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Nonlinear Boundary Value Problems : Existence, Estimates, Symmetry

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Introduction

Differential equations present, among different points of view, many interests. Let us mention here some of them: their *fundamental* aspect (what is more fundamental than an equation?), their *accessibility* (any student can understand very early the meaning of such a problem), the *diversity* of their study (differential and integral calculus, geometry, topology, measure theory, ...), their applications (the differential equations are often an *interface* between pure and applied mathematics), ...

A differential equation can be approached in two ways: by looking for its solutions (*Existence Results*), or by studying properties of some of its already known solutions (*Property Results*). In this thesis we present different works in these two directions. We are aware that our investigations are just a drop in the ocean of their study, but we hope (and think) that they have some interest.

A First Order Quasilinear Problem

For the “Existence Results part”, let us begin by considering the following problem studied with J. Mawhin,

$$(\phi(u))' = f(t, u), \quad u(T) = bu(0), \quad (\mathcal{P})$$

where $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $\phi : \mathbb{R} \rightarrow (-a, a)$ a homeomorphism, and a, b, T three real numbers with $a, T > 0$ and $b \neq 0$. This study was motivated by the one of C. Bereanu and J. Mawhin about the second order problem

$$(\phi(u'))' = f(t, u, u'), \quad l(u, u') = 0,$$

where $l(u, u') = 0$ stands for the Dirichlet, Neumann or periodic boundary conditions. In their work, the authors showed the difficulties to generalize the classical results for the case $\phi = \text{id}$ to the case of a homeomorphism ϕ for which $a < \infty$. They worked with *topological methods*, consisting of imbedding Problem $(\mathcal{P})$ in a continuous family of problems $(\mathcal{P}_\lambda)_{\lambda \in [0, 1]}$ ($\mathcal{P} = \mathcal{P}_1$) and of developing the two following steps:

- (a) the search of a *topologically strong* solution of $(\mathcal{P}_0)$,
- (b) the search of *a priori bounds* for solutions of $(\mathcal{P}_\lambda)_{\lambda \in [0, 1]}$.

We recall that (a) must be understood in the sense of *degree theory*, and that (b) means the research of a real number $K > 0$ such that any possible solution has its norm (here, the uniform norm $\|\cdot\|_\infty$) smaller than this number.

Let us introduce our work with the periodic case ($b = 1$) of Problem $(\mathcal{P})$ when $\phi = \text{id}$, by speaking about the following three assumptions, the first one for Step (a), and the other two for Step (b):

$$\exists R > 0, \forall u \in \mathcal{C}([0, T]) : \begin{cases} \int_0^T f(t, u(t)) dt > 0 & \text{when } u \geq R, \\ \int_0^T f(t, u(t)) dt < 0 & \text{when } u \leq -R, \end{cases} \quad (A)$$

$$\exists c \in L^1([0, T], \mathbb{R}^+), \forall (t, u) \in [0, T] \times \mathbb{R} : |f(t, u)| \leq c(t), \quad (B_1)$$

$$\exists c \in L^1([0, T], \mathbb{R}^-), \forall (t, u) \in [0, T] \times \mathbb{R} : f(t, u) \geq c(t), \quad (B_2)$$

where $u \geq R$ and $u \leq -R$ can be inverted in (A), and where $(B_1), (B_2)$ must be considered separately. This first assumption is based on the necessary condition (NC) that must verify a periodic solution u of $(\mathcal{P})$: $\int_0^T f(t, u(t)) dt = 0$, and on the simple Bolzano Theorem. Assumption (B_1) , is based on the following idea:

$$|u(t)| = \left| u(t_0) + \int_{t_0}^t f(s, u(s)) ds \right| \leq R + \int_0^T |f(s, u(s))| ds \leq R + \|c\|_{L^1},$$

where t_0 is the point in $[0, T]$ where $|u(t_0)| \leq R$ whose existence is ensured by (A) and (NC). Assumption (B_2) is similar but a little bit more subtle: since $r_1 \geq r_2$ implies $|r_1| \leq r_1 + 2|r_2|$, we deduce thanks to (NC) that

$$|u(t)| \leq R + \int_0^T |f(s, u(s))| ds \leq R + \int_0^T f(s, u(s)) ds + 2\|c\|_{L^1} = R + 2\|c\|_{L^1}.$$

In fact, J. Mawhin and J.R. Ward showed in [42] that Assumption (A) was also efficient for Step (b). Indeed, let us suppose, by contradiction, the existence, for any $K > R$, of a periodic solution u such that $\|u\|_\infty \geq K$. Without loss of generality, we shall suppose $u(0) = \sup u \geq K$. Because of the existence of t_0 , mentioned above, forbidding u to verify $u \geq K$ on all $[0, T]$, we must have for example the following kind of graph for u :

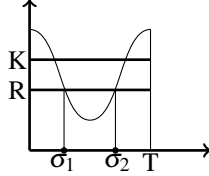


Figure 1

Assumption (A) ensures $\int_0^T f(t, \xi(t)) dt > 0$ where ξ is defined by u (resp. R) on $[0, \sigma_1]$ (resp. $[\sigma_1, T]$). This inequality leads to

$$-(u(0) - R) + \int_{\sigma_1}^T f(s, R) ds > 0 \quad (1)$$

forcing $(u(0) - R)$ to be smaller than $\sup_{\sigma_1 \in [0, T]} \int_{\sigma_1}^T f(s, R) ds =: K'$ which contradicts our assumption with $K = R + K'$. In [8, 9], C. Bereanu and J. Mawhin used the assumptions (A)-(B₂) which were adapted for a second order differential equation. Here, for the first order problem $(\mathcal{P})$, using only (A) is sufficient.

Let us speak about the observed particularities of working with ϕ bounded. We can give a bad and a good impact:

- the presence of ϕ replaces (1) by

$$-(\phi(u(0)) - \phi(R)) + \int_{\sigma_1}^T f(s, R) ds > 0$$

leading to $\phi(u(0)) < \phi(R) + K'$ which is always true if $\phi(R) + K' \geq a$. This leads to the failure of the previous method without assumption about K' and $\phi(R)$,

- the configuration (σ_2, T) in Figure 1 is inefficient for Inequality (1) at the contrary of the configuration $(0, \sigma_1)$. In the periodic case, that is not a problem because we always have the two configurations, which is not true for other boundary conditions. While it is a real problem when ϕ is unbounded, it can be bypassed when ϕ is bounded, in the case $b > 0$.

Let us also note another difficulty due to ϕ : the family of problems used for the steps (a) and (b) is the very classical one

$$(\phi(u))' = \lambda f(t, u) + (1 - \lambda) \frac{1}{T} \int_0^T f(s, u(s)) ds, \quad u(T) = bu(0). \quad (\mathcal{P}_\lambda)$$

If $b = 1$ and $\phi = \text{id}$, the solutions of problem $(\mathcal{P}_0)$ are in $\mathbb{R}$. When b, ϕ are different, the solutions of $(\mathcal{P}_0)$ are in a two-dimensional manifold, leading to some technical difficulties for Step (A).

So, despite of the problems due to the boundedness of ϕ , we succeeded to find some positive aspect of it permitting to generalize, with an additional assumption, traditional periodic case results for other boundary conditions. We illustrated our assumptions with numerical experiments.

REMARK 1. Before closing this paragraph, let us give a general remark about the study of the differential equations. Such a problem can always be denoted by $g(u) = 0$ with some operator $g : E \rightarrow F$ and some functional spaces E, F . It was the case for Problem $(\mathcal{P})$ with $g = L_\phi - N_f$ where

$$L_\phi(u) = (\phi(u))', \quad N_f(u) = f(\cdot, u),$$

$$E = \{u \in \mathcal{C}([0, T]) : \phi(u) \in \mathcal{C}^1([0, T])\}, \quad F = \mathcal{C}([0, T]).$$

Finding a zero of the operator g is in general divided in three steps:

- (a') studying the topology of g to construct a sequence $(u_n)_n$ such that $g(u_n) \rightarrow 0$,
- (b') proving, up to a subsequence, the existence of u such that $u_n \rightharpoonup u$,
- (b'') studying the compactness of g to deduce $u_n \rightarrow u$ from the two previous steps.

The step (a') is related to the previous step (a), and the step (b') to the previous step (b). With the topological methods mentioned previously, the step (b'') must be verified at first (we have not insisted on this aspect in the previous paragraphs). With other procedure like the *variational methods*, there is more freedom with it.

Sobolev Inequalities With Mean Value Zero

Again for the “Existence Results part”, we studied, with J. Van Schaftingen, the Sobolev inequalities with mean value zero condition, saying that for any $p \in [1, N)$ there exists $c > 0$ such that

$$\forall u \in \mathcal{D}_\#(\overline{\Omega}) : \left(\int_\Omega |u|^{p^*} \right)^{\frac{1}{p^*}} \leq c \left(\int_\Omega |\nabla u|^p \right)^{\frac{1}{p}},$$

where $\Omega \subset \mathbb{R}^N$ is a bounded *domain* (open and connected set) of class $\mathcal{C}^1$, $\mathcal{D}_\#(\overline{\Omega})$ is the set of the restriction on Ω of smooth compactly supported functions $u : \mathbb{R}^N \rightarrow \mathbb{R}$ with mean value $\bar{u} := \int_\Omega u / |\Omega|$ equal to 0, and $p^* := Np/(N - p)$. The problem is to know whether there exists a function u achieving the optimal constant c . For technical reason, we shall

consider the power $-p$ of this constant:

$$\mathcal{S}_p(\Omega) := \inf_{\substack{u \in \mathcal{D}_\#(\bar{\Omega}) \\ u \neq 0}} \frac{\|\nabla u\|_{L^p}^p}{\|u\|_{L^{p^*}}^p} = \inf_{\substack{u \in \mathcal{D}_\#(\bar{\Omega}) \\ \int_\Omega |u|^{p^*} = 1}} \int_\Omega |\nabla u|^p. \quad (2)$$

Let us recall the famous works of T. Aubin and G. Talenti [5, 52], where it is proved that when $\mathcal{D}_\#(\bar{\Omega})$ is replaced by the spaces

$$\mathcal{D}(\mathbb{R}^N) := \{u : \mathbb{R}^N \rightarrow \mathbb{R} : u \text{ is smooth and } \text{Supp } u \text{ is compact}\},$$

$$\mathcal{D}(\overline{\mathbb{R}_+^N}) := \{u|_{\mathbb{R}_+^N} : u \in \mathcal{D}(\mathbb{R}^N)\},$$

the optimal constant, denoted by $S_p(\mathbb{R}^N)$ and $S_p(\overline{\mathbb{R}_+^N})$, is achieved by the functions $aU(bx)$ for any $a, b \in \mathbb{R}$, where U is the so-called *instanton*:

$$U(x) = \frac{1}{(1 + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}} \text{ if } p > 1, \quad U(x) = 1_{B^N}(x) \text{ if } p = 1, \quad (3)$$

with B^N denoted for the unit ball in $\mathbb{R}^N$. This permits us to precise, what is clear in view of the expression of U , that the existence is searched in greater spaces than $\mathcal{D}_\#(\bar{\Omega})$, $\mathcal{D}(\mathbb{R}^N)$, $\mathcal{D}(\overline{\mathbb{R}_+^N})$. For example, when $p = 1$ and $N > 1$, the relevant spaces are respectively

$$\text{BV}_\#(\Omega) = \{u \in L^{1^*}(\Omega) : \partial_i u \in \mathcal{C}'_0(\Omega), \bar{u} = 0\},$$

$$\text{BV}(\mathbb{R}^N) = \{u \in L^{1^*}(\mathbb{R}^N) : \partial_i u \in \mathcal{C}'_0(\mathbb{R}^N)\},$$

$$\text{BV}(\mathbb{R}_+^N) = \{u \in L^{1^*}(\mathbb{R}_+^N) : \partial_i u \in \mathcal{C}'_0(\mathbb{R}_+^N)\},$$

where $\mathcal{C}'_0(\Omega)$ is the set of the continuous functions $v : \mathbb{R}^N \rightarrow \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a compact set K in Ω verifying $\sup_{\Omega \setminus K} |v| \leq \varepsilon$, and $\mathcal{C}'_0(\Omega)$ its dual space with respect to the uniform norm.

This problem with $p = 1$, was motivated by the paper [16] of H. Brezis and J. Van Schaftingen, in which it was presented among many other open questions about optimal constants. On an other hand, the case $p = 2$ had already been treated by P. Girão and T. Weth in [30], the cases $p \in [1, \frac{N+1}{2}]$ with $\Omega = B^N$, by M. Leckband in [33], and the cases $p \in (1, \frac{N+1}{2}]$ with any smooth Ω , by A.V. Demyanov and A.L. Nazarov in [26]. Our result generalize these three works respectively for other values of p than 2, for other Ω than the unit ball B^N , and for the delicate case of $p = 1$ needing the BV spaces. Let us still note that some of our results in BV can be found in [25] without any proof.

We shall describe our investigations in the case $p = 1$. If we replace $\mathcal{D}_\#(\bar{\Omega})$ by the space $\text{BV}_\#(\Omega)$ in (2), we can formulate our problem as $\inf_{I=1} J$ where

$$I, J : \text{BV}_\#(\Omega) \rightarrow \mathbb{R}, \quad J(u) = \int_\Omega |Du|, \quad I(u) = \int_\Omega |u|^{1^*},$$

Du corresponding to ∇u in $\text{BV}_\#(\Omega)$. The difficulty in solving this minimization problem with constraint concerns the compactness. Indeed, if $u_n \rightharpoonup u$ in $\text{BV}_\#(\Omega)$, as it is enough to have the property (4)(a) about J , we must have the property (4)(b) about I , which is not verified in general,

$$(a) J(u) \leq \liminf J(u_n), \quad (b) \lim I(u_n) = I(u). \quad (4)$$

Hence, the purpose is to know if (4)(b) is true when $(u_n)_n$ is a minimizing sequence, so to know if I has a partial compactness, related to some level depending of I and J . Let us note that, even this problem is not a differential equation, it has many common points with their

study: showing the partial compactness of the “structure” (I, J) is related to the step (b”) mentioned in Remark 1.

To study this partial compactness, we adapted the concentration-compactness lemma of P.L. Lions [35, 36] to the mean value zero functions on Ω . This result studies the measures $|Du_n|$ and the functions $|u_n|^{1^*}$ as measures of the space $\mathcal{C}'(\overline{\Omega})$, i.e., as continuous linear functionals of the space $\mathcal{C}(\overline{\Omega})$ equipped of the uniform norm. Two simple observations are the following:

- the natural norm of this dual space applied to the measures $|Du_n|, |u_n|^{1^*}$ coincides respectively with $J(u_n), I(u_n)$,
- on a bounded domain Ω , if $|Du_n| \rightharpoonup \mu, |u_n|^{1^*} \rightharpoonup \nu$, this norm verifies the property (4)(b):

$$\lim \| |Du_n| \| = \| \mu \|, \quad \lim \| |u_n|^{1^*} \| = \| \nu \|.$$

So, $\mathcal{C}'(\overline{\Omega})$ seems indicated to study the eventual difference between $I(u)$ and $\lim I(u_n)$. The two main results ensured by the concentration-compactness lemma are the following:

- if $u_n \rightharpoonup u$ in $BV_{\#}(\Omega)$ and $|u_n|^{1^*} \rightharpoonup \nu, |Du_n| \rightharpoonup \mu$ in $\mathcal{C}'(\overline{\Omega})$, then there exists $(x_j)_{j \in Q} \subset \overline{\Omega}, (v_j) \subset (0, \infty)$ with Q at most countable, such that $\nu = u + \sum_j v_j \delta_{x_j}$,
- moreover, there exists $(\mu_j)_{j \in Q} \subset (0, \infty)$ such that $\mu \geq |Du| + \sum_j \mu_j \delta_{x_j}$ with $v_j^{1/1^*} \leq c_j \mu_j$ and c_j equal to

$$S_1(\mathbb{R}^N)^{-1} \text{ if } x_j \in \Omega, \quad S_1(\overline{\mathbb{R}_+^N})^{-1} \text{ if } x_j \in \partial\Omega,$$

where δ_x , called *Dirac delta*, is defined by $\langle \delta_x | v \rangle = v(x)$, $\langle \cdot | \cdot \rangle$ denoting the dual product. A classical procedure using the homogeneity and the strict convexity of the “structure” (I, J) ensures the expected compactness, precisely $v_j = 0$ for any j , if the following condition holds:

$$\mathcal{S}_1(\Omega) < S_1(\overline{\mathbb{R}_+^N}). \quad (5)$$

To study this inequality, we considered the restriction on Ω of the instanton (3), perturbed in order to have a mean value zero, and concentrated it at a point $x_0 \in \partial\Omega$ where the mean curvature K was positive, point that we could suppose, up to a translation, equal to 0:

$$u_\varepsilon(x) = \varepsilon^{1-N} \left(1_{B_\varepsilon^N}(x) - \frac{|B_\varepsilon^N \cap \Omega|}{|\Omega|} \right),$$

with B_ε^N denoted for the ball $B(0, \varepsilon)$ in $\mathbb{R}^N$. We showed

$$\mathcal{S}_1(\Omega) \leq \frac{\|Du_\varepsilon\|_{L^1}}{\|u_\varepsilon\|_{L^{1^*}}} \leq S_1(\overline{\mathbb{R}_+^N}) \cdot \frac{1 - 2\varepsilon K \cdot \frac{\mu_{N-2}}{\mu_{N-1}} + o(\varepsilon)}{1 - 2\varepsilon K \cdot \frac{\mu_{N-2}}{\mu_{N-1}} \cdot \frac{N-1}{N+1} + o(\varepsilon)},$$

where μ_{N-1} is the $(N-1)$ -Hausdorff measure of ∂B^N and $\mu_0 = 2$. That proves (5) for ε small enough. These results remain true for any p in $[1, \frac{N+1}{2}]$.

The Lane–Emden Problem

To close this introduction about the existence results, let us give some words about one of the most used paradigm in the study of elliptic Differential Equations: the *Lane–Emden problem*. In this thesis, we studied it with Dirichlet and Neumann boundary conditions, and also a variant of it:

$$-\Delta u = |u|^{p-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (\text{LE}_1)$$

$$-\Delta u + u = |u|^{p-2}u \text{ in } \Omega, \quad \partial_\nu u = 0 \text{ on } \partial\Omega, \quad (\text{LE}_2)$$

$$-\Delta u - \lambda u = \varepsilon(|u|^{p-2}u - u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (\text{LE}_3)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p > 2, \varepsilon > 0$ and λ an eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

Many existence results about Problem (LE_1) and (LE_2) are already known. Indeed, the *variational methods* permit to prove the existence of *least energy* (or *ground states*) and *least energy nodal* solutions, which are solutions u verifying (6) respectively for any signed functions v or for any sign-changing functions v , and where J_p is defined by (7) with $\mu = 0$ (resp. $\mu = 1$) for the Dirichlet (resp. Neumann) case:

$$J_p(u) \leq J_p(v), \quad (6)$$

$$J_p(u) = \frac{1}{2} \int_{\Omega} [|\nabla u|^2 + \mu u^2] - \frac{1}{p} \int_{\Omega} |u|^p. \quad (7)$$

Despite of the mentioned knowledge of the previous problems, the structure of the set

$$\mathcal{S} := \{(p, u) \in [2, \infty) \times H^1(\Omega) : u \text{ solves } (\text{LE}_2) \text{ at } p\}$$

is not completely known. In a conference in Brussels in honor of P. Habets and J. Mawhin, after have recalled the inclusion of $\{u = 1\}$ in $\mathcal{S}$, D. Bonheure proposed, as a conjecture, the existence of bifurcation branches of solutions starting from each point $(p, u) = (\lambda_i + 1, 1)$ as illustrated in Figure 2.

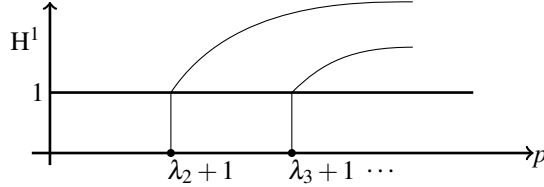


Figure 2

We proved this conjecture with the *Morse index* when $p < p^*$, with the possibility that the branch are pointwise. For Problem (LE_3) , we used the *Lyapunov-Schmidt reduction* and the *Brouwer degree* to prove an existence and uniqueness result whose interest will be explained later in this introduction. At last, our investigations about Problem (LE_1) were just devoted to introduce our technique for Problem (LE_3) .

Symmetries and Symmetry Breaking

Let us now speak about the “Property Results part”. In this way, we studied in [11, 10] with D. Bonheure, C. Grumiau and J. Van Schaftingen, the symmetry properties of some solutions of Problem

$$-\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (\mathcal{Q})$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain admitting some symmetry hyperplanes $T = \partial H$ where $H := \{x : h \cdot x \geq c\}$, $h \in \mathbb{R}^N, c \in \mathbb{R}$, and f is of class $\mathcal{C}^1$. The purpose is to know if a solution u is symmetric with respect to T . The antisymmetry can also be considered if f is antisymmetric. In this study, we shall suppose

$$\exists c > 0, p \in [2, 2^*), \forall t \in \mathbb{R} : |f(t)| \leq c(|t|^{p-1} + 1),$$

ensuring the variational structure of $(\mathcal{Q})$, permitting to consider as previously the functional

$$J : H_0^1(\Omega) \rightarrow \mathbb{R} : u \mapsto \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} F(u),$$

with $F(t) := \int_0^t f(s) ds$, and the notions of least energy and least energy nodal solutions.

The litterature on the study of the symmetries of solutions u is very rich. The most famous result is the one of B. Gidas, W.M. Ni and L. Nirenberg in [29]:

Let us suppose that Ω is convex and $u > 0$ is a solution of $(\mathcal{Q})$. If $T = \partial H$ as previously is a hyperplane of symmetry of Ω , u is symmetric with respect to T and $\partial_h u < 0$ in $\Omega^+ := \Omega \cap H$.

In particular, when Ω is a ball centered at 0, u is *Schwarz symmetric*, i.e., can be written as $\tilde{u}(r)$ with $r = |x|$ and $\tilde{u}$ non-increasing. When the domain is not convex, when the equation contains some singular “weights”, or when u is *nodal*, i.e., sign changing, some symmetry breaking can appear. The first two cases have been respectively proved by H. Brezis and L. Nirenberg in [15] and M. Willem and D. Smets in [50]. These proofs consist in giving an example where solutions are not symmetric. The last result has been obtained by A. Aftalion and F. Pacella in [2]:

When Ω is radial and u is nodal, under the assumption $f(t)t > 0$ (S_1), if the Morse index $\text{ind}_M(u, J)$ (i.e. the number of decreasing directions of J around u) is not bigger than N , then u is non-radial.

All these symmetry breaking results occur in a radial domain. In the sequel, we shall always suppose this radial assumption on Ω .

Even if there is symmetry breaking, it can be observed that solutions u preserve some symmetries: they are *foliated Schwarz symmetric*, i.e., can be written as $\tilde{u}(r, \theta_P)$ for some vector P of norm 1, where $r = |x|$, $\theta_P = |\arccos(\frac{x}{|x|}P)|$ and $\tilde{u}(r, \cdot)$ non-increasing. Let us mention the result of F. Pacella in [45]:

In a radial domain, under the assumption f convex (S_2), if the Morse index $\text{ind}_M(u, J)$ equals 1, then u is at least foliated Schwarz symmetric.

the one of T. Bartsch, T. Weth and M. Willem in [7]:

In a radial domain, under the assumption f non-decreasing (S_3), any least energy and least energy nodal solution u is at least foliated Schwarz symmetric.

and the one of F. Pacella and T. Weth in [46]:

In a radial domain, under the assumption f' convex (S_4), if the Morse index $\text{ind}_M(u, J)$ is not bigger than N , then u is foliated Schwarz symmetric.

The non-Radial Case In this thesis, we studied, with D. Bonheure, C. Grumiau and J. Van Schaftingen, to what extent it was possible to generalize the radial assumption of the domain in the nodal case to preserve foliated Schwarz symmetry. We worked firstly with the Lane–Emden problem (LE_1) previously mentioned, and secondly with other problems like

$$\begin{aligned} -\Delta u &= \lambda_2 u^3 + u^5 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \\ -\Delta u &= \lambda_2 u(e^{u^2} - 1)^{0.5} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \\ -\Delta u &= \lambda_2 u \left(\frac{1}{4}|u|^{0.25} + \frac{1}{4}|u|^{0.5} + \frac{1}{2}|u| \right) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{aligned}$$

where Ω is a rectangle in $\mathbb{R}^2$, and λ_2 the second eigenvalue of $-\Delta$.

We noted the difficulty of this purpose by bringing out examples of problems on non-radial domain Ω for which the foliated Schwarz symmetry is preserved for some nodal solutions, as well as examples of problems on such domains where some foliated Schwarz symmetry breaking appears.

The Pacella-Weth Result We studied also, with J. Van Schaftingen, the question of the optimality of the bound N in the result of F. Pacella and T. Weth mentioned earlier. Inspired by the previous work, we considered Problem (LE_3) with $\lambda = \lambda_3$ and $p \geq 3$, and attempted to prove the existence of solutions u having no foliated Schwarz symmetry and verifying

$$\text{ind}_M(u, J) = N + 2.$$

Currently, we do not know if such an example verifying $\text{ind}_M(u, J) = N + 1$ can be constructed. Let us give the abstract form of the deformation $(\mathcal{P}_\lambda)$ considered previously:

$$G(\lambda, \cdot) := L_\phi - [\lambda N_f(u) + (1 - \lambda)QN_f], \quad Q(u) = \bar{u}.$$

Inspired by it, we used the similar homotopy (8) where L, N_p, Q are defined by (9), in order to compute $\text{ind}_M(u, J)$:

$$G(\varepsilon, u) = L - [\varepsilon(\text{id} - Q)N_p(u) + QN_p(u)], \quad (8)$$

$$L(u) = -\Delta u - \lambda_3 u, \quad N_p(u) = (|u|^{p-2}u - u), \quad Q(g) = g|_{E(\lambda_3)}, \quad (9)$$

with $E(\lambda_3)$ the eigenspace associated to λ_3 , and $g \in (H_0^1)'$. For Problem $(\mathcal{P}_\lambda)$ we used the conservation of the Leray-Schauder degree of $G(\lambda, \cdot)$. Here, we used the conservation of the Morse index applied to the (primitive of) $G(\varepsilon, \cdot)$. Let us still precise that we worked with $p \geq 3$ to verify (S_4) and $\lambda = \lambda_3$ because it is the first eigenvalue of $-\Delta$ admitting eigenfunctions without the foliated Schwarz symmetry.

The CKN Inequalities At last, we studied in [14] with M. Willem, the symmetries of the functions giving the optimal constant of the Caffarelli-Kohn-Nirenberg inequalities

$$\exists c > 0, \forall u \in \mathcal{D}(\mathbb{R}^N) : \left(\int_{\mathbb{R}^N} |x|^\alpha |u|^p \right)^{\frac{1}{p}} \leq c \left(\int_{\mathbb{R}^N} |x|^\beta |\nabla u|^2 \right)^{\frac{1}{2}}, \quad (\text{CKN})$$

where

$$\begin{aligned} N \geq 3, \quad \frac{2-N}{2} < \beta < \infty, \quad \beta - 1 \leq \alpha \leq \beta, \quad p &= \frac{2N}{N-2+2(\beta-\alpha)}, \\ N = 2, \quad 0 < \beta < \infty, \quad \beta - 1 \leq \alpha < \beta, \quad p &= \frac{2}{\beta-\alpha}, \\ N = 1, \quad \frac{1}{2} < \beta < \infty, \quad \beta - 1 \leq \alpha < \beta - \frac{1}{2}, \quad p &= \frac{2}{-1+2(\beta-\alpha)}. \end{aligned}$$

In the sequel, we shall refer to these functions as the CKN-optimal functions. As for the previously mentioned Sobolev inequalities, these solutions are in a greater space than $\mathcal{D}(\mathbb{R}^N)$.

It has been proved in [21] by F. Catrina and Z.Q. Wang that even if nonnegative, a CKN-optimal function can be non-radial. It is due to the singular “weight” $|x|^\alpha, |x|^\beta$ as mentioned previously. Nevertheless, C.S. Lin and Z.Q. Wang proved in [34] that these functions are foliated Schwarz symmetric. They used the technique of the previously mentioned result of F. Pacella. We proved here that the technique of T. Bartsch, T. Weth and M. Willem was also efficient.

Even if the Pacella-result (working with the local condition $\text{ind}_M(u, J) = 1$) is in some sense more general than the Bartsch-Weth-Willem-result (working with the global least energy condition about u), its proof is less simple. This remark joined to the fact that the CKN-optimal functions can be characterized, by the condition $\text{ind}_M(u, J) = 1$, as well as by the least energy condition, gives the interest of our demarch.

REMARK 2. In the previous paragraph, we compared the generality of the Pacella-result with the one of the Bartsch-Weth-Willem-result. Such a discussion is to be taken only “in some sense”. Indeed, the local assumption about $\text{ind}_M(u, J)$ seems more general than the global least energy assumption, but requires the non-degeneracy of u , what is not the case of the second assumption. Moreover, such a discussion must also compare the assumptions about f . For example, Assumption (S_2) can only be applied for the Lane–Emden problem (LE_1) for positive solutions, contrary to Assumption (S_3) . Another example, (S_2) is in contradiction with (S_1) for nodal solutions.

The Radial Assumption in the Nodal Case

Let us finish this introduction by giving some details about the results of [11] proved in collaboration with D. Bonheure, C. Grumiau and J. Van Schaftingen, and improved in [10] with D. Bonheure and C. Grumiau.

In these works, we considered a family of problems $(\mathcal{Q}_\theta)_{\theta>2}$ associated to a family of functions $(f_\theta)_{\theta>2}$ perturbations of $f_2(u) := \lambda u$ where λ is an eigenvalue of $-\Delta$,

$$-\Delta u = f_\theta(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \quad (\mathcal{Q}_\theta)$$

We studied the family of solutions $(u_\theta)_{\theta>2}$ of these problems, converging to some function u_* which, by clarifying what means “perturbation” mentioned above, are proved to be solution of $(\mathcal{Q}_2)$:

$$(\mathcal{Q}_\theta)_{\theta>2}, \quad (u_\theta)_{\theta>2} \text{ converging to } u_*, \quad E(\lambda) \equiv (\mathcal{Q}_2). \quad (10)$$

Our idea was to use the symmetries of the solutions of $(\mathcal{Q}_2)$ to study the one of the solutions of $(\mathcal{Q}_\theta)$, and is based on the following observation to which we shall refer later by $(\lambda_1\text{-}\lambda_2)$:

On a ball, the solutions of $(\mathcal{Q}_2)$ with $\lambda = \lambda_1$ are signed and Schwarz symmetric like in the Gidas-Ni-Nirenberg-result, and the solutions of $(\mathcal{Q}_2)$ with $\lambda = \lambda_2$ are nodal and foliated Schwarz symmetric like in the Bartsch-Weth-Willem and Pacella-Weth results.

On a rectangle in $\mathbb{R}^2$ where T_1, T_2 are the medians, the solutions of $(\mathcal{Q}_2)$ with $\lambda = \lambda_1$ are signed and symmetric with respect to T_1, T_2 like in the Gidas-Ni-Nirenberg-result, and the solutions of $(\mathcal{Q}_2)$ with $\lambda = \lambda_2$ are nodal and symmetric and antisymmetric with respect to T_1, T_2 as we want to prove.

Because of that, it is natural to work with $\lambda = \lambda_2$. We shall do this in the sequel. Let us denote by $\bar{u}, \tilde{u}$ the orthogonal projections of a function u on $E(\lambda), E(\lambda)^\perp$.

Under some assumptions about $(f_\theta)_{\theta>2}$, we proved the following uniqueness result

$$\begin{aligned} &\text{Let } (u_\theta)_{\theta>2}, (v_\theta)_{\theta>2}, u_* \neq 0 \text{ like in (10),} \\ &(\bar{u}_\theta = \bar{v}_\theta) \Rightarrow (\exists \theta_0 > 2 : \forall \theta \in (2, \theta_0) : u_\theta = v_\theta), \end{aligned} \quad (11)$$

leading to the symmetry result

$$\begin{aligned} &\text{Let } (u_\theta)_{\theta>2}, u_* \text{ like in (10) and } (T_\theta)_{\theta>2} \in \mathcal{S}_\theta(H_0^1), \\ &(T_\theta(\bar{u}_\theta) = \bar{u}_\theta) \Rightarrow (\forall \theta \in (2, \theta_0) : T_\theta(u_\theta) = u_\theta) \end{aligned} \quad (12)$$

where we denoted by $\mathcal{S}_\theta(H_0^1)$ the set of any linear continuous bijection $T_\theta : H_0^1 \rightarrow H_0^1$ verifying

$$T_\theta(E(\lambda)) = E(\lambda), \quad T_\theta(E(\lambda)^\perp) = E(\lambda)^\perp, \quad J_\theta \circ T_\theta = J_\theta.$$

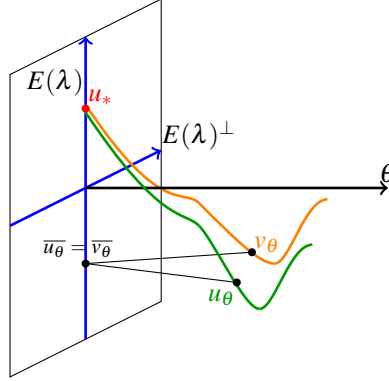


Figure 3

Indeed, Figure 3 illustrates the negation of (12) and is clearly in contradiction with (11). The idea is based on the fact that $v_\theta := T_\theta(u_\theta)$ has the same projection $\bar{v}_\theta$ as u_θ and verifies also $(\mathcal{Q}_\theta)$, what is ensured by the definition of $\mathcal{S}_\theta$. The considered operators T_θ must be seen as in Figure 4, and the condition $T_\theta(\bar{u}_\theta) = \bar{u}_\theta$ as encoding some symmetries of $\bar{u}_\theta$.

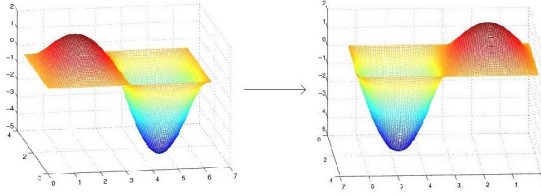


Figure 4

In regard of the observation $(\lambda_1 - \lambda_2)$, this symmetry result ensures the expected symmetry result on a rectangle for nodal solutions u_θ converging to u_* like in (10) with $u_* \neq 0$.

After these results, our aim was to find a continuum of nodal solutions $(u_\theta)_{\theta > 2}$ like in (10) having non-zero limit u_* . We succeeded with the least energy nodal solutions.

We do not want to give technical computations here, but just an idea of the asymptotic study that we made to know, once the previous statement obtained, which kind of eigenfunction is u_* . This seems important, for example on a square in $\mathbb{R}^2$, since in such a domain, the functions of $E(\lambda)$ are sometimes symmetric with respect to the medians, sometimes to the diagonals, and sometimes non-symmetric. Before, let us recall the definition of the Nehari manifold (13), (14), and the characterization of the least energy nodal solutions (15),

$$\mathcal{N}_\theta := \{u \neq 0 : I_\theta(u) = 0\}, \quad (13)$$

$$I_\theta(u) := \langle J'_\theta(u) | u \rangle = \int_\Omega |\nabla u|^2 - \int_\Omega f_\theta(u)u, \quad (14)$$

$$(v^\pm \in \mathcal{N}_\theta) \Rightarrow (J_\theta(u_\theta) \leq J_\theta(v)), \quad (15)$$

where $v^+ := \max\{v, 0\}$, $v^- := \min\{v, 0\}$. Let us also notice that

$$v^\pm \in \mathcal{N}_\theta \Rightarrow v \in \mathcal{N}_\theta.$$

We shall consider $u_*, v_* \in E(\lambda)$, a continuum $(u_\theta)_{\theta > 2}$ of least energy nodal solutions verifying (16), a continuum $(v_\theta)_{\theta > 2}$ verifying (17) such that $v_\theta^\pm \in \mathcal{N}_\theta$, and the assumption

(18) about the functions f_θ :

$$u_\theta = u_* + (\theta - 2)w + o(\theta - 2), \quad (16)$$

$$v_\theta = v_* + (\theta - 2)z + o(\theta - 2), \quad (17)$$

$$f_\theta = f_2 + (\theta - 2)f_* + o(\theta - 2). \quad (18)$$

The term $o(\theta - 2)$ is understood in the H_0^1 sense in the two first cases, and in the pointwise sense in the last one, with some controls that we do not want to clarify here. Later, we shall denote by g_θ the difference

$$f_\theta - [f_2 + f_*(\theta - 2)].$$

A heuristic first order development of I_θ with respect to $(\theta - 2)$ joined to the assumptions of v_θ and v_* ensures

$$\begin{aligned} I_\theta(v_\theta) &= \left(\int_\Omega |\nabla v_*|^2 - \lambda \int_\Omega v_*^2 \right) + \left(- \int_\Omega f_*(v_*)v_* + \langle I_2'(v_*)|z \rangle \right) (\theta - 2) + o(\theta - 2), \\ &= \left(- \int_\Omega f_*(v_*)v_* \right) (\theta - 2) + o(\theta - 2), \end{aligned}$$

and the same for J_θ joined to the assumptions of u_θ and u_* ensures

$$\begin{aligned} J_\theta(u_\theta) &= \left(\frac{1}{2} \int_\Omega |\nabla u_*|^2 - \frac{\lambda}{2} \int_\Omega u_*^2 \right) + \left(- \int_\Omega F_*(u_*) + \langle J_2'(u_*)|w \rangle \right) (\theta - 2) + o(\theta - 2), \\ &= \left(- \int_\Omega F_*(u_*) \right) (\theta - 2) + o(\theta - 2). \end{aligned}$$

Because of the characterization (15), we have $J_\theta(u_\theta) \leq J_\theta(v_\theta)$ which, joined to the previous computations, leads to

$$J_*(u_*) \leq J_*(v_*), \quad v_* \in \mathcal{N}_*,$$

where

$$J_* : E(\lambda) \rightarrow \mathbb{R} : v \mapsto - \int_\Omega F_*(v),$$

$$\mathcal{N}_* := \{v \neq 0 : \langle J_*'(v)|v \rangle = 0\} = \{v \neq 0 : \int_\Omega f_*(v)v = 0\}.$$

Hence the accumulation points of the continuum of least energy nodal solutions $(u_\theta)_{\theta > 2}$ seem to be the functions of $E(\lambda)$ minimizing the functional J_* on the manifold $\mathcal{N}_*$.

Let us consider a square Ω of $\mathbb{R}^2$ and the Lane–Emden problem or the other examples mentioned previously. The previous characterization led us to prove that the functions u_* limit of a continuum of least energy nodal solutions, are the second eigenfunctions symmetric with respect to one diagonal, and antisymmetric with respect to the other one.

Let us note that the main assumption that we used above is the existence of the continuum $(v_\theta)_{\theta > 2}$ converging to v_* . We may only consider that for a minimizer v_* of J_* on $\mathcal{N}_*$. This fact is ensured by $g'_\theta = o(\theta - 2)$ joined to

$$\int_\Omega f'_*(v_*)v_*^2 \neq 0.$$

Indeed, the last inequality ensures that v_* is a non-degenerate minimizer of J_* on $\mathcal{N}_*$, what is a traditional statement to prove that it generates a continuum of solutions, or here, a continuum of functions whose positive and negative parts are in $\mathcal{N}_\theta$.

Joined to the previous uniqueness result, this does not prove the symmetry of u_θ , but strangely, a symmetry breaking result. Indeed, replacing (10) by

$$(\Omega)_{\theta>2}, \quad (\mathcal{Q}_\theta)_{\theta>2}, \quad (u_\theta)_{\theta>2} \text{ converging to } u_*, \quad E(\lambda) \equiv (\mathcal{Q}_2),$$

where Ω_θ is a rectangle in $\mathbb{R}^2$ and Ω_2 a square, we proved, with some constraint on the convergence $\Omega_\theta \rightarrow \Omega_2$, that the least energy nodal solutions u_θ were neither symmetric nor antisymmetric with respect to the medians of Ω_θ , for θ close enough to 2. This fact is illustrated by Figure 5.

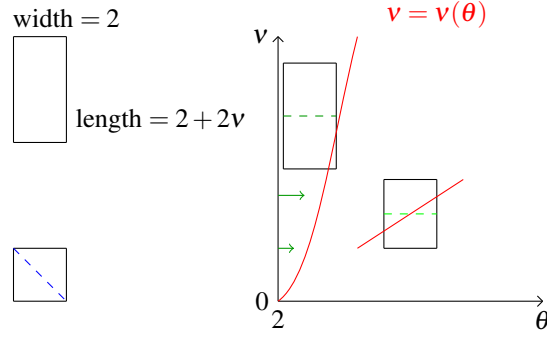


Figure 5

Planning

Chapter 1 is devoted to the main tools which are essential for the sequel (existence and symmetry results, weak and strong differential calculus on $\mathbb{R}^N$ and on smooth manifolds). In Chapter 2, we present our results about the first order quasilinear problems mentioned at the beginning of this introduction. We continue in Chapter 3 by studying the Sobolev and Caffarelli-Kohn-Nirenberg inequalities. Chapter 4 concerns our work about the radial assumption in the nodal case of the Lane-Emden problem with Dirichlet boundary conditions. At last, in Chapter 5, we improve some results of the previous chapter, in particular we give existence results about the Neumann's version of the Lane-Emden problem, a study of the Pacella-Weth result, a generalization of Chapter 4 to other non-linearities than the one of the Lane-Emden problem.

Appendix A completes the existence tools introduced in Chapter 1. In Appendices B and D, we give some details about the numerics of Chapters 2 et 4. Some technical computations associated to Chapter 3 are presented in Appendix C, and Appendix E is devoted to the Beta and Gamma Euler functions, and Bessel functions which are used in Chapters 3 and 5.

The Main Tools

Differential Equations, like any subject of science, require some important tools that we propose to recall here. Our goal is not to give a very detailed report of these ones (any introduction book of nonlinear analysis would be a better reading for it) but a brief survey which will permit this thesis to be self-contained. For more details, the reader can consult Appendix A or, among other references, [3, 17, 27, 47, 55, 43].

1 Solutions of Equations and Associated Spaces

In the next chapters, we shall study ordinary differential equations and partial differential equations with boundary conditions. These *boundary value problems* will have one of the following form:

$$(\phi(u))' = f(t, u) \text{ in } (0, T), \quad u(T) = bu(0), \quad (\mathcal{P})$$

$$(-\Delta + \mu \text{id}) u = f(x, u) \text{ in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega, \quad (\mathcal{Q})$$

where

- T is positive and $\Omega \subseteq \mathbb{R}^N$ a domain, i.e., an open connected set,
- $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ (resp. $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$) is continuous,
- $\phi : \mathbb{R} \rightarrow I$ is a homeomorphism from $\mathbb{R}$ to an open interval I ,
- $Bu = u$ -Dirichlet conditions- or $\partial_\nu u$ -Neumann conditions- the outward normal derivative of u on $\partial\Omega$, and $\mu \in \mathbb{R}, b \in \mathbb{R}_0$.

We shall always work with classical solutions for Problem $(\mathcal{P})$ and classical as well as weak solutions for Problem $(\mathcal{Q})$,

- functions $u \in \mathcal{C}([0, T])$, such that $\phi(u) \in \mathcal{C}^1(0, T)$, verifying equations of $(\mathcal{P})$,
- functions $u \in \mathcal{C}(\bar{\Omega}) \cap \mathcal{C}^2(\Omega)$ or $\mathcal{C}^1(\bar{\Omega}) \cap \mathcal{C}^2(\Omega)$ according to whether $Bu = u$ or $\partial_\nu u$, verifying equations of $(\mathcal{Q})$,
- functions $u \in E$, where E is $H_0^1(\Omega)$ or $H^1(\Omega)$ according to whether $Bu = u$ or $\partial_\nu u$, such that

$$\forall v \in E : \int_{\Omega} [\nabla u \cdot \nabla v + \mu uv] = \int_{\Omega} f(x, u)v. \quad (19)$$

The weak formulation (19) of $(\mathcal{Q})$ requires some conditions on f that we shall recall later.

The concept of weak solution is in relationship with the concept of distributional solution. Let us recall the notion of distribution and some other related notions. The functional spaces

$$\mathcal{D}(\Omega) \subset \mathcal{K}(\Omega) \subset \mathcal{C}_0(\Omega)$$

contain respectively $\mathcal{C}^\infty$ -functions with compact support in Ω , continuous functions with compact support in Ω , and bounded continuous functions v such that for all $\varepsilon > 0$, there exists a compact $K \subset \Omega$ such that $\sup_{\Omega \setminus K} |v| < \varepsilon$. They are generally equipped with the

following convergence systems: $v_n \rightarrow v$ in $\mathcal{D}(\Omega)$ if all the supports of the functions v_n are in a same compact of Ω and

$$\forall \alpha \in \mathbb{N}^N : \lim_{n \rightarrow \infty} \|\partial^\alpha (v_n - v)\|_\infty = 0;$$

$v_n \rightarrow v$ in $\mathcal{K}(\Omega)$ if all the supports of functions v_n are in a same compact of Ω and $\lim_{n \rightarrow \infty} \|v_n - v\|_\infty = 0$; $v_n \rightarrow v$ in $\mathcal{C}_0(\Omega)$ only if $\lim_{n \rightarrow \infty} \|v_n - v\|_\infty = 0$. We can consider the spaces

$$\mathcal{D}'(\Omega) \supset \mathcal{K}'(\Omega) \supset \mathcal{C}'_0(\Omega)$$

of linear functionals u defined on the previous spaces which are continuous in the sense that if $v_n \rightarrow v$ then $\langle u|v_n \rangle \rightarrow \langle u|v \rangle$, where $\langle \cdot | \cdot \rangle$ is the duality product. These functionals are respectively called *distributions*, *Radon measures*, *finite Borel measures*. If $u \in \mathcal{D}'(\Omega)$ then the *distributional partial derivative* $\partial^\alpha u \in \mathcal{D}'(\Omega)$ with $\alpha \in \mathbb{N}^N$ is defined by the equality

$$\langle \partial^\alpha u | v \rangle = (-1)^{|\alpha|} \langle u | \partial^\alpha v \rangle, \quad (20)$$

where $|\alpha| = \sum_i \alpha_i$. Let us give the following classical example of a finite Borel measure: the *Dirac delta* δ_x where $x \in \Omega$, defined by $\langle \delta_x | v \rangle := v(x)$.

REMARK 1.1. The concept of distribution generalizes the one of function in the following sense: the functions $u \in L^1_{\text{loc}}(\Omega)$ determine a unique distribution by the equality $\langle u, v \rangle := \int_\Omega uv$. The concept of distributional derivative generalizes the one of classical derivative because the functions $u \in \mathcal{C}^{|\alpha|}(\Omega)$ have their classical derivative $\partial^\alpha u$ determining the same distribution as their distributional derivative. Radon and finite Borel measures are particular distributions.

The weak solutions mentioned previously are functions such that their distributional derivatives are again functions, which are called their *weak derivatives*. By remark 1.1, classical solutions are weak solutions, and a weak solution which is in the space $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ (or $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$) is a classical solution. The following spaces are usual spaces E such that

$$\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega) \subset E \subset \mathcal{D}'(\Omega).$$

These are very useful to study the previous boundary value problems:

- $W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \partial_i u \in L^p(\Omega)\}$, $\|u\|^p = \int_\Omega |u|^p + \int_\Omega |\nabla u|^p$,
- $\mathcal{D}^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : \partial_i u \in L^p(\mathbb{R}^N)\}$, $\|u\|^p = \int_\Omega |\nabla u|^p$, with $p < N$,
- $BV(\mathbb{R}^N) = \{u \in L^{1^*}(\mathbb{R}^N) : \partial_i u \in \mathcal{C}'_0(\mathbb{R}^N)\}$, $\|u\| = \sum_i \|\partial_i u\|_{\mathcal{C}'_0(\Omega)}$, with $N > 1$,
- $W^{1,p}_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$,
- $\mathcal{D}^{1,p}_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, with $p < N$,
- $BV_0(\Omega) = \{u \in L^{1^*}(\Omega) : \partial_i u \in \mathcal{C}'_0(\Omega) \text{ and } u \text{ satisfies } \star\}$, $\|u\| = \sum_i \|\partial_i u\|_{\mathcal{C}'_0(\Omega)}$, with $N > 1$, where $\star$ means $\|P(u)\|_{BV(\mathbb{R}^N)} = \|u\|_{BV(\Omega)}$, $P(u)$ being the extension of u by 0 on $\mathbb{R}^N$,
- $\mathcal{D}^{1,p}(\mathbb{R}^N_+) = \{u \in L^{p^*}(\mathbb{R}^N_+) : \partial_i u \in L^p(\mathbb{R}^N_+)\}$, $\|u\|^p = \int_{\mathbb{R}^N_+} |\nabla u|^p$, with $p < N$,
- $BV(\mathbb{R}^N_+) = \{u \in L^{1^*}(\mathbb{R}^N_+) : \partial_i u \in \mathcal{C}'_0(\mathbb{R}^N_+)\}$, $\|u\| = \sum_i \|\partial_i u\|_{\mathcal{C}'_0(\mathbb{R}^N_+)}$, with $N > 1$,
- $\mathcal{D}^{1,p}_\#(\Omega) = \{u \in L^{p^*}(\Omega) : \partial_i u \in L^p(\Omega), \bar{u} = 0\}$, $\|u\|^p = \int_\Omega |\nabla u|^p$, with $p < N$,
- $BV_\#(\Omega) = \{u \in L^{1^*}(\Omega) : \partial_i u \in \mathcal{C}'_0(\Omega), \bar{u} = 0\}$, $\|u\| = \sum_i \|\partial_i u\|_{\mathcal{C}'_0(\Omega)}$, with $N > 1$,

where $p^* := \frac{Np}{N-p}$ with $p < N$, and $\bar{u} := \frac{1}{|\Omega|} \int_{\Omega} u$. Let us also define the spaces

$$\mathcal{D}(\overline{\mathbb{R}_+^N}) = \{u|_{\mathbb{R}_+^N} : u \in \mathcal{D}(\mathbb{R}^N)\}, \quad \mathcal{D}_{\#}(\overline{\Omega}) = \{u|_{\Omega} : u \in \mathcal{D}(\mathbb{R}^N), \bar{u} = 0\},$$

note that $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are called *Sobolev Spaces* and are denoted by $H^1(\Omega)$ and $H_0^1(\Omega)$ when $p = 2$, and that functions of $\mathcal{D}(\Omega)$ are called *test functions*.

REMARK 1.2. Usually, the BV space is introduced by considering the following functional and the following norm:

$$\begin{aligned} Du : \mathcal{C}_0(\Omega, \mathbb{R}^N) &\rightarrow \mathbb{R} : v \mapsto \sum_i \langle \partial_i u, v_i \rangle, \\ \|Du\| &= \sup\{\langle Du, v \rangle : v \in \mathcal{C}_0(\Omega, \mathbb{R}^N), \|v\|_{\infty} \leq 1\}, \quad \|v\|_{\infty} := \sup(\sum_i v_i^2)^{\frac{1}{2}}. \end{aligned}$$

This norm is equivalent to the previous one but is more practical because it can be shown that if B is a regular set of $\mathbb{R}^N$, then $1_B \in \text{BV}(\mathbb{R}^N)$ and $\|D1_B\| = p(B)$ is the perimeter of B .

REMARK 1.3. The definitions of $\mathcal{D}^{1,p}$ and BV are natural in view of the Sobolev inequalities (see Section 14) and the absence of definition of such spaces on Ω without null boundary conditions is natural because of the absence of such Sobolev inequalities in this case. Let us notice nevertheless that BV is generally introduced on Ω without null boundary condition by

$$\text{BV}(\Omega) = \{u \in L^1(\Omega) : \partial_i u \in \mathcal{C}_0^l(\Omega)\}, \quad \|u\| = \int_{\Omega} |u| + \sum_i \|\partial_i u\|_{\mathcal{C}_0^l(\Omega)},$$

but we shall not use this space in the sequel.

2 Existence Tools in Finite Dimension

In this section, let us recall briefly some methods of resolution of a nonlinear equation $g(u) = 0$ with $g : \mathbb{R}^N \rightarrow \mathbb{R}^M$ continuous. More details about that can be found in Appendix A.

Let us begin by the *degree theory*. The *Brouwer degree* associates to any pair (g, D) made of an open bounded set $D \subset \mathbb{R}^N$ and a function $g \in \mathcal{C}(\overline{D}, \mathbb{R}^N)$ such that $0 \notin g(\partial D)$, an integer denoted by $\deg_B(g, D)$, in such a way that this correspondence verifies several properties, like *existence* and *Homotopy invariance* properties:

$$\begin{aligned} (\deg_B(g, D) \neq 0) &\Rightarrow (\exists u \in D : g(u) = 0), \\ (G \in \mathcal{C}([0, 1] \times \overline{D}), \quad 0 \notin G([0, 1] \times \partial D)) &\Rightarrow (\partial_{\lambda} \deg_B(G(\lambda, \cdot), D) = 0). \end{aligned}$$

The *variational methods* study problems $g(u) = 0$ for which there exists $h \in \mathcal{C}^1(\mathbb{R}^N)$ such that $g = \nabla h$. Solving the equation consists now in finding critical points of h . A brief discussion about usefulness of this formulation is given in Appendix A. Let us mention here three concepts and two properties: the *Nehari manifold* of h , denoted by $\mathcal{N}$, the *critical groups of Morse* and the *Morse index* of a point u with respect to h respectively when $h \in \mathcal{C}^1(\mathbb{R}^N)$ or $\mathcal{C}^2(\mathbb{R}^N)$ and denoted by $C_n(u, h)$ or $\text{ind}_M(u, h)$, and the *homotopy invariance* property of these ones:

$$\begin{aligned} \mathcal{N} &:= \{u \neq 0 : \langle h'(u), u \rangle = 0\}, \\ \forall n \in \mathbb{N} : C_n(u, h) &= H^n(\{h \leq c\}, \{h \leq c\} \setminus \{u\}), \\ \text{ind}_M(u, h) &:= \sup\{\dim V : \dim V < \infty, \langle h''(u)|_V, v \rangle < 0 \text{ if } v \in V \setminus \{0\}\}, \\ \begin{cases} H \in \mathcal{C}^1([0, 1] \times D), \\ \text{resp. } H \in \mathcal{C}^{\#}([0, 1] \times D), \end{cases} &\quad \begin{cases} (\partial_u H)^{-1}([0, 1] \times \{0\}) \cap D = \{u_0\}, \\ \text{resp. the same with } u_0 \text{ non-degenerate,} \end{cases} \end{aligned}$$

$$\Rightarrow \begin{cases} \forall n \in \mathbb{N} : \partial_\lambda \dim C_n(u, h) = 0, \\ \text{resp. } \partial_\lambda \text{ind}_M(u_0, H(\lambda, \cdot)) = 0, \end{cases}$$

where $H \in \mathcal{C}^\#$ means $H \in \mathcal{C}^1$ and the continuity of $\partial_u^2 H$ with respect to u , and u_0 *non-degenerate* means for any $v \neq 0$, there existence of $w \neq 0$ such that $\langle h''(u_0)|v, w \rangle \neq 0$, with $h := H(\lambda, \cdot)$. We precise that $H^n(A, B)$ means the n^{th} group of relative homology of A with respect to B , where $B \subseteq A$, and that $\dim C_n(u, h) = \delta_{n,k}$ when $h \in \mathcal{C}^2$, where $k := \text{ind}_M(u, h)$ and $\delta_{n,k} = 1$ (resp. 0) if $n = k$ (resp. $n \neq k$).

Let us still mention the *Implicit Function Theorem*, which can be viewed as an “ancestor” of the Brouwer degree, and the Böhme, Krasnoselskii and Marino’s theorems, which studies *bifurcation Problems* with all the tools mentioned previously. In these theorems, the problem consists to study solutions of a family of equations: $G(\lambda, u) = 0$ with $\lambda \in \Lambda$.

Theorem 2.1 (Implicit Function Theorem): *Let $G \in \mathcal{C}^1(\Lambda \times \mathbb{R}^N, \mathbb{R}^N)$ with $\Lambda \subseteq \mathbb{R}^M$. If $\partial_u G(\lambda_0, u_0)$ is invertible, there exists $r, R > 0$ and $\gamma \in \mathcal{C}^1(B(\lambda_0, r), B(u_0, R))$ verifying*

$$\forall (\lambda, u) \in B(\lambda_0, r) \times B(u_0, R) : G(\lambda, u) = 0 \Leftrightarrow u = \gamma(\lambda).$$

For the second theorem, let us suppose that $\Lambda = \mathbb{R}$ and $G(\lambda, 0) = 0$ for each λ . We shall say that 0 is a trivial solution of the equation and shall denote by $\mathcal{S}$ the set of non-trivial solutions

$$\mathcal{S} := \{(\lambda, u) : G(\lambda, u) = 0, u \neq 0\}.$$

A real λ_* is a *bifurcation point* of G if $(\lambda_*, 0)$ is adherent to $\mathcal{S}$.

Theorem 2.2: *Let $G \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$ such that $G(\lambda, 0) = 0$ for any λ . We have the three following results:*

(NC) *Let us suppose that G is of class $\mathcal{C}^1$ in a neighborhood of $(\lambda_*, 0)$. If λ_* is a bifurcation point of G , then $\partial_u G(\lambda_*, 0)$ is not invertible.*

(K) *Let us suppose that for each λ :*

$$G(\lambda, u) = L(\lambda, u) - N(\lambda, u), \quad L(\lambda, \cdot) = \lambda \text{id} - T, \quad N(\lambda, u) = o(\|u\|),$$

with T linear, N continuous, and with the last equality uniform on each compact set of λ . If λ_ is an eigenvalue of T with odd algebraic multiplicity, then it is a bifurcation point of G .*

(BKM) *Let us suppose that for each λ , $G(\lambda, u) = \nabla_u H(\lambda, u)$ where*

$$H(\lambda, u) = \frac{1}{2} (L(\lambda, u)|u) - P(\lambda, u), \quad L(\lambda, \cdot) = \lambda \text{id} - T, \quad \nabla_u P(\lambda, u) = o(\|u\|),$$

with T linear and symmetric, P of class $\mathcal{C}^\#$, the last equality uniform on each compact set of λ , and where $(\cdot|\cdot)$ is the scalar product of $\mathbb{R}^N$ inducing $\|\cdot\|$. If λ_ is an eigenvalue of T , then it is a bifurcation point of G .*

The necessary condition (NC) is a trivial consequence of the Implicit Function Theorem. The sufficient condition (K), proved by Krasnoselskii, follows from the degree theory and has been clarified by Rabinowitz who studied the global behavior of the bifurcations branches. The sufficient condition (BKM), which is the generalization of the case $P(\lambda, u) = P(u)$ proved independently by Böhme, Krasnoselskii and Marino, follows from the Morse theory. Let us note that when G has the form given for the two previous cases, λ_* is an eigenvalue of T if and only if $\partial_u G(\lambda_*, 0)$ is not invertible. We recall also that

- λ is an *eigenvalue* of the linear operator $T : \mathbb{R}^N \rightarrow \mathbb{R}^N$ if $\ker(\lambda \text{id} - T) \neq \{0\}$,

- let us denote by $v(\lambda)$ the integer between 1 and N , called *order* of λ , such that

$$\begin{aligned} \ker(\lambda \text{id} - T) &\subsetneq \ker((\lambda \text{id} - T)^2) \subsetneq \dots \\ &\subsetneq \ker((\lambda \text{id} - T)^{v(\lambda)}) = \ker((\lambda \text{id} - T)^{v(\lambda)+1}) = \dots, \end{aligned} \quad (21)$$

- the *geometric multiplicity* of λ , denoted by $m_g(\lambda)$, and the *algebraic multiplicity* of λ , denoted by $m_a(\lambda)$, are respectively defined by $\dim \ker(\lambda \text{id} - T)$ and $\dim \ker((\lambda \text{id} - T)^{v(\lambda)})$,
- T is *symmetric* if for any $u, v \in \mathbb{R}^N : (T(u)|v) = (u|T(v))$; in this case, the algebraic multiplicity coincides with the geometric multiplicity.

At last, let us speak about the *Lyapunov-Schmidt reduction* which is a method consisting to simplify equations of the form $L(u) - N(u) = 0$ where $L, N : \mathbb{R}^N \rightarrow \mathbb{R}^M$ are continuous operators and L is linear. If $E_1, E_2 \subset \mathbb{R}^N$ and $F_1, F_2 \subset \mathbb{R}^M$ are two pairs of subspaces satisfying (22) and $p, q : \mathbb{R}^N \rightarrow \mathbb{R}^N$ and $P, Q : \mathbb{R}^M \rightarrow \mathbb{R}^M$ are two pairs of linear operators satisfying (23), we have the equivalence (24),

$$E_1 \oplus E_2 = \mathbb{R}^N, E_2 = \ker L, \quad F_1 \oplus F_2 = \mathbb{R}^M, F_1 = \text{ran } L, \quad (22)$$

$$\text{ran } p = E_1, \text{ran } q = E_2, p + q = \text{id}_{\mathbb{R}^N}, \quad \text{ran } P = F_1, \text{ran } Q = F_2, P + Q = \text{id}_{\mathbb{R}^M}, \quad (23)$$

$$L(u) - N(u) = 0 \Leftrightarrow \begin{cases} L(u) = PN(u), \\ QN(u) = 0, \end{cases} \Leftrightarrow \begin{cases} p(u) = L^{-1}PN(p(u) + q(u)), \\ QN(p(u) + q(u)) = 0, \end{cases} \quad (24)$$

where L^{-1} is the inverse of the bijection $L|_{E_1} : E_1 \rightarrow F_1$, $\ker L$ is the set of zeros of L , and $\text{ran } L$ the sets of any $L(u)$. This method is used when the first equation of the last equivalence of (24) leads to solve $p(u)$ with respect to $q(u)$ like a parameter, say $p(u) = A(q(u))$. The sequel consists to inject this expression in the second equation of the last equivalence of (24) and to look for $q(u)$. This reduction is particularly used when (25) is verified since in this case we have (26),

$$i(L) := \dim \ker L - \text{codim } \text{ran } L = 0, \quad (25)$$

$$q(u) \in E_2, \quad \text{ran } Q = F_2, \quad \dim E_2 = \dim F_2, \quad (26)$$

giving the possibility to use degree theory or variational methods to solve this last equation.

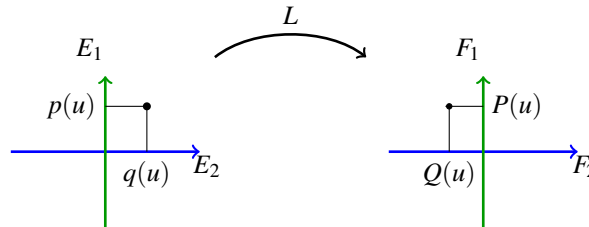


Figure 2.1

Let us finish this section by the following procedure inspired by the bifurcation theory. In this theory, the equation $G(\lambda, u) = 0$ is degenerate when $u = 0$, in our case, we shall suppose this equation degenerate when $\lambda = 0$:

Procedure 2.3: If G is of class $\mathcal{C}^1$, let us write $G(\lambda, u)$ like

$$L(u) - N(\lambda, u)$$

where $L := G'_u(0, 0)$ and let us suppose that $N(0, \cdot) \equiv 0$. We can solve the first equation of (24) by the Implicit Function Theorem to obtain

$$p(u) = \gamma(\lambda, q(u))$$

for some function γ defined on a neighborhood of $(0, \alpha)$ with $\alpha \in \ker L$. Then, we can imbed this formula in the second equation of (24). To avoid the degeneracy at $\lambda = 0$, we replace this equation by the following one that we suppose having sense:

$$\begin{cases} \lambda^{-1} QN(\lambda, \gamma(\lambda, q(u)) + q(u)) = 0 & \text{if } \lambda > 0, \\ \partial_\lambda QN(\lambda, \gamma(\lambda, q(u)) + q(u))|_{\lambda=0} = 0 & \text{if } \lambda = 0. \end{cases}$$

If $i(L) = 0$, we can try to solve this family of equations with degree theory, variational methods, or the Implicit Function Theorem.

3 Existence Tools in Infinite Dimension

Boundary value problems are in general written as a nonlinear equation $g(u) = 0$ with an operator $g : E \rightarrow F$ between infinite dimensional Banach spaces E, F . Let us recall briefly how the techniques of the previous section can be generalized in infinite dimension and applied in differential equations. Appendix A clarifies what is said here.

The Brouwer degree can be generalized with the *Leray-Schauder degree*, denoted by $\deg_{LS}$ and defined on pairs $(D, \text{id} - \chi)$ where $D \subseteq E$ is an open bounded set and $\chi : \overline{D} \rightarrow E$ a compact operator, i.e. $\chi(\overline{D})$ is relatively compact.

In variational methods, a compactness assumption is always required in order to have existence results. Let us mention the *Palais-Smale condition*, the *Palais-Smale condition at level c*, and the *sequential weak lower semi-continuity*, respectively defined by

$$\forall (u_n)_n \subset E \text{ s.t. } (\sup |h(u_n)| < \infty, \lim_{n \rightarrow \infty} h'(u_n) = 0), \quad \exists (\tilde{u}_n)_n \subset (u_n)_n \text{ convergent,} \quad (\text{PS})$$

$$\forall (u_n)_n \subset E \text{ s.t. } (\lim_{n \rightarrow \infty} h(u_n) = c, \lim_{n \rightarrow \infty} h'(u_n) = 0), \quad \exists (\tilde{u}_n)_n \subset (u_n)_n \text{ convergent,} \quad (\text{PS}_c)$$

$$\forall (u_n)_n : u_n \rightharpoonup u \Rightarrow \liminf_{n \rightarrow \infty} h(u_n) \geq h(u),$$

where $u_n \rightharpoonup u$ means that $(u_n)_n$ converges weakly to u . We can also mention the “light” Palais-smale conditions, denoted by $(\tilde{\text{PS}})$ and $(\tilde{\text{PS}}_c)$, where the existence of a convergent subsequence is replaced by the existence of a critical point u of h (verifying $h(u) = c$ for the second case). For the homotopy invariance of the Morse index, $H(\lambda, \cdot)$ must verify $(\tilde{\text{PS}})$ on D for any λ .

Theorem 2.1 is true in the infinite dimensional case (with the restriction of Remark 3.1) and also Theorem 2.2 with,

- for the result (K), T, N completely continuous, i.e., with a compact restriction to any bounded set. Let us note that under this assumption about T , the algebraic multiplicity $m_a(\lambda)$ defined keeps sense because there is a stabilization phenomenon in (21).
- for the result (BKM), the assumption that λ_* is an isolated eigenvalue of finite multiplicity, and that $H(\lambda, \cdot)$ verify Condition (PS) for any λ .

REMARK 3.1. The sets $\mathcal{L}(E, F)$ and $\text{Iso}(E, F)$ (denoted by $\mathcal{L}(E)$ and $\text{Iso}(E)$ if $F = E$) contain respectively any continuous linear operators and any bijective continuous linear operator from E to F . The fact that continuity of linear operators is not immediate in the infinite dimensional case, justifies the previous definitions. Let us recall that if $T \in \text{Iso}(E, F)$, T^{-1} is also continuous.

In the infinite dimensional case, the Lyapunov-Schmidt reduction needs to work with particular operator L : a linear operator $L : E \rightarrow F$ is a *Fredholm operator* if it is continuous and such that $\text{ran } L$ is closed, $\dim \ker L < \infty$ and $\text{codim } \text{ran } L < \infty$. The *Fredholm index* of L , denoted by $i(L)$, is defined like in (25) by

$$i(L) := \dim \ker L - \text{codim } \text{ran } L.$$

The set of Fredholm operators from E to F is denoted by $\Phi(E, F)$ or $\Phi(E)$ if $F = E$, and the subset of such operators with index equal to n , by $\Phi_n(E, F)$ or $\Phi_n(E)$.

Let us finish this section by some concrete existence (and uniqueness) results which will be very useful in this thesis. The first theorem studies Problem $(\mathcal{P})$ where $\phi = \text{id}$ and f is replaced by λf . The second theorem studies Problem $(\mathcal{Q})$ by using the following tools:

- the *energy functional* $J : E \rightarrow \mathbb{R} : u \mapsto \frac{1}{2} \int_{\Omega} [|\nabla u|^2 + \mu u^2] - \int_{\Omega} F(x, u)$,
- the *Nehari manifold* $\mathcal{N}$ of J defined as in the previous section:

$$\begin{aligned} \mathcal{N} &:= \{u \neq 0 : \langle J'(u) | u \rangle = 0\}, \\ &= \{u \neq 0 : \int_{\Omega} [|\nabla u|^2 + \mu u^2] = \int_{\Omega} f(x, u)u\}, \end{aligned}$$

- the *nodal Nehari set* $\mathcal{M}$ of J defined by

$$\begin{aligned} \mathcal{M} &:= \{u \neq 0 : u^{\pm} \in \mathcal{N}\}, \\ &= \{u \neq 0 : \int_{\Omega} [|\nabla u^{\pm}|^2 + \mu (u^{\pm})^2] = \int_{\Omega} f(x, u^{\pm})u^{\pm}\}, \end{aligned}$$

where $F(x, t) := \int_0^t f(x, s) ds$ and $E = H^1(\Omega)$ or $H_0^1(\Omega)$ according to whether $Bu = u$ or $\partial_{\nu} u$. In the sequel, we shall suppose $\mu \geq 0$ when $Bu = u$ and $\mu > 0$ when $Bu = \partial_{\nu} u$.

Theorem 3.2: *Let us consider the following problem, where f is continuous and $b \neq 0$:*

$$u' = \lambda f(t, u), \text{ in } (0, T), \quad u(T) = bu(0). \quad (\mathcal{R}_{\lambda})$$

- If $b \neq 1$, there exist $r, R > 0$ such that Problem $(\mathcal{R}_{\lambda})$ admits a unique solution u verifying $\|u\|_{\infty} < R$ when $|\lambda| < r$.
- If $b = 1$, f is of class $\mathcal{C}^1$ with respect to u , and $u_* \in \mathbb{R}$ satisfies

$$(a) \int_0^T f(t, u_*) dt = 0, \quad (b) \int_0^T \partial_u f(t, u_*) dt \neq 0, \quad (27)$$

there exists $r, R > 0$ such that Problem $(\mathcal{R}_{\lambda})$ admits a unique solution u verifying $\|u - u_*\|_{\infty} < R$ when $|\lambda| < r$.

- If $b = 1$ and if there exists $M_1 < M_2$ such that for any $u \in \mathcal{C}([0, T])$:

$$\begin{aligned} \int_0^T f(t, u(t)) dt &> 0 \quad \text{if } u \geq M_2, \\ \int_0^T f(t, u(t)) dt &< 0 \quad \text{if } u \leq M_1, \end{aligned} \quad (28)$$

(or the converse), Problem $(\mathcal{R}_{\lambda})$ admits a solution for any λ .

The first result follows from the Implicit Function Theorem. The second one studies the periodic case $b = 1$ for which Problem $(\mathcal{R}_0)$ is degenerate (any constant solves

it). (27)(a,b) are respectively necessary and sufficient conditions ensuring that u_* is non-degenerate and induces a solution. The third result works with the degree theory and improves the existence part of the second result (but loses the uniqueness part) by ensuring the thesis for any λ and replacing (27) by (28). The reader can consult [39] and Appendix A for a proof.

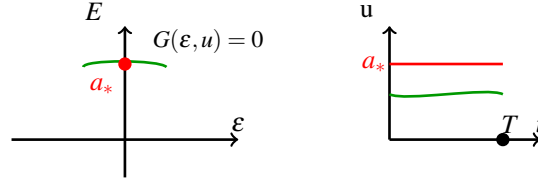


Figure 3.1

Theorem 3.3: *Let us consider Problem $(\mathcal{Q})$ and the following assumptions:*

- $(A_1)(a)$ f is continuous,
- $(A_1)(b)$ f is Hölder continuous on $\Omega \times [-R, R]$ for every $R > 0$,
- $(A_2)(a)$ $\exists p \in I, \exists c > 0, \forall (x, t) \in \Omega \times \mathbb{R} : |f(x, t)| \leq c(1 + |t|^{p-1})$,
 $I = [2, 2^*]$ if $N \geq 3$ and $[2, 2^* = \infty]$ if $N = 1, 2$,
- $(A_2)(b)$ $\exists p \in I, \exists c > 0, \forall (x, t) \in \Omega \times \mathbb{R} : |f(x, t)| \leq c(|t| + |t|^{p-1})$,
 $I = (2, 2^*]$ if $N \geq 3$ and $(2, 2^* = \infty)$ if $N = 1, 2$,
- $(A_3)(a)$ $\lim_{t \rightarrow 0} \frac{f(x, t)}{|t|} = 0$ and $\lim_{t \rightarrow \infty} \frac{f(x, t)}{t} = \infty$ for any $x \in \Omega$,
- $(A_3)(b)$ $t \mapsto \frac{f(x, t)}{|t|}$ is strictly increasing on $\mathbb{R} \setminus \{0\}$ for any $x \in \Omega$,

In what follows, (A_2) means $(A_2)(a)$ (resp. (b)) when Ω is bounded (resp. unbounded):

- Under $(A_1)(a)$ -(A_2), J is well defined, is of class $\mathcal{C}^1$ and its critical points are weak solutions of $(\mathcal{Q})$.
- Under $(A_1)(b)$ -(A_2), classical and weak solutions of $(\mathcal{Q})$ coincide.
- Under $(A_1)(a)$ -(A_2)-(A_3)(b), J is bounded from below on $\mathcal{N}$ by 0 and every minimizer $u \in \mathcal{N}$ of $J|_{\mathcal{N}}$ is a critical point of J .
- Under $(A_1)(a)$ -(A_2)-(A_3)(a, b) with $p < 2^*$ in (A_2) , J is bounded from below on $\mathcal{N}$ by 0 and reaches its minimum at at least one point $u \in \mathcal{N}$.
- Both previous results are true with $\mathcal{N}$ replaced by $\mathcal{M}$.

The two first results are very classical, see for example [53]. The third and fourth results have been obtained for the first time in [4] by Ambrosetti and Rabinowitz, but not on this form. We can mention the papers [7] of Bartsch, Weth and Willem and [48] of Van Schaftingen and Willem for the formulation used here. The last result has been given for the first time in [19] by Castro, Cossion and Neuberger, proved with another method in [7] by Bartsch, Weth and Willem and improved in [48] by Van Schaftingen and Willem. Theorem 2.1, proved in Appendix A, gives an abstract form of these results. Let us note that under the given assumptions, the minimizers of $J|_{\mathcal{N}}$ are signed solutions called *ground state* (shortly g.s.) solutions, what is not true for minimizers of $J|_{\mathcal{M}}$ called *least energy nodal* (shortly l.e.n.) solutions.

REMARK 3.4. Assumption $(A_3)(b)$ ensures that

$$\mathbb{R}_0^+ \rightarrow \mathbb{R} : t \mapsto t^{-1} \partial_t J(tu)$$

is strictly decreasing for any $u \in E$. Together with $(A_3)(a)$, that infers to the graph of J a mountain pass geometry, and leads to property $(\tilde{P}S_c)$ with $c := \inf_{\mathcal{N}} J$ or $\inf_{\mathcal{M}} J$. An interpretation of (A_3) on $f(x, \cdot)$ is easy. For more details, the reader can consult [48].

In view of that, let us mention another classical assumption. Inequality (29), called *Ambrosetti-Rabinowitch Assumption*, ensuring inequality (30) on $\mathcal{N}$, together with the compact injections of Rellich, leads to property (PS),

$$\exists \theta > 2, \forall (x, t) \in \Omega \times \mathbb{R} : tf(x, t) \geq \theta F(x, t), \quad (29)$$

$$J(u) = \frac{1}{2} \|u\|^2 - \int_{\Omega} F(x, u) \geq \frac{1}{2} \|u\|^2 - \frac{1}{\theta} \int_{\Omega} f(x, u) u = \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u\|^2, \quad (30)$$

where $\|u\|^2 := \int_{\Omega} [|\nabla u|^2 + \mu u^2]$. An interpretation of this condition is the existence of $c, R : \Omega \rightarrow \mathbb{R}^+$ such that

$$|F(x, u)| \leq c(x) |u|^\theta \text{ if } |u| \leq R(x), \quad |F(x, u)| \geq c(x) |u|^\theta \text{ if } |u| \geq R(x).$$

4 Manifold and Eigenvalues Problems

The functional spaces defined in the first section can sometimes be defined on manifolds. Let us consider a smooth manifold $\mathcal{M} \subseteq \mathbb{R}^N$ of dimension $1 \leq M \leq N$. We shall suppose that its smooth structure verifies the following property: $u : \mathcal{M} \rightarrow \mathbb{R}$ is of class $\mathcal{C}^k$ if and only if for any $x \in \mathcal{M}$, there exists an open set $U \subset \mathbb{R}^N$ containing x and a function $\tilde{u} \in \mathcal{C}^k(U, \mathbb{R})$ such that $\tilde{u}|_{\mathcal{M}} = u$ on $U \cap \mathcal{M}$. Let us denote by $T_x \mathcal{M}$ the *tangent space* of $\mathcal{M}$ at x , defined by

$$T_x \mathcal{M} := \{c'(0) : c \in \mathcal{C}^1((-\delta, \delta), \mathcal{M}), c(0) = x\}.$$

For any pair (u, x) as above, there exists a pair $(\tilde{u}, U)$ as above, such that

$$\exists \varepsilon_0 > 0, \forall x' \in U, \forall y \in (T_{x'} \mathcal{M})^\perp : 0 < \varepsilon < \varepsilon_0 \Rightarrow \tilde{u}(x' + \varepsilon y) = u(x').$$

A such pair $(\tilde{u}, U)$ is called *canonical representation* of (u, x) . Let us consider Ω an open set of $\mathcal{M}$.

- the M -Hausdorff measure $\mathcal{H}^M$ defines a measure on $\mathcal{M}$ which permits the definition of Lebesgue spaces $L^p(\Omega)$ for any $p \in [1, \infty)$.
- If $u \in \mathcal{C}^k(\Omega, \mathbb{R})$ then $\partial^\alpha u$ with $\alpha \in \mathbb{N}^N$ is defined at $x \in \Omega$ by $\partial^\alpha \tilde{u}(x)$ where $(\tilde{u}, U)$ is a canonical representation of (u, x) . This definition is independent of the choice of $(\tilde{u}, U)$.
- The definition at $x \in \Omega$ of ∇u for a function $u \in \mathcal{C}^1(\Omega, \mathbb{R})$, and of Δu for a function $u \in \mathcal{C}^2(\Omega, \mathbb{R})$, are respectively $\nabla \tilde{u}(x)$ and $\Delta \tilde{u}(x)$.

With these tools, definitions of spaces mentioned in the first section can be extended to $\mathcal{M}$. Two manifolds will be useful in this thesis: the sphere $S^{N-1} \subset \mathbb{R}^N$, boundary of the unit ball B^N , and the cylinder $\mathcal{C} := S^{N-1} \times \mathbb{R}$.

Theorem 4.1 (Strong Maximum Principle): *Under the following assumptions,*

- $\Omega \subseteq \mathcal{M}$ is an open connected set, possibly unbounded,
- $c \in \mathcal{C}(\Omega)$ is a non-negative function,
- $v \in \mathcal{C}^2(\Omega)$ is such that $-\Delta v + c(x)v \leq 0$,
- $\sup_{\Omega} v \geq 0$ when $c \not\equiv 0$,

we have:

$$(\exists x_0 \in \Omega : \sup_{\Omega} v = v(x_0)) \Rightarrow (v \equiv v(x_0)).$$

Theorem 4.2: *Let us consider the eigenvalue problem*

$$-\Delta v = \mu v \text{ on } \mathcal{S}^{N-1}.$$

Any real number μ admitting a nonzero solution v of this problem in $H^1(\mathcal{S}^{N-1})$, i.e., any eigenvalue of $-\Delta$ in $H^1(\mathcal{S}^{N-1})$, is given by

$$\mu_k = k(k + N - 2),$$

with $k \geq 0$. The associated solutions, i.e., eigenfunctions of $-\Delta$ in $H^1(\mathcal{S}^{N-1})$, are given by $H_k|_{\mathcal{S}^{N-1}}$ where $H_k : \mathbb{R}^N \rightarrow \mathbb{R}$ is a harmonic homogenous polynomial of degree k .

Theorem 4.3: *Let us consider the eigenvalue problem*

$$-\Delta u = \lambda u \text{ on } B^N, \quad Bu = 0 \text{ on } \partial B^N,$$

with $B(u)$ like in (2). Any real number λ admitting a nonzero solution u of this problem in $H_0^1(B^N)$ (resp. $H^1(B^N)$) if $Bu = u$ (resp. $Bu = \partial_{\nu} u$), i.e., any eigenvalue of $-\Delta$ in $H_0^1(B^N)$ (resp. $H^1(B^N)$), is given by

$$\lambda_{k,q} = j_{v_k,q}^2, \quad \begin{array}{l} j_{v_k,q} \text{ the } q^{\text{th}} \text{ solution of } J_{v_k}(s) = 0, \\ \text{(resp. } sJ'_{v_k}(s) = \frac{N-2}{2}J_{v_k}(s) \text{) in } \mathbb{R}_0^+, \end{array} \quad v_k = (k + \frac{N-2}{2}),$$

where $k \geq 0, q \geq 1$. The associated solutions, i.e., eigenfunctions of $-\Delta$ in $H_0^1(B^N)$ (resp. $H^1(B^N)$), are given by

$$u(x) = U_q^k(|x|)H_k\left(\frac{x}{|x|}\right), \quad U_q^k(r) := J_{v_k}(j_{v_k,q}r)r^{\frac{2-N}{2}}. \quad (31)$$

In the Neumann case, 0 is also an eigenvalue and $u = C^{\text{te}}$ the associated eigenfunctions.

The reader can find a proof of a more general version of the Strong Maximum Principle on manifolds in [6], can consult [49, 32] for the proof of Theorem 4.2 and Appendix E for information about the *Bessel functions* J_{ν} .

REMARK 4.4. Even if Theorem 4.3 is only true on B^N , any open bounded connected set $\Omega \subset \mathbb{R}^N$ is such that the eigenvalues of $-\Delta + \mu \text{ id}$, with $\mu = 0$ (resp. 1) in the Dirichlet case (resp. Neumann case), are strictly positive, countable, and converging to ∞ :

$$0 < \lambda_1 < \lambda_2 < \dots$$

Let us denote by $E(\lambda_1), E(\lambda_2), \dots$ the associated eigenspaces. In the sequel, we shall denote by $P_{E(\lambda_i)}$ the orthogonal projection on $E(\lambda_i)$. Since the differential operator considered is symmetric in H_0^1 (resp. H^1), we can define $m(\lambda_i) := m_a(\lambda_i) = m_g(\lambda_i)$ for any i and remark that $E(\lambda_i) \perp E(\lambda_j)$ if $i \neq j$. A simple regularity study ensures that all the eigenfunctions are of class $\mathcal{C}^\infty$. The first eigenvalue λ_1 is simple (so verifies $m(\lambda_1) = 1$) and such that

$$u \in E(\lambda_1) \Rightarrow |u| > 0 \text{ in } \Omega, \quad (\text{resp. } u = C^{\text{te}}).$$

At last, the Poincaré principle ensures

$$\forall k \in \mathbb{N}, \forall u \perp E_0 \oplus E(\lambda_1) \oplus \cdots \oplus E(\lambda_k) : \lambda_{k+1} \int_{\Omega} u^2 \leq \int_{\Omega} |\nabla u|^2 + \mu u^2,$$

where $E_0 := \{0\}$. In particular, when $\mu = 0 = k$ we obtain the case $p = 2$ of the *Poincaré inequalities*, and when $\mu = 1 = k$ we obtain the case $p = 2$ of the *Poincaré-Wirtinger inequalities*.

5 Symmetry Properties

Let us consider Ω an open set of $\mathbb{R}^N, \mathcal{S}^{N-1}$ or $\mathcal{C}$ and $u : \Omega \rightarrow \mathbb{R}$ a Lebesgue or Hausdorff-measurable function. Let us fix $P \in \mathcal{S}^{N-1}$, denote by d the geodesic distance in $\mathcal{S}^{N-1}$, and by (x', x_{N+1}) a point $x \in \mathbb{R}^{N+1}$.

DEFINITION 5.1. Let $\Omega = B(0, R)$ in $\mathbb{R}^N$ or $\{x : d(x, P) < R\}$ in $\mathcal{S}^{N-1}$. u is respectively said $\mathbb{R}^N$ -Schwarz symmetric or $\mathcal{S}^{N-1}$ -Schwarz symmetric with respect to P , if there exists a measurable function $\tilde{u} : \tilde{\Omega} \rightarrow \mathbb{R}$ such that $u(x) = \tilde{u}(r)$ for a.e. x in Ω , where $\tilde{\Omega} = [0, R)$ and $r = |x|$ or $d(x, P)$.

DEFINITION 5.2. Let $\Omega = \{R_1 < |x| < R_2\}$ in $\mathbb{R}^N$ or $\mathcal{S}^{N-1} \times (R_1, R_2)$ in $\mathcal{C}$. u is respectively said $\mathbb{R}^N$ -foliated Schwarz symmetric or $\mathcal{C}$ -foliated Schwarz symmetric with respect to P , if there exists a measurable function $\tilde{u} : \tilde{\Omega} \rightarrow \mathbb{R}$, such that $u(x) = \tilde{u}(r, s)$ for a.e. x in Ω , where

$$\begin{aligned} \tilde{\Omega} = \{(r, s) : r \in (R_1, R_2), s \in [0, \pi r]\} & \quad \text{or} \quad \tilde{\Omega} = [0, \pi) \times (R_1, R_2), \\ r = |x|, \quad s = d(x, rP) & \quad \text{or} \quad r = d(x', P), \quad s = x_{N+1}. \end{aligned}$$

The case $\Omega = \{R_1 < |x| < R_2\}$ can be extended to $B(0, R)$.

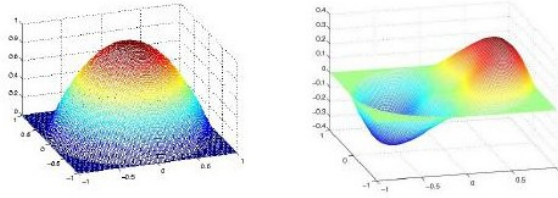


Figure 5.1

These symmetries can be characterized by *polarization*:

DEFINITION 5.3. H is a Half-space of $\mathbb{R}^N$ if there exists $h \in \mathcal{S}^{N-1}$ and $c \in \mathbb{R}$ such that $H = \{x \in \mathbb{R}^N : h \cdot x \geq c\}$. Let us denote by S_H the orthogonal symmetry associated to ∂H . If Ω invariant with respect to S_H , the *polarization* of u is the function $u_H : \Omega \rightarrow \mathbb{R}$ defined by

$$u_H(x) := \begin{cases} \max\{u(x), u(S_H(x))\} & \text{if } x \in \Omega^+ \\ \min\{u(x), u(S_H(x))\} & \text{if } x \in \Omega^- \end{cases},$$

where $\Omega^+ := \Omega \cap H, \Omega^- := \Omega \cap (\mathbb{R}^N \setminus H)$. We shall denote by $\mathcal{H}, \mathcal{H}^P, \mathcal{H}_0$ the sets of any half-space H verifying respectively no condition, condition $P \in \text{int } H$, and condition $0 \in \partial H$. In the case $\Omega \subseteq \mathcal{C} \subset \mathbb{R}^{N+1}$, we shall denote by $\mathcal{H}_{N+1}$ the sets of half-spaces H verifying condition $\{x' = 0\} \subset \partial H$.

Proposition 5.4: *Let Ω , u and P be like previously.*

- u is $\mathbb{R}^N$ -Schwarz symmetric $\Leftrightarrow \forall H \in \mathcal{H}^0 : u = u_H$,
- u is S^{N-1} -Schwarz symmetric with respect to $P \Leftrightarrow \forall H \in \mathcal{H}_0^P : u = u_H$,
- u is $\mathbb{R}^N$ -foliated Schwarz symmetric with respect to $P \Leftrightarrow \forall H \in \mathcal{H}_0^P : u = u_H$,
- u is $\mathcal{C}$ -foliated Schwarz symmetric with respect to $P \Leftrightarrow \forall H \in \mathcal{H}_{N+1}^P : u = u_H$.

PROOF. We consider the case of Schwarz symmetry around P on S^{N-1} , the other cases being similar. Let us define $\varphi(x) := \arccos(|P \cdot x|)$ the geodesic distance between x and P .

$\Rightarrow$ Let $H \in \mathcal{H}_0^P$ and $x \in \Omega^+$. As $\varphi(x) \leq \varphi(S_H(x))$ because $P \in \text{int } H$, by definition we have $u(x) \geq u(S_H(x))$. The thesis follows.

$\Leftarrow$ Let us prove that if $H \in \mathcal{H}_0$ is such that $P \in \partial H$, then $u = u_H$. Let us note that proving this fact for all such H is equivalent to prove that

$$\forall x \in \Omega : u(x) = u(S_H(x)). \quad (32)$$

Let $x \in \Omega$. We have to see that $u(x) = u(S_H(x))$. It is not difficult to notice that there exists $(H_n^1)_{n \in \mathbb{N}}$ in $\mathcal{H}_0^P$, and $(H_n^2)_{n \in \mathbb{N}}$ in $(\mathcal{H} \setminus \mathcal{H}_0^P)$ such that

$$\lim_{n \rightarrow \infty} S_{H_n^1}(x) = \lim_{n \rightarrow \infty} S_{H_n^2}(x) = S_H(x),$$

$$\forall n \in \mathbb{N} : u(S_{H_n^1}(x)) \leq u(x) \leq u(S_{H_n^2}(x)).$$

We can then conclude by continuity of u . Hence by (32) there exists a function $\tilde{u} : [0, R) \rightarrow \mathbb{R}$ such that $u(x) = \tilde{u}(\varphi(x))$. If we prove that $\tilde{u}$ is non-increasing, we have the thesis. If $\varphi, \varphi' \in (0, R)$ with $\varphi < \varphi'$ then

$$\exists H \in \mathcal{H}_0^P, \exists x \in \Omega^+ : \varphi(x) = \varphi, \varphi(S_H(x)) = \varphi',$$

and hence, because $u = u_H$, we have

$$\tilde{u}(\varphi) = u(x) \geq u(S_H(x)) = \tilde{u}(\varphi').$$

We conclude for $\varphi, \varphi' \in [0, R)$ by continuity of $\tilde{u}$ (which follows from the one of u). $\square$

Proposition 5.5: *Let u, Ω be like previously, $H \in \mathcal{H}$ and Ω invariant with respect to H . If $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$, then $u_H \in W^{1,p}(\Omega)$ with*

$$\int_{\Omega} |u_H|^p = \int_{\Omega} |u|^p, \quad \int_{\Omega} |\nabla u_H|^p = \int_{\Omega} |\nabla u|^p.$$

REMARK 5.6. The reader can consult [55, 54, 7] for more details about symmetry properties. In particular, if Ω is like in definitions 5.1 and 5.2 and u is measurable without any symmetry properties, these references mention the *symmetrization* which associates to u a function $u^* : \Omega \rightarrow \mathbb{R}$ having symmetries. A very important property of this operation is that Proposition 5.5 is true with u_H replaced by u^* and with the conclusion about integrals replaced by

$$\int_{\Omega} |u^*|^p = \int_{\Omega} |u|^p, \quad \int_{\Omega} |\nabla u^*|^p \leq \int_{\Omega} |\nabla u|^p.$$

At last, let us mention the following very important results studying symmetries of solutions of boundary value problems:

Theorem 5.7: *Let us consider Problem $(\mathcal{Q})$ with $B = \text{id}$, and the following assumptions:*

- (A₄)(a) f, f'_t are continuous,
- (A₄)(b) f is continuous and f'_t locally $\mathcal{C}^{1,\alpha}$ for some $\alpha \in (0, 1)$,
- (A₄)(c) Ω is of class $\mathcal{C}^\infty$ or is a path,
- (A₅)(a) $(\forall x \in \Omega, \forall t > 0)(x + th \in \Omega \Rightarrow [x, x + th] \subset \Omega)$,
- (A₅)(b) $(\forall x \in \Omega, \forall t \in \mathbb{R})(S_H(x) \in \Omega, f(S_H(x), t) = f(x, t))$,
- (A₅)(c) $(\forall S \in \mathcal{O}(N), \forall x \in \Omega, \forall t \in \mathbb{R})(S(x) \in \Omega, f(S(x), t) = f(x, t))$,
- (A₆)(a) $x \mapsto f(x, t)$ is non-increasing for every $t \in \mathbb{R}$,
- (A₆)(b) $t \mapsto f(x, t)$ has the sign of t and is independent of x ,
- (A₆)(c) $t \mapsto f(x, t)$ is non-decreasing for every $x \in \Omega$,
- (A₆)(d) $t \mapsto f(x, t)$ is strictly convex for every $x \in \Omega$,
- (A₆)(e) $t \mapsto f'_t(x, t)$ is convex for every $x \in \Omega$,

where $H \in \mathcal{H}_0$ and $[x, x + h]$ are respectively defined by $\{x : h \cdot x \geq 0\}$ and $\{x + sh : s \in [0, 1]\}$. Moreover, let us join to that, Assumptions (A₁) – (A₃) considered in Theorem 3.3. We have the following results:

- (GNN) Under (A₄)(a,c)-(A₅)(a,b) and (A₆)(a) on $\mathbb{R}^+$, if $u \in \mathcal{C}^2(\overline{\Omega})$ is a solution of $(\mathcal{Q})$ such that $u > 0$ in Ω , then for any $x \in \Omega^+$ we have $u(S_H(x)) = u(x)$ and $\partial_h u(x) < 0$.
- (BWW) Under (A₁)(a)-(A₅)(b)-(A₆)(c), if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is a solution of $(\mathcal{Q})$ such that $u_H \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is also a solution of $(\mathcal{Q})$, then, one of the three following conclusions turns to be true on Ω^+ :
 - (i) $u \circ S_H < u$, (ii) $u \circ S_H = u$, (iii) $u \circ S_H > u$.
- (BWW) Under (A₁)(b)-(A₂)-(A₃)(b)-(A₅)(c)-(A₆)(c), if $u \in \mathcal{N}$ is a minimizer of $J|_{\mathcal{N}}$, then $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is a solution of $(\mathcal{Q})$ and is $\mathbb{R}^N$ -foliated Schwarz symmetric. The same is true for $u \in \mathcal{M}$ a minimizer of $J|_{\mathcal{M}}$.
- (P) Under (A₄)(a)-(A₅)(c)-(A₆)(d), if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is a non-degenerate solution of $(\mathcal{Q})$ such that $u(x_0) := \sup_{\Omega} u > 0$ and $\text{ind}_{\mathcal{M}}(u, J) = 1$, then u is $\mathbb{R}^N$ -Schwarz symmetric if $x_0 = 0$ and $\mathbb{R}^N$ -foliated Schwarz symmetric with respect to $P = \frac{x_0}{|x_0|}$ if $x_0 \neq 0$.
- (AP) under (A₄)(a)-(A₅)(c)-(A₆)(b), if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is a non-degenerate solution of $(\mathcal{Q})$ such that $u(S_{H_i}(x)) = u(x)$ and $\{u = 0\} \cap \Omega = \emptyset$, where the vectors $h_1, \dots, h_k$ associated to the half-spaces H_i are linearly independent, then $\text{ind}_{\mathcal{M}}(u, J) \geq k + 1$.
- (PW) Under (A₄)(b)-(A₅)(c)-(A₆)(e), if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is a non-degenerate solution of $(\mathcal{Q})$ such that $1 \leq \text{ind}_{\mathcal{M}}(u, J) \leq N$, then u is $\mathbb{R}^N$ -foliated Schwarz symmetric.

The result (GNN) has been proved by Gidas, Ni and Nirenberg with the now celebrated moving plane's method [29]. The results (BWW) have been proved by Bartsch, Weth and Willem with the polarizations defined previously [7]. At last, the results (P), (AP) and

(PW) have been proved respectively by Pacella, Aftalion and Pacella, Pacella and Weth [45, 2, 46]. They work all the three with the Morse index defined in Section 2.

Boundary Value Problems For a Class of First Order Quasilinear Ordinary Differential Equations

Using continuation theorems of Leray-Schauder degree theory, we obtain existence results for boundary value Problems $(\mathcal{P})$ mentioned in Section 1. This work has been made in collaboration with J. Mawhin.

6 Introduction

We consider in this paper the boundary value problem

$$(\phi(u))' = f(t, u), \quad u(T) = bu(0), \quad (\mathcal{P}_b)$$

where $\phi : \mathbb{R} \rightarrow (-a, a)$ is an homeomorphism such that $\phi(0) = 0$, $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, a and T are positive real number and b is a real nonzero number. We call solution of this problem any function $u : [0, T] \rightarrow \mathbb{R}$ such that $\phi(u)$ is continuously differentiable, satisfying the two conditions of $(\mathcal{P}_b)$.

Several papers have been recently devoted to the study of the second order version of this problem

$$(\phi(u'))' = f(t, u, u'), \quad l(u, u') = 0, \quad (\mathcal{Q})$$

where $l(u, u') = 0$ denotes the periodic, Neumann or Dirichlet boundary conditions. In [38, 37], one can find a study of the vector case where $\phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is an homeomorphism verifying some monotonicity and coercivity properties, and $f : [0, T] \times \mathbb{R}^{2N} \rightarrow \mathbb{R}^N$ is a Carathéodory function. Are called solution of this problem, any function $u : [0, T] \rightarrow \mathbb{R}^N$ of class $\mathcal{C}^1$ such that $\phi(u')$ is absolutely continuous, satisfying the two conditions of $(\mathcal{Q})$ a.e. on $[0, T]$. For example, they proved that if $p > 1$,

$$\phi(u) = |u|^{p-2}u \text{ if } u \neq 0, \quad \phi(0) = 0,$$

and if f is, in some sense, asymptotically homogeneous, Problem $(\mathcal{Q})$ has at least one solution. For this ϕ , the left-member of $(\mathcal{Q})$ is the so-called p -laplacian of u . In [8, 9], the scalar case with ϕ like in $(\mathcal{P}_b)$ and f continuous is considered, and it is observed that the non-surjectivity of ϕ on $\mathbb{R}$ led to some difficulties. Among other ones the following theorems are proved in [8, 9] :

Theorem 6.1: *If the function f satisfies the following conditions:*

- (1) *There exists $c \in \mathcal{C}([0, T])$ such that $\|c^-\|_{L^1} < \frac{a}{2}$ and $f(t, u, v) \geq c(t)$ for all $(t, u, v) \in [0, T] \times \mathbb{R}^2$,*
- (2) *There exists $R > 0$ and $\varepsilon \in \{\pm 1\}$ such that for all $u \in \mathcal{C}([0, T])$,*

$$\begin{aligned} \varepsilon \int_0^T f(t, u(t), u'(t)) dt &> 0 \quad \text{if} \quad \min u \geq R, \quad \|u'\|_\infty \leq M, \\ \varepsilon \int_0^T f(t, u(t), u'(t)) dt &< 0 \quad \text{if} \quad \max u \leq -R, \quad \|u'\|_\infty \leq M, \end{aligned}$$

$$\text{where } M := \max\{|\phi^{-1}(2\|c^-\|_{L^1})|, |\phi^{-1}(-2\|c^-\|_{L^1})|\},$$

then, problem $(\mathcal{Q})$ with $l(u, u')$ denoting the periodic or Neumann boundary conditions, has at least one solution.

Theorem 6.2: *If the function f satisfies the condition*

$$\exists c > 0, \forall (t, u, v) \in [0, T] \times \mathbb{R}^2 : |f(t, u, v)| \leq c < \frac{a}{2T},$$

then, Problem $(\mathcal{Q})$ with $l(u, u')$ denoting the Dirichlet boundary conditions, has at least one solution.

The authors gave some applications of those theorems with $\phi(u) = u/\sqrt{1+u^2}$. For this ϕ , the left-member of $(\mathcal{Q})$ is the curvature of the graph of u .

Inspired by those results, we study the problem $(\mathcal{P}_b)$ by using similar topological methods based upon Leray-Schauder degree. Let us recall in a few words the main ideas of those methods. The first step consists to write the boundary value problem as a fixed-point problem $\chi(u) = u$ for some completely continuous operator $\chi : \mathcal{C}([0, T]) \rightarrow \mathcal{C}([0, T])$. The second step consists in imbedding the equation $\chi(u) = u$ in a compact family of equations $\zeta(\lambda, u) = u$ (with $\zeta(1, \cdot) = \chi$) indexed by $\lambda \in [0, 1]$. The third step is the search of an open bounded set $\Omega \subset \mathcal{C}([0, T])$ such that

$$\forall \lambda \in [0, 1] : u \in \partial\Omega \Rightarrow \zeta(\lambda, u) \neq u. \quad (33)$$

The last step consists in computing the Leray-Schauder degree of the operator $\text{id} - \zeta(0, \cdot)$ on Ω at 0, and to prove that it is different of 0. The homotopy invariance's property and the existence's property of the Leray-Schauder degree ensure then that χ has a fixed-point in Ω .

In this chapter, space $\mathcal{C}([0, T])$ will be equipped with the uniform norm $\|\cdot\|_\infty$. If $h \in \mathcal{C}([0, T])$, $\bar{h}$ will be denoted for the mean value $\frac{1}{T} \int_0^T h$ of h , and h^+, h^- for respectively $\max\{h, 0\}$ and $\min\{h, 0\}$. At last, if $t \in \mathbb{R}_0$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is injective, $\text{sgn}(t)$ and $\sigma(\phi)$ will be denoted respectively for $t/|t|$, and 1 if ϕ is increasing and -1 if ϕ is decreasing. Finally, let us define

$$B_{\phi, b} : \mathbb{R} \rightarrow \mathbb{R} : u \mapsto \phi(bu) - \phi(u). \quad (34)$$

The main result of this chapter is the following:

Theorem 6.3: *If the function f satisfies the assumptions (i), (ii) (resp. (i'), (ii')),*

(i) *There exists $M_1 < M_2$ such that for all $u \in \mathcal{C}([0, T])$,*

$$\int_0^T f(t, u(t)) dt > \sup B_{\phi, b} \quad \text{if } u \geq M_2,$$

$$\int_0^T f(t, u(t)) dt < \inf B_{\phi, b} \quad \text{if } u \leq M_1,$$

(i') *There exists $M_1 < M_2$ such that for all $u \in \mathcal{C}([0, T])$,*

$$\int_0^T f(t, u(t)) dt > \sup B_{\phi, b} \quad \text{if } u \leq M_1,$$

$$\int_0^T f(t, u(t)) dt < \inf B_{\phi, b} \quad \text{if } u \geq M_2,$$

(ii) $|\phi(M_2) + \sigma(\phi) \int_0^T f^+(t, M_2) dt| < a, \quad |\phi(M_1) + \sigma(\phi) \int_0^T f^-(t, M_1) dt| < a,$

(ii') $|\phi(M_2) - \sigma(\phi) \int_0^T f^-(t, M_2) dt| < a, \quad |\phi(M_1) - \sigma(\phi) \int_0^T f^+(t, M_1) dt| < a,$

then Problem $(\mathcal{P}_b)$ with $b > 0$ has at least one solution.

To prove the difficult property (33), we use a technique introduced in [42] to prove the fourth point of Theorem 3.2 (see Section 2) studying Problem $(\mathcal{R}_\lambda)$ with $b = 1$. We adapted that here for more general boundary conditions than just periodic, by using the

boundedness of ϕ , more precisely, by using the property

$$\text{If } u \text{ solves } (\mathcal{P}_b) \text{ then: } \forall 0 \leq t_1 < t_2 \leq 1 : \left| \int_{t_1}^{t_2} f(t, u(t)) dt \right| < 2a. \quad (35)$$

To be complete in the study of Problem $(\mathcal{P}_b)$, we give two more results, which do not use property (35).

Theorem 6.4: *If there exists a function $c \in L^1([0, T], \mathbb{R}^+)$ such that $\|c\|_{L^1} < a$ and*

$$\forall t \in [0, T], \forall u \in \mathbb{R} : |f(t, u)| \leq c(t),$$

then Problem $(\mathcal{P}_b)$ with $b < 0$ has at least one solution.

Theorem 6.5: *Let us distinguish the six cases:*

$$(a_1) \quad b < -1, \quad (a_2) \quad -1 < b < 0, \quad (a_3) \quad b = -1,$$

$$(b_1) \quad 0 < b < 1, \quad (b_2) \quad 1 < b, \quad (b_3) \quad b = 1,$$

and the six following assumptions:

$$(A_1) \quad \exists M > 0, \quad \forall t \in [0, T] : \sigma(\phi) \cdot f(t, -M) > 0 > \sigma(\phi) \cdot f(t, M),$$

$$(A_2) \quad \exists M > 0, \quad \forall t \in [0, T] : \sigma(\phi) \cdot f(t, -M) < 0 < \sigma(\phi) \cdot f(t, M),$$

$$(A_3) \quad \exists M > 0, \exists \varepsilon \in \{\pm 1\}, \quad \forall t \in [0, T] : \varepsilon \cdot f(t, -M) < 0 < \varepsilon \cdot f(t, M),$$

$$(B_1) \quad \exists M_1 < 0 < M_2, \quad \forall t \in [0, T] : \sigma(\phi) \cdot f(t, M_1) < 0 < \sigma(\phi) \cdot f(t, M_2),$$

$$(B_2) \quad \exists M_1 < 0 < M_2, \quad \forall t \in [0, T] : \sigma(\phi) \cdot f(t, M_1) > 0 > \sigma(\phi) \cdot f(t, M_2),$$

$$(B_3) \quad \exists M_1 < M_2, \exists \varepsilon \in \{\pm 1\}, \quad \forall t \in [0, T] : \varepsilon \cdot f(t, M_1) < 0 < \varepsilon \cdot f(t, M_2).$$

If Assumption (A_i) (resp. (B_i)) is satisfied, then Problem $(\mathcal{P}_b)$ in the case (a_i) (resp. (b_i)) has at least one solution with values in $(-M, M)$ (resp. (M_1, M_2)).

The chapter is organized as follows. We start with the search of the fixed point operator χ in Section 7. Theorem 6.3 is proved in Section 8. We proceed then to the study of Theorems 6.4 and 6.5 in Section 9. At last, Section 10 deals with examples and numerical experiments.

7 Fixed-Point Formulations of the Problem

7.1 Forced Problem We call *forced Problem* the special case of Problem $(\mathcal{P}_b)$ where the right member is a continuous function h independent of u :

$$(\phi(u))' = h(t), \quad u(T) = bu(0). \quad (\mathcal{P}_{b,h})$$

Because

$$\begin{aligned} u(T) = bu(0) &\Leftrightarrow \phi(u(T)) = \phi(bu(0)) \\ &\Leftrightarrow \phi(u(0)) + \int_0^T h(s) ds = \phi(bu(0)) \\ &\Leftrightarrow B_{\phi,b}(u(0)) = \int_0^T h(s) ds \end{aligned}$$

where $B_{\phi,b}(u)$ is defined in (34), we can formulate $(\mathcal{P}_{b,h})$ by:

$$\begin{aligned} \phi(u(t)) &= \phi(B_{\phi,b}^{-1}(\int_0^T h(s) ds)) + \int_0^t h(s) ds, \quad \text{if } b < 0, \\ \phi(u(t)) &= \phi(B_{\phi,b}^{-1}(\int_0^T h(s) ds)) + \int_0^t h(s) ds, \quad \text{if } b > 0, \end{aligned}$$

where “ $B_{\phi,b}^{-1}$ ” in the case $b > 0$ indicates that for this values of b , $B_{\phi,b}$ may be non-injective, and where the injectivity of $B_{\phi,b}$ in the case $b < 0$ is ensured by the fact that $\phi(\cdot)$ and $-\phi$ are simultaneously increasing or decreasing. From this form, we deduce necessary and sufficient conditions on h for ensuring solvability of $(\mathcal{P}_{b,h})$:

$$\begin{cases} \int_0^T h(s) ds \in \text{ran } B_{\phi,b} \\ \forall t \in [0, T] : | \phi(B_{\phi,b}^{-1}(\int_0^T h(s) ds)) + \int_0^t h(s) ds | < a \end{cases} \quad \text{if } b < 0,$$

$$\begin{cases} \int_0^T h(s) ds \in \text{ran } B_{\phi,b} \\ \forall t \in [0, T] : | \phi(“B_{\phi,b}^{-1}”(\int_0^T h(s) ds)) + \int_0^t h(s) ds | < a \end{cases} \quad \text{if } b > 0.$$

Let us specify that $B_{\phi,b}$ being bounded, continuous and defined on $\mathbb{R}$, $\text{ran } B_{\phi,b}$ will be a bounded interval. In particular,

$$\forall b < 0 : \text{ran } B_{\phi,b} = (-2a, 2a) \quad (36)$$

because

$$\begin{cases} \sup B_{\phi,b} = \lim_{u \rightarrow -\infty} B_{\phi,b}(u) = a - (-a) = 2a \\ \inf B_{\phi,b} = \lim_{u \rightarrow \infty} B_{\phi,b}(u) = -a - a = -2a \end{cases} \quad \text{if } \phi \text{ (and so } -B_{\phi,b}) \text{ is increasing,}$$

$$\begin{cases} \sup B_{\phi,b} = \lim_{u \rightarrow -\infty} B_{\phi,b}(u) = a - (-a) = 2a \\ \inf B_{\phi,b} = \lim_{u \rightarrow \infty} B_{\phi,b}(u) = -a - a = -2a \end{cases} \quad \text{if } \phi \text{ (and so } -B_{\phi,b}) \text{ is decreasing.}$$

If $b > 0$, excepted for $b = 1$ where $B_{\phi,b} \equiv 0$, $\text{ran } B_{\phi,b}$ depends of ϕ but we can nevertheless have an idea of the graph of $B_{\phi,b}$. If $b \in (0, \infty) \setminus \{1\}$, then $B_{\phi,b}$ equals zero only at 0 and has opposite signs on the left and on the right of 0. Indeed,

$$B_{\phi,b}(u) = 0 \Leftrightarrow \phi(u) = \phi(bu) \Leftrightarrow u = bu \Leftrightarrow u = 0$$

and

$$\begin{cases} u \cdot B_{\phi,b}(u) < 0 & \text{if } b \in A \\ u \cdot B_{\phi,b}(u) > 0 & \text{if } b \in A^c \end{cases} \Leftrightarrow \begin{cases} u \cdot [\phi(bu) - \phi(u)] < 0 & \text{if } b \in A \\ u \cdot [\phi(bu) - \phi(u)] > 0 & \text{if } b \in A^c \end{cases} \Leftrightarrow$$

$$\begin{cases} \begin{cases} \phi(u) > \phi(bu) & \text{if } u > 0 \\ \phi(u) < \phi(bu) & \text{if } u < 0 \end{cases} & \text{if } b \in A \\ \begin{cases} \phi(u) < \phi(bu) & \text{if } u > 0 \\ \phi(u) > \phi(bu) & \text{if } u < 0 \end{cases} & \text{if } b \in A^c \end{cases} \Leftrightarrow \begin{cases} A = (0, 1) & \text{if } \phi \text{ is increasing} \\ A = (1, \infty) & \text{if } \phi \text{ is decreasing} \end{cases}$$

(where A^c denotes $[(0, \infty) \setminus \{1\}] \setminus A$). Moreover, we have $\|B_{\phi,b}\|_\infty < a$, and

$$\lim_{u \rightarrow \pm\infty} B_{\phi,b}(u) = \lim_{u \rightarrow \pm\infty} \phi(bu) - \phi(u) = 0,$$

which combined with Dini's theorem ensures $\lim_{b \rightarrow 1} \|B_{\phi,b}\|_\infty = 0$.

EXAMPLE 7.1. If $b = -1$, $(\mathcal{P}_b)$ is called *antiperiodic problem*. Because of (36) and the injectivity of $B_{\phi,-1}$, h must satisfy (37) in order to have a solution of $(\mathcal{P}_{b,h})$ which will be given by (38):

$$| \int_0^T h(s) ds | < 2a, \quad \forall t \in [0, T] : | B_{\phi,-1}^{-1}(\int_0^T h(s) ds) + \int_0^t h(s) ds | < a \quad (37)$$

$$u(t) = \phi^{-1}[B_{\phi,-1}^{-1}(\int_0^T h(s) ds) + \int_0^t h(s) ds]. \quad (38)$$

Let us remark that ϕ odd implies $B_{\phi,-1}^{-1}(v) = \phi^{-1}(\frac{-v}{2})$. Condition (37) becomes then (39) where G is given by (40), and Solution (38) becomes (41):

$$|\int_0^T h(s)ds| < 2a, \quad \forall t \in [0, T] : |\int_0^T G(s, t)h(s)ds| < a \quad (39)$$

$$G(s, t) = \frac{1}{2} \text{ if } 0 \leq s \leq t, \quad G(s, t) = \frac{-1}{2} \text{ if } t \leq s \leq T, \quad (40)$$

$$u(t) = \phi^{-1}[\int_0^T G(s, t)h(s)ds]. \quad (41)$$

Hence, (39) is ensured by $\|h\|_{L^1} < 2a$ because we have for all $t \in [0, T]$:

$$|\int_0^T h(s)ds| \leq \int_0^T |h(s)|ds < 2a, \quad (42)$$

$$|\int_0^T G(s, t)h(s)ds| \leq \int_0^T |G(s, t)h(s)|ds = \frac{1}{2} \int_0^T |h(s)|ds < a. \quad (43)$$

EXAMPLE 7.2. If $b = 1$, $(\mathcal{P}_b)$ is called *periodic problem*. In this case, $B_{\phi,1} \equiv 0$ so “ $B_{\phi,1}^{-1}$ ”(0) denotes any real. Hence, h must satisfy (44) in order to have a solution of $(\mathcal{P}_{b,h})$ which will be given by (45):

$$\int_0^T h(s)ds = 0, \quad \exists r \in \mathbb{R}, \forall t \in [0, T] : |\phi(r) + \int_0^t h(s)ds| < a \quad (44)$$

$$u(t) = \phi^{-1}[\phi(r) + \int_0^t h(s)ds]. \quad (45)$$

7.2 Fixed-Point Operators

We can write Problem $(\mathcal{P}_b)$ by the abstract form

$$L_\phi(u) = N_f(u), \quad u \in \mathcal{C}_b([0, T]), \quad (46)$$

where

$$\begin{aligned} L_\phi &: \text{dom } L_\phi \subset \mathcal{C}_b([0, T]) \rightarrow \mathcal{C}([0, T]) : u \mapsto (\phi(u))', \\ N_f &: \mathcal{C}([0, T]) \rightarrow \mathcal{C}([0, T]) : u \mapsto f(., u(.)), \\ \mathcal{C}_b([0, T]) &= \{u \in \mathcal{C}([0, T]) : u(T) = bu(0)\}, \\ \text{dom } L_\phi &= \{u \in \mathcal{C}_b([0, T]) : \phi(u) \in \mathcal{C}^1([0, T])\}. \end{aligned}$$

If $b < 0$, L_ϕ has an inverse which is

$$L_\phi^{-1} : \text{ran } L_\phi \subset \mathcal{C}([0, T]) \rightarrow \mathcal{C}_b([0, T]) : h \mapsto \phi^{-1}[\phi(B_{\phi,b}^{-1}(\int_0^T h(s)ds)) + \int_0^t h(s)ds],$$

and (46) is equivalent to $u = L_\phi^{-1}N_f(u)$ with $u \in \mathcal{C}([0, T])$, hence our problem is finding a fixed-point of the operator

$$\chi_1 := L_\phi^{-1}N_f : \text{dom } L_\phi^{-1}N_f \subseteq \mathcal{C}([0, T]) \rightarrow \mathcal{C}_b([0, T]) \subset \mathcal{C}([0, T]).$$

In the case $b > 0$, we can not do the same because the operator L_ϕ is not injective. Like in [8, 9, 37], themselves inspired by [40], we consider the following operators:

$$P : \mathcal{C}([0, T]) \rightarrow \mathcal{C}([0, T]) : u \mapsto u(0),$$

$$Q : \mathcal{C}([0, T]) \rightarrow \mathcal{C}([0, T]) : h \mapsto \bar{h} = \frac{1}{T} \int_0^T h(s)ds,$$

$$H : \mathcal{C}([0, T]) \rightarrow \mathcal{C}^1([0, T]) \subset \mathcal{C}([0, T]) : u \mapsto \int_0^\cdot u(s)ds.$$

Lemma 7.3: *Let us consider the operator*

$$\chi_2 : \text{dom } \chi_2 \subset \mathcal{C}([0, T]) \rightarrow \mathcal{C}([0, T]) : u \mapsto$$

$$QN_f(u) - \frac{1}{T}B_{\phi,b}(Pu) + \phi^{-1}[\phi(Pu) + H(id - Q)N_f(u) + \frac{t}{T}B_{\phi,b}(Pu)]$$

where t denotes the function which sends t on t . Then u is a solution of Problem $(\mathcal{P}_b)$ if and only if u is a fixed-point of the operator χ_2 .

PROOF. Let $u \in \mathcal{C}([0, T])$, we have the following equivalences:

$$\begin{aligned} & (\phi(u))' = f(t, u), \quad u(T) = bu(0) \\ \Leftrightarrow & \begin{cases} (\phi(u))' = f(t, u) \\ B_{\phi,b}(u(0)) = \int_0^T f(s, u(s)) \, ds \end{cases} \\ \Leftrightarrow & \begin{cases} (\phi(u))' = f(t, u) - [\overline{f(t, u)} - \frac{1}{T}B_{\phi,b}(u(0))] \\ \frac{1}{T} \int_0^T f(s, u(s)) \, ds - \frac{1}{T}B_{\phi,b}(u(0)) = 0 \end{cases} \\ \Leftrightarrow & \begin{cases} \phi(u(t)) = \phi(u(0)) + \int_0^t (f(s, u(s)) - [\overline{f(t, u)} - \frac{1}{T}B_{\phi,b}(u(0))]) \, ds \\ \frac{1}{T} \int_0^T f(s, u(s)) \, ds - \frac{1}{T}B_{\phi,b}(u(0)) = 0 \end{cases} \\ \Leftrightarrow & \begin{cases} u = \phi^{-1}[\phi(Pu) + H(id - Q)N_f(u) + \frac{t}{T}B_{\phi,b}(Pu)] \\ QN_f(u) - \frac{1}{T}B_{\phi,b}(Pu) = 0 \end{cases} \\ \Leftrightarrow & u = QN_f(u) - \frac{1}{T}B_{\phi,b}(Pu) + \phi^{-1}[\phi(Pu) + H(id - Q)N_f(u) + \frac{t}{T}B_{\phi,b}(Pu)]. \end{aligned}$$

□

REMARK 7.4. We shall also consider in the sequel the case where ϕ is onto on $\mathbb{R}$. In this case, we can well define $B_{\phi,b}$ which has the same monotonicity than in the case studied up if $b < 0$ with this time

$$\sup B_{\phi,b} = \infty, \quad \inf B_{\phi,b} = -\infty,$$

and with the same sign component if $b > 0$ but without necessarily this time the property to converge to 0 in $\pm\infty$ because

$$\lim_{u \rightarrow \pm\infty} B_{\phi,b}(u) = (\pm\infty) - (\pm\infty).$$

Hence, in the case $b < 0$, we can again consider Operators $L_\phi, L_\phi^{-1}, \chi_1$ and χ_2 , which are, this time, defined everywhere.

8 The Main Result

In this section, we shall prove the main result of the paper using the particularity (35) of Problem $(\mathcal{P}_b)$.

Theorem 8.1: *If the function f satisfies Assumptions (i),(ii) (resp. (i'),(ii')),*

(i) *There exists $M_1 < M_2$ such that for all $u \in \mathcal{C}([0, T])$,*

$$\begin{aligned} \int_0^T f(t, u(t)) \, dt &> \sup B_{\phi,b} \quad \text{if } u \geq M_2, \\ \int_0^T f(t, u(t)) \, dt &< \inf B_{\phi,b} \quad \text{if } u \leq M_1, \end{aligned}$$

(i') There exists $M_1 < M_2$ such that for all $u \in \mathcal{C}([0, T])$,

$$\int_0^T f(t, u(t)) dt > \sup B_{\phi, b} \quad \text{if } u \leq M_1,$$

$$\int_0^T f(t, u(t)) dt < \inf B_{\phi, b} \quad \text{if } u \geq M_2,$$

(ii) $|\phi(M_2) + \sigma(\phi) \int_0^T f^+(t, M_2) dt| < a, \quad |\phi(M_1) + \sigma(\phi) \int_0^T f^-(t, M_1) dt| < a,$

(ii') $|\phi(M_2) - \sigma(\phi) \int_0^T f^-(t, M_2) dt| < a, \quad |\phi(M_1) - \sigma(\phi) \int_0^T f^+(t, M_1) dt| < a,$

then Problem $(\mathcal{P}_b)$ with $b > 0$ has at least one solution.

REMARK 8.2. Under condition (i) (resp. (i')), condition (ii) (resp. (ii')) induces necessary condition (47) (resp. (48)):

$$|\phi(M_2) + \sigma(\phi) \sup B_{\phi, b}| < a, \quad |\phi(M_1) + \sigma(\phi) \inf B_{\phi, b}| < a, \quad (47)$$

$$|\phi(M_2) - \sigma(\phi) \inf B_{\phi, b}| < a, \quad |\phi(M_1) - \sigma(\phi) \sup B_{\phi, b}| < a. \quad (48)$$

Because of (36), we understand why we do not work with the case $b < 0$ for this theorem.

REMARK 8.3. In this paper we have decided to work directly with Problem $(\mathcal{P}_b)$ and not use the reformulation (49) when ϕ is a diffeomorphism, or (50) where $v := \phi(u)$:

$$u' = f(t, u) \cdot (\phi'(u))^{-1} =: g(t, u), \quad u(T) = bu(0), \quad (49)$$

$$v' = f(t, \phi^{-1}[v]) =: h(t, u), \quad \phi^{-1}[v(T)] = b\phi^{-1}[v(0)], \quad (50)$$

Those problems are more classical but, in the second one, we have nonlinear boundary conditions (except in the periodical case), and in the first one, working with g depending of f and ϕ' has not seemed to us more interesting than directly deal with f and ϕ .

8.1 A Priori Bound for Problem $(\mathcal{P}_b)$

Lemma 8.4: *Under Assumptions of Theorem 8.1, there exists an a priori bound for the solutions of Problem $(\mathcal{P}_b)$ with $b > 0$.*

PROOF. Let us begin by the case ϕ increasing and with the conditions (i),(ii). If u is a solution of Problem $(\mathcal{P}_b)$, the boundary conditions lead to

$$\int_0^T f(s, u(s)) ds = B_{\phi, b}(u(0))$$

which implies following (i) the existence of $t_0 \in [0, T]$ verifying $M_1 < u(t_0) < M_2$. Let us consider $\tau \in [0, T]$ such that $u(\tau) \notin [M_1, M_2]$. If such a τ does not exist, the proof is finished, so let us suppose that it exists. We shall consider four configurations

$$(1) \quad u(\tau) > M_2, \quad \tau < t_0, \quad (2) \quad u(\tau) > M_2, \quad \tau > t_0,$$

$$(3) \quad u(\tau) < M_1, \quad \tau < t_0, \quad (4) \quad u(\tau) < M_1, \quad \tau > t_0,$$

for which we shall prove

$$(I_1) \quad \phi(u(\tau)) \leq \phi(M_2) + \int_{[\tau, \sigma]^c} f(t, M_2) dt - \sup B_{\phi, b},$$

$$(I_2) \quad \phi(u(\tau)) \leq \phi(M_2) + \int_{[\sigma, \tau]} f(t, M_2) dt + B_{\phi, b}(u(0)) - \sup B_{\phi, b},$$

$$(I_3) \quad \phi(u(\tau)) \geq \phi(M_1) + \int_{[\tau, \sigma]^c} f(t, M_1) dt - \inf B_{\phi, b},$$

$$(I_4) \quad \phi(u(\tau)) \geq \phi(M_1) + \int_{[\sigma, \tau]} f(t, M_1) dt + B_{\phi, b}(u(0)) - \inf B_{\phi, b}.$$

We shall only treat the case (1) (resp. (2)) because the case (3) (resp. (4)) is exactly the same. In the case (1), there exists $\sigma \in [0, T]$ such that

$$\tau < \sigma < t_0, \quad \tau \leq t < \sigma \Rightarrow u(t) > M_2, \quad u(\sigma) = M_2.$$

Let us consider the function ξ defined on $[0, T]$ by

$$\xi(t) := \begin{cases} M_2 & \text{if } t \notin [\tau, \sigma], \\ u(t) & \text{if } t \in [\tau, \sigma]. \end{cases} \quad (51)$$

Like $\xi \geq M_2$, we can define the following functions ξ_n defined on $[0, T]$ by

$$\xi_n(t) := \begin{cases} M_2 & \text{if } 0 \leq t \leq \tau - \frac{1}{n}, \\ M_2 + n(u(\tau) - M_2)(t - \tau + \frac{1}{n}) & \text{if } \tau - \frac{1}{n} \leq t \leq \tau, \\ u(t) & \text{if } \tau \leq t \leq \sigma, \\ M_2 & \text{if } \sigma \leq t \leq T. \end{cases}$$

and verifying

$$\xi_n \in \mathcal{C}([0, T]), \quad \xi_n \geq M_2, \quad \int_0^T f(t, \xi_n(t)) dt > \sup B_{\phi, b}.$$

Lebesgue's Dominated Convergence Theorem leads to $\int_0^T f(t, \xi(t)) dt \geq \sup B_{\phi, b}$ ensuring (I_1) . In the case (2), like if there exists $\tau < t'_0 \leq T$ such that $u(t'_0) \in [M_1, M_2]$ we are in the previous case, we shall suppose that such t'_0 does not exist. On the other hand, there exists $\sigma \in [0, T]$ verifying (52) or (53):

$$0 < \sigma < \tau, \quad 0 \leq t < \sigma \Rightarrow u(t) > M_2, \quad u(\sigma) = M_2, \quad (52)$$

$$0 = \sigma, \quad u(0) = u(\sigma) \leq M_2. \quad (53)$$

Let us consider a function ξ like previously, with M_2 and $u(t)$ interverted in (51). We have again $\int_0^T f(t, \xi(t)) dt \geq \sup B_{\phi, b}$ leading to (I_2) (by noting that $\phi(u(\sigma)) \leq \phi(M_2)$ because ϕ is increasing).

By (ii), Configurations (1), (2) (resp. (3), (4)) ensure (54) (resp. (55)) for all τ such that $u(\tau) > M_2$ (resp. $u(\tau) < M_1$), leading to (56) for all $t \in [0, T]$ so the thesis:

$$\phi(u(\tau)) \leq K_2 := \phi(M_2) + \int_0^T f^+(t, M_2) dt < a \quad (54)$$

$$\phi(u(\tau)) \geq K_1 := \phi(M_1) + \int_0^T f^-(t, M_1) dt > -a. \quad (55)$$

$$-\infty < K'_1 := \phi^{-1}[K_1] \leq u(t) \leq \phi^{-1}[K_2] =: K'_2 < \infty, \quad (56)$$

With Conditions (i'), (ii'), we have again four configurations:

$$(1') \quad u(\tau) > M_2, \quad \tau > t_0, \quad (2') \quad u(\tau) > M_2, \quad \tau < t_0,$$

$$(3') \quad u(\tau) < M_1, \quad \tau > t_0, \quad (4') \quad u(\tau) < M_1, \quad \tau < t_0,$$

that we can treat by the same reasoning as their correspondent. If ϕ is decreasing, we can put $\psi := -\phi$ and $g := -f$, consider Problem $(\mathcal{P}_b)$ with ϕ, f replaced by ψ, g , whose Solutions coincide with Solutions of original Problem $(\mathcal{P}_b)$, remark that $B_{\psi, b} = -B_{\phi, b}$ so

$$\sup B_{\psi, b} = -\inf B_{\phi, b}, \quad \inf B_{\psi, b} = -\sup B_{\phi, b},$$

and that ψ is increasing, and use the same proof as previously with conditions (i'), (ii') (resp. (i), (ii)) for this new problem if we worked with assumptions (i), (ii) (resp. (i'), (ii')) for the initial problem. $\square$

8.2 A Modified Problem

Let us consider

$$\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R} : u \mapsto \begin{cases} -\sigma(\phi)\kappa \log(-u-A+1) + \phi(-A) & \text{if } u \leq -A \\ \phi(u) & \text{if } -A < u < A \\ \sigma(\phi)\kappa \log(u-A+1) + \phi(A) & \text{if } A \leq u, \end{cases}$$

with $A > K$, K an a priori bound of the solutions of $(\mathcal{P}_b)$, and κ a positif constant that we shall fix later. $\tilde{\phi}$ is an homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. We can study Problem

$$(\tilde{\phi}(u))' = f(t, u), \quad u(T) = bu(0), \quad (\tilde{\mathcal{P}}_b)$$

corresponding to the research of fixed-points of the operator

$$\tilde{\chi}_2 : \mathcal{C}([0, T]) \rightarrow \mathcal{C}([0, T]) : u \mapsto QN_f(u) - \frac{1}{T}B_{\tilde{\phi},b}(Pu) + \tilde{\phi}^{-1}[\tilde{\phi}(Pu) + H(id - Q)N_f(u) + \frac{t}{T}B_{\tilde{\phi},b}(Pu)],$$

the proof of this fact being a consequence of Remark 7.4. This operator is defined on all $\mathcal{C}([0, T])$.

Lemma 8.5: *Under Assumptions of Theorem 8.1, and if κ is small enough, the solutions of Problem $(\tilde{\mathcal{P}}_b)$ with $b > 0$ take values in $[-K, K]$ with K defined above.*

PROOF. We will suppose in this proof that ϕ is increasing and that $b \in (0, 1)$; the proof is similar in the other cases. If we want to apply the same idea as in the proof of Lemma 8.4, we must have

$$\text{ran } B_{\tilde{\phi},b} \subseteq \text{ran } B_{\phi,b}. \quad (57)$$

Inasmuch as

$$B_{\tilde{\phi},b}(u) = \tilde{\phi}(bu) - \tilde{\phi}(u), \quad B_{\phi,b}(u) = \phi(bu) - \phi(u),$$

and as $\tilde{\phi}$ coincide with ϕ on $(-A, A)$, we have

$$|B_{\tilde{\phi},b}(u)| = \begin{cases} \kappa \left| \log\left(\frac{b|u|-A+1}{|u|-A+1}\right) \right| & \text{if } |u| \geq \frac{1}{b}A, \\ \kappa \log(u-A+1) + \phi(A) - \phi(bu) & \text{if } A \leq u < \frac{1}{b}A, \\ -[-\kappa \log(-u-A+1) + \phi(-A) - \phi(bu)] & \text{if } -\frac{1}{b}A < u \leq -A, \\ |B_{\phi,b}(u)| & \text{if } |u| < A. \end{cases}$$

So, if κ is chosen small enough such that

$$\begin{cases} \kappa \log(u-A+1) + \phi(A) \leq \phi(u) & \text{if } A \leq u < \frac{1}{b}A, \\ -\kappa \log(-u-A+1) + \phi(-A) \geq \phi(u) & \text{if } -\frac{1}{b}A < u \leq -A, \end{cases}$$

what is possible, we have

$$|B_{\tilde{\phi},b}(u)| \leq \begin{cases} \kappa B & \text{if } |u| \geq \frac{1}{b}A, \\ |B_{\phi,b}(u)| & \text{if } |u| < \frac{1}{b}A, \end{cases}$$

where

$$B := \sup\left\{\left|\log\left(\frac{b|u|-A+1}{|u|-A+1}\right)\right| : |u| \geq \frac{1}{b}A\right\} < \infty,$$

and so we obtain (57) if we reduce κ for that $[-\kappa B, \kappa B] \subseteq \text{ran } B_{\phi,b}$. $\square$

Corollary 8.6: *Let $b > 0$ and $u \in \mathcal{C}([0, T])$. u is a solution of Problem $(\mathcal{P}_b)$ if and only if u is a solution of Problem $(\tilde{\mathcal{P}}_b)$.*

8.3 Imbedding in a Family of Problems Let us consider the family of problems $(\tilde{\mathcal{P}}_{b,\lambda})$ corresponding to the research of the couples (λ, u) such that $u = \zeta(\lambda, u)$, with ζ defined by (58):

$$\begin{aligned} (\tilde{\phi}(u))' &= \lambda f(t, u) + (1 - \lambda) \frac{1}{T} \int_0^T f(s, u(s)) ds, \quad u(T) = bu(0), \quad (\tilde{\mathcal{P}}_{b,\lambda}) \\ \zeta : [0, 1] \times \mathcal{C}([0, T]) &\rightarrow \mathcal{C}([0, T]) : (\lambda, u) \mapsto \\ QN_f(u) - \frac{1}{T} B_{\tilde{\phi},b}(Pu) + \tilde{\phi}^{-1}[\tilde{\phi}(Pu) + \lambda H(id - Q)N_f(u) + \frac{\mathbf{t}}{T} B_{\tilde{\phi},b}(Pu)] \end{aligned} \quad (58)$$

Lemma 8.7: *Under Assumptions of Theorem 8.1, there exists a priori bounds for the solutions of Problem $(\tilde{\mathcal{P}}_{b,\lambda})$ uniform on $\lambda \in [0, 1]$.*

PROOF. The proof of this result is very close to the proof of Lemma 8.4. We shall just work in the case ϕ increasing and with Assumptions (i),(ii).

Let us begin by the case $\lambda \neq 0$. If u is a solution of Problem $(\tilde{\mathcal{P}}_{b,\lambda})$, the boundary conditions lead to

$$\int_0^T f(s, u(s)) ds = B_{\tilde{\phi},b}(u(0))$$

what implies, by (i), the existence of $t_0 \in [0, T]$ verifying $M_1 < u(t_0) < M_2$. Let us consider $\tau \in [0, T]$ such that $u(\tau) \notin [M_1, M_2]$. If such a τ does not exist, the proof is finished, so let us suppose that it exists. We have again the configurations (1) – (4) mentioned in Lemma 8.4 for which we can prove

$$\begin{aligned} (I_1) \quad \tilde{\phi}(u(\tau)) &\leq \tilde{\phi}(M_2) + \lambda \int_{[0,\tau] \cup [\sigma,T]} f(t, M_2) dt - (1 - \lambda) B_{\tilde{\phi},b}(u(0)) \frac{\sigma - \tau}{T} - \lambda \sup B_{\tilde{\phi},b}, \\ (I_2) \quad \tilde{\phi}(u(\tau)) &\leq \tilde{\phi}(M_2) + \lambda \int_{[\sigma,\tau]} f(t, M_2) dt + B_{\tilde{\phi},b}(u(0)) - (1 - \lambda) B_{\tilde{\phi},b}(u(0)) \frac{\sigma + T - \tau}{T} \\ &\quad - \lambda \sup B_{\tilde{\phi},b}, \\ (I_3) \quad \tilde{\phi}(u(\tau)) &\geq \tilde{\phi}(M_1) + \lambda \int_{[0,\tau] \cup [\sigma,T]} f(t, M_1) dt - (1 - \lambda) B_{\tilde{\phi},b}(u(0)) \frac{\sigma - \tau}{T} - \lambda \inf B_{\tilde{\phi},b}, \\ (I_4) \quad \tilde{\phi}(u(\tau)) &\geq \tilde{\phi}(M_1) + \lambda \int_{[\sigma,\tau]} f(t, M_1) dt + B_{\tilde{\phi},b}(u(0)) - (1 - \lambda) B_{\tilde{\phi},b}(u(0)) \frac{\sigma + T - \tau}{T} \\ &\quad - \lambda \inf B_{\tilde{\phi},b}. \end{aligned}$$

We shall only consider the case (2), other ones being similar or simpler. Like in Lemma 8.4, let us suppose that there does not exist $\tau < t'_0 \leq T$ such that $u(t'_0) \in [M_1, M_2]$, observe that there exists $\sigma \in [0, T]$ such that (52) or (53), and consider the same function ξ which verifies again

$$\int_0^T f(t, \xi(t)) dt \geq \sup B_{\tilde{\phi},b}.$$

By definition of ξ and because $\tilde{\phi}(u(\sigma)) \leq \tilde{\phi}(M_2)$ (like $\tilde{\phi}$ is increasing), we deduce (I_2) .

Configurations (1),(2) (resp. (3),(4)) ensure (59) (resp. (60)) for all τ such that $u(\tau) > M_2$ (resp. $u(\tau) < M_1$), leading to (61) for all $t \in [0, T]$ so the thesis:

$$\tilde{\phi}(u(\tau)) \leq \tilde{\phi}(M_2) + \int_0^T f^+(t, M_2) dt + \sup B_{\tilde{\phi},b} + |\inf B_{\tilde{\phi},b}| =: \tilde{K}_2, \quad (59)$$

$$\tilde{\phi}(u(\tau)) \geq \tilde{\phi}(M_1) + \int_0^T f^-(t, M_1) dt - \sup B_{\tilde{\phi},b} - |\inf B_{\tilde{\phi},b}| =: \tilde{K}_1. \quad (60)$$

$$-\infty < \tilde{K}'_1 := \tilde{\phi}^{-1}[\tilde{K}_1] \leq u(t) \leq \tilde{\phi}^{-1}[\tilde{K}_2] =: \tilde{K}'_2 < \infty. \quad (61)$$

For the case $\lambda = 0$, because we have

$$(\tilde{\phi}(u))' = \frac{1}{T} \int_0^T f(s, u(s)) ds = \frac{1}{T} B_{\tilde{\phi},b}(u(0)) =: c \in \left(\frac{-a}{T}, \frac{a}{T} \right),$$

and because there exists $t_0 \in [0, T]$ such that $M_1 < u(t_0) < M_2$, we can write (62) and hence deduce (63) for all $t \in [0, T]$, ensuring the thesis:

$$\tilde{\phi}(u(t)) = \tilde{\phi}(u(t_0)) + c(t - t_0), \quad (62)$$

$$\tilde{\phi}(M_1) - a < \tilde{\phi}(u(t)) < \tilde{\phi}(M_2) + a. \quad (63)$$

□

8.4 Existence Result for the Modified Problem In order to prove that Problem $(\mathcal{P}_b)$ has at least one solution, we shall compute de Leray-Schauder degree of $id - \tilde{\chi}_2$ on $B(0, r)$, where r is strictly greater than the a priori bounds of the last lemma. By noting that $\tilde{\chi}_2 = \zeta(1, \cdot)$ and

$$\deg_{\text{LS}}(id - \zeta(1, \cdot), B(0, r)) = \deg_{\text{LS}}(id - \zeta(0, \cdot), B(0, r)),$$

we have to prove $\deg_{\text{LS}}(id - \zeta(0, \cdot), B(0, r)) \neq 0$, i.e., that Problem (64) has a “strong” solution in the sense of degree theory:

$$(\tilde{\phi}(u))' = \frac{1}{T} \int_0^T f(s, u(s)) ds, \quad u(T) = bu(0). \quad (64)$$

Let us note that in order to follow this way, we need some compactness of ζ . This fact being very classical, we shall just recall it without proof.

Lemma 8.8: *The operator $\zeta : [0, 1] \times \mathcal{C}([0, T]) \rightarrow \mathcal{C}([0, T])$ is completely continuous.*

The thesis about (64) is easy in the periodic case because then, a solution u must be a constant and we can use the Brouwer degree for functions of one variable. In the general case, we shall use a new deformation:

$$(\tilde{\phi}(u))' = \frac{\lambda}{T} \int_0^T f(s, u(s)) ds, \quad \int_0^T f(t, u(t)) dt = \tilde{\phi}(bu(0)) - \tilde{\phi}(u(0)). \quad (65)$$

Lemma 8.9: *Under Assumptions of Theorem 8.1, we have $\deg_{\text{LS}}(id - \zeta(0, \cdot), B(0, r)) \neq 0$.*

PROOF. Let ξ the following operator for which $\xi(1, \cdot)$ coincide with $\zeta(0, \cdot)$:

$$\begin{aligned} \xi : [0, 1] \times \mathcal{C}([0, T]) &\rightarrow \mathcal{C}([0, T]) : (\lambda, u) \mapsto \\ QN_f(u) - \frac{1}{T} B_{\tilde{\phi}, b}(Pu) + \tilde{\phi}^{-1}[\tilde{\phi}(Pu) + \lambda \frac{\mathbf{t}}{T} B_{\tilde{\phi}, b}(Pu)]. \end{aligned}$$

If u is a fixed point of $\xi(\lambda, \cdot)$, by evaluation of u at 0, we have $QN_f(u) = \frac{1}{T} B_{\tilde{\phi}, b}(Pu)$ and we can then easily conclude (65). To deduce the existence of a $r > 0$ such that $\|u\|_\infty < r$, we proceed like in proof of Lemma 8.7 (in the case $\lambda = 0$).

Because ξ is completely continuous, we can compute the Leray-Schauder degree of $id - \xi(\lambda, \cdot)$ on $B(0, r)$ and conclude:

$$\begin{aligned} \deg_{\text{LS}}(id - \xi(1, \cdot), B(0, r)) &= \deg_{\text{LS}}(id - \xi(0, \cdot), B(0, r)) \\ &= \deg_{\text{LS}}(id - QN_f + \frac{1}{T} B_{\tilde{\phi}, b}P - P, B(0, r)) = \deg_{\text{B}}(\frac{1}{T} B_{\tilde{\phi}, b} - QN_f, (-r, r)) \\ &= \deg_{\text{B}}(\frac{1}{T} [B_{\tilde{\phi}, b} - \int_0^T f(s, \cdot) ds], (-r, r)) = \pm 1 \neq 0, \end{aligned}$$

where the last equality comes from Assumption (ii) and the fact that

$$-r < -\tilde{K}' < M_1 < M_2 < \tilde{K}' < r$$

where $\tilde{K}'$ is a bound of the solutions of $(\mathcal{P}_{b, \lambda})$.

□

9 Other Results

We shall give other results about Problem $(\mathcal{P}_b)$, which do not use the particularity (35). In the proof of those results we use an extension of ϕ which does not need to be as sophisticated as the one used in the previous section, for example

$$\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R} : u \mapsto \begin{cases} \sigma(\phi)(u+A) + \phi(-A) & \text{if } u \leq -A, \\ \phi(u) & \text{if } -A < u < A, \\ \sigma(\phi)(u-A) + \phi(A) & \text{if } A \leq u. \end{cases}$$

with $A > 0$ fixed.

9.1 Results with a Priori Bounds

Theorem 9.1: *If there exists a function $c \in L^1([0, T], \mathbb{R}^+)$ such that $\|c\|_{L^1} < a$ and*

$$\forall t \in [0, T], \forall u \in \mathbb{R} : |f(t, u)| \leq c(t),$$

then Problem $(\mathcal{P}_b)$ with $b < 0$ has at least one solution.

PROOF. Under the assumptions of the theorem, the operator χ_1 defined in Section 7 is bounded. Indeed, if $v = \chi_1(u)$ then $v(T) = bv(0)$ and so v has at least one zero t_0 . We have therefore

$$\phi(v(t)) = \phi(v(t_0)) + \int_{t_0}^t f(s, u(s)) \, ds = \int_{t_0}^t f(s, u(s)) \, ds$$

because $\phi(0) = 0$, and hence

$$|\phi(v(t))| = \left| \int_{t_0}^t f(s, u(s)) \, ds \right| \leq \int_{t_0}^t |f(s, u(s))| \, ds \leq \|c\|_{L^1} < a$$

which implies

$$\|v\|_{\infty} \leq \max\{|\phi^{-1}[\pm\|c\|_{L^1}]|\} =: K < \infty.$$

Let us consider now the extension $\tilde{\phi}$, explained previously, with $A > K$. By the same reasoning, the operator $\tilde{\chi}_1$ (see Remark 7.4) is bounded with the same bound. This boundedness jointed to its definition implies that its fixed points coincide with the ones of χ_1 . The Schauder's fixed point theorem leads to the conclusion. $\square$

Let us notice that we can improve the bound of f in the previous theorem by a factor 2 by supposing that ϕ is odd and $b = -1$.

Theorem 9.2: *If ϕ is odd and if there exists a function $c \in L^1([0, T], \mathbb{R}^+)$ such that $\|c\|_{L^1} < 2a$ and*

$$\forall t \in [0, T], \forall u \in \mathbb{R} : |f(t, u)| \leq c(t),$$

then Problem $(\mathcal{P}_b)$ with $b = -1$ has at least one solution.

PROOF. Let us remark that Inequalities (39), (42), (43) of Example 7.1, jointed to the assumptions of the theorem, imply that the operator χ_1 is defined on all $\mathcal{C}([0, T])$ by

$$\chi_1(u) = \phi^{-1}\left(\int_0^T G(s, \cdot) f(s, u(s)) \, ds\right)$$

(with the function G defined in Example 7.1). χ_1 is again bounded because if $u \in \mathcal{C}([0, T])$ then

$$\left| \int_0^T G(s, t) f(s, u(s)) ds \right| \leq \int_0^T |G(s, t) f(s, u(s))| ds \leq \frac{\|c\|_{L^1}}{2} < a$$

and so

$$\|\chi_1(u)\|_\infty = \|\phi^{-1}[\int_0^T G(s, \cdot) f(s, u(s)) ds]\|_\infty \leq \|\phi^{-1}[\frac{\|c\|_{L^1}}{2}]\| =: K.$$

We can then conclude like in the previous theorem without use an extension of ϕ . $\square$

REMARK 9.3. If ϕ is a diffeomorphism, we can rewrite Problem $(\mathcal{P}_b)$ in the form (66) and give an analog result as the two previous one, with the assumption (67):

$$u' = f(t, u) \cdot (\phi'(u))^{-1}, \quad u(T) = bu(0), \quad (66)$$

$$\exists c \in L^1([0, T], \mathbb{R}^+), \forall t \in [0, T] : \left| \frac{f(t, u)}{\phi'(u)} \right| \leq c(t). \quad (67)$$

Because $\lim_{u \rightarrow \infty} \phi'(u) = 0$ this condition is restrictive; in fact, it is as restrictive as the conditions $\|c\|_{L^1} < a$ or $\|c\|_{L^1} < 2a$ of our theorems.

9.2 Results without a Priori Bounds

Theorem 9.4: *Let us distinguish the six cases:*

$$(a_1) \quad b < -1, \quad (a_2) \quad -1 < b < 0, \quad (a_3) \quad b = -1, \\ (b_1) \quad 0 < b < 1, \quad (b_2) \quad 1 < b, \quad (b_3) \quad b = 1,$$

and the six following assumptions:

$$(A_1) \quad \exists M > 0, \quad \forall t \in [0, T] : \quad \sigma(\phi) \cdot f(t, -M) > 0 > \sigma(\phi) \cdot f(t, M), \\ (A_2) \quad \exists M > 0, \quad \forall t \in [0, T] : \quad \sigma(\phi) \cdot f(t, -M) < 0 < \sigma(\phi) \cdot f(t, M), \\ (A_3) \quad \exists M > 0, \exists \varepsilon \in \{\pm 1\}, \quad \forall t \in [0, T] : \quad \varepsilon \cdot f(t, -M) < 0 < \varepsilon \cdot f(t, M), \\ (B_1) \quad \exists M_1 < 0 < M_2, \quad \forall t \in [0, T] : \quad \sigma(\phi) \cdot f(t, M_1) < 0 < \sigma(\phi) \cdot f(t, M_2), \\ (B_2) \quad \exists M_1 < 0 < M_2, \quad \forall t \in [0, T] : \quad \sigma(\phi) \cdot f(t, M_1) > 0 > \sigma(\phi) \cdot f(t, M_2), \\ (B_3) \quad \exists M_1 < M_2, \exists \varepsilon \in \{\pm 1\}, \quad \forall t \in [0, T] : \quad \varepsilon \cdot f(t, M_1) < 0 < \varepsilon \cdot f(t, M_2).$$

If Assumption (A_i) (resp. (B_i)) is satisfied, then Problem $(\mathcal{P}_b)$ in the case (a_i) (resp. (b_i)) has at least one solution with values in $(-M, M)$ (resp. (M_1, M_2)).

PROOF. Let us consider the family of problems $(\mathcal{P}_{b, \lambda})$ (with Condition $u(T) = u(0)$ joined) in the (b_3) -case, and $(\mathcal{P}'_{b, \lambda})$ in any other case:

$$(\phi(u))' = \lambda f(t, u) + (1 - \lambda) \int_0^T f(s, u(s)) ds, \quad \lambda \in [0, 1], \quad (\mathcal{P}_{b, \lambda})$$

$$(\phi(u))' = \lambda f(t, u), \quad u(T) = bu(0), \quad \lambda \in [0, 1]. \quad (\mathcal{P}'_{b, \lambda})$$

Let us remark that in the periodic case we have $\int_0^T f(s, u(s)) ds = 0$ and so u verifies also the equation $(\phi(u))' = \lambda f(t, u)$ on $[0, T]$.

If u is a solution of those problems with values in $[-M, M]$ (resp. $[M_1, M_2]$) in the case (a_i) (resp. (b_i)), then u takes its values in $(-M, M)$ (resp. (M_1, M_2)). Indeed, if it is not true, u has at least one value, say $u(t_0)$, on $\partial[-M, M]$ (resp. $\partial[M_1, M_2]$), say $-M$ (resp. M_1).

Let us begin with the case $\lambda \neq 0$. If $t_0 \in (0, T)$, by monotonicity of ϕ , we have the following equation which is in contradiction with the assumption:

$$f(t_0, -M) = \lambda^{-1}(\phi(u))'(t_0) = 0,$$

$$(\text{resp. } f(t_0, M_1) = \lambda^{-1}(\phi(u))'(t_0) = 0).$$

If $t_0 \in \partial[0, T]$, we have to consider the problems separately. We shall just examine the antiperiodic case (a_3) and the periodic case (b_3) , the other ones being similar. In the case (b_3) , t_0 is 0 and T and we conclude the contradiction by 68, and in the case (a_3) , say $t_0 = 0$, the boundary conditions lead to 69 ensuring the contradiction:

$$\begin{aligned} \sigma(\phi) \cdot f(T, M_1) &= \sigma(\phi) \lambda^{-1}(\phi(u))'(T) \leq 0 \\ &\leq \sigma(\phi) \lambda^{-1}(\phi(u))'(0) = \sigma(\phi) \cdot f(0, M_1) \end{aligned} \quad (68)$$

$$\begin{aligned} \sigma(\phi) \cdot f(0, -M) &= \sigma(\phi) \lambda^{-1}(\phi(u))'(0) \geq 0, \\ \sigma(\phi) \cdot f(T, M) &= \sigma(\phi) \lambda^{-1}(\phi(u))'(T) \geq 0. \end{aligned} \quad (69)$$

In the case $\lambda = 0$, u is the constant $-M$ (resp. M_1) which is a contradiction with the boundary conditions in all cases except the periodic one for which $\int_0^T f(s, M_1) ds = 0$ ensures the contradiction with the assumption.

We can work with the previous extension of ϕ where A is chosen such that $[-A, A]$ contains strictly $[-M, M]$ (resp. $[M_1, M_2]$), and with the correspondent problems $(\tilde{\mathcal{P}}_{b,\lambda})$ and $(\tilde{\mathcal{P}}'_{b,\lambda})$ whose solutions with values in $[-M, M]$ (resp. $[M_1, M_2]$) take in fact their values in $(-M, M)$ (resp. (M_1, M_2)) for the same reason as above, and whose solutions are also solutions of Problems $(\mathcal{P}_{b,\lambda})$ or $(\mathcal{P}'_{b,\lambda})$ by definition of $\tilde{\phi}$.

We can consider the Leray-Schauder degree of $\text{id} - \zeta(\lambda, \cdot)$ on Ω defined by (70) and ζ defined by (71) in the case (b_3) , and (72) in the other ones:

$$\begin{aligned} \Omega &= \{u \in \mathcal{C}([0, T]) : -M < u < M\} \\ (\text{resp. } \Omega &= \{u \in \mathcal{C}([0, T]) : M_1 < u < M_2\}), \end{aligned} \quad (70)$$

$$\begin{aligned} \zeta : [0, 1] \times \mathcal{C}([0, T]) &\rightarrow \mathcal{C}([0, T]) : u \mapsto \\ QN_f(u) + \tilde{\phi}^{-1}[\tilde{\phi}(Pu) + \lambda H(id - Q)N_f(u)], \end{aligned} \quad (71)$$

$$\begin{aligned} \zeta : [0, 1] \times \mathcal{C}([0, T]) &\rightarrow \mathcal{C}([0, T]) : u \mapsto \\ \lambda QN_f(u) - \frac{1}{T}B_{\phi,b}(Pu) + \tilde{\phi}^{-1}[\tilde{\phi}(Pu) + \lambda H(id - Q)N_f(u) + \frac{t}{T}B_{\tilde{\phi},b}(Pu)]. \end{aligned} \quad (72)$$

Since Condition (33) leads to $\deg_{\text{LS}}(\text{id} - \zeta(1, \cdot), \Omega) = \deg_{\text{LS}}(\text{id} - \zeta(\cdot, 0), \Omega)$, we conclude the proof in the case (b_3) by the computation of $\deg_{\text{LS}}(\text{id} - \zeta(\cdot, 0), \Omega)$:

$$\deg_{\text{LS}}(\text{id} - QN_f - P, \Omega) = \deg_{\text{B}}(-\frac{1}{T} \int_0^T f(s, \cdot) ds, (M_1, M_2)) = \pm 1 \neq 0,$$

where the last equality comes from (B_3) . In the other cases, for which we have $\zeta(0, \cdot) = -\frac{1}{T}B_{\tilde{\phi},b}P + \tilde{\phi}^{-1}[\tilde{\phi}P + \frac{t}{T}B_{\tilde{\phi},b}P]$, we consider the homotopy

$$\begin{aligned} \xi : [0, 1] \times \mathcal{C}([0, T]) &\rightarrow \mathcal{C}([0, T]) : \\ (\lambda, u) &\mapsto -\frac{1}{T}B_{\tilde{\phi},b}(Pu) + \tilde{\phi}^{-1}[\tilde{\phi}(Pu) + \lambda \frac{t}{T}B_{\tilde{\phi},b}(Pu)]. \end{aligned}$$

We have again (33) as $\xi(\lambda, u) = u$ ensures $u = 0$, and we can conclude

$$\begin{aligned} \deg_{\text{LS}}(\text{id} - \zeta(0, \cdot), \Omega) &= \deg_{\text{LS}}(\text{id} - \zeta(1, \cdot), \Omega) = \deg_{\text{LS}}(\text{id} - \zeta(0, \cdot), \Omega) \\ &= \deg_{\text{LS}}(\text{id} + \frac{1}{T} B_{\tilde{\phi}, b} P - P, \Omega) = \deg_{\text{B}}(\frac{1}{T} B_{\tilde{\phi}, b}, (-M, M) \text{ (resp. } (M_1, M_2))) = \pm 1 \neq 0, \end{aligned}$$

where the last equality comes from the study of $B_{\tilde{\phi}, b}$ done in Section 7, and the fact that $-M < 0 < M$ (resp. $M_1 < 0 < M_2$). $\square$

REMARK 9.5. If ϕ is a diffeomorphism, we can rewrite Problems $(\mathcal{P}_{b, \lambda})$ and $(\mathcal{P}'_{b, \lambda})$ in the form (73) and notice, because of the sign of ϕ' , that Assumption (A_i) or (B_i) are still true:

$$u' = \lambda f(t, u) \cdot (\phi'(u))^{-1} =: \lambda g(t, u), \quad u(T) = bu(0). \quad (73)$$

10 Examples and Numerical Experiments

10.1 Application of the Main Result

EXAMPLE 10.1. Let us consider the problem

$$\left(\frac{u}{\sqrt{1+u^2}} \right)' = h(t) + e^u, \quad u(T) = bu(0). \quad (74)$$

with $b > 0$. Let us define $\alpha \in (0, 1)$ by $\alpha = \sup B_{\phi, b} = -\inf B_{\phi, b}$. If we suppose $\int_0^T h = T\bar{h} < -\alpha$, then we have

$$\int_0^T (h(t) + e^{M_1}) = T\bar{h} + Te^{M_1} < -\alpha, \quad \int_0^T (h(t) + e^{M_2}) = T\bar{h} + Te^{M_2} > \alpha,$$

where

$$M_1 = \log\left(-\frac{\alpha}{T} - \bar{h}\right) - \varepsilon, \quad M_2 = \log\left(\frac{\alpha}{T} - \bar{h}\right) + \varepsilon,$$

with any $\varepsilon > 0$, ensuring Assumption (i) of Theorem 8.1. Hence, if we suppose

$$\left| \phi \left(\log\left(\pm \frac{\alpha}{T} - \bar{h}\right) \right) + \int_0^T \left[h(t) + \left(\pm \frac{\alpha}{T} - \bar{h} \right) \right]^\pm \right| < 1, \quad (75)$$

we deduce Assumption (ii) and the existence of at least one solution of Problem (74).

EXAMPLE 10.2. Let us consider the periodical case of Problem (74). We have $b = 1$, $\alpha = 0$, and will consider $T = 1$ and h given by (76). Let us suppose that $\int_0^1 h_{p,q} = p + \frac{q}{2}$ equals -1 , so that $q = -2p - 2$.

$$h_{p,q}(t) = p + q|2t - 1|. \quad (76)$$

Let us observe for which values of p the condition (75), given here by (77), is verified. The sign of the function $h_{p,q} + 1$ is given by the next table when $p \neq -1$, and $h_{p,q} \equiv -1$ when $p = -1$,

$$\left| \int_0^1 [h_{p,q}(t) + 1]^\pm \right| < 1. \quad (77)$$

	0		$\frac{1}{4}$		$\frac{1}{2}$		$\frac{3}{4}$		1
$p > -1$	+	+	0	-	-	-	0	+	+
$p < -1$	-	-	0	+	+	+	0	-	-

Because

$$\left| \int_0^1 [h_{p,q}(t) + 1]^\pm dt \right| = \frac{|p+1|}{4},$$

we can apply Theorem 8.1 for $-5 < p < 3$. Let us note that if $p \in (-2, 0)$, the function $h_{p,q}$ is always negative and we can apply Theorem 9.4. But in the other cases, the function $h_{p,q}$ changes its sign and we may only use Theorem 8.1.

Let us make a numerical study of this example in order to compare the answer with the theoretical results. We shall work with the equation's form mentioned in Remark 8.3:

$$v'(t) = h_{p,q}(t) + e^{\frac{v}{\sqrt{1-v^2}}}, \quad v(0) = v(1),$$

where $v = \phi(u)$. Because the non-linearity of this equation is of class $\mathcal{C}^1$, we can apply the existence and uniqueness result of Cauchy. This leads us to work with the functions V and P , respectively called *flow-function* and the *Poincaré-function*, and respectively defined by the function associating to (t, s) the value at t of the solution of the Cauchy's problem with initial condition $v(0) = s$, and by $P(s) = V(1, s)$. We can note also that the solutions can not cruse itself, what infers to the sets of solutions v a simple structure because we are in dimension 1.

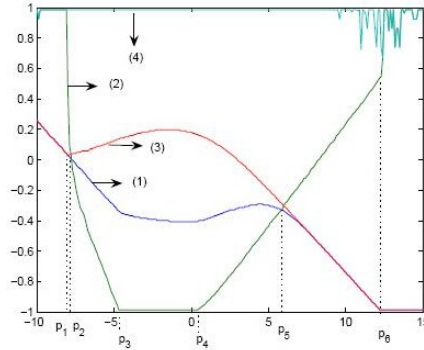


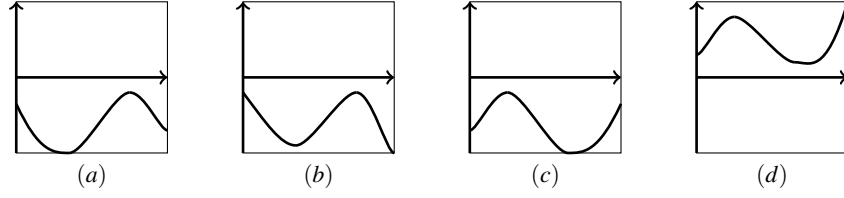
Figure 10.1 (1) $\alpha(0)$, (2) $\alpha(1)$, (3) $\beta(0)$, (4) $\beta(1)$.

p_1	p_2	p_3	p_4	p_5	p_6
-7.98	-7.83	-4.7	0.45	5.85	12.34

Let us begin by studying the *global-solutions* of the equation, i.e., solutions which live on all $[0, 1]$. It is easy to see that such solutions live upper (resp. less) than a function α (resp. β) with values in $[-1, 1]$, verifying the equation at the values of t whose their image is in $(-1, 1)$, and admitting at least one value of t whose their image equals ± 1 . Figure 10.1, obtained by Routine 4.1 exposed in Appendix B, gives for each p the values of α and β at $t = 0$ and $t = 1$.

We see on this figure the existence of p_1 and p_6 such that out of $[p_1, p_6]$, configurations of $\alpha(0), \beta(0), \alpha(1), \beta(1)$ implies the non existence of global solutions. Let us consider the three cases for which the graph of α is given by Figure 10.2 :

$$(a) p \in (p_1, p_3), \quad (b) p \in (p_3, p_4), \quad (c) p \in (p_4, p_6).$$

Figure 10.2 Pictures of α and β .

In the case (a), the graph of α reaches -1 in $(0, 1)$ at the first root r_1 of $h_{p,q}$; this is explained by the differential equation whose the right member equals $h_{p,q}(r_1) + 0$ and by the sign of $h_{p,q}$. In the case (b), the graph of α reaches -1 at $t = 1$. Finally, in the case (c), the graph of α reaches -1 at the second root r_2 of $h_{p,q}$, what is explained like in the case (a). In all cases, the graph of β is given by Figure 10.2 (d) (the differential equation ensures $\beta(1) = 1$, so the right part of the curve $\beta(1)$ on Figure 10.1 comes from numerical errors) up to the convexity of the beginning of the graph which depends of p .

By these observations, we see how the global solutions disappear out of (p_1, p_6) : by superposition of α and β . Let us note that by using condition (35) with t_1 and t_2 given by r_1 and r_2 for $p < -2$ and given by 0 and r_1 (or r_2 and 1) for $p > 0$ (i.e. working in a connex part of the domain of $h_{p,q}^+$), we can prove that that global solutions do not exist out of

$$(-6 - 2\sqrt{6}, 8 + 8\sqrt{2}) \simeq (-10.89, 19.31).$$

Like $s_1 \leq s_2$ ensures for any $t \in [0, 1]$ that $h_{p,q}(t) + e^{s_1} \leq h_{p,q}(t) + e^{s_2}$, we deduce easily that the functions P and $P - \text{id}$ are increasing. Hence, the existence of a periodic solution requires the following configuration:

$$\alpha(1) - \alpha(0) < 0, \quad \beta(1) - \beta(0) > 0.$$

The second condition being always true, Figure 10.1 ensures that there exists one and only one periodic solution for $p \in (p_2, p_5)$. We can hence compare the theoretic and numeric bounds obtained for p to ensure existence of periodic solutions.

The graph of the periodic solution v_p of our equation is given at Figure 10.3 for some values of p in the three cases

$$(i) p < -1, \quad (ii) p = -1, \quad (iii) p > -1.$$

The case (i) (resp. (iii)) concerns functions with the same comportment as the one of Figure 10.2 (a) (resp. (c)), and the case (ii) is the constant solution 0. These graphs are obtained by Routine 5.1 given in Appendix B. Let us try to explain the shape of the graph of v_p with the differential equation when $p > 0$. The shape of $h_{p,q}$ is given by

0		$\frac{1}{4} + \frac{1}{2q}$		$\frac{1}{4}$		$\frac{1}{2}$		$\frac{3}{4}$		$\frac{3}{4} - \frac{1}{2q}$		1
	+	0	-	-1	-	min	-	-1	-	0	+	

When $t = r_1$, the equation ensures $v_p' > 0$. This is to compare with the fact that the numerical experiments show functions v_p reaching their maximum after $1/4$, at a point more and more close of r_1 when p increases. When $t = 1/2$, the function v_p seems to be close of 0, leading thanks to the differential equation to

$$v_p' \simeq e^0 + p + q = -p - 1.$$

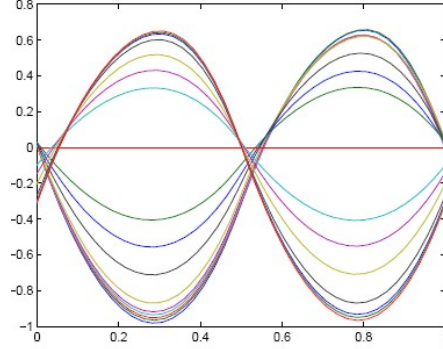


Figure 10.3 Case (a): $-7.7 \leq p \leq -7.3$ by step of 0.1 and $-7 \leq p \leq -4$ by step of 1. Case (c): $2 \leq p \leq 5$ by step of 1 and $5.4 \leq p \leq 5.7$ by step of 0.1.

This is to compare with the fact that numerical experiments show functions v_p such that v'_p decreases in the negative reals when p increases. At $t = 3/4$ we observe $v_p < 0$, what, combined with the differential equation, ensures $v'_p < 0$. When $t = r_2$, the equation ensures that $v'_p > 0$. So, v_p reaches its minimum between $3/4$ and r_2 which are more and more close when p increases. When $t > r_2$, the differential equation ensures $v'_p > 0$, what corresponds to Figure 10.3.

This description is not available for $-2 < p < 0$ because the function $h_{p,q}$ is always negative in these cases. When $p = -1$, the function v_p is identically 0, what corresponds with the differential equation. When $p < -2$, Figure 10.3 shows a similar comportment for v_p but with an inversion of the t where v_p reaches its minimum and maximum, what is to compare with explain with the differential equation jointed to the changing of the sign-comportment of $h_{p,q}$.

A discussion about the possible numerical errors for the graphs of v_p can be found in Appendix B.

10.2 Application of the Others Results

EXAMPLE 10.3. Let us consider the following problem:

$$\left(\frac{-\gamma_1 u^- + \gamma_2 u^+}{1 + \gamma_1 u^- + \gamma_2 u^+} \right)' = \alpha \cdot \frac{\sin(\beta t + u)}{\sqrt{t} + e^{-|u|}}, \quad u(T) = bu(0), \quad (78)$$

where $\alpha, \beta \in \mathbb{R}$ and $\gamma_1, \gamma_2 > 0$. Because

$$|f(t, u)| := \left| \alpha \cdot \frac{\sin(\beta t + u)}{\sqrt{t} + e^{-|u|}} \right| \leq \frac{|\alpha|}{\sqrt{t} + e^{-|u|}} \leq \frac{|\alpha|}{\sqrt{t}},$$

Theorem 9.1 ensures that Problem (78) with $b < 0$ has at least one solution if

$$\left\| \frac{|\alpha|}{\sqrt{t}} \right\|_{L^1} = 2 |\alpha| \sqrt{T} < 1.$$

If moreover $\gamma_1 = \gamma_2$ (ϕ odd) and $b = -1$ (antiperiodic problem), Theorem 9.2 ensures the same with condition $|\alpha| \sqrt{T} < 1$.

EXAMPLE 10.4. Let us consider the following problem:

$$\left(\frac{u}{\sqrt{1+u^2}} \right)' = h(t) + e^{|u|} \sin(u), \quad u(T) = bu(0), \quad (79)$$

where $h \in \mathcal{C}([0, T])$. We can chose $k > 0$ large enough in order to have (80) and 81 for any $t \in [0, 1]$ with $M = \frac{\pi}{2} + 2k\pi$ and $M' = \frac{3\pi}{2} + 2k\pi$,

$$h(t) + e^{|-M|} \sin(-M) < 0 < h(t) + e^{|M|} \sin(M), \quad (80)$$

$$h(t) + e^{|-M'|} \sin(-M') > 0 > h(t) + e^{|-M'|} \sin(M'). \quad (81)$$

Hence, Theorem 9.4 ensures that Problem (79) with $b \neq 0$ has at least one solution.

Two Inequalities

We present here a work about Sobolev inequalities for functions with mean value zero, and a work about Caffarelli-Kohn-Nirenberg inequalities for functions in $\mathbb{R}^N$ vanishing at infinity. The first one, made in collaboration with J. Van Schaftingen, is an existence result and the second one, made in collaboration with M. Willem in [14], a symmetry result. We have decided to gather these topics in spite of their differences because they both concern already known results and are devoted to give here an alternative proof or a generalization.

11 Sobolev Inequalities: Introduction

For $1 \leq p < N$, the Sobolev Inequalities ensure

$$\exists c > 0, \forall u \in \mathcal{D}(\mathbb{R}^N) : \left(\int_{\mathbb{R}^N} |u|^{p^*} \right)^{\frac{1}{p^*}} \leq c \left(\int_{\mathbb{R}^N} |\nabla u|^p \right)^{\frac{1}{p}}.$$

The same inequalities are also true for functions of $\mathcal{D}(\overline{\mathbb{R}_+^N})$ with a similar proof. These results can be extended for less regular functions, i.e. for functions in

$$\mathcal{D}^{1,p}(\mathbb{R}^N), \quad \mathcal{D}_0^{1,p}(\Omega), \quad \mathcal{D}^{1,p}(\mathbb{R}_+^N), \quad \text{BV}(\mathbb{R}^N), \quad \text{BV}_0(\Omega), \quad \text{BV}(\mathbb{R}_+^N),$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and where ∇u is replaced by Du in the inequalities for the BV-cases. Another generalization is known for functions with a mean value zero, precisely, for functions in the spaces

$$\mathcal{D}_\#(\overline{\Omega}), \quad \mathcal{D}_\#^{1,p}(\Omega), \quad \text{BV}_\#(\Omega),$$

defined in Chapter 1. The proof in $\mathcal{D}_\#(\overline{\Omega})$ is obtained by contradiction as follows: let us suppose that there exists a sequence $(u_n)_n$ in the space $\mathcal{D}_\#(\overline{\Omega})$ such that $\|\nabla u_n\| \rightarrow 0$, $\|u_n\|_{L^{p^*}} = 1$, Poincaré-Wirtinger Inequalities (see Remark 4.4) ensure that $u_n \rightarrow 0$ in $W^{1,p}(\Omega)$, which implies, by the previous Sobolev inequalities, that $\|u_n\|_{L^{p^*}} \rightarrow 0$, leading to the contradiction.

Let us study the optimal constants of these inequalities. For some practical reasons, we shall consider the power $-p$ of these constants:

$$S_p(\mathbb{R}^N) := \inf_{\substack{u \in \mathcal{D}(\mathbb{R}^N) \\ u \neq 0}} \frac{\|\nabla u\|_{L^p}^p}{\|u\|_{L^{p^*}}^p}, \quad S_p(\overline{\mathbb{R}_+^N}) := \inf_{\substack{u \in \mathcal{D}(\mathbb{R}_+^N) \\ u \neq 0}} \frac{\|\nabla u\|_{L^p}^p}{\|u\|_{L^{p^*}}^p},$$

$$S_p(\Omega) := \inf_{\substack{u \in \mathcal{D}(\Omega) \\ u \neq 0}} \frac{\|\nabla u\|_{L^p}^p}{\|u\|_{L^{p^*}}^p}, \quad \mathcal{S}_p(\Omega) := \inf_{\substack{u \in \mathcal{D}_\#(\overline{\Omega}) \\ u \neq 0}} \frac{\|\nabla u\|_{L^p}^p}{\|u\|_{L^{p^*}}^p}.$$

REMARK 11.1. We have the following equalities: $S_p(\mathbb{R}^N) = 2^{\frac{N}{p}} S_p(\overline{\mathbb{R}_+^N}) = S_p(\Omega)$. They are directly deduced by considering the following operators:

$$\mathcal{D}(\overline{\mathbb{R}_+^N}) \rightarrow \mathcal{D}(\mathbb{R}^N) : u \mapsto \bar{u}(x) = \begin{cases} u(x', x_N) & \text{if } x_N \geq 0, \\ u(x', -x_N) & \text{if } x_N < 0, \end{cases} \quad (82)$$

$$\mathcal{D}(\mathbb{R}^N) \rightarrow \mathcal{D}(\mathbb{R}^N) : u \mapsto u_{R,y,t}(x) = u(tRx + y), \quad (83)$$

where R is an isometry, $y \in \mathbb{R}^N$ and $t \in \mathbb{R}$. For the second operators, R, y, t are chosen for each u in order to have $\|u_{R,y}\|_{L^{p^*}(\mathbb{R}_+^N)} = \|u_{R,y}\|_{L^{p^*}(\mathbb{R}_+^N)}$ for the first equality, and $u_{y,t} \in \mathcal{D}(\Omega)$ for the second one, where $u_{R,y} := u_{R,y,1}$ and $u_{y,t} := u_{\text{id},y,t}$.

As the spaces (84) are dense in the corresponding spaces of (85), and since the spaces (86) verify Property (88) in the spaces (87) (the so-called *tight convergence in BV*),

$$\mathcal{D}(\mathbb{R}^N), \quad \mathcal{D}(\overline{\mathbb{R}_+^N}), \quad \mathcal{D}(\Omega), \quad \mathcal{D}_\#(\overline{\Omega}), \quad (84)$$

$$\mathcal{D}^{1,p}(\mathbb{R}^N), \quad \mathcal{D}^{1,p}(\mathbb{R}_+^N), \quad \mathcal{D}_0^{1,p}(\Omega), \quad \mathcal{D}_\#^{1,p}(\Omega), \quad (85)$$

$$\mathcal{D}^{1,1}(\mathbb{R}^N), \quad \mathcal{D}^{1,1}(\mathbb{R}_+^N), \quad \mathcal{D}_0^{1,1}(\Omega), \quad \mathcal{D}_\#^{1,1}(\overline{\Omega}), \quad (86)$$

$$\text{BV}(\mathbb{R}^N), \quad \text{BV}(\mathbb{R}_+^N), \quad \text{BV}_0(\Omega), \quad \text{BV}_\#(\overline{\Omega}), \quad (87)$$

$$\forall u \in \text{BV}, \exists (u_n)_n \subset \mathcal{D}^{1,1} : \|\nabla u_n\|_{L^1} \rightarrow \|Du\|, u_n \rightharpoonup u \text{ in BV}, \quad (88)$$

up to the power $-p$, the real numbers in the first line of the following table are the optimal constants of Sobolev inequalities in the spaces of the second line of the table:

$S_p(\mathbb{R}^N)$	$S_p(\Omega)$	$S_1(\mathbb{R}^N)$	$S_1(\Omega)$	$S_p(\overline{\mathbb{R}_+^N})$	$S_1(\overline{\mathbb{R}_+^N})$	$\mathcal{S}_p(\Omega)$	$\mathcal{S}_1(\Omega)$
$\mathcal{D}^{1,p}(\mathbb{R}^N)$	$\mathcal{D}_0^{1,p}(\Omega)$	$\text{BV}(\mathbb{R}^N)$	$\text{BV}_0(\Omega)$	$\mathcal{D}^{1,p}(\mathbb{R}_+^N)$	$\text{BV}(\mathbb{R}_+^N)$	$\mathcal{D}_\#^{1,p}(\Omega)$	$\text{BV}_\#(\Omega)$

The unique functions reaching $S_p(\mathbb{R}^N)$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ are given by $aU(bx)$ where $a, b \in \mathbb{R}$ and U is the *instanton*,

$$U(x) = \frac{1}{(1 + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}} \text{ if } p > 1, \quad U(x) = 1_{B^N}(x) \text{ if } p = 1, \quad (89)$$

and their translations. See for example [53, 55, 5, 52, 23]. There is no function reaching $S_p(\Omega)$ in $\mathcal{D}_0^{1,p}(\Omega)$ because that would be in contradiction with Remark 11.1 and the uniqueness of the instanton mentioned above. The case $p = 1$ is studied with BV spaces as suggested by the instanton in this case. The $\mathbb{R}_+^N$ -case is considered in Remark 11.1.

In order to study the mean value zero case, let us recall briefly the $\mathcal{D}^{1,p}(\mathbb{R}^N)$ -case. The idea is to study the following problem

$$\mathcal{M}_{\lambda,p}(\mathbb{R}^N) := \left\{ u \in \mathcal{D}^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{p^*} = \lambda \right\},$$

$$J(u) := \|\nabla u\|_{L^p}^p, \quad S_{\lambda,p}(\mathbb{R}^N) := \inf_{u \in \mathcal{M}_{\lambda,p}} J(u),$$

to consider a minimizing sequence $(u_n)_n$, to extract a weakly convergent subsequence of it that we rename $u_n : u_n \rightharpoonup u$, and to observe that $J(u) \leq S_{\lambda,p}$ by the sequential weak lower semi-continuity of J . Let us still note that

$$S_{\lambda,p}(\mathbb{R}^N) = \lambda^{p/p^*} S_p(\mathbb{R}^N). \quad (90)$$

It is sufficient to prove $u \in \mathcal{M}_{\lambda,p}$ to ensure that u is a solution of the problem. This last proof can be divided in two steps using the two famous *concentration-compactness lemmas* of Lions [35, 36, Lemma I.1] recalled below. Let us define $\rho_n := |u_n|^{p^*}$ and $\rho := |u|^{p^*}$.

- Lemma 11.2 ensures three behaviors for sequences $(\rho_n)_n$. It is proved in [35] that, up to operation (83), this sequence verifies the first behavior ensuring existence of $\mathbf{v} \in \mathcal{C}'_0(\mathbb{R}^N)$ such that $\rho_n \rightharpoonup \mathbf{v}$ in $\mathcal{C}'_0(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} \rho_n \rightarrow \|\mathbf{v}\|_{\mathcal{C}'_0(\mathbb{R}^N)}$. The weak convergence is said *tight* or *without loss of mass* in $\mathcal{C}'_0(\mathbb{R}^N)$.
- The thesis, equivalent to $\mathbf{v} = \rho \, dx$, is a consequence, up to operation (83), of Lemma 11.3 ensuring there is no dichotomy in $\mathcal{C}'_0(\mathbb{R}^N)$. Indeed, if it is not the case, i.e. two terms at least of the first equality in (93) are different of 0, the two inequalities of (93) ensure (91), where $\tilde{\lambda} := \int_{\mathbb{R}^N} |u|^{p^*}$. Since the last one is equivalent to (92) by (90), we contradict the concavity of $t \mapsto t^{p/p^*}$,

$$S_{\lambda,p}(\mathbb{R}^N) \geq \|\mu\| \geq \int_{\mathbb{R}^N} |\nabla u|^p + \sum_{j \in Q} \mu_j \geq S_{\tilde{\lambda},p}(\mathbb{R}^N) + S_p(\mathbb{R}^N) \sum_{j \in Q} v_j^{\frac{p}{p^*}}, \quad (91)$$

$$\lambda^{\frac{p}{p^*}} \geq \left(\tilde{\lambda}^{\frac{p}{p^*}} + \sum_{j \in Q} v_j^{\frac{p}{p^*}} \right). \quad (92)$$

Lemma 11.2 (Lions [35]): Let $(\rho_n)_n$ a sequence in $L^1_+(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} \rho_n \rightarrow \alpha$. Up to a subsequence, $(\rho_n)_n$ verify one of the three following possibilities:

- (i) (compactness) there exists $y_n \in \mathbb{R}^N$ such that $\rho_n(\cdot + y_n)$ is tight, i.e.,

$$\forall \varepsilon > 0, \exists R > 0 : \int_{B(y_n, R)} \rho_n \geq \alpha - \varepsilon,$$

- (ii) (vanishing) $\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, R)} \rho_n = 0$ for all $0 < R < \infty$,

- (iii) (dichotomy) there exists $\beta \in (0, \alpha)$ such that for all $\varepsilon > 0$:

$$\exists (y_n)_n \subset \mathbb{R}^N, \exists r_0, (r_n)_n \subset \mathbb{R}^+, \quad r_n > r_0 \text{ and } r_n \rightarrow +\infty,$$

$$|\beta - \int_{B(y_n, r_0)} \rho_n| \leq \varepsilon, \quad \int_{r_0 \leq |x - y_n| \leq r_n} \rho_n \leq \varepsilon$$

Lemma 11.3 (Lions [36]): Let $(u_n)_n$ and u in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ (resp. $BV(\mathbb{R}^N)$) and $\mu, \mathbf{v}$ in $\mathcal{C}'_0(\mathbb{R}^N)$ two positive finite Borel measures. If u_n converges weakly to u in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ (resp. $BV(\mathbb{R}^N)$), and if $|\nabla u_n|^p$ (resp. $|Du_n|$) and $|u_n|^{p^*}$ converge tightly respectively to μ and $\mathbf{v}$ in $\mathcal{C}'_0(\mathbb{R}^N)$, i.e. converge weakly with the moreover convergence

$$\|\nabla u_n\|^p_{\mathcal{C}'_0(\mathbb{R}^N)} \rightarrow \|\mu\|, \quad \| |u_n|^{p^*} \|_{\mathcal{C}'_0(\mathbb{R}^N)} \rightarrow \|\mathbf{v}\|_{\mathcal{C}'_0(\mathbb{R}^N)},$$

then there exists some at most countable set Q and

$$(x_j)_{j \in Q} \subset \mathbb{R}^N, \quad (v_j)_{j \in Q}, (\mu_j)_{j \in Q} \subset (0, \infty)$$

such that

$$\mathbf{v} = |u|^{p^*} + \sum_{j \in Q} v_j \delta_{x_j}, \quad \mu \geq |\nabla u|^p + \sum_{j \in Q} \mu_j \delta_{x_j}, \quad v_j^{\frac{p}{p^*}} \leq S_p^{-1}(\mathbb{R}^N) \mu_j, \quad (93)$$

where $|\nabla u|^p$ is replaced by $|Du|$ in the BV-case.

The proof that $(\rho_n)_n$ verifies the case (i) of Lemma 11.2 is slightly more difficult than explained previously. In [35], Lions uses the functions

$$\tilde{\rho}_n := |u_n|^{p^*} + |\nabla u_n|^p.$$

Application of this lemma, which was taking $\alpha = \lambda$, consists now to take $\alpha = \lambda + S_{\lambda,p}(\mathbb{R}^N)$. In the BV-case, $\tilde{\rho}_n := |u_n|^{p^*} + |Du_n|$ is a measure requiring a measure-version of the same lemma. Let us recall that if $u \in \text{BV}(\mathbb{R}^N)$, then $Du \in \mathcal{C}'_0(\mathbb{R}^N, \mathbb{R}^N)$. If μ is such a measure, $|\mu|$ is the positive finite Borel measure defined at $v \geq 0$ by

$$\int_{\mathbb{R}^N} v d|\mu| = \sup \left\{ \left| \int_{\mathbb{R}^N} w v d\mu \right| : w \in \mathcal{D}(\mathbb{R}^N, B^N) \right\}.$$

The proof, up to Operation (83), of $v = \rho dx$, is slightly longer than suggested previously. The absence of dichotomy in $\mathcal{C}'_0(\mathbb{R}^N)$ proved above ensures only two possible cases: thesis or $v = \lambda \delta_{x_j}$ for some $j \in Q$. Let us suppose, by contradiction, the second one. In [36], Lions considers the function

$$Q_n : (0, \infty) \rightarrow (0, \infty) : s \mapsto \sup_{z \in \mathbb{R}^N} \int_{B(z,s)} |u_n|^{p^*},$$

called the *concentration function* of u_n . The concentration function $Q_{n,t}$ of $u_{n,y,t}$ verifies $Q_{n,t}(s) = Q_n(st)$ for all s . Because $\lim_{s \rightarrow 0} Q_n(s) = 0$ and $\lim_{s \rightarrow \infty} Q_n(s) = \lambda$, it is clear that for any $\alpha \in (0, \lambda)$ and $r > 0$, there exists $t_n > 0$ such that $Q_{n,t_n}(r) = \alpha$. Let us rename u_{n,y,t_n} by u_n . If we consider $v \in \mathcal{C}_0(\mathbb{R}^N)$ verifying $0 \leq v \leq 1$ with $\text{Supp } v \subset B(x_j, r)$ and $v(x_j) = 1$, our assumption about v gives the following contradiction:

$$\alpha = Q_n(r) \geq \int_{B(x_j,r)} |u_n|^{p^*} \geq \int_{\mathbb{R}^N} |u_n|^{p^*} v \rightarrow \lambda.$$

We have seen that $S_p(\Omega)$ is not reached in $\mathcal{D}_0^{1,p}(\Omega)$. Let us observe why the previous method does not work in this case:

- The first step disappears because the sequence $(\rho_n)_n$ is clearly in the case (i) where $B(y_n, r)$ is replaced by Ω itself. Nevertheless, that does not imply tight convergence in $\mathcal{C}'_0(\Omega)$. Indeed, the following example illustrates a loss of mass:

$$\rho_n \rightharpoonup \delta_x = 0 \text{ in } \mathcal{C}'_0(\Omega) \text{ if } x \in \partial\Omega.$$

A solution is to work with the greater space $\mathcal{C}'(\bar{\Omega})$, dual of $\mathcal{C}(\bar{\Omega})$ equipped with the uniform norm. In this space,

$$\forall (\mu_n)_n, \mu \in \mathcal{C}'_+(\bar{\Omega}) : \mu_n \rightharpoonup \mu \Rightarrow \|\mu_n\|_{\mathcal{C}'(\bar{\Omega})} \rightarrow \|\mu\|_{\mathcal{C}'(\bar{\Omega})}$$

which follows by testing the sequence with $v \equiv 1$ as the measures are positive.

- The second step, consisting to prove there is no loss of mass in Space $\mathcal{D}_0^{1,p}(\Omega)$ itself, fails. Indeed, the $\mathcal{D}_0^{1,p}(\Omega)$ -BV $_0(\Omega)$ -version of Lemma 11.3, which consists just to replace $\mathcal{C}'_0(\mathbb{R}^N)$ by $\mathcal{C}'(\bar{\Omega})$, ensures (92). Concavity of the function $t \mapsto t^{p/p^*}$ ensures again the uniqueness of the nonzero term in the right hand side, but we can not avoid the case $v = \lambda \delta_x$. Indeed, if $x \in \partial\Omega$, the operations $u \mapsto u_{y,t}$ considered in Remark 11.1 is not defined anymore.

In the mean value zero case, we can hope a better end. Indeed, by applying the mean value zero version of the previous lemma, we again have inequality (97) with $S_{\lambda,p}, S_{\tilde{\lambda},p}$ replaced by $\mathcal{S}_{\lambda,p}, \mathcal{S}_{\tilde{\lambda},p}$, and by (90) applied to $\mathcal{S}$ and Remark 11.1, we obtain

$$\lambda^{\frac{p}{p^*}} \mathcal{S}_p(\Omega) \geq \tilde{\lambda}^{\frac{p}{p^*}} \mathcal{S}_p(\Omega) + S_p(\mathbb{R}^N) \left(\sum_{j \in Q_1} v_j^{\frac{p}{p^*}} + 2^{-\frac{p}{N}} \sum_{j \in Q_2} v_j^{\frac{p}{p^*}} \right)$$

that we can again formulate by

$$\lambda^{\frac{p}{p^*}} \geq \tilde{\lambda}^{\frac{p}{p^*}} + \sum_{j \in Q} v_j^{\frac{p}{p^*}} + \sum_{j \in Q_1} \left(\frac{S_p(\mathbb{R}^N)}{\mathcal{S}_p(\Omega)} - 1 \right) v_j^{\frac{p}{p^*}} + \sum_{j \in Q_2} \left(2^{\frac{p}{N}} \frac{S_p(\mathbb{R}^N)}{\mathcal{S}_p(\Omega)} - 1 \right) v_j^{\frac{p}{p^*}}.$$

As $\lambda = \tilde{\lambda} + \sum_{j \in Q} v_j$, we could avoid a dichotomy in $\mathcal{C}'(\overline{\Omega})$ by the concavity of function $t \mapsto t^{p/p^*}$ if we had

$$2^{\frac{p}{N}} S_p(\mathbb{R}^N) \geq \mathcal{S}_p(\Omega).$$

So in this case, there is no dichotomy in $\mathcal{C}'(\overline{\Omega})$. But, we can not avoid anymore, as done for Lemma 11.3 in $\mathbb{R}^N$, the case $v = \lambda \delta_x$, because the domain does not permit to consider Operation (83). Hence, a sufficient condition to be sure that there is no loss of mass is

$$2^{\frac{p}{N}} S_p(\mathbb{R}^N) > \mathcal{S}_p(\Omega). \quad (94)$$

In the sequel, we shall prove a mean value zero-version of the Concentration-compactness lemma 11.3, and the previous inequalities for $1 \leq p < (N+1)/2$. Like said previously, these results are not new. Indeed, Leckband obtained the same one on a ball in [33], and Girão and Weth for the case $p = 2$ in [30]. In the first article, symmetrization arguments were considered leading to the impossibility to generalize the result on other domain than a ball. In the second one, the Brezis-Lieb lemma was applied on minimizing sequences u_n , and its “weak version” in a Hilbert space on functions ∇u_n ,

$$\begin{cases} u_n \rightarrow u \text{ a.e.} \\ \sup \|u_n\|_{L^p} < \infty \end{cases} \Rightarrow \lim (\|u - u_n\|_{L^p}^p + \|u_n\|_{L^p}^p) = \|u\|_{L^p}^p,$$

$$\nabla u_n \rightharpoonup \nabla u \text{ in } L^2 \Rightarrow \lim (\|\nabla u - \nabla u_n\|_{L^2}^2 + \|\nabla u_n\|_{L^2}^2) = \|\nabla u\|_{L^2}^2,$$

leading to the impossibility to generalize the result for other values than $p = 2$. Moreover, but we were not aware of that at the beginning of our work, Demyanov and Nazarov considered in [26] all the cases when $p > 1$ with a very similar technique as us.

12 Sobolev Inequalities: Concentration-Compactness Lemma

In this section, we shall prove the mean value zero version of Lemma 11.3. Let us begin by the cases where the weak limit u of the considered sequence $(u_n)_n$ equals 0:

Theorem 12.1: *Let $(u_n)_n$ in $\mathcal{D}_\#^{1,p}(\Omega)$ (resp. $\text{BV}_\#(\Omega)$) and μ, v in $\mathcal{C}'(\overline{\Omega})$ two positive finite Borel measures. If u_n converges weakly to 0 in $\mathcal{D}_\#^{1,p}(\Omega)$ (resp. $\text{BV}_\#(\Omega)$), and if $|\nabla u_n|^p$ (resp. $|Du_n|$) and $|u_n|^{p^*}$ converges tightly respectively to μ and v in $\mathcal{C}'(\overline{\Omega})$, then there exists some at most countable set Q and*

$$(x_j)_{j \in Q} \subset \overline{\Omega}, \quad (v_j)_{j \in Q}, (\mu_j)_{j \in Q} \subset (0, \infty)$$

such that

$$v = \sum_{j \in Q} v_j \delta_{x_j}, \quad \mu \geq \sum_{j \in Q} \mu_j \delta_{x_j}, \quad v_j^{\frac{p}{p^*}} \leq c_j \mu_j,$$

where c_j equals

$$S_p(\mathbb{R}^N)^{-1} \text{ if } x_j \in \Omega, \quad S_p(\overline{\mathbb{R}_+^N})^{-1} \text{ if } x_j \in \partial\Omega,$$

(with p replaced by 1 in the BV-case).

REMARK 12.2. Let us note that the Sobolev inequalities in $\mathbb{R}^N$ or $\mathbb{R}_+^N$ can be expressed by (95) and (96) what explains the last inequality in Lemmas 11.3 and 12.1:

$$\|u\|_{L^{p^*}}^p \leq S_p^{-1}(\mathbb{R}^N) \|\nabla u\|_{L^p}^p \quad (95)$$

$$\|u\|_{L^{p^*}}^p \leq S_p^{-1}(\overline{\mathbb{R}_+^N}) \|\nabla u\|_{L^p}^p. \quad (96)$$

Like in [36], we shall work in several steps.

Lemma 12.3: *Let $x_0 \in \overline{\Omega}$ and $\varepsilon_0 > 0$ such that $B(x_0, 2\varepsilon_0) \subset \Omega$ if $x_0 \in \Omega$, and such that $B(x_0, 2\varepsilon_0) \cap \overline{\Omega}$ admits a rectification $r : B(0, 2\varepsilon_0) \cap \overline{\mathbb{R}_+^N} \rightarrow B(x_0, 2\varepsilon_0) \cap \overline{\Omega}$ if $x_0 \in \partial\Omega$. Let us denote by R the rotation verifying*

$$\lim_{\varepsilon \rightarrow 0} \|r - (x_0 + R)\|_{\mathcal{C}^1(B[0, \varepsilon] \cap \overline{\mathbb{R}_+^N})} = 0. \quad (97)$$

If φ is a Borel-measurable function defined on $\overline{\Omega}$ and null out of $B[x_0, \varepsilon] \cap \overline{\Omega}$ with $0 < \varepsilon \leq \varepsilon_0$, then

$$\left(\int_{\overline{\Omega}} |\varphi|^{p^*} d\nu \right)^{\frac{p}{p^*}} \leq c \int_{\overline{\Omega}} |\varphi|^p d\mu, \quad (98)$$

where $c = S_p(\mathbb{R}^N)^{-1}$ in the first case, and $c = dS_p^{-1}(\overline{\mathbb{R}_+^N})$ with $d = d(\varepsilon)$ equal to

$$1 + O\left(\|r' - R\|_{\mathcal{C}^1(B[0, \varepsilon] \cap \overline{\mathbb{R}_+^N})}\right), \quad (99)$$

in the second one (with p replaced by 1 in the BV-case).

PROOF. We shall make the proof in the case $p = 1$ and shall generalize for cases $p > 1$ and BV after; also, we shall suppose that φ is of class $\mathcal{C}^\infty$, the general case being a direct consequence of the Lebesgue's Dominated Convergence Theorem.

Let $d = \alpha^{p/p^*} \beta \gamma^p$ where α, β, γ are respectively given by

$$\sup_{x \in B[x_0, \varepsilon]} |\det r'(x)|, \quad \left(\inf_{x \in B[x_0, \varepsilon]} |\det r'(x)| \right)^{-1}, \quad \|r'\|_\infty = \sup_{\substack{x \in B[x_0, \varepsilon] \\ h \in B(0, 1)}} |\langle r'(x) | h \rangle|.$$

The thesis about d is clear.

When $x_0 \in \Omega$, we have

$$\begin{aligned} \left(\int_{\Omega} |\varphi u_n|^{1^*} \right)^{\frac{1}{1^*}} &\leq S_1(\mathbb{R}^N)^{-1} \int_{\Omega} |\nabla(\varphi u_n)|, \\ &\leq S_1(\mathbb{R}^N)^{-1} \left[\int_{\Omega} |\nabla \varphi| \cdot |u_n| + \int_{\Omega} |\varphi| \cdot |\nabla u_n| \right], \end{aligned} \quad (100)$$

which converges to the inequality of the thesis when n goes to ∞ because $\|u_n\|_{L^1} \rightarrow 0$ by the Rellich's theorem. When $x_0 \in \partial\Omega$, by the change of domain formula, we have

$$\begin{aligned} \left(\int_{\Omega} |\varphi u_n|^{1^*} \right)^{\frac{1}{1^*}} &= \left(\int_{\mathbb{R}_+^N} |(\varphi \circ r)(u_n \circ r)|^{1^*} |\det r'| \right)^{\frac{1}{1^*}}, \\ &\leq \alpha^{\frac{1}{1^*}} \left(\int_{\mathbb{R}_+^N} |(\varphi \circ r)(u_n \circ r)|^{1^*} \right)^{\frac{1}{1^*}}, \\ &\leq S_1(\overline{\mathbb{R}_+^N})^{-1} \alpha^{\frac{1}{1^*}} \int_{\mathbb{R}_+^N} |\nabla[(\varphi \circ r) \cdot (u_n \circ r)]|, \\ &\leq S_1(\overline{\mathbb{R}_+^N})^{-1} \alpha^{\frac{1}{1^*}} \beta \gamma \left[\int_{\Omega} |\nabla \varphi| \cdot |u_n| + \int_{\Omega} |\varphi| \cdot |\nabla u_n| \right], \end{aligned} \quad (101)$$

and we conclude as previously. Let us note that the previous integrals have in fact to be done on $\overline{\mathbb{R}_+^N} \cap B[0, \varepsilon]$ or $\overline{\Omega} \cap B[x_0, \varepsilon]$. When $p > 1$, inequality (100) becomes (102) whose the main factor in the right member converges to $\int_{\Omega} |\varphi|^p d\mu$ by (103) and (104),

$$\left(\int_{\Omega} |\varphi u_n|^{p^*} \right)^{\frac{p}{p^*}} \leq S_p(\mathbb{R}^N)^{-1} \int_{\Omega} [|\nabla \varphi| \cdot |u_n| + |\varphi| \cdot |\nabla u_n|]^p, \quad (102)$$

$$\int_{\Omega} [|\nabla \varphi| \cdot |u_n| + |\varphi| \cdot |\nabla u_n|]^p = \int_{\Omega} |\varphi|^p |\nabla u_n|^p + p \int_{\Omega} [|\nabla \varphi| \cdot |u_n| + \theta_n |\varphi| \cdot |\nabla u_n|]^{p-1} |\nabla \varphi| |u_n|, \quad (103)$$

$$\int_{\Omega} [|\nabla \varphi| \cdot |u_n| + |\varphi| \cdot |\nabla u_n|]^{p-1} |\nabla \varphi| |u_n| \leq c \|u_n\|_{L^p} \cdot [\|\nabla u_n\|_{L^{p^*}} + \|u_n\|_{L^p}]^{p-1}, \quad (104)$$

where θ_n are functions from Ω to $(0, 1)$. Same considerations can be done for (101). In the BV-case, inequalities (100) and (101) become (105) and (106),

$$\left(\int_{\tilde{\Omega}} |(|\varphi| u_n)_k|^{1^*} \right)^{\frac{1}{1^*}} \leq S_1(\mathbb{R}^N)^{-1} \int_{\tilde{\Omega}} |\nabla (|\varphi| u_n)_k|, \quad (105)$$

$$\begin{aligned} \left(\int_{\tilde{\Omega}} |(|\varphi| u_n)_k|^{1^*} \right)^{\frac{1}{1^*}} &\leq S_1(\overline{\mathbb{R}_+^N})^{-1} \tilde{\alpha}^{\frac{1}{1^*}} \int_{\mathbb{R}_+^N} |\nabla [(|\varphi| u_n)_k \circ \tilde{r}]|, \\ &\leq S_1(\overline{\mathbb{R}_+^N})^{-1} \tilde{\alpha}^{\frac{1}{1^*}} \tilde{\beta} \tilde{\gamma} \int_{\tilde{\Omega}} |\nabla (|\varphi| u_n)_k|, \end{aligned} \quad (106)$$

where

$$(|\varphi| u_n)_k : \Omega_k \rightarrow \mathbb{R}, \quad \Omega_k := \{x \in \Omega : d(x, \partial\Omega) > 1/k\}, \quad (|\varphi| u_n)_k := \rho_k * (|\varphi| u_n),$$

$$\tilde{\Omega} \subset \subset \Omega_k, \quad \tilde{r} : B(0, 2\varepsilon_0) \cap \overline{\mathbb{R}_+^N} \rightarrow B(\tilde{x}_0, 2\varepsilon_0) \cap \tilde{\Omega},$$

with $(\rho_k)_k$ a sequence of regularization, $\tilde{r}$ a rectification of $B(\tilde{x}_0, 2\varepsilon_0) \cap \tilde{\Omega}$, and $\tilde{x}_0$ a point of $\partial\tilde{\Omega}$ such that $d(x_0, \tilde{x}_0) \leq 2/k$. By definition of $(|\varphi| u_n)_k$, we have

$$\begin{aligned} \int_{\tilde{\Omega}} |\nabla (|\varphi| u_n)_k| &= \sup_{w \in \mathcal{D}(\tilde{\Omega}, B^N)} \left| \int_{\tilde{\Omega}} w \nabla (|\varphi| u_n)_k \right| = \sup_{w \in \mathcal{D}(\tilde{\Omega}, B^N)} \left| \int_{\tilde{\Omega}} \operatorname{div} w (|\varphi| u_n)_k \right| \\ &= \sup_{w \in \mathcal{D}(\tilde{\Omega}, B^N)} \left| \int_{\tilde{\Omega}} \operatorname{div} w_k (|\varphi| u_n) \right| \leq \sup_{w \in \mathcal{D}(\Omega, B^N)} \left| \int_{\tilde{\Omega}} \operatorname{div} w (|\varphi| u_n) \right| \end{aligned}$$

where the last inequality comes from $w_k \in \mathcal{D}(\Omega, B^N)$ with $w_k := \rho_k * w$. Moreover, as

$$-\int_{\tilde{\Omega}} \operatorname{div} w (|\varphi| u_n) = -\int_{\tilde{\Omega}} \operatorname{div} (w|\varphi|) u_n + \int_{\tilde{\Omega}} (w \nabla |\varphi|) u_n = \int_{\tilde{\Omega}} w |\varphi| d\mathbf{D}u_n + \int_{\tilde{\Omega}} w (\nabla |\varphi|) u_n$$

we deduce from the previous inequalities joined to (105), (106), the following one:

$$\left(\int_{\tilde{\Omega}} |\varphi u_n|^{1^*} \right)^{\frac{1}{1^*}} = \lim_k \left(\int_{\tilde{\Omega}} |(|\varphi| u_n)_k|^{1^*} \right)^{\frac{1}{1^*}} \leq \tilde{c} \left(\int_{\tilde{\Omega}} |\varphi| d\mathbf{D}u_n + \int_{\tilde{\Omega}} |\nabla \varphi| |u_n| \right),$$

where $\tilde{c} = \tilde{\alpha}^{\frac{1}{1^*}} \tilde{\beta} \tilde{\gamma} \cdot S_1(\overline{\mathbb{R}_+^N})^{-1}$ or $S_1(\mathbb{R}^N)^{-1}$. Replacing $\tilde{\Omega}$ by Ω_k and so $\tilde{r}$ by r_k , choosing r_k in order to have $\lim_k c_k = c$, and using the Levi's Theorem, leads to the same inequality with $\tilde{\Omega}$ replaced by Ω , what concludes the proof. $\square$

Lemma 12.4: *There exists some at most countable set Q and*

$$(x_j)_{j \in Q} \subset \overline{\Omega}, \quad (v_j)_{j \in Q}, (\mu_j)_{j \in Q} \subset (0, \infty)$$

such that

$$v = \sum_{j \in Q} v_j \delta_{x_j}, \quad \mu \geq \sum_{j \in Q} \mu_j \delta_{x_j}, \quad v_j^{\frac{p}{p^*}} \leq c_j \mu_j,$$

(with p replaced by 1 in the BV-case) where c_j is defined in Lemma 12.1.

PROOF. Let us first suppose $\mu << v$ so the existence of $g \in L_+^1(dv)$ such that $d\mu = g dv$. By defining $v_k := g^{N/p} \cdot 1_{\{g \leq k\}} \cdot v$ and taking $\varphi = g^{(N-p)/p^2} \cdot 1_{\{g \leq k\}} \cdot \psi$ with $\psi : B[x_0, \varepsilon] \cap \overline{\Omega} \rightarrow \mathbb{R}$ a Borel-measurable function and x_0, ε like in the previous lemma, Inequality (98) ensures

$$\left(\int_{\overline{\Omega}} |\psi|^{p^*} dv_k \right)^{\frac{p}{p^*}} \leq c \int_{\overline{\Omega}} |\psi|^p dv_k. \quad (107)$$

Hence, for any Borel-measurable set $A \subset \overline{\Omega} \cap B[x_0, \varepsilon]$, we have $v_k(A)^{\frac{p}{p^*}} \leq c v_k(A)$ leading $v_k(A) \geq \delta := c^{-N/p}$ or $v_k(A) = 0$. Therefore, if $x \in \overline{\Omega} \cap B[x_0, \varepsilon]$, we deduce $v_k(\{x\}) \geq \delta$ or the existence of $\eta > 0$ such that $v_k(B(x, \eta)) = 0$. Compactness of $\overline{\Omega} \cap B[x_0, \varepsilon]$ leads to finite sets $\{c_1^k, \dots, c_{n(k)}^k\}$ in $\mathbb{R}^+$ and $\{x_1^k, \dots, x_{n(k)}^k\}$ in $\overline{\Omega} \cap B[x_0, \varepsilon]$, such that

$$v_k = \sum_{i=1}^{n(k)} c_i^k \delta_{x_i^k},$$

and so the same for $1_{\{g \leq k\}} v$ like $g \neq 0$ v -a.e. because of $v << \mu$. Since $1_{\{g \leq k\}} v$ converges to v in $\mathcal{C}'(\overline{\Omega})$ when k goes to ∞ , we deduce the thesis. If μ is not supposed to be such that $\mu << v$, there exists $g \in L_+^1(dv)$ and σ such that

$$d\mu = g dv + \sigma, \quad v(\text{Supp } \sigma) = 0.$$

It is sufficient to multiply φ by $1_{\overline{\Omega} \setminus \text{Supp } \sigma}$ and observe that we have again (107) to conclude.

Inequality (98) applied to the functions $1_{\{x_i\}}$ and joined to (97) and (99) ensures (108) where $\mu_j := \mu(\{x_j\})$ and $c_j(\varepsilon)$ is defined like c in Lemma 12.3:

$$\forall \varepsilon > 0 : v_j^{\frac{p}{p^*}} \leq c_j(\varepsilon) \mu_j. \quad (108)$$

The thesis follows from (97) and (99), and compactness of $\overline{\Omega}$. $\square$

We can deduce the general case where the weak limit of $(u_n)_n$ is not necessarily 0.

Theorem 12.5: *Let $(u_n)_n$ and u in $\mathcal{D}_\#^{1,p}(\Omega)$ (resp. $\text{BV}_\#(\Omega)$), and μ, v in $\mathcal{C}'(\overline{\Omega})$ two positive finite Borel measures. If u_n converges weakly to u in $\mathcal{D}_\#^{1,p}(\Omega)$ (resp. $\text{BV}_\#(\Omega)$), and if $|\nabla u_n|^p$ (resp. $|Du_n|$) and $|u_n|^{p^*}$ converge tightly respectively to μ and v in $\mathcal{C}'(\overline{\Omega})$, then there exists some at most countable set Q and*

$$(x_j)_{j \in Q} \subset \overline{\Omega}, \quad (v_j)_{j \in Q}, (\mu_j)_{j \in Q} \subset (0, \infty)$$

such that

$$v = |u|^{p^*} + \sum_{j \in Q} v_j \delta_{x_j}, \quad \mu \geq |\nabla u|^p + \sum_{j \in Q} \mu_j \delta_{x_j}, \quad v_j^{\frac{p}{p^*}} \leq c_j \mu_j,$$

where c_j equals

$$S_p(\mathbb{R}^N)^{-1} \text{ if } x_j \in \Omega, \quad S_p(\overline{\mathbb{R}_+^N})^{-1} \text{ if } x_j \in \partial\Omega,$$

where $|\nabla u|^p$ is replaced by $|\mathrm{D}u|$ in the BV-case.

PROOF. As in the proof of Lemma 12.3, we shall treat the case $p = 1$ and shall generalize for cases $p > 1$ and BV after.

Let us consider $(u_n - u)$. The Brezis-Lieb lemma joined to $|u_n|^{1^*} \rightharpoonup v$ in $\mathcal{C}'(\overline{\Omega})$ and to $u_n \rightharpoonup u$ in $\mathcal{D}^{1,1}(\Omega)$, leads to

$$|u_n - u|^{1^*} \rightharpoonup v - |u|^{1^*} \text{ in } \mathcal{C}'(\overline{\Omega}).$$

By applying Theorem 12.1 to $(u_n - u)$, we deduce $v - |u|^{1^*} = \sum_{j \in Q} v_j \delta_{x_j}$. Unfortunately, we can not do the same for $|\nabla(u_n - u)|^p, |\nabla u_n|^p, |\nabla u|^p, \mu$ because we have no pointwise convergence in this case. We shall avoid this problem by adapting the proof of Theorem 12.1 about μ .

We have not Inequality (98), but if we observe inequalities (100), (101), we note that

$$\left(\int_{\Omega} |\varphi|^{1^*} d\nu \right)^{\frac{1}{1^*}} \leq c \left[\int_{\Omega} |\nabla \varphi| |u| + \int_{\Omega} |\varphi| d\mu \right] \quad (109)$$

for all φ of class $\mathcal{C}^\infty$ null out of $\overline{\Omega} \cap B[x_0, \varepsilon_0]$ with x_0, ε_0 chosen like previously. If we take $x_0 = x_i$ for some $i \in Q, \varphi \in \mathcal{D}(B(0, 1))$ such that $0 \leq \varphi \leq 1, \varphi(0) = 1$, and if we define $\varphi_n(x) = \varphi(n(x - x_i))$, then we deduce from $v_i \delta_{x_i} \leq v$ that

$$\begin{aligned} v_i^{\frac{1}{1^*}} &\leq c \left[n \int_{B(x_i, n^{-1})} |\nabla \varphi|(n(x - x_i)) |u| + \mu(B(x_i, n^{-1})) \right] \\ &\leq c \mu(B(x_i, n^{-1})) + cn \left(\int_{B(x_i, n^{-1})} |\nabla \varphi|^N(n(x - x_i)) \right)^{\frac{1}{N}} \left(\int_{B(x_i, n^{-1})} |u|^{1^*} \right)^{\frac{1}{1^*}} \\ &\leq c \mu(B(x_i, n^{-1})) + c \left(\int_{B(0, 1)} |\nabla \varphi|^N \right)^{\frac{1}{N}} \left(\int_{B(x_i, n^{-1})} |u|^{1^*} \right)^{\frac{1}{1^*}} \end{aligned}$$

leading to $v_i^{1/1^*} \leq c \mu(\{x_i\})$ with $c = S(\mathbb{R}^N)^{-1}$ or $S(\overline{\mathbb{R}_+^N}) d(\varepsilon)$ for any ε , where $d(\varepsilon) = 1 + O(\varepsilon)$ like in Lemma 12.3. The thesis follows by defining $\mu_i := \mu(\{x_i\})$ and by taking the limit on ε to 0.

The BV-case is completely the same. For the case $p > 1$, Inequality (109) is replaced by (102) treated with (103), (104) like above. $\square$

13 Sobolev Inequalities: Estimate of $\mathcal{S}_p(\Omega)$

In order to estimate $\mathcal{S}_p(\Omega)$ for Ω bounded and of class $\mathcal{C}^2$, we shall consider an instanton perturbed in such a way that its mean value is null, and concentrate it at a point of the boundary of Ω where the mean curvature is positive. So we shall work with U defined at (89) and consider $U_\varepsilon - \overline{U_\varepsilon}$, where $\overline{u} := \frac{1}{|\Omega|} \int_{\Omega} u$ and

$$U_\varepsilon := \varepsilon^{-\frac{N-p}{p}} U\left(\frac{\cdot}{\varepsilon}\right).$$

In the sequel, we shall suppose, without loss of generality,

- $0 \in \partial\Omega$ and $\exists f \in \mathcal{C}^2(B^{N-1}(0, R)) : \Omega \cap B(0, R) = \text{epi}(f) \cap B(0, R)$,

- $f'(0) = 0$ and $H(x') := f''(0; x', x')$ is given by $\sum_{1 \leq i \leq N-1} \lambda_i (x'_i)^2$,
- $\forall \delta > 0, \exists r > 0, \forall x' \in B^{N-1}(0, r) : H(x') - \delta |x'|^2 \leq f(x') \leq H(x') + \delta |x'|^2$,

where we have written a point $x \in \mathbb{R}^N$ by $(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and $\text{epi}(f)$ for the epigraph of f . Let us recall that the mean curvature K of $\partial\Omega$ at 0 is $(N-1)^{-1} \sum \lambda_i$.

Lemma 13.1: *Let $g : [0, \infty) \rightarrow \mathbb{R}$ a measurable function such that $g(s)s^N \in L^1([0, \infty))$, and h the associated radial function on $\mathbb{R}^N$. There exists $c > 0$ such that for any $\delta > 0$:*

$$\int_{\Omega/\varepsilon} h = \int_{\mathbb{R}_+^N} h - \varepsilon K \int_{\mathbb{R}^{N-1}} h(x') |x'|^2 dx' + o(\varepsilon).$$

PROOF. We shall prove the existence of $c > 0$ such that for any $\delta > 0$ there exists a function $B_c(\varepsilon)$ bounded by c verifying

$$\int_{\Omega/\varepsilon} h = \int_{\mathbb{R}_+^N} h - \varepsilon K \int_{\mathbb{R}^{N-1}} h(x') |x'|^2 dx' + \delta \varepsilon B_c(\varepsilon) + o(\varepsilon). \quad (110)$$

The first step is to observe

$$\int_{\Omega/\varepsilon} h = \int_{B(0, r/\varepsilon) \cap \Omega/\varepsilon} h + o(\varepsilon) \quad (111)$$

where r is defined previously. Indeed, let us denote by (a) the left-member of (111) and by (b) the first term of its right-member. There exists $R > 0$ such that

$$|(a) - (b)| \leq \int_{r/\varepsilon}^{R/\varepsilon} |g(s)| s^{N-1} ds \leq \frac{\varepsilon}{r} \int_{r/\varepsilon}^{R/\varepsilon} |g(s)| s^N ds. \quad (112)$$

By assumption, this last expression divided by ε converges to 0 when $\varepsilon \rightarrow 0$.

The second step is to observe

$$\int_{B(0, r/\varepsilon) \cap \Omega/\varepsilon} h = \int_{B(0, r/\varepsilon)_+} h - \varepsilon K \int_{B_{r/\varepsilon}^{N-1}} h(x') |x'|^2 dx' + \delta \varepsilon B_c(\varepsilon) + o(\varepsilon), \quad (113)$$

where $A_+ := A \cap \mathbb{R}_+^N$ and $B_s^{N-1} := B(0, s) \cap \mathbb{R}^{N-1}$ for any set A and any $s > 0$. Indeed, let us again denote by (a) the left-member of (113) and by (b) the first term of its right-member. It is clear that

$$(a) = (b) - \int_{B(0, r/\varepsilon)_+ \setminus \Omega/\varepsilon} h.$$

Figure 13.1 (a) shows

$$\left| \int_{B(0, r/\varepsilon)_+ \setminus \Omega/\varepsilon} h - \int_{B(0, r/\varepsilon)_+ \setminus U} h \right| \leq \int_{B(0, r/\varepsilon)_+ \cap (U_\delta^+ \setminus U_\delta^-)} |h|,$$

where

$$U := \text{epi}(\varepsilon H), \quad U_\delta^\pm := \text{epi}(\varepsilon H_\delta^\pm), \quad H_\delta^\pm(x') := H(x') \pm \delta |x'|^2.$$

Since, by the same reasoning as (112), the difference between the right-member terms of (113) and the ones of (110) equals $o(\varepsilon)$, it is sufficient to prove

$$\int_{B(0, r/\varepsilon)_+ \setminus U} h = \varepsilon K \int_{B_{r/\varepsilon}^{N-1}} h(x') |x'|^2 dx' + o(\varepsilon), \quad (114)$$

$$\int_{B(0, r/\varepsilon)_+ \cap (U_\delta^+ \setminus U_\delta^-)} |h| \leq 2\varepsilon \delta \int_{B_{r/\varepsilon}^{N-1}} h(x') |x'|^2 dx'. \quad (115)$$

leading to

$$c = 2 \int_{\mathbb{R}^{N-1}} h(x') |x'|^2 dx'.$$

Figure 13.1 (a) shows clearly that

$$\mathcal{H}^{N-1}(\partial B_s^N \cap (U_\delta^+ \setminus U_\delta^-)) \leq \mathcal{H}^{N-1}(A(s, \delta, \varepsilon))$$

where

$$A(s, \delta, \varepsilon) := \{x : |x'| = s, \varepsilon H(x') - \delta \varepsilon \leq x_N \leq \varepsilon H(x') + \delta \varepsilon\},$$

what leads to (115) by the following considerations:

$$\begin{aligned} \int_{B(0, r/\varepsilon) \cap (U_\delta^+ \setminus U_\delta^-)} |h| &= \int_0^{r/\varepsilon} |g(s)| \mathcal{H}^{N-1}(\partial B_s^N \cap (U_\delta^+ \setminus U_\delta^-)) ds \\ &\leq \int_0^{r/\varepsilon} |g(s)| \mathcal{H}^{N-1}(A(s, \delta, \varepsilon)) ds \\ &= 2\varepsilon \delta \int_0^{r/\varepsilon} |g(s)| \mathcal{H}^{N-2}(\partial B_s^{N-1}) ds \\ &= 2\varepsilon \delta \int_{B_{r/\varepsilon}^{N-1}} |h(x')| |x'|^2 dx'. \end{aligned}$$

Proving (114) is obtained, on one hand, by the same considerations with $A(s, \delta, \varepsilon)$ replaced by

$$B(s, \varepsilon) := \{x : |x'| = s, 0 \leq x_N \leq \varepsilon H(x')\},$$

and by noting that

$$\mathcal{H}^{N-1}(B(s, \varepsilon)) = \varepsilon \int_{\partial B_s^{N-1}} H(x') d\mathcal{H}^{N-2} = K\varepsilon \int_{\partial B_s^{N-1}} |x'|^2 d\mathcal{H}^{N-2},$$

and on an other hand, by proving

$$\varepsilon^{-1} \lim_{\varepsilon \rightarrow 0} \int_0^{r/\varepsilon} g(s) [\mathcal{H}^{N-1}(B(s, \varepsilon)) - \mathcal{H}^{N-1}(\partial B(0, s)_+ \setminus U)] ds = 0. \quad (116)$$

Figure 13.1 (b) shows that the integrand of (116) without $g(s)$, is smaller than (117) in which σ is given by (118) obtained by the intersection of the two surfaces $x_N = \varepsilon \lambda_m |x'|^2$ and $x_N = (s^2 - |x'|^2)^{1/2}$,

$$\int_{\partial B_s^{N-1}} \varepsilon H(x') d\mathcal{H}^{N-2} - \int_{\partial B_\sigma^{N-1}} \varepsilon H(x') d\mathcal{H}^{N-2}, \quad (117)$$

$$\sigma = \sqrt{2}s \left(1 + \sqrt{1 + 4\varepsilon^2 \lambda_m^2 s^2} \right)^{-1/2}, \quad (118)$$

and where $\lambda_m := \min \lambda_i$. Because (117) multiplied by ε^{-1} equals (119) converging to 0 as $\lim_{\varepsilon \rightarrow 0} \sigma = s$, and is majorated on $[0, \frac{r}{\varepsilon}]$ by (120),

$$(s^N - \sigma^N) \int_{\partial B^{N-1}} H(x') d\mathcal{H}^{N-2}, \quad (119)$$

$$s^N \left[1 - \left(\frac{\sqrt{2}}{\sqrt{1 + \sqrt{1 + 4\lambda_m^2 r^2}}} \right)^N \right] \int_{\partial B^{N-1}} H(x') d\mathcal{H}^{N-2}, \quad (120)$$

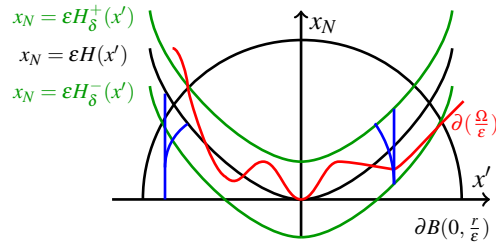


Figure 13.1 (a)

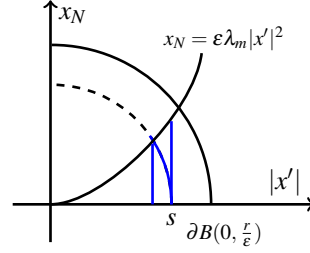


Figure 13.1 (b)

we can conclude (116) by the assumption about g joined to the Lebesgue's Dominated Convergence Theorem. $\square$

This lemma is a very good tool in order to obtain estimations of the integrals of U_ε at the power p^* , or of its gradient at the power p :

Proposition 13.2: *If $1 \leq p < N$ (resp. $1 \leq p < \frac{N+1}{2}$), there exists $c > 0$ such that:*

$$\int_{\Omega} |U_\varepsilon|^{\frac{Np}{N-p}} = \frac{1}{2} \int_{\mathbb{R}^N} |U|^{\frac{Np}{N-p}} \cdot A_1 \quad (\text{resp.} \quad \int_{\Omega} |\nabla U_\varepsilon|^p = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla U|^p \cdot A_2),$$

where ∇u is replaced by Du when $p = 1$,

$$A_i = 1 - 2\varepsilon K \cdot \frac{\mu_{N-2}}{\mu_{N-1}} \cdot a_i + o(\varepsilon),$$

with $\mu_{N-1} = \mathcal{H}^{N-1}(\mathcal{S}^{N-1})$, $\mu_0 = 2$, and a_i equals

$$\frac{\Gamma(N+i-\frac{N+1}{p})\Gamma(\frac{N+1}{p}-i)}{\Gamma(N+1-i-\frac{N}{p})\Gamma(\frac{N}{p}+1-i)} \text{ if } p > 1, \quad \frac{N-1+i}{N+1} \text{ if } p = 1.$$

PROOF. Let us consider the case $i = 1$. Because we have

$$\int_{\Omega} |U_\varepsilon|^{p^*} = \int_{\Omega/\varepsilon} |U|^{p^*}$$

and because $g(s) := |U(sx/|x|)|^{p^*}$ is such that $g(s)s^N \in L^1(0, \infty)$ for any p in $[1, N)$, we can use the previous lemma to conclude the thesis.

Let us consider the case $i = 2$ when $p > 1$. Because we have

$$\int_{\Omega} |\nabla U_\varepsilon|^p = \int_{\Omega/\varepsilon} |\nabla U|^p$$

and $g(s) := |(\nabla U)(sx/|x|)|^p$ is such that $g(s)s^N \in L^1(0, \infty)$ if and only if $p < \frac{N+1}{2}$, we can use the previous lemma to conclude the thesis.

When $p = 1$, by definition of U_ε we have

$$\int_{\Omega} |DU_\varepsilon| = \varepsilon^{1-N} \mathcal{H}^{N-1}(\partial B_{1/\varepsilon}^N \cap \Omega) = \mathcal{H}^{N-1}(\partial B^N \cap \Omega/\varepsilon).$$

The details of the proof of the previous lemma ensure that the last expression equals

$$\frac{1}{2} \mathcal{H}^{N-1}(\partial B^N) - \varepsilon K \mu_{N-2} + 2\varepsilon \delta \mu_{N-2} + o(\varepsilon).$$

In the three cases, we can use integral calculus (in particular formula (182) in the case $p > 1$, see Appendix E) to conclude the thesis. $\square$

Lemma 13.3: For any $\varepsilon > 0$, we have

$$\int_{\Omega} |U_{\varepsilon} - \overline{U}_{\varepsilon}|^{\frac{Np}{N-p}} \geq \int_{\Omega} |U_{\varepsilon}|^{\frac{Np}{N-p}} + c(\varepsilon),$$

where $c(\varepsilon)$ equals

$$O(\varepsilon^N) \text{ if } p < \frac{2N}{N+1}, \quad O(\varepsilon^N |\log(\varepsilon)|) \text{ if } p = \frac{2N}{N+1}, \quad O(\varepsilon^{\frac{N-p}{p-1}}) \text{ if } p > \frac{2N}{N+1}.$$

PROOF. The change of variable formula joined to $U \in L^{\frac{Np}{N-p}-1}(\mathbb{R}^N)$ ensures

$$\int_{\Omega} |U_{\varepsilon}|^{\frac{Np}{N-p}-1} \leq \varepsilon^{\frac{N-p}{p}} \int_{\mathbb{R}^N} |U|^{\frac{Np}{N-p}-1} \quad (121)$$

and the integrability (resp. integration by polar coordinates) of U when $p < \frac{2N}{N+1}$ (resp. $p \geq \frac{2N}{N+1}$) ensures the existence of $c > 0$ such that for any $\varepsilon > 0$:

$$\int_{\Omega} |U_{\varepsilon}| \leq c(\varepsilon), \quad c(\varepsilon) := \begin{cases} c\varepsilon^{N-\frac{N-p}{p}} & \text{if } p < \frac{2N}{N+1} \\ c\varepsilon^{\frac{N+1}{2}} |\log \varepsilon| & \text{if } p = \frac{2N}{N+1} \\ c\varepsilon^{\frac{N-p}{p(p-1)}} & \text{if } p > \frac{2N}{N+1} \end{cases}. \quad (122)$$

These inequalities joined to

$$|U_{\varepsilon} - \overline{U}_{\varepsilon}|^{\frac{Np}{N-p}} \geq |U_{\varepsilon}|^{\frac{Np}{N-p}} - \frac{Np}{N-p} |U_{\varepsilon}|^{\frac{Np}{N-p}-1} \overline{U}_{\varepsilon},$$

obtained by convexity of $t \mapsto t^{\frac{Np}{N-p}}$, lead to the conclusion. $\square$

The discussion at the end of Section 11 joined to the previous results lead to:

Theorem 13.4: When $1 \leq p \leq \frac{N+1}{2}$, the constant $\mathcal{S}_p(\Omega)$ is achieved.

PROOF. Because there always exists a point $x \in \partial\Omega$ where $K > 0$, we can use the previous estimates and deduce

$$\mathcal{S}_p(\Omega) \leq \frac{\int_{\Omega} |\nabla U_{\varepsilon}|^p}{\left(\int_{\Omega} |U_{\varepsilon} - \overline{U}_{\varepsilon}|^{\frac{Np}{N-p}} \right)^{\frac{N-p}{N}}} \leq \frac{\int_{\Omega} |\nabla U_{\varepsilon}|^p}{\left(\int_{\Omega} |U_{\varepsilon}|^{\frac{Np}{N-p}} + c(\varepsilon) \right)^{\frac{N-p}{N}}}, \quad (123)$$

where $c(\varepsilon)$ is given by Lemma 13.3 and ∇U_{ε} is replaced by DU_{ε} when $p = 1$.

Let us begin by the case $p < \frac{N+1}{2}$. By Proposition 13.2 and (124) we deduce (125) in which the term $c(\varepsilon)$ is added to the term $o(\varepsilon)$ in the definition of A_2 :

$$\frac{2N}{N+1} < \frac{N+1}{2}, \quad p < \frac{N+1}{2} \Leftrightarrow \frac{N-p}{p-1} > 1, \quad (124)$$

$$\mathcal{S}_p(\Omega) \leq 2^{\frac{-p}{N}} S_p(\mathbb{R}^N) \cdot (A_2/A_1^{\frac{N-p}{N}}). \quad (125)$$

In order to conclude (94) and so the thesis, we must show that $A_2/A_1^{\frac{N-p}{N}} < 1$. Since $(1+h)^q = 1 + qh + o(h)$, it is equivalent to prove $a_2 > a_1 \cdot (N-p)/N$. The case $p = 1$ is trivial and $p > 1$ equivalent to

$$\frac{N+1 - \frac{N+1}{p}}{N - \frac{N}{p}} > \frac{N-p}{N} \cdot \frac{\frac{N+1}{p} - 2}{\frac{N}{p} - 1}$$

by (181) (see Appendix E), leading to the conclusion.

When $p = \frac{N+1}{2}$, if Q is the last fraction of (123), (124) shows that the term $c(\varepsilon)$ in the denominator of Q equals $O(\varepsilon)$, and Proposition 13.2 can not treat its numerator. We can avoid these problems. Indeed, for the considered value of p , for any $\delta > 0$ there exists $r > 0$ such that the numerator and denominator of $Q' := S_p(\overline{\mathbb{R}_+^N})^{-1}Q$ are given respectively by

$$\gamma(\varepsilon) - \varepsilon K \int_{B_{r/\varepsilon}^{N-1}} h(x') |x'|^2 dx' + \delta \varepsilon \beta(\varepsilon) + \alpha(\varepsilon), \quad 1 - \frac{N-p}{N} a_1 \varepsilon + O(\varepsilon),$$

where

$$h(x) := |\nabla U(x)|, \quad \gamma(\varepsilon) := S_p(\overline{\mathbb{R}_+^N})^{-1} \int_{B(0, r/\varepsilon)_+} h \leq 1,$$

$$|\beta(\varepsilon)| \leq 2 \int_{B_{r/\varepsilon}^{N-1}} |h(x')| |x'|^2 dx', \quad \left| \frac{\alpha(\varepsilon)}{\varepsilon} \right| \leq \eta \int_{B_{r/\varepsilon}^{N-1}} |h(x')| |x'|^2 dx',$$

with η given by (120) without s^N . This fact is obtained by studying the term $o(\varepsilon), B_c(\varepsilon)$ in (111) and (114), replaced here by $\alpha(\varepsilon), \beta(\varepsilon)$. The thesis follows by proving $Q' < 1$, that is

$$K \int_{B_{r/\varepsilon}^{N-1}} h(x') |x'|^2 dx' + \delta \beta(\varepsilon) + \frac{\alpha(\varepsilon)}{\varepsilon} > a_1 + \frac{O(\varepsilon)}{\varepsilon} + \frac{\gamma(\varepsilon) - 1}{\varepsilon}. \quad (126)$$

If δ and r are chosen such that $2\delta + \eta < K$, then we have (126) since its left member is greater than the following expression converging to ∞ :

$$(K - (\eta + 2\delta)) \int_{B_{r/\varepsilon}^{N-1}} h(x') |x'|^2 dx'.$$

□

14 CKN Inequalities: Introduction

Particular cases of the Caffarelli-Kohn-Nirenberg inequalities, shortly CKN Inequalities, ensure

$$\exists c > 0, \forall u \in \mathcal{D}(\mathbb{R}^N) : \left(\int_{\mathbb{R}^N} |x|^\alpha |u|^p \right)^{\frac{1}{p}} \leq c \left(\int_{\mathbb{R}^N} |x|^\beta |\nabla u|^q \right)^{\frac{1}{q}} \quad (\text{CKN})$$

where $1 \leq p, q < \infty$ and $\alpha, \beta \in \mathbb{R}$ verify:

$$\frac{1}{p} + \frac{\alpha}{N}, \frac{1}{q} + \frac{\beta}{N} > 0, \quad \frac{1}{p} + \frac{\alpha}{N} = \frac{1}{q} + \frac{\beta - 1}{N}, \quad \beta - 1 \leq \alpha \leq \beta. \quad (127)$$

These inequalities generalize Sobolev and Hardy Inequalities. The conditions about p, q, α, β are necessary conditions, what can be seen by considering some particular functions u and some particular operations on each u like dilatation and translation. The reader can consult [18] for the original and complete proof of those inequalities, and also [54] for a simple proof of the particular case $p = q$ (and so $\beta - 1 = \alpha$).

Let us consider the optimal constants of these inequalities. For practical reasons, we shall work with the power $-q$ of these constants:

$$S_{\alpha, \beta, p, q} := \inf_{\substack{u \in \mathcal{D}(\mathbb{R}^N) \\ u \neq 0}} \frac{\| |x|^\beta \nabla u \|_{L^q}^q}{\| |x|^\alpha u \|_{L^p}^q}.$$

In [20, 21], Catrina and Wang study the case $q = 2$. In Figure 14.1, we recall some existence results of minimizers, i.e. functions reaching $S_{\alpha, \beta} := S_{\alpha, \beta, p, 2}$, and symmetry breaking

results about these functions. For the clarity of these tables, let us rewrite (127) in the case $q = 2$ as follows:

$$\begin{aligned} N \geq 3, \quad \frac{2-N}{2} < \beta < \infty, \quad \beta - 1 \leq \alpha \leq \beta, \quad p = \frac{2N}{N-2+2(\beta-\alpha)}, \\ N = 2, \quad 0 < \beta < \infty, \quad \beta - 1 \leq \alpha < \beta, \quad p = \frac{2}{\beta-\alpha}, \\ N = 1, \quad \frac{1}{2} < \beta < \infty, \quad \beta - 1 \leq \alpha < \beta - \frac{1}{2}, \quad p = \frac{2}{-1+2(\beta-\alpha)}, \end{aligned} \quad (128)$$

and let us precise that existence results work with minimizers in a bigger space than $\mathcal{D}(\mathbb{R}^N)$: CKN Inequalities ensure that the following application

$$\mathcal{D}(\mathbb{R}^N) \times \mathcal{D}(\mathbb{R}^N) \rightarrow \mathbb{R} : (u_1, u_2) \mapsto (u_1 | u_2)_\beta = \int_{\mathbb{R}^N} |x|^{2\beta} \nabla u_1 \cdot \nabla u_2 \, dx$$

is a scalar product on $\mathcal{D}(\mathbb{R}^N)$, what permits to consider the norm $\|\cdot\|_\beta$ induced by it and the space $\mathcal{D}_\beta^{1,2}(\mathbb{R}^N)$ which is the completion of $\mathcal{D}(\mathbb{R}^N)$ by this norm. Let us still note that the following injections are continuous:

$$\begin{aligned} \forall (\alpha, p) \text{ as in (128)} : \mathcal{D}_\beta^{1,2}(\mathbb{R}^N) &\subset L_\alpha^p(\mathbb{R}^N), \\ L_\alpha^p(\mathbb{R}^N) &:= \{u \in L_{\text{loc}}^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^{\alpha p} |u|^p \, dx < \infty\}, \end{aligned}$$

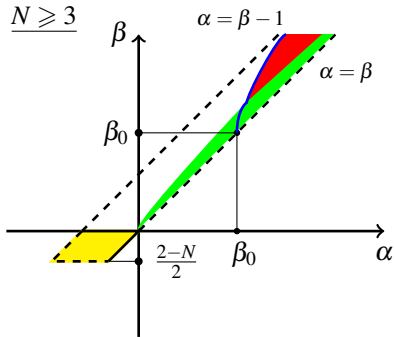
(it is direct a consequence of (CKN)) and that $\mathcal{D}(\mathbb{R}^N \setminus \{0\})$ is dense in $\mathcal{D}_\beta^{1,2}(\mathbb{R}^N)$ for the norm $\|\cdot\|_\beta$ (see Appendix C).

In [34], Lin and Wang proved that if a regular function u reaches the constant $S_{\alpha,\beta}$, in spite of the symmetry breaking mentioned previously saying that u is not necessarily radial, u is foliated Schwarz symmetric (see Section 4). Because of the homogeneity, we can formulate this result as follows:

Theorem 14.1: *If $N \geq 2$, every $\mathcal{C}^2$ -solution u of the minimization problem*

$$\inf_{\mathcal{M}} \|\cdot\|_\beta^2, \quad \mathcal{M} := \{u \in \mathcal{D}_\beta^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^{\alpha p} |u|^p \, dx = 1\},$$

is foliated Schwarz symmetric.



$$A := \{\beta - 1 < \alpha < \beta, (2-N)/2 < \beta\} \cup \{\alpha = \beta, \beta \leq 0\},$$

$$B := \{\alpha = \beta - 1, (2-N)/2 < \beta\} \cup \{\alpha = \beta, \beta > 0\},$$

$$C := \{(2-N)/2 < \beta \leq 0, \beta - 1 < \alpha \leq \beta\},$$

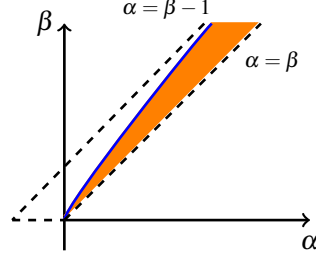
$$D \text{ is an open set in } \{\alpha, \beta > 0\} \text{ containing } \{\alpha = \beta\},$$

$$h : [\beta_0, \infty) \rightarrow \mathbb{R}, \text{ where } \beta_0 > 0, \text{ verifies } h(\beta_0) = \beta_0 \text{ and}$$

$$\beta - 1 < h(\beta) < \beta, \quad \lim_{\beta \rightarrow \infty} [h(\beta) - (\beta - 1)] = 0,$$

$$E := \{h(\beta) < \alpha < \beta, \beta > \beta_0\}.$$

- There exists at least one minimizer in A ,
- There does not exist any minimizer in B ,
- Any non-negative minimizer in C is radial,
- There exists D, β_0 and h like previously, such that any minimizer in $D \cup E$ is non-radial.

$N = 2$ 

$$A := \{\beta - 1 < \alpha < \beta, 0 < \beta\},$$

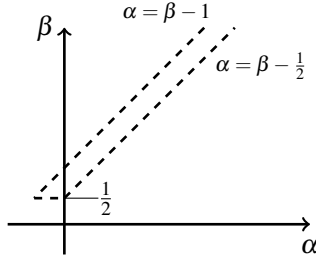
$$B := \{\alpha = \beta - 1, 0 < \beta\},$$

$h : [0, \infty) \rightarrow \mathbb{R}$ verifies $h(0) = 0$ and

$$\beta - 1 < h(\beta) < \beta, \quad \lim_{\beta \rightarrow \infty} [h(\beta) - (\beta - 1)] = 0,$$

$$C := \{h(\beta) < \alpha < \beta, \beta > 0\}.$$

- There exists at least one minimizer in A ,
- There does not exist any minimizer in B ,
- There exists h like previously, such that any minimizer in C is non-radial.

 $N = 1$ 

$$A := \{\beta - 1 < \alpha < \beta - \frac{1}{2}, \frac{1}{2} < \beta\},$$

$$B := \{\alpha = \beta - 1, \frac{1}{2} < \beta\},$$

- There exists at least one minimizer in A ,
- There does not exist any minimizer in B ,
- Any minimizer in A is non-radial.

Figure 14.1

Lin and Wang worked with the Morse index. Our purpose is to give a new proof working with polarization (see Section 4) based on a method of Bartsch, Weth and Willem in [7]. The interest of this work, made in collaboration with M. Willem in [14], is the simplicity of the technique permitting to consider a more general problem.

15 CKN Inequalities: Formulation on the Cylinder

Let $\mathcal{C}$ be the cylinder $\mathcal{S}^{N-1} \times \mathbb{R}$. We shall use the generic notations $\sigma = (\theta, t) = (\theta_1, \dots, \theta_N, t)$, for a point of $\mathcal{C}$, $y = (y_1, \dots, y_{N+1})$ for a point of $\mathbb{R}^{N+1}$, and v for a function from $\mathcal{C}$ to $\mathbb{R}$. Since $\mathcal{C}$ is a smooth manifold in the sense of Section 4, we can consider spaces $\mathcal{D}(\mathcal{C})$ and $H_0^1(\mathcal{C})$ equipped with the scalar product $(v|w)_{\mathcal{C}} := \int_{\mathcal{C}} \{vw + \nabla v \cdot \nabla w\}$ and with the norm induced by it.

As in [20, 21, 34], let us consider the map

$$T : \mathcal{D}(\mathbb{R}^N \setminus \{0\}) \rightarrow \mathcal{D}(\mathcal{C}) : u \mapsto v, \quad (129)$$

$$u(x) = |x|^{-\frac{N-2+2\beta}{2}} \cdot v\left(\frac{x}{|x|}, -\log |x|\right), \quad v(\theta, t) = e^{-\frac{N-2+2\beta}{2}t} \cdot u(\theta \cdot e^{-t}).$$

A direct computation shows that T satisfies (130), (131) (see Appendix C), what leads to define the scalar product (132), consider $\|\cdot\|_{\beta, \mathcal{C}}$ the norm induced by it, and to reformulate Theorem 14.1 on $\mathcal{C}$:

$$\int_{\mathbb{R}^N} |x|^{2\beta} \nabla u_1 \nabla u_2 \, dx = \int_{\mathcal{C}} \{\nabla v_1 \nabla v_2 + (\frac{N-2+2\beta}{2})^2 v_1 v_2\}, \quad (130)$$

$$\int_{\mathbb{R}^N} |x|^{\alpha p} |u|^p \, dx = \int_{\mathcal{C}} |v|^p. \quad (131)$$

$$(v_1|v_2)_{\beta,C} = \int_C \{ \nabla v_1 \nabla v_2 + (\frac{N-2+2\beta}{2})^2 v_1 v_2 \}, \quad (132)$$

Theorem 15.1: *If $N \geq 2$, every $\mathcal{C}^2$ -solution v of the minimization problem*

$$\inf_{\mathcal{M}_C} \|\cdot\|_{\beta,C}^2, \quad \mathcal{M}_C := \{v \in H_0^1(C) : \int_C |v|^p = 1\},$$

is foliated Schwarz symmetric.

Equivalence with Theorem 14.1 is ensured by the density of $\mathcal{D}(\mathbb{R}^N \setminus \{0\})$ in $\mathcal{D}_\beta^{1,2}(\mathbb{R}^N)$ mentioned in the previous section. The Lagrange's multiplier rule ensures that functions u and v mentioned in Theorem 14.1 and 15.1, verify equations (134) and (133):

$$-\Delta v + (\frac{N-2+2\beta}{2})^2 v = S_{\alpha,\beta} |v|^{p-2} v, \quad (133)$$

$$-\operatorname{div}(|x|^{2\beta} \nabla u) = S_{\alpha,\beta} |x|^{\alpha p} \cdot |u|^{p-2} u. \quad (134)$$

REMARK 15.2. Let us note that

$$S_{\alpha,\beta} = \inf_{\substack{v \in \mathcal{D}(C) \\ v \neq 0}} \frac{\|v\|_{\beta,C}^2}{\|v\|_{L^p}^2} > 0,$$

what is equivalent to continuous Sobolev injections:

$$\begin{aligned} H_0^1(C) &\subset L^p(C), \quad 2 \leq p < \infty, \quad N = 2, \\ H_0^1(C) &\subset L^p(C), \quad 2 \leq p \leq 2^* := \frac{2N}{N-2}, \quad N \geq 3. \end{aligned}$$

This can be regarded in parallel to the injections of $\mathcal{D}_\beta^{1,2}(\mathbb{R}^N)$ in $L_\alpha^p(\mathbb{R}^N)$ mentioned in the previous section, with $\beta > (2-N)/2$ fixed and α (and so p) taking all its values obtained by (128).

16 CKN Inequalities: Proof of the Result

Let Ω an open subset of $\mathcal{C}$, μ a non-negative real and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ a continuous function. Let us define $F(\sigma, t) := \int_0^t f(\sigma, s) ds$, consider Problem $(\mathcal{Q})$ defined in Section 1, and the functional $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ and the Nehari manifold $\mathcal{N}$ defined in Section 4.

The theorems 3.3 and 5.7 mention different important results. We shall prove a version on $\mathcal{C}$ of the results (BWW) of Theorem 5.7. Let us recall some assumptions about f and Ω of these two theorems, adapted for what we want to do here:

(A₁)(a) f is continuous,

(A₁)(b) f is Hölder continuous on $\Omega \times [-R, R]$ for every $R > 0$,

(A₂) $\exists p \in I, \exists c_1 > 0, \forall (\sigma, t) \in \Omega \times \mathbb{R} : |f(\sigma, t)| \leq c_1(|t| + |t|^{p-1}),$
 $I = (2, 2^*]$ if $N \geq 3$ and $(2, 2^* = \infty)$ if $N = 1, 2$,

(A₃)(a) the function $\mathbb{R} \rightarrow \mathbb{R} : t \rightarrow f(\sigma, t)$ is non-decreasing on $\mathbb{R}$ for every $\sigma \in \mathcal{C}$,

(A₃)(b) the function $\mathbb{R} \rightarrow \mathbb{R} : t \rightarrow \frac{f(\sigma, t)}{|t|}$ is strictly increasing on $\mathbb{R} \setminus \{0\}$ for every $\sigma \in \mathcal{C}$,

(A₄)(a) $(\forall \sigma \in \Omega : S_H(\sigma) \in \Omega), (\forall \sigma \in \Omega, \forall t \in \mathbb{R} : f(S_H(\sigma), t) = f(\sigma, t))$,

(A₄)(b) $(\Omega = \mathcal{S}^{N-1} \times (a, b)), (\forall S \in O(N), \forall \sigma \in \Omega, \forall t \in \mathbb{R} : f(S(\sigma), t) = f(\sigma, t))$,

where $H \in \mathcal{H}_{N+1}$ and $O(N)$ is the set of the rotations of $\mathbb{R}^{N+1}$ following the first N variables.

We shall prove the main Theorem of this Section, ensuring foliated Schwarz symmetries of regular minimizers of J on $\mathcal{N}$, by proceeding like in [7]. Let us begin by the two following lemmas:

Lemma 16.1: *Under Assumptions $(A_1)(a)$ - (A_2) - $(A_4)(a)$, if $v \in \mathcal{N}$ is such that $J(v) = \inf_{\mathcal{N}} J$ and if H is a half-space, then v_H is also such that $J(v_H) = \inf_{\mathcal{N}} J$ and $v_H \in \mathcal{N}$.*

PROOF. Let us note that by Proposition 5.5 we know that $v_H \in H_0^1(\Omega)$ and that $\int_{\Omega} v^2 = \int_{\Omega} v_H^2$ and $\int_{\Omega} |\nabla_{\sigma} v|^2 = \int_{\Omega} |\nabla_{\sigma} v_H|^2$ hold. Hence, it suffices to prove that

$$v_H \in \mathcal{N}, \quad J(v_H) = J(v),$$

what is equivalent to

$$\int_{\Omega} F(\sigma, v_H) = \int_{\Omega} F(\sigma, v), \quad \int_{\Omega} f(\sigma, v_H) v_H = \int_{\Omega} f(\sigma, v) v.$$

The proof is exactly the same for $\int_{\Omega} v^2 = \int_{\Omega} v_H^2$. □

Lemma 16.2: *Under Assumptions $(A_1)(a)$ - (A_2) - $(A_3)(a)$ - $(A_4)(a)$, if $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\Omega)$ is a solution of $(\mathcal{Q})$ such that*

$$v_H \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\Omega) \text{ and } v_H \text{ is a solution of } (\mathcal{Q}),$$

then we have one of the three following possibilities: $\forall \sigma \in \text{int } \Omega \cap H$,

$$(a) \ v(\sigma) > v(S_H(\sigma)), \quad (b) \ v(\sigma) < v(S_H(\sigma)), \quad (c) \ v(\sigma) = v(S_H(\sigma)).$$

PROOF. Let us consider the function $w \in \mathcal{C}^2(V) \cap \mathcal{C}_0(\bar{V})$ defined by the restriction of $v - \bar{v}$ to $V := \text{int } (\Omega \cap H)$, where $\bar{v} := v \circ S_H$. We have

$$\begin{aligned} -\Delta |w| + \lambda |w| &= -\Delta(2v_H - (v + \bar{v})) + \mu(2v_H - (v + \bar{v})) \\ &= 2f(\sigma, v_H) - f(\sigma, v) - f(\sigma, \bar{v}) \\ &= [f(\sigma, v_H) - f(\sigma, v)] + [f(\sigma, v_H) - f(\sigma, \bar{v})] \\ &\geq 0 \end{aligned}$$

where the inequality is due to $(A_3)(a)$. Hence, by the Strong Maximum Principle on the cylinder applied to the function $-|w|$, we have $-|w| \equiv 0$ or $-|w| < 0$ on V . The thesis follows. □

A direct consequence of the second lemma is the following theorem:

Theorem 16.3: *Under Assumptions $(A_1)(b)$ - (A_2) - $(A_3)(a)$ - $(A_4)(b)$, if $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\Omega)$ is a solution of $(\mathcal{Q})$ such that*

$$\forall H \in \mathcal{H} : v_H \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\Omega) \text{ and } v_H \text{ is a solution of } (\mathcal{Q}),$$

then there exists P in $\mathcal{S}^{N-1} \times \{0\}$ such that $v = v^P$.

This Theorem joined to the first leads to the main result mentioned above:

Theorem 16.4: *Under Assumptions $(A_1)(b)$ - (A_2) - $(A_3)(a, b)$ - $(A_4)(b)$, every minimizer $v \in \mathcal{N}$ of $J|_{\mathcal{N}}$ is foliated Schwarz symmetric.*

We can now prove Theorem 15.1:

Proof of Theorem 15.1. Because of (133), it is clear that we shall work with

$$\mu = \left(\frac{N-2+2\beta}{2}\right)^2, \quad f(\sigma, v) = S_{\alpha, \beta} |v|^{p-2} v.$$

The homogeneity of the problem ensures that

$$v \in \mathcal{M}_C, \quad \|v\|_{\beta, C}^2 = \inf_{\mathcal{M}_C} \|\cdot\|_{\beta, C}^2 \Leftrightarrow \lambda v \in \mathcal{N}, \quad J(\lambda v) = \inf_{\mathcal{N}} J,$$

with

$$\lambda := \left(S_{\alpha, \beta}^{-1} \|v\|_{\beta, C}^2\right)^{\frac{1}{p-2}},$$

which, together with Theorem 16.4, permits to conclude that if v is a minimizer of $\|\cdot\|_{\beta, C}^2$ on $\mathcal{M}_C$, then $\lambda^{-1}v$ is foliated Schwarz symmetric. Hence the same holds for v . $\square$

REMARK 16.5. Let us notice that this proof does not work if $N = 1$ because $\mathcal{C}$ is not connected in this case.

Asymptotics and Symmetries of Least Energy Nodal Solutions of Lane–Emden Problems with Slow Growth

In this chapter, we consider the Lane–Emden problem with Dirichlet boundary conditions, what is Problem $(\mathcal{Q})$ defined in Section 1 where $\mu = 0, B = \text{id}$ and

$$f(x, u) = |u|^{p-2}u, \quad p > 2.$$

First, we prove that, for p close to 2, the solution is unique up to the projection on the second eigenspace. From this uniqueness property, we deduce partial symmetries of least energy nodal solutions. We also analyze the asymptotic behavior of least energy nodal solutions as p goes to 2 and deduce an example of symmetry breaking from this. These results have been obtained in collaboration with D. Bonheure, C. Grumiau and J. Van Schaftingen in [11].

17 Introduction

We consider the super-linear elliptic boundary value problem

$$-\Delta u = |u|^{p-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (\mathcal{P}_p)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$ and $p > 2$. It is well-known that if $p < 2^*$, Problem $(\mathcal{P}_p)$ has a positive solution with least energy among positive solutions, referred as the *ground state* (shortly *g.s.*) solution, and a nodal solution with least energy among nodal solutions, referred as the *least energy nodal* (shortly *l.e.n.*) solution. See Theorem 3.3 and its references.

The result (GNN) of Theorem 5.7 proved by Gidas, Ni and Nirenberg, shows, using the elegant and now celebrated moving planes technique, that, on a convex domain, the *g.s.* solutions inherits all the symmetries of the domain. In view of that, a natural question is whether it is the case for *l.e.n.* solutions. The result (AP) of Theorem 5.7 proved by Aftalion and Pacella, shows that, on a ball, a *l.e.n.* solution cannot be radial. On the other hand, the results (BWW) of Theorem 5.7 proved by Bartsch, Weth and Willem, ensures partial symmetry results: on a radial domain, a *l.e.n.* solution u has the so-called $\mathbb{R}^N$ -foliated Schwarz symmetry defined in Section 4.

Motivated by (BWW), we study in this chapter, the question of the symmetry of *l.e.n.* solutions of Problem $(\mathcal{P}_p)$ on more general domains. Indeed, the method of Bartsch, Weth and Willem fails when the group of the symmetries of the domain is discrete. As a first step, we study the problem when p is close to 2, taking a special care for specific domains.

In $H_0^1(\Omega)$, we work with the scalar product $(u|v) = \int_{\Omega} \nabla u \cdot \nabla v$. We use the notations $\lambda_i, E(\lambda_i), P_{E(\lambda_i)}$ like in Remark 4.4, and, when λ_i is simple, e_i for an i^{th} eigenfunction such that $\|\nabla e_i\|_{L^2} = 1$.

Our first result basically shows that, for p close to 2, a priori bounded solutions of $(\mathcal{P}_p)$ can be distinguished by their projections on the second eigenspace.

Proposition 17.1: *For every $M > 0$, there exists $\bar{p} > 2$ such that, for every $\alpha \in E(\lambda_2) \setminus \{0\}$, for every $p \in (2, \bar{p})$, Problem $(\mathcal{P}_p)$ has at most one solution in the set $\{u \in B_M : P_{E(\lambda_2)}u = \alpha\}$.*

As readily expected, this uniqueness property immediately implies partial symmetries of l.e.n. solutions when the second eigenvalue λ_2 is simple (as for example on a rectangle) and p is close enough to 2.

Theorem 17.2: *Assume that λ_2 is simple. Then, for p close to 2 and any reflection R such that $R(\Omega) = \Omega$, l.e.n. solutions of Problem $(\mathcal{P}_p)$ respect the symmetry or antisymmetry of e_2 with respect to R .*

In the particular case of a rectangle, we infer that the nodal line is the small median.

When λ_2 is not simple, the situation is more delicate. Indeed, one can already figure out the difficulties on a square as the second eigenfunctions do not necessarily have an axis of symmetry whereas one would expect so for a l.e.n solution. When Ω is a ball, despite the degeneracy of the second eigenspace, we are still able to deduce a satisfactory statement.

Theorem 17.3: *Assume that Ω is a ball. Then, for p close to 2, l.e.n. solutions of Problem $(\mathcal{P}_p)$ are radially symmetric with respect to $N - 1$ independent directions and antisymmetric with respect to the orthogonal one.*

The first part of this result holds in fact for any subcritical $p > 2$ as proved in (BWW). To our knowledge, the antisymmetry property of the l.e.n. solutions is not known in general. In particular, we obtain that the nodal line is a diameter. It should be pointed out that the result (AP) ensures that, for a more general non-linearity, on a ball or an annulus in dimension two, the nodal line of a l.e.n. solution intersects the boundary of Ω .

The uniqueness property provided by Proposition 17.1 still allows to deduce a partial statement if all the second eigenfunctions enjoy some common symmetry. For instance, on a square, we deduce that l.e.n. solutions are antisymmetric with respect to the barycenter. In order to get further insight of symmetry properties of l.e.n. solutions, we study their asymptotic behavior as $p \rightarrow 2$.

Theorem 17.4: *If $(u_p)_{p>2}$ are l.e.n. solutions of Problem $(\mathcal{P}_p)$, then*

$$\|\nabla u_p\|_{L^2} \leq c \lambda_2^{\frac{1}{p-2}}.$$

If $p_n \rightarrow 2$ and $\lambda_2^{\frac{1}{2-p_n}} u_{p_n} \rightharpoonup u_$ in $H_0^1(\Omega)$, then $\lambda_2^{\frac{1}{2-p_n}} u_{p_n} \rightarrow u_*$ in $H_0^1(\Omega)$, u_* satisfies*

$$-\Delta u_* = \lambda_2 u_* \text{ in } \Omega, \quad u_* = 0 \text{ on } \partial\Omega$$

and

$$J_*(u_*) = \inf\{J_*(u) : u \in E(\lambda_2) \setminus \{0\}, \langle J'_*(u), u \rangle = 0\},$$

where

$$J_* : E(\lambda_2) \rightarrow \mathbb{R} : u \mapsto \frac{\lambda_2}{2} \int_{\Omega} u^2 - u^2 \log u^2.$$

Beyond its own interest, Theorem 17.4 leads to the following conjecture.

CONJECTURE 17.5. *If Ω is a square and p is close to 2, l.e.n. solutions are symmetric with respect to a diagonal and antisymmetric in the orthogonal direction.*

Theorem 17.4 also highlights an example of symmetry breaking by carefully exploiting the degeneracy of the square and playing with the eccentricity of the rectangle. Namely, we prove the following result.

Theorem 17.6: *For every $p > 2$ sufficiently close to 2, there exists a rectangle Ω such that any l.e.n. solutions of Problem $(\mathcal{P}_p)$ is neither symmetric nor antisymmetric with respect to the medians of Ω .*

The chapter is organized as follows. We start with some preliminaries in Section 18. Proposition 17.1 is proved in Section 19 where we also work out an abstract symmetry result. We then proceed to the study of the asymptotics in Section 20 where Theorem 17.4 is proved. Section 21 deals with specific domains. It contains in particular the proofs of Theorem 17.2, Theorem 17.3 and numerical evidence of Conjecture 17.5. The proof of Theorem 17.6 relies on the study of a perturbation of Problem $(\mathcal{P}_p)$:

$$-\operatorname{div}(A_p \nabla u) = |u|^{p-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (\tilde{\mathcal{P}}_p)$$

where A_p is a perturbation of the identity. This class of problems is analyzed in Section 22 where the proof of Theorem 17.6 is completed.

Throughout the paper, we use the Modified Mountain-Pass algorithm (MMPA) [24] to numerically illustrate our results. We point out that we are not aware of a proof of the convergence of this algorithm. Details on the computations are given in Appendix D.

18 Preliminaries

By scaling, and to avoid further renormalizations, Problem $(\mathcal{P}_p)$ can be rewritten as

$$-\Delta u = \lambda_2 |u|^{p-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \quad (\mathcal{P}_p)$$

18.1 Variational Setting Solutions of Problem $(\mathcal{P}_p)$ are critical points of the energy functional J_p defined on $H_0^1(\Omega)$ by

$$J_p(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{\lambda_2}{p} \int_{\Omega} |u|^p,$$

like in Section 2. Let us denote by $\mathcal{N}_p$ and $\mathcal{M}_p$ respectively the Nehari manifold and the nodal Nehari set defined in Section 2, and recall that, if $u \in H_0^1(\Omega)$, $u^+ \neq 0$ and $u^- \neq 0$, then $u \in \mathcal{M}_p$ if and only if

$$\int_{\Omega} |\nabla u^+|^2 = \lambda_2 \int_{\Omega} |u^+|^p \text{ and } \int_{\Omega} |\nabla u^-|^2 = \lambda_2 \int_{\Omega} |u^-|^p. \quad (135)$$

where $u^+(x) := \max\{0, u(x)\}$ and $u^-(x) := \min\{0, u(x)\}$.

The interest of $\mathcal{M}_p$ comes from the fact that it contains all the sign-changing critical points of J_p . If u minimizes J_p on $\mathcal{M}_p$ then u is a nodal solution of $(\mathcal{P}_p)$ usually referred to as the *least energy nodal solution* (shortly *l.e.n. solution*). Theorem 3.3 ensures the existence of such solutions.

Let us notice that, for $u \in H_0^1(\Omega) \setminus \{0\}$, there exists a unique scalar $t_u > 0$ such that $t_u u \in \mathcal{N}_p$ and moreover the function $t_u u$ maximizes the energy functional in the direction of u .

18.2 Reduced Problem As mentioned in the introduction, see Theorem 17.4, we shall consider the *reduced functional*

$$J_* : E(\lambda_2) \rightarrow \mathbb{R} : u \mapsto \frac{\lambda_2}{2} \int_{\Omega} u^2 - u^2 \log u^2$$

(where $t^2 \log t^2$ is extended continuously by 0 at $t = 0$). The critical points of J_* are the functions u_* such that

$$\forall v \in E(\lambda_2) : \int_{\Omega} v u_* \log u_*^2 = 0. \quad (136)$$

Any non-trivial critical points again belong to the *reduced Nehari manifold*

$$\mathcal{N}_* = \{u \in E(\lambda_2) \setminus \{0\} : \langle J'_*(u) | u \rangle = 0\}.$$

This manifold is compact and such that $u \in \mathcal{N}_*$ if and only if

$$\int_{\Omega} u^2 \log u^2 = 0,$$

or equivalently if and only if

$$J_*(u) = \frac{\lambda_2}{2} \|u\|_{L^2}^2.$$

Observe that, for any $u \in E(\lambda_2) \setminus \{0\}$, there exists again a unique constant $t_u^* > 0$ such that $t_u^* u \in \mathcal{N}_*$.

19 Abstract Symmetry Result

In this section, we present an abstract symmetry result which is the basis to prove Theorem 17.2 and Theorem 17.3.

19.1 Uniqueness Result As discussed in the introduction, for p close to 2, a priori bounded solutions of $(\mathcal{P}_p)$ can be distinguished by their projections on the second eigenspace. This will follow from Proposition 19.2 below. We begin with a standard preliminary lemma. We provide a proof for completeness.

Lemma 19.1: *Let $N \geq 3$. There exists $\varepsilon > 0$ such that if $\|a(x) - \lambda_2\|_{L^{N/2}} < \varepsilon$ and u solves the boundary value problem*

$$-\Delta u = a(x)u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (137)$$

then either $u = 0$ or $P_{E(\lambda_2)} u \neq 0$.

PROOF. Assume by contradiction that there exists a nontrivial solution u such that $P_{E(\lambda_2)} u = 0$. Let $w = P_{E(\lambda_1)} u$ and $z = P_{(E(\lambda_1) + E(\lambda_2))^{\perp}} u$. Taking successively w and z as test functions and using Poincaré and Sobolev inequalities, we infer that

$$\begin{aligned} \|\nabla w\|_{L^2}^2 &= \lambda_2 \int_{\Omega} w^2 + \int_{\Omega} (a(x) - \lambda_2)uw \\ &\geq \frac{\lambda_2}{\lambda_1} \|\nabla w\|_{L^2}^2 - c \|a(x) - \lambda_2\|_{L^{\frac{N}{2}}} \|\nabla w\|_{L^2} \|\nabla u\|_{L^2}, \\ \|\nabla z\|_{L^2}^2 &= \lambda_2 \int_{\Omega} z^2 + \int_{\Omega} (a(x) - \lambda_2)uz \\ &\leq \frac{\lambda_2}{\lambda_3} \|\nabla z\|_{L^2}^2 + c \|a(x) - \lambda_2\|_{L^{\frac{N}{2}}} \|\nabla z\|_{L^2} \|\nabla u\|_{L^2}. \end{aligned}$$

We deduce that

$$\begin{aligned}\|\nabla w\|_{L^2} &\leq \frac{\lambda_1 c}{\lambda_2 - \lambda_1} \|a(x) - \lambda_2\|_{L^{\frac{N}{2}}} \|\nabla u\|_{L^2}, \\ \|\nabla z\|_{L^2} &\leq \frac{\lambda_3 c}{\lambda_3 - \lambda_2} \|a(x) - \lambda_2\|_{L^{\frac{N}{2}}} \|\nabla u\|_{L^2}.\end{aligned}$$

Since $P_{E(\lambda_2)}u = 0$, we now conclude that

$$\|\nabla u\|_{L^2}^2 = \|\nabla w\|_{L^2}^2 + \|\nabla z\|_{L^2}^2 \leq c \|a(x) - \lambda_2\|_{L^{\frac{N}{2}}}^2 \|\nabla u\|_{L^2}^2.$$

When $\|a(x) - \lambda_2\|_{L^{\frac{N}{2}}}$ is small enough, this leads to a contradiction, so that the conclusion follows. $\square$

Observe that, if $N = 2$, the same statement can be formulated with the $L^{\frac{N}{2}}$ -norm replaced by any L^q -norm with $1 < q < \infty$. We only need to replace in the proof the use of the Sobolev inequality by the imbedding in any L^p with $1 < p < \infty$.

As a byproduct of the previous lemma, we deduce the following proposition.

Proposition 19.2: *For every $M > 0$, there exists $\bar{p} > 2$ such that, for every $p \in (2, \bar{p})$, if $u_p, v_p \in \{u \in B_M : P_{E(\lambda_2)}(u) \notin B_{\frac{1}{M}}\}$ solve the boundary value problem*

$$-\Delta u = \lambda_2 |u|^{p-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

then either $u_p = v_p$ or $P_{E(\lambda_2)}u_p \neq P_{E(\lambda_2)}v_p$.

PROOF. Let $p_n \rightarrow 2$, $(u_n)_n \subset B_M$ and $(v_n)_n \subset B_M$ be two sequences such that u_n and v_n solve Problem $(\mathcal{P}_p)$ with $p = p_n$ and $P_{E(\lambda_2)}u_n = P_{E(\lambda_2)}v_n$.

Since u_n and v_n are bounded in H_0^1 , up to subsequences, there exist $\alpha, \beta \in H_0^1(\Omega)$ such that $u_n \rightharpoonup \alpha$ and $v_n \rightharpoonup \beta$ in $H_0^1(\Omega)$. By assumption $P_{E(\lambda_2)}\alpha = P_{E(\lambda_2)}\beta$. By Rellich's Theorem, $u_n \rightarrow \alpha$ and $v_n \rightarrow \beta$ in $L^q(\Omega)$ for every $q < 2^*$. Therefore, $\alpha, \beta \in E(\lambda_2) \setminus \{0\}$, so that $\alpha = \beta$.

Observe that $(u_n - v_n)$ verifies (137) where a is given by

$$a_n(x) = \begin{cases} \lambda_2 \frac{|u_n(x)|^{p_n-2} u_n(x) - |v_n(x)|^{p_n-2} v_n(x)}{u_n(x) - v_n(x)}, & \text{if } u_n(x) \neq v_n(x), \\ \lambda_2(p_n - 1)|u_n(x)|^{p_n-2}, & \text{if } u_n(x) = v_n(x). \end{cases}$$

Noting that

$$\frac{|t|^{p-2}t - |s|^{p-2}s}{t - s} \leq (p-1)(|t|^{p-2} + |s|^{p-2}),$$

we can apply Lebesgue's Dominated Convergence Theorem, implying, since $\alpha = \beta \neq 0$ a.e., that $a_n \rightarrow \lambda_2$ in $L^q(\Omega)$, for every $q < \infty$. In particular, for every $\varepsilon > 0$ and n large enough, we have

$$\|a_n(x) - \lambda_2\|_{L^{\frac{N}{2}}} < \varepsilon.$$

Since $P_{E(\lambda_2)}(u_n - v_n) = 0$, Lemma 19.1 implies $u_n = v_n$ for large n . This concludes the proof. $\square$

REMARK 19.3. It could seem more natural to apply the Implicit Function Theorem to prove Proposition 19.2 as done for example in [51] where the authors deal with a super-linear term with slow growth close to the first eigenvalue. However, the application of the

Implicit Function Theorem to our situation is more delicate than in [51] due to a lack of smoothness in p . Hence, we preferred a direct argument which leads to a weaker conclusion but is enough to our purpose.

19.2 Abstract Symmetry We now apply the previous uniqueness result to deduce partial symmetries of the l.e.n. solutions of Problem $(\mathcal{P}_p)$ when p is close to 2. The next Lemma is the key ingredient.

Lemma 19.4: *Let $M > 0, \alpha \in E(\lambda_2) \setminus B_{\frac{1}{M}}$ and $\bar{p}$ be given by Proposition 19.2. If u_p is a solution of Problem $(\mathcal{P}_p)$ such that $P_{E(\lambda_2)}u_p = \alpha$ and $u_p \in B_M$, with $p < \bar{p}$, and if $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is a continuous isomorphism satisfying, for every $u \in H_0^1(\Omega)$ and $2 < p < \bar{p}$, the conditions*

$$(i) T(E(\lambda_2)^\perp) \subseteq E(\lambda_2)^\perp, \quad (ii) T\alpha = \alpha, \quad (iii) J_p(Tu) = J_p(u),$$

then $Tu_p = u_p$.

PROOF. Let us begin by noting that the assumptions on T can be formulated by

$$(i') T(E(\lambda_2)) = E(\lambda_2), \quad (ii') T(E(\lambda_2)^\perp) = E(\lambda_2)^\perp, \quad (iii') T\alpha = \alpha, \quad (iv') J_p(Tu) = J_p(u).$$

By Proposition 19.2, it is sufficient to prove that Tu_p is a solution of Problem $(\mathcal{P}_p)$ and $P_{E(\lambda_2)}Tu_p = \alpha$. On the one hand, it follows from (iv') that, for all $v \in H_0^1(\Omega)$, $\langle J'_p(Tu_p)|v \rangle = \langle J'_p(u_p)|T^{-1}v \rangle$, so that we infer that Tu_p is a solution of $(\mathcal{P}_p)$. On the other hand, since

$$Tu_p = TP_{E(\lambda_2)}u_p + TP_{E(\lambda_2)^\perp}u_p = P_{E(\lambda_2)}Tu_p + P_{E(\lambda_2)^\perp}Tu_p,$$

and since the conditions (i') and (ii') ensure $TP_{E(\lambda_2)}u_p \in E(\lambda_2)$ and $TP_{E(\lambda_2)^\perp}u_p \in E(\lambda_2)^\perp$, we deduce $P_{E(\lambda_2)}Tu_p = TP_{E(\lambda_2)}u_p$. We then conclude the proof by using the condition (iii') . $\square$

Let G be a group with identity 1. Recall that a group action on $H_0^1(\Omega)$ is a continuous application $G \times H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ such that, for all $u \in H_0^1(\Omega)$ and for all $g, h \in G$,

$$(i) 1u = u, \quad (ii) (gh)u = g(hu), \quad (iii) u \mapsto gu \text{ is linear.}$$

EXAMPLE 19.5. To any group $G \subset O(N)$ such that $g(\Omega) = \Omega$ for every $g \in G$, one can associate the two following actions

$$gu(x) := u(gx) \text{ and } gu(x) := (\det g)u(gx).$$

When $G = \{1, R\}$, with R the reflection with respect to a hyperplane H , the fixed points of the action of G are, in the first case, the symmetric functions with respect to H , and, in the second case, the antisymmetric functions with respect to H .

The next theorem is a straightforward consequence of Lemma 19.4.

Theorem 19.6: *Let $(G_\alpha)_{\alpha \in E(\lambda_2)}$ be groups acting on $H_0^1(\Omega)$ in such a way that $\sup_{\alpha, g \in G_\alpha} \|g\| < \infty$, and that, for every $g \in G_\alpha$, $2 < p < \bar{p}$ and $u \in H_0^1(\Omega)$,*

$$(i) g(E(\lambda_2)^\perp) \subseteq E(\lambda_2)^\perp, \quad (ii) g\alpha = \alpha, \quad (iii) J_p(gu) = J_p(u).$$

Then, for all $M > 0$, if p is close enough to 2, any l.e.n. solutions $u_p \in \{u \in B_M : P_{E(\lambda_2)}(u) \notin B_{\frac{1}{M}}\}$ of Problem $(\mathcal{P}_p)$ belongs to the fixator of G_{α_p} where $\alpha_p := P_{E(\lambda_2)}u_p$.

It is worth pointing out that in particular, if G_α describes the symmetries (or antisymmetries) of α , we deduce that, for p close to 2, u_p respects the symmetries of its orthogonal projection α_p . Specific cases will be discussed in Section 21.

20 Asymptotic Behavior

We now turn to the proof of Theorem 17.4. We proceed in several steps.

20.1 Upper Bound By a suitable choice of test functions, we first obtain an upper bound of the energy of l.e.n. solutions of Problem $(\mathcal{P}_p)$.

Proposition 20.1: *Let $(u_p)_{p>2}$ be a family of l.e.n. solutions of $(\mathcal{P}_p)$. Then we have*

$$\limsup_{p \rightarrow 2} \|\nabla u_p\|_{L^2}^2 = \limsup_{p \rightarrow 2} \left(\frac{J_p(u_p)}{\frac{1}{2} - \frac{1}{p}} \right) \leq \|\nabla u_*\|_{L^2}^2,$$

where $u_* \in E(\lambda_2)$ minimizes the functional J_* on $\mathcal{N}_*$.

PROOF. Let $w \in H_0^1(\Omega)$ solving

$$-\Delta w - \lambda_2 w = \lambda_2 u_* \log|u_*|, \quad P_{E(\lambda_2)} w = 0. \quad (138)$$

Since u_* verifies (136), by the Fredholm alternative, w is well-defined. Set

$$v_p := u_* + (p-2)w$$

and

$$\hat{v}_p := t_p^+ v_p^+ + t_p^- v_p^-, \quad \text{where } t_p^\pm = \left(\frac{\|\nabla v_p^\pm\|_{L^2}^2}{\lambda_2 \|v_p^\pm\|_{L^p}^p} \right)^{\frac{1}{p-2}},$$

so that $\hat{v}_p \in \mathcal{M}_p$. We claim that $t_p^\pm \rightarrow 1$. By definition of v_p , we have

$$\begin{aligned} \int_{\Omega} |\nabla v_p^\pm|^2 &= \int_{\Omega} \nabla v_p \cdot \nabla v_p^\pm \\ &= \int_{\Omega} (\lambda_2 u_* + \lambda_2 (p-2)w + \lambda_2 (p-2)u_* \log|u_*|) v_p^\pm \\ &= \lambda_2 \int_{\Omega} (v_p + (p-2)u_* \log|u_*|) v_p^\pm \\ &= \lambda_2 \int_{\Omega} |v_p|^{p-2} v_p v_p^\pm \\ &\quad + \lambda_2 (p-2) \left(\int_{\Omega} \frac{v_p - |v_p|^{p-2} v_p}{p-2} v_p^\pm + \int_{\Omega} u_* \log|u_*| v_p^\pm \right). \end{aligned}$$

Since

$$\begin{aligned} \left| \frac{t - |t|^{p-2} t}{p-2} \right| &= \frac{1}{p-2} \int_2^p |t \log|t|| |t|^{q-2} dq \\ &\leq |\log|t|| (|t| + |t|^{p-1}) \\ &\leq \frac{1}{s} (|t|^{1-s} + |t|^{p-1+s}), \end{aligned} \quad (139)$$

where $s > 0$ has been chosen small enough, applying Lebesgue's Dominated Convergence Theorem, we infer that

$$\lim_{p \rightarrow 2} \int_{\Omega} \left(\frac{v_p - |v_p|^{p-2} v_p}{p-2} + u_* \log |u_*| \right) v_p^{\pm} = 0.$$

Therefore, we deduce that

$$\int_{\Omega} |\nabla v_p^{\pm}|^2 = \lambda_2 \int_{\Omega} |v_p^{\pm}|^p + o(p-2), \quad (140)$$

so that $\lim_{p \rightarrow 2} t_p^{\pm} = 1$. At last, since

$$J_p(u_p) \leq J_p(\hat{v}_p)$$

and

$$\left(\frac{1}{2} - \frac{1}{p} \right) J_p(\hat{v}_p) = \|\nabla \hat{v}_p\|_{L^2}^2 = \|\nabla u_*\|_{L^2}^2 + o(1),$$

the conclusion follows easily. $\square$

REMARK 20.2. When working with g.s. solutions on the Nehari manifold, one obtains an optimal upper bound by evaluating the functional on eigenfunctions that are in the Nehari manifold. This is not anymore the case for l.e.n. solutions. Such a choice leads to a coarser estimate:

$$\|\nabla u_*^+\|_{L^2}^2 \exp\left(\frac{-\int_{\Omega} |u_*^+|^2 \log |u_*^+|^2}{\int_{\Omega} |u_*^+|^2}\right) + \|\nabla u_*^-\|_{L^2}^2 \exp\left(\frac{-\int_{\Omega} |u_*^-|^2 \log |u_*^-|^2}{\int_{\Omega} |u_*^-|^2}\right),$$

This is why we define $v_p = u_* + (p-2)w$. Geometrically, this can be pictured as follows: the natural projection of u_* on the nodal Nehari set is far from u_* , but u_* gets nearer to the Nehari manifold as $p \rightarrow 2$.

20.2 Limit Equation In this section, we shall consider the weak accumulation points u_* of a bounded family $(u_p)_{p>2}$ of solutions of Problem $(\mathcal{P}_p)$, and prove that those functions verify a limit equation.

Proposition 20.3: *Let $(u_p)_{p>2}$ be a bounded family of solutions of $(\mathcal{P}_p)$. If $p_n \rightarrow 2$ and $u_{p_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$, then u_* solves*

$$\begin{cases} -\Delta u_* = \lambda_2 u_*, & \text{in } \Omega, \\ u_* = 0, & \text{on } \partial\Omega, \\ \int_{\Omega} u_* \log |u_*| v = 0, & \forall v \in E(\lambda_2). \end{cases}$$

PROOF. Let $v \in H_0^1(\Omega)$. By Rellich's theorem and Lebesgue's Dominated Convergence Theorem, we deduce that

$$|u_{p_n}|^{p_n-2} u_{p_n} \rightarrow u_* \text{ in } L^2(\Omega),$$

so that

$$\begin{aligned} \int_{\Omega} \nabla u_* \cdot \nabla v &= \lim_{n \rightarrow \infty} \int_{\Omega} \nabla u_{p_n} \cdot \nabla v \\ &= \lim_{n \rightarrow \infty} \lambda_2 \int_{\Omega} |u_{p_n}|^{p_n-2} u_{p_n} v \\ &= \lambda_2 \int_{\Omega} u_* v. \end{aligned}$$

Henceforth, $u_* \in E(\lambda_2)$. To prove the second statement, taking $v \in E(\lambda_2)$ and multiplying the equation in $(\mathcal{P}_p)$ by v leads to

$$\int_{\Omega} (|u_{p_n}|^{p_n-2} u_{p_n} - u_{p_n}) v = 0. \quad (141)$$

Finally, arguing as in the previous theorem, we conclude using Lebesgue's Dominated Convergence Theorem and (139) that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{(|u_{p_n}|^{p_n-2} u_{p_n} - u_{p_n}) v}{p_n - 2} = \int_{\Omega} u_* \log |u_*| v.$$

Taking (141) into account, this completes the proof. $\square$

20.3 Lower Bound We now prove that l.e.n. solutions of problem $(\mathcal{P}_p)$ stay away from 0 when $p \rightarrow 2$.

Proposition 20.4: *Let $(u_p)_{p>2}$ be a family of l.e.n. solutions of Problem $(\mathcal{P}_p)$. Then we have*

$$\liminf_{p \rightarrow 2} \|\nabla u_p\|_{L^2} > 0.$$

PROOF. For each p , we define the function v_p as follows. Let $r = \frac{\int_{\Omega} u_p^+ e_1}{\int_{\Omega} |u_p| e_1}$ and take

$$\tilde{v}_p = (1-r)u_p^+ + ru_p^-.$$

Our choice of r implies that $\tilde{v}_p$ is orthogonal to e_1 . Choose now $t \geq 0$ in such a way that $v_p := t\tilde{v}_p \in \mathcal{N}_p$. By construction, $v_p \in E(\lambda_1)^\perp \cap \mathcal{N}_p$. Since $u_p \in \mathcal{M}_p$, we have

$$J_p(v_p) \leq J_p(u_p),$$

so that $\|\nabla v_p\|_{L^2} \leq \|\nabla u_p\|_{L^2}$. It is now sufficient to show that v_p is bounded away from 0 in $H_0^1(\Omega)$ when $p \rightarrow 2$.

By interpolation of Hölder's inequality between 2 and $q < 2^*$ with $p < q$, we obtain

$$\|v_p\|_{L^p}^2 \leq \|v_p\|_{L^2}^{2-2s} \|v_p\|_{L^q}^{2s},$$

where $s := \frac{q}{2-q} \frac{2-p}{p}$. On the other hand, as v_p is orthogonal to e_1 , the variational characterization of λ_2 (see Remark 4.4) ensures

$$\lambda_2 \int_{\Omega} v_p^2 \leq \int_{\Omega} |\nabla v_p|^2.$$

Therefore, by Sobolev Inequalities (with S its optimal constant), we obtain

$$\|v_p\|_{L^p}^2 \leq (\lambda_2^{-1} \|\nabla v_p\|_{L^2}^2)^{1-s} (S \|\nabla v_p\|_{L^2}^2)^s.$$

As v_p belongs to the Nehari manifold $\mathcal{N}_p$, we have $\|\nabla v_p\|_{L^2}^2 = \lambda_2 \|v_p\|_{L^p}^p$. It then follows that

$$\|\nabla v_p\|_{L^2} \geq (c\lambda_2)^{\frac{sp}{2(2-p)}} \lambda_2^{\frac{1}{2}}$$

and from our choice of s , we conclude that

$$\|\nabla v_p\|_{L^2} \geq (c\lambda_2)^{\frac{q}{2(2-q)}} \lambda_2^{\frac{1}{2}}.$$

$\square$

20.4 Conclusion

Bringing together the previous propositions, we now deduce

Theorem 20.5: *Let $(u_p)_{p>2}$ be a family of l.e.n. solutions of Problem $(\mathcal{P}_p)$. If $p_n \rightarrow 2$ and $u_{p_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$, then $u_{p_n} \rightarrow u_*$ in $H_0^1(\Omega)$, u_* is such that and*

$$\begin{aligned} -\Delta u_* &= \lambda_2 u_* \text{ in } \Omega, \quad u_* = 0 \text{ on } \partial\Omega, \\ J_*(u_*) &= c := \inf\{J_*(u) : u \in E(\lambda_2) \setminus \{0\}, \langle J'_*(u), u \rangle = 0\}. \end{aligned}$$

PROOF. Observe first that

$$\inf\{\|\nabla u\|_{L^2}^2 : u \in E(\lambda_2) \setminus \{0\}, \int_{\Omega} u^2 \log u^2 = 0\} = 2c$$

and by Lemma 20.3, $u_* \in E(\lambda_2)$. Applying successively Lemma 20.1, the weak lower semi-continuity of the norm and Lemma 20.3, we deduce that

$$2c \geq \limsup_{n \rightarrow \infty} \|\nabla u_{p_n}\|_{L^2}^2 \geq \liminf_{n \rightarrow \infty} \|\nabla u_{p_n}\|_{L^2}^2 \geq \|\nabla u_*\|_{L^2}^2 \geq 2c.$$

We hence conclude that $\lim_{n \rightarrow \infty} \|\nabla u_{p_n}\|_{L^2}^2 = \|\nabla u_*\|_{L^2}^2 = 2c$. This also implies immediately the strong convergence of the sequence $(u_{p_n})_n$. $\square$

21 Symmetries on Specific Domains

The abstract symmetry result presented in Section 19 allows to prove that, if some symmetry is shared by all the second eigenfunctions, then it is inherited by the solutions u_p . We illustrate our purposes with some numerical experiments, made by C. Grumiau, using the mountain pass algorithm (MPA) and the modified mountain pass algorithm (MMPA) to compute one-signed and sign-changing solutions of Problem $(\mathcal{P}_p)$. We refer the reader to Appendix D for some technical details on these algorithms, and for some precision about the numerical experiments done here.

21.1 Symmetries in the non-Degenerate Case

Corollary 21.1: *If λ_2 is simple, for any reflection R such that $R(\Omega) = \Omega$ and p close to 2, l.e.n. solutions of Problem $(\mathcal{P}_p)$ respect the symmetry or antisymmetry of e_2 with respect to R .*

PROOF. Consider the first group action of Example 19.5 if $e_2 \circ R = e_2$ or the second one if $e_2 \circ R = -e_2$, and conclude by Theorem 19.6. $\square$

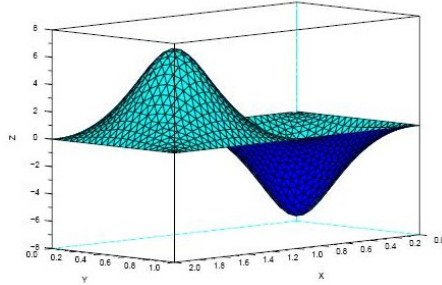
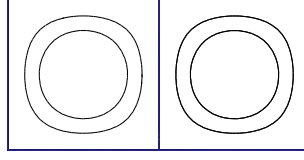


Figure 21.1 MMPA solution of the Lane–Emden problem on a rectangle.

Figure 21.2 Level lines at levels $-1, -0.5, 0, 0.5$ and 1 .

Corollary 21.1 can be illustrated numerically. Let us consider a rectangle of side-lengths 1 and 2. Figure 21.1 (and Table 1 and 2 in Appendix D) shows a nodal solution u of the problem $-\Delta u = u^3$ with Dirichlet boundary conditions obtained by the MPA. While it is not proved that this solution has least energy, all the other nodal solutions that we have found numerically have a larger energy. Moreover, the Mountain-Pass Algorithm (MPA) suggests that u^+ and u^- are g.s. solutions of Problem $(\mathcal{P}_p)$ on the squares defined by the nodal regions. One sees in Figure 21.2 that u is antisymmetric with respect to the shortest median, which is thus also the nodal line. More examples can be found in [31].

21.2 Symmetries on the Ball

Corollary 21.2: *If Ω is a ball, for p close to 2, i.e.n. solutions of Problem $(\mathcal{P}_p)$ are radially symmetric with respect to $N - 1$ independent directions and antisymmetric with respect to the orthogonal one.*

PROOF. By classical separation of variables and properties of the zeroes of Bessel functions, any second eigenfunction can be written as $e(x) = R(|x|)(\xi, x)$, $\xi \in \mathbb{R}^N$ (see Theorem 4.3 and Appendix E). Therefore, up to rotation, it belongs to the fixator of the group $G = O(N - 1) \times \{1, -1\}$ acting as $gu(x', x_N) = ku(hx', kx_N)$, for $g := (h, k)$. Previous arguments allow to conclude. $\square$

Let us illustrate this on the unit ball in $\mathbb{R}^2$. Figure 21.3 (and Table 1 and 2 in Appendix D) depicts a nodal solution u of the problem $-\Delta u = u^3$ with Dirichlet boundary conditions. As previously, comparison with the energy of other numerically computed nodal solutions of the Dirichlet problem and the application of the MPA to u^+ and u^- suggest that the algorithm indeed caught a least energy nodal solution. One should note that, as shown by the result (AP) of Theorem 5.7, u is not radial, u is symmetric with respect to a direction as obtained by the result (BWW) of Theorem 5.7 and, as suggested by Corollary 21.2, antisymmetric in the orthogonal one. In particular, the nodal line is a diameter.

21.3 Symmetries on the Square

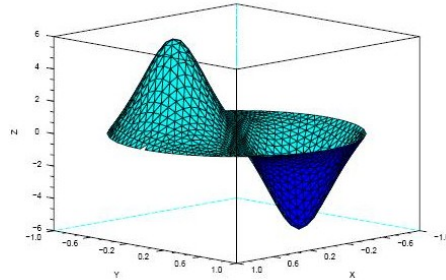
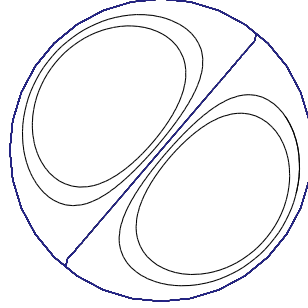


Figure 21.3 MPA solution of the Lane-Emden problem on a ball.

Figure 21.4 Level lines at levels $-1, -0.5, 0, 0.5$ and 1 .

Corollary 21.3: *If Ω is a square, then, for p close to 2, l.e.n. solutions of Problem $(\mathcal{P}_p)$ are odd with respect to the center of the square.*

PROOF. Without loss of generality, we can work on square $[-1, 1]^2$. The functions

$$v_1(x, y) = \cos\left(\frac{\pi}{2}x\right) \sin(\pi y), \quad v_2(x, y) = \sin(\pi x) \cos\left(\frac{\pi}{2}y\right),$$

form an orthonormal basis of $E(\lambda_2)$. One checks directly that all the second eigenfunctions of $-\Delta$ are odd functions. They all belong to the fixator of the group $G := \{1, -1\}$ acting in such a way that $(-1)u(x) = -u(-x)$. One concludes by Theorem 19.6. $\square$

A nodal solution u of the problem $-\Delta u = u^3$ with Dirichlet boundary conditions on the square $[-1, 1]^2$ is depicted in Figure 21.5 (and Table 1 and 2 in Appendix D). As explained in the previous cases, it is expected that u is a l.e.n. solution. Figure 21.6 suggests

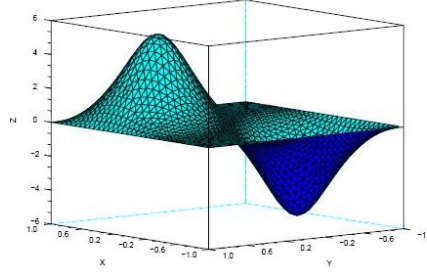
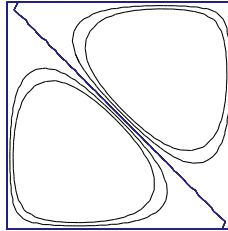


Figure 21.5 MPA solution of the Lane-Emden problem on a square.

Figure 21.6 Level lines at levels $-1, -0.5, 0, 0.5$ and 1 .

that u is odd.

Moreover, the nodal line of u seems to be a diagonal, u is antisymmetric with respect to that diagonal and symmetric with respect to the other diagonal. Further numerical

computations confirm our guess. Indeed, for $\theta \in \mathbb{R}$, set

$$v_\theta = \cos \theta v_1 + \sin \theta v_2$$

and consider the function

$$E_*(\theta) = \sup_{t>0} J_*(tv_\theta).$$

Since $\int_\Omega |v_\theta|^2 = 1$, one easily computes

$$S_*(\theta) := \log \frac{2}{\lambda_2} E_*(\theta) = - \int_\Omega |v_\theta|^2 \log |v_\theta|^2.$$

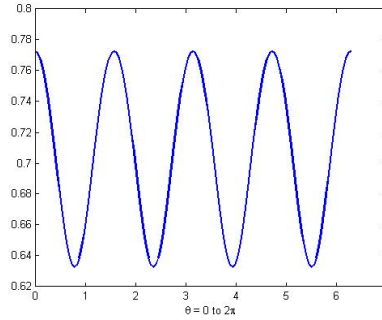


Figure 21.7 The function S_* .

This function can be thought as the entropy associated to the density $|v|^2$, and the accumulation points of the minimal energy nodal solutions are the eigenfunctions with minimal entropy. The numerical computation of S_* , see Figure 21.7, suggests that these eigenfunctions are the functions v_θ with $\theta = \frac{\pi}{4} + k\frac{\pi}{2}$. Up to rotations of $k\frac{\pi}{2}$, these can be written as

$$4 \cos \frac{\pi}{2} x \cos \frac{\pi}{2} y \cos \frac{\pi}{2} (x-y) \sin \frac{\pi}{2} (x+y).$$

These computations legitimate the following conjecture.

CONJECTURE 21.4. If Ω is a square, then, for p close to 2, any l.e.n. solution of Problem $(\mathcal{P}_p)$ is symmetric with respect to one diagonal and antisymmetric with respect to the orthogonal one.

22 Symmetry Breaking

In the previous section, the study of accumulation points of l.e.n. solutions $(u_p)_{p>2}$ allowed to exhibit the symmetries of l.e.n. solutions for p close to 2. This analysis can also be used to prove some symmetry breaking. We begin by studying small perturbations of the Laplacian.

22.1 Perturbation of the Laplacian Operator

The previous results can be extended to the problem

$$-\operatorname{div}(A_p \nabla u) = \lambda_2 |u|^{p-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (\tilde{\mathcal{P}}_p)$$

with $A_p \in \mathcal{C}(\Omega, S^{N \times N})$, where $S^{N \times N}$ is the set of symmetric $N \times N$ matrices, such that $A_2 = \operatorname{id}$ and $p \mapsto A_p$ is uniformly differentiable at $p = 2$, i.e.

$$\left\| \frac{A_p - A_2 - A'_2(p-2)}{p-2} \right\|_\infty \rightarrow 0$$

as $p \rightarrow 2$.

We are interested in critical points of the functional

$$\tilde{J}_p : H_0^1(\Omega) \rightarrow \mathbb{R} : u \mapsto \frac{1}{2} \int_{\Omega} (A_p \nabla u) \cdot \nabla u - \frac{\lambda_2}{p} \int_{\Omega} |u|^p.$$

Theorem 22.1: *Let $(u_p)_{p>2}$ be a family of l.e.n. solutions of Problem $(\tilde{\mathcal{P}}_p)$. If $p_n \rightarrow 2$ and $u_{p_n} \rightarrow u_*$ in $H_0^1(\Omega)$, then $u_{p_n} \rightarrow u_*$ in $H_0^1(\Omega)$, u_* satisfies*

$$-\Delta u_* = \lambda_2 u_* \text{ in } \Omega, \quad u_* = 0 \text{ on } \partial\Omega,$$

and

$$\tilde{J}_*(u_*) = \inf\{\tilde{J}_*(u) : u \in E(\lambda_2) \setminus \{0\}, \langle J'_*(u), u \rangle = 0\},$$

where

$$\tilde{J}_* : E(\lambda_2) \rightarrow \mathbb{R} : u \mapsto \int_{\Omega} A'_2 \nabla u \cdot \nabla u + \frac{\lambda_2}{2} (u^2 - u^2 \log u^2).$$

SKETCH OF THE PROOF. All the arguments developed in Section 20.1 hold for $(\tilde{\mathcal{P}}_p)$, up to some small differences. Indeed, the equations (138) and (140) become respectively

$$-\Delta w - \lambda_2 w = \operatorname{div}(A'_2 \nabla u_*) + \lambda_2 u_* \log |u_*|, \quad P_{E(\lambda_2)} w = 0,$$

and

$$\begin{aligned} \int_{\Omega} |\nabla v_p^{\pm}|^2 &= \lambda_2 \int_{\Omega} (v_p + (p-2)u_* \log |u_*|) v_p^{\pm} - (p-2) \int_{\Omega} (A'_2 \nabla u_*) \cdot \nabla v_p^{\pm} \\ &= \lambda_2 \int_{\Omega} |v_p^{\pm}|^p - (p-2) \int_{\Omega} (A'_2 \nabla u_*) \cdot \nabla v_p^{\pm} + o(p-2). \end{aligned}$$

The computation of $t_p^{\pm}$ gives here

$$(t_p^{\pm})^{p-2} = \frac{\|A_p \nabla v_p^{\pm}\|_{L^2}^2}{\lambda_2 \|v_p^{\pm}\|_{L^p}^p} = \frac{\|\nabla v_p^{\pm}\|_{L^2}^2 + (p-2) \int_{\Omega} \left(\frac{A_p - \operatorname{id}}{p-2} \nabla v_p \right) \cdot \nabla v_p^{\pm}}{\lambda_2 \|v_p^{\pm}\|_{L^p}^p},$$

and the proof that $t_p^{\pm} \rightarrow 1$ follows from the fact that

$$\begin{aligned} &(p-2) \int_{\Omega} \left(\frac{A_p - \operatorname{id}}{p-2} \nabla v_p \right) \cdot \nabla v_p^{\pm} - (p-2) \int_{\Omega} (A'_2 \nabla u_*) \cdot \nabla v_p^{\pm} \\ &= (p-2) \int_{\Omega} \left(\left(\frac{A_p - \operatorname{id}}{p-2} - A'_2 \right) \nabla v_p \right) \cdot \nabla v_p^{\pm} + (p-2)^2 \int_{\Omega} (A'_2 \nabla w) \cdot \nabla v_p^{\pm} \\ &= o(p-2), \end{aligned}$$

giving then the same conclusion as for Problem $(\mathcal{P}_p)$.

In a similar fashion, all the results of Sections 20.2 and 20.3 extend to the present framework, so that we can conclude as in Section 20.4. $\square$

Going to abstract symmetry results, Proposition 19.2 about the uniqueness of the solution with a given projection on $E(\lambda_2)$ extends immediately to Problem $(\tilde{\mathcal{P}}_p)$.

Proposition 22.2: *For every $M > 0$, there exists $\bar{p} > 2$ such that, for every $p \in (2, \bar{p})$, if $u_p, v_p \in \{u \in B_M : P_{E(\lambda_2)}(u) \notin B_{\frac{1}{M}}\}$ solve the boundary value problem*

$$-\operatorname{div}(A_p \nabla u) = \lambda_2 |u|^{p-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

then either $u_p = v_p$ or $P_{E(\lambda_2)} u_p \neq P_{E(\lambda_2)} v_p$.

So, we obtain

Theorem 22.3: *Let $(G_\alpha)_{\alpha \in E(\lambda_2)}$ be groups acting on $H_0^1(\Omega)$ in such a way that $\sup_{\alpha, g \in G_\alpha} \|g\| < \infty$, and that, for every $g \in G_\alpha$, $2 < p < \bar{p}$ and $u \in H_0^1(\Omega)$,*

$$(i) g(E(\lambda_2)^\perp) \subseteq E(\lambda_2)^\perp, \quad (ii) g\alpha = \alpha, \quad (iii) J_p(gu) = J_p(u).$$

Then, for p close to 2, any l.e.n. solutions u_p of Problem $(\tilde{\mathcal{P}}_p)$ belongs to the fixator of G_{α_p} where $\alpha_p := P_{E(\lambda_2)} u_p$.

22.2 Symmetry Breaking on the Rectangle On a rectangle R_v with sides of lengths 2 and $(1+v)2$, we consider the problem

$$-\Delta u = \lambda_2 |u|^{p-2} u \text{ in } R_v, \quad u = 0 \text{ on } \partial R_v. \quad (\mathcal{P}_v)$$

The change of variable $\tilde{u}(x, y) = u(x, (1+v)y)$, leads to the equivalent problem on the square $Q = (-1, 1)^2$

$$-\partial_x^2 \tilde{u} - (1+v)^{-2} \partial_y^2 \tilde{u} = \lambda_2 |\tilde{u}|^{p-2} \tilde{u} \text{ in } Q, \quad \tilde{u} = 0 \text{ on } \partial Q.$$

In both problems, $\lambda_2 = \lambda_2(Q)$. Observe also that u is a least energy nodal solution of $(\mathcal{P}_v)$ if and only if $\tilde{u}$ is a least energy nodal solution on the square Q .

Theorem 22.4: *There exists $c > 0$ and $\bar{p}$ such that if $2 < p < \bar{p}$ and $|v| \leq c(p-2)$, then any l.e.n. solution of problem $(\mathcal{P}_v)$ is neither symmetric nor antisymmetric with respect to one of the medians of R_v .*

PROOF. Assume by contradiction, that there exists $p_n \rightarrow 2$ and $v_n = o(p_n - 2)$ such that Problem $(\mathcal{P}_{v_n})$ has a solution u_n that is symmetric or antisymmetric to one of the medians. Define $\tilde{u}_n$ by the change of variables given above. The functions $\tilde{u}_n$ are least energy nodal solutions of the problem $(\tilde{\mathcal{P}}_{p_n})$ with $\Omega = Q$ and

$$A_{p_n} = \text{id} - (1 - (1 + v_n)^{-2}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The corresponding functional $\tilde{J}_*$ reads as J_* for Problem $(\mathcal{P}_p)$. In particular, the accumulation points of the sequence are symmetric or antisymmetric with respect to one of the medians. On the other hand, numerical computations illustrated by Figure 21.7 shows that such eigenfunctions cannot be minimizers of $\tilde{J}_*$ on $\tilde{\mathcal{N}}_*$ (one would indeed conjecture that they are maximizers on $\tilde{\mathcal{N}}_*$). $\square$

Now, consider the case where $v = \gamma(p-2)$. Arguing as in section 21.3, we can compute

$$\tilde{E}_*(\theta) = \sup_{t>0} \tilde{J}_*(t\theta)$$

and

$$\begin{aligned} \tilde{S}_{*,\gamma}(\theta) &:= \log \frac{2}{\lambda_2} \tilde{E}_*(\theta) \\ &= S_*(\theta) - \frac{4\gamma}{\lambda_2} \int_Q (\cos \theta \partial_y v_1 + \sin \theta \partial_y v_2)^2 \\ &= S_*(\theta) - \frac{4\gamma}{5} (3 \cos^2 \theta + 1). \end{aligned}$$

The function S achieves a global minimum at $\frac{\pi}{4} + k\frac{\pi}{2}$, see Figure 21.7, while the function $\frac{-4\gamma}{5}(3\cos^2\theta + 1)$ achieves its global minimum at $k\pi$. Since these points are critical points of S_* , the global minimizers of $\tilde{S}_{*,\gamma}(\theta)$ are $k\pi$ when γ is large enough. The functions v_θ for these values of θ being symmetric with respect to the medians of Q , we naturally conjecture

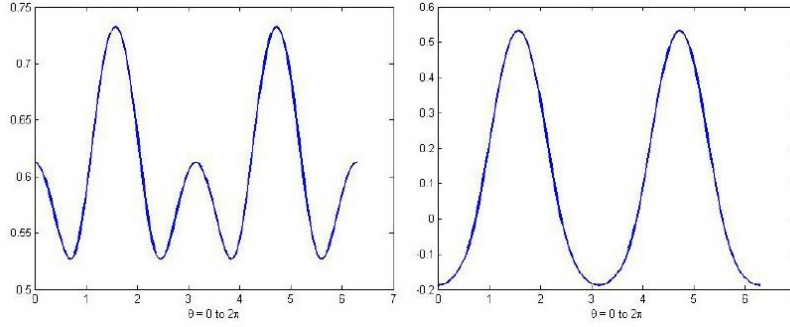


Figure 22.1 Computation of $\tilde{S}_{*,\gamma}$ with $\gamma = 0.05$ on the left, $\gamma = 0.3$ on the right.

CONJECTURE 22.5. There exists $c > 0$ and $\bar{p} > 2$, such that if $v > c(p-2)$ and $2 < p < \bar{p}$, then any l.e.n. solution to problem $(\mathcal{P}_v)$ is symmetric with respect to the longest median and antisymmetric with respect to the shortest one.

The shapes of the graphs of $\tilde{S}_{*,0.05}$ and $\tilde{S}_{*,0.5}$ (see Figure 22.1) seem to indicate that the threshold value for γ corresponds to a degenerate minimum. Hence, it is expected that the first γ for which $k\pi$ is a global minimum should verify $\tilde{S}_{*,\gamma}''(k\pi) = S_*''(k\pi) + \frac{24}{5}\gamma = 0$. A numerical approximation of the second derivative of S_* gives

$$\gamma = 0.207.$$

Generalizations and Improvements

In this chapter, we improve the study of the symmetries and asymptotics of the Lane–Emden problem. We begin in Section 23 by studying some existence results about Problem

$$A_\mu u = \lambda |u|^{p-2} u \text{ in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega, \quad (\mathcal{P}_{\mu,\lambda,p})$$

we continue in Sections 24–26 with a discussion, made in collaboration with J. Van Schaftingen, about the result (PW) of Theorem 5.7 which concerns Dirichlet problems

$$-\Delta u = f(x, u) \text{ in } B^N, \quad u = 0 \text{ on } \partial B^N, \quad (\mathcal{Q})$$

and we finish in Sections 27–31 with the generalization of the previous chapter made in collaboration with D. Bonheure and C. Grumiau [10], by working with problems

$$A_\mu u = f_\theta(u) \text{ in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega, \quad (\mathcal{P}_{\mu,\lambda,\theta})$$

where

- Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$,
- $A_\mu := -\Delta + \mu \text{ id}$, $\lambda > 0$ and $(Bu, \mu) = (u, 0)$ or $(\partial_\nu u, 1)$,
- $f : \Omega \times \mathbb{R} : (x, t) \mapsto f(x, t)$ is continuous and f'_t is convex continuous,
- $(f_\theta)_{\theta > 2} \subset \mathcal{C}^1(\mathbb{R})$ is a family of perturbations of $\lambda \text{ id}$.

We shall use the same notations as in Remark 4.4 for $\lambda_i, E(\lambda_i), P_{E(\lambda_i)}$ with λ_i an eigenvalue of A_μ , and with the same notations as in Section 3 for the functionals and manifolds $J_p, \mathcal{N}_p, \mathcal{M}_p$ for Problem $(\mathcal{P}_{\mu,\lambda,p})$, $J_\theta, \mathcal{N}_\theta, \mathcal{M}_\theta$ for Problem $(\mathcal{P}_{\mu,\lambda,\theta})$.

The Dirichlet and Neumann cases will be referred later only by writing $\mu = 0$ or $\mu = 1$. In this two cases, the domain E of J_p or J_θ is respectively H_0^1 or H^1 .

23 Existence Results About Problem (LE)

In this section, we shall work with the Lane–Emden (LE) problem $(\mathcal{P}_{\mu,\lambda,p})$. The previous chapter was concerned with the g.s. and the l.e.n. solutions. The existence of these solutions is a direct consequence of Theorem 3.3. Let us show how Procedure 2.3 gives a simple alternative proof of an existence result.

Theorem 23.1: *Let λ be an eigenvalue of A_μ , and $J_*, \mathcal{N}_*$ defined by:*

$$J_* : E(\lambda) \rightarrow \mathbb{R} : u \mapsto \frac{\lambda}{2} \int_\Omega u^2 - u^2 \log u^2, \quad \mathcal{N}_* := \{u \neq 0 : \langle J'_*(u) | u \rangle = 0\}.$$

If a minimizer of J_ on $\mathcal{N}_*$ is isolated, there exists $\bar{p} > 2$ such that for any $p \in [2, \bar{p})$, Problem $(\mathcal{P}_{\mu,\lambda,p})$ has at least one solution.*

PROOF. Let us write Problem $(\mathcal{P}_{\mu,\lambda,p})$ as $L(u) - N(p, u) = 0$ where

$$L : E \rightarrow E' : u \mapsto (A_\mu - \lambda \text{ id})u = \int_\Omega [\nabla u \nabla \cdot + (\mu - \lambda)u],$$

$$N(p, \cdot) : E \rightarrow E' : u \mapsto \lambda(|u|^{p-2}u - u) = \lambda \int_{\Omega} (|u|^{p-2}u - u) \cdot.$$

L is a Fredholm operator with index 0 since we can consider subspaces and operators like in (22), (23), defined by

$$E_1 = E(\lambda)^\perp, \quad E_2 = E(\lambda),$$

$$F_1 = \{f \in E' : f|_{E_2} = 0\}, \quad F_2 = \{f \in E' : f|_{E_1} = 0\},$$

and with p, q, P, Q defined as the orthogonal projections on E_1, E_2, F_1, F_2 .

Let $\alpha \in E_2$. Let us consider the first equation of the last equivalence of (24):

$$p(u) = \lambda(A_\mu - \lambda \text{id})^{-1} [\{|p(u) + \alpha|^{p-2}(p(u) + \alpha) - p(u) - \alpha\}|_{E_1}] =: \tilde{N}(p, \alpha, p(u)).$$

This equation can be solved with the Implicit Function Theorem applied to the operator $\text{id}_{E_1} - \tilde{N}(p, \alpha, \cdot) : E_1 \rightarrow E_1$ at the point $p = 2, \alpha = \alpha_0, p(u) = 0$, as the derivative of this operator at this point equals id_{E_1} . So, we can consider the function γ defined by (142) and the second equation of the last equivalence of (24) given by (143),

$$\gamma : [2, \bar{p}) \times B(\alpha_0, r) \cap E_2 \rightarrow E_1 : (p, \alpha) \mapsto p(u), \quad p(u) = \tilde{N}(p, \alpha, p(u)), \quad (142)$$

$$\forall v \in E_2 : \int_{\Omega} [|\gamma + \alpha|^{p-2}(\gamma + \alpha) - \gamma - \alpha] v = 0, \quad (143)$$

(where γ is denoted for $\gamma(p, \alpha)$) that we shall denote by $g(p, \alpha) = 0$. As this last equation is degenerate when $p = 2$, we shall work with equation $\tilde{g}(p, \alpha) = 0$ where $\tilde{g} : [2, \bar{p}) \times B(\alpha_0, r) \cap E_2 \rightarrow F_2$ is defined by

$$\tilde{g}(p, \alpha) = \begin{cases} \frac{1}{p-2} \int_{\Omega} [|\gamma + \alpha|^{p-2}(\gamma + \alpha) - \gamma - \alpha] \cdot & \text{if } p > 2, \\ \int_{\Omega} \log|\gamma + \alpha|(\gamma + \alpha) \cdot & \text{if } p = 2. \end{cases}$$

Let us note that the zeros of $\tilde{g}(2, \cdot)$ are the critical points of J_* . Under the assumption of the theorem, it can be shown that $\deg_B(\tilde{g}(2, \cdot), B(\alpha_*, r)) \neq 0$ with α_* isolated minimizer of J_* on $\mathcal{N}_*$ (see for example [27]). The homotopy invariance and existence properties of the Brouwer degree lead to the conclusion. $\square$

REMARK 23.2. We can do the same comments as in Remark 3.1 about an alternative proof using the Leray-Schauder degree. On an other hand, we can not use the Implicit Function Theorem as in the section 3 of Appendix A, due to the lack of smoothness of the operators considered here. Indeed, the candidate for the derivative of $\tilde{g}(2, \cdot)$ at α_* is

$$E(\lambda) \times E(\lambda) \rightarrow \mathbb{R} : (v, w) \mapsto \int_{\Omega} (\log|\alpha_*| + 1)vw,$$

whose integrand may not be integrable.

Let us now work with $\mu = 1$. While it is not the case for the Dirichlet boundary conditions, the Neumann problem has a line of trivial solutions: $u \equiv 1$. This observation led D. Bonheure to study the Neumann case with the bifurcation theory.

Proposition 23.3 (Bonheure): *We have:*

- $\exists \bar{p} > 2 : 2 < p \leq \bar{p} \Rightarrow 1$ is the only positive solution of $(\mathcal{P}_{1,1,p})$,
- $\forall i \in \mathbb{N}_0 : \lambda_i < p - 1 < \lambda_{i+1} \Rightarrow \text{ind}_M(1, J_p) = \sum_{k \leq i} \dim E(\lambda_k)$.

PROOF. We shall give a sketch of the proof by showing the first part of the thesis on any ball B_R . Let us consider u a positive solution of $(\mathcal{P}_{1,1,p})$ in B_R , and denote by u_1, u_2

the projections of u respectively on $E(\lambda_1)$ and $E(\lambda_1)^\perp$. By testing the problem with u_2 we have

$$\begin{aligned} \lambda_2 \int_{\Omega} u_2^2 &\leq \int_{\Omega} |\nabla u_2|^2 + u_2^2 = \int_{\Omega} u^{p-1} u_2 = \int_{\Omega} [(u_1 + u_2)^{p-1} - u_1^{p-1}] u_2 \\ &= \int_{\Omega} (p-1)(u_1 + \theta u_2)^{p-2} u_2^2 \leq (p-1) \|u\|_{\infty}^{p-2} \int_{\Omega} u_2^2 \end{aligned}$$

where θ is a measurable function from Ω to $(0, 1)$. This is a consequence of Remark 4.4, $u_2 \perp C^{\text{te}}$ and $u \geq 0$ solution of Problem $(\mathcal{P}_{1,1,p})$ whose weak solutions coincide with classical ones. These inequalities, joined to $\|u\|_{\infty} \leq c \|u\|^{n(p-1)}$ for some $c > 0$ and $n \in \mathbb{N}$ independent of p and obtained by classical bootstrap arguments, lead to $u_2 = 0$ if p is close enough to 2 because of $\|u\| \leq R$. For a uniform estimate independent of R , Bonheure used, on one hand an integration of the equation leading to

$$\int_{\Omega} u = \int_{\Omega} u^{p-1} \Rightarrow \int_{\Omega} u = \int_{\Omega} u^{p-1} \leq |\Omega|, \quad (144)$$

and on an other hand, a result of Ni and Takagi which begin the bootstrap with an estimate of $\|u\|_{W^{1,q}}$ by the norm L^1 of the right member of the equation, so $\int_{\Omega} u^{p-1}$. Inequality (144) permits then to conclude the first part of the proposition.

For the second part, let us recall that the Morse index of u with respect to J_p is the sum of the dimensions of the eigenspaces associated to the negative eigenvalues $\tilde{\lambda}$ of the following weak boundary value problem:

$$-\Delta v + v - (p-1)|u|^{p-2}v = \tilde{\lambda}v, \quad v \in H^1.$$

If $u = 1$, there exists an eigenvalue λ of $-\Delta + \text{id}$ such that $\tilde{\lambda} = \lambda - p + 1$. We conclude the proof by the following equality:

$$\text{ind}_M(1, J_p) = \sum_{\lambda_k - p + 1 < 0} \dim E(\lambda_k).$$

□

In fact, as conjectured by Bonheure, this changing of Morse index is a sufficient condition to prove bifurcations:

Theorem 23.4: *For any $i \in \mathbb{N}_0$, $(p, u) = (\lambda_i + 1, 1)$ is a bifurcation point of the family of problems $(\mathcal{P}_{1,1,p})$ with $2 < p < 2^*$.*

PROOF. Let us define $u = 1 + v$ and $p = 1 + q$. The research of zeros of $J'_p : H^1 \rightarrow (H^1)'$ can be formulated by $L(q, v) - N(q, v) = 0$ where

$$L : [1, 2^* - 1] \times H^1 \rightarrow (H^1)' : (q, v) \mapsto (-\Delta + \text{id})v - qv,$$

$$N : [1, 2^* - 1] \times H^1 \rightarrow (H^1)' : (q, v) \mapsto |1 + v|^{q-1}(1 + v) - (qv + 1).$$

Up to the isomorphism $H^1 \equiv (H^1)'$ and to the factorization by q , L and N are like in Result (KBM) of Theorem 2.2. Indeed, while it is clear for L , let us show $\lim_{v \rightarrow 0} N(q, v)/\|v\| = 0$ locally uniformly in q . We shall prove that if $(v_n)_n$ is a sequence of $\partial B(0, 1)$ in H^1 , $(t_n)_n$ a sequence converging to 0, and $(q_n)_n$ a bounded sequence in $(1, 2^* - 1)$, then

$$\lim_{n \rightarrow \infty} \frac{1}{t_n} \int_{\Omega} [|1 + t_n v_n|^{q_n-1}(1 + t_n v_n) - 1 - q_n t_n v_n] w = 0$$

uniformly with respect to $w \in B(0, 1)$ in H^1 . This integral is smaller than (145) whose the norm's argument of the first factor equals (146) and verifies (147):

$$\frac{1}{t_n} \left\| |1 + t_n v_n|^{q_n-1} (1 + t_n v_n) - 1 - q_n t_n v_n \right\|_{L^{q+1/q}} \cdot \|w\|_{L^{q+1}}, \quad (145)$$

$$f_n := q_n |1 + \theta_n t_n v_n|^{q_n-1} v_n - q_n v_n, \quad (146)$$

$$|f_n| \leq 2q |1 + |v_n||^q, \quad f_n \rightarrow 0, \quad (147)$$

where $q := \max q_n$. Since (up to a subsequence) v_n converges to some v in L^{q+1} , there exists $h \in L^{q+1}(\Omega)$ such that $|v_n| \leq h$ and we conclude by the Lebesgue's Dominated Convergence Theorem.

Since J_p verifies the condition (PS), the previous reasoning joined to the result (KBM) of Theorem 2.2, leads to the conclusion. $\square$

REMARK 23.5. In the previous proof, we could also consider the result (K) of Theorem 2.2. The thesis would be proved only for eigenvalues λ_i such that $m(\lambda_i)$ is odd, but could be clarified by the Global Bifurcation Theorem of Rabinowitz saying that the bifurcation branches are either unbounded, either linked together by pairs of λ_i .

The purpose of Bonheure was to study the symmetry breaking of Problem $(\mathcal{P}_{1,1,p})$. Indeed, it is known that in a radial domain Ω , for p close enough to 2^* , the g.s. solutions are not radial. As Proposition 23.3 ensures that for p close enough to 2 the g.s. solutions equal 1, and since the Morse index of a g.s. solution is 1, the following conjecture, verified numerically by Grumiau, can be formulated:

CONJECTURE 23.6 (Bonheure). Let us consider the family of problems $(\mathcal{P}_{1,1,p})$ with $2 < p < 2^*$. There is a symmetry breaking of the g.s. solutions when p crosses $\lambda_2 + 1$.

24 A Study of Result (PW): Statements

In this section, we shall work with the Dirichlet boundary value problem $(\mathcal{Q})$ defined on the unit ball B^N . On this domain, Theorem 4.3 ensures

$$E(\lambda_1) = \{U_1^0(|x|)\}, \quad E(\lambda_2) = \{U_1^1(|x|)H_1\left(\frac{x}{|x|}\right)\}, \quad E(\lambda_3) = \{U_1^2(|x|)H_2\left(\frac{x}{|x|}\right)\},$$

where H_k and U_q^k are respectively defined in Theorem 4.2 and 4.3. Let us note that

$$\dim E(\lambda_1) = 1, \quad \dim E(\lambda_2) = N, \quad \dim E(\lambda_3) = (N-1) + \frac{N(N-1)}{2}. \quad (148)$$

While the first two equalities are clear, the last one is due to the fact that the functions $|x|^2 - Nx_i^2$, called *zonal functions*, forming a space of dimension $N-1$, joined to the functions $\delta_{i,j}x_i x_j$, forming a space of dimension $N(N-1)/2$, generate any function H_2 .

In the previous chapter, we were motivated by the results (GNN), (BWW) and (PW) of Theorem 5.7. We used the fact that for some nonlinearities:

- the g.s. (resp. l.e.n.) solutions have the same symmetries (or non-symmetries) as functions of $E(\lambda_1)$ (resp. $E(\lambda_2)$),
- the g.s. (resp. l.e.n.) solutions converge to functions of $E(\lambda_1)$ (resp. $E(\lambda_2)$) having, with respect to some functionals, a common “topological structure”.

Let us focus here on the result (PW) proved by Pacella and Weth, concerning Morse index and symmetry results about Problem $(\mathcal{Q})$. Inspired by the previous procedure, we shall attempt to construct a solution u of such a problem, which is comparable to some functions of $E(\lambda_3)$

- with respect to its symmetries: u shall be not $\mathbb{R}^N$ -foliated Schwarz symmetric like the functions of $E(\lambda_3)$,
- with respect to its Morse index: the equality (148) will lead to $\text{ind}_M(u, J) = 1 + N + k$ with $k \geq 1$.

It will be a proof of the optimality of $N + 1$ instead of N in the result (PW).

Theorem 24.1: *Let us consider $L \in \Phi_0(E)$ symmetric, $J_* \in \mathcal{C}^2(E)$ where E is a Hilbert space, and $J_\varepsilon \in \mathcal{C}^2(E)$ defined by*

$$J_\varepsilon(u) := \frac{1}{2}(L(u)|u) + \varepsilon J_*(u). \quad (149)$$

Let us consider the subspaces E_1, E_2, F_1, F_2 and the operators p, q, P, Q like in (22), (23), with $F_i = E_i$ (so $P = p, Q = q$) and $E_1 = E_2^\perp$. If $u_ \in E_2$ is a non-degenerate critical point of $J_*|_{E_2}$, there exists $\bar{\varepsilon}, r, R > 0$ and*

$$\gamma \in \mathcal{C}^1((-\bar{\varepsilon}, \bar{\varepsilon}) \times B(u_*, r) \cap E_2, B(0, R) \cap E_1), \quad \eta \in \mathcal{C}^1((-\bar{\varepsilon}, \bar{\varepsilon}), B(u_*, r) \cap E_2),$$

such that

$$\begin{aligned} \forall (\varepsilon, q(u), p(u)) &\in (-\bar{\varepsilon}, \bar{\varepsilon}) \times B(u_*, r) \cap E_2 \times B(u_*, r) \cap E_1 : \\ \nabla J_\varepsilon(p(u) + q(u)) &= 0 \Leftrightarrow p(u) + q(u) = \gamma(\varepsilon, \eta(\varepsilon)) + \eta(\varepsilon), \end{aligned}$$

with

$$\eta(0) = u_*, \quad \forall \alpha \in B(u_*, r) \cap E_2 : \gamma(0, \alpha) = 0.$$

Moreover, for any $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$ different of 0, the function $u_\varepsilon := \gamma(\varepsilon, \eta(\varepsilon)) + \eta(\varepsilon)$ is a non-degenerate critical point of ∇J_ε , and if the eigenvalues of L are given by $\lambda_1 < \lambda_2 < \dots$ and J_ε verifies (PS), then we have

$$\text{ind}_M(u_\varepsilon, J_\varepsilon) = \text{ind}_M(u_*, \text{sgn}(\varepsilon) \cdot J_*|_{E_2}) + \sum_{\lambda_k < 0} \dim E(\lambda_k).$$

PROOF. Let us work with Procedure 2.3. The Implicit Function Theorem applied at the point

$$(\varepsilon, q(u), p(u)) = (0, u_*, 0)$$

to the first equation of the last equivalence of (24) ensures the thesis about γ . If we imbed this function in the second equation of the last equivalence of (24) and divide by ε , we obtain an equation $T(\varepsilon, \alpha) = 0$ like in Procedure 2.3 where $T : [0, \bar{\varepsilon}) \times E_2 \rightarrow F_2 = E_2$ is defined by

$$T(\varepsilon, \alpha) := \nabla_u(J_*|_{E_2})(\gamma(\varepsilon, \alpha) + \alpha),$$

and where we used the equality $Q\nabla J_* = \nabla(J_*|_{E_2})$. The assumption about u_* permits to apply the Implicit Function Theorem at the point $(\varepsilon, \alpha) = (0, u_*)$ to T and conclude the thesis about η .

As in Remark 3.1, we could apply the infinite dimensional version of the Implicit Function Theorem to the operator $p\nabla J_\varepsilon(u) + q\nabla J_*(u)$. We can also consider the functionals

$$\tilde{J}_\varepsilon(h) := J_\varepsilon(u_\varepsilon + h) + \text{sgn}(\varepsilon)J_*(u_\varepsilon + q(h)).$$

whose 0 is the only critical point in $D := B(0, R) \cap E_1 + B(u_*, r) \cap E_2$ for any $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$, which is non-degenerate by the assumption on u_* .

The homotopy invariance property of the Morse index applied to $H(\varepsilon, h) := \tilde{J}_\varepsilon(h)$ on D ensures

$$\text{ind}_M(0, \tilde{J}_\varepsilon) = \text{ind}_M(0, \tilde{J}_0).$$

The computation of $\tilde{J}_0''$, the homotopy invariance applied to the $\bar{H}(t, \cdot)$ and the definition of $\text{sgn}(0)$ by 1 (resp. -1) lead to the thesis when $\varepsilon > 0$ (resp. $\varepsilon < 0$),

$$\langle \tilde{J}''(0)|v, w \rangle = (L(p(v)|p(w)) + \text{sgn}(0) \langle J''_*(u_*)|q(v), q(w) \rangle),$$

$$\bar{H}(t, h) = J_\varepsilon(u_\varepsilon + h) + t \text{sgn}(\varepsilon) J_*(u_\varepsilon + q(h)).$$

Let us note that, in the assumptions of the homotopy invariance property of the Morse index mentioned in Section 2, we asked $\mathcal{C}^\#$ for the regularity of H , and the non-degeneracy of the critical point u_0 for any ε . Here, we just know the non-degeneracy for $\varepsilon = 0$, but we know more regularity, as $\mathcal{C}^2$ with respect to (ε, u) , which compensates. $\square$

25 A Study of Result (PW): Example

Let us apply the previous result on Problem

$$-\Delta u = \lambda_3 u + \varepsilon(|u|^{p-2}u - u) \text{ in } B^N, \quad u = 0 \text{ on } \partial B^N. \quad (\mathcal{Q}_\varepsilon)$$

with $\varepsilon > 0$ and $p \geq 3$ in order to have a nonlinearity verifying the assumptions of Result (PW). This problem can be written as the research of the critical points of $J_\varepsilon : H_0^1(B^N) \rightarrow \mathbb{R}$ defined like in (149) (up to the isomorphism $(H_0^1)' \equiv H_0^1$ in the definition of L) by (150) and (151):

$$L : H_0^1 \rightarrow (H_0^1)' : u \mapsto -\Delta u - \lambda_3 u = \int_{B^N} [\nabla u \nabla \cdot - \lambda_3 u \cdot], \quad (150)$$

$$J_* : H_0^1 \rightarrow \mathbb{R} : u \mapsto \frac{1}{2} \int_{B^N} u^2 - \frac{1}{p} \int_{B^N} |u|^p. \quad (151)$$

We can restrict J_* to $E(\lambda_3)$ and consider its Nehari manifold $\mathcal{N}_*$:

$$\mathcal{N}_* = \{u \in E(\lambda_3) \setminus \{0\} : \int_{B^N} u^2 = \int_{B^N} |u|^p\},$$

Theorem 25.1: *If u_* is a non-degenerate minimizer of $J_*|_{E(\lambda_3)}$, there exists, for $\varepsilon > 0$ small enough, solutions of Problem $(\mathcal{Q}_\varepsilon)$ without $\mathbb{R}^N$ -foliated Schwarz symmetry and with Morse index $N + 2$.*

PROOF. Under the assumption of u_* , Theorem 24.1 ensures the existence of a unique continuum of non-degenerate solutions $(u_\varepsilon)_{\varepsilon > 0}$ of $(\mathcal{Q}_\varepsilon)_{\varepsilon > 0}$ converging to u_* such that

$$\text{ind}_M(u_\varepsilon, J_\varepsilon) = \text{ind}_M(u_*, J_*|_{E(\lambda_3)}) + N + 1 = N + 2.$$

The fact that the functions of $E(\lambda_3)$ are not $\mathbb{R}^N$ -foliated Schwarz symmetric, joined to the uniqueness obtained with the Implicit Function Theorem, we conclude the proof. $\square$

26 A Study of Result (PW): Observations

Since there exists a constant $a_p = a_p(N) > 0$ such that for any $u = U_1^2 \cdot H_2$ in $E(\lambda_3)$, we have $\int_\Omega |u|^p = a_p \int_{S^{N-1}} |H_2|^p$, up to a rescaling, we can consider

$$\tilde{J}_* : \tilde{E}(\lambda_3) \rightarrow \mathbb{R} : u \mapsto \frac{1}{2} \int_{S^{N-1}} u^2 - \frac{1}{p} \int_{S^{N-1}} |u|^p$$

where $\tilde{E}(\lambda_3)$ is the space of harmonic homogeneous polynomials of degree 2 defined on $\mathcal{S}^{N-1}$, and

$$\begin{aligned}\tilde{\mathcal{N}}_* &:= \{u \in \tilde{E}(\lambda_3) \setminus \{0\} : \langle \tilde{J}_*(u) | u \rangle = 0\} \\ &= \{u \in \tilde{E}(\lambda_3) \setminus \{0\} : \int_{\mathcal{S}^{N-1}} u^2 = \int_{\mathcal{S}^{N-1}} |u|^p\}.\end{aligned}$$

By diagonalization, we can only consider functions $u : \mathcal{S}^{N-1} \rightarrow \mathbb{R}$ of the form $\sum a_i \sigma_i^2$ with $\sum a_i = 0$. We can then work with

$$\tilde{J}_* : \tilde{E}(\lambda_3) \rightarrow \mathbb{R}, \quad \tilde{E}(\lambda_3) = \{a \in \mathbb{R}^N : \sum a_i = 0\}$$

and $\tilde{\mathcal{N}}_*$ a sub-manifold of $\tilde{E}(\lambda_3)$ of codimension 1 (hence of dimension $N - 2$). By making a parametrization of $\tilde{\mathcal{N}}_*$ and an approximation of $\tilde{J}_*$ using the Simpson technique with the software Matlab, we have obtained in the cases $N = 3, 4, 5$, some approximations of the points a on which $\tilde{J}$ reaches its minimum or maximum.

These observations seem to indicate that if p is great enough, the minimum is reached by the most symmetrical elements of $\tilde{E}(\lambda_3)$, the previously mentioned zonal functions, and the maximum by elements having symmetries and anti-symmetries. Hence, the obtained functions u_* are not $\mathbb{R}^N$ -foliated Schwarz symmetric, but have, in many cases, a similar symmetry comportment.

26.1 Dimension 3

	Minimum	Maximum
$p = 3$	(0.01, 1.37, -1.38)	(-0.81, 1.63, -0.82)
$p = 4$	$\tilde{\mathcal{N}}_*$	$\tilde{\mathcal{N}}_*$
$p = 5$	(0.71, 0.73, -1.44) (-0.71, -0.73, 1.44)	(1.29, -0.01, -1.28) (-1.29, 0.01, 1.28)
$p = 6$	(-0.68, -0.70, 1.39)	(1.26, -0.01, -1.25)

For the case $p = 4$, $\tilde{J}_*$ is constant on $\tilde{\mathcal{N}}_*$. This seems coincide with the fact that the symmetries for the observed minimum and maximum in the cases $p < 4$ and $p > 4$ are inverted. In fact, the software Matlab does not observe that $\tilde{J}_*$ was constant but that its variation was very little (between 1.459 and 1.464). But the software Mathematica, working with symbolics computations, has exactly observed that $\tilde{J}_*$ was always $\frac{7\pi}{15}$ on $\tilde{\mathcal{N}}_*$.

26.2 Dimension 4

	Minimum	Maximum
$p = 3$	(-0.62, -0.62, -0.65, 1.89)	(1.33, -1.33, -1.33, 1.33)
$p = 4$	(0.57, 0.57, 0.58, -1.72)	(1.29, -1.29, -1.29, 1.29)

26.3 Dimension 5

	Minimum
$p = 3$	(0.5230, 0.5227, 0.5225, 0.5745, -2.1427) (2.1524, -0.5378, -0.5375, -0.5595, -0.5176)

	Maximum
$p = 3$	$(1.1195, -1.6803, 1.1178, 1.1206, -1.6775)$
	$(1.6805, -1.1196, -1.1198, 1.6920, -1.1332)$
	$(1.1211, -1.6791, -1.6793, 1.1086, 1.1287)$

27 Generalization of Problem (LE): Assumptions and Statements

Let us consider the problems $(\mathcal{P}_{\mu,\lambda,\theta})$. In the previous chapter, we studied the Dirichlet case of the Lane–Emden (LE) Problem $(f_\theta(t) = \lambda t^{\theta-2}t)$ and considered the l.e.n. solutions. In [10], we generalized, in collaboration with D. Bonheure and C. Grumiau, the previous results for other nonlinearities, the g.s. solutions and the Neumann case.

Let us denote by $\mathcal{M}(\mathbb{R})$ the set of Borel-measurable functions from $\mathbb{R}$ to $\mathbb{R}$. We refer respectively to a polynomial or an exponential growth condition by (P_p) and (E_γ) for a function $h \in \mathcal{M}(\mathbb{R})$, and introduce the classical subcritical growth conditions (A_1) – (A'_1) :

$$\begin{aligned} (P_q) \quad & \exists c > 0, \forall t \in \mathbb{R} : |h(t)| \leq c(|t|^q + 1), \\ (E_\gamma) \quad & \exists c > 0, \forall t \in \mathbb{R} : |h(t)| \leq ce^{\gamma t^2}, \\ (A_1) \quad & \forall \theta > 2, \exists q \in (2, 2^*) \text{ such that } f_\theta \text{ satisfies } (P_{q-1}), \\ (A'_1) \quad & \forall \theta > 2, \exists \gamma > 0 \text{ such that } f_\theta \text{ satisfies } (E_\gamma). \end{aligned}$$

As usual, the condition (A_1) is motivated by the classical Sobolev imbedding theorem for $N \geq 2$ while the latter is related to the *Trudinger-Moser inequality* when $N = 2$ [44]:

$$\begin{aligned} & \forall \gamma > 0, u \in H^1(\Omega) : \int_\Omega e^{\gamma u^2} < \infty, \\ & \exists \gamma_0 = \gamma_0(\Omega) > 0 : \sup \left\{ \int_\Omega e^{\gamma u^2} : \|u\| \leq 1 \right\} < \infty \Leftrightarrow \gamma < \gamma_0. \end{aligned}$$

These assumptions ensure that the functional J_θ is well defined and of class $\mathcal{C}^1$ on E the domain of J_θ .

The following standard assumption (A_2) gives a Nehari geometry (see Section 2 of Appendix A) to the functional J_θ :

$$\begin{aligned} (A_2) \quad & \text{(a) } \forall \theta > 2 : \lim_{t \rightarrow 0} \frac{f_\theta(t)}{|t|} = 0, \\ & \text{(b) } \forall \theta > 2 : \lim_{|t| \rightarrow +\infty} \frac{f_\theta(t)}{t} = +\infty, \\ & \text{(c) } \forall \theta > 2 : \forall t \neq 0 : \frac{d}{dt} \left(\frac{f_\theta(t)}{|t|} \right) > 0, \end{aligned}$$

The g.s. (resp. l.e.n.) solutions of $(\mathcal{P}_{\mu,\lambda,\theta})$ are the critical points of J_θ with minimum energy on $\mathcal{N}_\theta$ (resp. $\mathcal{M}_\theta$). Theorem 3.3 ensures the existence of g.s. and l.e.n. solutions of Problem $(\mathcal{P}_{\mu,\lambda,\theta})$ under the previous assumptions, that we shall always suppose in the sequel. We next consider slowly growing superlinear nonlinearities:

$$\begin{aligned} (A_3) \quad & \text{(a) } \exists \lambda, \forall t_0 \in \mathbb{R} : \lim_{(\theta,t) \rightarrow (2,t_0)} f_\theta(t) =: f_2(t_0) = \lambda t_0, \lambda \text{ an eigenvalue of } A_\mu, \\ & \text{(b) if } f_\theta \text{ satisfy } (A_1) \text{ (resp. } (A'_1)), \text{ there exists } q \in (2, 2^*) \text{ (resp. } \gamma > 0) \text{ and } h \in \mathcal{M}(\mathbb{R}) \text{ verifying } (P_q) \text{ (resp. } (E_\gamma)) \text{ such that } \forall \theta > 2, \forall t \in \mathbb{R}, |f_\theta(t)| \leq h(t), \\ (A_4) \quad & \text{(a) } \forall u_0 \neq 0 : \lim_{(t,s,\theta) \rightarrow (u_0,u_0,2)} \frac{f_\theta(t) - f_\theta(s)}{t-s} = \lambda, \end{aligned}$$

- (b) there exists $q \in [2, 2^*]$ (resp. $\gamma > 0$) and $h_a, h_b \in \mathcal{M}(\mathbb{R})$ verifying (P_{q-2}) (resp. (A'_1)) such that $\forall \theta > 2, \forall s \neq t \in \mathbb{R} : \left| \frac{f_\theta(t) - f_\theta(s)}{t-s} \right| \leq h_a(s) + h_b(t)$.

Assume $(u_\theta)_{\theta > 2}$ a family of solutions such that $a \leq \|u_\theta\| \leq b$ uniformly in θ for some $a, b > 0$. Under the additional assumptions (A_3) and (A_4) , u_θ respects for θ close enough to 2, the symmetries of its projection on the eigenspace associated to λ .

Theorem 27.1: *Let $(G_\alpha)_{\alpha \in E(\lambda)}$ be some groups acting on E in such a way that, $\sup_{\alpha, g \in G_\alpha} \|g\| < \infty$, and that for every $g \in G_\alpha, 2 < \theta < \bar{\theta}$ and $u \in E$,*

$$(i) g(E(\lambda)^\perp) \subseteq E(\lambda)^\perp, \quad (ii) g\alpha = \alpha, \quad (iii) J_\theta(gu) = J_\theta(u),$$

for some $\bar{\theta} > 2$. Under the assumptions $(A_1) - (A_4)$, for all $R > 0$, if θ is close enough to 2, any solution

$$u_\theta \in \{u \in B_R : P_{E(\lambda)}(u) \notin B_{\frac{R}{\theta}}\}$$

of Problem $(\mathcal{P}_{\mu, \lambda, \theta})$ belongs to the fixator of G_{α_θ} where $\alpha_\theta := P_{E(\lambda)}u_\theta$.

This abstract result becomes more interesting once we can prove a priori bounds on a family of solutions. This is indeed the case for g.s. and l.e.n. solutions under additional assumptions on the family $(f_\theta)_{\theta > 2}$. First of all, we work with the classical super-quadraticity assumption,

$$(A_5) \quad \forall \theta > 2, \forall t \in \mathbb{R} : f_\theta(t)t \geq \theta F_\theta(t),$$

where $F_\theta(t) := \int_0^t f_\theta(s) ds$. Let us refer the reader to Inequality (30) in Remark 3.4 about this assumption. Next, we consider the assumptions

- (A₆) (a) $\forall t \in \mathbb{R}, f_2(t) = \lambda t, \lambda \in \{\lambda_1, \lambda_2\}$, and $\lim_{(\theta, t) \rightarrow (2, t_0)} \frac{f_\theta(t) - f_2(t)}{\theta - 2} =: f_*(t_0) \in \mathbb{R}$,
 (b) $\exists q \in [2, 2^*]$ (resp. $\gamma > 0$) and $h \in \mathcal{M}(\mathbb{R})$ verifying (P_{q-1}) (resp. (E_γ)) such that $\forall \theta > 2, \forall t \in \mathbb{R} : \left| \frac{f_\theta(t) - f_2(t)}{\theta - 2} \right| \leq h(t)$,
 (c) (i) $\lim_{t \rightarrow 0} \frac{f_*(t)}{t} < 0$, (ii) $\lim_{|t| \rightarrow \infty} \frac{f_*(t)}{t} > 0$, (iii) $t \mapsto \frac{f_*(t)}{|t|}$ strictly increasing.

Roughly, it mainly means

$$f_\theta(t) = \lambda t + f_*(t)(\theta - 2) + g_\theta(t), \quad (152)$$

where $g_\theta(t)$ behaves like $o(\theta - 2)$. We then define the functional J_* and the associated Nehari manifold $\mathcal{N}_*$:

$$J_* : E(\lambda) \rightarrow \mathbb{R} : u \mapsto - \int_\Omega F_*(u), \quad F_*(t) := \int_0^t f_*(s) ds,$$

$$\mathcal{N}_* := \{u \in E(\lambda) \setminus \{0\} : \langle J'_*(u) | u \rangle = 0\}.$$

One can easily check that, assuming (A_6) ensures that J_* is well defined and has a Nehari geometry. This assumption also implies that any possible accumulation points u_* of the family $(u_\theta)_{\theta > 2}$ is a critical point of the functional J_* .

In Section 29.3, we prove the boundedness of $(u_\theta)_{\theta > 2}$ under the assumption

- (A₇) (a) $\forall t_0 \in \mathbb{R} : \lim_{(\theta, t) \rightarrow (2, t_0)} \frac{f'_\theta(t)t^2 - f_\theta(t)t}{\theta - 2} = H_*(t_0)$,
 (b) $\exists q \in [2, 2^*]$ (resp. $\gamma > 0$) and $h \in \mathcal{M}(\mathbb{R})$ verifying (P_{q-1}) (resp. (E_γ)) such that $\forall \theta > 2, \forall t \in \mathbb{R} : \left| \frac{f'_\theta(t)t - f_\theta(t)}{\theta - 2} \right| \leq h(t)$,
 (c) there exists a critical point u_* of J_* such that $\int_\Omega H_*(u_*) \neq 0$.

Roughly, it means, in regard to (152), that g'_θ behaves like $o(\theta - 2)$ and that u_* is a non-degenerate critical point of J_* . Let us also note that $(A_3)(a)-(A_4)(a)-(A_6)(a)-(A_7)(a)$ can be rephrased as a convergence in θ locally uniform with respect to t .

At last, in Section 29.4, we prove that if u_* is a weak accumulation point of $(u_\theta)_{\theta>2}$, then it is different from 0 under one of the two following assumptions:

$$(A_8) \quad \exists q \in [2, 2^*], \forall \theta > 2, \exists c_1(\theta), c_2(\theta) > 0, \forall t \in \mathbb{R} : |f_\theta(t)| \leq \lambda |t|^{1+c_1(\theta)} + c_2(\theta) |t|^{p-1},$$

$$\text{with } \lim_{\theta \rightarrow 2} c_i(\theta) = 0 \text{ and } \exists k \in \mathbb{R}, \forall \theta > 2, \frac{\log(1-c_2(\theta))}{c_1(\theta)} \geq k,$$

$$(A'_8) \quad \exists \eta > 0, \forall \theta < 2 + \eta, \forall t \in (-\eta, \eta) \setminus \{0\} : \frac{f_\theta(t)}{t} < \lambda, \text{ with } \Omega \in \mathcal{C}^2.$$

Assumption (A'_8) requires some smoothness of the domain. However, when more information is known about its geometry, this smoothness requirement can be weakened as we can deal e.g. with Lipschitz domains such as a product of intervals, see Remark 29.9 below.

Summing up, we deduce the following result:

Theorem 27.2: *Under Assumptions $(A_1) - (A_8)$ (with $(A_1)-(A_8)$ possibly replaced by $(A'_1)-(A'_8)$), if $(u_\theta)_{\theta>2}$ is a family of g.s. or l.e.n. solutions of Problem $(\mathcal{P}_{\mu,\lambda,\theta})$, then, for θ close enough to 2:*

- (i) *there exists $0 < a < b$ such that $a \leq \|u_\theta\| \leq b$ for any $\theta > 2$. Moreover, if $\theta_n \rightarrow 2$ and $u_{\theta_n} \rightharpoonup u_*$, then*

$$\lim_{n \rightarrow \infty} \left(\frac{J_{\theta_n}(u_{\theta_n})}{\theta_n - 2} \right) = J_*(u_*) \leq J_*(v_*),$$

for any critical point $v_ \in E(\lambda)$ of J_* such that $\int_\Omega H_*(v_*) \neq 0$,*

- (ii) *if $\lambda = \lambda_1$ or λ_2 with λ_2 simple, u_θ has respectively the symmetries of the first or the second eigenfunctions of $-\Delta + \mu \text{id}$ in $E(\lambda)$, the same being true for the antisymmetries if the functions f_θ are antisymmetric,*
- (iii) *if $\lambda = \lambda_2$, Ω is a ball, u_θ is invariant by rotation with respect to $N-1$ directions and, if the functions f_θ are antisymmetric, antisymmetric with respect to the orthogonal one,*
- (iv) *if $\lambda = \lambda_2$, Ω is a square and the functions f_θ are antisymmetric, u_θ is antisymmetric with respect to the center of the square.*

It is worth pointing out that the preceding result is not uniform with respect to the domain. Namely, we are able to construct a sequence of rectangles $(R_n)_{n \in \mathbb{N}}$ approaching a square and a sequence of l.e.n. solutions which are neither symmetric nor antisymmetric with respect to the medians of R_n . This was already observed in Chapter 4. We show here that this symmetry breaking also occurs when dealing with Neumann boundary conditions and non-homogeneous nonlinearities.

Our result can be seen as an alternative proof to the result (GNN) and (BWW) of Theorem 5.7 mentioned many times. Moreover, it works in some case where the precedent results does not, like for non-convex or non-radial domain. Moreover, our technique lead to some breaking of symmetry in the l.e.n. case.

As the assumptions $(A_1) - (A_8)$ could seem quite technical and ad hoc, we give some explicit model examples in Section 31. We shall analyse the three following cases where

f_θ is given by

$$\lambda t|t|^{\theta-2} + (\theta-2)t|t|^{q-2}, \quad \lambda t(e^{t^2} - 1)^{\theta-2}, \quad \lambda t \left(\sum_{i=1}^k \alpha_i |t|^{\beta_i(\theta)} \right),$$

i.e. the case of a subcritical super-linear ($2 < q < 2^*$) perturbation of a slowly growing homogeneous super-linearity, of a slowly exponentially growing subcritical nonlinearity in dimension two, and of a sum of small powers. These cases will be numerically illustrated by using mountain pass algorithms.

28 Generalization of Problem (LE): Symmetries and Eigenfunctions

In this section, we prove that, under Assumptions (A_3) – (A_4) , for θ close to 2, any bounded family $(u_\theta)_{\theta>2}$ of solutions uniformly bounded away from zero, respects the symmetries of the family projected in $E(\lambda)$. This follows from a uniqueness result (up to projection) which is our next concern.

The main idea has been used in Chapter 4. We include a sketch of the proof for completeness. We emphasize that the result holds at any eigenvalue λ_i of A_μ .

Lemma 28.1: *Let $N \geq 2$. There exists $\eta > 0$ such that, if $\|a(x) - \lambda_i\|_{L^{N/2}} < \eta$ and u solves the boundary value problem*

$$A_\mu u = a(x)u \text{ in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega, \quad (\mathcal{R}_\theta)$$

then either $u = 0$ or $P_{E(\lambda_i)}u \neq 0$.

PROOF. For $N \geq 3$, the proof is completely the same as the one of Lemma 19.1 with w and z defined by

$$w := P_{E(\lambda_1) \oplus \dots \oplus E(\lambda_{i-1})}u, \quad z := P_{(E(\lambda_1) \oplus \dots \oplus E(\lambda_i))^\perp}u,$$

and the inequalities about them given by

$$\begin{aligned} \|w\|^2 &\geq \frac{\lambda_i}{\lambda_{i-1}} \|w\|^2 - c \|a(x) - \lambda_i\|_{L^{\frac{N}{2}}} \|w\| \|u\|, \\ \|z\|^2 &\leq \frac{\lambda_i}{\lambda_{i+1}} \|z\|^2 + c \|a(x) - \lambda_i\|_{L^{\frac{N}{2}}} \|z\| \|u\|, \end{aligned}$$

For $N = 2$, the same statement can be formulated with the $L^{\frac{N}{2}}$ -norm replaced by L^q -norm for any $1 < q < \infty$. $\square$

As a consequence, we obtain

Proposition 28.2: *Assume that $(A_1) - (A_4)$ hold and $(u_{1,\theta})_{\theta>2}, (u_{2,\theta})_{\theta>2}$ are families of solutions of Problem $(\mathcal{P}_{\mu,\lambda,\theta})$. For every $R > 0$, there exists $\theta_R > 2$ such that, if*

$$\|u_{i,\theta}\| \leq R, \quad \|P_{E(\lambda)}(u_{i,\theta})\| \geq 1/R,$$

for $2 < \theta < \theta_R$, then either $u_{1,\theta} = u_{2,\theta}$ or $P_{E(\lambda)}u_{1,\theta} \neq P_{E(\lambda)}u_{2,\theta}$ for every $2 < \theta < \theta_R$, where λ is the eigenvalue of A_μ present in Assumption (A_3) .

PROOF. Let $\theta_n \rightarrow 2$ and $(u_{i,\theta_n})_n$ be two sequences of solutions such that

$$\|u_{i,\theta_n}\| \leq R, \quad \|P_{E(\lambda)}(u_{i,\theta_n})\| \geq 1/R, \quad P_{E(\lambda)}u_{1,\theta_n} = P_{E(\lambda)}u_{2,\theta_n}.$$

Since these sequences are bounded in E , up to subsequences, there exist $\alpha_i \in E$ such that $u_{i,\theta_n} \rightharpoonup \alpha_i$. By assumption, we have $P_{E(\lambda)}\alpha_1 = P_{E(\lambda)}\alpha_2$, and by (A_3) , a compactness argument and Lebesgue's Dominated Convergence Theorem, we conclude easily that $\alpha_i \in E(\lambda) \setminus \{0\}$, so that $\alpha_1 = \alpha_2$.

Observe that $u_{1,\theta_n} - u_{2,\theta_n}$ solves Problem $(\mathcal{R}_\theta)$ where $a(x) = a_n(x)$ is defined by

$$a_n(x) = \begin{cases} \frac{f_{\theta_n}(u_{1,\theta_n}(x)) - f_{\theta_n}(u_{2,\theta_n}(x))}{u_{1,\theta_n}(x) - u_{2,\theta_n}(x)}, & \text{if } u_{1,\theta_n}(x) \neq u_{2,\theta_n}(x), \\ \lambda, & \text{else.} \end{cases}$$

Assumption (A_4) ensures Lebesgue's Dominated Convergence Theorem applies and since $\alpha_i \neq 0$ almost everywhere, we deduce that $a_n \rightarrow \lambda$ in $L^{\frac{N}{2}}(\Omega)$. The proof now follows from Proposition 28.1. $\square$

Let us note that in the case of Assumption (A'_1) in dimension 2, the following lemma, proved in [12], plays the role of the standard compactness argument when assuming a subcritical polynomial growth.

Lemma 28.3: *Let $u_n, v_n \in H^1(\Omega)$ such that $\|u_n\|, \|v_n\| \leq 1$. If $u_n \rightharpoonup u, v_n \rightharpoonup v$ in $H^1(\Omega)$ and $\gamma_n \rightarrow \gamma < \gamma_0$, then, for every $p \in [1, \infty)$ and $q \in [1, \frac{\gamma_0}{\gamma})$, $v_n^p e^{\gamma_n u_n^2} \rightarrow v^p e^{\gamma u^2}$ in $L^q(\Omega)$.*

From what precedes, we can deduce the following abstract symmetry result dealing with group actions. The proof of this Theorem is exactly the same as the one of Theorem 19.6. The action group concept is recalled in Section 19 and illustrated by example 19.5 whose fixed points are the symmetric and antisymmetric functions with respect to a hyperplane.

Theorem 28.4: *Let $(G_\alpha)_{\alpha \in E(\lambda)}$ be groups acting on E in such a way that, $\sup_{\alpha, g \in G_\alpha} \|g\| < \infty$, and that for every $g \in G_\alpha$, $2 < \theta < \bar{\theta}$ and $u \in E$,*

$$(i) g(E(\lambda)^\perp) \subseteq E(\lambda)^\perp, \quad (ii) g\alpha = \alpha, \quad (iii) J_\theta(gu) = J_\theta(u),$$

for some $\bar{\theta} > 2$. Then, under the assumptions $(A_1) - (A_4)$, for all $R > 0$, if θ is close enough to 2, any solution u_θ of Problem $(\mathcal{Q}_\varepsilon)$ satisfying

$$\|u_\theta\| \leq R, \quad \|P_{E(\lambda)}(u_\theta)\| \geq 1/R$$

belongs to the fixator of G_{α_θ} where $\alpha_\theta := P_{E(\lambda)}u_\theta$.

29 Generalization of Problem (LE): Asymptotic Behaviors

In this Section, we work out a priori estimates for g.s. and l.e.n. solutions, and we analyze their asymptotic behavior, when $\theta \rightarrow 2$. In the sequel, λ exclusively takes the values λ_1 , when we discuss the case of g.s. solutions, and λ_2 when dealing with l.e.n. solutions. Let us denote by $(u_\theta)_{\theta > 2}$ a family of g.s. (resp. l.e.n.) solutions.

29.1 Limit Functional We first define the limit functional

$$J_* : E(\lambda) \rightarrow \mathbb{R} : u \rightarrow - \int_{\Omega} F_*(u)$$

where $F_*(t) := \int_0^t f_*(s) ds$. It follows from (A_6) that this functional is well defined and of class $\mathcal{C}^1$. The critical points u of this functional verify

$$\int_{\Omega} f_*(u)v = 0, \quad (153)$$

for every $v \in E(\lambda)$. The assumption $(A_6)(c)$ ensures a mountain pass geometry. Indeed, for each $u \neq 0$ there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_*$, and the function $u \mapsto t_u$ is continuous.

29.2 Limit Equation We next turn our attention to the candidate accumulation points of $(u_{\theta})_{\theta > 2}$, as $\theta \rightarrow 2$.

Proposition 29.1: *Let $(u_{\theta})_{\theta > 2}$ be a family of g.s. (resp. l.e.n.) solutions of $(\mathcal{P}_{\mu, \lambda, \theta})$. If $\theta_n \rightarrow 2$ and $u_{\theta_n} \rightharpoonup u_*$ in E , then, under the assumptions (A_1) - (A_2) and (A_5) - (A_6) , u_* solves*

$$\begin{cases} A_{\mu} u_* = \lambda u_*, & \text{in } \Omega, \\ B u_* = 0, & \text{on } \partial\Omega, \\ \int_{\Omega} f_*(u_*)v = 0, & \forall v \in E. \end{cases}$$

PROOF. The proof of this result is completely the same as the one of Lemma 20.3. Let us nevertheless note that, in dimension 2 when (A_1) is replaced by (A'_1) , the Rellich Theorem is replaced by Lemma 28.3 to deduce, together with Lebesgue Dominated Convergence Theorem and (A_3) , that $f_{\theta_n}(u_{\theta_n}) \rightarrow f_2(u_*)$ in $L^2(\Omega)$. $\square$

REMARK 29.2. Let us point out that our assumptions imply that if $u_{\theta} \in \mathcal{N}_{\theta}$ and $u_{\theta} \rightharpoonup u_*$ then $u_{\theta} \rightarrow u_*$ in E . Indeed,

$$\|u_{\theta} - u_*\|^2 = \|u_{\theta}\|^2 - 2(u_{\theta}|u_*) + \|u_*\|^2 = \int_{\Omega} f_{\theta}(u_{\theta})u_{\theta} - 2(u_{\theta}|u_*) + \lambda \int_{\Omega} u_*^2$$

which converges to $\lambda \int_{\Omega} u_*^2 - 2(u_*, u_*) + \lambda \int_{\Omega} u_*^2 = 0$.

29.3 Upper Bound The strategy we use to obtain an upper-bound is similar to the one of Chapter 4: we construct appropriate test functions v_{θ} leading to an estimate of the energy and the norm of g.s. (resp. l.e.n.) solutions of Problem $(\mathcal{P}_{\mu, \lambda, \theta})$. In the sequel we shall consider the following function

$$h_{\theta} : \mathbb{R}^+ \rightarrow \mathbb{R} : t \mapsto \int_{\Omega} t^{-1} f_{\theta}(tv_{\theta})v_{\theta}$$

associated to v_{θ} , denote by $\hat{v}_{\theta}$ the projection of v_{θ} on $\mathcal{N}_{\theta}$ (resp. $\mathcal{M}_{\theta}$) along rays, i.e. $\hat{v}_{\theta} = t_{\theta} v_{\theta}$ (resp. $t_{\theta}^+ v_{\theta}^+ + t_{\theta}^- v_{\theta}^-$) where $h_{\theta}(t_{\theta}) = \|v_{\theta}\|^2$ (resp. $h_{\theta}(t_{\theta}^{\pm}) = \|v_{\theta}^{\pm}\|^2$).

Proposition 29.3: *Let $v_{\theta} := v_* + (\theta - 2)w$, where w is the unique function in $E(\lambda)^{\perp}$ verifying*

$$A_{\mu} w = f_*(v_*) + \lambda w \text{ in } \Omega, \quad B(w) = 0 \text{ on } \partial\Omega,$$

and v_ is a critical point of J_* such that $\int_{\Omega} H_*(v_*) \neq 0$. Then, under Assumptions (A_1) - (A_2) and (A_5) - (A_7) , the family $(\hat{v}_{\theta})_{\theta > 2}$ converges to v_* .*

PROOF. We focus on the case $\lambda = \lambda_1$, the argument being similar if $\lambda = \lambda_2$. Since v_* verifies (153), by the Fredholm alternative, w is well defined. We then compute

$$\frac{d}{dt}h_\theta(t) = \int_{\Omega} t^{-1} f'_\theta(tv_\theta) v_\theta^2 - t^{-2} f_\theta(tv_\theta) v_\theta \quad (154)$$

and

$$\begin{aligned} \|v_\theta\|^2 &= \int_{\Omega} (\lambda v_* + \lambda(\theta - 2)w + (\theta - 2)f_*(v_*))v_\theta \\ &= \int_{\Omega} (\lambda v_\theta + (\theta - 2)f_*(v_*))v_\theta \\ &= \int_{\Omega} f_\theta(v_\theta)v_\theta + (\theta - 2) \left(\int_{\Omega} \frac{\lambda v_\theta - f_\theta(v_\theta)}{\theta - 2} v_\theta + \int_{\Omega} f_*(v_*)v_\theta \right) \\ &= \int_{\Omega} f_\theta(v_\theta)v_\theta + m_\theta(\theta - 2), \end{aligned}$$

where, by (A_6) , $m_\theta \rightarrow 0$, as $\theta \rightarrow 2$. We therefore deduce, thanks to (A_2) , that

$$t_\theta = h_\theta^{-1} \left(\int_{\Omega} f_\theta(v_\theta)v_\theta + m_\theta(\theta - 2) \right).$$

By defining $k_\theta := \int_{\Omega} f_\theta(v_\theta)v_\theta$ such that $h_\theta^{-1}(k_\theta) = 1$, we need to prove that

$$\lim_{\theta \rightarrow 2} h_\theta^{-1}(k_\theta + m_\theta(\theta - 2)) - h_\theta^{-1}(k_\theta) = 0,$$

which holds true as $(\theta - 2)(h_\theta^{-1})'$ is uniformly bounded (with respect to θ) in a neighbourhood of k_θ , by Assumption (A_7) and the fact that $\int_{\Omega} H_*(v_*) \neq 0$. $\square$

We now deduce from the previous propositions the expected upper bound.

Theorem 29.4: *Let $(u_\theta)_{\theta > 2}$ be a family of g.s. (resp. l.e.n.) solutions of Problem $(\mathcal{P}_{\mu, \lambda, \theta})$. Then, under Assumptions (A_1) - (A_2) and (A_5) - (A_7) , the family $(u_\theta)_{\theta > 2}$ is bounded and if $\theta_n \rightarrow 2$ and $u_{\theta_n} \rightharpoonup u_*$, then we have*

$$\lim_{n \rightarrow \infty} \left(\frac{J_{\theta_n}(u_{\theta_n})}{\theta_n - 2} \right) = J_*(u_*) \leq J_*(v_*),$$

for any critical point v_ of the functional J_* such that $\int_{\Omega} H_*(v_*) \neq 0$.*

PROOF. Let us consider the family $(\hat{v}_\theta)_{\theta > 2}$ of the previous Lemma. By definition of $\hat{v}_\theta$, we know that $\|\hat{v}_\theta\|^2 = \lambda \|\hat{v}_\theta\|_{L^2}^2 + o(\theta - 2)$, so that

$$\frac{J_\theta(\hat{v}_\theta)}{\theta - 2} = \frac{\int_{\Omega} (\frac{\lambda}{2} \hat{v}_\theta^2 - F_\theta(\hat{v}_\theta))}{\theta - 2} + o(1) \xrightarrow{\theta \rightarrow 2} J_*(v_*),$$

where the last convergence is due to (A_6) and to the definitions of F_θ and F_* .

Inequality (30) ensures the boundedness of $(u_\theta)_{\theta > 2}$ and hence its weak convergence up to a subsequence. As observed in Remark 29.2, the convergence is strong. Observe also that

$$\|u_\theta\|^2 = \lambda \|u_\theta\|_{L^2}^2 + o(\theta - 2).$$

Indeed, using Assumption (A_6) and Proposition 29.1, we infer

$$\frac{\|u_\theta\|^2 - \int_{\Omega} f_2(u_\theta)u_\theta}{\theta - 2} = \int_{\Omega} \frac{f_\theta(u_\theta) - f_2(u_\theta)}{\theta - 2} u_\theta \rightarrow \int_{\Omega} f_*(u_*)u_* = 0.$$

Hence, arguing as for the sequence $(\hat{v}_\theta)_{\theta>2}$, we infer that

$$\lim_{n \rightarrow +\infty} \frac{J_{\theta_n}(u_{\theta_n})}{\theta_n - 2} = J_*(u_*)$$

and by definition of $(u_{\theta_n})_{n \in \mathbb{N}}$ we have $J_{\theta_n}(u_{\theta_n}) \leq J_{\theta_n}(\hat{v}_{\theta_n})$. This ends the proof. $\square$

REMARK 29.5. Assumption (A_6) essentially ensures

$$f_\theta(t) = \lambda t + f_*(t)(\theta - 2) + g_\theta(t)$$

with $g_\theta(\cdot) = o(\theta - 2)$. Rewriting the quotient $\frac{f'_\theta(t)t^2 - f_\theta(t)t}{\theta - 2}$ having this in mind, we obtain a natural sufficient condition ensuring (A_7) which is $g'_\theta(\cdot) = o(\theta - 2)$. In this case, the function H_* turns to be

$$f'_*(t)t^2 - f_*(t)t$$

and since $u_* \in \mathcal{N}_*$, the condition $\int_\Omega H_*(u_*) \neq 0$ is equivalent to $\int_\Omega f'_*(u_*)u_*^2 \neq 0$ which implies that $\langle J''_*(u_*)|u_*, u_* \rangle \neq 0$. This last condition is the natural starting point of an approach based on the Implicit Function Theorem or degree theory to ensure the existence of a continuum of solutions of $(\mathcal{P}_{\mu, \lambda, \theta})$ emanating from u_* . Let us emphasize that this remark is very rough as we have supposed f_* to be derivable, while in some simple model example, it is not the case everywhere on $\mathbb{R}$.

29.4 Lower bound We next consider two sets of conditions which imply that the accumulation points u_* of a family $(u_\theta)_{\theta>2}$ of g.s. or l.e.n. solutions, are different from 0.

Theorem 29.6: *Under the assumptions (A_1) - (A_2) and (A_8) , any family $(u_\theta)_{\theta>2}$ of g.s. (resp. l.e.n.) solutions is bounded away from 0.*

PROOF. Let us give the proof in the case $\lambda = \lambda_2$, the other case being easier.

For each $\theta > 2$, we define the function v_θ as follows. Let $r := (u_\theta^+|e_1)/(|u_\theta||e_1)$ and take

$$\tilde{v}_\theta = (1 - r)u_\theta^+ + ru_\theta^-.$$

Let us choose $t \geq 0$ such that $v_\theta := t\tilde{v}_\theta \in \mathcal{N}_\theta$. By construction, $v_\theta \in E(\lambda_1)^\perp \cap \mathcal{N}_\theta$. Since $\|v_\theta\|^2 = \int_\Omega f_\theta(v_\theta)v_\theta$, we deduce by interpolation and Assumption (A_8) that

$$\|v_\theta\|^2 \leq \lambda \left(\int_\Omega |v_\theta|^2 \right)^{\frac{(1-\sigma)(c_1(\theta)+2)}{2}} \left(\int_\Omega |v_\theta|^{2^*} \right)^{\frac{\sigma(c_1(\theta)+2)}{2^*}} + c_2(\theta) \left(\int_\Omega |v_\theta|^2 \right)^{\frac{(1-s)p}{2}} \left(\int_\Omega |v_\theta|^{2^*} \right)^{\frac{sp}{2^*}}$$

where $s, \sigma \in [0, 1]$ are respectively given by $\frac{p-2}{p} \cdot \frac{2^*}{2^*-2}$ and the same with p replaced by $c_1(\theta) + 2$. Using Sobolev Inequalities (with S the optimal constant), the variational characterization of λ_2 (see Remark 4.4), and the fact that $v_\theta \in E(\lambda_1)^\perp$, we infer that

$$\|v_\theta\|^2 \leq \lambda^{1 - \frac{(1-\sigma)(c_1(\theta)+2)}{2}} S^{\sigma(c_1(\theta)+2)} \|v_\theta\|^{c_1(\theta)+2} + c_2(\theta) \lambda^{\frac{(s-1)p}{2}} S^{sp} \|v_\theta\|^p$$

We deduce

$$1 \leq \alpha\beta \|v_\theta\|^{c_1(\theta)} + c_2(\theta)ab \|v_\theta\|^{p-2}$$

where

$$\alpha(\theta) := \lambda^{1 - \frac{(1-\sigma)(c_1(\theta)+2)}{2}}, \quad \beta(\theta) := S^{\sigma(c_1(\theta)+2)}, \quad a := \lambda^{\frac{(s-1)p}{2}}, \quad b := S^{sp},$$

and so

$$\left(\frac{1 - c_2(\theta)ab \|v_\theta\|^{p-2}}{\alpha\beta} \right)^{\frac{1}{c_1(\theta)}} \leq \|v_\theta\|$$

which leads to the conclusion if the left-hand side of the inequality does not converge to 0. Since Assumption (A_8) ensures that the numerator does not converge to 0, the claim follows. $\square$

For the second theorem using Assumption (A'_8) , we need the following property, which has its own interest:

Proposition 29.7: *If $u \in E$, then either $\lambda_2 \|u^+\|_{L^2}^2 \leq \|u^+\|^2$ or $\lambda_2 \|u^-\|_{L^2}^2 \leq \|u^-\|^2$.*

PROOF. Let us recall the Riesz-Fisher characterization of λ_2

$$\lambda_2 = \inf_{\dim F=2} \sup_{u \in F \setminus \{0\}} \frac{\|u\|^2}{\|u\|_{L^2}^2}.$$

If we define

$$F := \{v \in E \setminus \{0\} : v = \alpha u^+ + \beta u^-, (\alpha, \beta) \neq (0, 0)\},$$

then

$$\lambda_2 \leq \max_{(\alpha, \beta) \neq (0, 0)} \frac{\alpha^2 \|u^+\|^2 + \beta^2 \|u^-\|^2}{\alpha^2 \|u^+\|_{L^2}^2 + \beta^2 \|u^-\|_{L^2}^2} = \max \left\{ \frac{\|u^+\|^2}{\|u^+\|_{L^2}^2}, \frac{\|u^-\|^2}{\|u^-\|_{L^2}^2} \right\}$$

ensuring the thesis, the last equality following from

$$\frac{a+b}{c+d} \leq \frac{a}{c} \Leftrightarrow \frac{b}{d} \leq \frac{a}{c} \text{ for any } a, b, c, d > 0.$$

$\square$

Theorem 29.8: *Under Assumptions (A_1) - (A_2) and (A'_8) , any bounded family $(u_\theta)_{\theta > 2}$ of g.s. (resp. l.e.n.) solutions is bounded away from 0.*

PROOF. Let us denote by u_{θ_n} a weak convergent subsequence of the family $(u_\theta)_{\theta > 2}$, where $\theta_n \rightarrow 2$, and by u_* its limit. Classical bootstrap arguments using Sobolev injections and elliptic regularity on smooth domain combined with Remark 29.2 lead to the uniform convergence of u_{θ_n} to u_* . Let us show that this convergence implies that $u_* \neq 0$ and more precisely that $\|u_{\theta_n}\|_{L^\infty} > \eta$ where η is defined in (A'_8) .

In the case $\lambda = \lambda_1$, as $u_{\theta_n} \in \mathcal{N}_{\theta_n}$, we have

$$\lambda_1 \int_{\Omega} u_{\theta_n}^2 \leq \|u_{\theta_n}\|^2 = \int_{\Omega} f_{\theta_n}(u_{\theta_n}) u_{\theta_n} = \int_{\{u_{\theta_n} \neq 0\}} \frac{f_{\theta_n}(u_{\theta_n})}{u_{\theta_n}} u_{\theta_n}^2, \quad (155)$$

which is impossible if $\|u_{\theta_n}\|_{L^\infty} \leq \eta$.

In the case $\lambda = \lambda_2$, by Proposition 29.7 and since $u_{\theta_n}^\pm \in \mathcal{N}_{\theta_n}$, one of the two following inequalities hold

$$\lambda_2 \int_{\Omega} (u_{\theta_n}^+)^2 \leq \|u_{\theta_n}^+\|^2 = \int_{\Omega} f_{\theta_n}(u_{\theta_n}^+) u_{\theta_n}^+ = \int_{\{u_{\theta_n}^+ \neq 0\}} \frac{f_{\theta_n}(u_{\theta_n}^+)}{u_{\theta_n}^+} (u_{\theta_n}^+)^2, \quad (156)$$

$$\lambda_2 \int_{\Omega} (u_{\theta_n}^-)^2 \leq \|u_{\theta_n}^-\|^2 = \int_{\Omega} f_{\theta_n}(u_{\theta_n}^-) u_{\theta_n}^- = \int_{\{u_{\theta_n}^- \neq 0\}} \frac{f_{\theta_n}(u_{\theta_n}^-)}{u_{\theta_n}^-} (u_{\theta_n}^-)^2, \quad (157)$$

which is again impossible if $\|u_{\theta_n}\|_{L^\infty} \leq \eta$. $\square$

REMARK 29.9. It is worth pointing out that for polygonal domains in the plane, Assumption (A'_8) does not apply because of the lack of smoothness. However, at least for a square or a rectangle, one can give an alternative proof of the a priori estimates required by Theorem 29.8 which does not rely on the smoothness of the domain but on some extension arguments which work fine even for a product of intervals in $\mathbb{R}^N$.

29.5 Conclusion Combining the previous estimates of this section with the abstract symmetry result of Section 28, we can conclude the Theorem 27.2.

30 Generalization of Problem (LE): Symmetry Breaking

We proved for the Lane–Emden problem in Chapter 4, that the previous results can not be extended to large θ when $\lambda = \lambda_2$. The results are not uniform with respect to Ω and by adjusting the dimensions of a rectangle with respect to θ , we exhibited l.e.n. solutions that were neither symmetric nor antisymmetric with respect to the medians.

The main idea goes as follows:

- in Section 22, it is shown numerically that, on a square, the minimizers of J_* on $\mathcal{N}_*$, are symmetric with respect to the diagonal and not to the medians,
- it follows that l.e.n. solutions do accumulate on the second eigenfunctions which are symmetric with respect to the diagonals,
- on the contrary, on a rectangle, l.e.n. solutions do accumulate on the second eigenfunctions which are symmetric with respect to the medians,
- taking a sequence of l.e.n. solutions on oblong rectangles approximating a square leads to a non-symmetric regime.

Arguing like that requires to extend the previous results to a more general second order elliptic operator. Let us consider

$$A_\theta \in \mathcal{C}(\Omega, S^{N \times N}), \quad A_0 = \text{id}, \quad \theta \mapsto A_\theta \text{ uniformly differentiable at } \theta = 0,$$

$S^{N \times N}$ being the set of symmetric $N \times N$ matrices, and let us denote by $\tilde{A}_{\mu, \theta}$ the operator $-\text{div}(A_\theta \nabla) + \mu \text{id}$. One can first prove that the previous results about Problem $(\mathcal{P}_{\mu, \lambda, \theta})$ extend to the similar boundary value problem with A_μ by $\tilde{A}_{\mu, \theta}$. There are not changes in the previous proofs to obtain the same conclusions, except for the non-convergence to 0. Indeed, the modifications in the proof of Theorem 29.6 are straightforward as soon as we assume the extra condition

$$\exists k \in \mathbb{R} : \forall \theta > 2 : \frac{\log(1 - c_2(\theta) - O(\theta - 2))}{c_1(\theta)} \geq k.$$

When dealing with (A'_8) in Theorem 29.8, we impose the fact that $A_\theta - \text{id}$ is semi-positive definite. Then, we control the sign of the additional term in the right-hand side of (155) (or (156) and (157)):

$$\|u_{\theta_n}\|^2 = \int_{\{u_{\theta_n} \neq 0\}} \frac{f_{\theta_n}(u_{\theta_n})}{u_{\theta_n}} u_{\theta_n}^2 - \int_{\Omega} \nabla u_{\theta_n} (A_{\theta_n} - \text{id}) \nabla u_{\theta_n}.$$

We next quickly sketch the argument of Chapter 4 and check that the last extra assumption is fulfilled in this situation. Consider sequences $(\theta_n)_n$ and $(v_n)_n$ such that $0 < v_n = o(\theta_n - 2)$. Let R_{v_n} be rectangles with sides of length 2 and $(1 - v_n)2$. We then consider the sequence of problems

$$(-\Delta + \mu \text{id})u = f_{\theta_n}(u) \text{ in } R_{v_n}, \quad Bu = 0 \text{ on } \partial R_{v_n}. \quad (\mathcal{P}_{v_n})$$

The change of variable $\tilde{u}(x, y) = u(x, (1 - v)y)$ leads to the equivalent sequence of problems

$$-\partial_x^2 \tilde{u} - (1 - v_n)^{-2} \partial_y^2 \tilde{u} + \mu \tilde{u} = f_{\theta_n}(\tilde{u}) \text{ in } Q, \quad B\tilde{u} = 0 \text{ on } \partial Q,$$

where $\bar{Q} = [-1, 1]^2$. In both cases, $\lambda = \lambda_2(Q)$. Observe that the differential operator can be written in divergence form with

$$A_{\theta_n} = \begin{pmatrix} 1 & 0 \\ 0 & (1 - v_n)^{-2} \end{pmatrix}$$

so that $A_{\theta_n} - \text{id}$ is semi-positive definite.

Theorem 30.1: Assume $\lambda = \lambda_2$ and f_θ fulfills the assumptions $(A_1) - (A_8)$ (with $(A_1) - (A_8)$ possibly replaced by $(A'_1) - (A'_8)$). Then, if Ω is a square and the minimizers u_* of J_* on $\mathcal{N}_*$ verify $\int_\Omega H_*(u_*) \neq 0$ and are neither symmetric, neither antisymmetric with respect to the medians, there exists a rectangle R (close to Ω) and $\theta > 2$ such that any l.e.n. solutions of the problem

$$(-\Delta + \mu \text{id})u = f_\theta(u) \text{ in } R, \quad Bu = 0 \text{ on } \partial R$$

is neither symmetric nor antisymmetric with respect to the medians of R .

31 Generalization of Problem (LE): Examples

In this section, we give three examples of problems different from the Lane–Emden one, where the nonlinearity verifies $(A_1) - (A_8)$, with $(A_1) - (A_8)$ possibly replaced by $(A'_1) - (A'_8)$. Like in Chapter 4, we mention some numerical experiments made by C. Grumiau, with the mountain pass algorithm (MPA) and the modified mountain pass algorithm (MMPA). We refer the reader to Appendix D for technical details about these algorithms, and for some precision about the numerical experiments done here.

All cases will be considered on the square $[-1, 1]^2$ in $\mathbb{R}^2$. Let us recall that

$$\lambda_1(D) = \frac{\pi^2}{2}, \quad \lambda_2(D) = \frac{5\pi^2}{4}, \quad \lambda_1(N) = 1, \quad \lambda_2(N) = 1 + \frac{\pi^2}{4},$$

where “D” means Dirichlet and “N” refers to Neumann.

We shall also study the presence of a symmetry breaking as described in Theorem 30.1. To this aim, we compute, with the software Matlab, the function J_* on $\mathcal{N}_*$. At last, let us recall that $E(\lambda_2) = \langle v_1, v_2 \rangle$, where, in the case of Dirichlet and of Neumann conditions:

$$v_1(x, y) = \cos\left(\frac{\pi}{2}x\right) \sin(\pi y), \quad v_2(x, y) = \sin(\pi x) \cos\left(\frac{\pi}{2}y\right), \quad (D)$$

$$v_1(x, y) = \sin\left(\frac{\pi}{2}x\right), \quad v_2(x, y) = \sin\left(\frac{\pi}{2}y\right). \quad (N)$$

31.1 Super-linear Perturbation of Problem (LE) The first example is a very natural nonlinearity in view of Problem (LE). Let us consider the function

$$f_\theta(t) = \lambda t|t|^{\theta-2} + (\theta - 2)t|t|^{q-2}$$

for some $q \in (2, 2^*)$.

- $(A_1) - (A_2)$ are verified;
- (A_5) is verified;

- $(A_3)-(A_4), (A_6)-(A_7)$ are verified by $f_*(t) = \lambda t \log|t| + t|t|^{q-2}$, $H_*(t) = \lambda t^2 + (q-2)|t|^q$ and $h(t) = (2\lambda + q - 1)(|t|^s + 1)$, with $s = q - 1$ or $q - 2$ accordingly;
- (A_8) is verified for $c_1(\theta) = c_2(\theta) = \theta - 2$.

In order to illustrate this example numerically, let us consider the following problems

$$(-\Delta + \mu \text{id})u = \lambda u^3 + 2u^5 \text{ in } \Omega, \quad B(u) = 0 \text{ on } \partial\Omega, \quad (158)$$

where $\lambda = \lambda_1$ or λ_2 .

Figure 31.1 (and Table 3 in Appendix D) shows the results of the numerical experiments: one-signed (resp. nodal) numerical solutions have the expected symmetries. While it is not sure that those solutions have least energy, all the other solutions that we have found numerically have a larger energy. Let us remark that for the ground state solutions in the Neumann case, the MPA gives the constant solution for θ close to 2 so that we do not consider it in our figures and tables. For both Dirichlet and Neumann boundary conditions, the nodal line seems to be a diagonal.

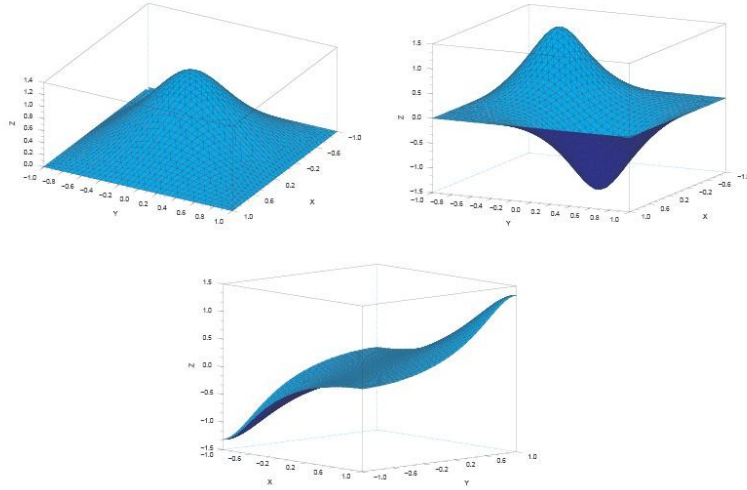


Figure 31.1 Numerical solutions in the case of a superlinear perturbation of the Lane–Emden Problem.

In the case of l.e.n. solutions, minimizers of J_* on $\mathcal{N}_*$ are symmetric with respect to a diagonal. Indeed, any second eigenfunctions can be written as a multiple of $v_\alpha = \cos(\alpha)v_1 + \sin(\alpha)v_2$, for $\alpha \in [0, 2\pi]$, where v_1 and v_2 have been defined previously. Figure 5.2 displays the graph of the function

$$S_* : [0, 2\pi] \rightarrow \mathbb{R} : \alpha \mapsto J_*(t_\alpha v_\alpha),$$

where $t_\alpha u_\alpha \in \mathcal{N}_*$. As the minima are attained for $\alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$, the corresponding minimizers are symmetric functions with respect to a diagonal.

As a consequence of the previous computation, the symmetry breaking described in Theorem 30.1 occurs in this example.

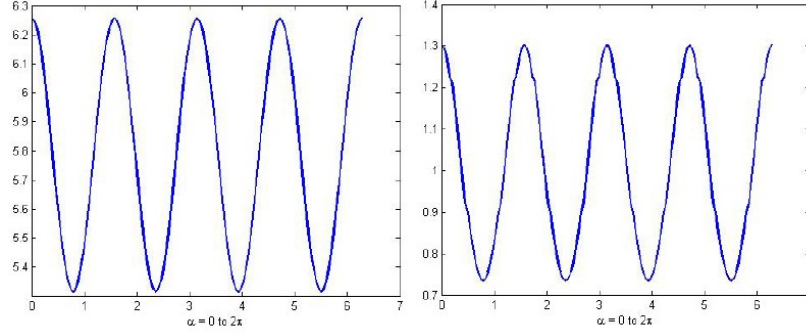


Figure 31.2 Computation of S_* in the superlinear perturbation of the Lane–Emden Problem. The graph on the left (resp. right) corresponds to Dirichlet (resp. Neumann) boundary conditions.

31.2 Exponential Growth The second example is a nonlinearity with an exponential growth in dimension 2:

$$f_\theta(t) = \lambda t(e^{t^2} - 1)^{\theta-2}.$$

- $(A'_1) - (A_2)$ are verified for some $\gamma > 0$;
- (A_5) is verified. Indeed, we have

$$\begin{aligned} (f_\theta(t)t)' - (\theta F_\theta(t))' &= t(f_\theta(t))' - (\theta - 1)f_\theta(t) \\ &= \lambda t(e^{t^2} - 1)^{\theta-2} \left(2(\theta - 2) \frac{t^2 e^{t^2}}{e^{t^2} - 1} + 2 - \theta \right) \end{aligned}$$

which proves the claim for t such that $g(t) := t^2 \frac{e^{t^2}}{e^{t^2} - 1} \geq \frac{1}{2}$ (then observe that $g > \frac{1}{2}$ on $\mathbb{R}$);

- $(A_3) - (A_4)$, $(A_6) - (A_7)$ are verified by $f_*(t) = \lambda t \log(e^{t^2} - 1)$, $H_*(t) = 2\lambda \frac{t^4 e^{t^2}}{e^{t^2} - 1}$ (extended by 0 at 0) and $h(t) = c(\delta) e^{(\delta+1)t^2}$, for some $\delta > 0$ and $c(\delta) > 0$;
- (A'_8) is verified (except the smoothness of the domain, see Remark 29.9) by taking for example $\eta = \sqrt{\log(a)}$ with $1 < a < 2$.

We numerically illustrate this example with the model problem

$$(-\Delta + \mu \text{id})u = \lambda u(e^{u^2} - 1)^{0.5} \text{ in } \Omega, \quad B(u) = 0 \text{ on } \partial\Omega.$$

Figure 31.3 (and Table 4 in Appendix D) shows the results of the numerical experiments. The same comments as in the previous example can be rephrased. In particular, in the case of l.e.n. solutions, minimizers of J_* on $\mathcal{N}_*$ are symmetric with respect to a diagonal as shown in Figure 314 and the symmetry breaking described in Theorem 30.1 still occurs in this example.

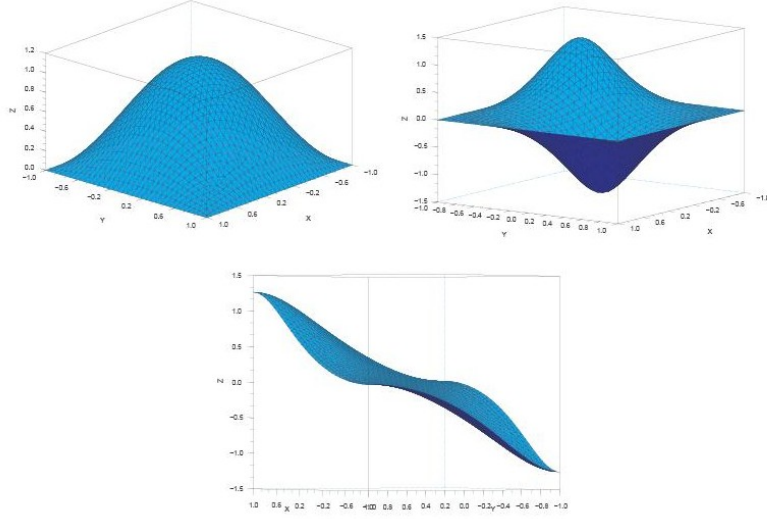
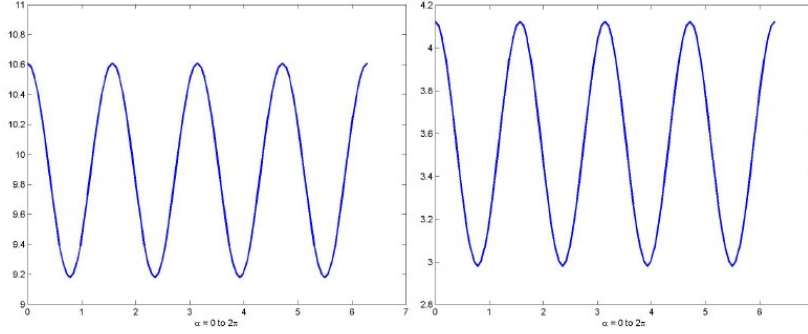


Figure 31.3 Numerical solutions in the case of an exponential growth.

Figure 31.4 Computation of S_k in the case of an exponential growth. The graph on the left (resp. right) corresponds to Dirichlet (resp. Neumann) boundary conditions.

31.3 Sum of Powers In this example, for some technical reasons, we consider a family $(\theta_\varepsilon)_{\varepsilon>0}$ converging to 2 when $\varepsilon \rightarrow 0$ and rename $f_{\theta_\varepsilon}, J_{\theta_\varepsilon}$ by $f_\varepsilon, J_\varepsilon$. We consider the functions

$$f_\varepsilon(t) = \lambda t \left(\sum_{i=1}^k \alpha_i |t|^{\beta_i(\varepsilon)} \right)$$

with $\alpha_i, \beta_i(\varepsilon) > 0$ such that

$$\min \beta_i(\varepsilon) = \beta_1(\varepsilon), \quad \max \beta_i(\varepsilon) = \beta_k(\varepsilon),$$

$$\sum_i \alpha_i = 1, \quad \lim_{\varepsilon \rightarrow 0} \beta_i(\varepsilon) = 0, \quad \lim_{\varepsilon \rightarrow 0} \frac{\beta_k(\varepsilon)}{\beta_1(\varepsilon)} = 1.$$

- (A_1) - (A_2) are verified with $p = \beta_k(\varepsilon)$;
- (A_5) is verified with $\theta_\varepsilon = \beta_1(\varepsilon) + 2$;
- (A_3) - (A_4) , (A_6) - (A_7) are verified by $f_*(t) = \lambda t \log|t|$, $H_*(t) = \lambda t^2$ and $p = 1 + \delta$ for some $\delta > 0$;

- (A'_8) is verified (except the smoothness of the domain, see Remark 29.9) for some $\eta < 1$.

Let us point out that in this example, we have a better control on the Nehari manifold, namely

$$\left(\frac{1}{2} - \frac{1}{\beta_1(\varepsilon) + 2}\right) \|u\|^2 \leq J_\varepsilon(u) \leq \left(\frac{1}{2} - \frac{1}{\beta_k(\varepsilon) + 2}\right) \|u\|^2$$

for every $u \in \mathcal{N}_\varepsilon$. This enables a very simple proof of the boundedness of g.s. (resp. l.e.n.) solutions.

We illustrate this example numerically by considering

$$(-\Delta + \mu \text{id})u = \lambda u \left(\frac{1}{4}|u|^{0.25} + \frac{1}{4}|u|^{0.5} + \frac{1}{2}|u| \right) \text{ in } \Omega, \quad B(u) = 0 \text{ on } \partial\Omega.$$

Figure 31.5 (and Table 5 in Appendix D) shows the results of the numerical experiments. We can do the same comments as in the previous cases.

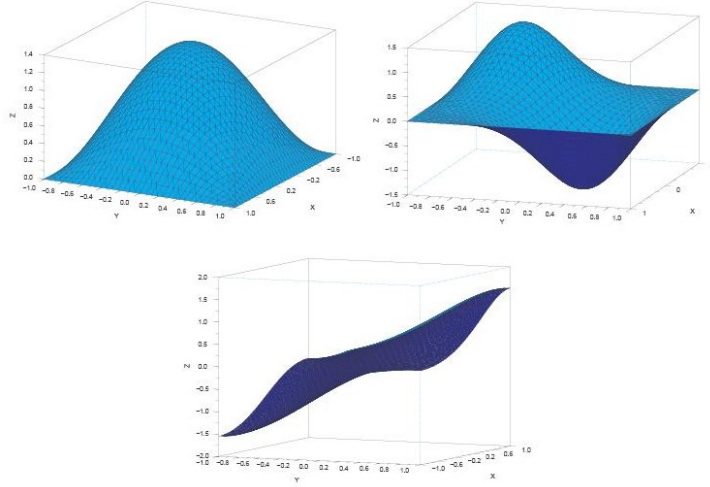


Figure 31.5 Numerical solutions in the case of a sum of powers.

Since the limit functional for this example is the same as the limit functional for the Lane–Emden problem, the symmetry breaking phenomenon occurs.

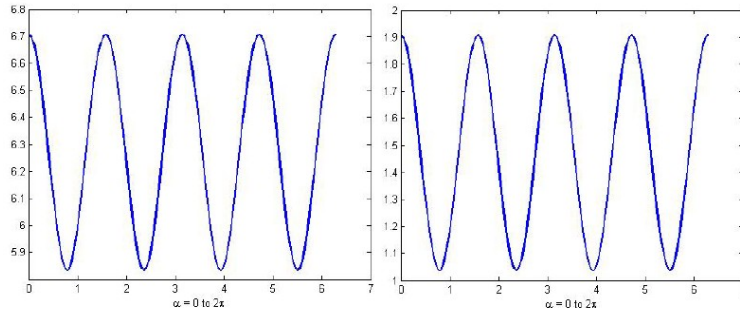


Figure 31.6 Computation of S_* in the case of a sum of powers. The graph on the left (resp. right) corresponds to Dirichlet (resp. Neumann) boundary conditions.

Notes on Degree Theory, Variational Methods and Lyapunov-Schmidt Reduction

In this appendix, let us give some precisions about *degree theory*, *variational methods* and the *Lyapunov-Schmidt reduction* given in Chapter 1. The goal of these techniques is studying nonlinear equations $g(u) = 0$. The two following very simple examples illustrate these three techniques:

- Let $a, b \in \mathcal{C}([-1, 1]^2)$. If for all $\sigma \in \{\pm 1\}$, conditions $\inf a(\sigma, \cdot)\sigma > 0$ and $\inf b(\cdot, \sigma)\sigma > 0$ are verified, there exists at least one solution of $(a, b) = (0, 0)$.
- $$\begin{cases} x + y = 1, \\ y + z = 2, \\ xyz = 3, \end{cases} \Leftrightarrow \begin{cases} L(x, y, z) = (1, 2), \\ N(x, y, z) = 3, \end{cases} \Leftrightarrow \begin{cases} (x, y) = (z - 1, 2 - z), \\ (z - 1)(2 - z)z = 3. \end{cases}$$

1 Degree Theory

Degree theory consists to associate to each pair (g, D) made of an open bounded set $D \subset \mathbb{R}^N$ and a function $g \in \mathcal{C}(\overline{D}, \mathbb{R}^N)$ such that $0 \notin g(\partial D)$, an integer written $\deg_B(g, D)$ and called *Brouwer degree of g on D* , in such a way that the application $\deg_B$ verifies the three following properties:

- **(Normalization)** $\deg_B(\text{id}, D) = 1$ if $0 \in D$,
- **(Additivity)** $\deg_B(g, D) = \deg_B(g, D_1) + \deg_B(g, D_2)$ if D_1, D_2 are disjoint open subsets of D such that $0 \notin g(\overline{D} \setminus (D_1 \cup D_2))$,
- **(Homotopy invariance)** if $G \in \mathcal{C}([0, 1] \times \overline{D}, \mathbb{R}^N)$ is such that $0 \notin G([0, 1] \times \partial D)$, then $\deg_B(G(\lambda, \cdot), D)$ is independent of $\lambda \in [0, 1]$.

There are different ways to define the Brouwer degree, for example proving that there exists one and only one application (159) verifying the three previous properties,

$$A : \{g \in \mathcal{C}(\overline{D}, \mathbb{R}^N) : 0 \notin g(\partial D)\} \rightarrow \mathbb{Z}, \quad (159)$$

or working with (160), the “algebraic number” of zeros of g in D , under the “removable” (see Remark 1.1) conditions (161),

$$\deg_B(g, D) = \sum_{u \in g^{-1}(0)} \text{sgn}(\det g'(u)), \quad (160)$$

$$g \in \mathcal{C}(\overline{D}, \mathbb{R}^N) \cap \mathcal{C}^2(D, \mathbb{R}^N), \quad \forall u \in g^{-1}(0) : g'(u) \in \text{Iso}(\mathbb{R}^N), \quad (161)$$

or considering the differential N -form defined by (162), (163) and working with the integral (164) under the “removable” condition $g \in \mathcal{C}(\overline{D}, \mathbb{R}^N) \cap \mathcal{C}^2(D, \mathbb{R}^N)$,

$$\omega_a^N(x) = a(|x|) dx_1 \wedge \cdots \wedge dx_N, \quad (162)$$

$$\begin{aligned} a &\in \mathcal{C}([0, \infty), \mathbb{R}), \quad \frac{1}{\mu_{N-1}} \int_{\mathbb{R}^N} a(|x|) dx = 1, \\ \text{Supp } a &\subset [\varepsilon_0, \mu_0], \quad 0 < \varepsilon_0 < \mu_0 < \inf_{\partial D} |g|, \end{aligned} \quad (163)$$

$$\deg_B(g, D) = \frac{1}{\mu_{N-1}} \int_D g^* \omega_a^N, \quad (164)$$

where $\mu_{N-1} := \mathcal{H}^{N-1}(\mathcal{S}^{N-1})$ is the $(N-1)$ -Hausdorff measure of $\mathcal{S}^{N-1}$. Whatever the definition, this tool is very useful to study nonlinear equations because it verifies the two following properties which are simple consequences of the three previous ones:

- **(Excision)** $\deg_B(g, D) = \deg_B(g, D \setminus \overline{D_1})$ if D_1 is an open subset of D such that $0 \notin g(\overline{D} \setminus D_1)$,
- **(Existence)** if $\deg_B(g, D) \neq 0$, there exists a solution of $g(u) = 0$.

In the first example given previously, the function (a, b) from $[-1, 1]^2$ to $\mathbb{R}^2$ can be deformed to the identity by the homotopy $(1 - \lambda)(a, b) + \lambda \text{id}$ and we conclude the existence result by normalization, homotopy invariance and existence properties.

REMARK 1.1. Let us note that the sum (160) is finite under the assumption (161), and that the integral (164) is independent of a under the assumption (163). We used previously the adjective “removable” for Conditions (161). Indeed, $\deg_B$ can be extended to any function g by $\lim_{n \rightarrow \infty} \deg_B(g_n, D)$ with $\lim_{n \rightarrow \infty} \|g_n - g\|_\infty = 0$ and g_n verifying these conditions, which is possible as $\mathcal{C}(\overline{D}, \mathbb{R}^N) \cap \mathcal{C}^2(D, \mathbb{R}^N)$ is dense in $\mathcal{C}(\overline{D}, \mathbb{R}^N)$ with respect to the uniform norm, and as regular values of g are dense in $\mathbb{R}^N$. Let us recall that a *regular value* v of g is a vector of $\mathbb{R}^N$ verifying the condition $g'(u) \in \text{Iso}(\mathbb{R}^N)$ for all $u \in g^{-1}(v)$ (*Sard's theorem*).

REMARK 1.2. If (g, D) is a pair as previously having the properties that $g \in \mathcal{C}^2(\overline{D}, \mathbb{R}^N)$ and D is of class $\mathcal{C}^2$, we can introduce the *Kronecker index of g on ∂D* written $\text{ind}_K(g, \partial D)$, by considering the differential $(N-1)$ -form defined by (165), (166) and the integral (167),

$$\omega_A^{N-1}(x) = A(|x|) \sum_{j=1}^N (-1)^{j+1} x_j dx_1 \wedge \cdots \wedge \widehat{dx}_j \wedge \cdots \wedge dx_N, \quad (165)$$

$$A \in \mathcal{C}([0, \infty), \mathbb{R}), \quad A(r) = \begin{cases} 0 & \text{if } r < \varepsilon_0, \\ r^{-N} & \text{if } r > \mu_0, \end{cases} \quad (166)$$

$$\text{ind}_K(g, \partial D) := \frac{1}{\mu_{N-1}} \int_{\partial D} g^* \omega_A^{N-1}, \quad (167)$$

with ε_0 and μ_0 as in (163). If $A, a \in \mathcal{C}^1(0, \infty)$ are linked by the relation

$$a(r) = rA'(r) + NA(r),$$

the Stokes-Cartan's theorem shows the equality $\deg_B(g, D) = \text{ind}_K(g, \partial D)$. This remark is useful to see the independence of the integral (164) with respect to a verifying (163), and to see that the Brouwer degree of a function g on D depends only on the values of g on ∂D . Let us still note that in dimension 1, this shows

$$\deg_B(g, (a, b)) = \frac{\text{sgn}(g(b)) - \text{sgn}(g(a))}{2},$$

which is equivalent to the Bolzano's theorem, and that in dimension 2 with the assumption that g is $\mathbb{C}$ -derivable, the equality of the right members of (160) and (167) is equivalent to the *residue theorem*.

In an infinite dimensional Banach space E , it is impossible to define an application A , like in (159), from any pair (g, D) to $\mathbb{Z}$ having Normalization, Additivity and Homotopy-Invariance properties. Nevertheless, this process can be done on sub-families of pairs

(g, D) . Let us mention, among other ones, the *Leray-Schauder degree*, the *Coincidence degree* and the *Furi-Benevieri degree*, working respectively with pairs

$$(\text{id} - \chi, D), \quad (L - \chi, D), \quad (N - \chi, D),$$

where respectively χ is compact, $L \in \Phi_0(E)$ and χ is “ L -compact”, N is an operator of class $\mathcal{C}^1$ with any Frechet-derivative in $\Phi_0(E)$ and χ is “ N -compact”. To make these generalizations, some questions about *orientation* and *compactness* must be solved. The reader can consult [13] for a simple introduction.

The Leray-Schauder generalization is the first one, and maybe the most famous and used one. If $T : E \rightarrow E$ is a continuous and completely continuous linear operator, and if λ is one of its eigenvalues, then a stability phenomenon occurs in inclusions (21), leading to the definition of $m_a(\lambda)$ as in finite dimension. If $\chi : \bar{D} \subset E \rightarrow E$ is a compact operator, it can be approximated by operators

$$\chi_n : D_n := D \cap E_n \rightarrow E_n,$$

where $(E_n)_n$ is an increasing sequence of finite dimensional subspaces of E converging to E . The previous stability result is the key to understand the stability phenomenon of sequence $(\deg_B(\chi_n, D_n))_n$, and so the key to define the Leray-Schauder degree $\deg_{LS}(\chi, D)$ as the limit of the previous sequence.

2 Variational Methods

The variational methods concern the research of critical points, i.e. points $u \in \mathbb{R}^N$ such that $h'(u) = 0$ with $h \in \mathcal{C}^1(\mathbb{R}^N)$, so the particular case where $g = \nabla h$. Some kind of geometries ensuring existence of critical points can be found in many books: *mountain pass*, *saddle point*, *linking*... the simplest one is the *minimum* geometry. For example, if h is *coercive*, i.e.

$$\lim_{\|u\| \rightarrow \infty} h(u) = +\infty,$$

there exists a critical point which is a (global) minimum. This geometry can be illustrated by the first example $(a, b) = (0, 0)$. Let us define here the following set

$$\mathcal{N} := \{u \neq 0 : \langle h'(u) | u \rangle = 0\}$$

called *Nehari manifold*. When $h \in \mathcal{C}^2(\mathbb{R}^N)$, we can define the subspace

$$T_u \mathcal{N} := \{v : \langle h'(u) | v \rangle + \langle h''(u) | u, v \rangle = 0\}. \quad (168)$$

If for each u there exists $v \notin T_u \mathcal{N}$, the Nehari manifold is a manifold of class $\mathcal{C}^1$. Moreover, if for each u , the condition $u \notin T_u \mathcal{N}$ is verified, a critical point of $h|_{\mathcal{N}}$ (for example a minimum of h on $\mathcal{N}$) is a critical point of h on $\mathbb{R}^N$. Let us consider a closed convex *cone* K in $\mathbb{R}^N$, so a convex subset of $\mathbb{R}^N$ closed for the norm and for the multiplication by a positive real. We can introduce the following subset of $\mathcal{N}$:

$$\mathcal{M} := \{u : u^\pm \in \mathcal{N}\}$$

where u^+ is the orthogonal projection of u on K , and $u^- = u - u^+$. Because the orthogonal projection on K is in general not of class $\mathcal{C}^1$, the set $\mathcal{M}$ is not a $\mathcal{C}^1$ -manifold. This forbids us to define $T_u \mathcal{M}$ and to consider critical points of h restricted to $\mathcal{M}$. Nevertheless we can study the minimum of h on $\mathcal{M}$. Let us consider the very particular case where

- $\forall u \in \mathbb{R}^N : [0, \infty) \rightarrow \mathbb{R} : t \mapsto h(tu)$ is increasing, reaches its maximum at a unique value, and after that is decreasing,
- $\forall u \in \mathbb{R}^N : h(u^+ + u^-) = h(u^+) + h(u^-)$,

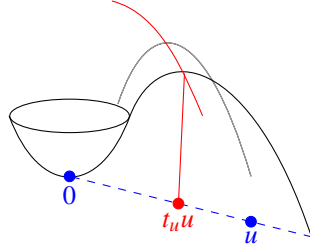


Figure 2.1 (a)

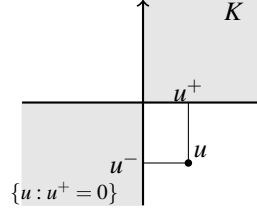


Figure 2.1 (b)

which describes a particular (and very usual in the theory of differential equations) case of the mountain pass geometry. In the sequel, these assumptions will be referred to as *Nehari geometry*. Let us note that under this geometry and because K is a convex cone, we have:

- $\forall u \in \mathcal{S}^{N-1}, \exists ! t > 0 : tu \in \mathcal{N}$,
- $\forall u \in \mathcal{S}^{N-1}, \exists ! t^\pm > 0 : t^+ u^+ + t^- u^- \in \mathcal{M}$.

Theorem 2.1: *Under the Nehari geometry assumption, a minimum value of $h|_{\mathcal{N}}$ and of $h|_{\mathcal{M}}$ is a critical value of h on $\mathbb{R}^N$.*

PROOF. Let us denote by u the point of $\mathcal{N}$ (resp. $\mathcal{M}$) where $h|_{\mathcal{N}}$ (resp. $h|_{\mathcal{M}}$) reaches its minimum. If we suppose by contradiction that $h'(u) \neq 0$, there exists $\varepsilon, \delta > 0$ such that

$$\forall v \in h^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta} : \|h'(v)\| \geq 8\varepsilon/\delta,$$

where $c := h(u)$, $S := \{u\}$, and $S_\delta := \{v : \|v - u\| \leq \delta\}$. Hence we can apply the *deformation's lemma* (see [53]) to deduce the existence of an homeomorphism $\eta : \mathbb{R}^N \rightarrow \mathbb{R}^N$ verifying

- (i) $\eta(v) = v$ if $v \notin h^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}$,
- (ii) $\eta(h^{c+\varepsilon} \cap S) \subset h^{c-\varepsilon}$,
- (iii) $h(\eta(v)) < c$ for any $v \in h^c \cap S_\delta$,
- (iv) $h(\eta(v)) \leq h(v)$ for any $v \in \mathbb{R}^N$.

where $h^d := h^{-1}((-\infty, d])$. It can be proved that

$$\eta(\pi) \cap \mathcal{N} \neq \emptyset \quad (\text{resp. } \eta(\pi) \cap \mathcal{M} \neq \emptyset) \quad (169)$$

where $\pi := \{tu : t > 0\}$ (resp. $\{su^+ + tu^- : s, t > 0\}$). Since the Nehari geometry ensures $h(\pi \setminus \{u\}) < c$, property (169) joined to the decreasing of $h \circ \eta$ with respect to h contradicts the definition of c . To prove (169), we must show the existence of a zero of

$$g : \mathbb{R}_0^+ \rightarrow \mathbb{R} : t \rightarrow \langle h'(\eta(tu)) | \eta(tu) \rangle,$$

$$(\text{resp. } g : (\mathbb{R}_0^+)^2 \rightarrow \mathbb{R}^2 : (s, t) \rightarrow \begin{pmatrix} \langle h'(\eta(su^+)) | \eta(su^+) \rangle \\ \langle h'(\eta(tu^-)) | \eta(tu^-) \rangle \end{pmatrix}).$$

Let us consider for that an interval (resp. a product of intervals) D of $\mathbb{R}_0^+$ (resp. $(\mathbb{R}_0^+)^2$) such that $(S_{2\delta} \cap \pi) \subset \tilde{D}$ where

$$\tilde{D} := \{su : s \in D\}, \quad (\text{resp. } \{su^+ + tu^- : s, t \in D\}).$$

Since property (i) ensures that $\eta = \text{id}$ on $\partial \tilde{D}$, we deduce, up to a translation $\tilde{t} = t - 1$ (resp. $\tilde{t} = t - 1, \tilde{s} = s - 1$), that $-g(t) \text{sgn}(t) > 0$ for $t \in \partial D$ (resp. $-g$ verifies the assumptions of function (a, b) of the first example at the beginning of this appendix) by the Nehari geometry, and hence the existence of a zero of g . $\square$

3 Lyapunov-Schmidt Reduction

The Lyapunov-Schmidt reduction is well detailed in Chapter 1. We just want to give here one comment and two illustrations.

The second example at the beginning of this appendix, illustrates very well the sequence of equivalences (24) of Lyapunov-Schmidt reduction given in Section 2. Nevertheless, let us precise that the natural operators L, N given in this example, must be replaced by operators from $\mathbb{R}^3$ to itself defined by

$$L(x, y, z) = (x + y, y + z, 0), \quad N(x, y, z) = (1, 2, 3 - xyz).$$

The coincidence degree, mentioned at the end of Section 1, can be described as a beautiful mix of the Leray-Schauder degree and the Lyapunov-Schmidt reduction. Indeed, let us consider the notations L, E_i, F_i and p, q, P, Q as in Section 2. If $B : E_2 \rightarrow F_2$ is a linear bijection, then we can extend the sequence of equivalences (24) by

$$(\text{id} - q)(u) = L^{-1}PN(u) - BQN(u).$$

We can then consider $k := \deg_{\text{LS}}(\text{id} - \chi, D)$ where $\chi := q + L^{-1}PN - BQN$. It can be proved that k is independent of E_i, F_i, p, q, P, Q and, up to the sign, of B , what explains the name of the “coincidence” degree. For more details, the lector can consult [40].

The proof of Theorem 3.2 given in Section 3 in the case $b = 1$, is a beautiful application of the process 2.3 given in Section 2. Indeed, the problem $\mathcal{R}_\lambda$ studied in this result, can be written by $Lu - N(\lambda, u) = 0$ where

$$L : \mathcal{C}_\#([0, T]) \rightarrow \mathcal{C}_0([0, T]) : u \mapsto u - u(0),$$

$$N(\lambda, \cdot) : \mathcal{C}_\#([0, T]) \rightarrow \mathcal{C}_0([0, T]) : u \mapsto \lambda \int_0^t f(s, u(s)) \, ds,$$

$$\mathcal{C}_\#([0, T]) := \{u \in \mathcal{C}([0, T]) : u(0) = u(T)\},$$

$$\mathcal{C}_0([0, T]) := \{u \in \mathcal{C}([0, T]) : u(0) = 0\},$$

with $\mathcal{C}_\#([0, T])$ and $\mathcal{C}_0([0, T])$ equipped of the uniform norm. We can consider spaces

E_1, E_2, F_1, F_2 and operators p, q, P, Q like in (22), (23), defined by

$$E_2 = \{u \in \mathcal{C}_\#([0, T]) : u = C^{\text{te}}\}, \quad F_1 = \{v \in \mathcal{C}_0([0, T]) : v(T) = 0\},$$

$$q(u) = u(0), \quad Q(v)(t) = \frac{v(T)}{T}t.$$

Let $\alpha \in E_2$. Let us consider the first equation of the last equivalence of (24):

$$p(u)(t) = \lambda \left(\int_0^t f(s, p(u)(s) + \alpha) \, ds - \frac{t}{T} \int_0^T f(s, p(u)(s) + \alpha) \, ds \right) =: \tilde{N}(\lambda, \alpha, p(u))(t).$$

This equation can be solved by the Implicit Function Theorem applied to the operator $\text{id}_{E_1} - \tilde{N}(\lambda, \alpha, \cdot) : E_1 \rightarrow E_1$ at the point $(\lambda, \alpha, p(u)) = (0, \alpha_0, 0)$, as the derivative of this operator at this point equals id_{E_1} . So, we can consider the function γ defined by (170) and hence the second equation of the last equivalence of (24) given by (171),

$$\begin{aligned} \gamma : [0, \bar{\lambda}] \times (\alpha_0 - r, \alpha_0 + r) &\rightarrow E_1 : (\lambda, \alpha) \mapsto p(u), \\ p(u) &= \tilde{N}(\lambda, \alpha, p(u)), \end{aligned} \tag{170}$$

$$\frac{\lambda}{T} \int_0^T f(s, \gamma(\lambda, \alpha) + \alpha) \, ds = 0, \tag{171}$$

and that we shall denote by $g(\lambda, \alpha) = 0$. Since this last equation is degenerate when $\lambda = 0$, we shall work with equation $\tilde{g}(\lambda, \alpha) := \lambda^{-1}g(\lambda, \alpha) = 0$. By the assumption

$$\int_0^T \partial_u f(s, \alpha_0) ds \neq 0, \quad (\text{resp. } \deg_B(\tilde{g}(0, \cdot), (-r, r)) \neq 0),$$

where α_0 is a zero of $\tilde{g}(0, \cdot)$ denoted by u_* in the assumption of the theorem (resp. $r > \max\{|M_1|, |M_2|\}$ with M_1, M_2 defined in the assumption of the Theorem), we can apply the Implicit Function Theorem (resp. the Homotopy invariance and the existence properties of the Brouwer degree) to conclude the first (resp. the second) thesis of the theorem for λ small enough. To complete the second thesis for any λ , we need also to prove a priori bounds r . This is a long work that we do not want to make here. The interested reader can consult [42] or Section 8.1 inspired by it.

REMARK 3.1. In fact, we could use the infinite dimensional version of the Implicit Function Theorem or the Leray-Schauder degree, applied to the operator

$$\text{id} - \zeta : [0, 1] \times \mathcal{C}_\#([0, T]) \rightarrow \mathcal{C}_\#([0, T]),$$

$$\zeta(\lambda, u)(t) := u(0) + \lambda \left(\int_0^t f(s, u(s)) ds - \frac{t}{T} \int_0^T f(s, u(s)) ds \right) + \frac{1}{T} \int_0^T f(s, u(s)) ds,$$

to conclude respectively the first and the second thesis of Theorem 3.2.

Numerical Experiments for Ordinary Differential Equation

In Chapter 2, several numerical experiments are given about Problem

$$v' = h_{p,q}(t) + e^{\frac{v}{\sqrt{1-v^2}}}, \quad v(0) = v(1).$$

The definition of $h_{p,q}$ (with $q = q(p)$) and the results of these experiments are given in Example 10.2. Let us describe here the different routines used to realize these ones.

4 First Routine: the Functions α, β

In order to obtain Figure 10.1 giving the values of the upper and less solutions α, β , we used the following routine, described here to compute $\alpha(0)$:

Routine 4.1: *Let us define*

$$a := -0.99, \quad b := 0.99, \quad c := 0, \quad \varepsilon := 2^{-7}.$$

While $(b - a) > \varepsilon$,

- (1) v_s is the solution of the Cauchy's Problem $\begin{cases} v' = (h_{p,q}(t) + e^{\phi^{-1}[v]}), \\ v(0) = s, \end{cases}$
- (2) $g : (-1, 1) \rightarrow [-1, 1] : s \rightarrow v_s(1)$,
- (3) if $g(c) > -1$ then $b = c$ and $c = \frac{a+b}{2}$, else $a = c$ and $c = \frac{a+b}{2}$.

Then we have

$$\alpha(0) \simeq c.$$

The precision of the results obtained by this routine can be improved if a (resp. b) is nearer than -1 (resp. 1), and if ε is nearer than 0 . Let us note also that $g(s)$ will be 1 (resp. -1) if the function v_s is not global, i.e. reaches 1 (resp. -1) before that t reaches 1 .

REMARK 4.2. Another way to obtain the graph of $\alpha(0)$ with respect to p , is to observe the value at 0 of the solutions of the Cauchy's Problem obtained with $e^{\phi^{-1}[v]}$ extended by 0 when $v = -1$, and with initial condition equal to $v(r_1) = -1$ (or $v(r_2) = -1$ depending of the values of p), where r_1, r_2 , defined in Section 10, are the roots of $h_{p,q}$.

5 Second Routine: the Solutions v_p

In order to obtain Figure 10.3 giving the solutions of the studied problems, we used the following routine working with Problem $(\tilde{\mathcal{P}}_{b,\lambda})$ (with $b = 1$) given in Section 8.3. This routine computes, from the Constant function 0 (solution of Problem $(\tilde{\mathcal{P}}_{b,\lambda})$ when $\lambda = 0$, and of equation $\int_0^1 (h_{p,q} + e^u) = 0$), the solution v_p of problem at indexed by p .

Routine 5.1: *Let us define*

$$v_0 := 0, \quad \forall i \in \{1, \dots, n\} : \quad \lambda_i := \frac{i}{n}.$$

For i equals 1 to n,

- (1) $v_{i,s}$ is the solution of the Cauchy's Problem $\begin{cases} v' = \lambda_i(h_{p,q}(t) + e^{\phi^{-1}[v]}), \\ v(0) = s, \end{cases}$
- (2) $g_i : \mathbb{R} \rightarrow \mathbb{R} : s \rightarrow s - v_{i,s}(1),$
- (3) $v_0 \leftarrow$ the closest zero of v_0 of the function g_i .

Then we have

$$v_p \simeq v_{n,v_0}.$$

The fact that the solutions of Problem $(\mathcal{P}_{b,\lambda})$ (with $b = 1$) when $\lambda = 1$ coincide with solutions of the original problem, mentioned at the beginning of this appendix, is a consequence of the proof of Theorem 8.1 (see Section 8.3, and note that $\tilde{K}_1, \tilde{K}_2$, defined at by (59) and (60), equal K_1, K_2 , defined by (54) and (55)).

The numerical errors of this routine can appear for 2 reasons:

- we used a numerical function in order to find the roots of g_i , giving just an approximation of the exact value,
- we used a numerical function to compute $v_{i,s}$, giving again just an approximation of the exact value.

We can say anything about the second error except that we worked with the software Matlab and the function *ode-45* of it. For the first kind of errors, when the routine run following λ , the errors made for λ do not increase when $\lambda \rightarrow 1$. Indeed, the function *fzero* of Matlab searching zero correct this error at each step. So we are not in a situation of error's explosion but instead the converse. All the error due to *fzero* will appear at the last step, when $\lambda = 1$. But even at this step, we think that the error is not bad.

Indeed, the function *fzero* looks for a root of $\text{id} - P$ (where P is the Poincaré operator defined by $P(s) = v_s(1)$ like at point (1) of the routine) with a better method than the Newton's algorithm. Let us recall that this one needs to be efficient, condition $(\text{id} - P)'(s_0) \neq 0$, where s_0 is the searched root. Like

$$(\text{id} - P)'(s_0) = 1 - \frac{d}{ds} v_s(1)|_{s_0} \neq 0 \Leftrightarrow \varepsilon(1, s_0) \neq 1, \quad (172)$$

where $\varepsilon(t, s) := \frac{d}{ds} v_s(t)$, we conclude that condition (172) is equivalent to

$$\int_0^1 \frac{\partial f}{\partial v}(x, v_{s_0}(x)) dx \neq 0 \quad (173)$$

where f is denoted for the nonlinearity $h_{p,q}(t) + e^{\phi^{-1}[v]}$ of our equation. That is a consequence of the general theory of ordinary differential equations ensuring that $\varepsilon(t, s)$ verifies (174) and so (175):

$$\varepsilon' = \frac{\partial f}{\partial v}(t, v_s(t))\varepsilon, \quad \varepsilon(0, s) = 1, \quad (174)$$

$$\varepsilon(t, s) = \exp\left(\int_0^t \frac{\partial f}{\partial v}(x, v_s(x)) dx\right). \quad (175)$$

Like Condition (173) is true because of

$$0 \neq \deg_{\text{LS}}(\text{id} - T, B(0, K)) = \int_0^1 \frac{\partial f}{\partial v}(x, v_{s_0}(x)) \, dx,$$

(where T and K are defined in Section 8, the inequality is due to the proof of Theorem 8.1, and the equality can be found for example in [39]) we can conclude what we said about the error.

Some Technical Results about CKN-Inequalities

In Chapter 3, it is proved that

$$\mathcal{D}(\mathbb{R}^N) \times \mathcal{D}(\mathbb{R}^N) \rightarrow \mathbb{R} : (u_1, u_2) \rightarrow (u_1 | u_2)_\beta = \int_{\mathbb{R}^N} |x|^{2\beta} \nabla u_1 \cdot \nabla u_2 \, dx$$

is a scalar product on $\mathcal{D}(\mathbb{R}^N)$, what permits to consider the norm $\|\cdot\|_\beta$ induced by it, and $\mathcal{D}_\beta^{1,2}(\mathbb{R}^N)$, the completion of $\mathcal{D}(\mathbb{R}^N)$ with this norm. Let us prove the following lemma, mentioned in [21] with few details:

Lemma 5.2: *If $\beta > \frac{2-N}{2}$, the space $\mathcal{D}(\mathbb{R}^N \setminus \{0\})$ is dense in $\mathcal{D}_\beta^{1,2}(\mathbb{R}^N)$ with respect to $\|\cdot\|_\beta$.*

PROOF. Let us consider an increasing cut off function $\eta \in \mathcal{C}^\infty(0, \infty)$ such that for any $s > 0$,

$$\eta(s) = 0 \text{ if } s \leq 1, \quad \eta(s) = 1 \text{ if } s \geq 2, \quad 0 \leq \eta(s) \leq 1.$$

We shall show that for any $u \in \mathcal{D}(\mathbb{R}^N)$, there exists $(u_n)_n \subset \mathcal{D}(\mathbb{R}^N \setminus \{0\})$ such that $\|u_n - u\|_\beta \rightarrow 0$. Let us define $\eta_n(x) := \eta(n|x|)$ and $u_n := \eta_n u \in \mathcal{D}(\mathbb{R}^N \setminus \{0\})$. Like

$$\|u_n - u\|_\beta \leq \left(\int_{\mathbb{R}^N} |x|^{2\beta} |u \nabla \eta_n|^2 \, dx \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^N} |x|^{2\beta} |\eta_n \nabla u - \nabla u|^2 \, dx \right)^{\frac{1}{2}},$$

we have to show that the two terms tend to 0. The second one is treated by the dominated convergence theorem of Lebesgues jointed to

$$|x|^{2\beta} |\eta_n \nabla u - \nabla u|^2 \rightarrow 0 \text{ a.e.,}$$

$$|x|^{2\beta} |\eta_n \nabla u - \nabla u|^2 \leq 4|x|^{2\beta} |\nabla u|^2 \in L^1(\mathbb{R}^N),$$

and the first one, by the same theorem jointed to

$$|x|^{2\beta} |u \nabla \eta_n|^2 = |x|^{2\beta} n^2 \eta'(n|x|)^2 u^2(x) \rightarrow 0 \text{ a.e.,}$$

$$|x|^{2\beta-2} (n|x|)^2 \eta'(n|x|)^2 u^2 \leq 4 \|\eta'\|_\infty^2 |x|^{2\beta-2} u^2 \in L^1(\mathbb{R}^N),$$

where the last inequality is due to $|x|^{2\beta-2} \in L^1_{\text{loc}}(\mathbb{R}^N)$ because $\beta > \frac{2-N}{2}$. $\square$

In the sequel, θ, t, σ will be the generic notations of a point on respectively $\mathcal{S}^{N-1}, \mathbb{R}, \mathcal{C}$, with $\mathcal{C} := \mathcal{S}^{N-1} \times \mathbb{R}$, and we shall denote by $\nabla, \nabla_\theta, \nabla_\sigma$ the gradient operator on respectively $\mathbb{R}^N, \mathcal{S}^{N-1}, \mathcal{C}$, with the gradient defined like in Section 4. Let us also recall the following application defined in Section 16:

$$T : \mathcal{D}(\mathbb{R}^N \setminus \{0\}) \rightarrow \mathcal{D}(\mathcal{C}) : u \mapsto v,$$

$$u(x) = |x|^{-\frac{N-2+2\beta}{2}} \cdot v\left(\frac{x}{|x|}, -\ln |x|\right), \quad v(\theta, t) = e^{-\frac{N-2+2\beta}{2}t} \cdot u(\theta \cdot e^{-t})$$

We shall prove the following lemma, mentioned in [21] with few details:

Lemma 5.3: for any $u_1, u_2 \in \mathcal{D}(\mathbb{R}^N \setminus \{0\})$ we have the following equalities:

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^{2\beta} \nabla u_1 \nabla u_2 \, dx &= \int_{\mathbb{C}} \left\{ \nabla v_1 \nabla v_2 + \left(\frac{N-2+2\beta}{2} \right)^2 v_1 v_2 \right\}, \\ \int_{\mathbb{R}^N} |x|^{\alpha p} |u|^p \, dx &= \int_{\mathbb{C}} |v|^p, \end{aligned}$$

where v_i equal $T(u_i)$ and $\alpha, \beta \in \mathbb{R}$ verify (128) given in Section 14.

PROOF. Because $u(x) = e^{\frac{N-2+2\beta}{2}t} v(\theta, t)$ with $\theta = \frac{x}{|x|}$ and $t = -\ln |x|$, we have for any $i = 1, \dots, N$:

$$\begin{aligned} \partial_{x_i} u(x) &= (\nabla_{\theta} v(\theta, t), D_i(\frac{x}{|x|})) e^{\frac{N-2+2\beta}{2}t} + \left\{ \partial_t v(\theta, t) + v(\theta, t) \frac{N-2+2\beta}{2} \right\} \frac{-x_i}{|x|^2} e^{\frac{N-2+2\beta}{2}t}, \\ &= (\nabla_{\theta} v(\theta, t), e_i - \theta_i \theta) e^t e^{\frac{N-2+2\beta}{2}t} + \left\{ \partial_t v(\theta, t) + v(\theta, t) \frac{N-2+2\beta}{2} \right\} (-\theta_i e^t) e^{\frac{N-2+2\beta}{2}t}, \\ &= (\nabla_{\theta} v(\theta, t), e_i) e^t e^{\frac{N-2+2\beta}{2}t} + \left\{ \partial_t v(\theta, t) + v(\theta, t) \frac{N-2+2\beta}{2} \right\} (-\theta_i e^t) e^{\frac{N-2+2\beta}{2}t}, \end{aligned}$$

where the last equality comes from $\nabla_{\theta} v(\theta, t) \cdot \theta = 0$, and so

$$\begin{aligned} |\nabla u(x)|^2 &= \left\{ |\nabla_{\theta} v(\theta, t)|^2 + \partial_t v(\theta, t)^2 + v^2(\theta, t) \left(\frac{N-2+2\beta}{2} \right)^2 + 2\partial_t v(\theta, t) v(\theta, t) \left(\frac{N-2+2\beta}{2} \right) \right\} \\ &\quad (\dots) \cdot e^{(N+2\beta)t} - 2 \left\{ \sum_{i=1}^N (\nabla_{\theta} v(\theta, t), \theta_i e_i) \left(\partial_t v(\theta, t) + v(\theta, t) \frac{N-2+2\beta}{2} \right) \right\} e^{(N+2\beta)t}, \end{aligned}$$

with the second term null again like $\nabla_{\theta} v(\theta, t) \cdot \theta = 0$. We can then conclude

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^{2\beta} |\nabla u(x)|^2 \, dx &= \int_{S^{N-1} \times \mathbb{R}} e^{-2\beta t} |(\nabla u)(\theta \cdot e^{-t})|^2 e^{-(N-1)t} e^{-t} \, d\theta \, dt, \\ &= \int_{S^{N-1} \times \mathbb{R}} e^{-(N+2\beta)t} |(\nabla u)(\theta \cdot e^{-t})|^2 \, d\theta \, dt, \\ &= \int_{S^{N-1} \times \mathbb{R}} \left\{ |\nabla_{\theta} v(\theta, t)|^2 + \partial_t v(\theta, t)^2 + v^2(\theta, t) \left(\frac{N-2+2\beta}{2} \right)^2 \right\} \, d\theta \, dt \\ &\quad + 2 \int_{S^{N-1} \times \mathbb{R}} \partial_t v(\theta, t) v(\theta, t) \left(\frac{N-2+2\beta}{2} \right) \, d\theta \, dt, \end{aligned}$$

whose the second term is null like

$$\int_{S^{N-1} \times \mathbb{R}} \partial_t v(\theta, t) v(\theta, t) \, d\theta \, dt = \int_{S^{N-1}} d\theta \int_{\mathbb{R}} \partial_t v(\theta, t) v(\theta, t) \, dt = \frac{1}{2} \int_{S^{N-1}} d\theta \int_{\mathbb{R}} \partial_t v^2(\theta, t) \, dt = 0.$$

Hence, we have the first part of the thesis when $u_1 = u_2$. The general case follows from the polarizations identities applied to $(\cdot|\cdot)_{\beta}$, $\|\cdot\|_{\beta}$ and $(\cdot|\cdot)_{\beta, \mathbb{C}}$, $\|\cdot\|_{\beta, \mathbb{C}}$ defined in Section 14 and 16.

The second part of the thesis follows from the following considerations:

$$\begin{aligned}
\int_{\mathbb{R}^N} |x|^{\alpha p} |u(x)|^p dx &= \int_{\mathcal{S}^{N-1} \times \mathbb{R}} e^{-\alpha p t} |u(\theta \cdot e^{-t})|^p e^{-(N-1)t} e^{-t} d\theta dt, \\
&= \int_{\mathcal{S}^{N-1} \times \mathbb{R}} e^{-(N+p\alpha)t} |u(\theta \cdot e^{-t})|^p d\theta dt, \\
&= \int_{\mathcal{S}^{N-1} \times \mathbb{R}} e^{-(N+p\alpha)t} \cdot e^{p \frac{N-2+2\beta}{2} t} |v(\theta, t)|^p d\theta dt, \\
&= \int_{\mathcal{S}^{N-1} \times \mathbb{R}} |v(\theta, t)|^p d\theta dt, \\
&= \int_{\mathcal{C}} |v|^p.
\end{aligned}$$

□

Numerical Computations

6 The MPA and MMPA

In Chapter 4 and 5, we mentioned the Mountain-Pass algorithm (MPA) and the modified Mountain-Pass algorithm (MMPA) to compute, respectively, one-signed and nodal solutions of problems having the Mountain-Pass geometry. All the numerical experiments using these methods, have been made by C. Grumiau. We refer the reader to [31].

The MPA was introduced in 1993 by Y. S. Choi and P. J. McKenna [22]. In 2001, J. Zhou and Y. Li [56, 57] proved the convergence of a variant of it. The MMPA is due to Costa, Ding and Neuberger, and was introduced in 2001 [24].

The algorithm relies at each step on the finite element method (see e.g. [28]). The domain Ω is triangulated with a Delaunay condition in such a manner that the distance between two nodes on the boundary of Ω is 0.05. C. Grumiau used the Easymesh software to do it.

The program stops when the gradient of the energy functional at the approximations has a norm strictly inferior than 1.0×10^{-2} (or after 20.000 steps). The Java language has been used to compute the MMPA and the Scilab software to graph numerical solutions.

6.1 The Problem (LE) The starting functions of the numerical experiments done in Chapter 4 for the study of the Lane–Emden problem (LE), are given in Table 1, and the values of minimum, maximum energy of the solutions in Table 2:

	domain	initial function
Rectangle	$(0, 2) \times (0, 1)$	$20 \cos(\pi x) \cos(\pi y) x(x-2)y(y-1)$
Disc	$B(0, 1)$	$-800 \cos\left(\frac{\pi}{2}(x^2 + y^2)^{\frac{1}{2}}\right)$
Square	$(-1, 1)^2$	$20 \sin(\pi x) \sin(\pi y)(x+y-0.4)(x+y-1.6)$ $(x-y-0.6)(x-y+0.6))$

TABLE 1. Equation $-\Delta u = u^3$

	$\min u$	$\max u$	$J(u^+)$	$J(u^-)$
Rectangle	-6.6	6.6	37.2	37.2
Disc	-5.85	5.85	29.7	29.7
Square	-5.2	5.2	23.9	23.9

TABLE 2. Characteristics of the approximate solutions

6.2 Generalization of the Problem (LE) The starting functions and the values of minimum, maximum energy of the solutions of the numerical experiments done in Chapter 5 for the generalization of the Lane–Emden problem (LE) study, are given in Table 3, 4, 5 for respectively, the following nonlinearities:

$$(a) \lambda t |t|^{\theta-2} + (\theta-2)t |t|^{q-2}, \quad (b) \lambda t (e^{t^2} - 1)^{\theta-2}, \quad (c) \lambda t \left(\sum_{i=1}^k \alpha_i |t|^{\beta_i(\theta)} \right). \quad (176)$$

Let us notice that all these experiments are done on the square $(-1, 1)^2$ in $\mathbb{R}^2$, with λ the first or the second eigenvalue of the operator $-\Delta + \mu \text{id}$, and with μ equal to 0 or 1 for Dirichlet (D) or Neumann (N) boundary conditions.

	Initial function	$\min u$	$\max u$	$J(u^+)$	$J(u^-)$
$\lambda_1(\text{D})$	$\sin(\frac{\pi(x+1)}{2}) \sin(\frac{\pi(y+1)}{2})$	0.0	1.29	1.47	0.0
$\lambda_2(\text{D})$	$\sin(\pi(x+1)) \sin(2\pi(y+1))$	-1.4	1.4	1.7	1.7
$\lambda_2(\text{N})$	$\cos(\pi(y+1))$	-1.31	1.33	0.78	0.77

TABLE 3. $-\Delta u + \mu u = \lambda u^3 + 2u^5$

	Initial function	$\min u$	$\max u$	$J(u^+)$	$J(u^-)$
$\lambda_1(\text{D})$	$\sin(\frac{\pi(x+1)}{2}) \sin(\frac{\pi(y+1)}{2})$	0.0	1.11	3.05	0.0
$\lambda_2(\text{D})$	$\sin(\pi(x+1)) \sin(2\pi(y+1))$	-1.2	1.2	2.79	2.79
$\lambda_2(\text{N})$	$\cos(\pi(y+1))$	-1.25	1.26	0.9	0.91

TABLE 4. $-\Delta u + \mu u = \lambda u(e^{u^2} - 1)^{0.5}$

	Initial Function	$\min u$	$\max u$	$J(u^+)$	$J(u^-)$
$\lambda_1(\text{D})$	$20 \cos(\frac{\pi}{2}(x^2 + y^2)^{0.5})$	0.0	1.39	4.79	0.0
$\lambda_2(\text{D})$	$-800 \cos(\frac{\pi}{2}(x^2 + y^2)^{0.5}) \cos(\pi(x^2 + y^2)^{0.5})$	-1.46	1.46	4.65	4.65
$\lambda_2(\text{N})$	$\cos(\pi y)$	-1.55	1.55	0.4	0.5

TABLE 5. $-\Delta u + \mu u = \lambda u(\frac{1}{4}|u|^{0.25} + \frac{1}{4}|u|^{0.5} + \frac{1}{2}|u|)$

7 The Entropy Functions

In Chapter 4 and 5, we computed the reduced functional

$$J_* : E(\lambda_2) \rightarrow \mathbb{R} : v \mapsto - \int_{\Omega} F_*(v)$$

on the reduced Nehari manifold $\mathcal{N}_*$, where $F_*(t) = \int_0^t f_*(s) ds$ and f_* are given by (176), and where $\Omega = (-1, 1)^2$. In order to do that, we considered a L^2 -orthonormal basis v_1, v_2 of the second eigenspace $E(\lambda_2)$, projected the functions

$$v_{\theta} := \cos(\theta)v_1 + \sin(\theta)v_2$$

on the manifold $\mathcal{N}_*$ by the operation $t_\theta v_\theta$ with $t_\theta > 0$ a root of

$$\frac{1}{t} \int_{\Omega} f_*(tu_\theta)u_\theta = 0, \quad (177)$$

For the nonlinearity (176)(a), the equation (177) and the functional J_* at the considered point, are respectively given by

$$\lambda_2 \log t + t^{q-2} \int_{\Omega} |u_\theta|^q + \frac{\lambda_2}{2} \int_{\Omega} u_\theta^2 \log(u_\theta^2) = 0, \quad (178)$$

$$\begin{aligned} J_*(t_\theta u_\theta) &= \frac{\lambda_2}{4} t_\theta^2 \left(1 - \int_{\Omega} u_\theta^2 \log(u_\theta^2) \right) - \frac{\lambda_2}{4} t_\theta^2 \log(t_\theta^2) - \frac{1}{q} t_\theta^q \int_{\Omega} |u_\theta|^q, \\ &= \frac{\lambda_2}{4} t_\theta^2 + \left(\frac{1}{2} - \frac{1}{q} \right) t_\theta^q \int_{\Omega} |u_\theta|^q, \end{aligned} \quad (179)$$

where the second line of (179) is obtained by embedding Equation (178) in its first line. For the nonlinearity (176)(b), the equation (177) is given by

$$\lambda_2 \int_{\Omega} u_\theta^2 \log(e^{t^2 u_\theta^2} - 1) = 0. \quad (180)$$

For the nonlinearity (176)(a) and the Lane–Emden problem, the equation (177) and the functional J_* at the considered point, are respectively given by (178) and (179) without the term $\int_{\Omega} |u_\theta|^q$. In this case, solving (177) was very easy, but in the other ones, we used the Secant method.

To approximate the different previous integrals, we applied the Simpson and the middle point method with the software Matlab and a step of $1/1024$. Let us estimate for example the error in the Lane–Emden case. On the one hand, we have

$$|\nabla(\cos(\theta)v_1 + \sin(\theta)v_2)| \leq \pi\sqrt{5}$$

and the function takes its values in $[-2, 2]$, whereas, on the other hand, it is straightforward that $4(1 + \log 4)$ is a Lipschitz constant for the function $t^2 \log t^2$ on $[-2, 2]$. We therefore infer that the approximation error is bounded by

$$\frac{4\pi\sqrt{5}(1 + \log 4)}{2048} = 0.0327$$

while the roundoff error is negligible (of the order 10^{-4}) in front of this. Let us still note that, in the exponential case where the roundoff error is more dangerous because of the argument of the function $\log$, we used the function `expm1`.

Special Functions

The following definitions are valid because of the integrability of the functions mentioned in their right hand sides:

$$B(a, b) := \int_0^1 x^{a-1} (1-x)^{b-1} dx, \quad \Gamma(c) := \int_0^\infty x^{c-1} e^{-x} dx,$$

where $a, b, c > 0$. These integrals are respectively called *Euler integral of the first kind* and *Euler integral of the second kind*, and these functions, *Euler's beta function* and *Euler's gamma function*. These functions verify, for any $n, m > 0$ integers, the interpolations formulas

$$B(n, m) = \frac{(n-1)!(m-1)!}{(n+m-1)!} = \frac{n+m}{nm} \cdot \binom{n+m}{n}^{-1}, \quad \Gamma(n) = (n-1)!,$$

easily deduced from the fact that $\Gamma(1) = 1$ and

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \Gamma(c+1) = c\Gamma(c), \quad (181)$$

for any $a, b, c > 0$. Two simple changes of variables and the previous link between B and Γ lead to the following formulas used in Section 13:

$$\int_0^\infty \frac{x^\gamma}{(1+x^\alpha)^\beta} dx = \frac{B(\frac{\gamma+1}{\alpha}, \beta - \frac{\gamma+1}{\alpha})}{\alpha} = \frac{\Gamma(\frac{\gamma+1}{\alpha})\Gamma(\beta - \frac{\gamma+1}{\alpha})}{\alpha\Gamma(\beta)}. \quad (182)$$

Let us still notice that these functions can be extended on $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. For further details, the reader can consult [1, 41].

The following definition is valid because of the convergence of the series mentioned in its right hand side:

$$J_\nu(z) := \left(\frac{z}{2}\right)^\nu \cdot \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} \Gamma(k+1) \Gamma(\nu+k+1)},$$

where $\nu, z \in \mathbb{C}$ and $\Re(z) > 0$ if $\Im(z) = 0$. Let us note that Γ is supposed infinite when its arguments has its real part negative. These series are called *Bessel function of order ν* . These functions verifies the $\mathbb{C}$ -differential equation

$$z^2 J_\nu''(z) + z J_\nu'(z) + (z^2 - \nu^2) J_\nu(z) = 0,$$

for any $\nu, z \in \mathbb{C}$ like above. Figure E.1 gives the graphs of these functions when ν goes from 0 to 3 by step of $1/2$, and illustrates the interlacing

$$j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < \dots$$

where $j_{\nu,q}$ is the q^{th} zero of J_ν when $\nu \geq 0$.

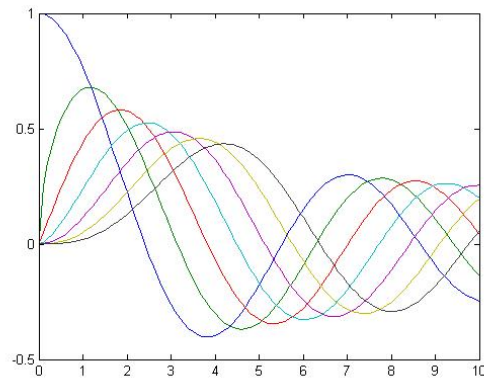


Figure E.1

Let us still notice that these functions are in fact called *Bessel functions of the first kind*, and that there exist *Bessel functions of the second kind* denoted by Y_ν . When $\Re(\nu) \geq 0$, the functions $J_\nu(z)$ are locally bounded around $z = 0$, what is not the case for the functions Y_ν . For further details, the reader can consult [1, 41].

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