Tensors via commutators

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joint work-in-progress with Manfred Hartl

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New trends in Hopf algebras and tensor categories
Brussels, 3rd of June 2015

Problem: the Ganea term in the exact homology sequence

Theorem [T Ganea, 1968]

Any central extension of groups

$$0 \longrightarrow K \rightarrowtail B \xrightarrow{f} A \longrightarrow 0 \qquad\qquad [K, B] = 0$$

induces a six-term exact sequence

$$K \otimes_{\mathbb{Z}} \frac{B}{[B,B]} \longrightarrow \mathrm{H}_2 B \longrightarrow \mathrm{H}_2 A \longrightarrow K \longrightarrow \mathrm{H}_1 B \longrightarrow \mathrm{H}_1 A \longrightarrow 0.$$

- Versions of this are known in several other categories, but… until recently, we had no categorical-algebraic interpretation
- cf. long exact homology sequence [T Everaert, 2010] in *semi-abelian categories* [G Janelidze, L Márki & W Tholen, 2002]

How to understand the tensor product on the left categorically?

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Semi-abelian categories

[G Janelidze, L Márki & W Tholen, 2002]

A category is called **semi-abelian** when it is

- finitely (co)complete
- pointed (initial = terminal $\rightsquigarrow$ notion of kernel and cokernel)
- Barr exact (image factorisations are pullback-stable, equivalence relations are effective)
- Bourn protomodular (the Split Short Five Lemma holds).

In a semi-abelian category, the basic lemmas of homological algebra such as the 3×3 Lemma, the Snake Lemma, the (Short) Five Lemma are valid.

$\rightsquigarrow$ General results in (co)homology, commutator theory, etc.

Some examples:

- all abelian categories;
- groups; (non-unitary) rings; loops; associative algebras; Lie, Leibniz, Poisson, Jacobi algebras, (pre)crossed modules;
- all *varieties of Ω-groups* in the sense of [Higgins, 1956].

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Co-smash products in semi-abelian categories

[A Carboni & G Janelidze, 2003]

Any two objects X and Y in a semi-abelian category
induce a short exact sequence

$$X + Y \xrightarrow{\begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix}} X \times Y$$

The kernel $X \diamond Y$ is called the **co-smash product** of X and Y.

Co-smash products may serve as (amongst other things)

1. formal (Higgins) commutators
   [S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013];
2. (intrinsic) tensor products
   (as shown for commutative rings by Carboni and Janelidze).

Our aim today is to explore 2.

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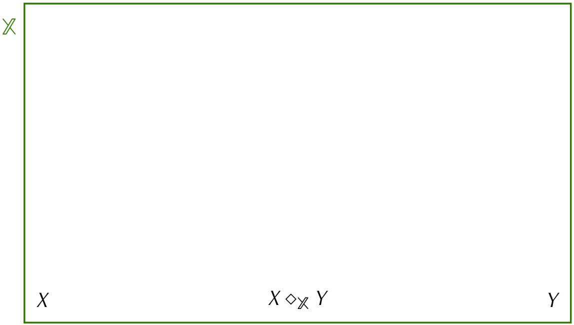
Where to take the co-smash product?



In $\mathbb{X}$ itself, the co-smash product $X \diamond_{\mathbb{X}} Y$ is a formal commutator.

Definition

Given objects X and Y in a semi-abelian category $\mathbb{X}$, their **bilinear product** $X \otimes_{\mathbb{X}} Y \in |\mathbf{Ab}(\mathbb{X})|$ is the co-smash product $\mathrm{nil}_2(X) \diamond_{\mathbf{Nil}_2(\mathbb{X})} \mathrm{nil}_2(Y)$ in the two-nilpotent core $\mathbf{Nil}_2(\mathbb{X})$ of $\mathbb{X}$.

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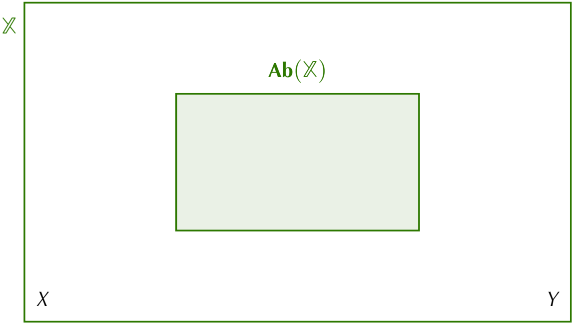

In $\mathbf{Ab}(\mathbb{X})$, the co-smash product $\mathrm{ab}(X) \diamond_{\mathbf{Ab}(\mathbb{X})} \mathrm{ab}(Y)$ is zero.

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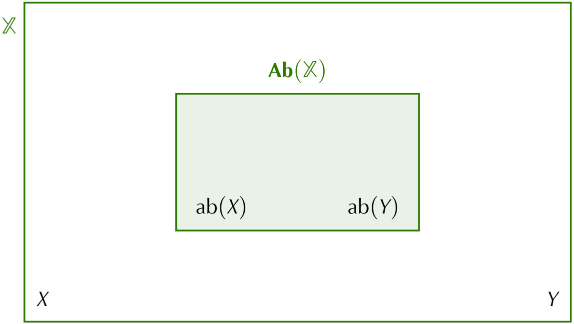

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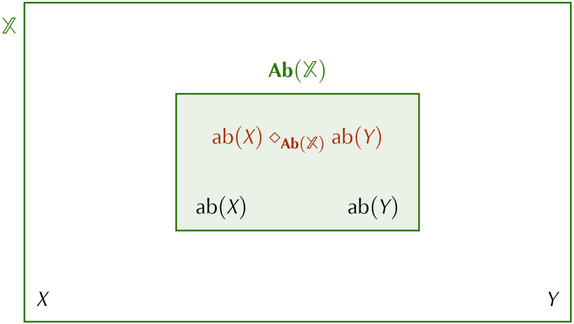

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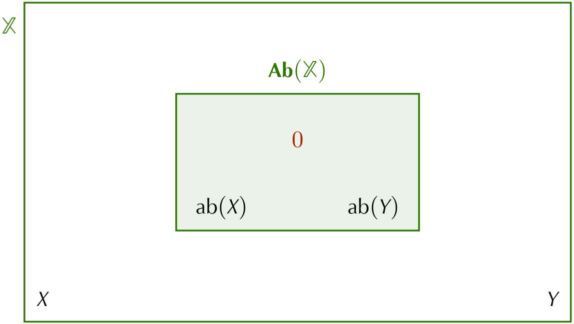

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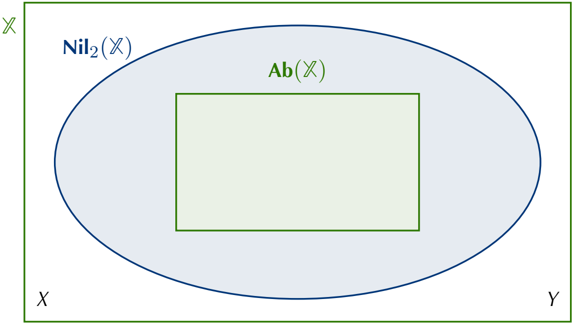

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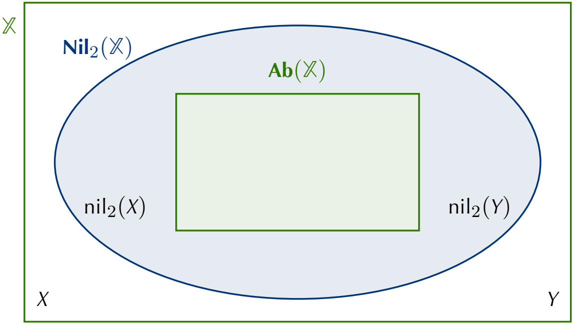

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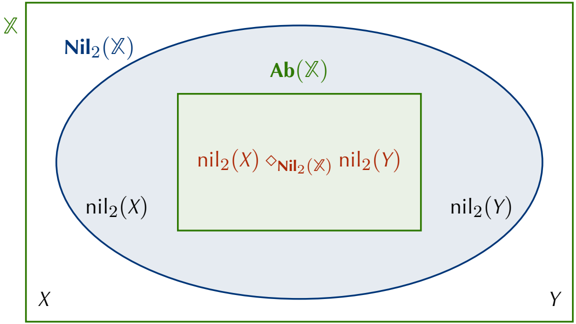

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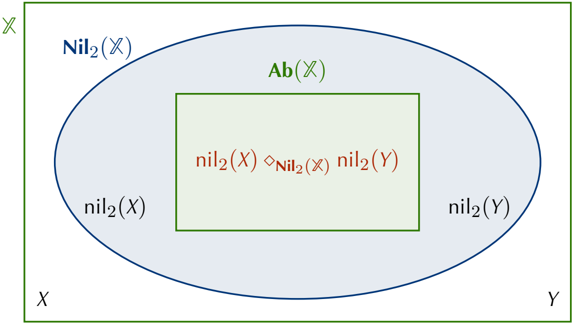

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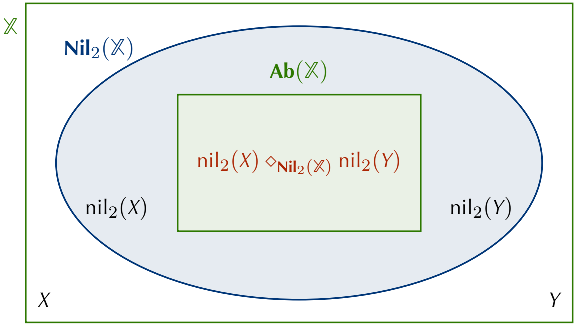

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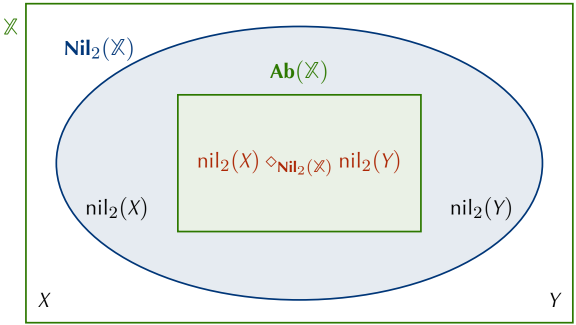

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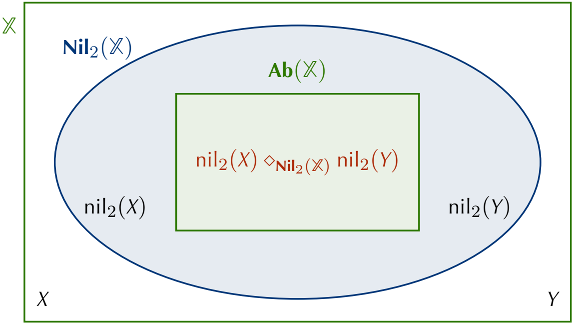

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Binary co-smash products and Higgins commutators

[P J Higgins, 1956] [S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

Consider $K, L \leqslant X$ in a semi-abelian category $\mathbb{X}$.

$$\begin{tikzcd} 0 \arrow[r] & K \diamond L \arrow[r, tail] & K + L \arrow[r, "\begin{pmatrix} 1_K & 0 \\ 0 & 1_L \end{pmatrix}"] \arrow[d, "\begin{pmatrix} k & l \end{pmatrix}"] & K \times L \arrow[r] & 0 \\ & & X & & \end{tikzcd}$$

- $[K, L]$ is called the **Higgins commutator** of K and L.
- $[K, L] = 0$ if and only if $\begin{pmatrix} k & l \end{pmatrix}$ lifts over $K \times L$, so that k and l *commute* [F Borceux & D Bourn, 2004].
- X in $\mathbb{X}$ is **abelian** if and only if $[X, X] = 0$. This determines the **abelian core** $\mathbf{Ab}(\mathbb{X})$ of $\mathbb{X}$ and the **abelianisation functor** $\mathrm{ab}\colon \mathbb{X} \to \mathbf{Ab}(\mathbb{X})$.

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Binary co-smash products and Higgins commutators

[P J Higgins, 1956] [S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

Consider $K, L \leqslant X$ in a semi-abelian category $\mathbb{X}$.

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Higher-order co-smash products and Higgins commutators

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The **higher-order co-smash product** $X_1 \diamond \cdots \diamond X_n$ defined as

$$\mathop{\Large\diamond}_{i=1}^{n} X_i \rightarrowtriangle \coprod_{i=1}^{n} X_i \xrightarrow{r} \prod_{i=1}^{n} \coprod_{j \neq i} X_j$$

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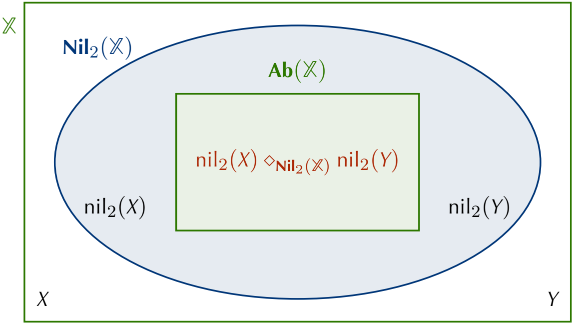
Where to take the co-smash product?



In $\mathbf{Nil}_2(\mathbb{X})$, the co-smash product $\mathrm{nil}_2(X) \diamond_{\mathbf{Nil}_2(\mathbb{X})} \mathrm{nil}_2(Y)$ is abelian.

Definition

Given objects X and Y in a semi-abelian category $\mathbb{X}$,
their **bilinear product** $X \otimes_{\mathbb{X}} Y \in |\mathbf{Ab}(\mathbb{X})|$ is the co-smash product $\mathrm{nil}_2(X) \diamond_{\mathbf{Nil}_2(\mathbb{X})} \mathrm{nil}_2(Y)$ in the two-nilpotent core $\mathbf{Nil}_2(\mathbb{X})$ of $\mathbb{X}$.

Examples

For groups, we get $X \otimes_{\mathbf{Gp}} Y \cong \mathrm{ab}(X) \otimes_{\mathbb{Z}} \mathrm{ab}(Y)$.
As far as I know, this phenomenon was first described in
[T MacHenry, 1960], then rediscovered in [M Hartl & C Vespa, 2011].

Considered as an operation on an abelian category $\mathbb{A} = \mathbf{Ab}(\mathbb{X})$, the bilinear product depends on $\mathbb{X}$. We write $M \otimes_{\mathbb{X}} N$ for M, N abelian.

category $\mathbb{X}$	objects	abelian core $\mathbf{Ab}(\mathbb{X})$	$M \otimes_{\mathbb{X}} N$
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$\mathbf{Loop}$	loops	$\mathbf{Ab}$	$M \otimes_{\mathbb{Z}} N$
$\mathbf{Nil}_2(\mathbf{Gp})$	two-nilpotent groups	$\mathbf{Ab}$	$M \otimes_{\mathbb{Z}} N$
$\mathbf{CRng}$	commutative rings	$\mathbf{Ab}$	$M \otimes_{\mathbb{Z}} N$
$\mathbf{Mod}_R$	R-modules	$\mathbf{Mod}_R$	0
$\mathbf{CAlg}_R$	commutative R-algebras	$\mathbf{Mod}_R$	$M \otimes_R N$
$\mathbf{Alg}_R$	R-algebras	$\mathbf{Mod}_R$	$(M \otimes_R N) \oplus (N \otimes_R M)$
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- Internal (pre)crossed modules (gives known concepts);
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R commutative ring with unit

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Basic properties of bilinear products

- bilinear products are symmetric: $X \otimes_{\mathbb{X}} Y \cong Y \otimes_{\mathbb{X}} X$
- bilinear products are **bilinear**: $X \otimes_{\mathbb{X}} 0 = 0$
  and $X \otimes_{\mathbb{X}} (Y + Z) \cong (X \otimes_{\mathbb{X}} Y) + (X \otimes_{\mathbb{X}} Z)$
- $X \otimes_{\mathbb{X}} (-)$ is sequentially right exact;
  furthermore, it preserves *all* colimits when $\mathbb{X}$ is a variety
- not monoidal in general:
  bilinear products may be non-associative and non-unitary

Suppose X is a unit for $\otimes_{\mathbb{X}}$ when $\mathbb{X} = \mathbf{Alg}_{\mathbb{R}}$; then in $\mathbf{Vect}_{\mathbb{R}}$ we have

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$$\mathbb{R} \cong \mathbb{R} \otimes_{\mathbb{X}} X = (\mathbb{R} \otimes_{\mathbb{R}} X) \oplus (X \otimes_{\mathbb{R}} \mathbb{R}) \cong X \oplus X.$$

Hence units cannot exist in general.

Basic properties of bilinear products

- bilinear products are symmetric: $X \otimes_{\mathbb{X}} Y \cong Y \otimes_{\mathbb{X}} X$
- bilinear products are **bilinear**: $X \otimes_{\mathbb{X}} 0 = 0$
  and $X \otimes_{\mathbb{X}} (Y + Z) \cong (X \otimes_{\mathbb{X}} Y) + (X \otimes_{\mathbb{X}} Z)$
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Ganea's Theorem

Theorem

Any central extension

$$0 \longrightarrow K \rightarrowtriangle^{k} B \xrightarrow{f} A \longrightarrow 0 \qquad [K, B] = 0$$

in a semi-abelian algebraically coherent category with enough projectives $\mathbb{X}$ induces a six-term exact sequence

$$K \otimes_{\mathbb{X}} B \longrightarrow \mathrm{H}_2 B \longrightarrow \mathrm{H}_2 A \longrightarrow K \longrightarrow \mathrm{H}_1 B \longrightarrow \mathrm{H}_1 A \longrightarrow 0.$$

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Idea of proof

$$K \otimes_{\mathbb{X}} B \overset{?}{\dashrightarrow} \mathrm{H}_2 B \xrightarrow{\mathrm{H}_2 f} \mathrm{H}_2 A$$

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Idea of proof

$$\begin{tikzcd} & & P \arrow[d, "p"', -open triangle] & & \\ 0 \arrow[r] & K \arrow[r, open triangle-] & B \arrow[r, "f"', -open triangle] & A \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} K \otimes_{\mathbb{X}} B \arrow[r, dotted, "?"'] & \mathrm{H}_2 B \arrow[r, "\mathrm{H}_2 f"'] & \mathrm{H}_2 A \end{tikzcd}$$

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 & & R \arrow[r, tail] \arrow[d, tail] & L \arrow[d, tail] & \\
 & & P \arrow[r, equal] \arrow[d, two heads, "p"'] & P \arrow[d, two heads, "f\circ p"] & \\
0 \arrow[r] & K \arrow[r, tail] & B \arrow[r, two heads, "f"'] & A \arrow[r] & 0
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\begin{tikzcd}
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 & 0 \arrow[r] \arrow[d] & P \arrow[r, equal] \arrow[d, "p", two heads] & P \arrow[r] \arrow[d, "f\circ p", two heads] & 0 \\
0 \arrow[r] & K \arrow[r, tail] \arrow[d, equal] & B \arrow[r, "f"', two heads] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
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\begin{tikzcd}
\frac{L}{R} \otimes_{\mathbb{X}} \frac{P}{R} \arrow[d, equal] & \frac{[P,P]\cap R}{[R,P]} \arrow[r] \arrow[d, equal] & \frac{[P,P]\cap L}{[L,P]} \arrow[d, equal] \\
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L \otimes_{\mathbb{X}} P \arrow[d, two heads] & \frac{[L,P]}{[R,P]} \arrow[d, equal] & & {[P,P] \cap R} \arrow[r, tail] \arrow[dl, two heads] \arrow[d, two heads] & {[P,P] \cap L} \arrow[d, two heads] \\
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\begin{tikzcd}
L \diamond P \arrow[r, two heads] \arrow[d, two heads] & {[L,P]} \arrow[d, two heads] & & {[L,P]} \arrow[r, equal] \arrow[d, tail] \arrow[dd, phantom] & {[L,P]} \arrow[d, tail] \\
L \otimes_{\mathbb{X}} P \arrow[d, two heads] & \frac{[L,P]}{[R,P]} \arrow[d, equal] & & {[P,P] \cap R} \arrow[r, tail] \arrow[dl, two heads] \arrow[d, two heads] & {[P,P] \cap L} \arrow[d, two heads] \\
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L \otimes_{\mathbb{X}} P \arrow[r, dotted, two heads, "\text{under (AC)}"'] \arrow[d, two heads] & \frac{[L,P]}{[R,P]} \arrow[d, equal] & & {[P,P] \cap R} \arrow[r, tail] \arrow[dl, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & {[P,P] \cap L} \arrow[d, two heads] \\
\frac{L}{R} \otimes_{\mathbb{X}} \frac{P}{R} \arrow[d, equal] & \frac{[L,P]}{[R,P]} \arrow[r, tail] & \frac{[P,P]\cap R}{[R,P]} \arrow[r, two heads] \arrow[d, equal] & \frac{[P,P]\cap R}{[L,P]} \arrow[r, tail] & \frac{[P,P]\cap L}{[L,P]} \arrow[d, equal] \\
K \otimes_{\mathbb{X}} B \arrow[rr, dotted, "?"'] & & \mathrm{H}_2 B \arrow[rr, "\mathrm{H}_2 f"'] & & \mathrm{H}_2 A
\end{tikzcd}
$$

- P projective, $B \cong P/R$, $A \cong P/L$, and $K \cong L/R$
- exact sequence and Hopf formula from [T Everaert & TVdL, 2004]
- $[L, P] \leqslant R$ because $p[L, P] = [pL, pP] \leqslant [K, B] = 0$
- under (AC), we have $[L, P, P] \leqslant [[L, P], P] \leqslant [R, P]$

Idea of proof

$$
\begin{tikzcd}
L \diamond P \arrow[r, two heads] \arrow[d, two heads] & {[L,P]} \arrow[d, two heads] & & {[L,P]} \arrow[r, equal] \arrow[d, tail] & {[L,P]} \arrow[d, tail] \\
L \otimes_{\mathbb{X}} P \arrow[r, dotted, two heads, "\text{under (AC)}"'] \arrow[d, two heads] & \frac{[L,P]}{[R,P]} \arrow[d, equal] & & {[P,P] \cap R} \arrow[r, tail] \arrow[dl, two heads] \arrow[d, two heads] & {[P,P] \cap L} \arrow[d, two heads] \\
\frac{L}{R} \otimes_{\mathbb{X}} \frac{P}{R} \arrow[r, dotted, two heads] \arrow[d, equal] & \frac{[L,P]}{[R,P]} \arrow[r, tail] & \frac{[P,P]\cap R}{[R,P]} \arrow[r, two heads] \arrow[d, equal] & \frac{[P,P]\cap R}{[L,P]} \arrow[r, tail] & \frac{[P,P]\cap L}{[L,P]} \arrow[d, equal] \\
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Ganea's Theorem

Theorem

Any central extension

$$0 \longrightarrow K \rightarrowtriangle^{k} B \xrightarrow{f} A \longrightarrow 0 \qquad\qquad [K, B] = 0$$

in a semi-abelian algebraically coherent category with enough projectives $\mathbb{X}$ *induces a six-term exact sequence*

$$K \otimes_{\mathbb{X}} B \longrightarrow \mathrm{H}_2 B \longrightarrow \mathrm{H}_2 A \longrightarrow K \longrightarrow \mathrm{H}_1 B \longrightarrow \mathrm{H}_1 A \longrightarrow 0.$$

- Algebraically coherent cats: [J R A Gray, A S Cigoli & TVdL, 2015]
- This captures the known results for groups, (pre)crossed modules, associative algebras, Lie and Leibniz algebras.
  [T Pirashvili, 2000] [J-M Casas & T Pirashvili, 2000]
  [D Arias & M Ladra, 2006]

Conclusion

Computed in the two-nilpotent core $\mathbf{Nil}_2(\mathbb{X})$ of a semi-abelian category $\mathbb{X}$, the co-smash product may play the role of intrinsic tensor product.

The Ganea term in the homology sequence of a central extension is such an intrinsic tensor product.

To do:

- Other applications of bilinear products?
- How does a bilinear product $\otimes$ on an abelian category $\mathbb{A}$ induce a semi-abelian category $\mathbb{X}$ such that $\mathbb{A} = \mathbf{Ab}(\mathbb{X})$ and $\otimes = \otimes_{\mathbb{X}}$?
  $\rightsquigarrow$ in the varietal case: algebras over an operad
- When is $\otimes_{\mathbb{X}}$ monoidal on $\mathbf{Ab}(\mathbb{X})$?
- How to capture the tensor product of associative algebras?
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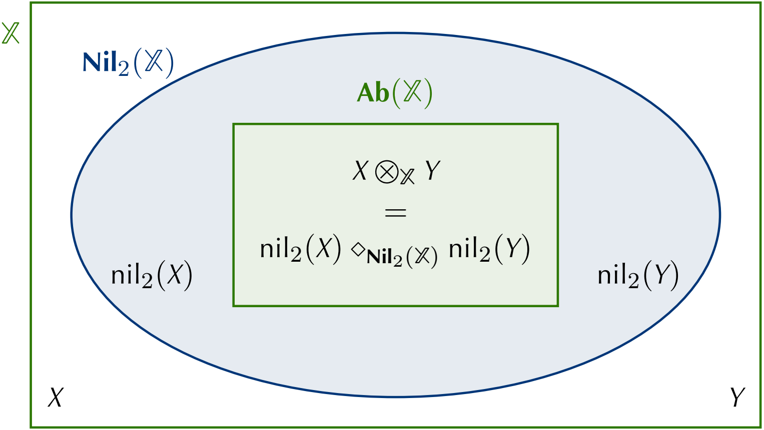

$\mathbb{X}$
$\mathbf{Nil}_2(\mathbb{X})$
$\mathbf{Ab}(\mathbb{X})$
$X \otimes_{\mathbb{X}} Y$
$=$
$\mathsf{nil}_2(X) \diamond_{\mathbf{Nil}_2(\mathbb{X})} \mathsf{nil}_2(Y)$
$\mathsf{nil}_2(X)$
$\mathsf{nil}_2(Y)$
X
Y