Sunspot Equilibria in Systems with Memory: an Introductory Presentation

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1 Introduction

This chapter aims at introducing the question of existence of sunspot equilibria in systems that have a more complex structure than the simple overlapping generation model in which the dynamical sunspot phenomena have been first studied: Azariadis (1981b), Azariadis and Guesnerie (1982a), Farmer and Woodford (1984), etc.... Indeed, starting from this literature, sunspot equilibria have become well understood in one-dimensional one-step-forward-looking infinite horizon models and reasonably well understood in multidimensional models that remain one-step forward looking (see Chiappori and Guesnerie [1991b], Guesnerie and Woodford [1992]).

Attention in the present chapter is focused on models that are again one-step forward looking but that have memory. Most of the argument holds in settings where the state of the system at each period is essentially one-dimensional, at the exception of the last section that considers multidimensional systems.

Our objective here is twofold. We first attempt to provide simple intuition on the nature of the self-fulfillment mechanisms that sustain sunspot equilibria in the models under consideration: It is why, for example, we discuss in detail what happens in linear models. Second, we only review existing results while providing, at best, sketches of proofs.

Finally, the chapter takes the option of concentrating on a particular type of sunspot equilibria that are the subject of Dávila's work (1994, 1997). Such sunspot equilibria are particularly appealing from an economic viewpoint: For example, the sunspot equilibria of Sections 1 to 3 look like periodic equilibria in the sense that they describe oscillations of the system between 2 or k states that recur for ever; the oscillations are however not deterministic, as for standard cycles, but stochastic. The stochastic quasi-cycles that are described are reminiscent of the 2-SSE or k-SSE that were first exhibited in the OLG model; they are much simpler than the general SSE whose existence was considered in Woodford (1984) and obtain under more severe conditions, as emphasized in Section 4.

The chapter proceeds as follows:

Section 1 presents, in simple models, the fulfillment mechanisms associated with sunspot equilibria with a binary support, first without memory and then with one-period memory.

Section 2 keeps a linear setting but extends the analysis to multi-period memory and finite support of any order.

Section 3 reassesses the sunspot existence problem in the non-linear version of the model previously introduced, and reviews the results obtained by Dávila (1997).

Section 4 discusses the relationship of the previous analysis with Woodford's contribution.

Section 5 provides insights on the sunspot existence problem in settings that are similar as far as memory is concerned, but that are multidimensional: results that are, however, weaker than in the one-dimensional setting are again borrowed from Dávila (1994).

Brief concluding comments are offered.

2 Sunspot Equilibria in Simple Linear Systems

Particularly simple sunspot equilibria appearing in dynamic economic models are the so-called finite Markovian stationary sunspot equilibrium (see Azariadis [1981b] and Azariadis and Guesnerie [1982a,b, 1986] for the first examples and Chiappori and Guesnerie [1991b] for an extensive survey covering these types of sunspot equilibria and related subjects). In order to provide a simple instance of such an equilibrium, let us consider the following one-dimensional dynamical system

$$x_{t+1} = f(x_t). \tag{2.1}$$

Equation (2.1) says that the state of the system considered is, at any time, a function of the previous date state. Let us simplify the example further considering f in (2.1) to be linear, the equation becoming thus:

$$x_{t+1} = ax_t. \tag{2.2}$$

Equation (2.2) says now that the state variable at any date is a fixed proportion of the state variable at the previous date. It reads, too:[1]

$$x_t = \frac{1}{a} x_{t+1}, \tag{2.3}$$

Equation (2.3) can be interpreted as follows: The state of the system at any date is determined by the perfectly foreseen next period state, i.e., the state expected to happen, with probability 1. Such backward dynamics obtains, for instance, as the linearization, around a steady state, of the reduced form of the perfect foresight equilibrium conditions of a simple overlapping generations economy: the state variable can be chosen to be the labor supply of the young or, equivalently, the first period consumption or the money balances depending on the particular details of the model (see for example Guesnerie and Woodford [1992]).

As a straightforward and natural extension, in the case where the state of the system in the next period is not known with certainty but only with some probability, equation (2.3) may be generalized to become

$$x_t = \frac{1}{a} E_t(x_{t+1}), \tag{2.4}$$

that is to say, the state of the system at any date is determined by the expectations at that time about the next period state. Again, it is the linearization, for small noise, of the equilibrium conditions involving stochastic forecasts.

For instance, at any date t, it may be foreseen that the next period state will be either x^1 with some probability π^{11} or a distinct state x^2 with the complementary probability π^{12} if at t the state is x^1, while the same distinct states will occur with some other probabilities π^{21} and π^{22} if at t the state is x^2. Then, according to (2.4), x^1 will actually be the equilibrium state at any given date, if it satisfies:

$$x^1 = \frac{1}{a}(\pi^{11}x^1 + \pi^{12}x^2) \tag{2.5a}$$

and similarly x^2 will actually be the equilibrium state at any given date if it satisfies

$$x^2 = \frac{1}{a}(\pi^{21}x^1 + \pi^{22}x^2). \tag{2.5b}$$

The equilibrium equations just described define the simplest example of a finite markovian stationary sunspot equilibrium of the system (2.4), from now on SSE, more specifically, of a markovian SSE of order 2 (a 2-SSE henceforth). Some comments on the equilibrium concept and the stochastic dynamics that it generates are now in order.

Think, as in a simple overlapping generations economy, that is taken provisionally as linear, of the state at any period as being the amount of labor supplied by the contemporary youngs. Thus the state at date t results from decisions that depend on the expectations about the next period's young labor supply,[2] i.e., about the next period state, say as in (2.4). Should there by any reason for generation t to believe that generation $t+1$ will supply x^1 or x^2 depending on the contemporary observation of either one value, say 1, or another value, say 2, of a random signal (the "sunspot") following a Markov chain governed by (π^{ij}), then the rational choice for generation t, as it comes from (2.4), is to choose to supply x^1 (resp. x^2) if it observes the value 1 (resp. 2) of the signal at time t. Beliefs of this type, if maintained through time will generate stochastic dynamics, where x^1, x^2, will alternate for ever, perfectly correlated with the exogenous random sunspot signal 1, 2.

Note that the randomness follows from beliefs triggered by sunspot events; however, a variety of stories involving a subtler chain of beliefs might be invoked in order to sustain the sunspot equilibrium.[3]

Formally, a 2-SSE of the kind under consideration, consists of two states x^1 and x^2 and a Markov matrix Π satisfying (2.5a), (2.5b). Even if the above remarks on interpretation should be reminded, the equations can simply read as follows: The beliefs: $1 \to x^1$, $2 \to x^2$, if universal, are self fulfilling.

The issue of the existence of such an equilibrium can now be briefly discussed. For the equations (2.5a), (2.5b) to be satisfied for some values of x^1, x^2 and Π, the parameter a has to be such that ax^1 and ax^2 can be written as convex linear combinations of x^1 and x^2. A careful inspection of this condition shows that: i) x^1, x^2 must have different signs, ii) a needs to have an absolute value smaller than one. Moreover, this is sufficient too, in the following sense: if $|a| < 1$, for any x^1, x^2 of different sign, one can find one Markov matrix, such that equations (2.5a) and (2.5b) are fulfilled.

Thus, *a necessary and sufficient condition for the existence of SSE of order 2, solution to (2.4), is* $|a| < 1$.[4]

Now, note an essential characteristic of the dynamics under consideration in (2.1): The state in the next period only depends upon the current state and not on the history of the system in the past. This is an obviously restrictive feature that leaves out of the scope of the example models of economies such as those in Woodford (1986) and Reichlin (1986), for instance, in which the reduced form of the equilibrium dynamics leads to a state determined at any date by both the expected next period state and the previous state. The simplest and most straightforward generalization of (2.1) in order to cope with the role of past history in determining the current state is the following:

$$x_{t+1} = f(x_t, x_{t-1}) \tag{2.6}$$

where, as in the previous example, we focus attention on the linearization of f (around some steady state), that is to say

$$x_{t+1} = a_0 x_t + a_1 x_{t-1}. \tag{2.7}$$

Again, equation (2.7) can be read as follows[5]:

$$x_t = \frac{1}{a_0} x_{t+1} - \frac{a_1}{a_0} x_{t-1} \tag{2.8}$$

meaning that the state is determined at any date by both the perfectly foreseen next period state and the previous state. Should there be any uncertainty about the next period state, equation (2.8) could be generalized to become

$$x_t = \frac{1}{a_0} E_t(x_{t+1}) - \frac{a_1}{a_0} x_{t-1}. \tag{2.9}$$

Our main concern now is the following: does such stochastic dynamics still have solutions similar to those just shown for equation (2.4)? Or, are there distinct states x^1 and x^2 and probabilities of transition between them π^{ij} such that x^i ($i = 1, 2$) is the equilibrium state if the next period state is expected to be either x^1 with probability π^{i1} or x^2 with probability π^{i2}, whichever, x^1 or x^2, may have been the previous state? If there is any such equilibrium it would be characterized by the equations

$$
\begin{aligned}
x^1 &= \frac{1}{a_0}(\pi^{11}x^1 + \pi^{12}x^2) - \frac{a_1}{a_0}x^1 \\
x^1 &= \frac{1}{a_0}(\pi^{11}x^1 + \pi^{12}x^2) - \frac{a_1}{a_0}x^2 \\
x^2 &= \frac{1}{a_0}(\pi^{21}x^1 + \pi^{22}x^2) - \frac{a_1}{a_0}x^1 \\
x^2 &= \frac{1}{a_0}(\pi^{21}x^1 + \pi^{22}x^2) - \frac{a_1}{a_0}x^2
\end{aligned}
\tag{2.10}
$$

From the first two equations (or the last two) the answer to this question is obviously no, since necessarily x^1 and x^2 would have to be equal at any solution.[6]

In order to overcome this difficulty, we may let the probabilities of transition to depend not only on the current state but on the previous state too, letting π^{ijk} denote the probability of x^k being the next state if the current one is x^j and the previous one has been x^i.

In this modified setting, the question becomes: Are there distinct states x^1, x^2 and four points in the relative interior of the unit simplex (π^{ij1}, π^{ij2}), $i, j = 1, 2$, satisfying the following equations?

$$
\begin{aligned}
x^1 &= \frac{1}{a_0}(\pi^{111}x^1 + \pi^{112}x^2) - \frac{a_1}{a_0}x^1 \\
x^1 &= \frac{1}{a_0}(\pi^{211}x^1 + \pi^{212}x^2) - \frac{a_1}{a_0}x^2 \\
x^2 &= \frac{1}{a_0}(\pi^{121}x^1 + \pi^{122}x^2) - \frac{a_1}{a_0}x^1 \\
x^2 &= \frac{1}{a_0}(\pi^{221}x^1 + \pi^{222}x^2) - \frac{a_1}{a_0}x^2
\end{aligned}
\tag{2.11}
$$

The first equation says that the equilibrium to-day is x^1, if the signals observed today and yesterday were 1 (in which case, the state observed yesterday was x^1), so that the transition probabilities for 1 and 2, tomorrow, will be π^{111}, π^{112}, respectively. And so on.

As above, one proceeds to an informal inspection of the compatibility of the four homogenous equations with two unknowns. We may first look at the case where a_0, a_1 have the same sign (say positive).

We first note that the signs of x^1, x^2 have to be different: If not, and if, for example, they were positive, $(a_0 + a_1)x^1$ and $(a_0 + a_1)x^2$ cannot be both convex combinations of x^1, x^2, as required by the first and the fourth equations. Now if $a_0 + a_1 > 1$, then $(a_0 + a_1)x^1 < x^1$, (if w.l.o.g $x^1 < 0$), so that neither the first nor the fourth equation can be satisfied. It follows that necessarily $a_0 + a_1 < 1$. We then check that, in this case, since both coefficients a_0, a_1 are positive and smaller than one, $a_0x^1 + a_1x^2$, as well as $a_0x^2 + a_1x^1$, belong to the segment $\{x^1, x^2\}$. It follows that if the latter condition is satisfied, for any x^1, x^2 of different signs, one can find Markov coefficients such that equations (2.12) hold true.

The argument could be pursued in the case where a_0 and a_1 have different signs. But the question can be settled in a more general and more efficient way by noticing that the existence of a solution $x^1 \neq x^2 \neq 0$, requires that the four planes in the x^1, x^2 space determined by (2.12) coincide, that is to say, that their four normal vectors coincide too,[7] for some probabilities of transition. It can be shown[8] that such a coincidence obtains if and only if, $|a_0| + |a_1| < 1$. Such stochastic dynamics solution to (2.10) is what we will refer to as a stationary sunspot equilibrium of order 2 of the system. Thus we can now state the following[9]:

PROPOSITION 2.1 There exist 2-SSE solution of (2.9) iff $|a_0| + |a_1| < 1$.

The characterization provided by the previous proposition shows that 2-SSE exist for a non-negligible set of dynamical systems (2.10).

Furthermore, following the line of argument of previous footnote (4), one can check that the dimensionality of the transition probabilities associated with SSE in a given system is one.

In Section 5, it will be seen how the set of dynamical systems that have SSE of the type under consideration relate to the set of systems of the same class with the steady state indeterminate in the perfect foresight dynamics.

3 More on Sunspot Equilibria in Linear Systems

While remaining in the model "with memory" governed by equation (2.9), we are going to consider successively the case where the sunspot phenomenon has arbitrarily many states (Subsection 3.1), and the case of "long memory" (Subsection 3.2).

3.1 Sunspot Equilibria with Arbitrarily Many States

The necessary and sufficient characterization of the systems (2.9) with 2-SSE in the previous paragraph turns out to be unchanged when we consider markovian stationary sunspot equilibria of any finite order k: as we shall show, the system (2.9) has what will be called k-SSE for any $k \geq 2$ if, and only if, it has 2-SSE.

A k-SSE solution of (2.9) will consist of k distinct states $x^1, \ldots, x^k$ and k^2 points π^{ij} (of coordinates π^{ijl}, $l = 1, \ldots, k$) in the relative interior of the $(k-1)$-dimensional unit simplex (one for each possible sequence of these states in the previous and current period) such that, for all i, j

$$x^j = \frac{1}{a_0}\left(\pi^{ij1}x^1 + \pi^{ijk}x^k\right) - \frac{a_1}{a_0}x^i \tag{3.1}$$

The equation says that state x^j occurs today, if state x^i has occurred yesterday, and if to-morrow x^l occurs with probability π^{ijl}. Again, when the equation holds true, the belief sunspot signal $i \to x^i$ is self-fulfilling, whenever the occurrence of the sunspot signal is governed by those transition probabilities π^{ijl}, that depend upon the two last values of the sunspot signal.

This can be rewritten as:

$$a_1x^i + a_0x^j = \pi^{ij1}x^1 + \ldots + \pi^{ijk}x^k \tag{3.2}$$

for all $i, j = 1, \ldots, k$.

Equivalently, a k-SSE consists of k distinct states $x^1, \ldots, x^k$ whose linear combinations $a_1x^i + a_0x^j$ are all in the interior of their convex hull. One can show now that if the system (2.9) has k-SSE for any given $k > 2$, it must also have a 2-SSE: let x^1 and x^2 be the smallest and the biggest of the x^l; the four linear combinations $a_1x^i + a_0x^j$, $i, j = 1, 2$ being in the convex hull of all the k states, they must necessarily be in the convex hull of the smallest and the biggest two as well. Hence, as can be checked from equations (3.1) x^1 and x^2 are the states of a 2-SSE.

Conversely, if a system has a 2-SSE, then it has k-SSE for all $k > 2$, since given any k let $y^1, \ldots, y^k$ be k states such that $y^1 = x^1$ and any other $y^i = x^2$, x^1 and x^2 being the states of the 2-SSE. Then every $a_1y^i + a_0y^i$ for all $i, j = 1, \ldots, k$, is in the convex hull of $y^1, \ldots, y^k$, since all of them coincide with some $a_1x^r + a_0x^s$ with $r, s = 1, 2$ and the convex hull of $y^1, \ldots, y^k$ is that of x^1, x^2. Of course these $y^1, \ldots, y^k$ are not the states of a k-SSE yet, since not all of them are distinct. But the values of $y^2, \ldots, y^k$ can be perturbed slightly in order to make all of them distinct and, by continuity, keep their linear combinations $a_1y^i + a_0y^j$ in their convex hull still. These newly perturbed states are now those of a k-SSE. Thus the previous proposition, 2.1, holds more generally and can be restated in the following terms.

PROPOSITION 3.1 There exist k-SSE solution of (2.9) for all k iff $|a_0| + |a_1| < 1$.

3.2 Linear Systems with Long Memory

The previous argument can also be modified in order to show that similar sunspot equilibria appear as solutions to dynamical systems with arbitrarily long memory. Indeed, consider a dynamical system like

$$x_t = \frac{1}{a_0} E_t(x_{t+1}) - \frac{a_1}{a_0} x_{t-1} - \ldots - \frac{a_p}{a_0} x_{t-p} \tag{3.3}$$

in which all the p most recent states (with the expectations) determine the current one. A stationary sunspot equilibrium of any order k will consist of k distinct states $x^1, \ldots, x^k$ and probabilities of transition between them necessarily depending on the current state and all the p immediate previous ones (otherwise, for the same reasons as before, no off-diagonal solutions can be found) satisfying conditions analogous to (3.1). But we have now k^{p+1} equations, one for each history of the realizations of the sunspot signals from $t-p$ until the current period t.

The existence of such k-SSE is equivalent, in this case as well, to the coincidence of all the normal vectors of the k^{p+1} hyperplanes representing the equilibrium conditions.

It can be shown[10] that such a condition can be met if, and only if,

$$\sum_{i=0}^{p} |a_i| < 1 \tag{3.4}$$

which actually encompasses the characterizations found in the previous cases (i.e., without memory, $p = 0$, and one-step backward memory, $p = 1$).

4 Local Sunspot Equilibria in Non-Linear Systems with Memory

The argument developed in the previous paragraph in order to show the existence of finite Markovian stationary sunspot equilibria of linear dynamical systems can be transposed to show the existence of similar equilibria in nonlinear systems. One proceeds as follows:

Assume the state at a given date, x_t, is determined by both the previous state x_{t-1} and the probability distribution of the state in the next period μ_{t+1}. The reduced (non-necessarily linear) form can then be written:

$$x_t = \tilde{f}(x_{t-1}, \mu_{t+1}) \tag{4.1}$$

In the case where μ_{t+1} is a lottery on, for example, two possible states x^1_{t+1}, x^2_{t+1}, we shall denote it as: $\{x^1_{t+1}, x^2_{t+1}, m^1, m^2\}$, so that

$$x_t = \tilde{f}(x_{t-1}, \{x^1_{t+1}, x^2_{t+1}, m^1, m^2\}) \tag{4.2}$$

A 2-SSE of such a system consists, by definition, of two distinct states x^1 and x^2 and four vectors of probabilities (m^{ij1}, m^{ij2}), $i, j = 1, 2$ satisfying

$$\begin{aligned}
&x^1 - \tilde{f}(x^1, \{x^1, x^2, m^{111}, m^{112}\}) = 0\\
&x^2 - \tilde{f}(x^1, \{x^1, x^2, m^{121}, m^{122}\}) = 0\\
&x^1 - \tilde{f}(x^2, \{x^1, x^2, m^{211}, m^{212}\}) = 0\\
&x^2 - \tilde{f}(x^2, \{x^1, x^2, m^{221}, m^{222}\}) = 0.
\end{aligned} \tag{4.3}$$

Under some conditions to be made precise later on, each equation of (4.3) can be thought of as determining a curve in the (x^1, x^2) plane, whose position is determined by the corresponding probabilities of transition. The existence of a 2-SSE is then equivalent to the existence of probabilities (m^{ij1}, m^{ij2}), $i, j = 1, 2$, such that the four curves have a common off-diagonal intersection. Therefore, we have to look for conditions on $\tilde{f}$ in (4.2) sufficient to guarantee such an off-diagonal intersection for some probabilities, and hence sufficient for the existence of 2-SSE of system (4.2).[11]

First assume that there is a steady state $\bar{x}$ of the system. This is a state from which the system will not depart if it comes from it and is expected with certainty to be in it in the next period, so that $\bar{x}$ satisfies:

$$\bar{x} = \tilde{f}(\bar{x}, \{\bar{x}, \bar{x}, \alpha^1, \alpha^2\}) \tag{4.4}$$

for any probabilities α^1, α^2 (the lottery $\{\bar{x}, \bar{x}, \alpha^1, \alpha^2\}$ is nothing but $\bar{x}$ with certainty whichever these probabilities are).

Hence, the point $(\bar{x}, \bar{x})$ in the (x^1, x^2) plane is an intersection point of all the four curves of equations (4.3), although it is not a 2-SSE since it is not off-diagonal.

Now, in order to be sure that these equations represent curves indeed (at least in some neighborhood of the point $(\bar{x}, \bar{x})$) some assumptions have to be made on $\tilde{f}$ which can be conveniently stated in terms of the restriction of $\tilde{f}$ to certainty lotteries, i.e., lotteries that can be written as $\{x_{t+1}, x_{t+1}, \alpha^1, \alpha^2\}$.[12] Such restriction f can thus be defined as

$$f(x_{t-1}, x_{t+1}) = \tilde{f}(x_{t-1}, \{x_{t+1}, x_{t+1}, \alpha^1, \alpha^2\}) \tag{4.5}$$

for any probabilities α^1, α^2.

Notice that using this restriction to pointwise expectations, the system (4.2) reduces to its deterministic counterpart:

$$x_t = f(x_{t-1}, x_{t+1}) \tag{4.6}$$

and that hence the steady state of the dynamics with uncertain expectations is also a steady state of the dynamics with certain expectations, i.e.,

$$\bar{x} = f(\bar{x}, \bar{x}) \tag{4.7}$$

(actually this is nothing but the equation (4.4) rewritten taking into account the definition of f in (4.5)).

Assume now that

$$\partial_{1i}\tilde{f}(\bar{x}, \{\bar{x}, \bar{x}, \alpha^1, \alpha^2\}) = \alpha^i \partial_1 f(\bar{x}, \bar{x}) \tag{4.8}$$

for $i = 1, 2$, where $\partial_{1i}\tilde{f}$ denotes the partial derivative of $\tilde{f}$ with respect to x^i_{t+1} and $\partial_1 f$ that of f with respect to x_{t+1}. Such an assumption amounts to say (in infinitesimal language) that the effect on $\tilde{f}$ of a small change in a future state that may happen with some probability is equal to the probability of the change multiplied by the deterministic effect on f of the same small change.[13] Then the gradients of the left-hand sides in (4.3) can be written as

$$\begin{aligned}
&(1-\partial_{-1} f(\bar{x},\bar{x}) - m^{111}\partial_1 f(\bar{x},\bar{x}), -m^{112}\partial_1 f(\bar{x},\bar{x}))\\
&(-\partial_{-1} f(\bar{x},\bar{x}) - m^{121}\partial_1 f(\bar{x},\bar{x}), 1 - m^{122}\partial_1 f(\bar{x},\bar{x}))\\
&(1 - m^{211}\partial_1 f(\bar{x},\bar{x}), -\partial_{-1} f(\bar{x},\bar{x}) - m^{212}\partial_1 f(\bar{x},\bar{x}))\\
&(-m^{211}\partial_1 f(\bar{x},\bar{x}), 1-\partial_{-1} f(\bar{x},\bar{x}) - m^{222}\partial_1 f(\bar{x},\bar{x}))
\end{aligned} \tag{4.9}$$

Where now $\partial_{-1} f$ is the partial derivative of f with respect to x_{t-1}.

Notice that the scalar product of any of these vectors with the vector (1, 1) is equal to $1 - \partial_{-1}f(\bar{x},\bar{x}) - \partial_1 f(\bar{x},\bar{x})$. If this is not zero, or equivalently, if

$$\partial_{-1}f(\bar{x},\bar{x}) + \partial_1 f(\bar{x},\bar{x}) \neq 1, \tag{4.10}$$

then surely none of the gradients in (4.9) is null and hence the equations (4.3) represent indeed four smooth curves in the (x^1, x^2) plane in some neighborhood of $(\bar{x},\bar{x})$. For the sake of consistency with the linear case let us denote $-\partial_{-1}f(\bar{x},\bar{x})/\partial_1 f(\bar{x},\bar{x}) = a_1$ and $1/\partial_1 f(\bar{x},\bar{x}) = a_0$. Then the condition (4.10) becomes

$$a_0 + a_1 \neq 1. \tag{4.11}$$

The assumptions made up to this point, namely the existence of $\bar{x}$ such that (4.7), (4.8) and (4.10) hold are not stringent. Actually they are straightforward generalizations to systems with memory of conditions holding in systems without memory, for instance, simple overlapping generations economies.

Now the key idea of the argument leading to the characterization of systems like (4.2) with an off-diagonal intersection of equations (4.3), i.e., with 2-SSE, can be stated as follows. If there exist probabilities such that all the four curves in (4.3) share a common tangent T at $(\overline{x}, \overline{x})$, then they can be slightly perturbed in order to make them intersect at any point in T close enough to $(\overline{x}, \overline{x})$.[14] Thus a sufficient condition for the existence of local sunspots, i.e., of uncountably many 2-SSE of (4.1) arbitrarily close to the steady state, is the existence of probabilities such that all the gradients in (4.9) are equal[15] and a necessary and sufficient condition for the existence of such probabilities is (as in the linear case)

$$|a_0| + |a_1| < 1. \tag{4.12}$$

Therefore, we can state the following

PROPOSITION 4.1 There exist local 2-SSE of (4.1) if $|a_0| + |a_1| < 1$.

Under the assumptions made and a_0, a_1 being defined as above, notice that this characterization was, in the linear case, not only sufficient but necessary, too. The converse implication cannot be obtained in the general case and examples of systems not satisfying $|a_0| + |a_1| < 1$ but with local 2-SSE can be produced.[16]

The same argument can be extended to characterize the existence of local k-SSE in order to obtain the next proposition.

PROPOSITION 4.2 There exist local k-SSE of (4.1) if $|a_0| + |a_1| < 1$.

The reasoning has to be done in this case in the space of k-tuples of states, i.e., in some piece of R^k, and our former curves become manifolds of dimension $k - 1$ in some neighborhood of $(\overline{x}, \ldots, \overline{x})$. The idea of the proof is essentially the same as before: A sufficient condition for producing an off-diagonal intersection of all the manifolds is the existence of probabilities making all of them tangent at $(\overline{x}, \ldots, \overline{x})$ to a nontrivial affine set.[17] Should there exist such probabilities, the manifolds could be perturbed to intersect at any point in that affine set close enough to $(\bar{x}, \ldots, \bar{x})$.[18] Now a necessary and sufficient condition for the existence of such an affine set is precisely $|a_0| + |a_1| < 1$.[19]

Moreover, as in the linear case, the argument can also be extended to characterize the existence of local k-SSE in systems with arbitrarily long memory

$$x_t = \tilde{f}(x_{t-p}, \ldots, x_{t-1}, \{x_{t+1}^1, \ldots, x_{t+1}^k, m^1, \ldots, m^k\}) \tag{4.13}$$

and to prove the following proposition.

PROPOSITION 4.3 There exist local k-SSE of (4.13) of if

$$\sum_{i=0}^{p} |a_i| < 1 \tag{4.14}$$

where $a_0 = \dfrac{1}{\partial_1 f(\bar{x}, \ldots, \bar{x})}$ and $a_i = \dfrac{\partial_{-i} f(\bar{x}, \ldots, \bar{x})}{\partial_1 f(\bar{x}, \ldots, \bar{x})}$, with the usual notation for partial derivatives.

The proof is essentially independent of the length of the memory: it is somewhat irrelevant for the argument that the number of manifolds, whose off-diagonal intersection is considered, varies with the length of the memory.[20]

5 The Connection with Woodford's Conjecture

On the one hand, the systems (4.1) for which the proposition 4.2 guarantees the existence of local k-SSE are actually systems for which the steady state $\bar{x}$ is indeterminate in the perfect foresight dynamics: Any trajectory starting close enough to the steady state converges to the steady state. On the other hand, the so-called Woodford conjecture (Woodford [1984]) links the existence of local sunspot equilibria to indeterminacy of the steady state, a fact that the recent work of Grandmont, Pintus, and de Vilder (1997) illustrate in a more geometric way.

Let us clarify the connections of the above results and of Woodford's conjectures.

Indeed, the dynamics

$$x_t = f(x_{t-1}, x_{t+1}) \tag{5.1}$$

if linearized around the steady state

$$x_t - \bar{x} = \partial_1 f(\bar{x}, \bar{x})(x_{t-1} - \bar{x}) + \partial_2 f(\bar{x}, \bar{x})(x_{t-1} - \bar{x}) \tag{5.2}$$

and written in the form

$$x_{t+1} - \bar{x} = \frac{1}{\partial_2 f(\bar{x}, \bar{x})}(x - \bar{x}) - \frac{\partial_1 f(\bar{x}, \bar{x})}{\partial_2 f(\bar{x}, \bar{x})}(x_{t-1} - \bar{x}) \tag{5.3}$$

can be restated as

$$\begin{pmatrix} x_{t+1} - \bar{x} \\ x_t - \bar{x} \end{pmatrix} = \begin{pmatrix} a_0 & a_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t - \bar{x} \\ x_{t-1} - \bar{x} \end{pmatrix} \tag{5.4}$$

where a_0 and a_1 denote $\frac{1}{\partial_2 f(\bar{x}, \bar{x})}$ and $-\frac{\partial_1 f(\bar{x}, \bar{x})}{\partial_2 f(\bar{x}, \bar{x})}$ respectively.

The point $(\bar{x}, \bar{x})$ is a globally stable fixed point of this dynamics (i.e., $\bar{x}$ is an indeterminate steady state) if no eigenvalue of the matrix of the linearization above is out of the unit circle in the complex plane. These eigenvalues are the solution to

$$\begin{vmatrix} a_0 - \lambda & a_1 \\ 1 & -\lambda \end{vmatrix} = 0 \tag{5.5}$$

i.e., they are the roots of the characteristic equation $\lambda^2 - a_0\lambda - a_1 = 0$, which are $\frac{1}{2}(a_0 + \sqrt{a_0^2 + 4a_1})$ and $\frac{1}{2}(a_0 - \sqrt{a_0^2 + 4a_1})$.

Simple algebra[21] shows that for the steady state $\bar{x}$ to be indeterminate in the perfect foresight dynamics a_0 and a_1 have to satisfy the inequalities $-1 < a_1$, $a_1 < 1 - a_0$ and $a_1 < 1 + a_0$.

Our condition for the existence of local k-SSE, $|a_0| + |a_1| < 1$, is sufficient for these inequalities to be satisfied, although it is certainly not a necessary one. In other words, the systems for which we have shown the existence of local k-SSE around the steady state constitute a proper subset of those for which this steady state is indeterminate in the corresponding perfect foresight dynamics.

This fact deserves some comments. Woodford (1984) established that the indeterminacy of the steady state is a necessary and sufficient condition for the existence of local sunspot equilibria. Here, we have focused on a special class of sunspot equilibria which have particularly appealing properties: The system oscillates forever between 2 or more states, and displays quasi-cycles; but the sufficient conditions that have been exhibited by Dávila and that have been reproduced here are more restrictive than those of Woodford and are not moreover necessary.[22]

However, and interestingly enough, our characterization, which is only sufficient in general, becomes necessary and sufficient indeed for the linear case, i.e. for dynamics of the form

$$x_{t+1} = a_0 x_t + a_1 x_{t-1}. \tag{5.6}$$

This implies that there are dynamics of this kind with the steady state indeterminate, and thus with local sunspot equilibria according to Woodford, but without local k-SSE. Therefore, the local sunspot equilibria detected by Woodford have to be for these linear dynamics either of non finite support or not markovian (at least with a memory whose length is of the order of magnitude of the cardinality of the support). This underlines the fact that the existence of local sunspot equilibria à la Woodford and that of k-SSE are clearly distinct questions in general: The results emphasized here, as they concern specially simple and appealing types of sunspot equilibria, are less powerful but may have a particular significance for the theory of endogenous fluctuations.

6 Sunspot Equilibria in Multidimensional Systems with Memory

Let us now reconsider the linear case when the dynamics take place in R^n. The dynamics (2.7) becomes thus

$$x_{t+1} = A_0 x_t + A_1 x_{t-1} \tag{6.1}$$

where A_0 and A_1 are $n \times n$ matrices. Assuming A_0 to be regular, these dynamics can be read as

$$x_t = A_0^{-1} x_{t+1} - A_0^{-1} A_1 x_{t-1}. \tag{6.2}$$

i.e., as the state in each period being determined by that of the previous one and the perfectly foreseen next state. Should the next period state be foreseen with uncertainty as a lottery $\{x_{t+1}^1, \ldots, x_{t+1}^k, m^1 \ldots, m^k\}$, and deterministic values replaced by expected values, the equation (6.2) could be rewritten as

$$x_t = A_0^{-1} \sum_{h=1}^{k} m^h x_{t+1}^h - A_0^{-1} A_1 x_{t-1}. \tag{6.3}$$

A k-SSE associated with the dynamics (6.1) will consist of a k-tuple of distinct states $x^1, \ldots, x^k$ and k^2 vectors of probabilities $(m^{ij1}, \ldots, m^{ijk})$, for $i, j = 1, \ldots, k$, such that

$$x^j = A_0^{-1} \sum_{h=1}^{k} m^{ijh} x^h - A_0^{-1} A_1 x^i \tag{6.4}$$

or equivalently,

$$A_1 x^i + A_0 x^j = \sum_{h=1}^{k} m^{ijh} x^h \tag{6.5}$$

for all $i, j = 1, \ldots, k$. Thus $x^1, \ldots, x^k$ and $\{m^{ij}\}$ will be a k-SSE if, and only if, all the vectors $A_1 x^i + A_0 x^j$ are in the convex hull of the states. Obviously, for any $x^1, \ldots, x^k$ such that *the full interior of their convex hull* contains the null vector, if A_1 and A_0 are regular are matrices close enough to the null matrix, then the conditions will be satisfied and $x^1, \ldots, x^k$ will certainly be the states of a k-SSE.

Therefore, A_1 and A_0 being strong enough contractions is sufficient to guarantee the existence of k-SSE of (6.1) of a high enough order ($k \geq n$).

More specifically, it suffices that the norms of A_1 and A_0 add up to less than 1, i.e.,

$$\|A_1\| + \|A_0\| < 1 \tag{6.6}$$

since in that case for any x^i, x^j such that $\|x^i\| = 1$ and $\|x^j\| = 1$ $\|A_1x^i\| \leq \|A_1\|$ and $\|A_0x^j\| \leq \|A_0\|$, and since $\|A_1x^i + A_0x^j\| \leq \|A_1x^i\| + \|A_0x^j\|$, then necessarily

$$\|A_1x^i + A_0x^j\| \leq \|A_1\| + \|A_0\| < 1, \tag{6.7}$$

that is to say, for any pair of points x^i, x^j on the unit sphere, $\|A_1x^i + A_0x^j\|$ is contained in a sphere strictly contained in the unit sphere. Thus there exist a high enough number[23] of points $x^1, \ldots,$ x^k on the unit sphere such that the sphere of radius $\|A_1\| + \|A_0\|$ is contained in its convex hull and therefore necessarily every $A_1x^i + A_0x^j$ too. This means that the equations (6.5) are fulfilled for some probabilities. Hence we have, on the one hand, the next sufficient condition for the existence of k-SSE of (6.1).

PROPOSITION 6.1 There exist k-SSE of (6.1) for k high enough (specifically, if $n = 2$, for any $k > \pi/\arccos(\|A_1\| + \|A_0\|)$ if

$$\|A_1\| + \|A_0\| < 1 \tag{6.8}$$

On the other hand, a necessary condition for the existence of k-SSE of any order k of (6.1) is provided by the following reasoning, for which, and for all what follows, it will be assumed, without loss of generality, that $A_1 + A_0$ has no eigenvalue with modulus 0 or 1.

Conditions (6.5) require the existence of a non-trivial intersection of the null spaces of the matrices

$$(m^{ij} \otimes I_n - (e^i \otimes A_1 + e^j \otimes A_0)) \tag{6.9}$$

$i, j = 1, \ldots, k$, and in particular of those where $i = j$. Therefore, should there exist a k-SSE of any order k, then the matrix

$$\begin{pmatrix} m^{11} \otimes I_n - e^1 \otimes (A_1 + A_0) \\ m^{22} \otimes I_n - e^2 \otimes (A_1 + A_0) \\ \vdots \\ m^{kk} \otimes I_n - e^k \otimes (A_1 + A_0) \end{pmatrix} = M \otimes I_n - I_k \otimes (A_1 + A_0) \tag{6.10}$$

(where M is the Markov matrix whose i-th row is m^{ii}) would have to be singular. In other words, $M \otimes (A_1 + A_0)^{-1} - I_{nk}$ would have to be singular (since $A_1 + A_0$ is regular because 0 is not in its spectrum), i.e., 0 would have to be one of its eigenvalues. But the eigenvalues of $M \otimes (A_1 + A_0)^{-1} - I_{nk}$ are of the form[24] $mb - 1$, m being any eigenvalue of the Markov matrix M and b any of $(A_1 + A_0)^{-1}$, hence $mb = 1$ would have to hold for some m and b. Now, since all the eigenvalues of a Markov matrix are within the unit circle, necessarily there ought to be an eigenvalue of $(A_1 + A_0)^{-1}$ within the unit circle, i.e., an eigenvalue of $A_1 + A_0$ outside the unit circle (off the circle since $A_1 + A_0$ has no eigenvalue on it). Thus we have the following:

PROPOSITION 6.2 If (6.1) has a k-SSE, then $A_1 + A_0$ has an eigenvalue within the unit circle.

Let us consider now the general nonlinear case

$$x_t = \tilde{f}(x_{t-1}, \{x_{t+1}^1, \ldots, x_{t+1}^k, m^1, \ldots, m^k\}) \tag{6.11}$$

with the state in R^n. We can now give a necessary condition for the existence of local k-SSE fluctuating around a steady state. Consider the function defined as

$$\tilde{F}(x^1,\ldots,x^k,\{\ldots,m^{ij},\ldots\}) = \begin{pmatrix} x^1 - \tilde{f}(x^1,\{x^1,\ldots,x^k,m^{111},\ldots,m^{11k}\}) \\ \vdots \\ x^k - \tilde{f}(x^1,\{x^1,\ldots,x^k,m^{1k1},\ldots,m^{1kk}\}) \\ \vdots \\ x^1 - \tilde{f}(x^k,\{x^1,\ldots,x^k,m^{k11},\ldots,m^{k1k}\}) \\ \vdots \\ x^k - \tilde{f}(x^k,\{x^1,\ldots,x^k,m^{kk1},\ldots,m^{kkk}\}) \end{pmatrix}. \tag{6.12}$$

The zeros of this function are either steady states or k-SSE.[25] Now, for any given probabilities $(m^{ij1},\ldots,m^{ijk})$, $i,j=1,\ldots,k$, should $D_{(\bar{x},\ldots,\bar{x})}\tilde{F}(\bar{x},\ldots,\bar{x},\{m^{ij}\})$ be full rank (kn), then in some neighborhood of $(\bar{x},\ldots,\bar{x},\{m^{ij}\})$ the set of zeros of $\tilde{F}$ would be a function giving $(x^1,\ldots,x^k)$ in terms of the probabilities. Since for any other probabilities $\tilde{m}^{ijh}$ close enough to m^{ijh} the point $(\bar{x},\ldots,\bar{x},\{\tilde{m}^{ij}\})$ is in that graph, no other point in that neighborhood with at least two states distinct can be a zero of $\tilde{F}$, otherwise the implicit function theorem would be contradicted. Therefore, there cannot be local k-SSE around $(\bar{x},\ldots,\bar{x},\{m^{ij}\})$. Thus if this is the case for all probabilities m^{ijh}, i.e., $D_{(\bar{x},\ldots,\bar{x})}\tilde{F}(\bar{x},\ldots,\bar{x},\{m^{ij}\})$ is full rank whichever the probabilities are, the system does not have local k-SSE. Hence a necessary condition for the existence of local k-SSE around $\bar{x}$ is that for some probabilities $D_{(\bar{x},\ldots,\bar{x})}\tilde{F}(\bar{x},\ldots,\bar{x},\{m^{ij}\})$ is not full rank.

Now, this jacobian can be written as

$$\begin{pmatrix} \vdots \\ -e^i \otimes D_{-1}f(\bar{x},\bar{x}) + e^j \otimes I_n - m^{ij} \otimes D_1 f(\bar{x},\bar{x}) \\ \vdots \end{pmatrix}. \tag{6.13}$$

where $D_h f$ denotes the partial derivatives of f with respect to x_{t+h}, $h=-1$, 1. Thus the lack of full rank of the jacobian asks for the non-trivial intersection of the null spaces of the matrices

$$(-e^i \otimes D_{-1}f(\bar{x},\bar{x}) + e^j \otimes I_n - m^{ij} \otimes D_1 f(\bar{x},\bar{x})) \tag{6.14}$$

or equivalently of those (since they are the same subspaces) of these matrices premultiplied by $-D_1f(\overline{x}, \overline{x})^{-1}$ (assuming $D_1f(\overline{x}, \overline{x})$ regular)

$$(m^{ij} \otimes I_n - (e^i \otimes A_1 + e^j \otimes A_0)) \tag{6.15}$$

for all $i, j = 1, \ldots, k$, where A_1 is $-D_1f(\overline{x}, \overline{x})^{-1} D_{-1}f(\overline{x}, \overline{x})$ and A_0 is $D_1f(\overline{x}, \overline{x})^{-1}$. But we have seen in the linear case that a necessary condition for having a non-trivial intersection of the null spaces of these matrices for any probabilities is that $A_1 + A_0$ has an eigenvalue within the unit disk. Hence we have the following:

PROPOSITION 6.3 If (6.1) has local k-SSE of any order k, then $A_1 + A_0$ has an eigenvalue within the unit circle, where A_1 is $-D_1f(\overline{x}, \overline{x})^{-1} D_{-1}f(\overline{x}, \overline{x})$ and A_0 is $D_1f(\overline{x}, \overline{x})^{-1}$.

7 Conclusion

We hope that this chapter has provided insights on sunspot equilibria in models with memory that both allow to emphasize, on the one hand, the robustness of the initial message of the sunspot literature (starting from Azariadis [1981b]) on the existence of kind of equilibrium stochastic quasi cycles, on the other hand the specificity of the mechanisms at work and then of the technical apparatus that is required for the analysis of the models under scrutiny here.

One may, from this brief overview, get the impression that we are not yet close to having a general enough understanding of the problem that was here under scrutiny. Naturally, one may expect no basic difficulty in extending the analysis of sunspot equilibria to linear systems that are much more complex than those considered here: multidimensional with long memory several steps forward looking. But the existence of corresponding local sunspot equilibria in nonlinear systems raises a priori difficult technical questions.[26] Such a difficulty is a reflection of a more general fact:

our understanding of rational expectations equilibria (of sunspot or nonsunspot type) of nonlinear dynamical systems, outside the class of one-step forward-looking models, is still rather limited.

Notes

1. Trivially assume $a \neq 0$.

2. The equilibrium future exchange value of the money balances obtained today in exchange for labor depends on future production, i.e., on the amount of labor supplied tomorrow.

3. Actually it is not needed, for the argument to to through, that generation t believes indeed that generation $t+1$ believes indeed that generation $t+1$ will make its decision following the rule "x^1 if I see 1, x^2 if I see 2". It suffices that t: i) believes that $t+1$ believes that $t+2$ will behave like this and ii) assumes rational behavior at $t+l$. Of course, following the same line of argument, it is not needed that any generation believes that any other generation will behave according to such a rule, but just that every generation believes that some subsequent generation believes that some other subsequent generation further in the future will behave like this, and moreover assumes all generations in between to be rational. Although no generation has, a priori, any reason for sticking to the sunspot behavior, given these beliefs about the beliefs of others concerning the behavior of others, everybody ends up adopting such a behavior.

4. More could be said, from the present inspection on the structure of SSE for example on the dimensionality of the set of sunspot Markov matrices, (dimension 1), but this is not our present purpose.

5. We assume both $a_0 \neq 0$ and $a_1 \neq 0$.

6. With supports of higher cardinality, if the dimension if 1 there is always as many equations, k^2, as unknowns k states and $k^2 - \mathrm{k}$ probabilities. In the linear case necessarily any solution has its support on the diagonal. In the nonlinear one, a common tangent space (which will be sufficient to produce k-SSE as it will be seen in section 4) can be obtained only if the partial derivative of f with respect to the past is null at the steady state! If the dimension n is 2 or more then the nonlinear system has more equations, k^2n, than unknowns, $kn + k^2 - k$.

7. Since all of them are in an affine set not containing a null vector.

8. See lemma 3 in Dávila (1997).

9. Assumptions implicitly used are $a_0 \neq 0$, which amounts to ask for a well-defined link between expectations and the current state, and $a_0 + a_1 \neq 1$, which is the equivalent to ask for the regularity of the steady state of the perfect foresight dynamics.

10. See Prop 4.6 in Dávila (1994).

11. More precisely, such conditions will be sufficient to guarantee the existence of so-called local k-SEE, i.e. uncountably many k-SEE arbitrarily close to a steady state of the system.

12. Trivially these lotteries represent x_{t+1} with certainty whichever the probabilities α^1 and α^2 are.

13. This is the consistency of derivatives condition of Guesnerie (1986) or Chiappori and Guesnerie (1989), who argue that he property should hold in a setting with Bayesian agents.

The property is in particular satisfied by the dynamics with memory of the Woodford (86) and Reichlin (86) economies and the simple overlapping genrations economies dynamics without memory, for instance.

14. For a proof of this fact see lemma 1 in Dávila (1997). Essentially, such a perturbation can be done because the gradients in (4.9) depend injectively upon their respective probabilities.

15. Since all of them are in an affine set not containing the null vector.

16. See a counterexample in Dávila (1997), section 3.

17. Proposition 3 in Dávila (1997).

18. See lemmas 1 and 2 in Dávila (1997).

19. Proposition 5 in Dávila (1997).

20. For details see chapter 4, section 5 in Dávila (1997).

21. These roots can be either both real or both complex.

If both of them are complex, i.e., $a_0^2 + 4a_1 < 0$, then they are $\frac{1}{2}(a_0 + \sqrt{-(a_0^2 + 4a_1)}i)$ and $\frac{1}{2}(a_0 - \sqrt{-(a_0^2 + 4a_1)}i)$, and their modulus $\frac{1}{4}(a_0^2 - (a_0^2 + 4a_1)) = -a_1$. Thus indeterminacy in this case is characterized by the condition $-1 < a_1 < 1$. Actually we just need to check the inequality of $-1 < a_1$ since, from $a_0^2 + 4a_1 < 0$, necessarily $a_1 < 0$ and hence $a_1 < 1$ too. In the case where both eigenvalues are real, i.e., $a_0^2 + 4a_1 \geq 0$, then we need that both $-1 < \frac{1}{2}(a_0 + \sqrt{a_0^2 + 4a_1}) < 1$ and $-1 < \frac{1}{2}(a_0 - \sqrt{a_0^2 + 4a_1}) < 1$ be satisfied or, equivalently, $-2 - a_0 < \sqrt{a_0^2 + 4a_1} < 2 - a_0$ and $-2 + a_0 < \sqrt{a_0^2 + 4a_1} < 2 + a_0$. Note that since the square roots are positive, the second inequalities in these conditions require that both $2 - a_0$ and $2 + a_0$ be positive, i.e., that $-2 < a_0 < 2$. More over for any a_0 between -2 and 2, the expressions $-2 - a_0$ and $-2 + a_0$ are negative and thus the first inequalities in both conditions are trivially satisfied. Therefore the only additional requirement to be fulfilled is that $\sqrt{a_0^2 + 4a_1} < 2 - a_0$ $\sqrt{a_0^2 + 4a_1}$ and $< 2 + a_0$ or, equivalently, $a_1 < 1 - a_0$ and $a_1 < 1 + a_0$.

22. That is to say, we cannot exclude that we have left some dynamics with local k-SSE out of our characterization.

23. Specifically, $k > \pi/\arccos(\|A_1\| + \|A_0\|)$ when the system is of dimension 2.

24. See lemma 2 in Chiappori, Geoffard, and Guesnerie (1992).

25. Actually, some of them may be k'-SSE with $k' < k$.

26. It might, however, be the case that the combination of the insights provided here and of the technical results of Chiappori, Geoffard, and Guesnerie (1992) provides clues for future progresses.