LOCALIZATION OF SPHERICAL AND RADIAL NONLOCAL ANISOTROPIC GRADIENTS

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Abstract. We establish nonlocal difference quotient representations for the L^p and total variation norms of distributions $\mathcal{A}u$, where u is a distribution and $\mathcal{A}$ is a constant coefficient homogeneous first-order linear differential operator. Our results provide a criterion for the boundedness of $\mathcal{A}$-gradients that does not require the use of distributional derivatives.

1. Introduction

In [1], Bourgain, Brezis and Mironescu established the following (BBM) difference quotient representation of Sobolev spaces. Let $\Omega \subset \mathbf{R}^n$ be an open set with Lipschitz boundary and let $u \in W^{1,p}(\Omega)$, then

$$\lim_{\varepsilon\to 0^+} \int_\Omega \int_\Omega \frac{|u(y)-u(x)|^p}{|y-x|^p}\, \rho_\varepsilon(y-x)\, \mathrm{d}y\, \mathrm{d}x = K_{p,n} \int_\Omega |\nabla u|^p,$$

where $K_{p,n}$ is a constant depending solely on $p \in [1,\infty)$ and the spatial dimension n. Here, $\{\rho_\varepsilon\}_{\varepsilon>0} \subset L^1(\mathbf{R}^n)$ is a family of probability non-negative radial functions, that is,

$$\|\rho_\varepsilon\|_{L^p(\mathbf{R}^n)} = 1, \tag{1}$$

which approximates the Dirac mass at zero in the sense that

$$\lim_{\varepsilon\to 0^+} \|\rho_\varepsilon\|_{L^p(\mathbf{R}^n\setminus B_\delta)} = 0 \qquad \text{for all } \delta > 0. \tag{2}$$

Their result states that the converse also holds in the range $1 < p < \infty$. In this regard, Dávila [2] showed that a related representation holds for $BV(\Omega)$ in the limiting case $p = 1$. It is well-known that a map $u \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ belongs to the homogeneous Sobolev space $u \in \dot{W}^{1,p}(\mathbf{R}^n)$ if and only if

$$\liminf_{h\to 0^+} \|\delta_{h\xi} u\|_{L^p} < \infty \quad \text{for all directions } \xi \in \mathbb{S}^{n-1}, \tag{3}$$

where

$$\delta_\xi u(x) \coloneqq \frac{u(x+\eta)-u(x)}{|\eta|}, \qquad x \in \mathbf{R}^n,$$

is the $|\eta|$-scale point-wise difference quotient of u on the direction $\frac{\eta}{|\eta|}$. When (3) holds, then the limit in (3) exists and

$$D_h u \longrightarrow Du \text{ strongly in } L^p(\mathbf{R}^n)^n, \quad \text{as } h \to^+ 0.$$

Motivated by the aforementioned BBM criterion and by the results contained in [3] criterion, where the first rigorous non-local vector calculus was developed for the gradient, curl, and divergence operators in Hilbert spaces, Mengesha and Spector [6] proved that the non-local gradient

$$\mathfrak{D}_\varepsilon u := n\left(\text{p.v.} \int \delta_\xi u \, \frac{\xi}{|\xi|} \, \rho_\varepsilon(\xi) \, \mathrm{d}\xi\right)$$

"localizes" towards the classical gradient D in the spirit of the Bourgain–Brezis–Mironescu characterization of Sobolev space above. More precisely, they showed that if u belongs to a topological function space (X, τ), then

$$\mathfrak{D}_\varepsilon u \xrightarrow{\tau} Du$$

for a wide selection of relevant topological pairs (X, τ); including the Sobolev space $\dot{W}^{1,p}(\mathbf{R}^n)$, and the space of functions with bounded variation $\dot{BV}(\mathbf{R}^n)$ endowed with area convergence topology (which is finer than the strict topology of measures). In particular, they obtained new derivative-free characterizations of Sobolev and BV spaces in the spirit of (BBM). This work is motivated by several the introduction of the peridynamics model of continuum mechanics, which is a model defined over nonlocal integral operators describing physically relevant quantities and laws. For a more detailed bibliographical account of this model, we refer the reader to [4, 6] and references therein.

The objective of this paper is twofold: Firstly, we provide a simple formula indenting the natural distributional radial nonlocal operators $\mathfrak{A}_\rho$ associated with any general constant-coefficient first-order differential operator $\mathcal{A}$, in terms of their associated principal symbol. Secondly, we introduce the nonlocal spherical operators $\mathcal{A}_s$, for which we establish smooth, distributional, L^p and area-convergence localizations. With the aid of the spherical kernels, we are able to give a straightforward proof of the localization of the nonlocal radial operators *on spaces of distributions*; for $\mathcal{D}, \mathcal{D}'$, and L^p topologies as well as the *area-convergence of measures*. In particular, we obtain a derivative-free characterization of L^p and measure-valued anisotropic gradients.

Our findings contain certain generalizations of the Du and Mengesha [4], where localization results were already introduced and established *on function spaces*, for nonlocal operators arising from general cubic third-order tensors; with respect to the strong L^p topology and the *coarser strict topology of measures* (amongst other interesting topologies). In this regard, our symbolic calculus formulas appear to substantially improve upon the transparency of the framework developed in [4]. Lastly, it is worth mentioning that our methods *differ significantly* from the ones developed in previous works, as they are based on the study of localization concerning spherical surface kernels (cf. SECTION 3). An essential advantage of this is that our proofs can be easily adapted to find localizations for kernels associated with iso-symmetric shapes (such as cubes, regular polygons, or rings) upon a simple modification of the kernels. This even applies to non-symmetric profiles (in this case, the localization would yield approximations for certain non-isotropic norms).

2. SET UP AND MAIN RESULTS

We consider a general constant coefficient first-order homogeneous distributional differential operator $\mathcal{A} : \mathcal{D}'(\mathbf{R}^n; V) \to \mathcal{D}'(\mathbf{R}^n; W)$, where V, W are finite-dimensional inner product spaces (identified with their own duals). Such operators can always be written in the form.

$$\mathcal{A} = \sum_{i=1}^{n} A_i \partial_i \,, \qquad A_i \in \mathrm{Hom}(V, W). \tag{4}$$

The gradient, divergence, curl, symmetric gradient, exterior differential, del-var are examples of such general operators. We recall that, in this form and up to a complex constant, the principal symbol associated with $\mathcal{A}$ is given by the $\mathrm{Hom}(V, W)$-valued linear polynomial

$$\mathbb{A}(\xi) \coloneqq \sum_{i=1}^{n} \xi_i A_i \,, \qquad \xi \in \mathbf{R}^n.$$

In light of our previous discussion, we formally define the nonlocal $\mathcal{A}$-gradient is defined as

$$\mathfrak{A}_\varepsilon u(x) = \text{ p.v. } n \int \frac{\mathbb{A}(h)[u(x) - u(x)]}{|y - x|^2} \, \rho_\varepsilon(y - x) \, dy$$

where $u : \mathbf{R}^n \to V$ is a sufficiently well-behaved vector field. By looking at each coordinate separately and invoking the results discussed in the introduction, we deduce that if $u \in \dot{W}^{1,p}(\mathbf{R}^n; V)$, then

$$\mathfrak{A}_\varepsilon u \longrightarrow \mathcal{A}u \quad \text{strongly in } L^p. \tag{5}$$

In general, the requirement $Du \in L^p$ is *too restrictive* to deduce this L^p localization, as one would expect the p-integrability of $\mathcal{A}u$ to be necessary and sufficient for its validity (cf. THEOREM 12). Indeed, the Calderon–Zygmund estimate $\|Du\|_{L^p} \cong \|\mathcal{A}u\|_{L^p}$ only holds for elliptic operators in the range $1 < p < \infty$. The situation worsens for $p = 1$, where the requirement $Du \in L^1$ is restrictive, even for elliptic operators (cf. [5, 7]).

2.1. **Nonlocal calculus.** Motivated by this discussion, our main object of study will be the following nonlocal spherical and radial (difference quotient) operators.

Definition 1 (Nonlocal spherical operator). For a given test function map $u \in \mathcal{D}(\mathbf{R}^n; V)$ and $s > 0$, we define the *s-spherical operator*

$$\begin{aligned} \mathcal{A}_s u(x) &\coloneqq \frac{n}{s} \fint_{\partial B_s} \Omega_{\mathbb{A}}(\omega)[\delta_\omega u(x)] \, dS(\omega) \\ &= \frac{n}{s} \fint_{\partial B_s} \mathbb{A}\left(\frac{\omega}{s}\right) \left[\frac{u(x+\omega) - u(x)}{s}\right] dS(\omega), \qquad x \in \mathbf{R}. \end{aligned}$$

Here, S denotes the uniform $(n-1)$-dimensional Hausdorff measure on ∂B_s and where $\Omega_{\mathbb{A}}$ is the zero-homogeneous profile of $\mathbb{A}$.

Hereinafter $\rho : \mathbf{R}^n \to [0, \infty)$ will denote an integrable radial Borel function with profile $\hat{\rho} : \mathbf{R} \to [0, \infty)$. With a slight abuse of notation (between ρ and ε), we define a nonlocal operator associated with $\mathcal{A}$ through the radial map ρ.

Definition 2 (Nonlocal radial operator). For a given test vector field $u \in \mathcal{D}(\mathbf{R}^n; V)$, we define the *nonlocal operator*

$$\begin{aligned} \mathfrak{A}_\rho u(x) &\coloneqq \text{p.v. } n \int \Omega_{\mathbb{A}}(h)[\delta_h u(x)] \, \rho(h) \, dh \\ &= \text{p.v. } n \int \frac{\mathbb{A}(y-x)[u(y) - u(x)]}{|y-x|^2} \, \rho(y-x) dy, \quad x \in \mathbf{R}^n. \end{aligned}$$

These nonlocal operators are linked through the Bochner integral identity

$$\mathfrak{A}_\rho u = \int_0^\infty n\omega_n \hat{\rho}(s) s^{n-1} \mathcal{A}_s u \, ds.$$

It is straightforward to verify that both $\mathcal{A}_s$ and $\mathfrak{A}_\rho$ are bounded integral operators from $\mathcal{D}(\mathbf{R}^n; V)$ into $\mathcal{D}(\mathbf{R}^n; W)$ —this will be formally discussed in Sections 3 and 4— and are also stable under integration by parts:

Proposition 3 (Integration by parts). *Let $u \in \mathcal{D}(\mathbf{R}^n; V)$ and $\varphi \in \mathcal{D}(\mathbf{R}^n, W)$ be test functions. Then*

$$\int \mathcal{A}_s u \cdot \varphi = \int u \cdot \mathcal{A}_s^* \varphi$$

and

$$\int \mathfrak{A}_\rho u \cdot \varphi = \int u \cdot \mathfrak{A}_\rho^* \varphi$$

Here $\mathcal{A}_s^*$ and $\mathfrak{A}_\rho^*$ are the respective spherical and nonlocal operators associated with $\mathcal{A}^* = -\sum_{i=1}^n A_i^t \partial_i$, the formal adjoint of $\mathcal{A}$.

Using duality, it is, therefore, possible to *uniquely and continuously* extend both $\mathcal{A}_s$ and $\mathfrak{A}_\rho$ to all distributions taking values on the vector-space V:

Definition 4 (Distributional operators). Let $T \in \mathcal{D}'(\mathbf{R}^n; V)$. The distributional spherical and nonlocal $\mathcal{A}$-gradient are defined respectively as the distributions acting on test functions $\varphi \in \mathcal{D}(\mathbf{R}^n; W)$ as

$$\mathcal{A}_s T(\varphi) = T(\mathcal{A}_s^* \varphi) \qquad \text{and} \qquad \mathfrak{A}_\rho T(\varphi) = T(\mathfrak{A}_\rho^* \varphi).$$

By construction, it stems that $\mathcal{A}_s, \mathfrak{A}_\rho : \mathcal{D}'(\mathbf{R}^n; V) \to \mathcal{D}'(\mathbf{R}^n; W)$ are bounded linear operators. Given that we shall often work directly with distributions, it will be useful to work with extended L^p-norms:

Definition 5 (Extended L^p norms). Let $1 \leq p \leq \infty$ and let q be the the Hölder conjugate of p. For a distribution $T \in \mathcal{D}'(\mathbf{R}^n; V)$, we define the extended L^p-norm of T as

$$\|T\|_p = \left(\int |T|^p \right)^{\frac{1}{p}} := \sup \{ T[\varphi] \, , \, : \varphi \in \mathcal{D}(\mathbf{R}^n; V), \|\varphi\|_{L^q} \leq 1 \}$$

where q is the Hölder conjugate of p satisfying $p^{-1} + q^{-1} = 1$.

Remark 6. If $p \in (1, \infty]$, then $\|T\|_p$ is finite if and only if T can be respresented by a p-integrable map (bounded map when $p = \infty$). Similarly, $\|T\|_1$ is finite if and only if T can be represented by a measure with bounded total variation.

Our first result establishes extended L^p-norm bounds and localizations of the spherical operators in terms of the $\mathcal{A}$-gradient (its proof is contained in Section 3):

Theorem 7 (Spherical localization).

(1) If $u \in \mathcal{D}(\mathbf{R}^n; V)$, then

$$\mathcal{A}_s u \xrightarrow{\mathcal{D}} \mathcal{A}u \qquad \text{as } s \to 0^+.$$

and

$$\int f(\mathcal{A}_s u) \leq \int f(\mathcal{A}u)$$

for all convex functions $f : W \to \mathbf{R}$.

(2) If $T \in \mathcal{D}'(\mathbf{R}^n; V)$, then

$$\mathcal{A}_s T \xrightarrow{\mathcal{D}'} \mathcal{A}T \qquad \text{as } s \to 0^+.$$

and

$$\|\mathcal{A}_s T\|_p \leq \|\mathcal{A}T\|_p \quad \text{for all } p \in [1, \infty],$$

(3) If $\mathcal{A}T \in L^p(\mathbf{R}^n; W)$ for some $p \in [1, \infty]$, then, in fact,

$$\mathcal{A}_s T \in (C \cap L^p)(\mathbf{R}^n; W).$$

In this case, and for $p \in [1, \infty)$, the distributional convergence improves to the strong convergence

$$\mathcal{A}_s T \xrightarrow{L^p} \mathcal{A}T \qquad \text{as } s \to 0^+.$$

(4) If $\mathcal{A}T \in \mathcal{M}(\mathbf{R}^n; W)$, then

$$\mathcal{A}_s T \in (L^\infty \cap L^1)(\mathbf{R}^n; W)$$

and the absolutely continuous measures $\mathcal{A}_s T \, \mathcal{L}^n$ converge to $\mathcal{A}u$ in the area sense of measures as $s \to 0$, that is,

$$\lim_{s \to 0^+} \int f(\mathcal{A}_s T) = \int f(\mathcal{A}^a T) + \int f^\infty(\mathcal{A}^s T)$$

for all convex maps $f : W \to \mathbf{R}$ with linear growth at infinity. ([1])

[1]Here,

$$f^\infty(w) := \lim_{t \to \infty} \frac{f(tw)}{t}, \qquad w \in W,$$

is the recession function of F and

$$\mathcal{A}u = \mathcal{A}^a u \, \mathcal{L}^n + \mathcal{A}^s u, \qquad |\mathcal{A}^s u| \perp \mathcal{L}^n,$$

is the Lebesgue–Radon–Nykodým decomposition of $\mathcal{A}u$.

Thanks to a simple measure-theoretic lemma discussed in the Appendix, we obtain the following representation for the spherical operators when acting on locally summable maps (cf. Eqn. 7).

LEMMA 8 (CONVOLUTION REPRESENATION). *Let $u \in L^1_{\mathrm{loc}}(\mathbf{R}^n; V)$ and further assume that $\mathcal{A}u \in \mathcal{M}(\mathbf{R}^n; W)$. Then,*

$$\mathcal{A}_s u(x) = \frac{\mathcal{A}u(B_s(x))}{\omega_n s^n}.$$

at almost every $x \in \mathbf{R}^n$.

The localization result for the spherical operators builds directly into a swift proof of the estimates and localization results for nonlocal operators, which will be described next. (The proof of this result is a direct consequence of Eqn. (10) and PROPOSITION 19.)

THEOREM 9. *Let $1 \leq p < \infty$ and let $u \in \mathcal{D}'(\mathbf{R}^n; V)$. Then*

$$\|\mathfrak{A}_\rho u\|_p \leq \|\rho\|_{L^1}^{\frac{1}{p}} \|\mathcal{A}u\|_p.$$

If moreover $\|\mathcal{A}u\|_p < \infty$, then

$$\mathfrak{A}_\rho u \in L^p(\mathbf{R}^n; W)$$

and the extended L^1-norm estimate

$$\int |f(\mathfrak{A}_\rho u)| \, dx \leq \int |f(\mathcal{A}^a u)| \, dx + \int |f^\infty(d\mathcal{A}^s u)|$$

holds for all convex maps $F : W \to \mathbf{R}$.

For locally summable maps with locally bounded measure $\mathcal{A}$-gradients, the operator $\mathfrak{A}_\rho$ is expressed, as one would expect, by the principal value integral (this follows from LEMMA 17):

LEMMA 10. *Let $u \in L^1_{\mathrm{loc}}(\mathbf{R}^n; V)$ with $\|\mathcal{A}u\|_p < \infty$. Then, the limit*

$$\mathrm{p.v.}\ n \int \frac{\mathbb{A}(y-x)[u(y)-u(x)]}{|y-x|^2} \, dy$$

exists and coincides with $\mathfrak{A}_\rho u(x)$ for almost every $x \in \mathbf{R}^n$. Here, $u(x)$ is understood as the Lebesgue representative of u at x.

Our first main localization result concerns both an *energy criterion* and an approximation result for the extended L^p-norms of the distribution $\mathcal{A}u$, in terms of $\{\mathfrak{A}_\varepsilon u\}_\varepsilon$, where $\{\rho_\varepsilon\}_{\varepsilon>0}$ is a family of radial probability functions satisfying (1)-(2). The precise statement, which follows by gathering the results contained in Eqn. (9), LEMMA 17 Proposition 19, is the following:

LEMMA 11. *Let $\varphi \in \mathcal{D}(\mathbf{R}^n; V)$ and let $u \in \mathcal{D}'(\mathbf{R}^n; V)$ a. Then*

$$\mathfrak{A}_\varepsilon \varphi \xrightarrow{\mathcal{D}} \mathfrak{A}\varphi \qquad \text{and} \qquad \mathfrak{A}_\varepsilon u \xrightarrow{\mathcal{D}'} \mathfrak{A}u.$$

The extended L^p-norm limit

$$\lim_{\varepsilon \to 0^+} \int |\mathfrak{A}_\varepsilon u|^p \in [0,\infty]$$

always exists.

If moreover $u \in L^1_{\mathrm{loc}}(\mathbf{R}^n; V)$, then

$$\lim_{\varepsilon \to 0^+} \int \left| \mathrm{p.v.}\, n \int \frac{\mathbb{A}(y-x)[u(y)-u(x)]}{|y-x|^2}\, \rho_\varepsilon(y-x)\, \mathrm{d}y \right|^p = \int |\mathcal{A}u|^p.$$

This result and COROLLARY 2 directly imply the following localization (and characterization) criterion:

THEOREM 12 (L^p-LOCALIZATION). *Let $1 < p < \infty$ and let $u \in \mathcal{D}'(\mathbf{R}^n; V)$. Then $\mathcal{A}u \in L^p(\mathbf{R}^n; W)$ if and only if*

$$\liminf_{\varepsilon \to 0^+} \|\mathfrak{A}_\varepsilon u\|_p < \infty.$$

Moreover, in this case

$$\mathfrak{A}_\varepsilon u \longrightarrow \mathcal{A}u \quad \text{strongly in } L^p(\mathbf{R}^n; W).$$

Remark 13. For $p = 1$, it can be shown that the assertion still holds in the following sense: if $\mathcal{A}u \in L^1(\mathbf{R}^n; W)$, then the representation of THEOREM 12 exists and agrees with the L^1-norm of $\mathcal{A}u$ (cf. Corollary 1 below). In general, the converse fails due to concentration effects appearing on the difference quotient integral.

Our second main result covers this case precisely when $\mathcal{A}u$ is a vector-valued Radon measure. Notice that, with our definition of extended norm, it holds that a W-valued distribution T belongs to $\mathcal{M}_b(\mathbf{R}^n; W)$, the space of W-valued Radon measures with bounded total variation, if and only if $\|T\|_1$, in which case it also holds $\|T\|_1 = |T|(\mathbf{R}^n)$. In particular, a direct consequence of THEOREM 9 and LEMMA 11 is the following criterion and approximation result:

THEOREM 14 (AREA CONVERGENCE LOCALIZATION). *Let $u \in L^1_{\mathrm{loc}}(\mathbf{R}^n; V)$. Then $\mathcal{A}u \in \mathcal{M}_b(\mathbf{R}^n; W)$ if and only if*

$$\liminf_{\varepsilon \to 0^+} \|\mathfrak{A}_\varepsilon u\|_1 < \infty.$$

Moreover, in this case

$$(\mathfrak{A}_\varepsilon u)\mathcal{L}^n \longrightarrow \mathcal{A}u \quad \text{in the area sense of measures.}$$

In particular,

$$\lim_{\varepsilon \to 0} \int f(\mathfrak{A}_\varepsilon u)\, dx = \int f(\mathcal{A}^a u)\, dx + f^\infty(\frac{\mathcal{A}u}{|\mathcal{A}u|})\, d|\mathcal{A}^s u|$$

for all continuous integrands $f : W \to \mathbf{R}$ with linear growth such that the limit

$$f^\infty(A) := \lim_{t \to \infty} \frac{f(tA)}{t}$$

exists for all $A \in W$.

The particular advantage of the area convergence of measures (over the usual strict convergence) is that it discriminates the appearance of L^1 concentrations. In particular, we obtain the following refinement of THEOREM 14:

COROLLARY 1. *If $\mathcal{A}u \in L^1(\mathbf{R}^n; W)$, then*

$$\mathfrak{A}_\varepsilon u \longrightarrow \mathcal{A}u \quad \text{strongly in } L^1.$$

Remark 15 (More general shapes). The results presented here can be deduced for more general shapes other than the sphere and radial symmetry. It is straightforward from our methods that the same results can be obtained by working with any boundary ∂U, where U is a simply connected Lipschitz set containing $0 \in \mathbf{R}^n$. In this particular case, the modified surface kernel takes the form

$$\mathcal{A}_{U,s} u(x) = \frac{1}{s^n \mathcal{L}^n(U)} \int_{\partial(sU)} \mathbb{A}(\nu(\omega)) \left[\frac{u(x+\omega) - u(x)}{s}\right] dS(\omega).$$

where ν is the exterior normal to sU at $\partial(sU)$.

3. THE SPHERICAL SURFACE KERNELS

In this section, we discuss localizations concerning spherical kernels. As we shall see below, working with these lower-dimensional measure kernels arises naturally from the Gauss-Green theorem. The analysis of these kernels will be fundamental in establishing similar estimates and localizations for the radial kernels discussed in the introductions. We shall from now on consider the uniform spherical surface measure

$$m_s = \frac{\mathcal{H}^{n-1} \llcorner \partial B_s}{|B_s|}, \qquad s > 0.$$

To it, we associate a *surface* kernel $K_s : \mathbf{R}^n \setminus \{0\} \to \mathrm{Hom}(V, W)$ by setting

$$K_s(d\omega) := \Omega_{\mathbb{A}}(\omega)\, m_s(d\omega),$$

where $\Omega_{\mathbb{A}}$ is the zero-homogeneous angular part of the symbol $\mathbb{A}$ map, i.e.,

$$\Omega_{\mathbb{A}}(x) := \mathbb{A}\left(\frac{x}{|x|}\right), \qquad x \in \mathbf{R}^n - \{0\}.$$

For each $s > 0$, K_s defines a finite $\mathrm{Hom}(V, W)$-valued measure on $\mathbf{R}^n$ and its therefore a tensor-valued compactly supported distribution. When applied on a test map $u \in \mathcal{D}(\mathbf{R}^n; V)$, the Gauss–Green theorem yields the identity

$$K_s \star u(x) = \frac{1}{\omega_n s^n} \int_{\partial B_s(x)} \Omega_{\mathbb{A}}(y - x) u(y)\, dS(y) = \frac{\mathcal{A}u(B_s(x))}{\omega_n s^n}.$$

The cancelation property

$$\int_{\mathbf{S}^{n-1}} \mathbb{A}(\omega)\, dS(\omega) = 0 \tag{6}$$

conveys the identity

$$\mathcal{A}_s u = K_s \star u \qquad \text{for all } \varphi \in \mathcal{D}(\mathbf{R}^n; V)$$

so that, in particular, we can reduce the study of $\mathcal{A}_s$ to the study of the K_s-convolution operator. Notice that a direct consequence of this identity is the integration by parts formula

$$\int \mathcal{A}_s \varphi \cdot \psi = \int (K_s \star \varphi) \cdot \psi = \int \varphi \cdot (\tilde{K}_s^t \star \psi) = \int \varphi \cdot \mathcal{A}_s^* \psi$$

for all test functions $\varphi \in \mathcal{D}(\mathbf{R}^n; V)$ and $\psi \in \mathcal{D}(\mathbf{R}^n; W)$. Here, the operator $\tilde{(\cdot)}$ denotes the reflection operation $x \mapsto -x$ so that indeed $\tilde{K}_s^t(h) = \Omega_{\mathbb{A}^*}\, dm_s$. By standard functional analysis theory, the convolution $K_s \star T$ defines a W-valued distribution on $\mathbf{R}^n$ for all $T \in \mathcal{D}'(\mathbf{R}^n; V)$ by setting

$$K_s \star T(\varphi) := T(\tilde{K}_s^t \star \varphi) \quad \forall \varphi \in \mathcal{D}(\mathbf{R}^n; W).$$

Here, the operator $\tilde{(\cdot)}$ denotes the operation $x \mapsto -x$. The previous convolution representation holds for all distributions T such that $\mathcal{A}T$ is a Radon measure. Indeed, a standard approximation argument implies that, in fact,

$$K_s \star T \in L^\infty \quad \text{with} \quad \|K_s \star T\|_\infty \le \frac{|\mathcal{A}u|(\mathbf{R}^n)}{\omega_n s^n}.$$

Therefore, by standard mollifier theory, we get that

$$K_s \star T(x) = \frac{1}{|B_s|} \lim_{\varepsilon \to 0^+} (\mathcal{A}T)_\varepsilon (B_s(x)) \quad \text{for a.e. } x \in \mathbf{R}^n,$$

where $(\bullet)_\varepsilon$ denotes mollification with a standard mollifier at scale ε. Lemma 20 implies that $|\mathcal{A}T|(\partial B_s(x)) = 0$ for almost every $x \in \mathbf{R}^n$ and therefore also the right-hand side above converges to $\mathcal{A}T(B_s(x))/|B_s|$ almost everywhere on $\mathbf{R}^n$, or equivalently,

$$K_s \star T = \frac{\mathcal{A}T(B_s(\bullet))}{|B_s|} \quad \text{as maps in } L^\infty. \tag{7}$$

Remark 16. In general, $K_s \star u$ needs not to be continuous for $u \in L^1_{\text{loc}}(\mathbf{R}^n; V)$ with $\mathcal{A}u \in \mathcal{M}(\mathbf{R}^n; W)$. Indeed, the vector field on the plane given by

$$u(x_1, x_2) = \frac{1}{2\pi} \left(\frac{(x_1, x_2 + 1)}{|(x_1, x_2 + 1)|^2} - \frac{(x_1 + 1, x_2)}{|(x_1 + 1, x_2)|^2} \right),$$

satisfies $\operatorname{div} u = \delta_{(0,1)} - \delta_{(1,0)}$. In particular, the map $x \mapsto \operatorname{div} u(B_1(x))$ is not continuous at $0 \in \mathbf{R}^2$.

We devise the following direct properties from this discussion, which validate the assertions contained in Theorem 7.

(i) If $u \in \mathcal{D}(\mathbf{R}^n; V)$, then

$$K_s \star u \xrightarrow{\mathcal{D}} \mathcal{A}u \qquad \text{as } s \to 0^+.$$

Indeed, by the properties of differentiation for convolutions, it suffices to observe the validity of the supremum norm estimate

$$|\partial^\alpha (K_s \star u - \mathcal{A}u)(x)| \le \fint_{B_s(x)} |\mathcal{A}\partial^\alpha u(y) - \mathcal{A}\partial^\alpha u(x)| \, dy \le s \|D^k(\mathcal{A}u)\|_\infty$$

where $k = |\alpha| + 1$.

(ii) If $T \in \mathcal{D}'(\mathbf{R}^n; V)$, then a byproduct of the previous convergence is the distributional convergence

$$K_s \star T \xrightarrow{\mathcal{D}'} \mathcal{A}T \qquad \text{as } s \to 0^+.$$

Indeed, observing that $-\mathbb{A}^t$ is the principal symbol associated with the formal adjoint $\mathcal{A}^*$ of $\mathcal{A}$, we deduce that

$$K_s \star T(u) = T(\tilde{K}_s^t \star u) \longrightarrow T(\mathcal{A}^* u) = \mathcal{A}T(u).$$

(iii) If $u \in \mathcal{D}(\mathbf{R}^n; V)$, then Jensen's inequality gives

$$F(K_s \star u) \leq F(\mathcal{A}u) \star \frac{\chi_{B_s}}{|B_s|} \tag{8}$$

for all convex maps $F : W \to \mathbf{R}$. Young's convolution inequality conveys the estimate

$$\|F(K_s \star u)\|_{L^1} \leq \|F(\mathcal{A}u)\|_{L^1} \quad \text{for all } p \in [1, \infty].$$

In particular,

$$\|K_s \star u\|_{L^p} \leq \|\mathcal{A}u\|_{L^p} \quad \text{for all } 1 \leq p < \infty.$$

(iv) The lower semicontinuity of the extended L^p-norms under distributional convergence and the previous estimate gives rise to the extended estimate

$$\forall p \in [1, \infty], \quad \|K_s \star T\|_p \leq \|\mathcal{A}T\|_p \quad \text{for all } T \in \mathcal{D}'(\mathbf{R}^n; V).$$

In particular, again from lower semicontinuity and distributional convergence, we obtain

$$\limsup_{s \to 0^+} \|K_s \star T\|_p \leq \|\mathcal{A}T\|_p \leq \liminf_{s \to 0^+} \|K_s \star T\|_p\,,$$

which means that the limit exists and equals $\|\mathcal{A}T\|_p$.

(v) If $\mathcal{A}T \in L^p$ for some $p \in [1, \infty]$, then the fact that $|\mathcal{A}u|(\partial B_s(x)) = 0$ for all $x \in \mathbf{R}^n$ implies that

$$K_s \star T \in C(\mathbf{R}^n; W) \quad \forall s > 0.$$

Moreover, for $1 < p < \infty$, the convergence of the L^p norms implies that the distributional converge improves to the strong convergence

$$K_s \star T \xrightarrow{L^p} \mathcal{A}T \quad \text{as } s \to 0^+.$$

(vi) Lastly, if $\mathcal{A}T \in \mathcal{M}$, then (7) and the fact that the approximation by convolution for measures implies convergence with respect to the area functional gives

$$(K_s \star T)\,\mathcal{L}^n \xrightarrow{\text{area}} \mathcal{A}T \quad \text{as } s \to 0^+.$$

4. The radial kernels

Let $\rho(x) = \hat{\rho}(|x|)$ be a non-negative radial summable function and consider the kernel $\mathcal{K}_\rho \in C^\infty(\mathbf{R}^n - \{0\}; \mathrm{Hom}(V, W))$ defined by

$$\mathcal{K}_\rho(z) := n\,\Omega_{\mathbb{A}}\left(\frac{z}{|z|}\right)\rho(z), \qquad z \in \mathbf{R}^n \setminus \{0\}.$$

The mean-value zero property

$$\int_{\partial B_s(0)} \mathcal{K}_\rho(\omega)\, dS(\omega) = 0 \qquad \text{for all } s > 0$$

ensures that the *principal value* convolution integral

$$\begin{aligned}\mathfrak{A}_\rho u(x) &= \text{p.v.} \int \mathcal{K}_\rho(y-x)u(y)\, dy \\ &:= \lim_{\delta \to 0} \int_{\mathbf{R}^n \setminus B_\delta(x)} \mathcal{K}_\rho(y-x)u(y)\, dy\end{aligned}$$

is well-defined for all test maps $u \in \mathcal{D}(\mathbf{R}^n; V)$ and all $x \in \mathbf{R}^n$, independently of the infinitesimal sequence $\delta \to 0^+$. Indeed, once more, the cancelation property (6) gives

$$\mathfrak{A}_\rho u(x) = \text{p.v.} \int \mathcal{K}_\rho(y-x)[u(y) - u(x)]\, dy,$$

Since the integrand on the right-hand is dominated by a multiple of $\|Du\|_\infty \tau^x \rho$, it follows that the limit in the principal value integral exists and $\mathfrak{A}_\rho u \in L^\infty(\mathbf{R}^n; W)$, with

$$\|\mathfrak{A}_\rho u\|_{L^\infty} \leq n\|\Omega_{\mathbb{A}}\|_{L^\infty}\|Du\|_{L^\infty}\|\rho\|_{L^1}.$$

Moreover, the absolute integrability of the integrand above and Fubini's theorem give

$$\begin{aligned}\langle \mathfrak{A}_\rho u, \varphi\rangle &= \int\int \langle u(x+h) - u(x), \mathcal{K}_\rho(h)^t \varphi(x)\rangle\, dx\, dh \\ &= \int\int \langle u(z), \mathcal{K}_\rho(h)^t[\varphi(z-h) - \varphi(z)]\rangle\, dz\, dh \\ &= -\int\int \langle u(z), \mathcal{K}_\rho(y-z)^t[\varphi(y) - \varphi(z)]\rangle\, dy\, dz = \langle u, \mathfrak{A}_\rho^* \varphi\rangle,\end{aligned}$$

for all $\varphi \in \mathcal{D}(\mathbf{R}^n; W)$. This shows that the formal adjoint $(\mathfrak{A}_\rho)^*$ of $\mathfrak{A}_\rho$ is precisely $\mathfrak{A}_\rho^*$. By a standard duality argument, we may extend $\mathfrak{A}_\rho$ uniquely and continuously to all distributions $T \in \mathcal{D}'(\mathbf{R}^n; V)$ by setting

$$\mathfrak{A}_\rho T(\varphi) := T(\mathfrak{A}_\rho^* \varphi), \qquad \varphi \in \mathcal{D}(\mathbf{R}^n; W).$$

In particular, this shows that, for a family $\{\rho_\varepsilon\}_{\varepsilon>0}$ of probability non-negative radial functions satisfying (1)-(2), it holds

$$\mathfrak{A}\rho_\varepsilon T \xrightarrow{\mathcal{D}'} \mathcal{A}T \qquad \text{as } \varepsilon \to 0^+ \tag{9}$$

for all $T \in \mathcal{D}'(\mathbf{R}^n; V)$.

Let $p \in [1, \infty]$ and let q be its associated Hölder conjugate exponent. If $T \in \mathcal{D}'(\mathbf{R}^n; W)$, then, using that the radial integral defining $\mathfrak{A}^*\varphi$ is absolutely convergent as a Bochner integral of smooth maps, we deduce from the spherical kernel estimates the following one:

$$\begin{aligned}
\mathfrak{A}_\rho T(\varphi) &= T(\mathfrak{A}_\rho^*\varphi) \\
&= T\left(-\int_0^\infty n\omega_n \hat{\rho}(r) r^{n-1} \tilde{K}_r^t \star \varphi \, dr\right) \\
&= \int_0^\infty n\omega_n \hat{\rho}(r) r^{n-1} T(K_r^* \star \varphi) \, dr \\
&= \int_0^\infty n\omega_n \hat{\rho}(r) r^{n-1} (K_r \star T)(\varphi) \, dr \\
&\le \int_0^\infty n\omega_n \hat{\rho}(r) r^{n-1} \|K_r \star T\|_p \|\varphi\|_{L^q} \, dr \le \|\rho\|_{L^1} \|\mathcal{A}T\|_p \|\varphi\|_{L^q}.
\end{aligned}$$

Taking the supremum over all φ with $\|\varphi\|_{L^q} = 1$ gives

$$\|\mathfrak{A}_\rho T\|_p \le \|\rho\|_{L^1} \|\mathcal{A}T\|_p \quad \text{for all } T \in \mathcal{D}'(\mathbf{R}^n; V). \tag{10}$$

LEMMA 17. *Let $u \in L^1_{\mathrm{loc}}(\mathbf{R}^n; V)$. Then*

$$\|\mathfrak{A}_\rho u\|_p^p \le \liminf_{\delta \to 0} \int \left| \int_{\mathbf{R}^n \setminus B_\delta(x)} K_\rho(y - x) u(y) \, dy \right|^p.$$

If moreover $\|\mathcal{A}u\|_p < \infty$, then $\mathfrak{A}_\rho u \in L^p(\mathbf{R}^n; W)$, the limit above exists, commutes with the integral and

$$\mathfrak{A}_\rho u(x) = \text{p.v.} \int K_\rho(y - x) u(y) \, dy \qquad \textit{for almost every } x \in \mathbf{R}^n.$$

Remark 18. Notice that $\mathfrak{A}_\rho u$ is absolutely continuous even if $\mathcal{A}u$ is only a finite measure.

Proof of Lemma 17. First, we show that $\mathfrak{A}_\rho u$ is given by the principal value convolution p.v. $\mathcal{K}_\rho \star u$. By construction, we have that $\mathfrak{A}_\rho u \in \mathcal{M}(\mathbf{R}^n; W)$. Since the integrand

$$(x, y) \mapsto \langle K_\rho(y - x) u(y), \varphi(x) - \varphi(y) \rangle$$

is absolutely integrable on $\mathbf{R}^n \times \mathbf{R}^n$, we deduce from Fubini's theorem and a few changes of variables that, for any infinitesimal sequence $\delta \to 0$, it holds

$$\begin{aligned}
\langle u, \mathfrak{A}_\rho^* \varphi\rangle &= \int \mathrm{p.v.} \int \langle K_\rho(y-x)u(y), \varphi(x)\rangle \, dx \, dy \\
&= \int \int \langle K_\rho(y-x)u(y), \varphi(x) - \varphi(y)\rangle \, dx \, dy \\
&= \int \int \langle K_\rho(h)u(y), \varphi(y-h) - \varphi(y)\rangle \, dh \, dy.
\end{aligned}$$

Now, the right-hand side can be expressed as the limit

$$\begin{aligned}
&= \lim_{\delta \to 0} \int_{B_\delta(0)^c} \int \langle K_\rho(h)[u(y) - u(y-h)], \varphi(y-h)\rangle \, dy \, dh \\
&= \lim_{\delta \to 0} \int \int_{B_\delta(0)^c} \langle K_\rho(h)[u(y) - u(y-h)], \varphi(y-h)\rangle \, dh \, dy. \\
&= \lim_{\delta \to 0} \int \int_{B_\delta(y)^c} \langle K_\rho(y-x)[u(y) - u(x)], \varphi(x)\rangle \, dx \, dy \\
&= \lim_{\delta \to 0} \int \langle \int_{B_\delta(x)^c} K_\rho(y-x)u(y) \, dy, \varphi(x)\rangle \, dx \\
&\leq \|\varphi\|_{L^q} \lim_{\delta \to 0} \left(\int \left| \int_{\mathbf{R}^n \setminus B_\delta(x)} K_\rho(y-x)u(y) \, dy \right|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Taking the supremum over all $\varphi \in \mathcal{D}(\mathbf{R}^n; W)$ with $\|\varphi\|_{L^q} = 1$, where q is the Hölder conjugate exponent of p, we conclude that

$$\|\mathfrak{A}_\rho u\|_p^p \leq \liminf_{\delta \to 0} \int \left| \int_{\mathbf{R}^n \setminus B_\delta(x)} K_\rho(y-x)u(y) \, dy \right|^p.$$

Let us now assume that $|\mathcal{A}u|$ is a finite measure. Since for every $r > 0$ the function $x \mapsto |\mathcal{A}u|(B_r(x))$ is measurable, it follows that the non-negative function defined by

$$f(x) := \int_0^\infty \hat{\rho}(r) r^{-1} |\mathcal{A}u|(B_r(x)) \varphi(x) \, dr \in [0, \infty], \qquad x \in \mathbf{R}^n,$$

is also Borel measurable on $\mathbf{R}^n$. Moreover, Fubini's theorem and Young's convolution inequality convey that f is p-integrable with $\|f\|_{L^p} \leq \|\rho\|_{L^1} \|\mathcal{A}u\|_p$. Let us now define the measurable maps

$$g_\delta(x) := \int_{\mathbf{R}^n \setminus B_\delta(x)} \mathcal{K}_\rho(y-x)u(y) \, dy, \quad x \in \mathbf{R}^n.$$

By Lemma XX and by the generalized Gauss–Green theorem it follows that f dominates the family $\{|g_\delta|^p\}_{\delta>0}$. Since also

$$g_\delta(x) \xrightarrow{\delta\to 0} \text{p.v.} \int K_\rho(y-x)u(y)\,dy \quad \text{for almost every } x \in \mathbf{R}^n,$$

the Dominated convergence theorem implies that $\mathfrak{A}_\rho u = \text{p.v.}(\mathcal{K}_\rho \star u)$. This finishes the proof. $\square$

PROPOSITION 19. *Let $u \in \mathcal{D}'(\mathbf{R}^n; V)$ and further suppose that $\mathcal{A}u \in \mathcal{M}_b(\mathbf{R}^n; W)$. If ρ is a probability density, then*

$$\|F(\mathfrak{A}_\rho u)\|_1 \leq \|F(\mathcal{A}^a u)\|_1 + \|F^\infty(\mathcal{A}^s u)\|_1$$

for all convex functions $F : W \to \mathbf{R}$. In particular,

$$\lim_{\varepsilon\to 0} \int |F(\mathfrak{A}_{\rho_\varepsilon} u)|\,dx = \int F(\mathcal{A}^a u)\,dx + \int F^\infty\left(\frac{\mathcal{A}u}{|\mathcal{A}u|}\right) |\mathcal{A}u|(dx)$$

whenever F has linear growth at infinity.

Proof. By the discussion above, we infer that $F(\mathfrak{A}_\rho u) \in L^1_{\text{loc}}$. Let K be a compact set of $\mathbf{R}^n$. Appealing to Jensen's inequality and Young's convolution inequality, we deduce that

$$\begin{aligned}
\int_K F(\mathfrak{A}_\rho u(x))\,dx &= \int_K F\left(\int_0^\infty K_s \star u(x)\, n\omega_n \hat{\rho}(s) s^{n-1}\,ds\right) dx \\
&\leq \int_K \int_0^\infty F(K_s \star u)\, n\omega_n \hat{\rho}(s) s^{n-1}\,ds\,dx \\
&= \int_0^\infty n\omega_n \hat{\rho}(s) s^{n-1}\,ds \int_K F(\mathcal{A}u \star \frac{\chi_{B_s}}{|B_s|})\,dx.
\end{aligned}$$

The inequality $F(a+b) \leq F(a) + F^\infty(b)$ and subsequent applications of Jensen's and Young's inequality permit us to estimate the right-hand side above by

$$\begin{aligned}
&\int_K F(\mathcal{A}^a u \star \frac{\chi_{B_s}}{|B_s|}) + F^\infty(\mathcal{A}^s u \star \frac{\chi_{B_s}}{|B_s|})\,dx \\
&\quad \leq \int_K F(\mathcal{A}^a u) \star \frac{\chi_{B_s}}{|B_s|}\,dx + \int_K F^\infty(\mathcal{A}^s u) \star \frac{\chi_{B_s}}{|B_s|}\,dx \\
&\quad \leq \|F(\mathcal{A}^a u)\|_1 + \|F^\infty(\mathcal{A}^s u)\|_1.
\end{aligned}$$

The first assertion follows this estimate by taking a sequence of compact sets $K_j \nearrow \mathbf{R}^n$. When F has linear growth at infinity, the recession function F^∞ is a positively homogeneous convex function taking values on $\mathbf{R}$. Notice that

$$F^\infty(\mu) := F^\infty(g_\mu)|\mu| \in \mathcal{M}(\mathbf{R}^n), \qquad \forall \mu \in \mathcal{M}(\mathbf{R}^n; W).$$

By (9), Reschetnyak's theorem (which holds under distributional convergence) and the previous proposition, we conclude that

$$\begin{aligned}\limsup_{\varepsilon\to 0}\|F(\mathfrak{A}_{\rho_\varepsilon}u)\|_1 &\le \|F(\mathcal{A}^a u)\|_1 + |F^\infty(\mathcal{A}^s u)|(\mathbf{R}^n)\\ &\le \liminf_{\varepsilon\to 0}\|F(\mathfrak{A}_{\rho_\varepsilon}u)\|_1.\end{aligned}$$

This proves the second assertion. □

Corollary 2. *Let $u \in L^1_{\mathrm{loc}}(\mathbf{R}^n;V)$ and let $1 \le p < \infty$.*

(1) If $\mathcal{A}u \in L^p$, then

$$\mathfrak{A}_{\rho_\varepsilon}u \xrightarrow{L^p} \mathcal{A}u \quad \text{as } \varepsilon \to 0.$$

(2) If $\mathcal{A}u \in \mathcal{M}(\mathbf{R}^n;W)$, then

$$\mathfrak{A}_{\rho_\varepsilon}u\,\mathcal{L}^d \xrightarrow{area} \mathcal{A}u \quad \text{as } \varepsilon \to 0.$$

Proof. By letting $F(t) = t^p$ in the previous proposition, we obtain that

$$\lim_{\varepsilon\to 0}\|\mathfrak{A}_{\rho_\varepsilon}u\|_p = \|\mathcal{A}u\|_p \in [0,\infty]. \tag{11}$$

For $1 < p \le \infty$, it follows that $\mathcal{A}u \in L^p(\mathbf{R}^n;V)$ if and only if $\liminf_{\varepsilon\to 0}\|\mathfrak{A}_{\rho_\varepsilon}u\|_p < \infty$. For $1 < p < \infty$, the Brezis–Lieb lemma implies that the distributional convergence in (9) improves to the convergence

$$\mathfrak{A}_{\rho_\varepsilon}u \longrightarrow \mathcal{A}u \quad \text{strongly in } L^p.$$

This proves (1) for $p > 1$. For $p = 1$, the previous proposition implies that $\mathfrak{A}_{\rho_\varepsilon}u$ indeed converges in measure to $\mathcal{A}u$. This proves (2). We are left to show (1) for $p = 1$. Notice that when $\mathcal{A}u \in L^1$, then the singular part $\mathcal{A}^s u$ vanishes, and hence from the area convergence, we get

$$\int \sqrt{1+|\mathfrak{A}_{\rho_\varepsilon}u|^2}\,dx \longrightarrow \int \sqrt{1+|\mathcal{A}u|^2}\,dx.$$

Since $\sqrt{1+|\bullet|^2}$ is uniformly strictly convex and larger or equal than $|\bullet|$, it follows that $\{\mathfrak{A}_{\rho_\varepsilon}u\}_\varepsilon$ is an equi-bounded family of L^1. The Dunford–Pettis criterion implies that the distributional convergence (9) lifts to L^1 strong convergence. □

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Appendix A. Measure-theoretic lemmata

Lemma 20. *Let $\mu \in \mathcal{M}^+(\mathbf{R}^n)$ be a finite measure and let $s > 0$ be given. Then, the set*

$$R \triangleq \{x \in \mathbf{R}^n : \mu(\partial B_s(x)) > 0\} = 0$$

is negligible with respect to the n-dimensional Lebesgue measure.

Proof. Since μ is a Borel measure, it follows that the map

$$g(x) \triangleq \mu(B_s(x)) \qquad x \in \mathbf{R}^n$$

is Borel measurable. Moreover, by Young's convolution inequality, it follows that

$$\|g\|_{L^1} \leq \|\eta_s \star \mu\|_{L^1} \leq \|\eta_s\|_{L^1}|\mu|(\mathbf{R}^n) < \infty$$

for any nonnegative test function $\eta \in C_c^\infty(B_{2s})$ such that $\eta \equiv 1$ on B_s. In particular, g is integrable; hence, it is Lebesgue continuous at almost every $x \in \mathbf{R}^n$. Notice however that if $|\mu|(B_s(x)) > 0$, then there exists a real $\delta > 0$ and an open cone $S \subset \mathbf{R}^n$ such that

$$\frac{\mathcal{L}^n(S \cap B_1)}{|B_1|} \geq \delta, \qquad g(y) \geq g(x) + \delta \quad \text{for every } y \in (B_\delta \cap S) + x.$$

In particular, by a scaling argument, it follows that

$$\limsup_{r \to 0} \fint_{B_r(x)} |g - g(x)| \geq \delta^2 > 0 \qquad \text{for all } r > 0.$$

From this basic analysis, we deduce that $R_\mu \subset S_g$ —where S_g is the set of Lebesgue discontinuity points of g. The Lebesgue differentiation theorem gives

$$\mathcal{L}^n(R) \leq \mathcal{L}^n(S_g) = 0,$$

which is what we wanted to show. □

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