rho)^2 + \frac{1}{2} \rho^2 (\partial_1 \varphi + e A_1)^2 \right. \\ \left. + \frac{1}{2} \frac{1}{\rho^2} \pi_\varphi^2 + \partial_1 (A_0 \pi^1) - A_0 (\partial_1 \pi^1 + e \pi_\varphi) + V(\rho) \right), \quad (8)$$

where the range of integration for the spatial variable is to be specified shortly.

3 Non-perturbative dynamics in the physical sector

Let us now choose to compactify the real spatial line into a circle of length $2L$, with $-L \leq x \leq L$. Hence the topology of spacetime becomes that of a cylinder, $\tau = \mathbb{R} \times S^1$. Such a choice improves the Fourier mode analysis of the degrees of freedom of the system and their subsequent quantisation. Moreover, being gauge invariant the model possesses a primary constraint. Indeed the momentum π^0 conjugate to A_0 leads to the primary constraint $\sigma_0 = \pi^0 = 0$. Hence one needs to study the dynamics of this constraint to see whether other constraint are generated. Indeed there should exist at least another one, namely Gauss' law constraint for this electromagnetic interaction. Using the Dirac formalism for constraints [2], it is readily shown that the primary constraint σ_0 generates only one more secondary constraint, σ_1 , namely Gauss' law; that the latter constraint does not generate any further constraint; and that the two constraints, σ_0 and σ_1 , are first-class constraints. The gauge transformations generated by the the first-class constraint σ_1 are

$$\begin{aligned} A_0' &= A_0 - \partial_0 \alpha, & A_1' &= A_1 - \partial_1 \alpha, & \pi_1' &= \pi_1, \\ \rho' &= \rho, & \pi_\rho' &= \pi_\rho, \\ \varphi' &= \varphi + e \alpha, & \pi_\varphi' &= \pi_\varphi, & \partial_1 \varphi' + e A_1' &= \partial_1 \varphi + e A_1, \end{aligned}$$

$\alpha(t, x)$ being an arbitrary function of spacetime defined modulo $2\pi/e$ and periodic in $x \rightarrow x + 2L$. Besides these small gauge transformations topologically connected to the identity transformation, the system also possesses an invariance under large gauge transformations leaving all variables invariant except for A_1 and φ which are shifted in a quantised fashion in such a manner as if the above gauge function $\alpha(t, x)$ took the value $\alpha_\ell(t, x) = x\pi\ell/(eL)$ with $\ell \in \mathbb{Z}$ and $\ell \neq 0$.

3.1 The physical Hamiltonian formulation

In order to obtain a physical Hamiltonian formulation of the system, we need a canonical redefinition of its degrees of freedom. Let us then introduce [1]

$$B = -\frac{1}{e}E, \quad \pi_B = \partial_1 \varphi + e A_1.$$

The first-order gauge invariant action then becomes

$$S_{\text{phys}} = \int_{\mathbb{R}} dx^0 \int_{-L}^{+L} dx^1 [\partial_0 B \pi_B + \partial_0 \rho \pi_\rho - \mathcal{H}_{\text{phys}}],$$

where

$$\mathcal{H}_{\text{phys}} = \frac{1}{2} \rho^2 \pi_B^2 + \frac{1}{2\rho^2} (\partial_1 B)^2 + \frac{1}{2} e^2 B^2 + \frac{1}{2} \pi_\rho^2 + \frac{1}{2} (\partial_1 \rho)^2 + V(\rho) + \partial_0 [B \pi_B] - \partial_1 [B (\partial_0 \varphi + e A_0)], \quad (9)$$

any remaining terms then cancelling on account of the gauge constraints.

3.2 The physical Lagrangian formulation

The Hamiltonian equations of motion in ρ and B are given by

$$\partial_0 \rho = \int_{-L}^L dy \{ \rho(t, x), \mathcal{H}_{\text{phys}} \} = \int_{-L}^L dy \left\{ \rho(t, x), \frac{1}{2} (\pi_\rho)^2(t, y) \right\} = \pi_\rho, \quad (10)$$

$$\partial_0 B = \int_{-L}^L dy \{ B(t, x), \mathcal{H}_{\text{phys}} \} = \int_{-L}^L dy \left\{ B(t, x), \frac{1}{2} (\rho^2 (\pi_B)^2)(t, y) \right\} = \rho^2 \pi_B. \quad (11)$$

Considering the equations of motion (10) in ρ , and (11) in B , and reducing their associated conjugate momenta in the expression for $\mathcal{L}_{\text{phys}} = \pi_\rho \rho + \pi_B B - \mathcal{H}_{\text{phys}}$, the Lagrangian formulation of the system in its gauge invariant physical sector is the following,

$$\mathcal{L}_{\text{phys}} = \frac{1}{2} \frac{1}{\rho^2} (\partial_\mu B)^2 - \frac{1}{2} e^2 B^2 + \frac{1}{2} (\partial_\mu \rho)^2 - V(\rho) - \partial_0 \left[\frac{1}{\rho^2} B \partial_0 B \right] + \partial_1 [B(\partial_0 \varphi + e A_0)]. \quad (12)$$

Note that even though the analysis was developed within the Hamiltonian formulation, this expression for the physical Lagrangian density is once again manifestly Lorentz invariant, up to surface terms.

3.3 The non-perturbative dynamics in the physical sector

In this sector, the model coincides with the coupling of a pseudoscalar field, namely the electric field E , to the real scalar Higgs field ρ , which are both neutral and $U(1)$ gauge invariant, since they lie within the physical sector. All the other and gauge non invariant degrees of freedom have indeed decoupled from the physical Lagrangian.

3.3.1 The electric field

Up to its normalisation, the field B is the electric field which acquires a mass proportional to the gauge coupling constant, e . This mass also depends on the vacuum expectation value of ρ . In fact, if ρ is constant, namely such that $\rho(x) = \rho_0$, the kinetic factor $1/(2\rho^2)(\partial_\mu B)^2$ for B is well defined and, by an appropriate change of normalisation in relation to the term in B^2 , corresponds to a pseudoscalar field of mass $|e\rho_0|$. The reason why the electric field, or B is pseudoscalar is that under the parity transformation in $1+1$ spacetime dimensions, the electric field changes its sign (while there is no notion of rotation in a space with a single dimension).

Let us now assume that $\rho(t, x)$ takes any real field configuration, not necessarily constant nor static. Since the kinetic energy term for B includes the factor $1/\rho^2$, this implies that the field B acquires a dynamical mass of which the value depends on the configuration $\rho(t, x)$ for the physical scalar field. Incidentally, it thus also follows that the system classifies as an effective representation for superconductivity, and indeed this model corresponds to the celebrated Landau–Ginzburg model in the stationary case when extending the number of space dimensions, or the Higgs model in the case of the $U(1)$ gauge symmetry in higher dimensions where the gauge boson then acquires a dynamical mass.

3.3.2 The radial field

The radial field ρ arises as a real field in the theory, while its mass value, as does that of the electric field, depends on the choice of scalar potential, $V(\rho)$. A choice of interest is the Higgs potential of the form

$$V(\rho) = \frac{1}{8} M^2 (\rho^2 - \rho_0^2)^2,$$

where M^2 , with $M > 0$, is a factor that possesses the physical dimension of a squared mass, and $\rho_0 > 0$ is the vacuum expectation value of the Higgs sector. Given this choice, the mass of the Higgs field ρ is given by $|M\rho_0|$.

3.4 Noether symmetries

According to the Noether theorem [2], applied to the present system in the case of spacetime translations,

$$x'^\mu = x^\mu + a^\mu, \quad \rho'(x') = \rho(x), \quad B'(x') = B(x),$$

it follows that the current density is given by,

$$\gamma^{\nu\mu} = \partial^\mu \rho \partial^\nu \rho + \frac{1}{\rho^2} \partial^\mu B \partial^\nu B - \eta^{\nu\mu} \mathcal{L}_{phys},$$

an expression thus defining the energy-momentum tensor of the system in its physical sector. The associated conserved charges are the total energy, E_{phys} , and momentum, P_{phys} , of the system. One finds

$$E_{phys} = \int_{-L}^L dx \left[\frac{1}{2\rho^2} (\partial_0 B)^2 + \frac{1}{2\rho^2} (\partial_1 B)^2 + \frac{1}{2} e^2 B^2 + \frac{1}{2} (\partial_0 \rho)^2 + \frac{1}{2} (\partial_1 \rho)^2 + V(\rho) \right],$$

and

$$P_{phys} = - \int_{-L}^L dx \left[\partial_0 \rho \partial_1 \rho + \frac{1}{\rho^2} \partial_0 B \partial_1 B \right]. \quad (13)$$

Furthermore, we have the Noether current associated to the local $U(1)$ gauge symmetry,

$$\begin{aligned} j_\mu &= -\frac{1}{2} i \frac{1}{|\phi|^2} [\phi^* D_\mu \phi - (D_\mu \phi)^* \phi] \\ &= -\frac{1}{2} i \frac{1}{\rho^2} [\rho (\partial_\mu \rho + i \rho (\partial_\mu \varphi + e A_\mu)) - \rho (\partial_\mu \rho - i \rho (\partial_\mu \varphi + e A_\mu))] \\ &= \partial_\mu \varphi + e A_\mu, \end{aligned}$$

as well as the identity

$$\begin{aligned} \epsilon^{\mu\nu} \partial_\mu [B j_\nu] &= \partial_0 [B (\partial_1 \varphi + e A_1)] - \partial_1 [B (\partial_0 \varphi + e A_0)] \\ &= \partial_0 [B \pi_B] - \partial_1 [B (\partial_0 \varphi + e A_0)]. \end{aligned}$$

4 Concluding Remarks

We have thus reached the following fundamental conclusion: at the classical level, the two dimensional scalar electrodynamics model, in the symmetry breaking phase, is physically equivalent to a coupled system of a neutral pseudo-scalar field and a neutral scalar field, both massive.

The Euler-Lagrange equations of motion in ρ and B are, respectively,

$$\partial_\mu^2 \rho + \frac{1}{\rho^3} (\partial_\mu B)^2 + \frac{\partial V}{\partial \rho} = 0, \quad \partial_\mu \left[\frac{1}{\rho^2} \partial^\mu B \right] + e^2 B = 0.$$

These equations constitute a system of coupled non-linear equations. We are interested in the configurations which are solutions to these equations [3]. It might be possible to have some analytical solutions.

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