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Information Advantage and Dominant Strategies in Second-Price Auctions

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Abstract

We study a general model of common-value second-price auctions with differential information. We show that one of the bidders has an information advantage over the other bidders if and only if he possesses a dominant strategy. A dominant strategy is, in fact, unique, and is given by the conditional expectation of the common value with respect to his information field. Furthermore, when a bidder has information advantage, other bidders cannot make a profit.

Keywords*:* common-value second-price auctions, differential information, dominant strategies, information advantage.

JEL Classification: C72, D44, D82.

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1 Introduction

In a second-price sealed-bid auction, each bidder independently submits a bid. The bidder with the highest bid wins the auctioned object and pays the second-highest bid. The other bidders receive nothing but do not incur any costs.[1] It is well known that in independent private-value second-price auctions, bidding his value of the object is a dominant strategy for every bidder (e.g., Vickrey (1961)). In common-value second-price auctions with asymmetric information (first studied by Wilson (1977)) the situation is more complicated and bidders do not usually have dominant strategies.

In this paper we consider common-value second-price auctions with differential information, and prove an equivalence between the existence of a dominant strategy of a bidder, and a possession by him of a certain information advantage over all other bidders. More precisely, we show (see Theorem 3) that a bidder possesses a dominant strategy in such an auction if and only if his information allows him to predict the value of the auctioned object just as well as all bidders can do collectively. The latter notion of information advantage also manifests itself in terms of resulting outcomes: when a bidder with information advantage uses his dominant strategy, other bidders cannot make a profit. That is, no matter what profile of strategies the other bidders use, the conditional expected payoff of any one of them (with respect to his information field) is not positive at any state of nature.

We use a general model of differential information and impose no restrictions on the space of states of nature and on the information fields of bidders. In particular, our framework allows for discrete as well as continuous private information, and it includes Savage (1954) information partitions model as well as Harsanyi (1967) types model.

The issue of information advantage is a recurrent topic in the literature of auctions. Milgrom (1979) and Milgrom and Weber (1982b, Theorem 3) have shown that in a first-price auction if Bidder A has a finer information than Bidder B, then in any Bayesian equilibrium the expected payoff of Bidder B is zero. However, the information advantage of a bidder in a common-value second-price auction is not always rewarded in a Bayesian equilibrium (see Example 2).

[1] See, for example, Vickrey (1961), McAfee and Mcmillan (1987), Wilson (1992) and Klemperer (1999).

2 The Model

Consider a group of *bidders* $N = \{1, 2, ..., n\}$ who compete to acquire an indivisible object through a second-price auction. The value of the object is the same for all the bidders but there is uncertainty about the common value of the object. This uncertainty is described by a probability space $(\Omega, \mathcal{F}, \mu)$, where Ω is the set of *states of nature*, $\mathcal{F}$ is σ-field of subsets of Ω, and μ is σ additive probability measure on $(\Omega, \mathcal{F})$ (we interpret μ as the *common prior* of the bidders in N). Once the state of nature is realized the value of the object is determined. To reduce the number of technicalities, it is assumed that the *value* function $v : \Omega \to R_+$ is bounded from above (i.e., v is a non-negative element of $L_\infty(\Omega, \mathcal{F}, \mu)$).

Let $x = (x_1, ..., x_n) \in R^n$. We write $m(x) = |\{i \in N \mid x_i = \max_{j \in N} x_j\}|$. The payoff function of bidder i in a second-price auction with a common value v is a function $u_i : \Omega \times R_+^n \to R$ given by

$$u_i(\omega, x) = \begin{cases} \frac{1}{m(x)}(v(\omega) - \max_{j \in N \setminus \{i\}} x_j) & \text{if } x_i = \max_{j \in N} x_j \\ 0 & \text{otherwise} \end{cases}$$

While the bidders do not observe the state of nature that actually occurs, they may have some information about it. The initial information of bidder i is described by a σ- subfield $\mathcal{F}_i$ of $\mathcal{F}$, his *information field*. That is, given any $\mathcal{F}_i$-measurable set, bidder i knows whether the realized state of nature is included in this set or not.

A common-value second-price auction with differential information is thus described by the collection $G = (N, (\Omega, \mathcal{F}, \mu), v, R_+, \{u_i\}_{i=1}^n, \{\mathcal{F}_i\}_{i=1}^n)$.

Let G be such an auction. A (pure) strategy for bidder i is an integrable function $b_i : \Omega \to R_+$ which is also $\mathcal{F}_i$-measurable. We denote by S_i the set of all pure strategies of bidder i, and by $S = \times_{j \in N} S_j$ the set of all strategy profiles.

Let X be an integrable random variable on Ω, and H is a σ-subfield of $\mathcal{F}$. We denote by $E(X \mid H)$ the conditional expectation of X with respect to H. Our assumption that each bidder's pure strategy is integrable guarantees that for all $i \in N$ and all $b \in S$, $E(u_i(\cdot, b(\cdot)) \mid H)$ is defined μ-almost everywhere for every σ-subfield H of $\mathcal{F}$.

A *Bayesian equilibrium* of G is a profile of strategies $b^* = (b_1^*, ..., b_n^*) \in S$ such that for all $i \in N$ and every $b_i \in S_i$ we have,

$$E(u_i(\cdot, b^*(\cdot)) \mid \mathcal{F}_i)(\omega) \geq E(u_i(\cdot, (b_i(\cdot), b_{-i}^*(\cdot)) \mid \mathcal{F}_i)(\omega)$$

for almost all $\omega \in \Omega$.

3 Dominant Strategies and Information Advantage

Let $G = (N, (\Omega, \mathcal{F}, \mu), v, R_+, \{u_i\}_{i=1}^n, \{\mathcal{F}_i\}_{i=1}^n)$ be a common-value second-price auction with differential information. Denote $S_{-i} = \times_{j \in N \setminus \{i\}} S_j$. We say that a pure strategy b_i of bidder i *dominates* his pure strategy c_i if for all $b_{-i} \in S_{-i}$ and almost all $\omega \in \Omega$,

$$E(u_i(\cdot, (b_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega) \geq E(u_i(\cdot, (c_i(\cdot), b_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega) \quad (1)$$

and there exist $\widehat{b}_{-i} \in S_{-i}$ and event A with $\mu(A) > 0$ such that for all $\omega \in A$,

$$E(u_i(\cdot, (b_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega) > E(u_i(\cdot, (c_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega). \quad (2)$$

A pure strategy b_i of bidder i is *dominant* if it dominates any other strategy of bidder i that differs from b_i on an event with a positive probability.

Note that if $b, c \in S$ and $b_i = c_i$ μ-almost everywhere, then $u_i(\cdot, b(\cdot)) = u_i(\cdot, c(\cdot))$ μ-almost everywhere. Thus we can identify between strategies which are equal almost everywhere when checking the dominance relation or verifying the conditions of Bayesian equilibrium. Since in the sequel we confine ourselves only to these tasks, almost everywhere equal strategies will never be distinguished. Thus, by $f = g$ or $f \geq g$ we will mean that $f(\omega) = g(\omega)$ or $f(\omega) \geq g(\omega)$ for almost every $\omega \in \Omega$; saying that $f = g$ or $f \geq g$ on some event A will mean that the equality or inequality holds pointwise for almost every $\omega \in A$.

From now we fix a common-value second-price auction with differential information, $G = (N, (\Omega, \mathcal{F}, \mu), v, R_+, \{u_i\}_{i=1}^n, \{\mathcal{F}_i\}_{i=1}^n)$. Denote by $\bigvee_{j \in N} \mathcal{F}_j$ the minimal σ-field generated by all $\mathcal{F}_j$.

We start by noting that dominant strategies of bidders in G, provided that they exist, have a very specific form: a bidder bids the expected value of the object, conditional on information that was revealed to him.

Proposition 1 *If $i \in N$ has a dominant strategy in G, then it is equal to $E(v \mid \mathcal{F}_i)$.*

Proof. Denote $b_i = E(v \mid \mathcal{F}_i)$. Let c_i be a dominant strategy of i, and suppose contrary to the assertion that $b_i \neq c_i$. We show that there is $\widehat{b}_{-i} \in S_{-i}$ and an event A with $\mu(A) > 0$ such that for all $\omega \in A$,

$$E(u_i(\cdot, (b_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega) > E(u_i(\cdot, (c_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega). \quad (3)$$

Assume that there is a set $C \in \mathcal{F}_i$ of positive measure on which $b_i > c_i$. Since the union of $\mathcal{F}_i$-measurable sets of the form $C_r = \{\omega \in C \mid b_i(\omega) > r > c_i(\omega)\}$, for all rational r, is equal to C, it follows that there exists r_0 for which $\mu(C_{r_0}) > 0$. Denote $A = C_{r_0}$.

Now take $\widehat{b}_j(\omega) \equiv r_0$ for every $j \in N \setminus \{i\}$, $\omega \in \Omega$. For all $\omega \in A$

$$u_i(\omega, (b_i(\omega), \widehat{b}_{-i}(\omega))) = v(\omega) - r_0$$

and

$$u_i(\omega, (c_i(\omega), \widehat{b}_{-i}(\omega))) = 0.$$

Thus

$$E(u_i(\cdot, (b_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega) = E(v \mid \mathcal{F}_i)(\omega) - r_0 > 0 =$$

$$= E(u_i(\cdot, (c_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega),$$

which establishes (3). But this contradicts our assumption that c_i is a dominant strategy of i.

Now, if there is a set $C \in \mathcal{F}_i$ of positive measure on which $c_i > b_i$, we can construct r_0, A, and $\widehat{b}_{-i}$ by the method we used before. We then get for all $\omega \in A$

$$E(u_i(\cdot, (c_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega) = E(v \mid \mathcal{F}_i)(\omega) - r_0 < 0 =$$

$$= E(u_i(\cdot, (c_i(\cdot), \widehat{b}_{-i}(\cdot))) \mid \mathcal{F}_i)(\omega),$$

and thus (3), as before, in contradiction to the assumed dominance of c_i. ■

Literally taken, the concept of information advantage of a bidder over other bidders can be understood in terms of inclusion of information fields: one is inclined to think of having the largest information field as the ultimate expression of informational advantage of that bidder. However, it leaves out other important cases. There may be no inclusion relation between any two of the information fields of bidders, and yet one of them can possess an undeniable information advantage, as the following example shows.

Example 1 Let $N = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $v(\omega_i) = i$ for $i = 1, 2$ and $v(\omega_3) = 2$. Let $\mathcal{F}_1$ be the minimal field on Ω containing $\{\omega_1\}, \{\omega_2, \omega_3\}$, and $\mathcal{F}_2$ be the minimal field on Ω containing $\{\omega_1, \omega_2\}, \{\omega_3\}$. Clearly $\mathcal{F}_1 \nsubseteq \mathcal{F}_2$. However, bidder 1 always knows the real value of the object, unlike bidder 2. It would be difficult not to interpret this fact as an information advantage of bidder 1.

This brings us to the following definition:

Definition 2 *We say that $i \in N$ has an information advantage over the other bidders in G if*

$$E\left(v \mid \bigvee_{j \in N} \mathcal{F}_j\right) = E(v \mid \mathcal{F}_i) \tag{4}$$

The interpretation of (4) is simple: it says that the information of bidder i allows him to predict the value of the auctioned object just as well as all bidders can do collectively. Having the largest information field implies (4), and thus is a particular form of information advantage. Theorem 3 below shows a somewhat surprising equivalence between the possessions of information advantage and of dominant strategy.

Theorem 3 *Let $G = (N, (\Omega, \mathcal{F}, \mu), v, R_+, \{u_i\}_{i=1}^n, \{\mathcal{F}_i\}_{i=1}^n)$ be a common-value second-price auction with differential information. Then bidder $i \in N$ has a dominant strategy in G if and only if he has information advantage over the other bidders in G.*

Proof. I. We start by treating the "if" direction. Assume throughout that $i = 1$, to make notations simpler. We show first that for every $c_1 \in S_1$ and $b_{-1} \in S_{-1}$,

$$\max\left\{E(v \mid \bigvee_{j \in N} \mathcal{F}_j) - d, 0\right\} \geq E(u_1(\cdot, (c_1(\cdot), b_{-1}(\cdot))) \mid \bigvee_{j \in N} \mathcal{F}_j), \tag{5}$$

where d is a $\bigvee_{j\in N}\mathcal{F}_j$-measurable function given by

$$d(\omega) = \max_{j\in N\setminus\{1\}} b_j(\omega).$$

Indeed, let $B = \{\omega \in \Omega \mid d(\omega) \le c_1(\omega)\} \in \bigvee_{j\in N}\mathcal{F}_j$. Now, for all $\omega \in B$,

$$u_1(\omega, (c_1(\omega), b_{-1}(\omega))) = \frac{1}{m\left((c_1(\omega), b_{-1}(\omega))\right)}\left[v(\omega) - d(\omega)\right].$$

Since $m\left((c_1(\omega), b_{-1}(\omega))\right)$ and $d(\omega)$ are $\bigvee_{j\in N}\mathcal{F}_j$-measurable,

$$E(u_1(\cdot, (c_1(\cdot), b_{-1}(\cdot)) \mid \bigvee_{j\in N}\mathcal{F}_j) =$$

$$= \frac{1}{m(c_1, b_{-1})}\left[E(v \mid \bigvee_{j\in N}\mathcal{F}_j) - d\right] \le \max\left\{E(v \mid \bigvee_{j\in N}\mathcal{F}_j) - d, 0\right\}. \tag{6}$$

If $D = \{\omega \in \Omega \mid d(\omega) > c_1(\omega)\} \in \bigvee_{j\in N}\mathcal{F}_j$, then bidder 1 loses the auction at every $\omega \in D$, and so $E(u_1(\cdot, (c_1(\cdot), b_{-1}(\cdot)) \mid \bigvee_{j\in N}\mathcal{F}_j) = 0$ on D. All in all, it implies (5).

Now, given $c_1 \in S_1$, $c_1 \ne E(v \mid \mathcal{F}_1)$, we have to show that $b_1 = E(v \mid \mathcal{F}_1)$ dominates c_1. Let $b_{-1} \in S_{-1}$. Looking at the proof of (5) with b_1 instead of c_1, we see that the last inequality in (6) holds as equality, since

$$E(v \mid \bigvee_{j\in N}\mathcal{F}_j) - d = E(v \mid \mathcal{F}_1) - d = b_1 - d \ge 0$$

on B, and if $b_1(\omega) - d(\omega) > 0$ then $m\left((c_1(\omega), b_{-1}(\omega))\right) = 1$. And on D, as before,

$$E(u_1(\cdot, (b_1(\cdot), b_{-1}(\cdot)) \mid \bigvee_{j\in N}\mathcal{F}_j) = 0 =$$

$$= \max\left\{E(v \mid \bigvee_{j\in N}\mathcal{F}_j) - d, 0\right\}.$$

Thus

$$E(u_1(\cdot,(b_1(\cdot),b_{-1}(\cdot)) \mid \bigvee_{j\in N} \mathcal{F}_j) = \max\left\{E(v \mid \bigvee_{j\in N} \mathcal{F}_j) - d, 0\right\},$$

which, together with (5), implies

$$E(u_1(\cdot,(b_1(\cdot),b_{-1}(\cdot)) \mid \bigvee_{j\in N} \mathcal{F}_j) \geq E(u_1(\cdot,(c_1(\cdot),b_{-1}(\cdot)) \mid \bigvee_{j\in N} \mathcal{F}_j)$$

Since for every $\mathcal{F}$-measurable and integrable random variable X

$$E\left(E\left(X \mid \bigvee_{j\in N} \mathcal{F}_j\right) \mid \mathcal{F}_i\right) = E\left(X \mid \mathcal{F}_i\right) \tag{7}$$

(Theorem 34.4 in Billingsley (1995)), the above inequality implies

$$E(u_1(\cdot,(b_1(\cdot),b_{-1}(\cdot))) \mid \mathcal{F}_1) \geq E(u_1(\cdot,(c_1(\cdot),b_{-1}(\cdot))) \mid \mathcal{F}_1),$$

which is (1) in the definition of dominance. In order to complete the proof that b_1 is a dominant strategy for bidder 1, we have to show (2). This, however, can be achieved precisely as (3) in the proof of Proposition 1.

II. Now we prove the "only if" direction of the theorem. Assume, on the contrary, that bidder 1 has a dominant strategy (which must equal $E\left(v \mid \mathcal{F}_1\right)$ by Proposition 1) and yet (4) does not hold. We claim that for all $\varepsilon, \delta > 0$ small enough there are sets $S^j \in \mathcal{F}_j$ for every $j \in N$ and a positive number a such that:

(i) $\left|E\left(v \mid \mathcal{F}_1\right)(\omega) - a\right| < \varepsilon$ for every $\omega \in S^1$;

(ii) $E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right)(\omega) < a - 10\varepsilon$ for ω in a certain $\bigvee_{j\in N} \mathcal{F}_j$-measurable subset of $S^1 \cap ... \cap S^n$ with measure of at least $(1-\delta)\,\mu\left(S^1 \cap ... \cap S^n\right)$;

(iii) $\mu\left(S^1 \cap ... \cap S^n\right) > 0$.

First note that, if for every ε and a set of the form $\left\{\omega \in \Omega \mid \left|E\left(v \mid \mathcal{F}_1\right)(\omega) - a\right| < \varepsilon\right\}$, the subset of $\omega \in \Omega$ for which $E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right)(\omega) < a - 10\varepsilon$ has the measure zero, then

$$E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right) - E\left(v \mid \mathcal{F}_1\right) \geq -11\varepsilon$$

(almost everywhere). Since it holds for every small $\varepsilon > 0$, actually $E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right) \geq E\left(v \mid \mathcal{F}_1\right)$. Since

$$E\left(E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right)\right) = E\left(v\right) = E\left(E\left(v \mid \mathcal{F}_1\right)\right),$$

we have $E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right) = E\left(v \mid \mathcal{F}_1\right)$, which contradicts our assumption that (4) does not hold.

Now let $S' = \{\omega \mid |E\left(v \mid \mathcal{F}_1\right)(\omega) - a| < \varepsilon\}$ be one such set, and also denote $T' = \left\{\omega \in S' \mid E\left(v \mid \bigvee_{j\in N} \mathcal{F}_j\right)(\omega) < a - 10\varepsilon\right\}$. We know that $\mu\left(T'\right) > 0$. Since T' is $\bigvee_{j\in N} \mathcal{F}_j$-measurable, it can be approximated in measure by sets generating the field $\bigvee_{j\in N} \mathcal{F}_j$ - unions (without loss of generality, disjoint) of intersections of N sets, from each one of the $\mathcal{F}_j$. Thus, for any $\delta > 0$ there are $S_1^j, ..., S_m^j \in \mathcal{F}_j$ (suppose also that $S_k^1 \subset S'$) such that

$$\mu\left(\left[T' \setminus \bigcup_{k=1}^{m} S_k^1 \cap ... \cap S_k^n\right] \cup \left[\bigcup_{k=1}^{m} S_k^1 \cap ... \cap S_k^n \setminus T\right]\right) < \delta\mu\left(\bigcup_{k=1}^{m} S_k^1 \cap ... \cap S_k^n\right),$$

and $\mu\left(S_k^1 \cap ... \cap S_k^n\right) > 0$ for every k. Thus there is k such that

$$\mu\left(\left(S_k^1 \cap ... \cap S_k^n\right) \setminus T'\right) < \delta\mu\left(S_k^1 \cap ... \cap S_k^n\right).$$

Define $S^j = S_k^j$; it is easy to see that these sets satisfy conditions (i)-(iii) above.

Now consider strategies b_j of $j \in N\setminus\{1\}$ given by $b_j(\omega) = a - 3\varepsilon$ for $\omega \in S^j$, and $b_j(\omega) = a + 10\varepsilon$ otherwise. Recall that the dominant strategy $b_1 = E\left(v \mid \mathcal{F}_1\right)$ of 1 takes values in the interval $(a - \varepsilon, a + \varepsilon)$ on the set S^1. It is clear that, given S^1, the dominant strategy of 1 only succeeds in winning the auction on $S^2 \cap ... \cap S^n$ against b_{-1}. Thus on a $\bigvee_{j\in N} \mathcal{F}_j$-measurable set B of $\omega \in S^1 \cap ... \cap S^N$ with $\mu(B) > (1-\delta)\,\mu\left(S^1 \cap ... \cap S^n\right)$,

$$E\left(u_1(\cdot, (b_1(\cdot), b_{-1}(\cdot))) \mid \bigvee_{j\in N} \mathcal{F}_j\right) \leq (a - 10\varepsilon) - (a - 3\varepsilon) = -7\varepsilon. \quad (8)$$

Given an $\mathcal{F}$-measurable and integrable random variable X and an event $M \in \mathcal{F}$ with a positive probability, denote by $E\left(X \mid M\right)$ the expectation of

X conditional on the event M. The definition of the conditional expectation and (8) thus yield

$$E\left(u_1(\cdot,(b_1(\cdot),b_{-1}(\cdot)))\mid B\right)=$$

$$=E\left(E\left(u_1(\cdot,(b_1(\cdot),b_{-1}(\cdot)))\mid\bigvee_{j\in N}\mathcal{F}_j\right)\mid B\right)\leq-7\varepsilon.$$

For small enough δ this implies that

$$E\left(u_1(\cdot,(b_1(\cdot),b_{-1}(\cdot)))\mid S^1\cap...\cap S^N\right)<0$$

(we use the fact that u_1 is bounded from above, because so is v). Since $u_1(\omega,(b_1(\omega),b_{-1}(\omega)))=0$ for $\omega\in S^1\backslash\left(S^1\cap...\cap S^N\right)$ (clearly bidder 1 loses the auction at those states), also

$$E\left(u_1(\cdot,(b_1(\cdot),b_{-1}(\cdot)))\mid S^1\right)<0.$$

However,

$$E\left(u_1(\cdot,(b_1(\cdot),b_{-1}(\cdot)))\mid S^1\right)=E\left(E\left(u_1(\cdot,(b_1(\cdot),b_{-1}(\cdot)))\mid\mathcal{F}_1\right)\mid S^1\right),$$

and so $E\left(u_1(\cdot,(b_1(\cdot),b_{-1}(\cdot)))\mid\mathcal{F}_1\right)$ must be negative on a subset A of S^1 with $\mu(A)>0$.

Now consider a strategy c_1 of bidder 1, which is equal to $a-4\varepsilon$ on S^1. Using this strategy against b_{-1} means a loss in the auction in all states of S^1. Thus $E\left(u_1(\cdot,c_1(\cdot),b_{-1}(\cdot)))\mid\mathcal{F}_1\right)=0$ on S^1, and so

$$E\left(u_1(\cdot,(c_1(\cdot),b_{-1}(\cdot)))\mid\mathcal{F}_1\right)(\omega)>E\left(u_1(\cdot,b_1(\cdot),b_{-1}(\cdot)))\mid\mathcal{F}_1\right)(\omega)$$

for all $\omega\in A$. This contradicts (1) in the definition of b_1 as a dominant strategy. ■

We already noted that (4) is satisfied when bidder i's information field is the largest. The following is thus a corollary of Theorem 3.

Corollary 4 *If $i\in N$ is such that $\mathcal{F}_j\subset\mathcal{F}_i$ for every $j\in N$, then i possesses a dominant strategy.*

The following result shows that the information advantage is reflected in outcomes of auctions where a bidder with information advantage uses his dominant strategy.

Theorem 5 *If $i \in N$ has information advantage over the other bidders in G then for every $b_{-i} \in S_{-i}$ and $j \in N \backslash \{i\}$*

$$E(u_j(\cdot,(b_i(\cdot),b_{-i}(\cdot)) \mid \mathcal{F}_j)(\omega) \leq 0,$$

where b_i denotes i's dominant strategy (that is, $b_i = E\left(v \mid \mathcal{F}_i\right)$).

Proof. Let $j \in N \backslash \{i\}$. For all $\omega \in \Omega$ we define

$$d(\omega) = \max_{k \in N \backslash \{i,j\}} (b_i(\omega), b_k(\omega)),$$

and let

$$B = \{\omega \in \Omega \mid b_j(\omega) \geq d(\omega)\}.$$

Then d and B are $\bigvee_{k \in N} \mathcal{F}_k$-measurable, and for all $\omega \in B$,

$$u_j(\omega,(b_i(\omega),b_{-i}(\omega))) = \frac{1}{m\left((b_i(\omega),b_{-i}(\omega))\right)}\left[v(\omega) - d(\omega)\right].$$

Therefore on B

$$E(u_j(\cdot,(b_i(\cdot),b_{-i}(\cdot))) \mid \bigvee_{k \in N} \mathcal{F}_k) =$$

$$= \frac{1}{m\left((b_i,b_{-i})\right)}\left[E(v \mid \bigvee_{k \in N} \mathcal{F}_k) - d\right].$$

By (4) in the definition of information advantage, this expression is equal to $\frac{1}{m((b_i,b_{-i}))}\left[b_i - d\right]$, which is nonpositive. Thus

$$E(u_j(\cdot,(b_i(\cdot),b_{-i}(\cdot))) \mid \bigvee_{k \in N} \mathcal{F}_k) \leq 0$$

on B. On the other hand, clearly

$$E(u_j(\cdot,(b_i(\cdot),b_{-i}(\cdot))) \mid \bigvee_{k \in N} \mathcal{F}_k)\,(\omega) = 0$$

if $b_j(\omega) < d(\omega)$. Therefore, by applying (7),

$$E(u_j(\cdot,(b_i(\cdot),b_{-i}(\cdot))) \mid \mathcal{F}_j) = E(E(u_j(\cdot,(b_i(\cdot),b_{-i}(\cdot))) \mid \bigvee_{k\in N} \mathcal{F}_k) \mid \mathcal{F}_j) \leq 0.$$

■

The result above implies that if all bidders have a symmetric information then the Bayesian game associated with the second-price auction has an equilibrium in dominant strategies in which each bidder's expected payoff is zero.

Corollary 6 *Let $G = (N,(\Omega,\mathcal{F},\mu),v,R_+,\{u_i\}_{i=1}^n,\{\mathcal{F}_i\}_{i=1}^n)$ be a second-price auction with a common value v. If the bidders have a symmetric information (i.e., $\mathcal{F}_1 = \mathcal{F}_2 = ... = \mathcal{F}_n$), then for all $i \in N$ the strategy $b_i^* = E(v \mid \mathcal{F}_i)$ is a dominant strategy for bidder i, and $E(u_i(\cdot,b^*(\cdot)) \mid \mathcal{F}_i)(\omega) = 0$ for all $i \in N$ and almost all $\omega \in \Omega$, where $b^* = (b_1^*,...,b_n^*)$.*

The following example shows that in a Bayesian equilibrium of a common-value second-price auction with differential information the information advantage of a bidder may not be rewarded. That is, bidder's conditional expected payoff at a Bayesian equilibrium may be below that of another bidder with (strictly) less information.

Example 2 Let G be a common-value second-price auction with differential information, in which $N = \{1,2\}$, $\Omega = \{\omega_1,\omega_2\}$, $\mathcal{F} = 2^\Omega$, $\mu(\{\omega_j\}) = \frac{1}{2}$ and $v(\omega_j) = j$ for $j = 1,2$. The information field of bidder 1 is $\mathcal{F}_1 = \mathcal{F}$ and the information field of bidder 2 is $\mathcal{F}_2 = \{\emptyset,\Omega\}$ (i.e., bidder 2 is completely uninformed).

Consider the pair of pure strategies $b = (b_1,b_2) \in S$, where $b_1(\omega) = 1$ and $b_2(\omega) = 2$ for all $\omega \in \Omega$. It is easy to verify that $b = (b_1,b_2)$ is a Bayesian equilibrium of G. It is clear that, under b, 1 loses the auction and 2 wins it at every $\omega \in \Omega$, paying 1. We obtain that for all $\omega \in \Omega$,

$$E(u_1(\cdot,b(\cdot)) \mid \mathcal{F}_1)(\omega) = 0,$$

and

$$E(u_2(\cdot,b(\cdot)) \mid \mathcal{F}_2)(\omega) = \frac{1}{2}((1-1)+(2-1)) = \frac{1}{2}.$$

That is, for all $\omega \in \Omega$,

$$E(u_2(\cdot, b(\cdot)) \mid \mathcal{F}_2)(\omega) > E(u_1(\cdot, b(\cdot)) \mid \mathcal{F}_1)(\omega).$$

Thus, although bidder 1 has an information advantage over bidder 2, his conditional expected payoff is lower than the conditional expected payoff of his opponent at every state of nature. Note, however, that the equilibrium strategy b_1 of bidder 1 is dominated by his pure strategy $b_1^* = E(v \mid \mathcal{F}_1)$; this is the reason that the assertion of Theorem 5 fails here.

4 References

1. Billingsley, P. (1995). Probability and Measure (third edition). John Wiley and Sons, New York.

2. Harsanyi, J.C. (1967). "Games with Incomplete Information Played by Bayesian players." *Management Science*, 14, 159-182.

3. Klemperer, P. (1999). "Auction Theory: A Guide to the Literature." *Journal of Economic Surveys*, 13, 227-286.

4. McAfee, P., and J. McMillan (1987). "Auctions and Bidding." *Journal of Economic Literature*, 25, 699-738.

5. Milgrom, P. (1979). "The Structure of Information in Competitive Bidding." Garland Press: New York.

6. Milgrom, P., and R.J. Weber (1982a). "A Theory of Auctions and Competitive Bidding." *Econometrica*, 50, 1089-1112.

7. Milgrom, P., and R.J. Weber (1982b). "The Value of Information in a Sealed Bid Auction." *Journal of Mathematical Economics*, 10, 105-114.

8. Savage, L.J. (1954). The Foundation of Statistics, Wiley: New York.

9. Vickrey, W. (1961). "Counterspeculation, Auctions and Competitive Sealed Tenders." *Journal of Finance*, 16, 8-37.

10. Wilson, R. (1977). "A Bidding Model of Perfect Competition." *Review of Economic Studies*, 44, 511-518.

11. Wilson, R. (1992). "Strategic Analysis of Auctions", in Handbook of Game Theory, Vol. 1 (R.J. Aumann and S. Hart, Eds.), Amsterdam/New York: Elsevier.