Disk counting statistics near hard edges of random normal matrices: the multi-component regime

Yacin Ameur*, Christophe Charlier*, Joakim Cronvall* and Jonatan Lenells†

January 28, 2024

Abstract

We consider a two-dimensional point process whose points are separated into two disjoint components by a hard wall, and study the multivariate moment generating function of the corresponding disk counting statistics. We investigate the “hard edge regime” where all disk boundaries are a distance of order $\frac{1}{n}$ away from the hard wall, where n is the number of points. We prove that as $n \to +\infty$, the asymptotics of the moment generating function are of the form

$$\exp\left(C_1 n + C_2 \ln n + C_3 + \mathcal{F}_n + \frac{C_4}{\sqrt{n}} + \mathcal{O}(n^{-\frac{3}{5}})\right),$$

and we determine the constants $C_1, \ldots, C_4$ explicitly. The oscillatory term $\mathcal{F}_n$ is of order 1 and is given in terms of the Jacobi theta function. Our theorem allows us to derive various precise results on the disk counting function. For example, we prove that the asymptotic fluctuations of the number of points in one component are of order 1 and are given by an oscillatory discrete Gaussian. Furthermore, the variance of this random variable enjoys asymptotics described by the Weierstrass $\wp$-function.

AMS Subject Classification (2020): 41A60, 60B20, 60G55.
Keywords: Oscillatory asymptotics, Moment generating functions, Random matrix theory.

1 Introduction and statement of results

In recent years there have been a lot of works on counting statistics of two dimensional point processes, see e.g. [45, 26, 47, 48, 58, 22, 27, 59, 3, 30, 19, 10, 38, 4] and the recent surveys [20, 21]. The common feature of these works is that they all deal exclusively with models for which the points condensate on a single connected component (“the one-component regime”). In this paper we deviate from these earlier works in that we study the disk counting statistics of a Coulomb gas (at inverse temperature $\beta = 2$) whose points are separated into two disjoint components by a hard wall. Let us now introduce the Coulomb gas model investigated in this work.

The Mittag-Leffler ensemble is the following joint probability density function

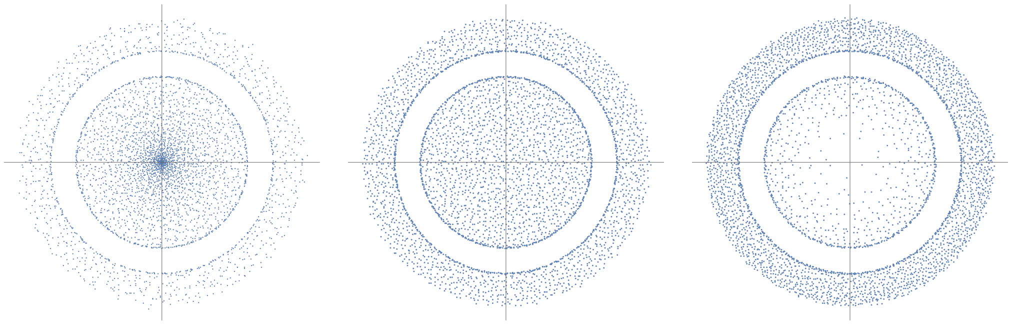
$$\frac{1}{n! Z_n} \prod_{1 \le j < k \le n} |\zeta_k - \zeta_j|^2 \prod_{j=1}^{n} |\zeta_j|^{2\alpha} e^{-n|\zeta_j|^{2b}}, \qquad \zeta_1, \ldots, \zeta_n \in \mathbb{C}, \tag{1.1}$$

where $b > 0$ and $\alpha > -1$ are fixed parameters and Z_n is the normalization constant. As $n \to +\infty$, with high probability the random points $\zeta_1, \ldots, \zeta_n$ accumulate on the disk centered at 0 of radius $b^{-\frac{1}{2b}}$ according to the probability measure $\mu(d^2 z) = \frac{b^2}{\pi}|z|^{2b-2} d^2 z$ [43, 55]. This determinantal point process generalizes the complex Ginibre process (which corresponds to $(b, \alpha) = (1, 0)$) and has attracted a lot of attention over the years, see e.g. [8, 11, 24].

In this paper we focus on the Mittag-Leffler ensemble with a hard wall that separates the random points into two disjoint components. To be precise, let $0 < \rho_1 < \rho_2 < b^{-\frac{1}{2b}}$. We consider the probability density

$$\frac{1}{n! \mathcal{Z}_n} \prod_{1 \le j < k \le n} |z_k - z_j|^2 \prod_{j=1}^{n} e^{-nQ(z_j)}, \qquad z_1, \ldots, z_n \in \mathbb{C}, \tag{1.2}$$

*Centre for Mathematical Sciences, Lund University, 22100 Lund, Sweden. e-mails: yacin.ameur@math.lu.se, christophe.charlier@math.lu.se, joakim.cronvall@math.lu.se
†Department of Mathematics, KTH Royal Institute of Technology, 10044 Stockholm, Sweden. e-mail: jlenells@kth.se



$b = \frac{1}{2}$ $\qquad$ $b = 1$ $\qquad$ $b = 2$

Figure 1: Illustration of the point processes corresponding to (1.2) with $n = 4096$, $\rho_1 = \frac{3}{5}b^{-\frac{1}{2b}}$, $\rho_2 = \frac{4}{5}b^{-\frac{1}{2b}}$, $\alpha = 0$ and the indicated values of b.

where $\mathcal{Z}_n$ is the normalization constant and

$$Q(z) = \begin{cases} |z|^{2b} - \frac{2\alpha}{n}\ln|z|, & \text{if } |z| \in [0,\rho_1] \cup [\rho_2, +\infty), \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.3}$$

Because $\rho_1, \rho_2 < b^{-\frac{1}{2b}}$, the macroscopic behavior of (1.2) is different from that of (1.1); it is described by a probability measure μ_h which is supported on $\{z \in \mathbb{C} : |z| \in [0,\rho_1] \cup [\rho_2, b^{-\frac{1}{2b}}]\}$ and has a singular component on the circles of radii ρ_1 and ρ_2. This measure can be computed using standard balayage techniques [55] (we provide the details of this computation in Appendix A) and is given by

$$\mu_h(d^2z) = 2b^2 r^{2b-1}\chi_{[0,\rho_1]\cup[\rho_2,b^{-\frac{1}{2b}}]}(r)dr\frac{d\theta}{2\pi} + \sigma_1\delta_{\rho_1}(r)dr\frac{d\theta}{2\pi} + \sigma_2\delta_{\rho_2}(r)dr\frac{d\theta}{2\pi}, \tag{1.4}$$

where $z = re^{i\theta}$, $r > 0$, $\theta \in (-\pi, \pi]$ and

$$\sigma_1 = \sigma_\star - b\rho_1^{2b}, \qquad \sigma_2 = b\rho_2^{2b} - \sigma_\star, \qquad \sigma_\star := \frac{\rho_2^{2b} - \rho_1^{2b}}{2\ln(\frac{\rho_2}{\rho_1})}. \tag{1.5}$$

The assumption $0 < \rho_1 < \rho_2 < b^{-\frac{1}{2b}}$ implies that $b\rho_1^{2b} < \sigma_\star < b\rho_2^{2b}$ and hence $\sigma_1, \sigma_2 > 0$. The quantity $\sigma_\star$ is the mass of μ_h on $\{|z| \le \rho_1\}$. Indeed, straightforward calculations show that

$$\int_{|z|\le\rho_1} \mu_h(d^2z) = \sigma_\star, \qquad \int_{|z|\ge\rho_2} \mu_h(d^2z) = 1 - \sigma_\star, \tag{1.6}$$

which means that for large n, the number of z_j's on $\{|z| \le \rho_1\}$ is roughly $\sigma_\star n$ with high probability (see also Corollary 1.7 and the asymptotics (1.29) and (1.32) below). The point process (1.2) is an example of a two-dimensional Coulomb gas that is rotation invariant (meaning that the density (1.2) remains unchanged if all z_j's are multiplied by $e^{i\beta}$, $\beta \in \mathbb{R}$). This ensemble can be seen as a conditional process where the points from (1.1) are conditioned on the hole event $\mathcal{H}$ that no ζ_j's lie in the annulus centered at 0 of radii ρ_1 and ρ_2. The partition function $\mathcal{Z}_n$ of (1.2) is precisely equal to $Z_n\mathbb{P}(\mathcal{H})$, and its large n asymptotics were investigated in [1, 2, 28]; see also [42, 39, 44, 6, 7, 5, 40, 48] for related works on the hole event. The process (1.2) can also be realized as the eigenvalues of an $n \times n$ random normal matrix M taken at random according to the probability density proportional to $e^{-n\,\mathrm{tr}Q(M)}dM$, where "tr" is the trace and dM is the measure on the set of $n \times n$ normal matrices induced by the flat Euclidian metric of $\mathbb{C}^{n\times n}$ [52, 31, 36]. Correlation kernels near hard edges have been studied in [65, 53, 56].

We will focus on "the hard edge regime", i.e. when all disk boundaries are a distance of order $\frac{1}{n}$ away from the hard edges $\{|z| = \rho_1\}$ and $\{|z| = \rho_2\}$. (Further away from the hard edges, it turns out that there are no oscillations and the situation gets similar to the one-component semi-hard edge regime considered in [10].) Let us now be more specific. Let $\mathrm{N}(y) := \#\{z_j : |z_j| < y\}$ be the random variable that counts the number of points of (1.2) in the disk centered at 0 of radius y. Our main result is a precise asymptotic formula as $n \to +\infty$ for the multivariate moment generating function (MGF)

$$\mathbb{E}\Big[\prod_{j=1}^{2m} e^{u_j \mathrm{N}(r_j)}\Big] \tag{1.7}$$

where $m \in \mathbb{N}_{>0}$ is arbitrary (but fixed), $u_1, \ldots, u_{2m} \in \mathbb{R}$, and the radii $r_1, \ldots, r_{2m}$ satisfy $r_1 < \cdots < r_{2m}$ and are merging at a critical speed in the following way

$$r_\ell = \rho_1 \Big(1 - \frac{t_\ell}{n}\Big)^{\frac{1}{2b}}, \qquad t_\ell \geq 0, \ \ell = 1, \ldots, m, \tag{1.8}$$

$$r_\ell = \rho_2 \Big(1 + \frac{t_\ell}{n}\Big)^{\frac{1}{2b}}, \qquad t_\ell \geq 0, \ \ell = m+1, \ldots, 2m. \tag{1.9}$$

As $n \to +\infty$, we prove that $\mathbb{E}\big[\prod_{j=1}^{2m} e^{u_j \mathrm{N}(r_j)}\big]$ enjoys asymptotics of the form

$$\exp\bigg(C_1 n + C_2 \ln n + C_3 + \mathcal{F}_n + \frac{C_4}{\sqrt{n}} + \mathcal{O}(n^{-\frac{3}{5}})\bigg), \tag{1.10}$$

and we determine $C_1, \ldots, C_4, \mathcal{F}_n$ explicitly. As corollaries of our various results on the generating function (1.7), we also provide a central limit theorem for the joint fluctuations of $\mathrm{N}(r_1), \ldots, \mathrm{N}(r_{2m})$, and precise asymptotic formulas for all cumulants of these random variables. Even for $m = 1$ our results are new.

We now introduce the necessary material to present our results. For $j \geq 0$, define

$$\mathsf{T}_j(x; \vec{t}, \vec{u}) = \sum_{\ell=1}^{m} \omega_\ell t_\ell^j e^{-\frac{t_\ell}{b}(x - b\rho_1^{2b})}, \qquad \hat{\mathsf{T}}_j(x; \vec{t}, \vec{u}) = \sum_{\ell=m+1}^{2m} \omega_\ell t_\ell^j e^{-\frac{t_\ell}{b}(b\rho_2^{2b} - x)}, \tag{1.11}$$

where

$$\omega_\ell = \begin{cases} e^{u_\ell + \cdots + u_{2m}} - e^{u_{\ell+1} + \cdots + u_{2m}}, & \text{if } \ell < 2m, \\ e^{u_{2m}} - 1, & \text{if } \ell = 2m, \\ 1, & \text{if } \ell = 2m+1, \end{cases} \tag{1.12}$$

and $\vec{t} = (t_1, \ldots, t_{2m})$, $\vec{u} = (u_1, \ldots, u_{2m})$. Since the quantities $\mathsf{T}_0(b\rho_1^{2b}; \vec{t}, \vec{u})$ and $\hat{\mathsf{T}}_0(b\rho_2^{2b}; \vec{t}, \vec{u})$ are independent of $\vec{t}$, we will simply write $\mathsf{T}_0(b\rho_1^{2b}; \vec{u})$ and $\hat{\mathsf{T}}_0(b\rho_2^{2b}; \vec{u})$ instead. Define

$$f(x; \vec{t}, \vec{u}) = \frac{-\big(\frac{b\rho_1^{2b}}{x - b\rho_1^{2b}} + \frac{\alpha}{b}\big)\mathsf{T}_1(x; \vec{t}, \vec{u}) - \frac{x}{2b}\mathsf{T}_2(x; \vec{t}, \vec{u})}{1 + \mathsf{T}_0(x; \vec{t}, \vec{u}) + \hat{\mathsf{T}}_0(b\rho_2^{2b}; \vec{u})}, \tag{1.13}$$

$$\hat{f}(x; \vec{t}, \vec{u}) = \frac{\big(\frac{b\rho_2^{2b}}{b\rho_2^{2b} - x} - \frac{\alpha}{b}\big)\hat{\mathsf{T}}_1(x; \vec{t}, \vec{u}) + \frac{x}{2b}\hat{\mathsf{T}}_2(x; \vec{t}, \vec{u})}{1 - \hat{\mathsf{T}}_0(x; \vec{t}, \vec{u}) + \hat{\mathsf{T}}_0(b\rho_2^{2b}; \vec{u})}. \tag{1.14}$$

Let $\Omega := e^{u_1 + \cdots + u_{2m}}$ and let

$$\mathsf{Q}(\vec{t}, \vec{u}) := \frac{1 + \mathsf{T}_0(\sigma_\star; \vec{t}, \vec{u}) + \hat{\mathsf{T}}_0(b\rho_2^{2b}; \vec{u})}{1 - \hat{\mathsf{T}}_0(\sigma_\star; \vec{t}, \vec{u}) + \hat{\mathsf{T}}_0(b\rho_2^{2b}; \vec{u})}. \tag{1.15}$$

Our main result involves $\ln \mathsf{Q}(\vec{t}, \vec{u})$ and the next lemma, whose proof is given in Appendix B, shows that this logarithm is well-defined. It also shows that f and $\hat{f}$ are smooth functions of $x \in (b\rho_1^{2b}, b\rho_2^{2b})$.

Lemma 1.1. *Suppose $\vec{u} \in \mathbb{R}^{2m}$, $t_1 > \cdots > t_m \geq 0$, and $0 \leq t_{m+1} < \cdots < t_{2m}$. For $x \in [b\rho_1^{2b}, b\rho_2^{2b}]$, it holds that*

$$1 + \mathsf{T}_0(x; \vec{t}, \vec{u}) + \hat{\mathsf{T}}_0(b\rho_2^{2b}; \vec{u}) > 0, \qquad 1 - \hat{\mathsf{T}}_0(x; \vec{t}, \vec{u}) + \hat{\mathsf{T}}_0(b\rho_2^{2b}; \vec{u}) > 0.$$

Because μ_h is supported on two disjoint components, there will be some oscillations in the large n asymptotics of (1.7). It turns out that these oscillations are described in terms of the Jacobi theta function $\theta(z; \tau)$, which we recall is defined for $z \in \mathbb{C}$ and $\operatorname{Im} \tau > 0$ by

$$\theta(z; \tau) = \sum_{\ell=-\infty}^{\infty} e^{\pi i \ell^2 \tau + 2\pi i \ell z}.$$

This function satisfies

$$\theta(z+1; \tau) = \theta(z; \tau), \qquad \theta(z+\tau; \tau) = e^{-2\pi i z} e^{-\pi i \tau} \theta(z), \qquad \theta(-z) = \theta(z), \qquad \text{for all } z \in \mathbb{C}, \tag{1.16}$$

see also [54, Chapter 20] for further properties. We are now ready to state our main result.

Theorem 1.2 (Merging radii at the hard edge: the multi-component regime)**.** *Let $m \in \mathbb{N}_{>0}$, $b > 0$, $0 < \rho_1 < \rho_2 < b^{-\frac{1}{2b}}$, $t_1, \ldots, t_{2m} \geq 0$, and $\alpha > -1$ be fixed parameters such that $t_1 > \cdots > t_m \geq 0$ and $0 \leq t_{m+1} < \cdots < t_{2m}$. For $n \in \mathbb{N}_{>0}$, define*

$$r_\ell = \begin{cases} \rho_1\big(1 - \frac{t_\ell}{n}\big)^{\frac{1}{2b}}, & \ell = 1, \ldots, m, \\ \rho_2\big(1 + \frac{t_\ell}{n}\big)^{\frac{1}{2b}}, & \ell = m+1, \ldots, 2m. \end{cases} \tag{1.17}$$

For any fixed $x_1, \ldots, x_{2m} \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\mathbb{E}\bigg[\prod_{j=1}^{2m} e^{u_j \mathrm{N}(r_j)}\bigg] = \exp\bigg(C_1 n + C_2 \ln n + C_3 + \mathcal{F}_n + \frac{C_4}{\sqrt{n}} + \mathcal{O}\big(n^{-\frac{3}{5}}\big)\bigg), \qquad \text{as } n \to +\infty \tag{1.18}$$

uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \ldots, u_{2m} \in \{z \in \mathbb{C} : |z - x_{2m}| \leq \delta\}$, where

$$\begin{aligned}
C_1 &= b\rho_1^{2b} \sum_{j=1}^{2m} u_j + \int_{b\rho_1^{2b}}^{\sigma_\star} \ln(1 + \mathsf{T}_0(x;\vec{t},\vec{u}) + \hat{\mathsf{T}}_0(b\rho_2^{2b};\vec{u}))dx + \int_{\sigma_\star}^{b\rho_2^{2b}} \ln(1 - \hat{\mathsf{T}}_0(x;\vec{t},\vec{u}) + \hat{\mathsf{T}}_0(b\rho_2^{2b};\vec{u}))dx, \\
C_2 &= -\frac{b\rho_1^{2b}}{2} \frac{\mathsf{T}_1(b\rho_1^{2b};\vec{t},\vec{u})}{\Omega} + \frac{b\rho_2^{2b}}{2} \hat{\mathsf{T}}_1(b\rho_2^{2b};\vec{t},\vec{u}), \\
C_3 &= -\frac{1}{2} \sum_{j=1}^{2m} u_j - \bigg(\alpha - \frac{2\ln(\sigma_2/\sigma_1) + \ln \mathsf{Q}(\vec{t},\vec{u})}{4\ln(\rho_2/\rho_1)}\bigg) \ln \mathsf{Q}(\vec{t},\vec{u}) \\
&\quad + \int_{b\rho_1^{2b}}^{\sigma_\star} \bigg\{ f(x;\vec{t},\vec{u}) + \frac{b\rho_1^{2b}\mathsf{T}_1(b\rho_1^{2b};\vec{t},\vec{u})}{\Omega(x - b\rho_1^{2b})} \bigg\} dx + \int_{\sigma_\star}^{b\rho_2^{2b}} \bigg\{ \hat{f}(x;\vec{t},\vec{u}) - \frac{b\rho_2^{2b}\hat{\mathsf{T}}_1(b\rho_2^{2b};\vec{t},\vec{u})}{b\rho_2^{2b} - x} \bigg\} dx \\
&\quad + b\rho_1^{2b} \frac{\mathsf{T}_1(b\rho_1^{2b};\vec{t},\vec{u})}{\Omega} \ln\bigg(\frac{b\rho_1^b}{\sqrt{2\pi}(\sigma_\star - b\rho_1^{2b})}\bigg) - b\rho_2^{2b}\hat{\mathsf{T}}_1(b\rho_2^{2b};\vec{t},\vec{u}) \ln\bigg(\frac{b\rho_2^b}{\sqrt{2\pi}(b\rho_2^{2b} - \sigma_\star)}\bigg), \\
C_4 &= \sqrt{2}\mathcal{I} b \bigg(\rho_1^{3b} \frac{\mathsf{T}_2(b\rho_1^{2b};\vec{t},\vec{u})}{\Omega} - \rho_1^b \frac{\mathsf{T}_1(b\rho_1^{2b};\vec{t},\vec{u})}{\Omega} - \rho_1^{3b} \frac{\mathsf{T}_1(b\rho_1^{2b};\vec{t},\vec{u})^2}{\Omega^2} \\
&\quad - \rho_2^{3b}\hat{\mathsf{T}}_2(b\rho_2^{2b};\vec{t},\vec{u}) - \rho_2^b\hat{\mathsf{T}}_1(b\rho_2^{2b};\vec{t},\vec{u}) - \rho_2^{3b}\hat{\mathsf{T}}_1(b\rho_2^{2b};\vec{t},\vec{u})^2 \bigg), \\
\mathcal{F}_n &= \ln \frac{\theta\big(\sigma_\star n + \frac{1}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1) + \ln \mathsf{Q}(\vec{t},\vec{u})}{2\ln(\rho_2/\rho_1)}; \frac{\pi i}{\ln(\rho_2/\rho_1)}\big)}{\theta\big(\sigma_\star n + \frac{1}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)}; \frac{\pi i}{\ln(\rho_2/\rho_1)}\big)},
\end{aligned} \tag{1.19}$$

and the constant $\mathcal{I} \in \mathbb{R}$ is given by

$$\mathcal{I} = \int_{-\infty}^{+\infty} \bigg\{ \frac{y\, e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y)\bigg[y^2 + \frac{1}{2}\bigg] \bigg\} dy \approx -0.81367. \tag{1.20}$$

In particular, since $\mathbb{E}\big[\prod_{j=1}^{2m} e^{u_j \mathrm{N}(r_j)}\big]$ is analytic in $u_1, \ldots, u_{2m} \in \mathbb{C}$ and is positive for $u_1, \ldots, u_{2m} \in \mathbb{R}$, the asymptotic formula (1.18) together with Cauchy's formula shows that

$$\partial_{u_1}^{k_1} \ldots \partial_{u_{2m}}^{k_{2m}} \bigg\{ \ln \mathbb{E}\bigg[\prod_{j=1}^{2m} e^{u_j \mathrm{N}(r_j)}\bigg] - \bigg(C_1 n + C_2 \ln n + C_3 + \frac{C_4}{\sqrt{n}}\bigg) \bigg\} = \mathcal{O}\big(n^{-\frac{3}{5}}\big), \qquad \text{as } n \to +\infty, \tag{1.21}$$

for any $k_1, \ldots, k_{2m} \in \mathbb{N}$, and $u_1, \ldots, u_{2m} \in \mathbb{R}$.

Remark 1.3. *We believe that the estimate $\mathcal{O}\big(n^{-\frac{3}{5}}\big)$ in (1.18) is not optimal and could be improved to $\mathcal{O}\big(\frac{\ln n}{n}\big)$. However, proving this is a very technical task that shall not be pursued here.*

Remark 1.4. *It is well-known that the theta function is a universal object of one-dimensional point processes in the multi-cut regime, see e.g. [34, 35, 17, 41, 57, 32, 18, 14, 15, 37, 16, 46, 29]. In dimension two, the emergence of this function was conjectured in [50, Section 1.5] and proved in [28] in the context of large gap problems (or equivalently, partition function asymptotics with hard edges). This function also appears in [9] in the study of microscopic correlations and smooth macroscopic statistics of two-dimensional rotation-invariant ensembles with soft edges. To our knowledge, Theorem 1.2 is the first result on counting statistics of a two-dimensional point process involving the θ-function.*

Remark 1.5. *(Periodicity of $\mathcal{F}_n$.) In the multi-cut regime of one-dimensional point processes, asymptotic formulas are, in general, only quasiperiodic in n, see e.g. [63, 35, 57, 18]. However, in the special case where the mass of the equilibrium measure on each interval of the support is a rational number, these asymptotic formulas become periodic, see e.g. [51] and [29, Corollary 2.2].*

Interestingly, Theorem 1.2 shows that an analogous phenomenon holds in our two-dimensional setting. To see this, recall from (1.16) that θ is periodic of period 1. Since n runs over the integers, the function $n \mapsto \mathcal{F}_n$ is, in general, only quasiperiodic in n. However, it follows from (1.19) and (1.6) that if the mass $\sigma_\star = \int_{|z|\leq \rho_1} \mu_h(d^2z)$ is rational, then $n \mapsto \mathcal{F}_n$ is periodic in n.

More generally, asymptotic formulas related to a given two-dimensional point process are expected to be periodic in n whenever the masses of the components of the equilibrium measure are all rational. This holds true in the setting of this paper, as well as in the two-dimensional soft edge setting of [9]. Recent works [33, 25] on the so-called lemniscate ensemble also support this belief. This model has a d-fold rotational symmetry, where d is a parameter that determines the number of components. The mass of the equilibrium measure on each component is $1/d$, and therefore one expects asymptotic formulas in this model to be periodic in n of period d. The formulas in [33, 25] are consistent with this expectation: these formulas involve $n = Nd$ points and are not oscillatory as $N \to +\infty$, $N \in \mathbb{N}_{>0}$.

Remark 1.6. *(Probabilistic interpretation of $\mathcal{F}_n$.) Using the well-known relation (see [54, eq 21.5.8])*

$$\theta(\tau^{-1}z; -\tau^{-1}) = e^{\pi i z^2 \tau^{-1}} \sqrt{-i\tau}\,\theta(z;\tau),$$

we can rewrite $\mathcal{F}_n$ as

$$e^{\mathcal{F}_n} = \exp\Big(- \frac{(\ln \mathsf{Q}(\vec{t},\vec{u}))^2}{4\ln(\frac{\rho_2}{\rho_1})}\Big) \frac{\sum_{\ell\in\mathbb{Z}} (\frac{\rho_1}{\rho_2})^{(\ell-\langle \Lambda_n\rangle)^2} \mathsf{Q}(\vec{t},\vec{u})^{\ell-\langle\Lambda_n\rangle}}{\sum_{\ell\in\mathbb{Z}}(\frac{\rho_1}{\rho_2})^{(\ell-\langle\Lambda_n\rangle)^2}}, \tag{1.22}$$

where $\langle \Lambda_n\rangle := \Lambda_n - \lfloor \Lambda_n \rfloor$ is the fractional part of Λ_n and

$$\Lambda_n := \sigma_\star n - \frac{1}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)}.$$

The right-most fraction in (1.22) is related to the random variable $v_n(\vec{t},\vec{u}) := (\mathcal{X}_n - \langle\Lambda_n\rangle)\ln \mathsf{Q}(\vec{t},\vec{u})$ via

$$\frac{\sum_{\ell\in\mathbb{Z}} (\frac{\rho_1}{\rho_2})^{(\ell-\langle \Lambda_n\rangle)^2} \mathsf{Q}(\vec{t},\vec{u})^{\ell-\langle\Lambda_n\rangle}}{\sum_{\ell\in\mathbb{Z}}(\frac{\rho_1}{\rho_2})^{(\ell-\langle\Lambda_n\rangle)^2}} = \mathbb{E}[e^{v_n(\vec{t},\vec{u})}], \tag{1.23}$$

where $\mathcal{X}_n$ is the discrete Gaussian random variable on $\mathbb{Z}$ defined by

$$\mathbb{P}(\mathcal{X}_n = x) = \frac{(\frac{\rho_1}{\rho_2})^{(x-\langle\Lambda_n\rangle)^2}}{\sum_{\ell\in\mathbb{Z}}(\frac{\rho_1}{\rho_2})^{(\ell-\langle\Lambda_n\rangle)^2}}, \qquad x \in \mathbb{Z}. \tag{1.24}$$

In the multi-cut regime of one-dimensional point processes, fluctuation formulas analogous to (1.22) involving an oscillatory discrete Gaussian can be found in [18, Section 8] and [29].

The following corollary gives a probabilistic interpretation of $\mathcal{X}_n$ in terms of counting statistics and is analogous to the earlier result [29, Corollary 1.4] obtained in dimension one.

Corollary 1.7. *Let $x \in \mathbb{Z}$ be fixed. As $n \to +\infty$,*

$$\mathbb{P}\big(\mathrm{N}(\rho_1) = \lfloor \Lambda_n \rfloor + x\big) = \mathbb{P}(\mathcal{X}_n = x) + o(1). \tag{1.25}$$

Proof. The proof is inspired by the proof of [29, Corollary 1.4]. Using Theorem 1.2 with $m = 1$, $u_2 = 0$ and $t_1 = 0 = t_2$, we have $C_1 = u_1\sigma_\star$, $C_2 = 0$, $\ln \mathsf{Q}(\vec{0},\vec{u}) = u_1$, $C_3 = -\frac{u_1}{2} - u_1(\alpha - \frac{2\ln(\sigma_2/\sigma_1)+u_1}{4\ln(\rho_2/\rho_1)})$, $C_4 = 0$, and thus

$$\mathbb{E}[e^{u_1 \mathrm{N}(\rho_1)}] = e^{u_1\Lambda_n}\mathbb{E}[e^{u_1(\mathcal{X}_n - \langle\Lambda_n\rangle)}]\big(1 + \mathcal{O}(n^{-\frac{3}{5}})\big), \qquad \text{as } n \to +\infty, \tag{1.26}$$

uniformly for $u_1 \in K$, where $K \subset \mathbb{R}$ is a compact subset containing 0. The rest of the proof proceeds by contradiction. Fix $x \in \mathbb{Z}$ and suppose that (1.25) does not hold. Then there exists a sequence $\{n_k\}_{k=1}^{+\infty} \subset \mathbb{N}$ such that $\mathbb{P}\big(\mathrm{N}(\rho_1) = \lfloor\Lambda_{n_k}\rfloor + x\big)$ remains bounded away from $\mathbb{P}(\mathcal{X}_{n_k} = x)$ for all k. Since $\langle\Lambda_{n_k}\rangle \in [0,1)$ for all k, there exists a subsequence $\{n_{k_j}\}_{j=1}^{+\infty}$ such that $\langle\Lambda_{n_{k_j}}\rangle \to y_* \in [0,1]$ as $j \to +\infty$.

Let $\mathcal{X}_*$ be the random variable defined as in (1.24) but with $\langle \Lambda_n \rangle$ replaced by y_*. Then (1.26) together with the fact that

$$\mathbb{E}[e^{u_1 \mathcal{X}_{n_{k_j}}}] = \mathbb{E}[e^{u_1 \mathcal{X}_*}] + o(1), \qquad \text{as } j \to +\infty \text{ uniformly for } u_1 \in K$$

implies that

$$\mathbb{E}[e^{u_1(\mathrm{N}(\rho_1) - \lfloor \Lambda_{n_{k_j}} \rfloor)}] = \mathbb{E}[e^{u_1 \mathcal{X}_*}] + o(1), \qquad \text{as } j \to +\infty \text{ uniformly for } u_1 \in K.$$

By [13, top of page 415], this implies that $\mathrm{N}(\rho_1) - \lfloor \Lambda_{n_{k_j}} \rfloor$ converges in distribution to $\mathcal{X}_*$ as $j \to +\infty$. Since $\mathbb{P}(\mathcal{X}_{n_{k_j}} = x) = \mathbb{P}(\mathcal{X}_* = x) + o(1)$ as $j \to +\infty$, we thus have $\mathbb{P}\big(\mathrm{N}(\rho_1) = \lfloor \Lambda_{n_{k_j}} \rfloor + x\big) = \mathbb{P}(\mathcal{X}_{n_{k_j}} = x) + o(1)$ as $j \to +\infty$. We have obtained our contradiction. □

For $\vec{j} \in (\mathbb{N}^{2m})_{>0} := \{\vec{j} = (j_1, \ldots, j_{2m}) \in \mathbb{N} : j_1 + \cdots + j_{2m} \geq 1\}$, the joint cumulant $\kappa_{\vec{j}} = \kappa_{\vec{j}}(r_1, \ldots, r_{2m}; n, b, \alpha)$ of $\mathrm{N}(r_1), \ldots, \mathrm{N}(r_{2m})$ is defined by

$$\kappa_{\vec{j}} = \kappa_{j_1, \ldots, j_{2m}} := \partial_{\vec{u}}^{\vec{j}} \ln \mathbb{E}[e^{u_1 \mathrm{N}(r_1) + \cdots + u_{2m} \mathrm{N}(r_{2m})}] \Big|_{\vec{u} = \vec{0}}, \tag{1.27}$$

where $\partial_{\vec{u}}^{\vec{j}} := \partial_{u_1}^{j_1} \ldots \partial_{u_{2m}}^{j_{2m}}$. As an immediate corollary of Theorem 1.2, we obtain the large n behavior of any cumulant $\kappa_{\vec{j}}$.

Corollary 1.8 (Asymptotics of cumulants)**.** *Let $m \in \mathbb{N}_{>0}$, $b > 0$, $0 < \rho_1 < \rho_2 < b^{-\frac{1}{2b}}$, $\vec{j} \in (\mathbb{N}^{2m})_{>0}$, $\alpha > -1$, $t_1 > \cdots > t_m \geq 0$ and $0 \leq t_{m+1} < \cdots < t_{2m}$ be fixed. For $n \in \mathbb{N}_{>0}$, define $\{r_\ell\}_{\ell=1}^{2m}$ by (1.17). As $n \to +\infty$, the joint cumulant $\kappa_{\vec{j}}$ satisfies*

$$\kappa_{\vec{j}} = \partial_{\vec{u}}^{\vec{j}} C_1 \big|_{\vec{u}=\vec{0}}\, n + \partial_{\vec{u}}^{\vec{j}} C_2 \big|_{\vec{u}=\vec{0}} \ln n + \partial_{\vec{u}}^{\vec{j}} C_3 \big|_{\vec{u}=\vec{0}} + \partial_{\vec{u}}^{\vec{j}} \mathcal{F}_n \big|_{\vec{u}=\vec{0}} + \frac{\partial_{\vec{u}}^{\vec{j}} C_4 \big|_{\vec{u}=\vec{0}}}{\sqrt{n}} + \mathcal{O}(n^{-\frac{3}{5}}), \tag{1.28}$$

where $C_1, \ldots, C_4, \mathcal{F}_n$ are as in Theorem 1.2.

Since $\mathbb{E}[\mathrm{N}(r)] = \kappa_1(r)$, we obtain the following result after setting $\vec{j} = 1$ in Corollary 1.8 and performing some long but straightforward calculations.

Corollary 1.9 (Asymptotics of $\mathbb{E}[\mathrm{N}(r_\ell)]$)**.** *Under the assumptions of Corollary 1.8, the expectation value $\mathbb{E}[\mathrm{N}(r_\ell)]$ obeys the following formula for any $1 \leq \ell \leq 2m$:*

$$\mathbb{E}[\mathrm{N}(r_\ell)] = b_1(t_\ell) n + c_1(t_\ell) \ln n + d_1(t_\ell) + \mathsf{f}_1(n, t_\ell) + e_1(t_\ell) n^{-\frac{1}{2}} + \mathcal{O}(n^{-\frac{3}{5}})$$

as $n \to +\infty$, where

$$b_1(t_\ell) = \begin{cases} b\rho_1^{2b} + b\frac{1 - e^{-\frac{t_\ell}{b}(\sigma_\star - b\rho_1^{2b})}}{t_\ell}, & \ell = 1, \ldots, m \text{ and } t_\ell > 0, \\ b\rho_2^{2b} - b\frac{1 - e^{-\frac{t_\ell}{b}(b\rho_2^{2b} - \sigma_\star)}}{t_\ell}, & \ell = m+1, \ldots, 2m \text{ and } t_\ell > 0, \\ \sigma_\star, & \ell = m, m+1 \text{ and } t_\ell = 0, \end{cases}$$

$$c_1(t_\ell) = \begin{cases} -\frac{b\rho_1^{2b} t_\ell}{2}, & \ell = 1, \ldots, m, \\ \frac{b\rho_2^{2b} t_\ell}{2}, & \ell = m+1, \ldots, 2m, \end{cases}$$

$$d_1(t_\ell) = -\frac{1}{2} - e^{-\frac{t_\ell}{b}(\sigma_\star - b\rho_1^{2b})} \left(\alpha - \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)} \right) + b\rho_1^{2b} t_\ell \ln \left(\frac{b\rho_1^b}{\sqrt{2\pi}(\sigma_\star - b\rho_1^{2b})} \right)$$
$$+ \int_{b\rho_1^{2b}}^{\sigma_\star} \left(\frac{b\rho_1^{2b} t_\ell (1 - e^{-\frac{t_\ell}{b}(x - b\rho_1^{2b})})}{x - b\rho_1^{2b}} - e^{-\frac{t_\ell}{b}(x - b\rho_1^{2b})} \frac{t_\ell(2\alpha + x t_\ell)}{2b} \right) dx, \qquad \ell = 1, \ldots, m,$$

$$d_1(t_\ell) = -\frac{1}{2} - e^{-\frac{t_\ell}{b}(b\rho_2^{2b} - \sigma_\star)} \left(\alpha - \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)} \right) - b\rho_2^{2b} t_\ell \ln \left(\frac{b\rho_2^b}{\sqrt{2\pi}(b\rho_2^{2b} - \sigma_\star)} \right)$$
$$- \int_{\sigma_\star}^{b\rho_2^{2b}} \left(\frac{b\rho_2^{2b} t_\ell (1 - e^{-\frac{t_\ell}{b}(b\rho_2^{2b} - x)})}{b\rho_2^{2b} - x} + e^{-\frac{t_\ell}{b}(b\rho_2^{2b} - x)} \frac{t_\ell(2\alpha - x t_\ell)}{2b} \right) dx, \qquad \ell = m+1, \ldots, 2m,$$

$$\mathsf{f}_1(n, t_\ell) = \begin{cases} \frac{e^{-\frac{t_\ell}{b}(\sigma_\star - b\rho_1^{2b})}}{2\ln(\rho_2/\rho_1)} (\ln \theta)'(n\sigma_\star + \frac{1}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)}; \frac{\pi i}{\ln(\rho_2/\rho_1)}), & \ell = 1, \ldots, m, \\ \frac{e^{-\frac{t_\ell}{b}(b\rho_2^{2b} - \sigma_\star)}}{2\ln(\rho_2/\rho_1)} (\ln \theta)'(n\sigma_\star + \frac{1}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)}; \frac{\pi i}{\ln(\rho_2/\rho_1)}), & \ell = m+1, \ldots, 2m, \end{cases}$$

$$e_1(t_\ell) = \begin{cases} \sqrt{2}\,\mathcal{I} b\rho_1^b t_\ell\big(\rho_1^{2b} t_\ell - 1\big), & \ell = 1,\ldots,m, \\ -\sqrt{2}\,\mathcal{I} b\rho_2^b t_\ell\big(\rho_2^{2b} t_\ell + 1\big), & \ell = m+1,\ldots,2m. \end{cases}$$

In particular, taking $m = 1$ and $t_1 = 0$, as $n \to +\infty$ we infer that

$$\begin{aligned} \mathbb{E}[N(\rho_1)] &= \Lambda_n + \frac{(\ln\theta)'(n\sigma_\star + \frac{1}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)}; \frac{\pi i}{\ln(\rho_2/\rho_1)})}{2\ln(\rho_2/\rho_1)} + \mathcal{O}(n^{-\frac{3}{5}}) \\ &= \sigma_\star n - \frac{1}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)} + \frac{(\ln\theta)'(n\sigma_\star + \frac{1}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)}; \frac{\pi i}{\ln(\rho_2/\rho_1)})}{2\ln(\rho_2/\rho_1)} + \mathcal{O}(n^{-\frac{3}{5}}). \end{aligned} \tag{1.29}$$

Using that

$$\mathrm{Cov}[\mathrm{N}(r_\ell), \mathrm{N}(r_k)] = \kappa_{(1,1)}(r_\ell, r_k), \qquad \mathrm{Var}[\mathrm{N}(r_\ell)] = \mathrm{Cov}[\mathrm{N}(r_\ell), \mathrm{N}(r_\ell)],$$

we can also obtain asymptotic formulas for the covariance and variance of the disk counting function by setting $\vec{j} = (1,1)$ in Corollary 1.8. We distinguish three cases depending the values of ℓ and k.

1. If $1 \le \ell \le k \le m$, then $\mathrm{N}(r_\ell)$ and $\mathrm{N}(r_k)$ count the number of points in disks whose radii r_ℓ and r_k are close to ρ_1.
2. If $m+1 \le \ell \le k \le 2m$, then $\mathrm{N}(r_\ell)$ and $\mathrm{N}(r_k)$ count the number of points in disks whose radii r_ℓ and r_k are close to ρ_2.
3. If $1 \le \ell \le m$ and $m+1 \le k \le 2m$, then $\mathrm{N}(r_\ell)$ counts the number of points in a disk of radius $r_\ell \approx \rho_1$, whereas $\mathrm{N}(r_k)$ counts the number of points in a disk of radius $r_k \approx \rho_2$.

The formulas for the covariance are naturally expressed in terms of the Weierstrass $\wp$-function. Indeed, these formulas involve the second derivative of $\ln\theta$. By [54, eq 23.6.14] ($\theta_1(\pi z)$ and $\theta_3(\pi z)$ in [54] correspond here to $\theta_1(z)$ and $\theta(z)$, respectively)

$$\wp(z;\tau) = \frac{\theta_1'''(0;\tau)}{3\theta_1'(0;\tau)} - \frac{d^2}{dz^2}\ln\theta_1(z;\tau), \quad \text{where} \quad \theta_1(z;\tau) = e^{i\pi z + \frac{\pi i \tau}{4} - \frac{\pi i}{2}}\theta\Big(z + \frac{1+\tau}{2};\tau\Big),$$

so we can express this second derivative in terms of the Weierstrass $\wp$-function as

$$-\frac{d^2}{dz^2}\ln\theta(z;\tau) = \wp\Big(z - \frac{1+\tau}{2};\tau\Big) - c,$$

where $c \in \mathbb{C}$ is defined by

$$c := \frac{\theta_1'''(0;\tau)}{3\theta_1'(0;\tau)}. \tag{1.30}$$

The function $z \mapsto \wp(z;\tau)$ is doubly periodic in the complex plane, of periods 1 and τ. In the following corollaries, we let $c = c(\rho_1,\rho_2)$ be given by (1.30) with τ defined by

$$\tau := \frac{\pi i}{\ln(\rho_2/\rho_1)}.$$

Corollary 1.10 (Asymptotics of the covariance for $1 \le \ell \le k \le m$)**.** *Under the assumptions of Corollary 1.8, the covariance $\mathrm{Cov}(\mathrm{N}(r_\ell),\mathrm{N}(r_k))$ obeys the following formula for any $1 \le \ell \le k \le m$:*

$$\begin{aligned} \mathrm{Cov}(\mathrm{N}(r_\ell),\mathrm{N}(r_k)) = b_{(1,1)}(t_\ell,t_k)n + c_{(1,1)}(t_\ell,t_k)\ln n + d_{(1,1)}(t_\ell,t_k) \\ + \mathsf{f}_{(1,1)}(n,t_\ell,t_k) + e_{(1,1)}(t_\ell,t_k)n^{-\frac{1}{2}} + \mathcal{O}\big(n^{-\frac{3}{5}}\big) \end{aligned}$$

as $n \to +\infty$, where

$$b_{(1,1)}(t_\ell,t_k) = b\frac{1 - e^{-\frac{t_\ell}{b}(\sigma_\star - b\rho_1^{2b})}}{t_\ell} - b\frac{1 - e^{-\frac{t_\ell+t_k}{b}(\sigma_\star - b\rho_1^{2b})}}{t_\ell + t_k} \quad \text{if } t_k > 0, \tag{1.31}$$

$$b_{(1,1)}(t_\ell,0) = 0, \qquad c_{(1,1)}(t_\ell,t_k) = \frac{b\rho_1^{2b}t_k}{2},$$

$$
\begin{aligned}
&d_{(1,1)}(t_\ell,t_k) = -e^{-\frac{t_\ell}{b}(\sigma_\star - b\rho_1^{2b})}\bigg(\alpha - \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)}\bigg) + e^{-\frac{t_\ell+t_k}{b}(\sigma_\star - b\rho_1^{2b})}\bigg(\alpha - \frac{\ln(\sigma_2/\sigma_1)-1}{2\ln(\rho_2/\rho_1)}\bigg) \\
&+ \int_{b\rho_1^{2b}}^{\sigma_\star} \bigg\{ - b\rho_1^{2b} t_k \frac{1-e^{-\frac{t_\ell+t_k}{b}(x-b\rho_1^{2b})}}{x-b\rho_1^{2b}} + \frac{x t_k^2 + 2\alpha t_k}{2b} e^{-\frac{t_\ell+t_k}{b}(x-b\rho_1^{2b})} \\
&- t_\ell (e^{-\frac{t_\ell}{b}(x-b\rho_1^{2b})} - e^{-\frac{t_\ell+t_k}{b}(x-b\rho_1^{2b})}) \bigg(\frac{b\rho_1^{2b}}{x-b\rho_1^{2b}} + \frac{2\alpha + x t_\ell}{2b} \bigg) \bigg\} dx - b\rho_1^{2b} t_k \ln\bigg(\frac{b\rho_1^b}{\sqrt{2\pi}(\sigma_\star - b\rho_1^{2b})} \bigg), \\
&\mathsf{f}_{(1,1)}(n,t_\ell,t_k) = -\frac{e^{-\frac{t_\ell+t_k}{b}(\sigma_\star - b\rho_1^{2b})}}{4\ln(\rho_2/\rho_1)^2} \bigg\{ \wp\bigg(n\sigma_\star + \frac{\tau}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)}; \tau \bigg) - c \\
&- 2(e^{\frac{t_k}{b}(\sigma_\star - b\rho_1^{2b})} - 1) \ln(\rho_2/\rho_1) (\ln\theta)' \bigg(n\sigma_\star + \frac{1}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)}; \tau \bigg) \bigg\}, \\
&e_{(1,1)}(t_\ell,t_k) = \sqrt{2}\,\mathcal{I} b \rho_1^b t_k \big(1 - \rho_1^{2b}(2t_\ell + t_k)\big).
\end{aligned}
$$

In particular, with $m=1$, $\ell=k=1$ and $t_1=0$, as $n\to+\infty$ we obtain

$$
\mathrm{Var}[N(\rho_1)] = \frac{1}{2\ln(\rho_2/\rho_1)} - \frac{\wp\big(n\sigma_\star + \frac{\tau}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)};\tau\big) - c}{4\ln(\rho_2/\rho_1)^2} + \mathcal{O}(n^{-\frac{3}{5}}). \tag{1.32}
$$

Remark 1.11. *It should be emphasized that the case $t_k > 0$ of Corollary 1.10 above is drastically different from the case $t_k = 0$. Indeed, if $t_k > 0$, then $\mathrm{Cov}(\mathrm{N}(r_\ell),\mathrm{N}(r_k))$ is of order n, while if $t_k = 0$, then $\mathrm{Cov}(\mathrm{N}(r_\ell),\mathrm{N}(r_k)) = \mathrm{Cov}(\mathrm{N}(r_\ell),\mathrm{N}(\rho_1))$ is of order 1.*

Corollary 1.12 (Asymptotics of the covariance for $m+1 \le \ell \le k \le 2m$)**.** *Under the assumptions of Corollary 1.8, the covariance $\mathrm{Cov}(\mathrm{N}(r_\ell),\mathrm{N}(r_k))$ obeys the following formula for any $m+1 \le \ell \le k \le 2m$:*

$$
\begin{aligned}
\mathrm{Cov}(\mathrm{N}(r_\ell),\mathrm{N}(r_k)) = {} & b_{(1,1)}(t_\ell,t_k) n + c_{(1,1)}(t_\ell,t_k)\ln n + d_{(1,1)}(t_\ell,t_k) \\
& + \mathsf{f}_{(1,1)}(n,t_\ell,t_k) + e_{(1,1)}(t_\ell,t_k) n^{-\frac{1}{2}} + \mathcal{O}\big(n^{-\frac{3}{5}}\big)
\end{aligned}
$$

as $n\to+\infty$, where

$$
b_{(1,1)}(t_\ell,t_k) = b\frac{1-e^{-\frac{t_k}{b}(b\rho_2^{2b}-\sigma_\star)}}{t_k} - b\frac{1-e^{-\frac{t_\ell+t_k}{b}(b\rho_2^{2b}-\sigma_\star)}}{t_\ell+t_k} \qquad \text{if } t_\ell > 0, \tag{1.33}
$$

$$
\begin{aligned}
&b_{(1,1)}(0,t_k) = 0, \qquad c_{(1,1)}(t_\ell,t_k) = \frac{b\rho_2^{2b} t_\ell}{2}, \\
&d_{(1,1)}(t_\ell,t_k) = e^{-\frac{t_k}{b}(b\rho_2^{2b}-\sigma_\star)}\bigg(\alpha - \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)}\bigg) - e^{-\frac{t_\ell+t_k}{b}(b\rho_2^{2b}-\sigma_\star)}\bigg(\alpha - \frac{\ln(\sigma_2/\sigma_1)+1}{2\ln(\rho_2/\rho_1)}\bigg) \\
&+ \int_{\sigma_\star}^{b\rho_2^{2b}} \bigg\{ - b\rho_2^{2b} t_\ell \frac{1-e^{-\frac{t_\ell+t_k}{b}(b\rho_2^{2b}-x)}}{b\rho_2^{2b}-x} + \frac{x t_\ell^2 - 2\alpha t_\ell}{2b} e^{-\frac{t_\ell+t_k}{b}(b\rho_2^{2b}-x)} \\
&+ t_k (e^{-\frac{t_k}{b}(b\rho_2^{2b}-x)} - e^{-\frac{t_\ell+t_k}{b}(b\rho_2^{2b}-x)}) \bigg(\frac{-b\rho_2^{2b}}{b\rho_2^{2b}-x} + \frac{2\alpha - x t_k}{2b} \bigg) \bigg\} dx - b\rho_2^{2b} t_\ell \ln\bigg(\frac{b\rho_2^b}{\sqrt{2\pi}(b\rho_2^{2b} - \sigma_\star)} \bigg), \\
&\mathsf{f}_{(1,1)}(n,t_\ell,t_k) = -\frac{e^{-\frac{t_\ell+t_k}{b}(b\rho_2^{2b}-\sigma_\star)}}{4\ln(\rho_2/\rho_1)^2} \bigg\{ \wp\bigg(n\sigma_\star + \frac{\tau}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)}; \tau \bigg) - c \\
&+ 2(e^{\frac{t_\ell}{b}(b\rho_2^{2b}-\sigma_\star)} - 1) \ln(\rho_2/\rho_1) (\ln\theta)' \bigg(n\sigma_\star + \frac{1}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)}; \tau \bigg) \bigg\}, \\
&e_{(1,1)}(t_\ell,t_k) = -\sqrt{2}\,\mathcal{I} b \rho_2^b t_\ell \big(1 + \rho_2^{2b}(t_\ell + 2t_k)\big).
\end{aligned}
$$

Corollary 1.13 (Asymptotics of the covariance for $1 \le \ell \le m$ and $m+1 \le k \le 2m$)**.** *Under the assumptions of Corollary 1.8, the covariance $\mathrm{Cov}(\mathrm{N}(r_\ell),\mathrm{N}(r_k))$ obeys the following formula for any $1 \le \ell \le m$ and $m+1 \le k \le 2m$:*

$$
\mathrm{Cov}(\mathrm{N}(r_\ell),\mathrm{N}(r_k)) = d_{(1,1)}(t_\ell,t_k) + \mathsf{f}_{(1,1)}(n,t_\ell,t_k) + \mathcal{O}\big(n^{-\frac{3}{5}}\big)
$$

as $n\to+\infty$, where

$$
\begin{aligned}
&d_{(1,1)}(t_\ell,t_k) = \frac{e^{-\frac{t_\ell}{b}(\sigma_\star - b\rho_1^{2b})} e^{-\frac{t_k}{b}(b\rho_2^{2b}-\sigma_\star)}}{2\ln(\rho_2/\rho_1)}, \\
&\mathsf{f}_{(1,1)}(n,t_\ell,t_k) = \frac{e^{-\frac{t_\ell}{b}(\sigma_\star - b\rho_1^{2b})} e^{-\frac{t_k}{b}(b\rho_2^{2b}-\sigma_\star)}}{4\ln(\rho_2/\rho_1)^2} \bigg\{ c - \wp\bigg(n\sigma_\star + \frac{\tau}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2\ln(\rho_2/\rho_1)}; \tau \bigg) \bigg\}.
\end{aligned}
$$

It is crucial for the next corollary that $t_m, t_{m+1} > 0$, so that $b_{(1,1)}(t_\ell, t_\ell) > 0$ for all $\ell \in \{1, \ldots, 2m\}$. In this case the asymptotic joint fluctuations of $\mathrm{N}(r_1), \ldots, \mathrm{N}(r_{2m})$ are of order $\sqrt{n}$ and are continuous Gaussians. This should be compared to Corollary 1.7 (corresponding to the case $m = 1$ and $t_m = 0$), which shows that the asymptotic fluctuations of $\mathrm{N}(\rho_1)$ are of order 1 and are described by a discrete Gaussian.

Corollary 1.14. *Suppose that the assumptions of Corollary 1.8 hold and that $t_m, t_{m+1} > 0$. As $n \to +\infty$, the random variable $(\mathcal{N}_1, \ldots, \mathcal{N}_{2m})$, where*

$$\mathcal{N}_\ell := \frac{\mathrm{N}(r_\ell) - (b_1(t_\ell)n + c_1(t_\ell)\ln n)}{\sqrt{b_{(1,1)}(t_\ell, t_\ell)n}}, \qquad \ell = 1, \ldots, 2m, \tag{1.34}$$

converges in distribution to a multivariate normal random variable of mean $(0, \ldots, 0)$ whose covariance matrix Σ is given by

$$\Sigma_{\ell,k} = \Sigma_{k,\ell} = \begin{cases} \frac{b_{(1,1)}(t_\ell, t_k)}{\sqrt{b_{(1,1)}(t_\ell,t_\ell)b_{(1,1)}(t_k,t_k)}}, & 1 \le \ell \le k \le m \text{ or } m+1 \le \ell \le k \le 2m, \\ 0, & 1 \le \ell \le m \text{ and } m+1 \le k \le 2m, \end{cases}$$

where $b_{(1,1)}(t_\ell, t_k)$ is given by (1.31) for $1 \le \ell \le k \le m$ and by (1.33) for $m+1 \le \ell \le k \le 2m$.

Proof. By Lévy's continuity theorem, the assertion will follow if we can show that the characteristic function $\mathbb{E}[e^{i\sum_{\ell=1}^{2m} v_\ell \mathcal{N}_\ell}]$ converges pointwise to $e^{-\frac{1}{2}\sum_{\ell,k=1}^{2m} v_\ell \Sigma_{\ell,k} v_k}$ for every $v_\ell \in \mathbb{R}^{2m}$ as $n \to +\infty$. Letting $u_\ell = \frac{iv_\ell}{\sqrt{b_{(1,1)}(t_\ell,t_\ell)n}}$, (1.34) and (1.18) show that

$$\begin{aligned} \mathbb{E}[e^{i\sum_{\ell=1}^{2m} v_\ell \mathcal{N}_\ell}] &= \mathbb{E}[e^{\sum_{\ell=1}^{2m} u_\ell \mathrm{N}(r_\ell)}]e^{-\sum_{\ell=1}^{2m} u_\ell(b_1(t_\ell)n + c_1(t_\ell)\ln n)} \\ &= e^{C_1(\vec{u})n + C_2(\vec{u})\ln n + C_3(\vec{u}) + \mathcal{F}_n(\vec{u}) + \mathcal{O}(n^{-\frac{1}{2}})} e^{-\sum_{\ell=1}^{2m} u_\ell(\partial_{u_\ell}C_1|_{\vec{u}=\vec{0}}n + \partial_{u_\ell}C_2|_{\vec{u}=\vec{0}}\ln n)} \end{aligned}$$

as $n \to +\infty$ for any fixed $v_\ell \in \mathbb{R}^{2m}$. Since $C_j|_{\vec{u}=\vec{0}} = 0$ for $j = 1,2,3$, $\mathcal{F}_n(\vec{u}) = \sum_{\ell=1}^{2m} \mathsf{f}_1(n, t_\ell)u_\ell + \mathcal{O}(|\vec{u}|^2) = \mathcal{O}(|\vec{u}|)$, and $|\vec{u}| = \mathcal{O}(n^{-1/2})$, we obtain

$$\begin{aligned} \mathbb{E}[e^{i\sum_{\ell=1}^{2m} v_\ell \mathcal{N}_\ell}] &= e^{\frac{1}{2}\sum_{\ell,k=1}^{2m} u_\ell u_k \partial_{u_\ell}\partial_{u_k} C_1|_{\vec{u}=\vec{0}} n + \mathcal{O}(|\vec{u}|^3 n + |\vec{u}|^2 \ln n + |\vec{u}| + n^{-1/2})} \\ &= e^{\frac{1}{2}\sum_{\ell,k=1}^{m} \frac{iv_\ell}{\sqrt{b_{(1,1)}(t_\ell,t_\ell)}} \frac{iv_k}{\sqrt{b_{(1,1)}(t_k,t_k)}} b_{(1,1)}(t_{\min(\ell,k)}, t_{\max(\ell,k)})} \\ &\quad \times e^{\frac{1}{2}\sum_{\ell,k=m+1}^{2m} \frac{iv_\ell}{\sqrt{b_{(1,1)}(t_\ell,t_\ell)}} \frac{iv_k}{\sqrt{b_{(1,1)}(t_k,t_k)}} b_{(1,1)}(t_{\min(\ell,k)}, t_{\max(\ell,k)}) + \mathcal{O}(n^{-1/2})} \to e^{-\frac{1}{2}\sum_{\ell,k=1}^{2m} v_\ell \Sigma_{\ell,k} v_k} \end{aligned}$$

as $n \to +\infty$, which completes the proof. □

Outline of proof. Using that $\prod_{1 \le j < k \le n} |z_k - z_j|^2$ is the product of two Vandermonde determinants, we obtain after standard manipulations that

$$\mathcal{E}_n := \mathbb{E}\Big[\prod_{\ell=1}^{2m} e^{u_\ell \mathrm{N}(r_\ell)}\Big] = \frac{1}{n!\mathcal{Z}_n} \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} \prod_{1 \le j < k \le n} |z_k - z_j|^2 \prod_{j=1}^{n} w(z_j) d^2 z_j \tag{1.35}$$

$$= \frac{1}{\mathcal{Z}_n} \det\left(\int_{\mathbb{C}} z^j \overline{z}^k w(z) d^2 z\right)_{j,k=0}^{n-1} \tag{1.36}$$

$$= \frac{1}{\mathcal{Z}_n} (2\pi)^n \prod_{j=0}^{n-1} \left(\int_0^{\rho_1} + \int_{\rho_2}^{+\infty}\right) u^{2j+1} w(u) du, \tag{1.37}$$

where the weight w is defined by

$$w(z) := e^{-nQ(z)}\omega(|z|), \qquad \omega(x) := \prod_{\ell=1}^{2m} \begin{cases} e^{u_\ell}, & \text{if } x < r_\ell, \\ 1, & \text{if } x \ge r_\ell. \end{cases} \tag{1.38}$$

Formula (1.37) directly follows from (1.36) and the fact that w is rotation-invariant. Indeed, since $w(z) = w(|z|)$, the integral $\int_{\mathbb{C}} z^j \overline{z}^k w(z) d^2 z$ is 0 for $j \neq k$ and is $2\pi \int_0^{+\infty} u^{2j+1} w(u) du$ for $j = k$. So only the main diagonal contributes for the determinants in (1.36).

Shifting $j \to j-1$ and then performing the change of variables $v = nu^{2b}$ in (1.37), we obtain

$$\begin{aligned}\mathcal{E}_n &= \frac{\prod_{j=1}^n \big(\int_0^{\rho_1} + \int_{\rho_2}^{+\infty} \big) u^{2j+2\alpha-1} e^{-nu^{2b}} \omega(u) du}{\prod_{j=1}^n \big(\int_0^{\rho_1} + \int_{\rho_2}^{+\infty} \big) u^{2j+2\alpha-1} e^{-nu^{2b}} du} \\ &= \prod_{j=1}^n \frac{\big(\int_0^{n\rho_1^{2b}} + \int_{n\rho_2^{2b}}^{+\infty} \big) v^{\frac{j+\alpha}{b}-1} e^{-v} \omega((v/n)^{\frac{1}{2b}}) dv}{\big(\int_0^{n\rho_1^{2b}} + \int_{n\rho_2^{2b}}^{+\infty} \big) v^{\frac{j+\alpha}{b}-1} e^{-v} dv}. \end{aligned} \tag{1.39}$$

At this stage it is convenient to write

$$\omega(x) = \sum_{\ell=1}^{2m+1} \omega_\ell \mathbf{1}_{[0,r_\ell)}(x), \tag{1.40}$$

where $\{\omega_\ell\}_1^{2m+1}$ are given by (1.12) and $r_{2m+1} := +\infty$. Using this representation for $\omega(x)$ in (1.39), we arrive at the following representation for $\ln \mathcal{E}_n$:

$$\ln \mathcal{E}_n = \sum_{j=1}^n \ln \bigg(1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell} \bigg), \tag{1.41}$$

$$F_{n,j,\ell} := \begin{cases} \frac{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b})}{\Gamma(\frac{j+\alpha}{b}) - \gamma(\frac{j+\alpha}{b}, n\rho_2^{2b}) + \gamma(\frac{j+\alpha}{b}, n\rho_1^{2b})}, & \ell = 1, \ldots, m, \\ \frac{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b}) - \gamma(\frac{j+\alpha}{b}, n\rho_2^{2b}) + \gamma(\frac{j+\alpha}{b}, n\rho_1^{2b})}{\Gamma(\frac{j+\alpha}{b}) - \gamma(\frac{j+\alpha}{b}, n\rho_2^{2b}) + \gamma(\frac{j+\alpha}{b}, n\rho_1^{2b})}, & \ell = m+1, \ldots, 2m, \end{cases} \tag{1.42}$$

where $\gamma(a,z)$ is the incomplete gamma function defined by

$$\gamma(a,z) = \int_0^z t^{a-1} e^{-t} dt.$$

Note from (1.35) that $\mathcal{E}_n$ can also be viewed as a partition function with discontinuities approaching hard edges; see also e.g. [64, 49, 12, 62, 33, 28, 19, 23] for other works on partition functions.

In (1.41) and below, ln always denotes the principal branch of the logarithm.

Remark 1.15. *Consider the probability density*

$$\frac{1}{n! \hat{\mathcal{Z}}_n} \prod_{1 \le j < k \le n} |\hat{z}_k - \hat{z}_j|^2 |\hat{z}_k - \overline{\hat{z}_j}|^2 \prod_{j=1}^n e^{-nQ(|\hat{z}_j|^2)} |\hat{z}_j - \overline{\hat{z}_j}|^2, \qquad \operatorname{Im} \hat{z}_j > 0, \tag{1.43}$$

where $\hat{\mathcal{Z}}_n$ is the normalization constant and Q is defined in (1.3). This Pfaffian point process corresponds to the law of the eigenvalues of the quaternion matrices from a symplectic Mittag-Leffler ensemble [21]. Let $\hat{\mathrm{N}}(y) := \#\{\hat{z}_j : |\hat{z}_j| < y\}$ be the random variable that counts the number of points of (1.43) in the disk centered at 0 of radius y. By (1.3), (1.35), (1.38) and [21, Proposition 5.10], we have

$$\mathbb{E}\bigg[\prod_{j=1}^{2m} e^{u_j \hat{\mathrm{N}}(\sqrt{r_j})} \bigg] = \mathbb{E}\bigg[\prod_{j=1}^{2m} e^{u_j \mathrm{N}(r_j)} \bigg].$$

Hence all our aforementioned results for the point process (1.2) also directly yield analogous results for the point process (1.43).

2 Proof of Theorem 1.2

In this section we will derive the precise large n behavior of $\ln \mathcal{E}_n$ using (1.41). To do so, one needs the asymptotics of $\gamma(a,z)$ as $z \to +\infty$ uniformly for $a \in [\frac{1+\alpha}{b}, \frac{z}{br_1^{2b}} + \frac{\alpha}{b}]$ — these asymptotic formulas are already known and we recall them in Appendix C. In particular, there is a critical transition in the large a asymptotics of $\gamma(a,z)$ when $z \to +\infty$ is such that $\lambda = \frac{z}{a} \approx 1$. Therefore, we define critical indices $j_{k,-}, j_{k,+}$ by

$$j_{k,-} := \lceil \tfrac{bn\rho_k^{2b}}{1+\epsilon} - \alpha \rceil, \qquad j_{k,+} := \lfloor \tfrac{bn\rho_k^{2b}}{1-\epsilon} - \alpha \rfloor, \qquad k = 1, 2,$$

where $\epsilon > 0$ is independent of n and sufficiently small such that

$$\frac{b\rho_1^{2b}}{1-\epsilon} < \frac{b\rho_2^{2b}}{1+\epsilon}, \qquad \frac{b\rho_2^{2b}}{1-\epsilon} < 1, \tag{2.1}$$

and we split the sum (1.41) into six parts (following ideas from [27, 28, 30, 10, 19]):

$$\ln \mathcal{E}_n = S_0 + S_1 + S_2 + S_3 + S_4 + S_5, \tag{2.2}$$

where

$$S_0 = \sum_{j=1}^{M'} \ln\bigg(1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell}\bigg), \qquad S_1 = \sum_{j=M'+1}^{j_{1,-}-1} \ln\bigg(1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell}\bigg), \tag{2.3}$$

$$S_2 = \sum_{j=j_{1,-}}^{j_{1,+}} \ln\bigg(1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell}\bigg), \qquad S_3 = \sum_{j=j_{1,+}+1}^{j_{2,-}-1} \ln\bigg(1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell}\bigg), \tag{2.4}$$

$$S_4 = \sum_{j=j_{2,-}}^{j_{2,+}} \ln\bigg(1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell}\bigg), \qquad S_5 = \sum_{j=j_{2,+}+1}^{n} \ln\bigg(1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell}\bigg), \tag{2.5}$$

and where $M' > 0$ is an integer independent of n. For $j = 1, \ldots, n$, we define $a_j := \frac{j+\alpha}{b}$ and

$$\hat{\lambda}_{j,\ell} := \frac{bnr_\ell^{2b}}{j+\alpha}, \qquad \hat{\eta}_{j,\ell} := (\hat{\lambda}_{j,\ell} - 1)\sqrt{\frac{2(\hat{\lambda}_{j,\ell} - 1 - \ln \hat{\lambda}_{j,\ell})}{(\hat{\lambda}_{j,\ell} - 1)^2}}, \qquad \ell = 1, \ldots, 2m, \tag{2.6}$$

$$\lambda_{j,k} := \frac{bn\rho_k^{2b}}{j+\alpha}, \qquad \eta_{j,k} := (\lambda_{j,k} - 1)\sqrt{\frac{2(\lambda_{j,k} - 1 - \ln \lambda_{j,k})}{(\lambda_{j,k} - 1)^2}}, \qquad k = 1, 2. \tag{2.7}$$

With this notation, we can write

$$F_{n,j,\ell} = \begin{cases} \frac{\gamma(a_j, a_j\hat{\lambda}_{j,\ell})}{\Gamma(a_j) - \gamma(a_j, a_j\lambda_{j,2}) + \gamma(a_j, a_j\lambda_{j,1})}, & \ell = 1, \ldots, m, \\ \frac{\gamma(a_j, a_j\hat{\lambda}_{j,\ell}) - \gamma(a_j, a_j\lambda_{j,2}) + \gamma(a_j, a_j\lambda_{j,1})}{\Gamma(a_j) - \gamma(a_j, a_j\lambda_{j,2}) + \gamma(a_j, a_j\lambda_{j,1})}, & \ell = m+1, \ldots, 2m. \end{cases} \tag{2.8}$$

The sums S_2, S_3, S_4 are the most difficult to analyze: S_2 and S_4 correspond to the critical regimes $\lambda_{j,1} \approx 1$ and $\lambda_{j,2} \approx 1$, respectively, and S_3 corresponds to the intermediate regime $\frac{\lambda_{j,1}}{1-\epsilon} < 1 + \mathcal{O}(n^{-1}) < \frac{\lambda_{j,2}}{1+\epsilon}$. It turns out that only the large n behavior of S_3 involves the Jacobi θ function (see Lemma 2.7). We start our analysis with the simpler sums S_0, S_1, S_5.

Lemma 2.1. *Let $x_1, \ldots, x_{2m} \in \mathbb{R}$ be fixed. There exists $\delta > 0$ such that*

$$S_0 = M' \ln \Omega + \mathcal{O}(e^{-cn}), \qquad \text{as } n \to +\infty, \tag{2.9}$$

uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \ldots, u_{2m} \in \{z \in \mathbb{C} : |z - x_{2m}| \leq \delta\}$.

Proof. Using (1.42) and Lemma C.1 we infer that

$$F_{n,j,\ell} = 1 + \mathcal{O}(e^{-cn}), \qquad \text{as } n \to +\infty$$

uniformly for $j \in \{1, \ldots, M'\}$ and $\ell \in \{1, \ldots, 2m\}$. Hence, since $1 + \sum_{\ell=1}^{2m} \omega_\ell = e^{u_1 + \cdots + u_{2m}} = \Omega$,

$$S_0 = \sum_{j=1}^{M'} \ln\bigg(1 + \sum_{\ell=1}^{2m} \omega_\ell \big[1 + \mathcal{O}(e^{-cn})\big]\bigg) = \sum_{j=1}^{M'} \ln \Omega + \mathcal{O}(e^{-cn}), \quad \text{as } n \to +\infty.$$

Since the above error terms on the left of the second equality are independent of $u_1, \ldots, u_{2m}$, the claim follows. ☐

Lemma 2.2. *We can choose M' sufficiently large such that the following holds. For any fixed $x_1, \ldots, x_{2m} \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_1 = (j_{1,-} - M' - 1) \ln \Omega + \mathcal{O}(e^{-cn}),$$

as $n \to +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \ldots, u_{2m} \in \{z \in \mathbb{C} : |z - x_{2m}| \leq \delta\}$.

Proof. By Lemma C.2 (i), for any $\tilde{\delta} > 0$ there exist $A = A(\tilde{\delta}), C = C(\tilde{\delta}) > 0$ such that $|\frac{\gamma(a,z)}{\Gamma(a)} - 1| \leq Ce^{-\frac{a\eta^2}{2}}$ for all $a \geq A$ and all $\lambda = \frac{z}{a} \geq 1 + \tilde{\delta}$, where η is defined by (C.1). For n large enough, we have $\lambda_{j,k}, \hat{\lambda}_{j,\ell} \geq 1 + \frac{\epsilon}{2}$ for all $j \in \{M'+1, \dots, j_{1,-} - 1\}$ and all k and ℓ. Thus, let us take $\tilde{\delta} = \frac{\epsilon}{2}$ and choose M' large enough so that $a_j = \frac{j+\alpha}{b} \geq A(\frac{\epsilon}{2})$ for all $j \in \{M'+1, \dots, j_{1,-} - 1\}$. From (2.8), we infer that, as $n \to +\infty$,

$$F_{n,j,\ell} = \frac{1 + \mathcal{O}(e^{-\frac{a_j \hat{\eta}_{j,\ell}^2}{2}})}{1 + \mathcal{O}(e^{-\frac{a_j \eta_{j,2}^2}{2}} + e^{-\frac{a_j \eta_{j,1}^2}{2}})} = 1 + \mathcal{O}(e^{-cn}), \qquad \ell = 1, \dots, m,$$

$$F_{n,j,\ell} = \frac{1 + \mathcal{O}(e^{-\frac{a_j \hat{\eta}_{j,\ell}^2}{2}} + e^{-\frac{a_j \eta_{j,2}^2}{2}} + e^{-\frac{a_j \eta_{j,1}^2}{2}})}{1 + \mathcal{O}(e^{-\frac{a_j \eta_{j,2}^2}{2}} + e^{-\frac{a_j \eta_{j,1}^2}{2}})} = 1 + \mathcal{O}(e^{-cn}), \qquad \ell = m+1, \dots, 2m,$$

uniformly for $j \in \{M'+1, \dots, j_{1,-} - 1\}$, and the claim follows. ☐

Lemma 2.3. *For any fixed $x_1, \dots, x_{2m} \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_5 = \mathcal{O}(e^{-cn}), \qquad \text{as } n \to +\infty, \tag{2.10}$$

uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_{2m} \in \{z \in \mathbb{C} : |z - x_{2m}| \leq \delta\}$.

Proof. We infer from (1.42) and Lemma C.2 (ii) that

$$F_{n,j,\ell} = \mathcal{O}(e^{-cn}), \qquad \text{as } n \to +\infty$$

uniformly for $j \in \{j_{2,+} + 1, \dots, n\}$ and $\ell \in \{1, \dots, 2m\}$. Hence

$$S_5 = \sum_{j=j_{2,+}+1}^{n} \ln\left(1 + \sum_{\ell=1}^{2m} \omega_\ell \mathcal{O}(e^{-cn})\right) = \mathcal{O}(e^{-cn}), \quad \text{as } n \to +\infty.$$

Since the above error terms on the left of the second equality are independent of $u_1, \dots, u_{2m}$, the claim follows. ☐

The following lemma will be used to obtain the large n asymptotics of S_3.

Lemma 2.4. *[Taken from [28, Lemma 3.4]] Let $A = A(n), a_0 = a_0(n), B = B(n), b_0 = b_0(n)$ be bounded functions of $n \in \{1, 2, \dots\}$, such that*

$$a_n := An + a_0 \qquad \text{and} \qquad b_n := Bn + b_0$$

are integers. Assume also that $B - A$ is positive and remains bounded away from 0. Let f be a function independent of n, and which is $C^4([\min\{\frac{a_n}{n}, A\}, \max\{\frac{b_n}{n}, B\}])$ for all $n \in \{1, 2, \dots\}$. Then as $n \to +\infty$, we have

$$\begin{aligned}\sum_{j=a_n}^{b_n} f(\tfrac{j}{n}) &= n\int_A^B f(x)dx + \frac{(1-2a_0)f(A) + (1+2b_0)f(B)}{2} \\ &+ \frac{(-1+6a_0-6a_0^2)f'(A) + (1+6b_0+6b_0^2)f'(B)}{12n} + \frac{(-a_0+3a_0^2-2a_0^3)f''(A) + (b_0+3b_0^2+2b_0^3)f''(B)}{12n^2} \\ &+ \mathcal{O}\bigg(\frac{\mathfrak{m}_{A,n}(f''') + \mathfrak{m}_{B,n}(f''')}{n^3} + \sum_{j=a_n}^{b_n-1} \frac{\mathfrak{m}_{j,n}(f'''')}{n^4}\bigg),\end{aligned} \tag{2.11}$$

where, for a given function g continuous on $[\min\{\frac{a_n}{n}, A\}, \max\{\frac{b_n}{n}, B\}]$,

$$\mathfrak{m}_{A,n}(g) := \max_{x \in [\min\{\frac{a_n}{n}, A\}, \max\{\frac{a_n}{n}, A\}]} |g(x)|, \quad \mathfrak{m}_{B,n}(g) := \max_{x \in [\min\{\frac{b_n}{n}, B\}, \max\{\frac{b_n}{n}, B\}]} |g(x)|,$$

and for $j \in \{a_n, \dots, b_n - 1\}$, $\mathfrak{m}_{j,n}(g) := \max_{x \in [\frac{j}{n}, \frac{j+1}{n}]} |g(x)|$.

Following the method of [28], we define

$$\theta_{k,+}^{(n,\epsilon)} = \left(\frac{bn\rho_k^{2b}}{1-\epsilon} - \alpha\right) - \left\lfloor \frac{bn\rho_k^{2b}}{1-\epsilon} - \alpha \right\rfloor, \quad \theta_{k,-}^{(n,\epsilon)} = \left\lceil \frac{bn\rho_k^{2b}}{1+\epsilon} - \alpha \right\rceil - \left(\frac{bn\rho_k^{2b}}{1+\epsilon} - \alpha\right), \quad k = 1, 2, \tag{2.12}$$

and we split S_3 in two parts

$$S_3 = S_3^{(1)} + S_3^{(2)}, \tag{2.13}$$

where

$$S_3^{(1)} = \sum_{j=j_{1,+}+1}^{\lfloor j_\star \rfloor} \ln\left(1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell}\right), \qquad S_3^{(2)} = \sum_{j=\lfloor j_\star \rfloor+1}^{j_{2,-}-1} \ln\left(1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell}\right), \tag{2.14}$$

with

$$j_\star := n\sigma_\star - \alpha, \tag{2.15}$$

and where $\sigma_\star$ is defined in (1.5). Define also

$$\theta_\star = j_\star - \lfloor j_\star \rfloor. \tag{2.16}$$

The identity

$$\frac{a_j(\eta_{j,2}^2 - \eta_{j,1}^2)}{2} = 2(j_\star - j)\ln\left(\frac{\rho_2}{\rho_1}\right), \tag{2.17}$$

implies that $\eta_{j,2}^2 - \eta_{j,1}^2$ is positive for $j \in \{j_{1,+}+1, \ldots, \lfloor j_\star \rfloor\}$ and negative for $j \in \{\lfloor j_\star \rfloor + 1, \ldots, j_{2,-} - 1\}$.

Lemma 2.5. *We can choose M' sufficiently large such that the following holds. For any fixed $x_1, \ldots, x_{2m} \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$\begin{aligned} S_3^{(1)} &= n\int_{\frac{b\rho_1^{2b}}{1-\epsilon}}^{\sigma_\star} f_1(x)dx + \left(\alpha - \frac{1}{2} + \theta_{1,+}^{(n,\epsilon)}\right) f_1\left(\frac{b\rho_1^{2b}}{1-\epsilon}\right) + \left(\frac{1}{2} - \alpha - \theta_\star\right) f_1(\sigma_\star) + \int_{\frac{b\rho_1^{2b}}{1-\epsilon}}^{\sigma_\star} f(x)dx \\ &+ \sum_{j=0}^{+\infty} \ln\left\{1 - \frac{\mathsf{T}_0(\sigma_\star) + \hat{\mathsf{T}}_0(\sigma_\star)}{1 + \mathsf{T}_0(\sigma_\star) + \hat{\mathsf{T}}_0(b\rho_2^{2b})} \frac{\frac{\sigma_\star - b\rho_1^{2b}}{b\rho_2^{2b} - \sigma_\star}\left(\frac{\rho_2}{\rho_1}\right)^{-2(\theta_\star + j)}}{1 + \frac{\sigma_\star - b\rho_1^{2b}}{b\rho_2^{2b} - \sigma_\star}\left(\frac{\rho_2}{\rho_1}\right)^{-2(\theta_\star + j)}}\right\} + \mathcal{O}\left(\frac{(\ln n)^2}{n}\right) \end{aligned} \tag{2.18}$$

as $n \to +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \ldots, u_{2m} \in \{z \in \mathbb{C} : |z - x_{2m}| \leq \delta\}$, where $f_1(x) := \ln\left(1 + \mathsf{T}_0(x) + \hat{\mathsf{T}}_0(b\rho_2^{2b})\right)$ and f and $\mathsf{T}_j, \hat{\mathsf{T}}_j$ are defined in (1.13) and (1.11).

Proof. Using (2.1), we see that for $j \in \{j_{1,+}+1, \ldots, j_{2,-}-1\}$ and $\ell \in \{1, \ldots, m\}$, $1 - \hat{\lambda}_{j,\ell}$ and $1 - \lambda_{j,1}$ are positive and remain bounded away from 0, while for $j \in \{j_{1,+}+1, \ldots, j_{2,-}-1\}$ and $\ell \in \{m+1, \ldots, 2m\}$, $\hat{\lambda}_{j,\ell} - 1$ and $\lambda_{j,2} - 1$ are positive and remain bounded away from 0.

Hence, using Lemma C.3 (i) and (ii), as $n \to +\infty$ we have

$$\begin{aligned} \frac{\gamma(a_j, a_j\hat{\lambda}_{j,\ell})}{\Gamma(a_j)} &= \frac{e^{-\frac{a_j\hat{\eta}_{j,\ell}^2}{2}}}{\sqrt{2\pi}}\left\{\frac{1}{1-\hat{\lambda}_{j,\ell}}\frac{1}{\sqrt{a_j}} + \frac{1 + 10\hat{\lambda}_{j,\ell} + \hat{\lambda}_{j,\ell}^2}{12(\hat{\lambda}_{j,\ell}-1)^3}\frac{1}{a_j^{3/2}} + \mathcal{O}(n^{-5/2})\right\}, \qquad \ell = 1, \ldots, m, \\ \frac{\gamma(a_j, a_j\hat{\lambda}_{j,\ell})}{\Gamma(a_j)} &= 1 + \frac{e^{-\frac{a_j\hat{\eta}_{j,\ell}^2}{2}}}{\sqrt{2\pi}}\left\{\frac{-1}{\hat{\lambda}_{j,\ell}-1}\frac{1}{\sqrt{a_j}} + \frac{1 + 10\hat{\lambda}_{j,\ell} + \hat{\lambda}_{j,\ell}^2}{12(\hat{\lambda}_{j,\ell}-1)^3}\frac{1}{a_j^{3/2}} + \mathcal{O}(n^{-5/2})\right\}, \qquad \ell = m+1, \ldots, 2m, \\ \frac{\gamma(a_j, a_j\lambda_{j,1})}{\Gamma(a_j)} &= \frac{e^{-\frac{a_j\eta_{j,1}^2}{2}}}{\sqrt{2\pi}}\left\{\frac{1}{1-\lambda_{j,1}}\frac{1}{\sqrt{a_j}} + \frac{1 + 10\lambda_{j,1} + \lambda_{j,1}^2}{12(\lambda_{j,1}-1)^3}\frac{1}{a_j^{3/2}} + \mathcal{O}(n^{-5/2})\right\}, \\ \frac{\gamma(a_j, a_j\lambda_{j,2})}{\Gamma(a_j)} &= 1 + \frac{e^{-\frac{a_j\eta_{j,2}^2}{2}}}{\sqrt{2\pi}}\left\{\frac{-1}{\lambda_{j,2}-1}\frac{1}{\sqrt{a_j}} + \frac{1 + 10\lambda_{j,2} + \lambda_{j,2}^2}{12(\lambda_{j,2}-1)^3}\frac{1}{a_j^{3/2}} + \mathcal{O}(n^{-5/2})\right\}, \end{aligned} \tag{2.19}$$

uniformly for $j \in \{j_{1,+}+1,\dots,j_{2,-}-1\}$. Moreover, as $n \to +\infty$,

$$e^{-\frac{a_j \hat{\eta}_{j,\ell}^2}{2}} = e^{-\frac{a_j \eta_{j,1}^2}{2}} e^{-\frac{t_\ell}{b}(\frac{j}{n}-b\rho_1^{2b})}\bigg\{1-\frac{t_\ell^2 j/n+2t_\ell\alpha}{2bn}+\mathcal{O}(n^{-2})\bigg\}, \qquad \ell=1,\dots,m,$$
$$e^{-\frac{a_j \hat{\eta}_{j,\ell}^2}{2}} = e^{-\frac{a_j \eta_{j,2}^2}{2}} e^{-\frac{t_\ell}{b}(b\rho_2^{2b}-\frac{j}{n})}\bigg\{1-\frac{t_\ell^2 j/n-2t_\ell\alpha}{2bn}+\mathcal{O}(n^{-2})\bigg\}, \qquad \ell=m+1,\dots,2m, \tag{2.20}$$

uniformly for $j \in \{j_{1,+}+1,\dots,j_{2,-}-1\}$. Hence, after a long but direct computation, we obtain

$$F_{n,j,l} = \frac{e^{-\frac{t_\ell}{b}(j/n-b\rho_1^{2b})}}{1+\frac{j/n-b\rho_1^{2b}}{b\rho_2^{2b}-j/n}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j_\star-j)}} + t_\ell e^{-\frac{t_\ell}{b}(j/n-b\rho_1^{2b})}\frac{(j/n)^2 t_\ell+2b^2\rho_1^{2b}-t_\ell b\rho_1^{2b}j/n+2\alpha(j/n-b\rho_1^{2b})}{-2b(j/n-b\rho_1^{2b})n},$$
$$+\mathcal{O}\bigg(\frac{\big(\frac{\rho_2}{\rho_1}\big)^{-2(j_\star-j)}}{n}+\frac{1}{n^2}\bigg), \qquad \ell=1,\dots,m,$$
$$F_{n,j,l} = 1-\frac{e^{-\frac{t_\ell}{b}(b\rho_2^{2b}-j/n)}\frac{j/n-b\rho_1^{2b}}{b\rho_2^{2b}-j/n}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j_\star-j)}}{1+\frac{j/n-b\rho_1^{2b}}{b\rho_2^{2b}-j/n}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j_\star-j)}}+\mathcal{O}\bigg(\frac{\big(\frac{\rho_2}{\rho_1}\big)^{-2(j_\star-j)}}{n}+\frac{1}{n^2}\bigg), \quad \ell=m+1,\dots,2m,$$

as $n \to +\infty$ uniformly for $j \in \{j_{1,+}+1,\dots,\lfloor j_\star\rfloor\}$. Substituting the above into (2.14) yields

$$S_3^{(1)} = \sum_{j=j_{1,+}+1}^{\lfloor j_\star\rfloor} \ln\bigg\{1+\frac{\mathsf{T}_0(j/n)}{1+\frac{j/n-b\rho_1^{2b}}{b\rho_2^{2b}-j/n}(\frac{\rho_2}{\rho_1})^{-2(j_\star-j)}}+\hat{\mathsf{T}}_0(b\rho_2^{2b})-\frac{\hat{\mathsf{T}}_0(j/n)\frac{j/n-b\rho_1^{2b}}{b\rho_2^{2b}-j/n}(\frac{\rho_2}{\rho_1})^{-2(j_\star-j)}}{1+\frac{j/n-b\rho_1^{2b}}{b\rho_2^{2b}-j/n}(\frac{\rho_2}{\rho_1})^{-2(j_\star-j)}}$$
$$+\frac{(j/n)^2\mathsf{T}_2(j/n)+2b^2\rho_1^{2b}\mathsf{T}_1(j/n)-b\rho_1^{2b}j/n\;\mathsf{T}_2(j/n)+2\alpha(j/n-b\rho_1^{2b})\mathsf{T}_1(j/n)}{-2b(j/n-b\rho_1^{2b})n}$$
$$+\sum_{\ell=1}^{2m}\omega_\ell\mathcal{O}\bigg(\frac{\big(\frac{\rho_2}{\rho_1}\big)^{-2(j_\star-j)}}{n}+\frac{1}{n^2}\bigg)\bigg\}$$

as $n \to +\infty$, where the above error terms are independent of $u_1,\dots,u_{2m}$ and uniform for $j \in \{j_{1,+}+1,\dots,\lfloor j_\star\rfloor\}$. Expanding further, we obtain

$$S_3^{(1)} = \mathcal{S}_1+\mathcal{S}_2+\frac{1}{n}\mathcal{S}_3+\sum_{j=j_{1,+}+1}^{\lfloor j_\star\rfloor}\mathcal{O}\bigg(\frac{\big(\frac{\rho_2}{\rho_1}\big)^{-2(j_\star-j)}}{n}+\frac{1}{n^2}\bigg), \qquad \text{as } n\to+\infty,$$

where

$$\mathcal{S}_1 = \sum_{j=j_{1,+}+1}^{\lfloor j_\star\rfloor}\ln\Big\{1+\mathsf{T}_0(j/n)+\hat{\mathsf{T}}_0(b\rho_2^{2b})\Big\},$$
$$\mathcal{S}_2 = \sum_{j=j_{1,+}+1}^{\lfloor j_\star\rfloor}\ln\bigg\{1-\frac{\mathsf{T}_0(j/n)+\hat{\mathsf{T}}_0(j/n)}{1+\mathsf{T}_0(j/n)+\hat{\mathsf{T}}_0(b\rho_2^{2b})}\frac{\frac{j/n-b\rho_1^{2b}}{b\rho_2^{2b}-j/n}(\frac{\rho_2}{\rho_1})^{-2(j_\star-j)}}{1+\frac{j/n-b\rho_1^{2b}}{b\rho_2^{2b}-j/n}(\frac{\rho_2}{\rho_1})^{-2(j_\star-j)}}\bigg\},$$
$$\mathcal{S}_3 = \sum_{j=j_{1,+}+1}^{\lfloor j_\star\rfloor}\frac{(j/n)^2\mathsf{T}_2(j/n)+2b^2\rho_1^{2b}\mathsf{T}_1(j/n)-b\rho_1^{2b}j/n\;\mathsf{T}_2(j/n)+2\alpha(j/n-b\rho_1^{2b})\mathsf{T}_1(j/n)}{-2b(j/n-b\rho_1^{2b})(1+\mathsf{T}_0(j/n)+\hat{\mathsf{T}}_0(b\rho_2^{2b}))},$$

and we have used that

$$\bigg(1-\frac{\mathsf{T}_0(j/n)+\hat{\mathsf{T}}_0(j/n)}{1+\mathsf{T}_0(j/n)+\hat{\mathsf{T}}_0(b\rho_2^{2b})}\frac{\frac{j/n-b\rho_1^{2b}}{b\rho_2^{2b}-j/n}(\frac{\rho_2}{\rho_1})^{-2(j_\star-j)}}{1+\frac{j/n-b\rho_1^{2b}}{b\rho_2^{2b}-j/n}(\frac{\rho_2}{\rho_1})^{-2(j_\star-j)}}\bigg)^{-1} = 1+\mathcal{O}\bigg(\Big(\frac{\rho_2}{\rho_1}\Big)^{-2(j_\star-j)}\bigg)$$

uniformly for $j \in \{j_{1,+}+1,\dots,\lfloor j_\star\rfloor\}$ to obtain the expression for $\mathcal{S}_3$. Clearly, M' can be chosen large enough such that

$$\sum_{j=j_{1,+}+1}^{\lfloor j_\star\rfloor}\mathcal{O}\bigg(\frac{\big(\frac{\rho_2}{\rho_1}\big)^{-2(j_\star-j)}}{n}+\frac{1}{n^2}\bigg) = \mathcal{O}(n^{-1})+\sum_{j=j_{1,+}+1}^{\lfloor j_\star\rfloor}\mathcal{O}\bigg(\frac{\big(\frac{\rho_2}{\rho_1}\big)^{-2(j_\star-j)}}{n}\bigg)$$

$$= \mathcal{O}(n^{-1}) + \mathcal{O}(n^{-100}) + \sum_{j=\lfloor j_\star - M'\ln n\rfloor}^{\lfloor j_\star\rfloor} \mathcal{O}\bigg(\frac{\big(\frac{\rho_2}{\rho_1}\big)^{-2(j_\star-j)}}{n}\bigg) = \mathcal{O}\Big(\frac{\ln n}{n}\Big).$$

Also, we can express $\mathcal{S}_1$ and $\mathcal{S}_3$ in terms of the functions f_1 and f as follows:

$$\mathcal{S}_1 = \sum_{j=j_{1,+}+1}^{\lfloor j_\star\rfloor} f_1(j/n), \qquad \mathcal{S}_3 = \sum_{j=j_{1,+}+1}^{\lfloor j_\star\rfloor} f(j/n).$$

These sums can be expanded using Lemma 2.4 (with $A = \frac{b\rho_1^{2b}}{1-\epsilon}$, $a_0 = 1-\alpha-\theta_{1,+}^{(n,\epsilon)}$, $B=\sigma_\star$ and $b_0 = -\alpha-\theta_\star$); this gives the terms involving f_1 and f on the right-hand side of (2.18). Thus it only remains to expand $\mathcal{S}_2$ (the analysis required for $\mathcal{S}_2$ is similar but different from [28, Lemma 3.6]). Let $s_{2,j}$ be the summand of $\mathcal{S}_2$. Since $s_{2,j} = \mathcal{O}((\frac{\rho_2}{\rho_1})^{-2(j_\star-j)})$ as $n\to+\infty$ uniformly for $j\in\{j_{1,+}+1,\ldots,\lfloor j_\star\rfloor\}$, we have

$$\mathcal{S}_2 = \sum_{j=\lfloor j_\star - M'\ln n\rfloor}^{\lfloor j_\star\rfloor} s_{2,j} + \mathcal{O}(n^{-100}), \qquad \text{as } n\to+\infty,$$

provided M' is chosen large enough. Furthermore, since T_0 and $\hat{\mathsf{T}}_0$ are analytic at $\sigma_\star$,

$$\begin{aligned}
\mathcal{S}_2 &= \sum_{j=\lfloor j_\star - M'\ln n\rfloor}^{\lfloor j_\star\rfloor} \Bigg[\ln\Bigg\{1 - \frac{\mathsf{T}_0(\sigma_\star)+\hat{\mathsf{T}}_0(\sigma_\star)}{1+\mathsf{T}_0(\sigma_\star)+\hat{\mathsf{T}}_0(b\rho_2^{2b})} \frac{\frac{\sigma_\star - b\rho_1^{2b}}{b\rho_2^{2b}-\sigma_\star}(\frac{\rho_2}{\rho_1})^{-2(j_\star-j)}}{1+\frac{\sigma_\star-b\rho_1^{2b}}{b\rho_2^{2b}-\sigma_\star}(\frac{\rho_2}{\rho_1})^{-2(j_\star-j)}}\Bigg\} + \mathcal{O}(\sigma_\star - j/n)\Bigg] + \mathcal{O}(n^{-100}) \\
&= \sum_{j=\lfloor j_\star - M'\ln n\rfloor}^{\lfloor j_\star\rfloor} \ln\Bigg\{1 - \frac{\mathsf{T}_0(\sigma_\star)+\hat{\mathsf{T}}_0(\sigma_\star)}{1+\mathsf{T}_0(\sigma_\star)+\hat{\mathsf{T}}_0(b\rho_2^{2b})} \frac{\frac{\sigma_\star - b\rho_1^{2b}}{b\rho_2^{2b}-\sigma_\star}(\frac{\rho_2}{\rho_1})^{-2(j_\star-j)}}{1+\frac{\sigma_\star-b\rho_1^{2b}}{b\rho_2^{2b}-\sigma_\star}(\frac{\rho_2}{\rho_1})^{-2(j_\star-j)}}\Bigg\} + \mathcal{O}\bigg(\frac{(\ln n)^2}{n}\bigg) \\
&= \sum_{j=-\infty}^{\lfloor j_\star\rfloor} \ln\Bigg\{1 - \frac{\mathsf{T}_0(\sigma_\star)+\hat{\mathsf{T}}_0(\sigma_\star)}{1+\mathsf{T}_0(\sigma_\star)+\hat{\mathsf{T}}_0(b\rho_2^{2b})} \frac{\frac{\sigma_\star - b\rho_1^{2b}}{b\rho_2^{2b}-\sigma_\star}(\frac{\rho_2}{\rho_1})^{-2(j_\star-j)}}{1+\frac{\sigma_\star-b\rho_1^{2b}}{b\rho_2^{2b}-\sigma_\star}(\frac{\rho_2}{\rho_1})^{-2(j_\star-j)}}\Bigg\} + \mathcal{O}\bigg(\frac{(\ln n)^2}{n}\bigg)
\end{aligned}$$

as $n\to+\infty$. Changing now indices, we find

$$\mathcal{S}_2 = \sum_{j=0}^{+\infty} \ln\Bigg\{1 - \frac{\mathsf{T}_0(\sigma_\star)+\hat{\mathsf{T}}_0(\sigma_\star)}{1+\mathsf{T}_0(\sigma_\star)+\hat{\mathsf{T}}_0(b\rho_2^{2b})} \frac{\frac{\sigma_\star - b\rho_1^{2b}}{b\rho_2^{2b}-\sigma_\star}(\frac{\rho_2}{\rho_1})^{-2(\theta_\star+j)}}{1+\frac{\sigma_\star-b\rho_1^{2b}}{b\rho_2^{2b}-\sigma_\star}(\frac{\rho_2}{\rho_1})^{-2(\theta_\star+j)}}\Bigg\} + \mathcal{O}\bigg(\frac{(\ln n)^2}{n}\bigg) \qquad \text{as } n\to+\infty.$$

The claim follows after a computation combining the asymptotics of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$. ☐

Lemma 2.6. *We can choose M' sufficiently large such that the following holds. For any fixed $x_1,\ldots,x_{2m}\in\mathbb{R}$, there exists $\delta>0$ such that*

$$\begin{aligned}
S_3^{(2)} &= n\int_{\sigma_\star}^{\frac{b\rho_2^{2b}}{1+\epsilon}} \hat{f}_1(x)dx + \Big(\alpha - \frac{1}{2} + \theta_\star\Big)\hat{f}_1(\sigma_\star) + \Big(\theta_{2,-}^{(n,\epsilon)} - \alpha - \frac{1}{2}\Big)\hat{f}_1\Big(\frac{b\rho_2^{2b}}{1+\epsilon}\Big) + \int_{\sigma_\star}^{\frac{b\rho_2^{2b}}{1+\epsilon}} \hat{f}(x)dx \\
&+ \sum_{j=0}^{+\infty} \ln\Bigg\{1 + \frac{\mathsf{T}_0(\sigma_\star)+\hat{\mathsf{T}}_0(\sigma_\star)}{1-\hat{\mathsf{T}}_0(\sigma_\star)+\hat{\mathsf{T}}_0(b\rho_2^{2b})} \frac{\frac{b\rho_2^{2b}-\sigma_\star}{\sigma_\star - b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j+1-\theta_\star)}}{1+\frac{b\rho_2^{2b}-\sigma_\star}{\sigma_\star-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j+1-\theta_\star)}}\Bigg\} + \mathcal{O}\bigg(\frac{(\ln n)^2}{n}\bigg)
\end{aligned} \tag{2.21}$$

as $n\to+\infty$ uniformly for $u_1\in\{z\in\mathbb{C}: |z-x_1|\le\delta\},\ldots,u_{2m}\in\{z\in\mathbb{C}:|z-x_{2m}|\le\delta\}$, where $\hat{f}_1(x) := \ln\big(1-\hat{\mathsf{T}}_0(x)+\hat{\mathsf{T}}_0(b\rho_2^{2b})\big)$ and $\hat{f}$ and $\mathsf{T}_j, \hat{\mathsf{T}}_j$ are defined in (1.14) and (1.11).

Proof. After a long but direct computation using (2.19) and (2.20), we obtain

$$F_{n,j,l} = e^{-\frac{t_\ell}{b}(j/n - b\rho_1^{2b})} \frac{\frac{b\rho_2^{2b}-j/n}{j/n-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}{1+\frac{b\rho_2^{2b}-j/n}{j/n-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}} + \mathcal{O}\bigg(\frac{\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}{n} + \frac{1}{n^2}\bigg), \qquad \ell = 1,\ldots,m,$$

$$F_{n,j,l} = 1 - \frac{e^{-\frac{t_\ell}{b}(b\rho_2^{2b}-j/n)}}{1+\frac{b\rho_2^{2b}-j/n}{j/n-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}} + \frac{t_\ell}{n}e^{-\frac{t_\ell}{b}(b\rho_2^{2b}-j/n)}\bigg(\frac{b\rho_2^{2b}}{b\rho_2^{2b}-j/n} - \frac{\alpha}{b} + \frac{t_\ell\, j/n}{2b}\bigg)$$

$$+\mathcal{O}\bigg(\frac{\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}{n}+\frac{1}{n^2}\bigg), \qquad \ell = m+1,\ldots,2m,$$

as $n\to+\infty$ uniformly for $j\in\{\lfloor j_\star\rfloor+1,\ldots,j_{2,-}-1\}$. Hence,

$$S_3^{(2)} = \sum_{j=\lfloor j_\star\rfloor+1}^{j_{2,-}-1}\ln\bigg\{1+\mathsf{T}_0(j/n)\frac{\frac{b\rho_2^{2b}-j/n}{j/n-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}{1+\frac{b\rho_2^{2b}-j/n}{j/n-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}+\hat{\mathsf{T}}_0(b\rho_2^{2b})-\frac{\hat{\mathsf{T}}_0(j/n)}{1+\frac{b\rho_2^{2b}-j/n}{j/n-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}$$
$$+\frac{1}{n}\bigg\{\bigg(\frac{b\rho_2^{2b}}{b\rho_2^{2b}-j/n}-\frac{\alpha}{b}\bigg)\hat{\mathsf{T}}_1(j/n)+\frac{j/n}{2b}\hat{\mathsf{T}}_2(j/n)\bigg\}+\sum_{\ell=1}^{2m}\omega_\ell\mathcal{O}\bigg(\frac{\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}{n}+\frac{1}{n^2}\bigg)\bigg\}$$

as $n\to+\infty$, where the above error terms are independent of $u_1,\ldots,u_{2m}$ and uniform for $j\in\{\lfloor j_\star\rfloor+1,\ldots,j_{2,-}-1\}$. Expanding further, we obtain

$$S_3^{(2)} = \mathcal{S}_4+\mathcal{S}_5+\frac{1}{n}\mathcal{S}_6+\mathcal{O}\bigg(\frac{\ln n}{n}\bigg), \qquad \text{as } n\to+\infty,$$

where

$$\mathcal{S}_4 = \sum_{j=\lfloor j_\star\rfloor+1}^{j_{2,-}-1}\ln\Big\{1-\hat{\mathsf{T}}_0(j/n)+\hat{\mathsf{T}}_0(b\rho_2^{2b})\Big\},$$
$$\mathcal{S}_5 = \sum_{j=\lfloor j_\star\rfloor+1}^{j_{2,-}-1}\ln\bigg\{1+\frac{\mathsf{T}_0(j/n)+\hat{\mathsf{T}}_0(j/n)}{1-\hat{\mathsf{T}}_0(j/n)+\hat{\mathsf{T}}_0(b\rho_2^{2b})}\frac{\frac{b\rho_2^{2b}-j/n}{j/n-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}{1+\frac{b\rho_2^{2b}-j/n}{j/n-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}\bigg\},$$
$$\mathcal{S}_6 = \sum_{j=\lfloor j_\star\rfloor+1}^{j_{2,-}-1}\frac{\big(\frac{b\rho_2^{2b}}{b\rho_2^{2b}-j/n}-\frac{\alpha}{b}\big)\hat{\mathsf{T}}_1(j/n)+\frac{j/n}{2b}\hat{\mathsf{T}}_2(j/n)}{1-\hat{\mathsf{T}}_0(j/n)+\hat{\mathsf{T}}_0(b\rho_2^{2b})}.$$

We can express $\mathcal{S}_4$ and $\mathcal{S}_6$ in terms of the functions $\hat{f}_1$ and $\hat{f}$ as follows:

$$\mathcal{S}_4 = \sum_{j=j_{1,+}+1}^{\lfloor j_\star\rfloor}\hat{f}_1(j/n), \qquad \mathcal{S}_6 = \sum_{j=j_{1,+}+1}^{\lfloor j_\star\rfloor}\hat{f}(j/n).$$

These sums can be expanded using Lemma 2.4 (with $A=\sigma_\star$, $a_0=1-\alpha-\theta_\star$, $B=\frac{b\rho_2^{2b}}{1+\epsilon}$ and $b_0=\theta_{2,-}^{(n,\epsilon)}-\alpha-1$); this gives the terms involving $\hat{f}_1$ and $\hat{f}$ on the right-hand side of (2.21). We now turn to the analysis of $\mathcal{S}_5$. Let $s_{5,j}$ be the summand of $\mathcal{S}_5$. Since $s_{5,j}=\mathcal{O}((\frac{\rho_2}{\rho_1})^{-2(j-j_\star)})$ as $n\to+\infty$ uniformly for $j\in\{\lfloor j_\star\rfloor+1,\ldots,j_{2,-}-1\}$, we have

$$\mathcal{S}_5 = \sum_{j=\lfloor j_\star\rfloor+1}^{\lfloor j_\star+M'\ln n\rfloor}s_{5,j}+\mathcal{O}(n^{-100}), \qquad \text{as } n\to+\infty,$$

provided M' is chosen large enough. Furthermore, since T_0 and $\hat{\mathsf{T}}_0$ are analytic at $\sigma_\star$,

$$\mathcal{S}_5 = \sum_{j=\lfloor j_\star\rfloor+1}^{\lfloor j_\star+M'\ln n\rfloor}\bigg[\ln\bigg\{1+\frac{\mathsf{T}_0(\sigma_\star)+\hat{\mathsf{T}}_0(\sigma_\star)}{1-\hat{\mathsf{T}}_0(\sigma_\star)+\hat{\mathsf{T}}_0(b\rho_2^{2b})}\frac{\frac{b\rho_2^{2b}-\sigma_\star}{\sigma_\star-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}{1+\frac{b\rho_2^{2b}-\sigma_\star}{\sigma_\star-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}\bigg\}+\mathcal{O}\big(j/n-\sigma_\star\big)\bigg]+\mathcal{O}(n^{-100})$$
$$= \sum_{j=\lfloor j_\star\rfloor+1}^{\lfloor j_\star+M'\ln n\rfloor}\ln\bigg\{1+\frac{\mathsf{T}_0(\sigma_\star)+\hat{\mathsf{T}}_0(\sigma_\star)}{1-\hat{\mathsf{T}}_0(\sigma_\star)+\hat{\mathsf{T}}_0(b\rho_2^{2b})}\frac{\frac{b\rho_2^{2b}-\sigma_\star}{\sigma_\star-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}{1+\frac{b\rho_2^{2b}-\sigma_\star}{\sigma_\star-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}\bigg\}+\mathcal{O}\bigg(\frac{(\ln n)^2}{n}\bigg)$$
$$= \sum_{j=\lfloor j_\star\rfloor+1}^{+\infty}\ln\bigg\{1+\frac{\mathsf{T}_0(\sigma_\star)+\hat{\mathsf{T}}_0(\sigma_\star)}{1-\hat{\mathsf{T}}_0(\sigma_\star)+\hat{\mathsf{T}}_0(b\rho_2^{2b})}\frac{\frac{b\rho_2^{2b}-\sigma_\star}{\sigma_\star-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}{1+\frac{b\rho_2^{2b}-\sigma_\star}{\sigma_\star-b\rho_1^{2b}}\big(\frac{\rho_2}{\rho_1}\big)^{-2(j-j_\star)}}\bigg\}+\mathcal{O}\bigg(\frac{(\ln n)^2}{n}\bigg)$$

as $n \to +\infty$. Changing indices, we get

$$\mathcal{S}_5 = \sum_{j=0}^{+\infty} \ln \left\{ 1 + \frac{\mathsf{T}_0(\sigma_\star) + \hat{\mathsf{T}}_0(\sigma_\star)}{1 - \hat{\mathsf{T}}_0(\sigma_\star) + \hat{\mathsf{T}}_0(b\rho_2^{2b})} \frac{\frac{b\rho_2^{2b}-\sigma_\star}{\sigma_\star - b\rho_1^{2b}} \left(\frac{\rho_2}{\rho_1}\right)^{-2(j+1-\theta_\star)}}{1 + \frac{b\rho_2^{2b}-\sigma_\star}{\sigma_\star - b\rho_1^{2b}} \left(\frac{\rho_2}{\rho_1}\right)^{-2(j+1-\theta_\star)}} \right\} + \mathcal{O}\left(\frac{(\ln n)^2}{n}\right), \qquad \text{as } n \to +\infty.$$

The claim follows after a computation combining the asymptotics of $\mathcal{S}_4, \mathcal{S}_5, \mathcal{S}_6$. $\square$

Recall that $\sigma_1, \sigma_2 > 0$ were defined in (1.5), $\mathsf{Q} = \mathsf{Q}(\vec{t}, \vec{u})$ was defined in (1.15), and $\mathcal{F}_n$ was defined in terms of the Jacobi theta function $\theta(z;\tau)$ in (1.19).

Lemma 2.7. *We can choose M' sufficiently large such that the following holds. For any fixed $x_1, \ldots, x_{2m} \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$\begin{aligned} S_3 = n\bigg\{ \int_{\frac{b\rho_1^{2b}}{1-\epsilon}}^{\sigma_\star} f_1(x)dx + \int_{\sigma_\star}^{\frac{b\rho_2^{2b}}{1+\epsilon}} \hat{f}_1(x)dx \bigg\} + \left(\alpha - \frac{1}{2} + \theta_{1,+}^{(n,\epsilon)}\right) f_1\left(\frac{b\rho_1^{2b}}{1-\epsilon}\right) + \left(\theta_{2,-}^{(n,\epsilon)} - \alpha - \frac{1}{2}\right) \hat{f}_1\left(\frac{b\rho_2^{2b}}{1+\epsilon}\right) \\ + \left(\frac{1}{2} - \alpha - \theta_\star\right)\left(f_1(\sigma_\star) - \hat{f}_1(\sigma_\star)\right) + \int_{\frac{b\rho_1^{2b}}{1-\epsilon}}^{\sigma_\star} f(x)dx + \int_{\sigma_\star}^{\frac{b\rho_2^{2b}}{1+\epsilon}} \hat{f}(x)dx \\ + \theta_\star \ln \mathsf{Q} - \frac{\ln \mathsf{Q}}{2}\left(1 - \frac{2\ln(\sigma_2/\sigma_1) + \ln \mathsf{Q}}{2\ln(\rho_2/\rho_1)}\right) + \mathcal{F}_n + \mathcal{O}\left(\frac{(\ln n)^2}{n}\right) \end{aligned}$$

as $n \to +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \le \delta\}, \ldots, u_{2m} \in \{z \in \mathbb{C} : |z - x_{2m}| \le \delta\}$, where $f_1(x) := \ln\left(1 + \mathsf{T}_0(x) + \hat{\mathsf{T}}_0(b\rho_2^{2b})\right)$, $\hat{f}_1(x) := \ln\left(1 - \hat{\mathsf{T}}_0(x) + \hat{\mathsf{T}}_0(b\rho_2^{2b})\right)$ and T_j, $\hat{\mathsf{T}}_j$, f and $\hat{f}$ are defined in (1.11)–(1.14).

Proof. Let

$$\begin{aligned} \Sigma_n := \sum_{j=0}^{+\infty} \ln \left\{ 1 - \frac{\mathsf{T}_0(\sigma_\star) + \hat{\mathsf{T}}_0(\sigma_\star)}{1 + \mathsf{T}_0(\sigma_\star) + \hat{\mathsf{T}}_0(b\rho_2^{2b})} \frac{\frac{\sigma_\star - b\rho_1^{2b}}{b\rho_2^{2b}-\sigma_\star}(\frac{\rho_2}{\rho_1})^{-2(\theta_\star+j)}}{1 + \frac{\sigma_\star - b\rho_1^{2b}}{b\rho_2^{2b}-\sigma_\star}(\frac{\rho_2}{\rho_1})^{-2(\theta_\star+j)}} \right\} \\ + \sum_{j=0}^{+\infty} \ln \left\{ 1 + \frac{\mathsf{T}_0(\sigma_\star) + \hat{\mathsf{T}}_0(\sigma_\star)}{1 - \hat{\mathsf{T}}_0(\sigma_\star) + \hat{\mathsf{T}}_0(b\rho_2^{2b})} \frac{\frac{b\rho_2^{2b}-\sigma_\star}{\sigma_\star - b\rho_1^{2b}} \left(\frac{\rho_2}{\rho_1}\right)^{-2(j+1-\theta_\star)}}{1 + \frac{b\rho_2^{2b}-\sigma_\star}{\sigma_\star - b\rho_1^{2b}} \left(\frac{\rho_2}{\rho_1}\right)^{-2(j+1-\theta_\star)}} \right\}. \end{aligned}$$

The desired conclusion is a direct consequence of Lemmas 2.5 and 2.6 if we can show that

$$\Sigma_n = \theta_\star \ln \mathsf{Q} - \frac{\ln \mathsf{Q}}{2}\left(1 - \frac{2\ln(\sigma_2/\sigma_1) + \ln \mathsf{Q}}{2\ln(\rho_2/\rho_1)}\right) + \mathcal{F}_n. \tag{2.22}$$

To show (2.22), we introduce the short-hand notation $v := \frac{b\rho_2^{2b}-\sigma_\star}{\sigma_\star - b\rho_1^{2b}} = \sigma_2/\sigma_1 > 0$ and $w := \rho_1/\rho_2 \in (0,1)$. Then we can write

$$\Sigma_n = \sum_{j=0}^{+\infty} \ln \left\{ 1 - \frac{\mathsf{Q}-1}{\mathsf{Q}} \frac{1}{v w^{-2(\theta_\star+j)} + 1} \right\} + \sum_{j=0}^{+\infty} \ln \left\{ 1 + (\mathsf{Q}-1) \frac{1}{\frac{w^{-2(j+1-\theta_\star)}}{v} + 1} \right\}.$$

Combining the logarithms in the two sums and then letting $j = \ell - 1$, this becomes

$$\Sigma_n = \sum_{j=0}^{+\infty} \ln \frac{(1 + w^{2j+2}\frac{v\mathsf{Q}}{w^{2\theta_\star}})(1 + w^{2j}\frac{w^{2\theta_\star}}{v\mathsf{Q}})}{(1 + w^{2j+2}\frac{v}{w^{2\theta_\star}})(1 + w^{2j}\frac{w^{2\theta_\star}}{v})} = \sum_{\ell=1}^{+\infty} \ln \frac{(1 + w^{2\ell-1}\frac{v\mathsf{Q}}{w^{2\theta_\star-1}})(1 + w^{2\ell-1}\frac{w^{2\theta_\star-1}}{v\mathsf{Q}})}{(1 + w^{2\ell-1}\frac{v}{w^{2\theta_\star-1}})(1 + w^{2\ell-1}\frac{w^{2\theta_\star-1}}{v})}.$$

Using the Jacobi triple product identity

$$\theta(z;\tau) = \prod_{\ell=1}^{\infty}(1 - e^{2\ell\pi i\tau})(1 + e^{(2\ell-1)\pi i\tau + 2\pi iz})(1 + e^{(2\ell-1)\pi i\tau - 2\pi iz}),$$

we obtain

$$\Sigma_n = \ln \frac{\theta(\frac{1}{2\pi i}\ln(\frac{v\mathsf{Q}}{w^{2\theta_\star-1}}); \frac{\ln w}{\pi i})}{\theta(\frac{1}{2\pi i}\ln(\frac{v}{w^{2\theta_\star-1}}); \frac{\ln w}{\pi i})},$$

where $\mathrm{Im}\,(\frac{\ln w}{\pi i})>0$ because $w\in(0,1)$. With the help of the Jacobi identity

$$\theta(z;\tau)=(-i\tau)^{-1/2}e^{-\frac{\pi i z^2}{\tau}}\theta\Big(\frac{z}{\tau};-\frac{1}{\tau}\Big)$$

this can be written as

$$\Sigma_n=\theta_\star\ln\mathsf{Q}-\frac{\ln\mathsf{Q}}{2}\bigg(1+\frac{2\ln v+\ln\mathsf{Q}}{2\ln w}\bigg)+\ln\frac{\theta(\frac{\ln(v\mathsf{Q})}{2\ln w}-\theta_\star+\frac{1}{2};-\frac{\pi i}{\ln w})}{\theta(\frac{\ln v}{2\ln w}-\theta_\star+\frac{1}{2};-\frac{\pi i}{\ln w})}.$$

Using the periodicity property $\theta(z+1;\tau)=\theta(z;\tau)$, we can replace $\theta_\star=j_\star-\lfloor j_\star\rfloor$ by $j_\star=n\sigma_\star-\alpha$ and $+\frac{1}{2}$ by $-\frac{1}{2}$ inside θ. Using also $\theta(-z;\tau)=\theta(z;\tau)$, $v=\sigma_2/\sigma_1$ and $w=\rho_1/\rho_2$, we arrive at (2.22). $\square$

We now turn our attention to S_2 and S_4. Let $M:=n^{\frac{1}{10}}$. We split S_{2k}, $k=1,2$, as follows:

$$S_{2k}=S_{2k}^{(1)}+S_{2k}^{(2)}+S_{2k}^{(3)},\qquad S_{2k}^{(v)}:=\sum_{j:\lambda_{j,k}\in I_v}\ln\bigg(1+\sum_{\ell=1}^{2m}\omega_\ell F_{n,j,\ell}\bigg),\qquad v=1,2,3,\tag{2.23}$$

where

$$I_1=[1-\epsilon,1-\tfrac{M}{\sqrt{n}}),\qquad I_2=[1-\tfrac{M}{\sqrt{n}},1+\tfrac{M}{\sqrt{n}}],\qquad I_3=(1+\tfrac{M}{\sqrt{n}},1+\epsilon].$$

From (2.23), we see that the large n asymptotics of $\{S_{2k}^{(v)}\}_{v=1,2,3}$ involve the asymptotics of $\gamma(a,z)$ when $a\to+\infty$, $z\to+\infty$ with $\lambda=\frac{z}{a}\in[1-\epsilon,1+\epsilon]$. These sums can also be rewritten using

$$\sum_{j:\lambda_{j,k}\in I_3}=\sum_{j=j_{k,-}}^{g_{k,-}-1},\qquad \sum_{j:\lambda_{j,k}\in I_2}=\sum_{j=g_{k,-}}^{g_{k,+}},\qquad \sum_{j:\lambda_{j,k}\in I_1}=\sum_{j=g_{k,+}+1}^{j_{k,+}},\tag{2.24}$$

where $g_{k,-}:=\lceil\frac{bn\rho_k^{2b}}{1+\frac{M}{\sqrt{n}}}-\alpha\rceil$ and $g_{k,+}:=\lfloor\frac{bn\rho_k^{2b}}{1-\frac{M}{\sqrt{n}}}-\alpha\rfloor$ for $k=1,2$. Let us also define

$$\theta_{k,-}^{(n,M)}:=g_{k,-}-\bigg(\frac{bn\rho_k^{2b}}{1+\frac{M}{\sqrt{n}}}-\alpha\bigg)=\bigg\lceil\frac{bn\rho_k^{2b}}{1+\frac{M}{\sqrt{n}}}-\alpha\bigg\rceil-\bigg(\frac{bn\rho_k^{2b}}{1+\frac{M}{\sqrt{n}}}-\alpha\bigg),$$
$$\theta_{k,+}^{(n,M)}:=\bigg(\frac{bn\rho_k^{2b}}{1-\frac{M}{\sqrt{n}}}-\alpha\bigg)-g_{k,+}=\bigg(\frac{bn\rho_k^{2b}}{1-\frac{M}{\sqrt{n}}}-\alpha\bigg)-\bigg\lfloor\frac{bn\rho_k^{2b}}{1-\frac{M}{\sqrt{n}}}-\alpha\bigg\rfloor.$$

Clearly, $\theta_{k,-}^{(n,M)},\theta_{k,+}^{(n,M)}\in[0,1)$.

Lemma 2.8. *For any fixed $x_1,\ldots,x_{2m}\in\mathbb{R}$, there exists $\delta>0$ such that*

$$S_2^{(3)}=\Big(b\rho_1^{2b}n-j_{1,-}-bM\rho_1^{2b}\sqrt{n}+bM^2\rho_1^{2b}-\alpha+\theta_{1,-}^{(n,M)}-bM^3\rho_1^{2b}n^{-\frac{1}{2}}\Big)\ln\Omega+\mathcal{O}(M^4n^{-1}),$$

as $n\to+\infty$ uniformly for $u_1\in\{z\in\mathbb{C}:|z-x_1|\leq\delta\},\ldots,u_{2m}\in\{z\in\mathbb{C}:|z-x_{2m}|\leq\delta\}$.

Proof. For all sufficiently large n, and uniformly for $j\in\{j:\lambda_{j,1}\in I_3\}$, we have that $\eta_{j,2},\hat{\eta}_{j,\ell}$, $\ell=m+1,\ldots,2m$ are positive and bounded away from 0, and furthermore

$$\min_{\ell\in\{1,\ldots,m\}}\{\eta_{j,1},\hat{\eta}_{j,\ell}\}\geq\tfrac{M}{\sqrt{n}}+\mathcal{O}(\tfrac{1}{\sqrt{n}}),\qquad -\sqrt{a_j/2}\min_{\ell\in\{1,\ldots,m\}}\{\eta_{j,1},\hat{\eta}_{j,\ell}\}\leq-\tfrac{Mr^b}{\sqrt{2}}+\mathcal{O}(1).$$

Using Lemma C.3 and the fact that $M=n^{\frac{1}{10}}$, we infer that

$$F_{n,j,\ell}=\frac{1+\mathcal{O}(e^{-\frac{a_j\hat{\eta}_{j,\ell}^2}{2}})}{1+\mathcal{O}(e^{-\frac{a_j\eta_{j,2}^2}{2}}+e^{-\frac{a_j\eta_{j,1}^2}{2}})}=1+\mathcal{O}(e^{-cn^{1/5}}),\qquad \ell=1,\ldots,m,$$
$$F_{n,j,\ell}=\frac{1+\mathcal{O}(e^{-\frac{a_j\hat{\eta}_{j,\ell}^2}{2}}+e^{-\frac{a_j\eta_{j,2}^2}{2}}+e^{-\frac{a_j\eta_{j,1}^2}{2}})}{1+\mathcal{O}(e^{-\frac{a_j\eta_{j,2}^2}{2}}+e^{-\frac{a_j\eta_{j,1}^2}{2}})}=1+\mathcal{O}(e^{-cn^{1/5}}),\qquad \ell=m+1,\ldots,2m,$$

uniformly for $j \in \{j_{1,-}, \dots, g_{1,-} - 1\}$. Hence, by (2.24),

$$S_2^{(3)} = \sum_{j=j_{1,-}}^{g_{1,-}-1} \ln\left(1 + \sum_{\ell=1}^{2m} \omega_\ell \big[1 + \mathcal{O}(e^{-cn^{1/5}})\big]\right) = (g_{1,-} - j_{1,-}) \ln \Omega + \mathcal{O}(ne^{-cn^{1/5}}), \quad \text{as } n \to +\infty.$$

In the above, the error terms before the second equality are independent of $u_1, \dots, u_{2m}$, so the expansion (see [27, Lemma 2.4])

$$g_{1,-} - j_{1,-} = b\rho_1^{2b} n - j_{1,-} - bM\rho_1^{2b}\sqrt{n} + bM^2\rho_1^{2b} - \alpha + \theta_{1,-}^{(n,M)} - bM^3\rho_1^{2b} n^{-\frac{1}{2}} + \mathcal{O}(M^4 n^{-1})$$

yields the claim. □

Lemma 2.9. *For any fixed $x_1, \dots, x_{2m} \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_4^{(1)} = \mathcal{O}(n^{-100}),$$

as $n \to +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_{2m} \in \{z \in \mathbb{C} : |z - x_{2m}| \leq \delta\}$.

Proof. For all sufficiently large n, and uniformly for $j \in \{j : \lambda_{j,2} \in I_1\}$, we have that $\eta_{j,1}, \hat{\eta}_{j,\ell}$, $\ell = 1, \dots, m$ are negative and bounded away from 0, and furthermore

$$\max_{\ell \in \{m+1,\dots,2m\}} \{\eta_{j,2}, \hat{\eta}_{j,\ell}\} \leq -\tfrac{M}{\sqrt{n}} + \mathcal{O}(\tfrac{1}{\sqrt{n}}), \qquad -\sqrt{a_j/2} \max_{\ell \in \{m+1,\dots,2m\}} \{\eta_{j,2}, \hat{\eta}_{j,\ell}\} \geq \tfrac{Mr^b}{\sqrt{2}} + \mathcal{O}(1).$$

Using Lemma C.3 and the fact that $M = n^{\frac{1}{10}}$, we infer that

$$F_{n,j,\ell} = \mathcal{O}(e^{-cn^{1/5}}) = \mathcal{O}(n^{-101}), \qquad \ell = 1, \dots, 2m,$$

uniformly for $j \in \{g_{2,+} + 1, \dots, j_{2,+}\}$. Hence, by (2.24),

$$S_4^{(1)} = \sum_{j=g_{2,+}+1}^{j_{2,+}} \ln\left(1 + \sum_{\ell=1}^{2m} \omega_\ell \mathcal{O}(n^{-101})\right) = \mathcal{O}(n^{-100}), \quad \text{as } n \to +\infty,$$

and the claim follows. □

Lemma 2.10. *For any fixed $x_1, \dots, x_{2m} \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_2^{(1)} = D_1^{(\epsilon)} n + D_2^{(M)}\sqrt{n} + D_3 \ln n + D_4^{(n,\epsilon,M)} + \frac{D_5^{(n,M)}}{\sqrt{n}} + \mathcal{O}\left(\frac{M^4}{n} + \frac{1}{\sqrt{n}M} + \frac{1}{M^6} + \frac{\sqrt{n}}{M^{11}}\right),$$

as $n \to +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \dots, u_{2m} \in \{z \in \mathbb{C} : |z - x_{2m}| \leq \delta\}$, where

$$D_1^{(\epsilon)} = \int_{b\rho_1^{2b}}^{\frac{b\rho_1^{2b}}{1-\epsilon}} f_1(x)dx, \qquad D_2^{(M)} = -b\rho_1^{2b} f_1(b\rho_1^{2b})M, \qquad D_3 = \frac{-b\rho_1^{2b}\mathsf{T}_1(b\rho_1^{2b})}{2(1 + \mathsf{T}_0(b\rho_1^{2b}) + \hat{\mathsf{T}}_0(b\rho_2^{2b}))},$$

$$D_4^{(n,\epsilon,M)} = -b\rho_1^{2b}M^2\left(f_1(b\rho_1^{2b}) + \frac{b\rho_1^{2b}}{2} f_1'(b\rho_1^{2b})\right) - \frac{b\rho_1^{2b}\mathsf{T}_1(b\rho_1^{2b})}{1 + \mathsf{T}_0(b\rho_1^{2b}) + \hat{\mathsf{T}}_0(b\rho_2^{2b})} \ln\left(\frac{\epsilon}{M(1-\epsilon)}\right)$$
$$+ \int_{b\rho_1^{2b}}^{\frac{b\rho_1^{2b}}{1-\epsilon}} \left\{ f(x) + \frac{b\rho_1^{2b}\mathsf{T}_1(b\rho_1^{2b})}{(1 + \mathsf{T}_0(b\rho_1^{2b}) + \hat{\mathsf{T}}_0(b\rho_2^{2b}))(x - b\rho_1^{2b})} \right\} dx + \left(\alpha - \frac{1}{2} + \theta_{1,+}^{(n,M)}\right) f_1(b\rho_1^{2b})$$
$$+ \left(\frac{1}{2} - \alpha - \theta_{1,+}^{(n,\epsilon)}\right) f_1\left(\frac{b\rho_1^{2b}}{1-\epsilon}\right) + \frac{b\mathsf{T}_1(b\rho_1^{2b})}{M^2(1 + \mathsf{T}_0(b\rho_1^{2b}) + \hat{\mathsf{T}}_0(b\rho_2^{2b}))} + \frac{-5b\mathsf{T}_1(b\rho_1^{2b})}{2\rho_1^{2b}M^4(1 + \mathsf{T}_0(b\rho_1^{2b}) + \hat{\mathsf{T}}_0(b\rho_2^{2b}))},$$

$$D_5^{(n,M)} = -M^3 b\rho_1^{2b}\left(f_1(b\rho_1^{2b}) + b\rho_1^{2b} f_1'(b\rho_1^{2b}) + \frac{(b\rho_1^{2b})^2}{6} f_1''(b\rho_1^{2b})\right) + Mb\rho_1^{2b} f_1'(b\rho_1^{2b})\left(\alpha - \frac{1}{2} + \theta_{1,+}^{(n,M)}\right)$$
$$+ M\left(\frac{(b+\alpha)\rho_1^{2b}\mathsf{T}_1(b\rho_1^{2b})}{1 + \mathsf{T}_0(b\rho_1^{2b}) + \hat{\mathsf{T}}_0(b\rho_2^{2b})} - \frac{b\rho_1^{4b}\mathsf{T}_2(b\rho_1^{2b})}{2(1 + \mathsf{T}_0(b\rho_1^{2b}) + \hat{\mathsf{T}}_0(b\rho_2^{2b}))} + \frac{b\rho_1^{4b}\mathsf{T}_1(b\rho_1^{2b})^2}{(1 + \mathsf{T}_0(b\rho_1^{2b}) + \hat{\mathsf{T}}_0(b\rho_2^{2b}))^2}\right),$$

where f_1 and f are as in the statement of Lemma 2.7.

Proof. Using (2.8) and Lemma C.2, we obtain

$$\begin{aligned}S_2^{(1)} &= \sum_{j=g_{1,+}+1}^{j_{1,+}} \ln\bigg(1+\sum_{\ell=1}^{m}\omega_\ell\bigg\{\frac{\frac{1}{2}\mathrm{erfc}\big(-\hat{\eta}_{j,\ell}\sqrt{a_j/2}\big)-R_{a_j}(\hat{\eta}_{j,\ell})}{\frac{1}{2}\mathrm{erfc}\big(-\eta_{j,1}\sqrt{a_j/2}\big)-R_{a_j}(\eta_{j,1})}+\mathcal{O}(e^{-cn^{\frac{1}{5}}})\bigg\}+\sum_{\ell=m+1}^{2m}\omega_\ell(1+\mathcal{O}(e^{-cn^{\frac{1}{5}}}))\bigg) \\ &= \sum_{j=g_{1,+}+1}^{j_{1,+}} \ln\left(1+\sum_{\ell=1}^{m}\omega_\ell\frac{\frac{1}{2}\mathrm{erfc}\big(-\hat{\eta}_{j,\ell}\sqrt{a_j/2}\big)-R_{a_j}(\hat{\eta}_{j,\ell})}{\frac{1}{2}\mathrm{erfc}\big(-\eta_{j,1}\sqrt{a_j/2}\big)-R_{a_j}(\eta_{j,1})}+\hat{\mathsf{T}}_0(b\rho_2^{2b})\right)+\mathcal{O}(e^{-cn^{1/5}}). \qquad (2.25)\end{aligned}$$

The computation and the resulting formulas appearing in the rest of this proof are the same as in the proof of [10, Lemma 2.6], except that ρ in [10] corresponds to ρ_1 here and the extra term $\hat{\mathsf{T}}_0(b\rho_2^{2b})$ in the above equation implies that $1+\mathsf{T}_0(x)$ in [10, Lemma 2.6] is replaced here by $1+\mathsf{T}_0(x)+\hat{\mathsf{T}}_0(b\rho_2^{2b})$. We nevertheless provide the details here for completeness. Using (C.3), we find

$$\begin{aligned}S_2^{(1)} = \sum_{j=g_{1,+}+1}^{j_{1,+}} \bigg(& f_1(j/n)+\frac{1}{n}f(j/n)+\frac{1}{n^2}\frac{2b^3\rho_1^{4b}\mathsf{T}_1(j/n)}{(1+\mathsf{T}_0(j/n)+\hat{\mathsf{T}}_0(b\rho_2^{2b}))(j/n-b\rho_1^{2b})^3} \\ &+\frac{1}{n^3}\frac{-10b^5\rho_1^{6b}\mathsf{T}_1(j/n)}{(1+\mathsf{T}_0(j/n)+\hat{\mathsf{T}}_0(b\rho_2^{2b}))(j/n-b\rho_1^{2b})^5} \\ &+\mathcal{O}\bigg(\frac{n^{-2}}{(j/n-b\rho_1^{2b})^2}+\frac{n^{-3}}{(j/n-b\rho_1^{2b})^4}+\frac{n^{-4}}{(j/n-b\rho_1^{2b})^7}+\frac{n^{-6}}{(j/n-b\rho_1^{2b})^{12}}\bigg)\bigg).\end{aligned}$$

Note that

$$\begin{aligned}&\sum_{j=g_{1,+}+1}^{j_{1,+}} \mathcal{O}\bigg(\frac{n^{-2}}{(j/n-b\rho_1^{2b})^2}+\frac{n^{-3}}{(j/n-b\rho_1^{2b})^4}+\frac{n^{-4}}{(j/n-b\rho_1^{2b})^7}+\frac{n^{-6}}{(j/n-b\rho_1^{2b})^{12}}\bigg) \\ &= \mathcal{O}\bigg(\frac{1}{M\sqrt{n}}+\frac{1}{M^3\sqrt{n}}+\frac{1}{M^6}+\frac{\sqrt{n}}{M^{11}}\bigg).\end{aligned}$$

Also, using Lemma 2.4 (with $A=\frac{b\rho_1^{2b}}{1-\frac{M}{\sqrt{n}}}$, $a_0=1-\alpha-\theta_{1,+}^{(n,M)}$, $B=\frac{b\rho_1^{2b}}{1-\epsilon}$ and $b_0=-\alpha-\theta_{1,+}^{(n,\epsilon)}$), we get

$$\sum_{j=g_{1,+}+1}^{j_{1,+}} f_1(j/n) = n\int_{\frac{b\rho_1^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho_1^{2b}}{1-\epsilon}} f_1(x)dx+\big(\alpha-\tfrac{1}{2}+\theta_{1,+}^{(n,M)}\big)f_1\big(\tfrac{b\rho_1^{2b}}{1-\frac{M}{\sqrt{n}}}\big)+\big(\tfrac{1}{2}-\alpha-\theta_{1,+}^{(n,\epsilon)}\big)f_1\big(\tfrac{b\rho_1^{2b}}{1-\epsilon}\big)+\mathcal{O}(n^{-1}),$$

$$\frac{1}{n}\sum_{j=g_{1,+}+1}^{j_{1,+}} f(j/n) = \int_{\frac{b\rho_1^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho_1^{2b}}{1-\epsilon}} f(x)dx+\mathcal{O}\bigg(\frac{1}{M\sqrt{n}}\bigg),$$

$$\frac{1}{n^2}\sum_{j=g_{1,+}+1}^{j_{1,+}} \frac{2b^3\rho_1^{4b}\mathsf{T}_1(j/n)(j/n-b\rho_1^{2b})^{-3}}{1+\mathsf{T}_0(j/n)+\hat{\mathsf{T}}_0(b\rho_2^{2b})} = \frac{1}{n}\int_{\frac{b\rho_1^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho_1^{2b}}{1-\epsilon}} \frac{2b^3\rho_1^{4b}\mathsf{T}_1(x)(x-b\rho_1^{2b})^{-3}}{1+\mathsf{T}_0(x)+\hat{\mathsf{T}}_0(b\rho_2^{2b})}dx+\mathcal{O}\bigg(\frac{1}{M^3\sqrt{n}}\bigg),$$

$$\frac{1}{n^3}\sum_{j=g_{1,+}+1}^{j_{1,+}} \frac{-10b^5\rho_1^{6b}\mathsf{T}_1(j/n)(j/n-b\rho_1^{2b})^{-5}}{1+\mathsf{T}_0(j/n)+\hat{\mathsf{T}}_0(b\rho_2^{2b})} = \frac{1}{n^2}\int_{\frac{b\rho_1^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho_1^{2b}}{1-\epsilon}} \frac{-10b^5\rho_1^{6b}\mathsf{T}_1(x)(x-b\rho_1^{2b})^{-5}}{1+\mathsf{T}_0(x)+\hat{\mathsf{T}}_0(b\rho_2^{2b})}dx+\mathcal{O}\bigg(\frac{1}{M^5\sqrt{n}}\bigg).$$

Furthermore, by a direct analysis,

$$\begin{aligned}n\int_{\frac{b\rho_1^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho_1^{2b}}{1-\epsilon}} f_1(x)dx &= n\int_{b\rho_1^{2b}}^{\frac{b\rho_1^{2b}}{1-\epsilon}} f_1(x)dx-b\rho_1^{2b}f_1(b\rho_1^{2b})M\sqrt{n}-M^2b\rho_1^{2b}\big(f_1(b\rho_1^{2b})+\frac{b\rho_1^{2b}}{2}f_1'(b\rho_1^{2b})\big) \\ &\quad -\frac{M^3}{\sqrt{n}}b\rho_1^{2b}\Big(f_1(b\rho_1^{2b})+b\rho_1^{2b}f_1'(b\rho_1^{2b})+\frac{(b\rho_1^{2b})^2}{6}f_1''(b\rho_1^{2b})\Big)+\mathcal{O}\bigg(\frac{M^4}{n}\bigg),\end{aligned}$$

$$f_1\big(\tfrac{b\rho_1^{2b}}{1-\frac{M}{\sqrt{n}}}\big) = f_1(b\rho_1^{2b})+\frac{M}{\sqrt{n}}b\rho_1^{2b}f_1'(b\rho_1^{2b})+\mathcal{O}\bigg(\frac{M^2}{n}\bigg),$$

$$\int_{\frac{b\rho_1^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho_1^{2b}}{1-\epsilon}} f(x)dx = \int_{b\rho_1^{2b}}^{\frac{b\rho_1^{2b}}{1-\epsilon}} \bigg\{f(x)+\frac{b\rho_1^{2b}\mathsf{T}_1(b\rho_1^{2b})}{(1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b}))(x-b\rho_1^{2b})}\bigg\}dx-\frac{b\rho_1^{2b}\mathsf{T}_1(b\rho_1^{2b})}{2(1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b}))}\ln n$$

$$
-\frac{b\rho_1^{2b}\mathsf{T}_1(b\rho_1^{2b})}{1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b})}\ln\frac{\epsilon}{M(1-\epsilon)}+\frac{M}{\sqrt{n}}\bigg\{\frac{(b+\alpha)\rho_1^{2b}\mathsf{T}_1(b\rho_1^{2b})}{1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b})}-\frac{b\rho_1^{4b}\mathsf{T}_2(b\rho_1^{2b})}{2(1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b}))}
$$

$$
+\frac{b\rho_1^{4b}\mathsf{T}_1(b\rho_1^{2b})^2}{(1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b}))^2}\bigg\}+\mathcal{O}\bigg(\frac{M^2}{n}\bigg),
$$

$$
\frac{1}{n}\int_{\frac{b\rho_1^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho_1^{2b}}{1-\epsilon}}\frac{2b^3\rho_1^{4b}\mathsf{T}_1(x)}{(1+\mathsf{T}_0(x)+\hat{\mathsf{T}}_0(b\rho_2^{2b}))(x-b\rho_1^{2b})^3}dx=\frac{b\mathsf{T}_1(b\rho_1^{2b})}{M^2(1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b}))}+\mathcal{O}\bigg(\frac{1}{M\sqrt{n}}\bigg),
$$

$$
\frac{1}{n^2}\int_{\frac{b\rho_1^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho_1^{2b}}{1-\epsilon}}\frac{-10b^5\rho_1^{6b}\mathsf{T}_1(x)}{(1+\mathsf{T}_0(x)+\hat{\mathsf{T}}_0(b\rho_2^{2b}))(x-b\rho_1^{2b})^5}dx=\frac{-5b\mathsf{T}_1(b\rho_1^{2b})}{2\rho_1^{2b}M^4(1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b}))}+\mathcal{O}\bigg(\frac{1}{M^3\sqrt{n}}\bigg).
$$

The claim now follows after a computation. □

Lemma 2.11. *For any fixed $x_1,\ldots,x_{2m}\in\mathbb{R}$, there exists $\delta>0$ such that*

$$
S_4^{(3)}=\hat{D}_1^{(\epsilon)}n+\hat{D}_2^{(M)}\sqrt{n}+\hat{D}_3\ln n+\hat{D}_4^{(n,\epsilon,M)}+\frac{\hat{D}_5^{(n,M)}}{\sqrt{n}}+\mathcal{O}\bigg(\frac{M^4}{n}+\frac{1}{\sqrt{n}M}+\frac{1}{M^6}+\frac{\sqrt{n}}{M^{11}}\bigg),
$$

as $n\to+\infty$ uniformly for $u_1\in\{z\in\mathbb{C}:|z-x_1|\leq\delta\},\ldots,u_{2m}\in\{z\in\mathbb{C}:|z-x_{2m}|\leq\delta\}$, where

$$
\hat{D}_1^{(\epsilon)}=\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{b\rho_2^{2b}}\hat{f}_1(x)dx,\qquad \hat{D}_2^{(M)}=0,\qquad \hat{D}_3=\frac{b\rho_2^{2b}\hat{\mathsf{T}}_1(b\rho_2^{2b})}{2},
$$

$$
\hat{D}_4^{(n,\epsilon,M)}=M^2\frac{b^2\rho_2^{4b}}{2}\hat{f}_1'(b\rho_2^{2b})+b\rho_2^{2b}\hat{\mathsf{T}}_1(b\rho_2^{2b})\ln\bigg(\frac{\epsilon}{M(1+\epsilon)}\bigg)+\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{b\rho_2^{2b}}\bigg\{\hat{f}(x)-\frac{b\rho_2^{2b}\hat{\mathsf{T}}_1(b\rho_2^{2b})}{b\rho_2^{2b}-x}\bigg\}dx
$$

$$
+\bigg(\frac{1}{2}+\alpha-\theta_{2,-}^{(n,\epsilon)}\bigg)\hat{f}_1\bigg(\frac{b\rho_2^{2b}}{1+\epsilon}\bigg)-\frac{b\hat{\mathsf{T}}_1(b\rho_2^{2b})}{M^2}+\frac{5b\hat{\mathsf{T}}_1(b\rho_2^{2b})}{2\rho_2^{2b}M^4},
$$

$$
\hat{D}_5^{(n,M)}=-M^3b^2\rho_2^{4b}\bigg(\hat{f}_1'(b\rho_2^{2b})+\frac{b\rho_2^{2b}}{6}\hat{f}_1''(b\rho_2^{2b})\bigg)+Mb\rho_2^{2b}\hat{f}_1'(b\rho_2^{2b})\bigg(\frac{1}{2}+\alpha-\theta_{2,-}^{(n,M)}\bigg)
$$

$$
+M\bigg((b+\alpha)\rho_2^{2b}\hat{\mathsf{T}}_1(b\rho_2^{2b})+\frac{b\rho_2^{4b}}{2}\hat{\mathsf{T}}_2(b\rho_2^{2b})+b\rho_2^{4b}\hat{\mathsf{T}}_1(b\rho_2^{2b})^2\bigg),
$$

where $\hat{f}_1$ and $\hat{f}$ are as in the statement of Lemma 2.7.

Proof. Using Lemma C.2, we obtain

$$
S_4^{(3)}=\sum_{j=j_{2,-}}^{g_{2,-}-1}\ln\bigg(1+\sum_{\ell=1}^{m}\omega_\ell\mathcal{O}(e^{-cn})
$$

$$
+\sum_{\ell=m+1}^{2m}\omega_\ell\bigg\{\frac{\frac{1}{2}\mathrm{erfc}\big(-\hat{\eta}_{j,\ell}\sqrt{a_j/2}\big)-R_{a_j}(\hat{\eta}_{j,\ell})-\big[\frac{1}{2}\mathrm{erfc}\big(-\eta_{j,2}\sqrt{a_j/2}\big)-R_{a_j}(\eta_{j,2})\big]}{1-\big[\frac{1}{2}\mathrm{erfc}\big(-\eta_{j,2}\sqrt{a_j/2}\big)-R_{a_j}(\eta_{j,2})\big]}+\mathcal{O}(e^{-cn})\bigg\}\bigg)
$$

$$
=\mathcal{O}(e^{-cn})+\sum_{j=j_{2,-}}^{g_{2,-}-1}\ln\bigg(1
$$

$$
+\sum_{\ell=m+1}^{2m}\omega_\ell\frac{\frac{1}{2}\mathrm{erfc}\big(-\hat{\eta}_{j,\ell}\sqrt{a_j/2}\big)-R_{a_j}(\hat{\eta}_{j,\ell})-\big[\frac{1}{2}\mathrm{erfc}\big(-\eta_{j,2}\sqrt{a_j/2}\big)-R_{a_j}(\eta_{j,2})\big]}{1-\big[\frac{1}{2}\mathrm{erfc}\big(-\eta_{j,2}\sqrt{a_j/2}\big)-R_{a_j}(\eta_{j,2})\big]}\bigg). \tag{2.26}
$$

Using then (C.3), we find

$$
S_4^{(3)}=\sum_{j=j_{2,-}}^{g_{2,-}-1}\bigg(\hat{f}_1(j/n)+\frac{1}{n}\hat{f}(j/n)+\frac{1}{n^2}\frac{-2b^3\rho_2^{4b}\hat{\mathsf{T}}_1(j/n)}{(1-\hat{\mathsf{T}}_0(j/n)+\hat{\mathsf{T}}_0(b\rho_2^{2b}))(b\rho_2^{2b}-j/n)^3}
$$

$$
+\frac{1}{n^3}\frac{10b^5\rho_2^{6b}\hat{\mathsf{T}}_1(j/n)(b\rho_2^{2b}-j/n)^{-5}}{1-\hat{\mathsf{T}}_0(j/n)+\hat{\mathsf{T}}_0(b\rho_2^{2b})}
$$

$$+\mathcal{O}\Big(\frac{n^{-2}}{(b\rho_2^{2b}-j/n)^2}+\frac{n^{-3}}{(b\rho_2^{2b}-j/n)^4}+\frac{n^{-4}}{(b\rho_2^{2b}-j/n)^7}+\frac{n^{-6}}{(b\rho_2^{2b}-j/n)^{12}}\Big)\Big).$$

Note that

$$\sum_{j=j_{2,-}}^{g_{2,-}-1}\mathcal{O}\Big(\frac{n^{-2}}{(b\rho_2^{2b}-j/n)^2}+\frac{n^{-3}}{(b\rho_2^{2b}-j/n)^4}+\frac{n^{-4}}{(b\rho_2^{2b}-j/n)^7}+\frac{n^{-6}}{(b\rho_2^{2b}-j/n)^{12}}\Big)$$
$$=\mathcal{O}\Big(\frac{1}{M\sqrt{n}}+\frac{1}{M^3\sqrt{n}}+\frac{1}{M^6}+\frac{\sqrt{n}}{M^{11}}\Big).$$

Also, using Lemma 2.4 (with $A=\frac{b\rho_2^{2b}}{1+\epsilon}$, $a_0=\theta_{2,-}^{(n,\epsilon)}-\alpha$, $B=\frac{b\rho_2^{2b}}{1+\frac{M}{\sqrt{n}}}$ and $b_0=\theta_{2,-}^{(n,M)}-1-\alpha$), we get

$$\sum_{j=j_{2,-}}^{g_{2,-}-1}\hat{f}_1(j/n)=n\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{\frac{b\rho_2^{2b}}{1+\frac{M}{\sqrt{n}}}}\hat{f}_1(x)dx+(\tfrac{1}{2}+\alpha-\theta_{2,-}^{(n,\epsilon)})\hat{f}_1(\tfrac{b\rho_2^{2b}}{1+\epsilon})+\big(\theta_{2,-}^{(n,M)}-\tfrac{1}{2}-\alpha\big)\hat{f}_1(\tfrac{b\rho_2^{2b}}{1+\frac{M}{\sqrt{n}}})+\mathcal{O}(n^{-1}),$$

$$\frac{1}{n}\sum_{j=j_{2,-}}^{g_{2,-}-1}\hat{f}(j/n)=\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{\frac{b\rho_2^{2b}}{1+\frac{M}{\sqrt{n}}}}\hat{f}(x)dx+\mathcal{O}\Big(\frac{1}{M\sqrt{n}}\Big),$$

$$\frac{1}{n^2}\sum_{j=j_{2,-}}^{g_{2,-}-1}\frac{-2b^3\rho_2^{4b}\hat{\mathrm{T}}_1(j/n)(b\rho_2^{2b}-j/n)^{-3}}{1-\hat{\mathrm{T}}_0(j/n)+\hat{\mathrm{T}}_0(b\rho_2^{2b})}=\frac{1}{n}\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{\frac{b\rho_2^{2b}}{1+\frac{M}{\sqrt{n}}}}\frac{-2b^3\rho_2^{4b}\hat{\mathrm{T}}_1(x)(b\rho_2^{2b}-x)^{-3}}{1-\hat{\mathrm{T}}_0(x)+\hat{\mathrm{T}}_0(b\rho_2^{2b})}dx+\mathcal{O}\Big(\frac{1}{M^3\sqrt{n}}\Big),$$

$$\frac{1}{n^3}\sum_{j=j_{2,-}}^{g_{2,-}-1}\frac{10b^5\rho_2^{6b}\hat{\mathrm{T}}_1(j/n)(b\rho_2^{2b}-j/n)^{-5}}{1-\hat{\mathrm{T}}_0(j/n)+\hat{\mathrm{T}}_0(b\rho_2^{2b})}=\frac{1}{n^2}\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{\frac{b\rho_2^{2b}}{1+\frac{M}{\sqrt{n}}}}\frac{10b^5\rho_2^{6b}\hat{\mathrm{T}}_1(x)(b\rho_2^{2b}-x)^{-5}}{1-\hat{\mathrm{T}}_0(x)+\hat{\mathrm{T}}_0(b\rho_2^{2b})}dx+\mathcal{O}\Big(\frac{1}{M^5\sqrt{n}}\Big).$$

Furthermore, by a direct analysis,

$$n\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{\frac{b\rho_2^{2b}}{1+\frac{M}{\sqrt{n}}}}\hat{f}_1(x)dx=n\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{b\rho_2^{2b}}\hat{f}_1(x)dx+M^2\frac{b^2\rho_2^{4b}}{2}\hat{f}_1'(b\rho_2^{2b}))$$
$$-\frac{M^3}{\sqrt{n}}b^2\rho_2^{4b}\Big(\hat{f}_1'(b\rho_2^{2b})+\frac{b\rho_2^{2b}}{6}\hat{f}_1''(b\rho_2^{2b})\Big)+\mathcal{O}\Big(\frac{M^4}{n}\Big),$$

$$\hat{f}_1(\tfrac{b\rho_2^{2b}}{1+\frac{M}{\sqrt{n}}})=\hat{f}_1(b\rho_2^{2b})-\frac{M}{\sqrt{n}}b\rho_2^{2b}\hat{f}_1'(b\rho_2^{2b})+\mathcal{O}\Big(\frac{M^2}{n}\Big),$$

$$\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{\frac{b\rho_2^{2b}}{1+\frac{M}{\sqrt{n}}}}\hat{f}(x)dx=\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{b\rho_2^{2b}}\Big\{\hat{f}(x)-\frac{b\rho_2^{2b}\hat{\mathrm{T}}_1(b\rho_2^{2b})}{b\rho_2^{2b}-x}\Big\}dx+\frac{b\rho_2^{2b}\hat{\mathrm{T}}_1(b\rho_2^{2b})}{2}\ln n-b\rho_2^{2b}\hat{\mathrm{T}}_2(b\rho_2^{2b})\ln\frac{M(1+\epsilon)}{\epsilon}$$
$$+\frac{M}{\sqrt{n}}\Big\{(b+\alpha)\rho_2^{2b}\hat{\mathrm{T}}_1(b\rho_2^{2b})+\frac{b\rho_2^{4b}}{2}\hat{\mathrm{T}}_2(b\rho_2^{2b})+b\rho_2^{4b}\hat{\mathrm{T}}_1(b\rho_2^{2b})^2\Big\}+\mathcal{O}\Big(\frac{M^2}{n}\Big),$$

$$\frac{1}{n}\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{\frac{b\rho_2^{2b}}{1+\frac{M}{\sqrt{n}}}}\frac{-2b^3\rho_2^{4b}\hat{\mathrm{T}}_1(x)(b\rho_2^{2b}-x)^{-3}}{1-\hat{\mathrm{T}}_0(x)+\hat{\mathrm{T}}_0(b\rho_2^{2b})}dx=-\frac{b}{M^2}\hat{\mathrm{T}}_1(b\rho_2^{2b})+\mathcal{O}\Big(\frac{1}{M\sqrt{n}}\Big),$$

$$\frac{1}{n^2}\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{\frac{b\rho_2^{2b}}{1+\frac{M}{\sqrt{n}}}}\frac{10b^5\rho_2^{6b}\hat{\mathrm{T}}_1(x)(b\rho_2^{2b}-x)^{-5}}{1-\hat{\mathrm{T}}_0(x)+\hat{\mathrm{T}}_0(b\rho_2^{2b})}dx=\frac{5b\hat{\mathrm{T}}_1(b\rho_2^{2b})}{2\rho_2^{2b}M^4}+\mathcal{O}\Big(\frac{1}{M^3\sqrt{n}}\Big).$$

The claim now follows after a computation. □

For $\ell\in\{1,\dots,2m\}$ and $j\in\{1,\dots,n\}$, we define $\hat{M}_{j,\ell}:=\sqrt{n}(\hat{\lambda}_{j,\ell}-1)$, $M_{j,1}:=\sqrt{n}(\lambda_{j,1}-1)$ and $M_{j,2}:=\sqrt{n}(\lambda_{j,2}-1)$. For the large n asymptotics of $S_2^{(2)}$ we will need the following lemma.

Lemma 2.12 (Taken from [28, Lemma 3.11]). *Let $h\in C^3(\mathbb{R})$ and $k\in\{1,2\}$. As $n\to+\infty$, we have*

$$\sum_{j=g_{k,-}}^{g_{k,+}}h(M_{j,k})=b\rho_k^{2b}\int_{-M}^{M}h(t)dt\,\sqrt{n}-2b\rho_k^{2b}\int_{-M}^{M}th(t)dt+\Big(\frac{1}{2}-\theta_{k,-}^{(n,M)}\Big)h(M)+\Big(\frac{1}{2}-\theta_{k,+}^{(n,M)}\Big)h(-M)$$

$$+\frac{1}{\sqrt{n}}\bigg[3b\rho_k^{2b}\int_{-M}^{M}t^2h(t)dt+\bigg(\frac{1}{12}+\frac{\theta_{k,-}^{(n,M)}(\theta_{k,-}^{(n,M)}-1)}{2}\bigg)\frac{h'(M)}{b\rho_k^{2b}}-\bigg(\frac{1}{12}+\frac{\theta_{k,+}^{(n,M)}(\theta_{k,+}^{(n,M)}-1)}{2}\bigg)\frac{h'(-M)}{b\rho_k^{2b}}\bigg]$$

$$+\mathcal{O}\Bigg(\frac{1}{n^{3/2}}\sum_{j=g_{k,-}+1}^{g_{k,+}}\bigg((1+|M_{j,k}|^3)\tilde{\mathfrak{m}}_{j,n,k}(h)+(1+M_{j,k}^2)\tilde{\mathfrak{m}}_{j,n,k}(h')+(1+|M_{j,k}|)\tilde{\mathfrak{m}}_{j,n,k}(h'')+\tilde{\mathfrak{m}}_{j,n,k}(h''')\bigg)\Bigg), \tag{2.27}$$

where, for $\tilde{h}\in C(\mathbb{R})$ and $j\in\{g_{k,-}+1,\ldots,g_{k,+}\}$, we define $\tilde{\mathfrak{m}}_{j,n,k}(\tilde{h}):=\max_{x\in[M_{j,k},M_{j-1,k}]}|\tilde{h}(x)|$.

Lemma 2.13. *Let $x_1,\ldots,x_{2m}\in\mathbb{R}$ be fixed. There exists $\delta>0$ such that*

$$\begin{aligned}
&S_2^{(2)}=E_2^{(M)}\sqrt{n}+E_4^{(M)}+\frac{E_5^{(M)}}{\sqrt{n}}+\mathcal{O}\bigg(\frac{M^4}{n}+\frac{M^{14}}{n^2}\bigg),\\
&E_2^{(M)}=2b\rho_1^{2b}M\ln(1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b})),\\
&E_4^{(M)}=\ln(1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b}))\big(1-\theta_{1,-}^{(n,M)}-\theta_{1,+}^{(n,M)}\big)+b\rho_1^{2b}\int_{-M}^{M}h_1(t)dt,\\
&E_5^{(M)}=2b\rho_1^{2b}M^3\ln(1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b}))+\bigg(\frac{1}{2}-\theta_{1,-}^{(n,M)}\bigg)h_1(M)+\bigg(\frac{1}{2}-\theta_{1,+}^{(n,M)}\bigg)h_1(-M)\\
&\qquad\quad+b\rho_1^{2b}\int_{-M}^{M}\big(h_2(t)-2th_1(t)\big)dt,
\end{aligned}$$

as $n\to+\infty$ uniformly for $u_1\in\{z\in\mathbb{C}:|z-x_1|\le\delta\},\ldots,u_{2m}\in\{z\in\mathbb{C}:|z-x_{2m}|\le\delta\}$, where h_1, h_2 are given by

$$\begin{aligned}
h_1(x)&=-\frac{2\rho_1^b\mathsf{T}_1(b\rho_1^{2b})}{1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b})}\frac{e^{-\frac{1}{2}x^2\rho_1^{2b}}}{\sqrt{2\pi}\,\mathrm{erfc}(-\frac{x\rho_1^b}{\sqrt{2}})},\\
h_2(x)&=-\frac{h_1(x)^2}{2}+\frac{1}{1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b})}\frac{e^{-\frac{1}{2}x^2\rho_1^{2b}}}{\sqrt{2\pi}\,\mathrm{erfc}(-\frac{x\rho_1^b}{\sqrt{2}})}\bigg\{\bigg(\rho_1^bx-\frac{5}{3}\rho_1^{3b}x^3\bigg)\mathsf{T}_1(b\rho_1^{2b})\\
&\qquad-\rho_1^{3b}x\mathsf{T}_2(b\rho_1^{2b})+\frac{4-10\rho_1^{2b}x^2}{3}\mathsf{T}_1(b\rho_1^{2b})\frac{e^{-\frac{1}{2}x^2\rho_1^{2b}}}{\sqrt{2\pi}\,\mathrm{erfc}(-\frac{x\rho_1^b}{\sqrt{2}})}\bigg\}.
\end{aligned}$$

Proof. In a similar way as in (2.25), we infer that there exists $\delta>0$ such that

$$S_2^{(2)}=\sum_{j:\lambda_{j,1}\in I_2}\ln\Bigg(1+\hat{\mathsf{T}}_0(b\rho_2^{2b})+\sum_{\ell=1}^{m}\omega_\ell\frac{\frac{1}{2}\mathrm{erfc}\big(-\hat{\eta}_{j,\ell}\sqrt{a_j/2}\big)-R_{a_j}(\hat{\eta}_{j,\ell})}{\frac{1}{2}\mathrm{erfc}\big(-\eta_{j,1}\sqrt{a_j/2}\big)-R_{a_j}(\eta_{j,1})}\Bigg)+\mathcal{O}(e^{-cn^{1/5}}), \tag{2.28}$$

as $n\to+\infty$ uniformly for $u_1\in\{z\in\mathbb{C}:|z-x_1|\le\delta\},\ldots,u_{2m}\in\{z\in\mathbb{C}:|z-x_{2m}|\le\delta\}$. For $j\in\{j:\lambda_{j,1}\in I_2\}$, we have $1-\frac{M}{\sqrt{n}}\le\lambda_{j,1}=\frac{bn\rho_1^{2b}}{j+\alpha}\le1+\frac{M}{\sqrt{n}}$, $-M\le M_{j,1}\le M$, and

$$\hat{M}_{j,\ell}=M_{j,1}-\frac{t_\ell}{\sqrt{n}}-\frac{t_\ell M_{j,1}}{n},\qquad \ell=1,\ldots,m.$$

Furthermore, for $\ell\in\{1,\ldots,m\}$, as $n\to+\infty$ we have

$$\begin{aligned}
\hat{\eta}_{j,\ell}&=\frac{M_{j,1}}{\sqrt{n}}-\frac{M_{j,1}^2+3t_\ell}{3n}+\frac{7M_{j,1}^3-12t_\ell M_{j,1}}{36n^{3/2}}+\mathcal{O}\bigg(\frac{1+M_{j,1}^4}{n^2}\bigg),\\
-\hat{\eta}_{j,\ell}\sqrt{a_j/2}&=-\frac{M_{j,1}\rho_1^b}{\sqrt{2}}+\frac{(5M_{j,1}^2+6t_\ell)\rho_1^b}{6\sqrt{2}\sqrt{n}}-\frac{\rho_1^bM_{j,1}(53M_{j,1}^2+12t_\ell)}{72\sqrt{2}n}+\mathcal{O}\bigg(\frac{1+M_{j,1}^4}{n^{3/2}}\bigg),
\end{aligned}$$

uniformly for $j\in\{j:\lambda_{j,1}\in I_2\}$. Hence, for $\ell\in\{1,\ldots,m\}$, by (C.3) as $n\to+\infty$ we have

$$R_{a_j}(\hat{\eta}_{j,\ell})=\frac{e^{-\frac{M_{j,1}^2\rho_1^{2b}}{2}}}{\sqrt{2\pi}}\bigg(\frac{-1}{3\rho_1^b\sqrt{n}}-\frac{M_{j,1}(3+10M_{j,1}^2\rho_1^{2b}+12t_\ell\rho_1^{2b})}{36\rho_1^bn}+\mathcal{O}((1+M_{j,1}^6)n^{-\frac{3}{2}})\bigg)$$

and

$$\frac{1}{2}\mathrm{erfc}\Big(-\hat{\eta}_{j,\ell}\sqrt{a_j/2}\Big) = \frac{1}{2}\mathrm{erfc}\Big(-\frac{\rho_1^b M_{j,1}}{\sqrt{2}}\Big) - \frac{e^{-\frac{M_{j,1}^2\rho_1^{2b}}{2}}\rho_1^b(5M_{j,1}^2+6t_\ell)}{6\sqrt{2\pi}\sqrt{n}}$$
$$+\frac{e^{-\frac{M_{j,1}^2\rho_1^{2b}}{2}}M_{j,1}\rho_1^b}{72\sqrt{2\pi}\,n}\Big(53M_{j,1}^2+12t_\ell-25M_{j,1}^4\rho_1^{2b}-60M_{j,1}^2t_\ell\rho_1^{2b}-36t_\ell^2\rho_1^{2b}\Big)+\mathcal{O}\Big(e^{-\frac{M_{j,1}^2\rho_1^{2b}}{2}}\frac{1+M_{j,1}^8}{n^{3/2}}\Big),$$

uniformly for $j\in\{j:\lambda_{j,1}\in I_2\}$. Extending the above asymptotic formulas to higher order (see [10, Lemma 2.8] for details in a similar situation), we deduce that the argument of ln in (2.28) enjoys the following asymptotics

$$1+\hat{\mathsf{T}}_0(b\rho_2^{2b})+\sum_{\ell=1}^m\omega_\ell\frac{\frac{1}{2}\mathrm{erfc}\big(-\hat{\eta}_{j,\ell}\sqrt{a_j/2}\big)-R_{a_j}(\hat{\eta}_{j,\ell})}{\frac{1}{2}\mathrm{erfc}\big(-\eta_{j,1}\sqrt{a_j/2}\big)-R_{a_j}(\eta_{j,1})}$$
$$= g_1(M_{j,1})+\frac{g_2(M_{j,1})}{\sqrt{n}}+\frac{g_3(M_{j,1})}{n}+\frac{g_4(M_{j,1})}{n^{3/2}}+\frac{g_5(M_{j,1})}{n^2}+\mathcal{O}\Big(\frac{1+|M_{j,1}|^{13}}{n^{5/2}}\Big), \tag{2.29}$$

as $n\to+\infty$, where

$$g_1(x)=1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b}),\qquad g_2(x)=-\frac{e^{-\frac{1}{2}x^2\rho_1^{2b}}2\rho_1^b\mathsf{T}_1(b\rho_1^{2b})}{\sqrt{2\pi}\mathrm{erfc}(-\frac{x\rho_1^b}{\sqrt{2}})},$$
$$g_3(x)=\frac{e^{-\frac{1}{2}x^2\rho_1^{2b}}}{3\sqrt{2\pi}\,\mathrm{erfc}(-\frac{x\rho_1^b}{\sqrt{2}})}\bigg\{\frac{e^{-\frac{1}{2}x^2\rho_1^{2b}}\mathsf{T}_1(b\rho_1^{2b})}{\sqrt{2\pi}\,\mathrm{erfc}(-\frac{x\rho_1^b}{\sqrt{2}})}(4-10x^2\rho_1^{2b})+\mathsf{T}_1(b\rho_1^{2b})\big(3x\rho_1^b-5x^3\rho_1^{3b}\big)$$
$$-3\rho_1^{3b}x\mathsf{T}_2(b\rho_1^{2b})\bigg\}.$$

We remark that equation (2.29) coincides with [10, eq (2.29)] except that (2.29) has the extra term $\hat{\mathsf{T}}_0(b\rho_2^{2b})$ on each side and $(\rho,M_j,\eta_j,\eta_{j,\ell})$ in [10, eq (2.29)] are given by $(\rho_1,M_{j,1},\eta_{j,1},\hat{\eta}_{j,\ell})$ here. The functions g_4 and g_5 can also be computed explicitly, but we do not write them down. Since $\frac{e^{-\frac{1}{2}x^2\rho_1^{2b}}}{\sqrt{2\pi}\,\mathrm{erfc}(-\frac{x\rho_1^b}{\sqrt{2}})}\sim -\frac{\rho_1^b x}{2}$ as $x\to-\infty$, $g_2(x)=\mathcal{O}(x)$ as $x\to-\infty$. At first glance, it seems that $g_3(x)=\mathcal{O}(x^4)$ as $x\to-\infty$. However, some cancellations take place and in fact $g_3(x)=\mathcal{O}(x^2)$ as $x\to-\infty$. Likewise, the expressions for g_4 and g_5 suggest a priori that $g_4(x)=\mathcal{O}(x^7)$ and $g_5(x)=\mathcal{O}(x^{10})$ as $x\to-\infty$, but here too, cancellations take place and in fact $g_4(x)=\mathcal{O}(x^3)$ and $g_5(x)=\mathcal{O}(x^4)$ as $x\to-\infty$. Using the above large x estimates for $\{g_j(x)\}_{j=2}^5$ and (2.29), we obtain after a computation that

$$S_2^{(2)}=\sum_{j=g_{1,-}}^{g_{1,+}}\bigg\{\ln(1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b}))+\frac{h_1(M_{j,1})}{\sqrt{n}}+\frac{h_2(M_{j,1})}{n}+\mathcal{O}\bigg(\frac{1+|M_{j,1}|^3}{n^{3/2}}+\frac{1+|M_{j,1}|^{13}}{n^{5/2}}\bigg)\bigg\}.$$

as $n\to+\infty$. The above error can be estimated as follows:

$$\sum_{j=g_{1,-}}^{g_{1,+}}\mathcal{O}\bigg(\frac{1+|M_{j,1}|^3}{n^{3/2}}+\frac{1+|M_{j,1}|^{13}}{n^{5/2}}\bigg)=\mathcal{O}\Big(\frac{M^4}{n}+\frac{M^{14}}{n^2}\Big),\qquad \text{as } n\to+\infty.$$

Using then Lemma 2.12, we obtain the claim. □

Lemma 2.14. *Let $x_1,\ldots,x_{2m}\in\mathbb{R}$ be fixed. There exists $\delta>0$ such that*

$$S_4^{(2)}=\hat{E}_2^{(M)}\sqrt{n}+\hat{E}_4^{(M)}+\frac{\hat{E}_5^{(M)}}{\sqrt{n}}+\mathcal{O}\Big(\frac{M^4}{n}+\frac{M^{14}}{n^2}\Big),$$
$$\hat{E}_2^{(M)}=0,\qquad \hat{E}_4^{(M)}=b\rho_2^{2b}\int_{-M}^{M}\hat{h}_1(t)dt,$$
$$\hat{E}_5^{(M)}=\Big(\frac{1}{2}-\theta_{2,-}^{(n,M)}\Big)\hat{h}_1(M)+\Big(\frac{1}{2}-\theta_{2,+}^{(n,M)}\Big)\hat{h}_1(-M)+b\rho_2^{2b}\int_{-M}^{M}\big(\hat{h}_2(t)-2t\hat{h}_1(t)\big)dt,$$

as $n \to +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \le \delta\}, \ldots, u_{2m} \in \{z \in \mathbb{C} : |z - x_{2m}| \le \delta\}$, where $\hat{h}_1$, $\hat{h}_2$ are given by

$$\hat{h}_1(x) = 2\rho_2^b \hat{\mathsf{T}}_1(b\rho_2^{2b}) \frac{e^{-\frac{1}{2}x^2\rho_2^{2b}}}{\sqrt{2\pi}\left(2 - \mathrm{erfc}(-\frac{x\rho_2^b}{\sqrt{2}})\right)},$$

$$\hat{h}_2(x) = -\frac{\hat{h}_1(x)^2}{2} + \frac{e^{-\frac{1}{2}x^2\rho_2^{2b}}}{\sqrt{2\pi}\left(2 - \mathrm{erfc}(-\frac{x\rho_2^b}{\sqrt{2}})\right)} \bigg\{ \Big(- \rho_2^b x + \frac{5}{3}\rho_2^{3b}x^3 \Big) \hat{\mathsf{T}}_1(b\rho_2^{2b})$$
$$- \rho_2^{3b} x \hat{\mathsf{T}}_2(b\rho_2^{2b}) + \frac{4 - 10\rho_2^{2b}x^2}{3} \hat{\mathsf{T}}_1(b\rho_2^{2b}) \frac{e^{-\frac{1}{2}x^2\rho_2^{2b}}}{\sqrt{2\pi}\left(2 - \mathrm{erfc}(-\frac{x\rho_2^b}{\sqrt{2}})\right)} \bigg\}.$$

Proof. In a similar way as in (2.26), we infer that there exists $\delta > 0$ such that

$$S_4^{(2)} = \mathcal{O}(e^{-cn}) + \sum_{j:\lambda_{j,2} \in I_2} \ln \bigg(1$$
$$+ \sum_{\ell=m+1}^{2m} \omega_\ell \frac{\frac{1}{2}\mathrm{erfc}\big(- \hat{\eta}_{j,\ell}\sqrt{a_j/2}\big) - R_{a_j}(\hat{\eta}_{j,\ell}) - \big[\frac{1}{2}\mathrm{erfc}\big(- \eta_{j,2}\sqrt{a_j/2}\big) - R_{a_j}(\eta_{j,2})\big]}{1 - \big[\frac{1}{2}\mathrm{erfc}\big(- \eta_{j,2}\sqrt{a_j/2}\big) - R_{a_j}(\eta_{j,2})\big]} \bigg) \tag{2.30}$$

as $n \to +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \le \delta\}, \ldots, u_{2m} \in \{z \in \mathbb{C} : |z - x_{2m}| \le \delta\}$. For $j \in \{j : \lambda_{j,2} \in I_2\}$, we have $1 - \frac{M}{\sqrt{n}} \le \lambda_{j,2} = \frac{bn\rho_2^{2b}}{j+\alpha} \le 1 + \frac{M}{\sqrt{n}}$, $-M \le M_{j,2} \le M$, and

$$\hat{M}_{j,k} = M_{j,2} + \frac{t_k}{\sqrt{n}} + \frac{t_k M_{j,2}}{n}, \qquad k = m+1, \ldots, 2m.$$

Furthermore, for $\ell \in \{m+1, \ldots, 2m\}$, as $n \to +\infty$ we have

$$\hat{\eta}_{j,\ell} = \frac{M_{j,2}}{\sqrt{n}} - \frac{M_{j,2}^2 - 3t_\ell}{3n} + \frac{7M_{j,2}^3 + 12t_\ell M_{j,2}}{36n^{3/2}} + \mathcal{O}\Big(\frac{1 + M_{j,2}^4}{n^2}\Big) \tag{2.31}$$

$$-\hat{\eta}_{j,\ell}\sqrt{a_j/2} = -\frac{M_{j,2}\rho_2^b}{\sqrt{2}} + \frac{(5M_{j,2}^2 - 6t_\ell)\rho_2^b}{6\sqrt{2}\sqrt{n}} - \frac{\rho_2^b M_{j,2}(53M_{j,2}^2 - 12t_\ell)}{72\sqrt{2}n} + \mathcal{O}\Big(\frac{1 + M_{j,2}^4}{n^{3/2}}\Big) \tag{2.32}$$

uniformly for $j \in \{j : \lambda_{j,2} \in I_2\}$. Hence, for $\ell \in \{m+1, \ldots, 2m\}$, by (C.3) as $n \to +\infty$ we have

$$R_{a_j}(\hat{\eta}_{j,\ell}) = \frac{e^{-\frac{M_{j,2}^2\rho_2^{2b}}{2}}}{\sqrt{2\pi}} \bigg(\frac{-1}{3\rho_2^b\sqrt{n}} - \frac{M_{j,2}(3 + 10M_{j,2}^2\rho_2^{2b} - 12t_\ell\rho_2^{2b})}{36\rho_2^b n} + \mathcal{O}((1 + M_{j,2}^6)n^{-\frac{3}{2}}) \bigg)$$

and

$$\frac{1}{2}\mathrm{erfc}\Big(- \hat{\eta}_{j,\ell}\sqrt{a_j/2}\Big) = \frac{1}{2}\mathrm{erfc}\Big(- \frac{\rho_2^b M_{j,2}}{\sqrt{2}}\Big) - \frac{e^{-\frac{M_{j,2}^2\rho_2^{2b}}{2}}\rho_2^b(5M_{j,2}^2 - 6t_\ell)}{6\sqrt{2\pi}\sqrt{n}}$$
$$+ \frac{e^{-\frac{M_{j,2}^2\rho_2^{2b}}{2}}M_{j,2}\rho_2^b}{72\sqrt{2\pi}\,n}\Big(53M_{j,2}^2 - 12t_\ell - 25M_{j,2}^4\rho_2^{2b} + 60M_{j,2}^2 t_\ell\rho_2^{2b} - 36t_\ell^2\rho_2^{2b}\Big) + \mathcal{O}\Big(e^{-\frac{M_{j,2}^2\rho_2^{2b}}{2}}\frac{1 + M_{j,2}^8}{n^{3/2}}\Big),$$

uniformly for $j \in \{j : \lambda_{j,2} \in I_2\}$. Extending the above asymptotic formulas to higher order (see [10, Lemma 2.8] for details in a similar situation), we deduce from (2.30) that

$$1 + \sum_{\ell=m+1}^{2m} \omega_\ell \frac{\frac{1}{2}\mathrm{erfc}\big(- \hat{\eta}_{j,\ell}\sqrt{a_j/2}\big) - R_{a_j}(\hat{\eta}_{j,\ell}) - \big[\frac{1}{2}\mathrm{erfc}\big(- \eta_{j,2}\sqrt{a_j/2}\big) - R_{a_j}(\eta_{j,2})\big]}{1 - \big[\frac{1}{2}\mathrm{erfc}\big(- \eta_{j,2}\sqrt{a_j/2}\big) - R_{a_j}(\eta_{j,2})\big]}$$
$$= 1 + \frac{\hat{g}_2(M_{j,2})}{\sqrt{n}} + \frac{\hat{g}_3(M_{j,2})}{n} + \frac{\hat{g}_4(M_{j,2})}{n^{3/2}} + \frac{\hat{g}_5(M_{j,2})}{n^2} + \mathcal{O}\Big(\frac{1 + |M_{j,2}|^{13}}{n^{5/2}}\Big)\bigg), \tag{2.33}$$

as $n \to +\infty$, where

$$\hat{g}_2(x) = \frac{e^{-\frac{1}{2}x^2\rho_2^{2b}} 2\rho_2^b \hat{\mathsf{T}}_1(b\rho_2^{2b})}{\sqrt{2\pi}(2 - \mathrm{erfc}(-\frac{x\rho_2^b}{\sqrt{2}}))},$$

$$\hat{g}_3(x) = \frac{e^{-\frac{1}{2}x^2\rho_2^{2b}}}{\sqrt{2\pi}\left(2-\mathrm{erfc}(-\frac{x\rho_2^b}{\sqrt{2}})\right)}\bigg\{\Big(-\rho_2^b x + \frac{5}{3}\rho_2^{3b}x^3\Big)\hat{\mathsf{T}}_1(b\rho_2^{2b}) - \rho_2^{3b}x\hat{\mathsf{T}}_2(b\rho_2^{2b}) + \frac{4-10\rho_2^{2b}x^2}{3}\hat{\mathsf{T}}_1(b\rho_2^{2b})\frac{e^{-\frac{1}{2}x^2\rho_2^{2b}}}{\sqrt{2\pi}\left(2-\mathrm{erfc}(-\frac{x\rho_2^b}{\sqrt{2}})\right)}\bigg\}.$$

The functions $\hat{g}_4$ and $\hat{g}_5$ can also be computed explicitly, but we do not write them down. Since $\frac{e^{-\frac{1}{2}x^2\rho_2^{2b}}}{\sqrt{2\pi}\,(2-\mathrm{erfc}(-\frac{x\rho_2^b}{\sqrt{2}}))} \sim \frac{\rho_2^b x}{2}$ as $x \to +\infty$, $\hat{g}_2(x) = \mathcal{O}(x)$ as $x \to +\infty$. At first glance, it seems that $\hat{g}_3(x) = \mathcal{O}(x^4)$ as $x \to +\infty$. However, some cancellations take place and in fact $\hat{g}_3(x) = \mathcal{O}(x^2)$ as $x \to +\infty$. Likewise, the expressions for $\hat{g}_4$ and $\hat{g}_5$ suggest a priori that $\hat{g}_4(x) = \mathcal{O}(x^7)$ and $\hat{g}_5(x) = \mathcal{O}(x^{10})$ as $x \to +\infty$, but here too, cancellations take place and in fact $\hat{g}_4(x) = \mathcal{O}(x^3)$ and $\hat{g}_5(x) = \mathcal{O}(x^4)$ as $x \to +\infty$. Using the above large x estimates for $\{\hat{g}_j(x)\}_{j=2}^5$ and (2.33), we obtain after a computation that

$$S_4^{(2)} = \sum_{j=g_{2,-}}^{g_{2,+}} \bigg\{\frac{\hat{h}_1(M_{j,2})}{\sqrt{n}} + \frac{\hat{h}_2(M_{j,2})}{n} + \mathcal{O}\Big(\frac{1+|M_{j,2}|^3}{n^{3/2}} + \frac{1+|M_{j,2}|^{13}}{n^{5/2}}\Big)\bigg\}.$$

as $n \to +\infty$, where $\hat{h}_1 = \hat{g}_2$ and $\hat{h}_2 = -\frac{\hat{h}_1^2}{2} + \hat{g}_3$. Note that

$$\sum_{j=g_{2,-}}^{g_{2,+}} \mathcal{O}\Big(\frac{1+|M_{j,2}|^3}{n^{3/2}} + \frac{1+|M_{j,2}|^{13}}{n^{5/2}}\Big) = \mathcal{O}\Big(\frac{M^4}{n} + \frac{M^{14}}{n^2}\Big), \qquad \text{as } n \to +\infty.$$

Using then Lemma 2.12, we find the claim. □

Define the real constants $\{\mathcal{I}_j\}_1^4 \subset \mathbb{R}$ by

$$\mathcal{I}_1 = \int_{-\infty}^{+\infty} \bigg\{\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y)\Big[y + \frac{y}{2(1+y^2)}\Big]\bigg\}dy, \tag{2.34}$$

$$\mathcal{I}_2 = \int_{-\infty}^{+\infty} \bigg\{\frac{y^3e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y)\Big[y^4 + \frac{y^2}{2} - \frac{1}{2}\Big]\bigg\}dy, \tag{2.35}$$

$$\mathcal{I}_3 = \int_{-\infty}^{+\infty} \bigg\{\Big(\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\Big)^2 - \chi_{(0,+\infty)}(y)\Big[y^2 + 1\Big]\bigg\}dy, \tag{2.36}$$

$$\mathcal{I}_4 = \int_{-\infty}^{+\infty} \bigg\{\Big(\frac{y\,e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\Big)^2 - \chi_{(0,+\infty)}(y)\Big[y^4 + y^2 - \frac{3}{4}\Big]\bigg\}dy. \tag{2.37}$$

Recall also that $\mathcal{I}$ is given by (1.20).

Lemma 2.15. *For any fixed $x_1, \ldots, x_{2m} \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_2 = -j_{1,-}\ln\Omega + F_1^{(\epsilon)}n + F_2\ln n + F_3^{(n,\epsilon)} + \frac{F_4}{\sqrt{n}} + \mathcal{O}\Big(\frac{\sqrt{n}}{M^{11}} + \frac{1}{M^6} + \frac{1}{\sqrt{n}M} + \frac{M^4}{n} + \frac{M^{14}}{n^2}\Big),$$

as $n \to +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \ldots, u_{2m} \in \{z \in \mathbb{C} : |z - x_{2m}| \leq \delta\}$, where

$$F_1^{(\epsilon)} = b\rho_1^{2b}\ln\Omega + \int_{b\rho_1^{2b}}^{\frac{b\rho_1^{2b}}{1-\epsilon}} f_1(x)dx,$$

$$F_2 = -\frac{b\rho_1^{2b}}{2}\frac{\mathsf{T}_1(b\rho_1^{2b})}{\Omega},$$

$$F_3^{(n,\epsilon)} = \frac{1}{2}\ln\Omega + \int_{b\rho_1^{2b}}^{\frac{b\rho_1^{2b}}{1-\epsilon}}\bigg\{f(x) + \frac{b\rho_1^{2b}\mathsf{T}_1(b\rho_1^{2b})}{\Omega(x - b\rho_1^{2b})}\bigg\}dx + \Big(\frac{1}{2} - \alpha - \theta_{1,+}^{(n,\epsilon)}\Big)f_1\Big(\frac{b\rho_1^{2b}}{1-\epsilon}\Big) - 2b\rho_1^{2b}\frac{\mathsf{T}_1(b\rho_1^{2b})}{\Omega}\mathcal{I}_1 + \frac{b\rho_1^{2b}}{2\Omega}\mathsf{T}_1(b\rho_1^{2b})\big(\ln 2 - 2b\ln(\rho_1)\big) - \frac{\mathsf{T}_1(b\rho_1^{2b})}{\Omega}b\rho_1^{2b}\ln\Big(\frac{\epsilon}{1-\epsilon}\Big),$$

$$F_4 = \sqrt{2}b\rho_1^b\frac{\rho_1^{2b}\mathsf{T}_2(b\rho_1^{2b}) - 5\mathsf{T}_1(b\rho_1^{2b})}{\Omega}\mathcal{I} + \frac{10\sqrt{2}b\rho_1^b}{3}\frac{\mathsf{T}_1(b\rho_1^{2b})}{\Omega}\mathcal{I}_2$$

$$+\sqrt{2}b\rho_1^b\frac{\mathsf{T}_1(b\rho_1^{2b})}{\Omega}\bigg(\frac{2}{3}-\rho_1^{2b}\frac{\mathsf{T}_1(b\rho_1^{2b})}{\Omega}\bigg)\mathcal{I}_3-\frac{10\sqrt{2}b\rho_1^b}{3}\frac{\mathsf{T}_1(b\rho_1^{2b})}{\Omega}\mathcal{I}_4,$$

and f_1 and f are as in the statement of Lemma 2.7.

Proof. By combining Lemmas 2.8, 2.10 and 2.13, we have

$$S_2=-j_{1,-}\ln\Omega+F_1^{(\epsilon)}n+\widetilde{F}_2\sqrt{n}+F_2\ln n+F_3^{(n,\epsilon,M)}+\frac{F_4^{(M)}}{\sqrt{n}}+\mathcal{O}\bigg(\frac{\sqrt{n}}{M^{11}}+\frac{1}{M^6}+\frac{1}{\sqrt{n}M}+\frac{M^4}{n}+\frac{M^{14}}{n^2}\bigg),$$

as $n\to+\infty$ uniformly for $u_1\in\{z\in\mathbb{C}:|z-x_1|\leq\delta\},\ldots,u_{2m}\in\{z\in\mathbb{C}:|z-x_{2m}|\leq\delta\}$, where $F_1^{(\epsilon)}$ and F_2 are as in the statement, and

$$\widetilde{F}_2=-bM\rho_1^{2b}\ln\Omega+D_2^{(M)}+E_2^{(M)},$$
$$F_3^{(n,\epsilon,M)}=\big(bM^2\rho_1^{2b}-\alpha+\theta_{1,-}^{(n,M)}\big)\ln\Omega+D_4^{(n,\epsilon,M)}+E_4^{(M)},$$
$$F_4^{(M)}=-bM^3\rho_1^{2b}\ln\Omega+D_5^{(n,M)}+E_5^{(M)}.$$

Using that $f_1(b\rho_1^{2b})=\ln(1+\mathsf{T}_0(b\rho_1^{2b})+\hat{\mathsf{T}}_0(b\rho_2^{2b}))=\ln\Omega$, we readily verify that $\widetilde{F}_2=0$. Furthermore, a long but direct computation shows that (see [10, Lemma 2.9] for a similar situation with more details provided)

$$F_3^{(n,\epsilon,M)}=F_3^{(n,\epsilon)}+\mathcal{O}(M^{-6}),\qquad F_4^{(M)}=F_4+\mathcal{O}(M^{-1}),$$

and the claim follows. □

Lemma 2.16. *For any fixed $x_1,\ldots,x_{2m}\in\mathbb{R}$, there exists $\delta>0$ such that*

$$S_4=\hat{F}_1^{(\epsilon)}n+\hat{F}_2\ln n+\hat{F}_3^{(n,\epsilon)}+\frac{\hat{F}_4}{\sqrt{n}}+\mathcal{O}\bigg(\frac{\sqrt{n}}{M^{11}}+\frac{1}{M^6}+\frac{1}{\sqrt{n}M}+\frac{M^4}{n}+\frac{M^{14}}{n^2}\bigg),$$

as $n\to+\infty$ uniformly for $u_1\in\{z\in\mathbb{C}:|z-x_1|\leq\delta\},\ldots,u_{2m}\in\{z\in\mathbb{C}:|z-x_{2m}|\leq\delta\}$, where

$$\hat{F}_1^{(\epsilon)}=\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{b\rho_2^{2b}}\hat{f}_1(x)dx,$$
$$\hat{F}_2=\frac{b\rho_2^{2b}}{2}\hat{\mathsf{T}}_1(b\rho_2^{2b}),$$
$$\hat{F}_3^{(n,\epsilon)}=\int_{\frac{b\rho_2^{2b}}{1+\epsilon}}^{b\rho_2^{2b}}\bigg\{\hat{f}(x)-\frac{b\rho_2^{2b}\hat{\mathsf{T}}_1(b\rho_2^{2b})}{b\rho_2^{2b}-x}\bigg\}dx+\bigg(\frac{1}{2}+\alpha-\theta_{2,-}^{(n,\epsilon)}\bigg)\hat{f}_1\bigg(\frac{b\rho_2^{2b}}{1+\epsilon}\bigg)$$
$$+2b\rho_2^{2b}\hat{\mathsf{T}}_1(b\rho_2^{2b})\mathcal{I}_1-\frac{b\rho_2^{2b}}{2}\hat{\mathsf{T}}_1(b\rho_2^{2b})\big(\ln 2-2b\ln(\rho_2)\big)+\hat{\mathsf{T}}_1(b\rho_2^{2b})b\rho_2^{2b}\ln\bigg(\frac{\epsilon}{1+\epsilon}\bigg),$$
$$\hat{F}_4=-\sqrt{2}b\rho_2^b\big(\rho_2^{2b}\hat{\mathsf{T}}_2(b\rho_2^{2b})+5\hat{\mathsf{T}}_1(b\rho_2^{2b})\big)\mathcal{I}+\frac{10\sqrt{2}b\rho_2^b}{3}\hat{\mathsf{T}}_1(b\rho_2^{2b})\mathcal{I}_2$$
$$+\sqrt{2}b\rho_2^b\hat{\mathsf{T}}_1(b\rho_2^{2b})\bigg(\frac{2}{3}-\rho_2^{2b}\hat{\mathsf{T}}_1(b\rho_2^{2b})\bigg)\mathcal{I}_3-\frac{10\sqrt{2}b\rho_2^b}{3}\hat{\mathsf{T}}_1(b\rho_2^{2b})\mathcal{I}_4,$$

and $\hat{f}_1$ and $\hat{f}$ are as in the statement of Lemma 2.7.

Proof. By combining Lemmas 2.9, 2.11 and 2.14, we have

$$S_4=\hat{F}_1^{(\epsilon)}n+\widetilde{\hat{F}}_2\sqrt{n}+\hat{F}_2\ln n+\hat{F}_3^{(n,\epsilon,M)}+\frac{\hat{F}_4^{(M)}}{\sqrt{n}}+\mathcal{O}\bigg(\frac{\sqrt{n}}{M^{11}}+\frac{1}{M^6}+\frac{1}{\sqrt{n}M}+\frac{M^4}{n}+\frac{M^{14}}{n^2}\bigg),$$

as $n\to+\infty$ uniformly for $u_1\in\{z\in\mathbb{C}:|z-x_1|\leq\delta\},\ldots,u_{2m}\in\{z\in\mathbb{C}:|z-x_{2m}|\leq\delta\}$, where $\hat{F}_1^{(\epsilon)}$ and $\hat{F}_2$ are as in the statement, and

$$\widetilde{\hat{F}}_2=\hat{D}_2^{(M)}+\hat{E}_2^{(M)},$$
$$\hat{F}_3^{(n,\epsilon,M)}=\hat{D}_4^{(n,\epsilon,M)}+\hat{E}_4^{(M)},$$

$$\hat{F}_4^{(M)} = \hat{D}_5^{(n,M)} + \hat{E}_5^{(M)}.$$

It is easy to check that $\widetilde{\hat{F}}_2 = 0$. Furthermore, a long but direct computation shows that (see e.g. [10, Proof of Lemma 2.9] for a similar analysis with more details provided)

$$\hat{F}_3^{(n,\epsilon,M)} = \hat{F}_3^{(n,\epsilon)} + \mathcal{O}(M^{-6}), \qquad \hat{F}_4^{(M)} = \hat{F}_4 + \mathcal{O}(M^{-1}),$$

and the claim follows. $\square$

End of the proof of Theorem 1.2. Let $M' > 0$ be sufficiently large such that Lemmas 2.2 and 2.15 hold. Using (2.2) and Lemmas 2.1, 2.2, 2.15, 2.7, 2.16 and 2.3, we conclude that for any $x_1, \ldots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\begin{aligned}
&\ln \mathcal{E}_n = S_0 + S_1 + S_2 + S_3 + S_4 + S_5 \\
&= M' \ln \Omega + (j_- - M' - 1) \ln \Omega - j_- \ln \Omega + C_1 n + C_2 \ln n + (C_3 + \ln \Omega) + \mathcal{F}_n + \frac{C_4}{\sqrt{n}} \\
&+ \mathcal{O}\bigg(\frac{\sqrt{n}}{M^{11}} + \frac{1}{M^6} + \frac{1}{\sqrt{n}M} + \frac{M^4}{n} + \frac{M^{14}}{n^2} \bigg),
\end{aligned}$$

as $n \to +\infty$ uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \le \delta\}, \ldots, u_m \in \{z \in \mathbb{C} : |z - x_m| \le \delta\}$, where

$$C_1 = F_1^{(\epsilon)} + \int_{\frac{b\rho_1^{2b}}{1-\epsilon}}^{\sigma_\star} f_1(x)dx + \int_{\sigma_\star}^{\frac{b\rho_2^{2b}}{1+\epsilon}} \hat{f}_1(x)dx + \hat{F}_1^{(\epsilon)}, \qquad C_2 = F_2 + \hat{F}_2, \qquad C_4 = F_4 + \hat{F}_4,$$

$$\begin{aligned}
C_3 + \ln \Omega = F_3^{(n,\epsilon)} &+ \Big(\alpha - \frac{1}{2} + \theta_{1,+}^{(n,\epsilon)} \Big) f_1\Big(\frac{b\rho_1^{2b}}{1-\epsilon} \Big) + \Big(\theta_{2,-}^{(n,\epsilon)} - \alpha - \frac{1}{2} \Big) \hat{f}_1\Big(\frac{b\rho_2^{2b}}{1+\epsilon} \Big) \\
&+ \Big(\frac{1}{2} - \alpha - \theta_\star \Big) \big(f_1(\sigma_\star) - \hat{f}_1(\sigma_\star) \big) + \int_{\frac{b\rho_1^{2b}}{1-\epsilon}}^{\sigma_\star} f(x)dx + \int_{\sigma_\star}^{\frac{b\rho_2^{2b}}{1+\epsilon}} \hat{f}(x)dx \\
&+ \theta_\star \ln \mathsf{Q} - \frac{\ln \mathsf{Q}}{2} \Big(1 - \frac{2\ln(\sigma_2/\sigma_1) + \ln \mathsf{Q}}{2\ln(\rho_2/\rho_1)} \Big) + \hat{F}_3^{(n,\epsilon)}.
\end{aligned}$$

It only remains to show that the constants $\{C_j\}_1^4$ can be expressed as in the statement of Theorem 1.2. This is easily verified for C_1 and C_2. Using the identities

$$f_1(\sigma_\star) - \hat{f}_1(\sigma_\star) = \ln \mathsf{Q}, \qquad \mathcal{I}_1 = \frac{\ln(2\sqrt{\pi})}{2},$$

$$\int_{\frac{b\rho_1^{2b}}{1-\epsilon}}^{\sigma_\star} \frac{b\rho_1^{2b}\mathsf{T}_1(b\rho_1^{2b};\vec{t},\vec{u})}{\Omega(x - b\rho_1^{2b})}dx = \frac{b\rho_1^{2b}\mathsf{T}_1(b\rho_1^{2b};\vec{t},\vec{u})}{\Omega}\Big(\ln \frac{\sigma_\star - b\rho_1^{2b}}{b\rho_1^{2b}} - \ln \frac{\epsilon}{1-\epsilon} \Big),$$

$$-\int_{\sigma_\star}^{\frac{b\rho_2^{2b}}{1+\epsilon}} \frac{b\rho_2^{2b}\hat{\mathsf{T}}_1(b\rho_2^{2b};\vec{t},\vec{u})}{b\rho_2^{2b} - x}dx = b\rho_2^{2b}\hat{\mathsf{T}}_1(b\rho_2^{2b};\vec{t},\vec{u})\Big(\ln \frac{b\rho_2^{2b}}{b\rho_2^{2b} - \sigma_\star} + \ln \frac{\epsilon}{1+\epsilon} \Big).$$

long but straightforward calculations show that C_3 also can be written as in Theorem 1.2. Moreover, substituting the expressions for F_4 and $\hat{F}_4$ obtained in Lemmas 2.15 and 2.16 into the relation $C_4 = F_4 + \hat{F}_4$, we infer that

$$\begin{aligned}
C_4 = \sqrt{2b}\Big(&\rho_1^b \frac{\rho_1^{2b}\mathsf{T}_2(b\rho_1^{2b};\vec{t},\vec{u}) - 5\mathsf{T}_1(b\rho_1^{2b};\vec{t},\vec{u})}{\Omega} - \rho_2^b\big(\rho_2^{2b}\hat{\mathsf{T}}_2(b\rho_2^{2b};\vec{t},\vec{u}) + 5\hat{\mathsf{T}}_1(b\rho_2^{2b};\vec{t},\vec{u}) \big) \Big)\mathcal{I} \\
+ \frac{10\sqrt{2b}}{3} &\Big(\rho_1^b \frac{\mathsf{T}_1(b\rho_1^{2b};\vec{t},\vec{u})}{\Omega} + \rho_2^b \hat{\mathsf{T}}_1(b\rho_2^{2b};\vec{t},\vec{u}) \Big)(\mathcal{I}_2 - \mathcal{I}_4) \\
+ \sqrt{2b}&\Big[\rho_1^b \frac{\mathsf{T}_1(b\rho_1^{2b};\vec{t},\vec{u})}{\Omega}\Big(\frac{2}{3} - \rho_1^{2b}\frac{\mathsf{T}_1(b\rho_1^{2b};\vec{t},\vec{u})}{\Omega} \Big) + \rho_2^b\hat{\mathsf{T}}_1(b\rho_2^{2b};\vec{t},\vec{u})\Big(\frac{2}{3} - \rho_2^{2b}\hat{\mathsf{T}}_1(b\rho_2^{2b};\vec{t},\vec{u}) \Big) \Big]\mathcal{I}_3.
\end{aligned}$$

Using the identities $\mathcal{I}_3 = \mathcal{I}$ and $\mathcal{I}_2 - \mathcal{I}_4 = \mathcal{I}$, which can be obtained via integration by parts (see [10, Lemma 2.10] for details), we conclude that C_4 also can be expressed as in Theorem 1.2. This finishes the proof of Theorem 1.2. $\square$

A Balayage of radially symmetric measures

The measure μ_h is defined as the unique minimizer of the energy functional

$$I[\nu] = \iint \ln \frac{1}{|z-w|} \nu(d^2z)\nu(d^2w) + \int \hat{Q}(z)\nu(d^2z), \quad \hat{Q}(z) := \begin{cases} |z|^{2b}, & \text{if } |z| \in [0,\rho_1] \cup [\rho_2, +\infty), \\ +\infty, & \text{otherwise,} \end{cases}$$

among all Borel probability measures ν on $\mathbb{C}$. In this appendix, we present a derivation of the expression (1.4) for μ_h.

Let $C = \{z \in \mathbb{C} : |z| \in [0,\rho_1] \cup [\rho_2, +\infty)\}$ and $G = \mathbb{C} \setminus C = \{z \in \mathbb{C} : |z| \in (\rho_1, \rho_2)\}$. The unique minimizer of

$$\tilde{I}[\nu] = \iint \ln \frac{1}{|z-w|} \nu(d^2z)\nu(d^2w) + \int |z|^{2b}\nu(d^2z)$$

among all Borel probability measures ν on $\mathbb{C}$ is given by

$$\mu(d^2z) = \chi_{[0,b^{-\frac{1}{2b}}]}(|z|)\frac{1}{4\pi}\Delta|z|^{2b}d^2z = \chi_{[0,b^{-\frac{1}{2b}}]}(|z|)\frac{b^2}{\pi}|z|^{2b-2}d^2z,$$

(see [55, Example IV.6.2]). We start with the following lemma.

Lemma A.1. *We have*

$$\mu_h = \mu \cdot \chi_C + \hat{\mu}, \tag{A.1}$$

where $\hat{\mu} := Bal(\mu \cdot \chi_G, \partial G)$ is the balayage of $\mu \cdot \chi_G$ onto $\partial G = \{|z| = \rho_1\} \cup \{|z| = \rho_2\}$.

Proof. Let us first introduce some notation and background from [55]. Given a Borel probability measure ν, define

$$U^\nu(z) := \int_{\mathbb{C}} \ln \frac{1}{|z-w|} \nu(d^2w), \qquad z \in \mathbb{C}.$$

By [55, Theorem I.1.3 (d) and (f)], μ and μ_h satisfy the following conditions: there exist $F, F_h \in \mathbb{R}$ such that

$$\begin{cases} U^\mu(z) + \frac{1}{2}|z|^{2b} \geq F, & z \in \mathbb{C}, \\ U^\mu(z) + \frac{1}{2}|z|^{2b} = F, & |z| \leq b^{-\frac{1}{2b}}, \end{cases} \tag{A.2}$$

$$\begin{cases} U^{\mu_h}(z) + \frac{1}{2}\hat{Q}(z) \geq F_h, & z \in \mathbb{C}, \\ U^{\mu_h}(z) + \frac{1}{2}\hat{Q}(z) = F_h, & z \in \operatorname{supp}(\mu_h). \end{cases} \tag{A.3}$$

Furthermore, by [55, Theorem I.3.3], the conditions (A.3) uniquely characterize μ_h in the sense that if ν is a Borel probability measure with compact support which satisfies $U^\nu(z) + \frac{1}{2}\hat{Q}(z) \geq F_\nu$ for some constant F_ν and all $z \in \mathbb{C}$ and if $U^\nu(z) + \frac{1}{2}\hat{Q}(z) = F_\nu$ for all $z \in \operatorname{supp}(\nu)$, then $\nu = \mu_h$.

Let $\tilde{\mu}_h$ denote the right-hand side of (A.1). We will show that $\tilde{\mu}_h = \mu_h$ using the aforementioned uniqueness theorem, namely [55, Theorem I.3.3]. Since $\hat{\mu}$ is the balayage of $\mu \cdot \chi_G$ onto ∂G, [55, Theorem II.4.1] shows that

$$\begin{cases} U^{\hat{\mu}}(z) \leq U^{\mu \cdot \chi_G}(z), & z \in \mathbb{C}, \\ U^{\hat{\mu}}(z) = U^{\mu \cdot \chi_G}(z), & z \in C. \end{cases}$$

Combining the above with (A.2) and using that $U^\mu = U^{\mu\cdot\chi_C} + U^{\mu\cdot\chi_G}$, we obtain (recall that $\hat{Q}(z) = +\infty$ for $z \in G$)

$$\begin{cases} U^{\mu\cdot\chi_C}(z) + U^{\hat{\mu}}(z) + \frac{1}{2}\hat{Q}(z) \geq F, & |z| \notin [0,\rho_1] \cup [\rho_2, b^{-\frac{1}{2b}}], \\ U^{\mu\cdot\chi_C}(z) + U^{\hat{\mu}}(z) + \frac{1}{2}\hat{Q}(z) = F, & |z| \in [0,\rho_1] \cup [\rho_2, b^{-\frac{1}{2b}}]. \end{cases}$$

Since $U^{\mu\cdot\chi_C} + U^{\hat{\mu}} = U^{\tilde{\mu}_h}$, [55, Theorem I.3.3] applies and gives $\tilde{\mu}_h = \mu_h$. ◻

It only remains to compute $\hat{\mu} := \mathrm{Bal}(\mu \cdot \chi_G\,, \partial G)$ explicitly. Recall that

$$\mu(d^2z) = \frac{b^2}{\pi}|z|^{2b-2}\chi_{[0,b^{-\frac{1}{2b}}]}(r)d^2z, \qquad z = re^{i\theta},\ r \geq 0,\ \theta \in (-\pi,\pi],$$

and that $0 < \rho_1 < \rho_2 < b^{-\frac{1}{2b}}$. Hence $\mu \cdot \chi_G$ is the radially symmetric measure given by

$$(\mu \cdot \chi_G)(d^2z) = f(r)\,\chi_{(\rho_1,\rho_2)}(r)\,rdr\frac{d\theta}{\pi},$$

where $f(r) := b^2 r^{2b-2}$. Fix $z \in \mathbb{C} \setminus \overline{G}$ and compute the logarithmic potential

$$U^{\mu\cdot\chi_G}(z) = \int_{\mathbb{C}} \ln\frac{1}{|z-w|}\,(\mu\cdot\chi_G)(d^2w) = 2\int_{\rho_1}^{\rho_2} rf(r)\,dr\frac{1}{2\pi}\int_0^{2\pi}\ln\frac{1}{|z-re^{i\theta}|}\,d\theta.$$

Now we use the familiar integral (e.g. [55, Chapter 0, page 22])

$$\frac{1}{2\pi}\int_0^{2\pi}\ln\frac{1}{|z-re^{i\theta}|}\,d\theta = \begin{cases}\ln\frac{1}{r}, & |z| \leq r,\\ \ln\frac{1}{|z|}, & |z| \geq r.\end{cases} \tag{A.4}$$

It follows that

$$U^{\mu\cdot\chi_G}(z) = \begin{cases}\mathcal{C}_1, & \text{if } |z| \leq \rho_1,\\ \mathcal{C}_2\ln\frac{1}{|z|}, & \text{if } |z| \geq \rho_2,\end{cases} \quad \text{where} \quad \mathcal{C}_1 = 2\int_{\rho_1}^{\rho_2} rf(r)\ln\frac{1}{r}\,dr, \quad \mathcal{C}_2 = 2\int_{\rho_1}^{\rho_2} rf(r)\,dr.$$

We make the ansatz that $\hat{\mu}$ has the form (see [55, Theorem II.4.1])

$$\hat{\mu}(d^2z) = \sigma_1\,\delta_{\rho_1}(r)\,dr\frac{d\theta}{2\pi} + \sigma_2\,\delta_{\rho_2}(r)\,dr\frac{d\theta}{2\pi}, \qquad z = re^{i\theta},\ r \geq 0,\ \theta \in (-\pi,\pi], \tag{A.5}$$

for some constants σ_1 and σ_2. Using again (A.4) we obtain

$$U^{\hat{\mu}}(z) = \begin{cases}\sigma_1\ln\frac{1}{\rho_1} + \sigma_2\ln\frac{1}{\rho_2}, & \text{if } |z| \leq \rho_1,\\ (\sigma_1+\sigma_2)\ln\frac{1}{|z|}, & \text{if } |z| \geq \rho_2.\end{cases}$$

By definition of $\hat{\mu}$, we must have $U^{\mu\cdot\chi_G}(z) = U^{\hat{\mu}}(z)$ for all $z \notin \overline{G}$, so we have the system

$$\begin{cases}\sigma_1\ln\frac{1}{\rho_1} + \sigma_2\ln\frac{1}{\rho_2} = \mathcal{C}_1,\\ \sigma_1 + \sigma_2 = \mathcal{C}_2.\end{cases}$$

The solution is

$$\sigma_1 = \frac{\mathcal{C}_1 - \mathcal{C}_2\ln\frac{1}{\rho_2}}{\ln\frac{\rho_2}{\rho_1}}, \qquad \sigma_2 = \frac{\mathcal{C}_2\ln\frac{1}{\rho_1} - \mathcal{C}_1}{\ln\frac{\rho_2}{\rho_1}}. \tag{A.6}$$

Recalling that $f(r) = b^2r^{2b-2}$, we get

$$\mathcal{C}_1 = 2\int_{\rho_1}^{\rho_2} rf(r)\ln\frac{1}{r}\,dr = -b\rho_2^{2b}\ln(\rho_2) + b\rho_1^{2b}\ln(\rho_1) + \tfrac{1}{2}(\rho_2^{2b} - \rho_1^{2b}),$$
$$\mathcal{C}_2 = 2\int_{\rho_1}^{\rho_2} rf(r)\,dr = b(\rho_2^{2b} - \rho_1^{2b}).$$

Substituting the above expressions into (A.6) and recalling (A.1) and (A.5), we arrive at (1.4).

B Proof of Lemma 1.1

Define ϕ_1 and ϕ_2 by

$$\phi_1(x;\vec{t},\vec{u}) := 1 + \mathsf{T}_0(x;\vec{t},\vec{u}) + \hat{\mathsf{T}}_0(b\rho_2^{2b};\vec{u}) = 1 + \sum_{\ell=1}^{m}\omega_\ell e^{-\frac{t_\ell}{b}(x-b\rho_1^{2b})} + \sum_{\ell=m+1}^{2m}\omega_\ell,$$

$$\phi_2(x;\vec{t},\vec{u}) := 1 - \hat{\mathsf{T}}_0(x;\vec{t},\vec{u}) + \hat{\mathsf{T}}_0(b\rho_2^{2b};\vec{u}) = 1 - \sum_{\ell=m+1}^{2m} \omega_\ell e^{-\frac{t_\ell}{b}(b\rho_2^{2b}-x)} + \sum_{\ell=m+1}^{2m} \omega_\ell.$$

A calculation gives

$$\begin{aligned}\partial_{u_{2m}}\phi_1 &= e^{u_1+\cdots+u_{2m}} e^{-\frac{t_1}{b}(x-b\rho_1^{2b})} + \sum_{\ell=2}^{m} e^{u_\ell+\cdots+u_{2m}}\big(e^{-\frac{t_\ell}{b}(x-b\rho_1^{2b})} - e^{-\frac{t_{\ell-1}}{b}(x-b\rho_1^{2b})}\big) \\ &\quad + e^{u_{m+1}+\cdots+u_{2m}}\big(1 - e^{-\frac{t_m}{b}(x-b\rho_1^{2b})}\big), \\ \partial_{u_{2m}}\phi_2 &= e^{u_{m+1}+\cdots+u_{2m}}\big(1 - e^{-\frac{t_{m+1}}{b}(b\rho_2^{2b}-x)}\big) + \sum_{\ell=m+2}^{2m} e^{u_\ell+\cdots+u_{2m}}\big(e^{-\frac{t_{\ell-1}}{b}(b\rho_2^{2b}-x)} - e^{-\frac{t_\ell}{b}(b\rho_2^{2b}-x)}\big).\end{aligned}$$

Using that $t_1 > \cdots > t_m \geq 0$, we see that $\partial_{u_{2m}}\phi_1(x;\vec{t},\vec{u}) > 0$ for any $x \geq b\rho_1^{2b}$, and using that $0 \leq t_{m+1} < \cdots < t_{2m}$, we see that $\partial_{u_{2m}}\phi_2(x;\vec{t},\vec{u}) \geq 0$ for any $x \leq b\rho_2^{2b}$. In view of the limits

$$\lim_{u_{2m}\to-\infty} \phi_1(x;\vec{t},\vec{u}) = 0, \qquad \lim_{u_{2m}\to-\infty} \phi_2(x;\vec{t},\vec{u}) = e^{-\frac{t_{2m}}{b}(b\rho_2^{2b}-x)} > 0,$$

the desired inequalities follow.

C Uniform asymptotics of the incomplete gamma function

Lemma C.1 (From [54, eq 8.11.2]). *Let $a > 0$ be fixed. As $z \to +\infty$,*

$$\gamma(a,z) = \Gamma(a) + \mathcal{O}(e^{-\frac{z}{2}}).$$

Lemma C.2 (From [60, Section 11.2.4]). *We have*

$$\frac{\gamma(a,z)}{\Gamma(a)} = \frac{1}{2}\mathrm{erfc}(-\eta\sqrt{a/2}) - R_a(\eta), \qquad R_a(\eta) = \frac{e^{-\frac{1}{2}a\eta^2}}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2} g(u)du,$$

where erfc *is the complementary error function,*

$$\lambda = \frac{z}{a}, \qquad \eta = (\lambda-1)\sqrt{\frac{2(\lambda-1-\ln\lambda)}{(\lambda-1)^2}}, \qquad g(u) := \frac{dt}{du}\frac{1}{\lambda-t} + \frac{1}{u+i\eta}. \tag{C.1}$$

The variables t and u are related by the bijection $t \mapsto u$ from $\mathcal{L} := \{\frac{\theta}{\sin\theta}e^{i\theta} : -\pi < \theta < \pi\}$ to $\mathbb{R}$ given by

$$u = -i(t-1)\sqrt{\frac{2(t-1-\ln t)}{(t-1)^2}}, \qquad t \in \mathcal{L}. \tag{C.2}$$

The principal branch is taken for the roots in (C.1) and (C.2). In addition,

$$R_a(\eta) \sim \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}}\sum_{j=0}^{\infty}\frac{c_j(\eta)}{a^j}, \qquad \text{as } a \to +\infty \tag{C.3}$$

uniformly for $z \in [0,\infty)$ where all coefficients $c_j(\eta)$ are bounded functions of $\eta \in \mathbb{R}$ (i.e. bounded for $\lambda \in (0,+\infty)$). The first two coefficients are given by (see [60, p. 312])

$$c_0(\eta) = \frac{1}{\lambda-1} - \frac{1}{\eta}, \qquad c_1(\eta) = \frac{1}{\eta^3} - \frac{1}{(\lambda-1)^3} - \frac{1}{(\lambda-1)^2} - \frac{1}{12(\lambda-1)}.$$

More generally, we have

$$c_j(\eta) = \frac{1}{\eta}\frac{d}{d\eta}c_{j-1}(\eta) + \frac{\gamma_j}{\lambda-1}, \qquad j \geq 1, \tag{C.4}$$

where the γ_j are the Stirling coefficients

$$\gamma_j = \frac{(-1)^j}{2^j\, j!}\left[\frac{d^{2j}}{dx^{2j}}\left(\frac{1}{2}\frac{x^2}{x-\ln(1+x)}\right)^{j+\frac{1}{2}}\right]_{x=0}.$$

In particular, the following hold:

(i) As $a \to +\infty$, $\gamma(a,\lambda a) = \Gamma(a)\big(1 + \mathcal{O}(e^{-\frac{a\eta^2}{2}})\big)$ uniformly for $\lambda \geq 1+\delta$ for each fixed $\delta > 0$.

(ii) As $a \to +\infty$, $\gamma(a,\lambda a) = \Gamma(a)\mathcal{O}(e^{-\frac{a\eta^2}{2}})$ uniformly for λ in compact subsets of $(0,1)$,

The following lemma is essentially a result of Tricomi [61]; however, the coefficients appearing in Lemma C.3 below are written in a non-recursive way using [10, Lemma A.4].

Lemma C.3 (From [10, Lemma A.4])**.** *Let $N \geq 0$ be an integer, let η be as in (C.1), let $\varphi_j(\lambda) := \frac{(-1)^{j+1}(2j-1)!!}{\eta^{2j+1}}$, and let $S(\varphi_j(\lambda))$ denote the singular part of $\varphi_j(\lambda)$ at $\lambda = 1$, i.e., $S(\varphi_j(\lambda))$ is the sum of the singular terms in the Laurent expansion of $\varphi_j(\lambda)$ at $\lambda = 1$. The first $S(\varphi_j(\lambda))$ are given by*

$$S(\varphi_0(\lambda)) = -\frac{1}{\lambda-1}, \qquad S(\varphi_1(\lambda)) = \frac{1}{(\lambda-1)^3} + \frac{1}{(\lambda-1)^2} + \frac{1}{12(\lambda-1)},$$
$$S(\varphi_2(\lambda)) = -\frac{3}{(\lambda-1)^5} - \frac{5}{(\lambda-1)^4} - \frac{25}{12(\lambda-1)^3} - \frac{1}{12(\lambda-1)^2} - \frac{1}{288(\lambda-1)}.$$

(i) As $a \to +\infty$, uniformly for $\lambda \geq 1 + \frac{1}{\sqrt{a}}$,

$$\frac{\gamma(a,\lambda a)}{\Gamma(a)} = 1 + \frac{e^{-\frac{a}{2}\eta^2}}{\sqrt{2\pi}}\left\{\sum_{j=0}^{N-1}\frac{S(\varphi_j(\lambda))}{a^{j+\frac{1}{2}}} + \mathcal{O}\left(\frac{1}{a^{N+\frac{1}{2}}}\right) + \mathcal{O}\left(\frac{1}{(a\eta^2)^{N+\frac{1}{2}}}\right)\right\}.$$

(ii) As $a \to +\infty$, uniformly for $\lambda \in [\epsilon, 1 - \frac{1}{\sqrt{a}}]$ for any fixed $\epsilon > 0$,

$$\frac{\gamma(a,\lambda a)}{\Gamma(a)} = \frac{e^{-\frac{a}{2}\eta^2}}{\sqrt{2\pi}}\left\{\sum_{j=0}^{N-1}\frac{S(\varphi_j(\lambda))}{a^{j+\frac{1}{2}}} + \mathcal{O}\left(\frac{1}{a^{N+\frac{1}{2}}}\right) + \mathcal{O}\left(\frac{1}{(a\eta^2)^{N+\frac{1}{2}}}\right)\right\}.$$

Acknowledgements. CC acknowledges support from the Swedish Research Council, Grant No. 2021-04626. JL acknowledges support from the Swedish Research Council, Grant No. 2021-03877, and the Ruth and Nils-Erik Stenbäck Foundation.

References

[1] K. Adhikari, Hole probabilities for β-ensembles and determinantal point processes in the complex plane, *Electron. J. Probab.* **23** (2018), 21 pp.

[2] K. Adhikari and N.K. Reddy, Hole probabilities for finite and infinite ginibre ensembles, *Int. Math. Res. Not. IMRN* **2017** (2017), no. 21, 6694–6730.

[3] G. Akemann, S.-S. Byun and M. Ebke, Universality of the number variance in rotational invariant two-dimensional Coulomb gases, *J. Stat. Phys.* **190** (2023), no.1, Paper No. 9, 34 pp.

[4] G. Akemann, S.-S. Byun, M. Ebke and G. Schehr, Universality in the number variance and counting statistics of the real and symplectic Ginibre ensemble, *J. Phys. A: Math. Theor.* **56** (2023), 495202.

[5] G. Akemann, J. Ipsen and E. Strahov, Permanental processes from products of complex and quaternionic induced Ginibre ensembles, *Random Matrices Theory Appl.* **3** (2014), no. 4, 1450014, 54 pp.

[6] G. Akemann, M. J. Phillips and L. Shifrin, Gap probabilities in non-Hermitian random matrix theory, *J. Math. Phys.* **50** (2009), no. 6, 063504, 32 pp.

[7] G. Akemann and E. Strahov, Hole probabilities and overcrowding estimates for products of complex Gaussian matrices, *J. Stat. Phys.* **151** (2013), no. 6, 987–1003.

[8] G. Akemann and G. Vernizzi, Characteristic polynomials of complex random matrix models, *Nuclear Phys. B* **660** (2003), no. 3, 532–556.

[9] Y. Ameur, C. Charlier and J. Cronwall, The two-dimensional Coulomb gas: fluctuations through a spectral gap, arXiv:2210.13959.

[10] Y. Ameur, C. Charlier, J. Cronwall and J. Lenells, Exponential moments for disk counting statistics at the hard edge of random normal matrices, *J. Spectr. Theory* **13** (2023), no.3, 841–902.

[11] Y. Ameur, N-G. Kang and S-M. Seo, The random normal matrix model: insertion of a point charge, *Potential Anal.* **58** (2023), no. 2, 331–372.

[12] R. Bauerschmidt, P. Bourgade, M. Nikula and H.-T. Yau, The two-dimensional Coulomb plasma: quasi-free approximation and central limit theorem, *Adv. Theor. Math. Phys.* **23** (2019), 841–1002.

[13] P. Billingsley, Probability and measure. Anniversary edition. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 2012.

[14] E. Blackstone, C. Charlier and J. Lenells, Oscillatory asymptotics for Airy kernel determinants on two intervals, *Int. Math. Res. Not. IMRN* **2022** (2022), no. 4, 2636–2687.

[15] E. Blackstone, C. Charlier and J. Lenells, Gap probabilities in the bulk of the Airy process, *Random Matrices Theory Appl.* **11** (2022), no. 2, Paper No. 2250022, 30 pp.

[16] E. Blackstone, C. Charlier and J. Lenells, The Bessel kernel determinant on large intervals and Birkhoff's ergodic theorem, *Comm. Pure Appl. Math.* **76** (2023), no.11, 3300–3345.

[17] G. Bonnet, F. David and B. Eynard, Breakdown of universality in multi-cut matrix models, *J. Phys. A* **33** (2000), no. 38, 6739–6768.

[18] G. Borot and A. Guionnet, Asymptotic expansion of beta matrix models in the multi-cut regime, arXiv:1303.1045.

[19] S.-S. Byun and C. Charlier, On the characteristic polynomial of the eigenvalue moduli of random normal matrices, arXiv:2205.04298.

[20] S.-S. Byun and P.J. Forrester, Progress on the study of the Ginibre ensembles I: GinUE, arXiv:2211.16223.

[21] S.-S. Byun and P.J. Forrester, Progress on the study of the Ginibre ensembles II: GinOE and GinSE, arXiv:2301.05022.

[22] S.-S. Byun, N.-G. Kang, J.O. Lee and J. Lee, Real Eigenvalues of Elliptic Random Matrices, *Int. Math. Res. Not. IMRN*, doi:rnab310, arXiv:2105.11110.

[23] S.-S. Byun, N.-G. Kang and S.-M. Seo, Partition functions of determinantal and Pfaffian Coulomb gases with radially symmetric potentials, *Comm. Math. Phys.* **401** (2023), no.2, 1627–1663.

[24] S.-S. Byun and S.-M. Seo, Random normal matrices in the almost-circular regime, *Bernoulli* **29** (2023), no.2, 1615–1637.

[25] S.-S. Byun and M. Yang, Determinantal Coulomb gas ensembles with a class of discrete rotational symmetric potentials, *SIAM J. Math. Anal.* **55** (2023), no.6, 6867–6897.

[26] L. Charles and B. Estienne, Entanglement entropy and Berezin-Toeplitz operators, *Comm. Math. Phys.* **376** (2020), no. 1, 521–554.

[27] C. Charlier, Asymptotics of determinants with a rotation-invariant weight and discontinuities along circles, *Adv. Math.* **408** (2022), Paper No. 108600.

[28] C. Charlier, Large gap asymptotics on annuli in the random normal matrix model, *Math. Ann.* (2023), `https://doi.org/10.1007/s00208-023-02603-z`.

[29] C. Charlier, B. Fahs, C. Webb and M.D. Wong, Asymptotics of Hankel determinants with a multi-cut regular potential and Fisher-Hartwig singularities, arXiv:2111.08395 (to appear in *Mem. Amer. Math. Soc.*).

[30] C. Charlier and J. Lenells, Exponential moments for disk counting statistics of random normal matrices in the critical regime, *Nonlinearity* **36** (2023), no.3, 1593–1616.

[31] L.-L. Chau and O. Zaboronsky, *On the structure of correlation functions in the normal matrix model*, *Comm. Math. Phys.* **196** (1998), no. 1, 203–247.

[32] T. Claeys, T. Grava and K.T.-R. McLaughlin, Asymptotics for the partition function in two-cut random matrix models, *Comm. Math. Phys.* **339** (2015), no. 2, 513–587.

[33] A. Deaño and N. Simm, Characteristic polynomials of complex random matrices and Painlevé transcendents, *Int. Math. Res. Not. IMRN* **2022** (2022), no. 1, 210–264.

[34] P. Deift, A. Its and X. Zhou, A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics, *Ann. of Math.* (2) **146** (1997), no. 1, 149–235.

[35] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Comm. Pure Appl. Math.* **52** (1999), no. 11, 1335–1425.

[36] P. Elbau and G. Felder, Density of eigenvalues of random normal matrices, *Comm. Math. Phys.* **259** (2005), no. 2, 433–450.

[37] B. Fahs and I. Krasovsky, Sine-kernel determinant on two large intervals, *Comm. Pure Appl. Math.* (2023), `https://doi.org/10.1002/cpa.22147`.

[38] W. FitzGerald and N. Simm, Fluctuations and correlations for products of real asymmetric random matrices, *Ann. Inst. Henri Poincaré Probab. Stat.* **59** (2023), no.4, 2308–2342.

[39] P.J. Forrester, Some statistical properties of the eigenvalues of complex random matrices, *Phys. Lett. A* **169** (1992), no. 1-2, 21–24.

[40] S. Ghosh and A. Nishry, Point Processes, Hole Events, and Large Deviations: Random Complex Zeros and Coulomb Gases, *Constr. Approx.* **48** (2018), no. 1, 101–136.

[41] T. Grava, Partition function for multi-cut matrix models, *J. Phys. A* **39** (2006), no. 28, 8905–8919.

[42] R. Grobe, F. Haake, H.-J. Sommers, Quantum distinction of regular and chaotic dissipative motion, *Phys. Rev. Lett.* **61** (1988), no. 17, 1899–1902.

[43] H. Hedenmalm and N. Makarov, Coulomb gas ensembles and Laplacian growth, *Proc. Lond. Math. Soc.* (3) **106** (2013), no. 4, 859–907.

[44] B. Jancovici, J. Lebowitz and G. Manificat, Large charge fluctuations in classical Coulomb systems, *J. Statist. Phys.* **72** (1993), no. 3-4, 773–787.

[45] E. Kanzieper, M. Poplavskyi, C. Timm, R. Tribe, O. Zaboronski, What is the probability that a large random matrix has no real eigenvalues?, *Ann. Appl. Probab.* **26** (2016), no.5, 2733–2753.

[46] I. Krasovsky and T.-H. Maroudas, Airy-kernel determinant on two large intervals, arXiv:2108.04495.

[47] B. Lacroix-A-Chez-Toine, S.N. Majumdar and G. Schehr, Rotating trapped fermions in two dimensions and the complex Ginibre ensemble: Exact results for the entanglement entropy and number variance, *Phys. Rev. A* **99** (2019), 021602.

[48] B. Lacroix-A-Chez-Toine, J.A.M. Garzón, C.S.H. Calva, I.P. Castillo, A. Kundu, S.N. Majumdar, and G. Schehr, Intermediate deviation regime for the full eigenvalue statistics in the complex Ginibre ensemble, *Phys. Rev. E* **100** (2019), 012137.

[49] T. Leblé and S. Serfaty, Large deviation principle for empirical fields of log and Riesz gases, *Invent. Math.* **210** (2017), no. 3, 645–757.

[50] T. Leblé and S. Serfaty, Fluctuations of two dimensional Coulomb gases, *Geom. Funct. Anal.* **28** (2018), no. 2, 443–508.

[51] O. Marchal, Asymptotic expansion of Toeplitz determinants of an indicator function with discrete rotational symmetry and powers of random unitary matrices, *Lett. Math. Phys.* **113** (2023), no.4, Paper No. 78, 16 pp.

[52] M.L. Mehta, Random matrices. *Pure and Applied Mathematics* (Amsterdam), Vol. 142, 3rd ed., Elsevier/Academic Press, Amsterdam, 2004.

[53] T. Nagao, G. Akemann, M. Kieburg and I. Parra, Families of two-dimensional Coulomb gases on an ellipse: correlation functions and universality, *J. Phys. A* **53** (2020), no. 7, 075201, 36 pp.

[54] F.W.J. Olver, A.B. Olde Daalhuis, D.W. Lozier, B.I. Schneider, R.F. Boisvert, C.W. Clark, B.R. Miller and B.V. Saunders, NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.13 of 2016-09-16.

[55] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1997.

[56] S.-M. Seo, Edge behavior of two-dimensional Coulomb gases near a hard wall, *Ann. Henri Poincaré* **23** (2022), no. 6, 2247–2275.

[57] M. Shcherbina, Fluctuations of Linear Eigenvalue Statistics of β Matrix Models in the Multi-cut Regime, *J. Stat. Phys.* **151** (2013), no. 6, 1004–1034.

[58] N.R. Smith, P. Le Doussal, S.N. Majumdar and G. Schehr, Counting statistics for non-interacting fermions in a d-dimensional potential, *Phys. Rev. E* **103**, no. 3, 8 pp.

[59] N.R. Smith, P. Le Doussal, S.N. Majumdar and G. Schehr, Counting statistics for non-interacting fermions in a rotating trap, *Phys. Rev. A* **105** (2022), 043315.

[60] N.M. Temme, *Special functions: An introduction to the classical functions of mathematical physics*, John Wiley & Sons (1996).

[61] F. G. Tricomi, Asymptotische Eigenschaften der unvollständigen Gammafunktion, *Math. Z.* **53** (1950), 136–148.

[62] C. Webb and M.D. Wong, On the moments of the characteristic polynomial of a Ginibre random matrix, *Proc. Lond. Math. Soc.* (3) **118** (2019), no. 5, 1017–1056.

[63] H. Widom, Asymptotics for the Fredholm determinant of the sine kernel on a union of intervals, *Comm. Math. Phys.* **171** (1995), 159–180.

[64] A. Zabrodin and P. Wiegmann, Large-N expansion for the 2D Dyson gas, *J. Phys. A* **39** (2006), no. 28, 8933–8963.

[65] K. Życzkowski and H.-J. Sommers, Truncations of random unitary matrices, *J. Phys. A* **33** (2000), no. 10, 2045–2057.