

Analysis and data-driven low-dimensional modelling of choking mechanisms in supersonic ejectors

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Freedom. This is the word that comes closest to condensing the past five years. During the PhD, I experienced the freedom to explore and to dig deep, to learn and to teach, to fail and to prevail. Now, standing at this turning point, about to finish, a whole new sort of freedom appears: one where different career paths lie ahead and where “the real world” awaits. Indeed, my work has been quite fundamental — a luxury we could afford thanks to the supporting grants of F.R.S.-FNRS FRiA, UCLouvain and VKI, for which I am deeply grateful.

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Thank you all for making these past years an unforgettable experience.

To Saskia and Louise

List of symbols

Latin Letters

A	Cross-section	$[\text{m}^2]$
a	Speed of sound or advection speed (only section 3.6)	$[\text{m}]$
c	Generic coefficient	
c_p	Specific heat at constant pressure	$[\text{J}/\text{kg}/\text{K}]$
c_v	Specific heat at constant volume	$[\text{J}/\text{kg}/\text{K}]$
D	Diameter	$[\text{m}]$
d	Euclidean distance	$[\text{m}]$
e	Specific internal energy or error function (chapter 3)	$[\text{J}/\text{kg}]$
F	Force per unit length	$[\text{N}/\text{m}]$
f	Friction coefficient or function (dynamical system)	$[-]$
$\mathbf{F}$	Flux vector	
g	Closure function	
h	Specific enthalpy or observation operator	$[\text{J}/\text{kg}]$
J	Cost function	
k	Turbulent kinetic energy	$[\text{m}^2/\text{s}^2]$
$\mathcal{L}$	Lagrangian cost function	
L	Length	$[\text{m}]$
l	Perimeter	$[\text{m}]$
l_{mix}	Turbulent length scale	$[\text{m}]$
$\dot{m}$	Mass flow rate	$[\text{kg}/\text{s}]$
Ma	Mach number	$[-]$
$\mathcal{M}$	Momentum flux	$[\text{N}]$
N	Numerator	
n	Normal coordinate or number of elements	$[\text{m}]$
p	Pressure	$[\text{Pa}]$

Pr	Prandle number	$[-]$
$\mathbf{p}$	Parameter vector	
Q	Energy exchange per unit length	$[W/m]$
q	Generic quantity	
$\mathbf{Q}$	Discretely orthogonal Chebyshev basis	
R	Radius	$[m]$
r	Radial coordinate	$[m]$
Re	Reynolds number	$[-]$
R_{gas}	Gas constant	$[m]$
$\mathbf{R}$	Orthogonalisation matrix for $\mathbf{Q}$	
$\mathbb{R}$	Real numbers	
$\mathcal{R}$	Residual	
S	Reference temperature in Sutherland's law	$[K]$
s	Specific entropy	$[J/kg/K]$
$\mathbf{S}$	Source vector	
T	Temperature	$[K]$
t	time	$[s]$
TI	Turbulence intensity	$[-]$
u	Axial velocity	$[m/s]$
$\mathbf{U}$	State vector $[\rho, \rho u, \rho e_t]$	
$\mathbf{u}$	Generic state vector	
v	Radial velocity	$[m/s]$
$\mathbf{v}$	Velocity vector	$[m/s]$
$\mathbf{w}$	Weight vector	
x	Axial coordinate	$[m]$
y^+	Dimensionless wall distance	$[-]$
$\mathbf{z}$	Vector of observations	
Greek Letters		
α	Angle	$[rad]$
β	Compound choking indicator	$[m^2]$
Δ	Difference	
δ	Uncertainty	
ε	Aspect ratio R/L	$[-]$
η	Isentropic or mixing efficiency	$[-]$
γ	Ratio of specific heats	$[-]$
λ	Adjoint state vector	
μ	Dynamic viscosity	$[Pa\ s]$
ν	Prandtl-Meyer angle	$[rad]$

ϕ	Entrainment ratio or basis function	$\begin{bmatrix} - \end{bmatrix}$
Π_{bp}	Ratio of back pressure to primary pressure	$\begin{bmatrix} - \end{bmatrix}$
Π_{sp}	Ratio of inlet pressures	$\begin{bmatrix} - \end{bmatrix}$
Π	Euler number	$\begin{bmatrix} - \end{bmatrix}$
Φ	Feature matrix	
ρ	Density	$[\text{kg}/\text{m}^3]$
σ	Shape factor (RBF) or activation function (ANN)	
τ	Reynolds stress	$[\text{Pa}]$
θ	Angle of the dividing streamline	$[\text{rad}]$
ω	Specific turbulence dissipation rate	$[1/\text{s}]$

Superscripts

*	Critical point, sonic point or star state (chapter 5)
0	Breakdown point
~	Reference data or Favre-average
-	Initial state or Reynolds-average
^	Scaled quantity
(<i>n</i>)	Discrete index for time or space

Subscripts

0	Initial guess (<i>w</i>) or inlet coordinate (<i>x</i>)
<i>b</i>	Back
<i>c</i>	Collocation point or convective (Mach)
<i>d</i>	Dividing (streamline)
<i>eff</i>	Effective or Eddy viscosity
<i>eq</i>	Equivalent
<i>i</i>	Inlet or placeholder for <i>p</i> or <i>s</i>
<i>L</i>	Left
<i>R</i>	Right
<i>m</i>	Mixing pipe
<i>o</i>	Outlet
<i>p</i>	Primary
<i>ps</i>	Interface primary-secondary stream (shear)
<i>ref</i>	Reference
<i>s</i>	Secondary
<i>sh</i>	Shock
<i>t</i>	Total / stagnation
τ	Frictional
<i>th</i>	Throat
θ	Tangential direction
<i>w</i>	Wall

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1

Introduction

1.1 Context

1.1.1 Ejector operation

Ejectors are passive devices for mixing, compression or vacuum creation. Their operating principle is based on expanding a high-pressure flow into a mixing chamber such that shear and pressure effects entrain a secondary flow as shown schematically in figure 1.1. Having no moving parts, ejectors need little maintenance and have low operational costs, no restriction on the working fluids, and a long service life [1]. They can be used to decrease the electricity-driven compression work and reduce throttling losses in refrigeration systems [2], recover low-grade waste heat [3], minimize jet noise and increase thrust in an aircraft [4]. The reader is referred to [5] for a comprehensive overview. Their performance is typically quantified in terms of the entrainment ratio $\phi = \dot{m}_s / \dot{m}_p$, defined as the ratio of secondary to primary mass flow rate, in terms of the compression ratio between the static back pressure p_b and the total pressure $p_{t,s}$ at the secondary inlet (figure 1.2). Their operational conditions further depend on the total pressure at the primary inlet, and the total temperatures $T_{t,i}$ at both inlets. The subscripts $i \in [p, s]$ are used throughout this document to indicate the primary and secondary streams respectively.

The evolution of the entrainment ratio in function of the back pressure

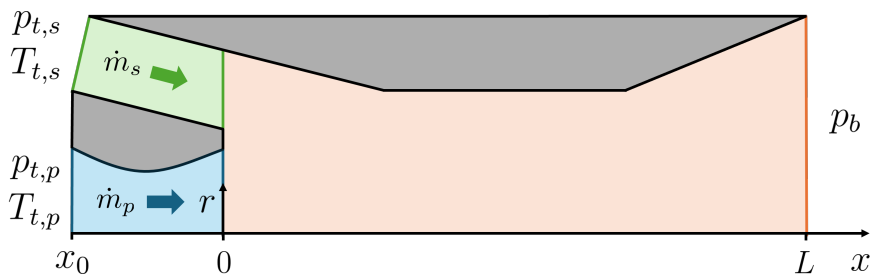


Fig. 1.1 Schematic representation of an ejector. A primary flow with a high stagnation pressure $p_{t,p}$ expands through a nozzle, creating a low pressure region at the inlet of the mixing pipe and consequently entraining a secondary flow with a lower stagnation pressure $p_{t,s}$. The flows mix and reach an intermediate static pressure p_b at the outlet.

is shown schematically in figure 1.2. The maximal flow rate is achieved in the choked on-design regime at sufficiently low back pressure. The entrainment decreases with the back pressure in the off-design regime, eventually reaching flow reversal beyond the so-called break-down point. The latter regime is normally considered as a malfunction mode. The choked regime is usually the most relevant because it determines the maximum achievable entrainment of the secondary flow for a given set of inlet conditions. Ejectors are usually designed to operate under choked conditions to maximise entrainment efficiency, prevent flow reversal, and ensure stable and robust performance against fluctuations in back pressure.

The seemingly simple operating principle at the global scale hides a complex interplay of fluid dynamical phenomena at the local scale [3], including the choking of parallel streams, a spatially developing turbulent shear layer and the corresponding mixing, oblique shock trains due to over- or under-expansion of the primary jet and boundary layer separation in the diffuser. Their interaction makes it difficult to discern their individual impact on the local flow field and the resulting global performance of the ejector.

1.1.2 Modelling options

These flows can be studied experimentally, simulated numerically with high-fidelity CFD simulations and modelled at lower fidelity with lumped parameters. Figure 1.3 shows the hierarchy of these modelling options, which are always based and validated on experimental evidence. Reynolds-

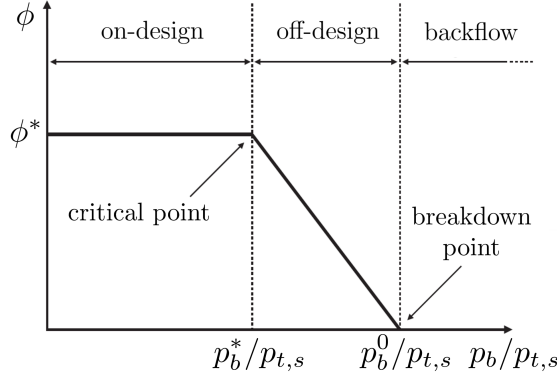


Fig. 1.2 Schematic representation of the operating curve of an ejector. The mixed flow is choked in on-design operation, reaching the maximal flow rate in the ejector for given inlet conditions. The entrainment decreases with increasing back pressure in off-design operation, eventually reversing the secondary flow direction at the breakdown point.

Averaged Navier Stokes (RANS) simulations remain the workhorse in industry and academics due to the high computational cost of Large Eddy Simulations (LES). This cost is a consequence of the high speeds in ejectors, often reaching Mach 2, and of the high Reynolds number of these flows due to the high pressure of the primary stream, of the order of 5 bar. This explains the relatively few studies in LES [6–8] compared to those employing RANS simulations [9–28].

Modelling-wise, four categories can be distinguished based on the treatment of the mixing pipe, as depicted in figure 1.4. By far the most common are the 0D models, which employ integral balances of mass, momentum and energy between key sections such as the primary throat and the constant area section in the mixing pipe (see figure 1.4a). These were pioneered by Keenan et al. in 1942 [29], with a long lineage of developments up to recent years (see [5] for a review). More recent variants of these models include the formulation of empirical or semi-empirical laws to model losses [30], the shock patterns during the mixing [31] and off-design operation [32].

While these models are valuable design tools due to their low computational cost, they rely on closure parameters (e.g., isentropic or mixing efficiencies; see [3, 33]) and are inherently zero-dimensional. As a result, they cannot capture the effects of nozzle position, mixing chamber geome-

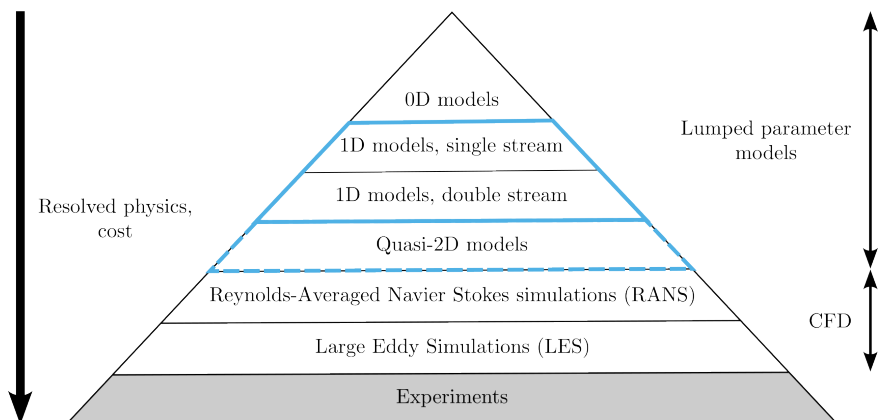


Fig. 1.3 Hierarchy of ejector studies, with increasingly detailed data towards the bottom. Which option is most effective depends on the application, e.g., CFD simulations are generally too costly at the application level, but are excellent tools for detailed design optimisation. This work focuses mostly on one-dimensional modelling.

try, or the downstream evolution of velocity, pressure, and Mach number. Moreover, these models must be calibrated using numerical or experimental data [34]. The resulting coefficients are not universal but instead depend on the specific geometry and operating conditions.

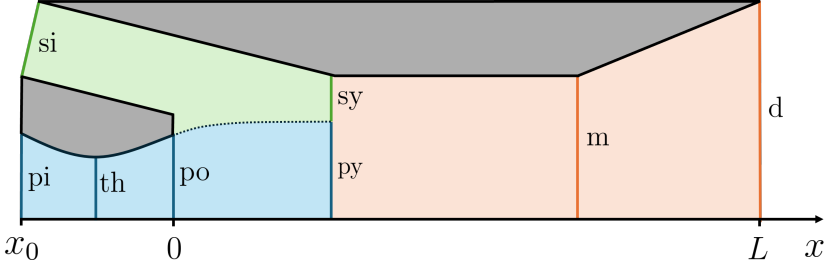
One-dimensional models are relatively scarce compared to their 0D counterparts, although they present many advantages. Ejectors are slender devices and have a clear principle axis, so axial distributions of averaged quantities are quite representative of the local state of the flow. These can be averaged over the entire cross-section or over the separate streams as in figures 1.4b and 1.4c. The former is developed for the first time in the framework of this thesis (see chapter 5 for more details). Moreover, these predictions come at a similar computational cost compared to 0D models; both run on a laptop in the order of a few seconds up to a minute. In contrast to the 0D models, the one-dimensional predictions provide more information, depend on the length of the ejector and generally require less calibration parameters. However, they do not capture important two-dimensional effects such as shocks and recirculation zones, which are essential to minimise losses during the design phase. One-dimensional models are therefore primarily investigative tools for improving the understanding of the physics underlying choking in ejectors. The models by

Clark [35], Papamoschou [36] and Grazzini et al. [15] are of the double-stream type (see figure 1.4c). They assume uniform static pressure profiles at a given axial position and consider no mass exchange between the streams. They include momentum and energy exchange, using friction forces at the wall and between the streams. An alternative approach including mass transfer was proposed by Banasiak and Hafner [37], who also considered phase change. Their hybrid 0D-1D model lets the static pressures equalise across a 0D control volume between the primary nozzle and the inlet of the constant-area section. None of these one-dimensional models tackle on-design operation and the related choking process.

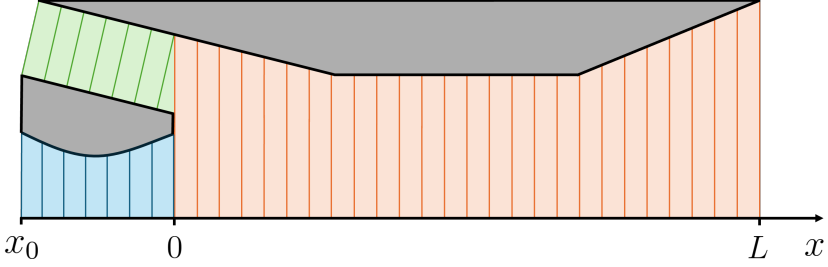
Finally, quasi-2D models constrain the radial velocity profiles in the mixing pipe to a prescribed functional shape, e.g., a cosine in the mixing layer to smoothly transition from the constant values in the primary and the secondary free-streams [25, 38, 39] (see figure 1.4d). The spatial development of the shear layer is thus tracked in these models, using correlations for its growth as proposed by Papamoschou [40]. Furthermore, these models use the same one-dimensional conservation equations as the models described above, also complemented with exchange terms, with the difference that the relevant fluxes are integrated analytically from the known radial profiles and the shear between the streams can be computed from the radial derivative of the local velocity profile. The model presented by del Valle et al. [41] is based on a one-dimensional primary mean flow that is perturbed in the axial and radial directions, and a one-dimensional secondary stream. The dividing streamline is found at the exit of the primary nozzle using Prandtl-Meyer expansion theory and downstream by imposing identical static pressures at either side. The two-dimensional perturbation is found using potential flow theory, hence limiting the model to isentropic conditions. It assumes choking occurs in the Fabri sense [42] and predicts the flow field only up to the sonic point of the secondary stream.

1.1.3 Open questions

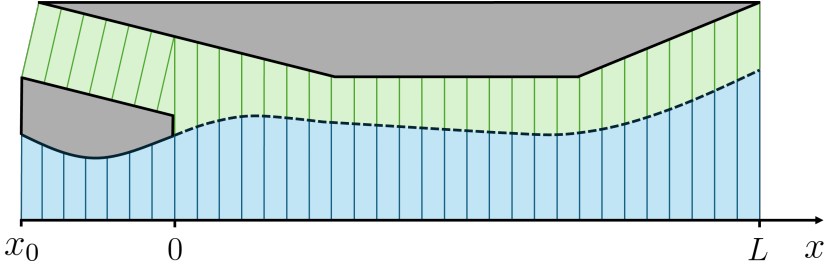
Despite the plethora of options for modelling and simulating ejector flows, many phenomena remain only partially understood due to their complex interaction. Even LES, while resolving most turbulent exchanges between the streams, still overestimates global entrainment by about 4 % [28], a level similar to RANS. The source of this discrepancy is difficult to identify; it could stem from the exchanges, but also from the wall treatment (e.g. adiabatic or constant temperature), two-phase or real gas effects, or from



(a) 0D models



(b) 1D models, single stream



(c) 1D models, double stream (continued below)

the choking mechanism. The latter in particular is difficult to interpret from two- or three-dimensional flow fields, since the two existing theories for choking in ejectors are based on one-dimensional formulations.

The Fabri choking theory, introduced in the seminal work by Fabri and Siestrunk [42], considers the sonic conditions of the primary and secondary streams as independent. According to this theory, an ejector is choked when both streams individually reach unitary Mach at their respective throats, accounting for the expansion of the primary stream within the mixing section. In contrast, the compound choking theory, originally proposed by Pearson [43], later refined by Hoge [44] and Bernstein [45], and firstly adopted in ejector modelling by Lamberts et al. [19] and Metsue et

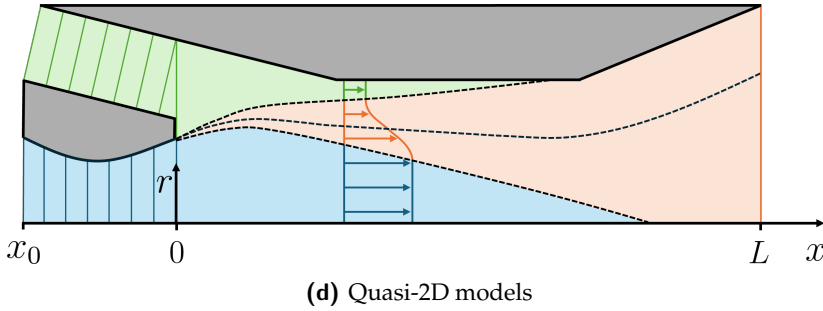


Fig. 1.4 Schematic representation of the lumped parameter modelling options from figure 1.3. Vertical lines indicate positions where the averaged flow is calculated (with arbitrary axial resolution in 1D and 2D).

al. [46], defines choking as a compound sonic condition in which the two streams are dynamically coupled. Under this framework, choking may occur even if one of the streams remains subsonic throughout the mixing chamber. A key assumption of this model is that the static pressure remains uniform across both streams during mixing, which implies that travelling pressure waves are planar and that information propagates at the same velocity in both streams. One of the main strengths of the compound choking theory is that, unlike the Fabri theory, it offers a predictive framework that can be used to estimate the critical back pressure and associated flow conditions.

To date, neither theory has been definitively proven superior or more general than the other. Computational fluid dynamics (CFD) has made it possible to reproduce the mass-flow limitations observed in ejectors [9,11], but these simulations have not provided conclusive evidence in favor of one choking mechanism over the other. In fact, both types of behavior have been reported in the literature: Lamberts et al. documented cases consistent with Fabri choking [17] as well as cases exhibiting compound choking [19]. The sonic line clearly penetrates the core of the secondary stream in the first case, while it stays within the shear layer between the streams in the latter case.

Both numerical [28] and experimental [44] works show that higher ratios of primary to secondary total pressures tend to favor Fabri choking. A possible explanation is the increasingly under-expanded nature of the primary stream, which results in stronger expansion within the mixing chamber and thus a restriction of the secondary stream akin to the throat of

a converging–diverging nozzle. On the other hand, lower inlet pressure ratios tend to produce over-expanded primary flows, which contract in the mixing channel and hence leave more space for the secondary stream, which then remains subsonic throughout. Wall friction has also been shown to play an important role: Kracik and Dvorak [26] numerically showed that smooth walls led to Fabri choking, while rough walls resulted in compound choking under the same operating conditions. However, the mechanism behind this switch remains poorly understood.

Historically, the Fabri condition has been the most widely used in integral 0D models (see [29, 32, 42, 47], to name but a few). In contrast, the compound flow theory has only gained attention more recently [23, 46, 48]. Notably, Croquer et al. [23] reported a reduction in prediction error from 17% with the Fabri theory to 5% using the compound theory. While this might suggest a superiority of the compound theory, it is important to note that 0D models rely heavily on the calibration of efficiency coefficients.

A comprehensive analysis of the choking mechanism requires at least one-dimensional models to capture the underlying wave propagation phenomena. Indeed, the sonic point occurs at the point where pressure waves no longer manage to propagate upstream and hence further reductions in downstream pressure no longer affect the flow rate. The different definitions of the sonic point in the Fabri and compound choking theory ultimately stem from a different pressure wave: the Fabri theory considers a separate wave in the secondary stream, whereas the compound wave is subject to the constraint of uniform static pressure and thus propagates at the same velocity in both streams. It is interesting to note that the presented one-dimensional models do not tackle on-design operation in either sense and that the quasi-2D models by del Valle et al. [41] and Chen [39] consider choking in the Fabri sense. Compound choking thus remains under-explored in the framework of 1D and quasi-2D ejector models.

1.2 Research objectives

Existing ejector models have been categorised in the natural hierarchy presented in figure 1.3 based on the amount of detail in their predictions. The first objective of this thesis is to answer the following question:

How far can data-driven calibration push zero-, one-, and quasi-two-dimensional models toward maximal accuracy and detail?

This question aims to identify the inherent limitations of physical models by

ruling out further improvement through closure modelling. Conversely, from the perspective of CFD simulations, it compels the reduction of complex, higher-dimensional flow interactions into lower-dimensional representations—or their omission altogether. Such exercises in simplification promote a deeper understanding of ejector physics, leading to the second central research question:

To what extent can these models, despite their previously identified limits, contribute to unveiling and understanding the physical mechanisms driving ejector operation?

Three axes of research have been put forward to address these questions.

Firstly, this thesis aims to develop one-dimensional models that provide the additional local information compared to 0D models at a similar cost, as mentioned in section 1.1, and that operate in both on- and off-design conditions. Both the single and double stream configurations from figures 1.4b and 1.4c are developed, calibrated and analysed. In particular, the compound and Fabri choking mechanisms are implemented and compared in the double stream configuration, to assess the impact of the choking mechanism on the model predictions. As such, the predictive models are used as analysis tools to better understand the physical mechanisms at play in ejectors.

Secondly, the analysis above requires calibration tools that are tailored to the ejector models. A general framework is proposed to discover relations between the closure coefficients and the physical quantities in dynamical systems, by combining the adjoint method with conventional regression techniques from machine learning.

Finally, a numerical and experimental database is built to validate the developed models and to analyse their performance.

The combination of these three aspects ultimately aims to reconcile engineering models with experiments and simulations.

1.3 Structure of the manuscript

The rest of the manuscript is organised as follows.

Chapter 2: Experimental and numerical data

The ejector models developed in this thesis are validated against experimental and high-fidelity numerical data introduced here.

The chapter begins with a description of the numerical setup in OpenFOAM, which is verified against the earlier results of Mazzelli [13]. The associated post-processing tools are then presented, including normalisation of mass flow rates and cross-stream averaging of the RANS simulations to obtain one-dimensional reference distributions. Next, two datasets are presented: one from an ejector installed at the von Karman Institute (VKI) and another at the Université Catholique de Louvain (UCLouvain).

The VKI ejector is an adapted version of the device used in the Ejemod project^{1 2} [27, 49–51], modified to feature a converging-diverging primary nozzle instead of the original converging design. This configuration was first analysed in the master theses of Angiero [52] and Venezia [53], and later extended through additional experimental and numerical campaigns to form the present database.

The UCLouvain ejector is a more classic ejector with a longer primary nozzle. It has been studied numerically in ANSYS Fluent 2022-R2 (student version release [54]) by Baguet and Querinjean [55] and experimentally by Brosteaux and Doucet [56].

For this chapter, many thanks are in place for Tani and Biagio, who have completed their master thesis under my supervision,

T. Angiero, “Numerical and experimental comparison between a convergent and a convergent-divergent primary nozzle for a supersonic axisymmetric ejector,” master thesis, Politecnico di Torino, 2023.

B. M. Venezia, “Numerical characterization of a supersonic ejector through rans simulations,” master thesis, Politecnico di Torino, 2025.

and to Nicolas, Guillaume, Loric and Quentin, who have numerically and experimentally studied the ejector at UCLouvain in their master thesis:

N. Baguet and G. Querinjean, “Experimental and numerical characterization of an axisymmetric supersonic ejector,” master thesis, Ecole polytechnique de Louvain, Université Catholique de Louvain, 2023.

L. Brosteaux and Q. Doucet, “Experimental characterization of an axisymmetric supersonic air ejector,” master thesis, Ecole polytechnique de Louvain, Université Catholique de Louvain, 2024.

¹<https://cordis.europa.eu/project/id/101008100>

²<https://cleansky-ejemod.eu/>

Chapter 3: Data-driven methods for calibration

The third chapter is dedicated to the general calibration and regression tools used throughout the following chapters. It introduces two approaches for identifying closure relations in physical systems governed by differential equations: a physics-separated and a physics-integrated approach. In the first, the model is not involved during calibration; instead, reference closure parameters are extracted from a sufficiently rich dataset, and the closure relation is learnt offline in a classic supervised fashion. In the physics-integrated approach, the model is included in the loop, allowing to penalise errors on any of the predicted quantities, but requiring knowledge of the model's sensitivity with respect to its closure coefficients.

To address this, the adjoint method is presented for first-order differential equations—the foundation of all models in this thesis—as an efficient way to compute derivatives of the cost function without explicitly evaluating these sensitivities. This method was used extensively in chapter 8 to assimilate velocity profiles in the mixing pipe. In the present chapter, it is combined with an artificial neural network in an academic test case to uncover the hidden dynamics of a Burgers-like equation with unknown advection speed.

Finally, both calibration approaches are applied on an early version of the double stream ejector model in chapter 7, illustrating their advantages and limitations in a more practical context. It is noteworthy to mention that the models in chapters 5 and 7 were ultimately calibrated with the physics-integrated method due to their low computational cost and small number of closure coefficients. The latter stems from the inclusion of extensive prior physical knowledge, particularly through the use of classical friction coefficients and existing correlations.

This chapter is largely based on the following conference paper:

J. Van den Berghe, J. B. Vemula, Y. Bartosiewicz, and M. A. Mendez, "A machine learning-based calibration of a 1D ejector model from CFD," in Proceedings of ECOS 2023 - the 36th international conference on efficiency, cost, optimization, simulation and environmental impact of energy systems, pp. 424–435, June 2023.

Chapter 4: Governing equations

To bridge the gap between RANS simulations and 0D lumped-parameter models (cf. figure 1.3), the RANS equations are progressively simplified in

this chapter. A leading-order analysis identifies the boundary layer equations as a reduced description of the physics in the mixing pipe, paving the way for the quasi-two-dimensional modelling presented in chapter 8. Integrating these equations over the radial coordinate yields the governing equations of the one-dimensional models, either over the entire cross-section (single-stream, chapter 5) or over the primary and secondary mass flow tubes (double-stream, chapters 6 and 7).

Chapter 5: Single stream one-dimensional model

The model based on the single stream configuration in figure 1.4b is presented in this chapter. It is largely inspired by pipe networks, where the primary inlet, secondary inlet and the mixing pipe are connected through a junction model. The resulting model solves the unsteady one-dimensional gas dynamics in these three domains, allowing to simulate transient operation and to capture the characteristic response time of ejectors (around 30 milliseconds). The model naturally captures choking in the mixing pipe in a compound sense, with an averaged Mach number over the entire cross-section equal to unity. The predictions are limited to global quantities in space and time, such as the local mass flow rate, but these are in excellent agreement with unsteady RANS simulations. This chapter is largely based on the following paper:

J. Van den Berghe, B. R. Dias, Y. Bartosiewicz, and M. A. Mendez, "A 1D model for the unsteady gas dynamics of ejectors," Energy, vol. 267, p. 126551, 2023.

Chapter 6: Extension of the compound flow theory

From the previous chapter, it is concluded that the response time of ejectors is fast enough to consider it as a quasi-steady process when placed in a system. Therefore, steady state operation is considered in the remainder of this thesis. The analysis shifts to the double stream configuration in figure 1.4c, in order to reveal the separate primary and secondary streams, which remained hidden under the averaging operation over the entire cross-section as in the previous chapter. This allows to predict and to analyse the axial distributions of relevant variables in both streams and to include exchanges between them. The compound flow theory proves a useful tool for this purpose and has been extended in this thesis with momentum exchange between the streams and at the wall. This leads to an

interesting mechanism that displaces the compound-sonic section up- or downstream depending of the balance between wall and inter-stream friction terms. Choking of parallel streams in a convergent section had been observed earlier by Lamberts et al. [19], but the underlying mechanism had not been discovered before. This observation inspired the following paper, on which this chapter is based, and forms an essential tool in the model described in the next chapter:

J. Van den Berghe, M. A. Mendez, and Y. Bartosiewicz, "An extension of the compound flow theory with friction between the streams and at the wall," Journal of Fluid Mechanics, vol. 1000, p. A85, 2024.

Chapter 7: Double stream one-dimensional model

This chapter presents the one-dimensional model in the double stream configuration from figure 1.4c, and compares its performance using a compound or a Fabri sonic condition. The advantages and limitations of both theories are analysed. The compound theory provides a predictive framework, allowing to compute the dividing streamline, but overly constrains the static pressures and therefore suffers lower accuracy if the primary stream is under-expanded. The Fabri mechanism yields more accurate results, but either requires the dividing streamline from an external source (the RANS simulations from chapter 3 in this work) or from a dedicated model. Moreover, the integration proves highly sensitive to errors on the dividing streamline and the closure modelling, especially near the sonic point. A hybridisation seems a promising avenue for future developments. This chapter is largely based on the following paper:

J. Van den Berghe, M. A. Mendez, and Y. Bartosiewicz, "On the choking mechanism in supersonic ejectors: a one-dimensional analysis of RANS simulations," submitted to Journal of Fluid Mechanics, 2025.

Chapter 8: Exploring quasi-two-dimensional modelling: early developments and perspectives

The one-dimensional formulations, in both single- and double-stream form, have been examined in the previous chapters, highlighting the limitations of cross-stream averaging. As the shear layer thickens and mixing intensifies, the averaged quantities become less representative of the true radial profiles (cf. schematic 1.4d). Explicit representation of these profiles could

enhance 1D models by embedding two-dimensional information into the averaged terms and by defining more accurate inter-stream exchange at their boundaries. Alternatively, quasi-2D models offer a path toward Reduced Order Models (ROMs) that approximate RANS behaviour at a fraction of the computational cost.

This chapter investigates both directions. Starting from a common theoretical basis, the compressible RANS equations are scaled and reduced to their leading-order form, which is verified a posteriori using the RANS data from chapter 2. The route toward improved 1D models is outlined only at a conceptual level, whereas the quasi-2D ROM development is supported by an initial numerical framework, enabling low-cost solution and calibration of a simplified linear system. Preliminary results are encouraging: despite its linearity, the model reproduces the shear-layer evolution with surprising accuracy.

Appendices

The appendices contain a practical table for the explicit ODEs of the relevant flow variables in the spirit of the book by Shapiro [57], which can be derived from the one-dimensional conservation equations for mass, momentum and energy. Next, the approximations of the static pressure gradients in and near the sonic point of a single and compound stream are listed. These allow to build Taylor expansions near the sonic point and to avoid numerical difficulties due to division by zero. An attempt at a compound shock model is also presented, which did not prove sufficiently representative of the flow downstream of the shock train in the diffuser, so did not end up in the model with the double stream configuration (see chapter 6 and [58]). Finally, the adjoint counterparts of the compound flow equations are presented. These have not been used in the calibration of the inter-stream friction due to the low number of tunable parameters (two) and the fast predictions, allowing to apply brute force finite differences or Nelder-Mead. A simple correction of the existing correlation by Papamoschou [40] proved sufficient, so no heavy-duty regression tools like neural networks were required. These equations may still prove helpful in future developments.

2

Experimental and numerical data

2.1 Introduction

This chapter presents the experimental and numerical reference data used throughout the following chapters to calibrate and validate the various ejector models. Section 2.2 presents the settings for the RANS simulations carried out in OpenFOAM and the postprocessing tools to convert the two-dimensional flow fields to axial distributions of averaged quantities. Sections 2.3 and 2.4 are dedicated to the specific ejectors installed at the von Karman Institute (VKI) and at the Université Catholique de Louvain (UCLouvain) respectively. The experiments and simulations of the former have been carried out in the framework of this thesis in direct supervision of Tani Angiero and Biagio Venezia [52, 53]. These simulations have been carried out with the tools from section 2.2. The ejector at UCLouvain has been studied in the master theses of Baguet & Querinjean [55] and Brosteaux & Doucet [56]. Their simulations have been carried out in ANSYS Fluent 2022-R2 [54], but the same post-processing tools have been used to obtain the one-dimensional reference data.

2.2 Reynolds-Averaged Navier Stokes simulations

2.2.1 Solver settings

The numerical simulations in this work include both planar and axisymmetric Reynolds-Averaged Navier–Stokes (RANS) computations. While Large Eddy Simulations (LES) become feasible for specific test cases [7,8], their high computational cost precludes simulations of a wide range of operating conditions. The RANS simulations use the k - ω SST turbulence model of Menter [59] which is well-established for ejectors [9]. The simulations of the ejector at VKI employed an adapted version of the transient, compressible solver *rhoCentralFoam* from *OpenFOAM v9*¹ (see section 2.3). The adapted solver does not solve for the tangential component of the velocity in the axisymmetric simulations, which is set to zero (no swirl).

The meshes were generated using Gmsh², coupled to Python to allow full control of the number of elements and of the refinement near the walls. The simulations are generally wall-resolved with $y^+ \approx 1$, unless stated otherwise. This is not the case in the primary nozzle where this condition leads to excessive refinement and consequently to severe limitations of the time step. OpenFOAM requires three-dimensional meshes, so the mesh is extruded with a single cell to perform two-dimensional simulations. This extrusion matches the width of the channel in planar geometries and consists of a wedge of 5 degrees in axisymmetric simulations. The resulting mass flow rates are multiplied by 72 in the axisymmetric case to be representative of the complete geometry.

Table 2.1 collects the kind of applied boundary conditions. The total pressure, the total temperature, the turbulence intensity (TI) and the turbulent length scale l_{mix} are imposed at the inlets, with zero gradient for the velocity. The mass flow rates are thus a result from the calculation. The turbulence intensity is assumed to equal 5 %, which corresponds to a fully developed pipe flow [9,11]. The turbulent mixing length was set equal to 7 % of the hydraulic diameter, as recommended by Versteeg [60]. Zero gradients are imposed at the outlet except for the pressure. The wave transmissive condition is used here to prevent excessive reflections. The walls have no slip and are adiabatic. For k and ω , the ‘kLowReWallFunction’ [61] and the ‘omegaWallFunction’ [62] are used, which despite their naming automatically adapt to low- or high-Reynolds number conditions, i.e., wall-resolved

¹<https://openfoam.org/version/9/>

²<https://gmsh.info/>

or wall-modelled approaches. The functions switch between the two formulations based on the locally computed y^+ value. The side walls receive the ‘empty’ condition in a planar geometry and the ‘wedge’ condition in an axisymmetric geometry to inform OpenFOAM of the two-dimensional nature of the simulations. The numerical values of the boundary conditions are presented where needed in sections dedicated to specific geometries and operating conditions.

The working fluid is air in all simulations, which is assumed to behave as an ideal gas with a gas constant $R_{gas} = 287 \text{ J/kg/K}$ and a ratio of specific heats $\gamma = 1.4$. The law of Sutherland is used for the viscosity [63]:

$$\mu = \mu_{ref} \left(\frac{T}{T_{ref}} \right)^{3/2} \frac{T_{ref} + S}{T + S}, \quad (2.1)$$

with a reference viscosity μ_{ref} of $1.716 \cdot 10^{-5} \text{ Pa s}$ at $T_{ref} = 273.2 \text{ K}$, and $S = 110.4 \text{ K}$. The turbulent Prandtl number is assumed to be equal to 0.85.

Table 2.1 Boundary conditions used for the 2D CFD. The values for the pressure and temperature follow from the operating conditions.

	primary inlet	secondary inlet	outlet	walls
p	$p_{t,p}$	$p_{t,s}$	transmissive	$\partial_n p = 0$
T	$T_{t,p}$	$T_{t,s}$	$\partial_n T = 0$	$\partial_n T = 0$
$\mathbf{v}$	$\partial_n \mathbf{v} = 0$	$\partial_n \mathbf{v} = 0$	$\partial_n \mathbf{v} = 0$	0
k	TI	TI	$\partial_n k = 0$	wall function
ω	l_{mix}	l_{mix}	$\partial_n \omega = 0$	wall function

2.2.2 Post-processing

The two-dimensional data from the RANS simulations are post-processed by extracting the dividing streamline and by averaging over the resulting cross-sections [16, 64]. The radial position r_d of the dividing streamline is found at any point x by integrating the axial mass flux:

$$\dot{m}_p(x) = \int_0^{r_d(x)} \rho(x, r) u(x, r) 2\pi r dr, \quad (2.2)$$

where u denotes the axial component of the velocity vector and the factor $2\pi r$ is replaced by the width of the channel in planar configurations. The

primary mass flow rate is known from its inlet. The second step consists of averaging the relevant flow variables across the ejector's section, i.e.

$$\begin{aligned}\hat{\rho}_i(x) &= \frac{1}{A_i} \int_{A_i} \rho dA_i, \quad \hat{u}_i(x) = \frac{1}{\hat{\rho}_i A_i} \int_{A_i} \rho u dA_i, \\ \text{and } \hat{e}_{t,i}(x) &= \frac{1}{\hat{\rho}_i A_i} \int_{A_i} \rho e_t dA_i,\end{aligned}\tag{2.3}$$

for $i \in [p, s]$, where $e_t = c_v T + u^2/2$ denotes the total specific internal energy and c_v is the specific heat at constant volume. The density-averaged velocity and internal energy ensure that the 1D mass flow rate and internal energy match their integrated 2D counterpart. The averaged static temperature, static pressure and Mach number follow from (2.3) and the specific gas constant R_{gas} :

$$\hat{T}_i = \frac{1}{c_v} \left(\hat{e}_{t,i} - \frac{1}{2} \hat{u}_i^2 \right), \quad \hat{p}_i = \hat{\rho}_i R_{gas} \hat{T}_i, \quad \text{and} \quad \hat{Ma}_i = \frac{\hat{u}_i}{\sqrt{\gamma R_{gas} \hat{T}_i}}.\tag{2.4}$$

The reported mass flow rates are normalised with the following reference flow rates:

$$\dot{m}_{p,ref} = p_{t,p} A_{th} \sqrt{\frac{\gamma}{R_{gas} T_{t,p}}} \left(\frac{\gamma + 1}{2} \right)^{-\frac{1}{2} \frac{\gamma+1}{\gamma-1}},\tag{2.5}$$

$$\dot{m}_{s,ref} = p_{t,s} (A_m - A_{th}) \sqrt{\frac{\gamma}{R_{gas} T_{t,s}}} \left(\frac{\gamma + 1}{2} \right)^{-\frac{1}{2} \frac{\gamma+1}{\gamma-1}}.\tag{2.6}$$

These correspond to sonic flows with the stagnation conditions $(p_{t,i}, T_{t,i})$ through cross-sections based on the primary throat A_{th} and the constant-area section A_m in the mixing pipe. These estimates form an upper limit for the mass flow rates: they disregard losses, and the secondary cross-section used in the equation above is generally larger than the one where the secondary stream actually reaches Mach one. The normalised mass flow rates are thus bounded between zero and one [65]. They are denoted as follows:

$$\hat{m}_i = \dot{m}_i / \dot{m}_{i,ref}.\tag{2.7}$$

2.3 The ejector facility at VKI

An ejector testing facility has been designed at the von Karman Institute in the framework of the Ejemod project ³. The ejector has been studied experimentally and numerically in order to provide reference data for the development of 0D and 1D models [27,49–51]. These test campaigns have been extended later, replacing the original convergent primary nozzle with a convergent-divergent nozzle [52,53].

The geometry of the ejectors with the different primary inlets are shown in figure 2.1. The ejectors are cylindrical, so the RANS simulations are axisymmetric. The outer shroud is identical in both cases, allowing to keep the same instrumentation. The total length of the ejector equals 0.740 m. The origin of the axes used in this work is located at the inlet of the constant area section, which has a length $L = 375$ mm. This length is used as reference for non-dimensionalising the axial coordinate x , so the diffuser starts at $x/L = 1$. The throat has a radius of 14 mm in both cases to keep the primary mass flow rate identical across the geometries with the same inlet conditions. The outlet of the converging-diverging nozzle has a radius of 15 mm. The diffuser has a half-angle of 2.5 degrees and connects the constant-area section with a radius of 25 mm to the outlet with a radius of 37 mm.

The wider converging-diverging nozzle narrows the passage for the secondary stream by 1 mm, but allows the primary flow to expand further, potentially entraining more fluid into the low pressure region. A detailed numerical comparison [52] showed that the narrowed flow passage is dominant over the increased suction, leading to a reduction of the entrainment. In the remainder of this thesis, the results on the converging-diverging geometry are used since this situation is more common for ejectors. The interested reader is referred to [27,49–52] for an analysis of the converging geometry. The RANS simulations have been carried out with the settings outlined in section 2.2.1; additional details can be found in [53].

Figure 2.2 shows a sketch of the experimental facility. Compressed air is supplied on the top left. The inlet pressures are then regulated independently, allowing to set inlet pressure ratios $\Pi_{sp} = p_{t,s}/p_{t,p}$ between 0.2 and 1, with a maximal primary pressure of 5 bar. A manual valve at the diffuser exit allows to manipulate the back pressure. A 150 kW heater can heat one of the inlets up to a temperature of 200 °C. The flow settles through 2.5

³<https://cleansky-ejemod.eu/>

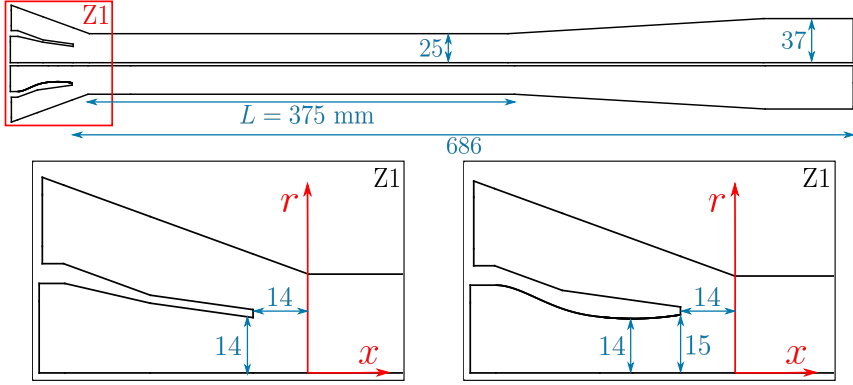


Fig. 2.1 The convergent (top) and convergent-divergent (bottom) configurations of the primary nozzle are tested in the same outer casing. The geometry is axisymmetric and the origin is located at the outlet of the primary nozzle. Dimensions are in millimeters.

m long straight pipes before passing through two vortex mass flow meters. The maximal total achievable flow rate of the facility is 0.7 kg/s. Two flexible tubes allow to connect the heated inlet either to the primary or the secondary inlet of the ejector, which is located on the far right. Both inlets and the outlet are instrumented with four thermocouples and a pressure transducer connected to four pressure taps. With the measured mass flow rates and the known cross-sections, these static quantities are converted to their total counterparts to serve as boundary conditions in the numerical model. The propagation of the uncertainties can be found in appendix D.

The experimental data are organised in campaigns number from 1 to 19 as listed in table 2.2. Only the campaigns with a numerical counterpart through RANS simulations have been listed. Not all experimental tests have been simulated, since the shown tests already cover a wide range of conditions through classic operating curves. The lowest back pressures were obtained by fully opening the manual valve at the outlet of the facility; a compression ratio below unity is not desirable in practical applications.

The secondary mass flow rates are normalised through equation (2.7), and are compared in figure 2.3 for campaigns 3, 8, 9 and 14. Other campaigns have been left out due to their low inlet pressure ratio. This induces a strongly over-expanded primary stream, which is not desired for efficient ejector operation. The secondary mass flow rates are consistently overestimated, as often observed in literature [11,28]. The predicted oper-

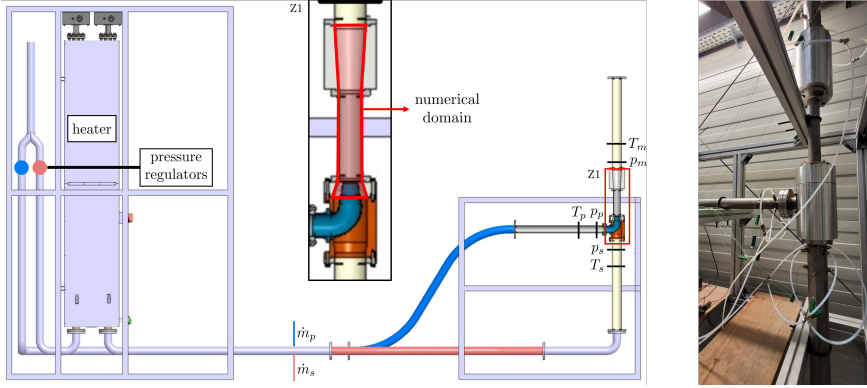


Fig. 2.2 Schematic overview of the experimental facility [49] with individual inlet pressure regulators, one heated line, vortex mass flow meters and pressure and temperature sensors at the ejector inlets and outlet.

Table 2.2 Overview of the inlet pressures of the experimental campaigns. The inlet temperatures are the ambient temperature (no heating). The lines indicated in bold have a numerical counterpart.

Campaign	$\Pi_{ps} = \frac{p_{t,p}}{p_{t,s}} [-]$	$\Pi_{bs} = \frac{p_b}{p_{t,s}} [-]$
3	3.46	0.85 - 1.61
8	4.53	0.68 - 2.13
9	4.19	0.81 - 1.95
14	2.78	0.48 - 1.51
15	2.14	0.57 - 1.31
16	1.73	0.62 - 1.22
18	1.36	0.67 - 1.06

ating curves are shifted towards the top right corner, leading to an error in on-design operation of the order of 5-10 % and a strongly overestimated entrainment in off-design operation. In particular, some off-design operating points are predicted as on-design conditions in the RANS simulations, for example the point at $p_b/p_{t,s} \approx 1.2$ for campaign 14. The reason for this poor agreement remains an open question. A comparison of experiments, RANS and LES simulations by Debroeyer et al. suggests inaccurate modelling of the exchanges between the streams as the most probable cause [28]. These exchanges indeed directly affect the pressure recovery in the mixing pipe and hence the critical back pressure. The overestimated flow rate in on-design operation is directly linked to the choking

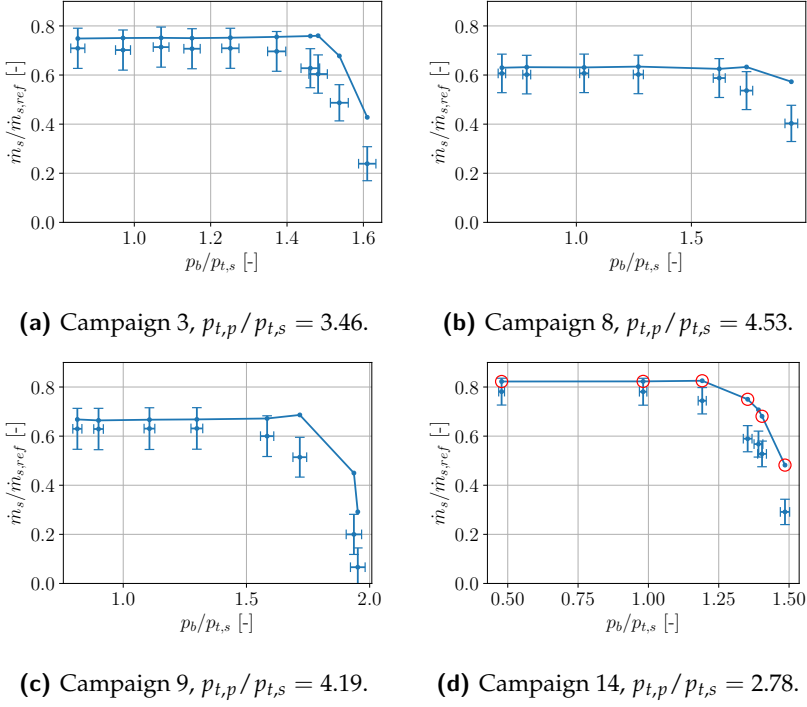


Fig. 2.3 Non-dimensional secondary mass flow rate from the experiments and RANS simulations on the VKI ejector. The wall pressure distributions of the circled points are shown in figure 2.4.

mechanism, which is also under debate and is further analysed in chapter 7. On the other hand, the possibility of inaccuracies in the experimental data should not be dismissed. Sensors may exhibit bias, and the measured quantities may be nonstationary.

The mismatch in entrainment is reflected in the wall pressure distributions, as shown in figure 2.4 for the circled test points of campaign 14 in figure 2.3. The overlapping curves of tests 1 and 4 indicate choking. The shock in the diffuser moves upstream with increasing back pressure until it reaches the sonic section, at which point communication is established from the outlet to the secondary inlet. In the resulting off-design regime, the static pressure increases axially due to momentum exchange in compound-subsonic conditions (see chapter 6 for a more detailed analysis). This allows to compress the secondary flow beyond its stagnation

pressure at the inlet ($p/p_{t,s} > 1$).

The critical point is encountered between tests 4 and 5 in the experiments. In contrast, the RANS simulation of test 5 still predicts on-design operation, as evidenced by the overlap with the curves at lower back pressure. The error on the critical back pressure is also visible in the off-design distributions: for example, the experimental distribution of test 5 and the numerical distribution of test 6 show good agreement despite the different back pressure. The reason is the similar mass flow rate of these points (the points with $\dot{m}_s/\dot{m}_{s,ref} \approx 0.75$ in figure 2.3d). The simulations thus predict a higher compression rate for the same entrainment ratio, which indicates an unknown pressure loss in reality that is not captured in the RANS framework. Investigating the source of this error is an interesting research question, but out of the scope of this thesis. Instead, both data sources are used as reference data to calibrate the models developed in the following chapters.

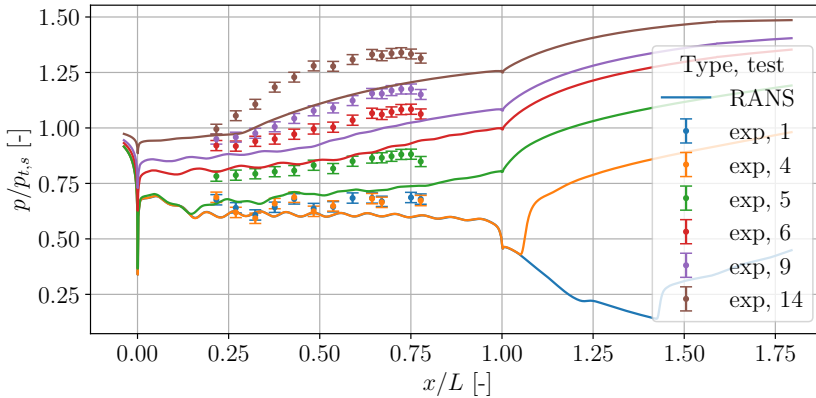


Fig. 2.4 Distributions of the wall pressure for campaign 14. The constant-area section extends from $x/L = 0$ to $x/L = 1$. The RANS simulations underestimate the static pressure since they overestimate the secondary flow rate, see figure 2.3.

2.4 The ejector facility at UCLouvain

Figure 2.5 shows the geometry of the ejector installed at UCLouvain. It has a more classical design with rounded corners and a primary nozzle with an

aspect ratio of the order of 13, as opposed to 3 in the VKI case. The experimental setup is displayed in figure 2.6. Air is supplied to the primary inlet from a compressor and its pressure is controlled by a pressure regulator (up to 10 bar). The secondary inlet is open to atmosphere and controlled by a manual valve, allowing to create a vacuum as the valve is progressively closed. These conditions have not been included in this work, so the secondary stagnation pressure corresponds to ambient pressure in all presented data points. The flows settle through pipes outside of view in figure 2.6, and their flow rate is measured using orifice plates. The secondary flow settles in a large plenum chamber, where its stagnation pressure is measured. The stagnation temperatures consist of the surrounding temperature since no heater is installed. The outer shroud of the ejector is covered with 64 pressure taps. Finally, the pressures and temperatures are measured at the inlets and outlets of the ejector.

The RANS simulations were carried out in ANSYS Fluent 2022-R2 [54], which has a dedicated implicit solver for steady compressible flows, allowing faster simulations compared to those with the explicit solver in OpenFOAM. Further, the settings were similar: the flow was assumed to be axisymmetric, the same type of boundary conditions were imposed and the $k - \omega$ SST turbulence model was used. The mesh is wall-resolving except in the primary inlet with y^+ values between 5 and 16. The same type of data is thus available as in the VKI case; the most important difference being the axial resolution of the experimental pressure taps (64 vs. 16).

Figure 2.7 shows the obtained operating curves from the experiments and the RANS simulations. They showcase the same shortcomings as the simulations of the VKI ejector (see figure 2.3): the on-design entrainment and the critical back pressure are overestimated. They serve as reference data for the double-stream model in chapter 7, in which the results are analysed in detail.

It is noteworthy to observe that the corner in the operating curve becomes sharper for higher inlet pressure ratios $p_{t,p}/p_{t,s}$. This is believed to be related to the Fabri- and compound-choking mechanisms [28,44,55,56]. The sharp corner corresponds to the Fabri pattern, which constitutes a clear ‘on-off’ switch at the individual sonic point of the secondary stream. This pattern is promoted in case of an under-expanded primary stream, which forms a narrowing passage for the secondary stream analogous to a converging-diverging nozzle. For sufficiently low back pressures, the secondary flow thus reaches a unitary Mach number at the throat of this channel, which is commonly referred to as the aerodynamic or hypothetical

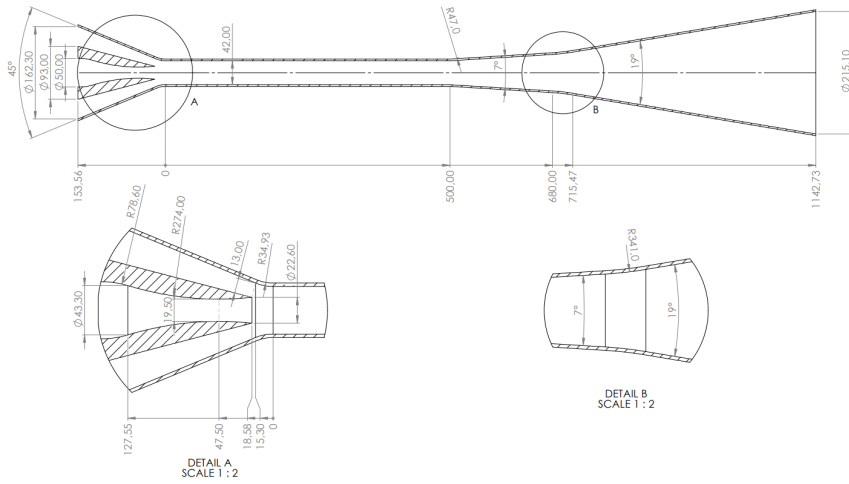


Fig. 2.5 The ejector at UCLouvain has a more classical design, in particular with a longer primary nozzle. Figure from [55].

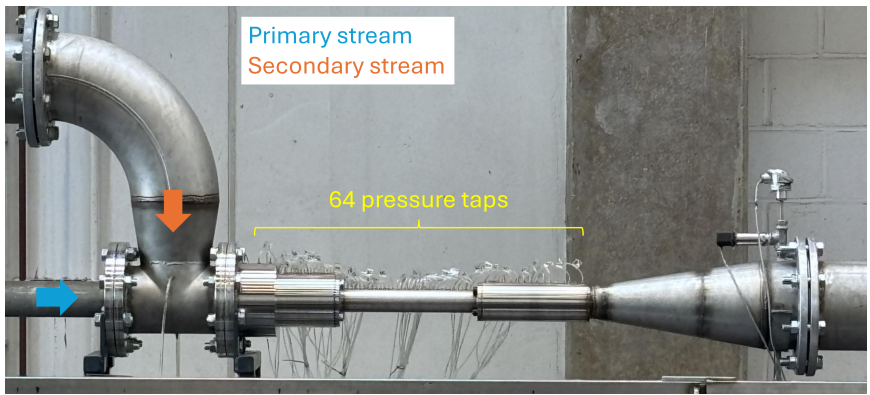


Fig. 2.6 The experimental setup at UCLouvain includes measurements of the pressure and temperature measurements at the inlets and outlets, the mass flow rates and 64 local wall pressures. Figure adapted from [56].

throat, critical section or effective area [66]. The compound-sonic section is more difficult to locate precisely, since the fundamental assumption of uniform static pressure is generally not met in reality (see chapter 7). This could explain the smoother transition in cases with lower inlet pressure ratios, although it remains an open question, especially when observing that

the sharp corner does not appear in the experimental data.

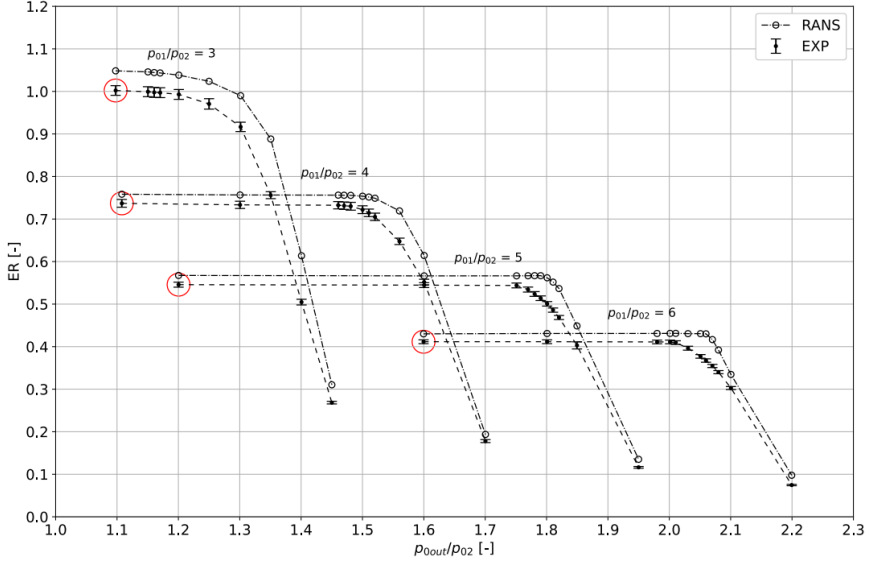


Fig. 2.7 Experimental and numerical operating curves from the ejector at UCLouvain in terms of the entrainment ratio. Figure from [56]. The total pressure is denoted here as p_0 , with subscripts 1,2 for the primary and secondary streams.

Figures 2.8–2.10 show the wall-pressure distributions for inlet pressure ratios 4, 5 and 6. For each operating curve of figure 2.7, five operating points were selected for static-pressure measurements, as reading the pressure taps with a scanning valve is time-intensive. These points correspond to two on-design conditions, the point closest to the critical point, and two off-design conditions (see Brosteaux and Doucet [56]).

The horizontal lines indicate the static pressures corresponding to the Fabri-sonic (dashed) and compound-sonic (dotted) conditions under the assumption of isentropic flow, conserving the inlet total pressures. The Fabri-sonic pressure ratio to achieve $Ma_s = 1$ is:

$$\frac{p_F^*}{p_{t,s}} = \left(\frac{\gamma + 1}{2} \right)^{-\frac{\gamma}{\gamma-1}} \approx 0.528 \text{ (air)}. \quad (2.8)$$

This ratio depends only on the gas properties. The compound-sonic pressure follows from the inlet total pressures and the cross-sections of the pri-

mary throat (th) and the mixing pipe (m) via the system

$$\text{Ma}_i^2 = \left[\left(\frac{p_{ti}}{p_c^*} \right)^{\frac{\gamma-1}{\gamma}} - 1 \right] \frac{2}{\gamma-1}, \quad (2.9)$$

$$A_p = A_{th} \left(\frac{\gamma+1}{2} \right)^{-\frac{1}{2} \frac{\gamma+1}{\gamma-1}} \frac{1}{\text{Ma}_p} \left(1 + \frac{\gamma-1}{2} \text{Ma}_p^2 \right)^{\frac{1}{2} \frac{\gamma+1}{\gamma-1}}, \quad (2.10)$$

$$A_s = A_m - A_p, \quad (2.11)$$

$$\beta = A_p \frac{1 - \text{Ma}_p^2}{\gamma \text{Ma}_p} + A_s \frac{1 - \text{Ma}_s^2}{\gamma \text{Ma}_s} = 0. \quad (2.12)$$

Here, the expression for A_p follows from mass conservation between the primary throat and the compound-sonic section under isentropic conditions. The static pressure p_c^* is found iteratively: a guessed value determines β , which must vanish at the sonic point (chapter 6). Because β decreases monotonically with p_c^* , the root is obtained in a few iterations. As $p_c^* \rightarrow p_{t,s}$, $\beta \rightarrow +\infty$ (with $\text{Ma}_s \rightarrow 0$), whereas compound-supersonic solutions produce $\beta < 0$. The Fabri-sonic point is always at least compound-sonic, since $\beta \leq 0$ for $\text{Ma}_s = 1$ and $\text{Ma}_p \geq 1$. Crucially, the Fabri-sonic pressure is always lower than the compound-sonic pressure unless the stagnation pressures are equal, in which case both criteria coincide ($\text{Ma}_p = \text{Ma}_s = 1$). This supports the broader applicability of compound-flow theory, discussed in chapter 7.

Across all three inlet-pressure ratios, the off-design distributions show similar behaviour: an acceleration up to the inlet of the constant-area section at $x/L = 0$, followed by a rise in static pressure as momentum is transferred from the primary to the secondary flow, eventually recovering the back pressure in the diffuser ($x/L \geq 1$). The accuracy of the RANS predictions for these profiles depends directly on the secondary mass-flow rate. For instance, for $p_{t,p}/p_{t,s} = 5$ (figure 2.9), the prediction is better for the point at $p_b/p_{t,s} = 1.85$ than for 1.80, reflecting the larger overestimation of the secondary mass flow at the latter (see figure 2.7). Lower predicted wall pressures are consistent with a higher entrained flow rate.

A similar trend appears in on-design operation: RANS consistently underestimates the static pressure relative to the experiments. The pressure level also depends on the inlet pressure ratio. For $p_{t,p}/p_{t,s} = 4$ (figure 2.8), the measurements remain well above both sonic pressure ratios, whereas for $p_{t,p}/p_{t,s} = 5$ (figure 2.9) the compound-sonic line accurately matches

the minimum pressure near $x/L = 0$. For $p_{t,p}/p_{t,s} = 6$ (figure 2.10), a Fabri-sonic pocket forms, followed downstream by a normal shock (the classical Y-pattern [17,28,55,56]). Increasing inlet pressure ratio therefore favours the Fabri pattern, most likely due to the stronger under-expansion of the primary jet, which tightens the secondary passage. Note, however, that the flow remains compound-sonic; the compound-sonic point lies farther upstream.

The discrepancies near the critical point are more challenging to interpret. The flow field at the entrance of the mixing pipe is highly complex: as back pressure increases, the diffuser shock train moves upstream and begins interacting with the shock system attached to the primary nozzle, itself sensitive to the degree of under- or over-expansion. Small changes in back pressure can thus produce large differences in the flow topology, as seen for $p_b/p_{t,s} = 2.00$ and 2.05 in the $p_{t,p}/p_{t,s} = 6$ case (figure 2.10). In addition, (de-)choking dynamics play a role, and these combined effects are difficult to isolate. This region also represents a particular challenge for one-dimensional modelling, which likewise exhibits high sensitivity near the sonic point (chapter 7).

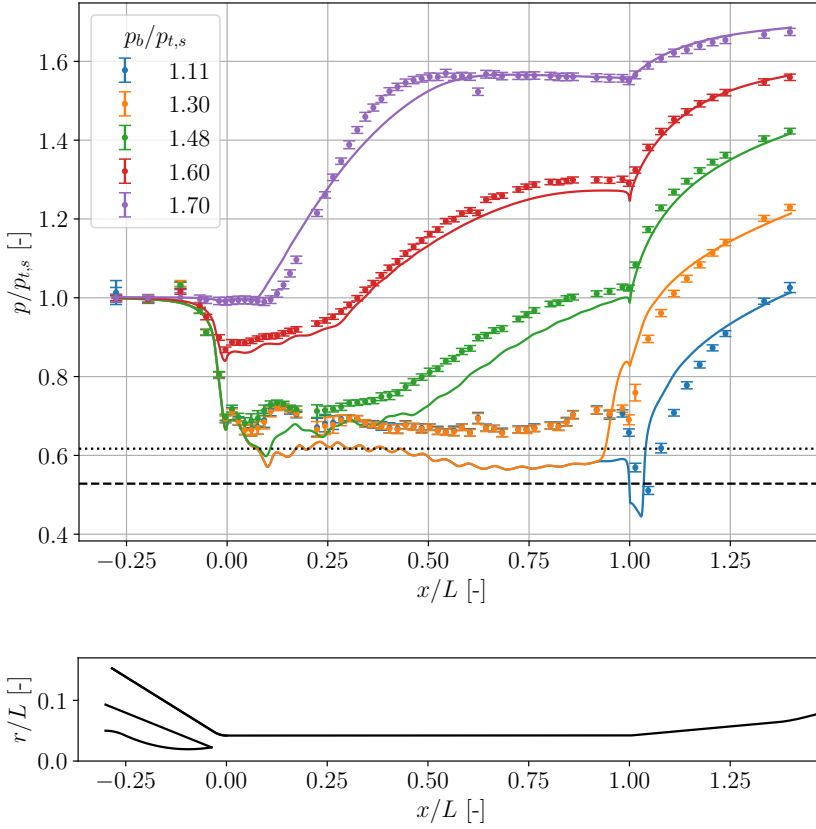


Fig. 2.8 Wall pressure distributions from experiments and RANS simulations on the ejector at UCLouvain, for $p_{t,p}/p_{t,s} = 4$. The dotted and dashed lines indicate the isentropic compound- and Fabri-sonic pressure ratios. Neither is reached experimentally.

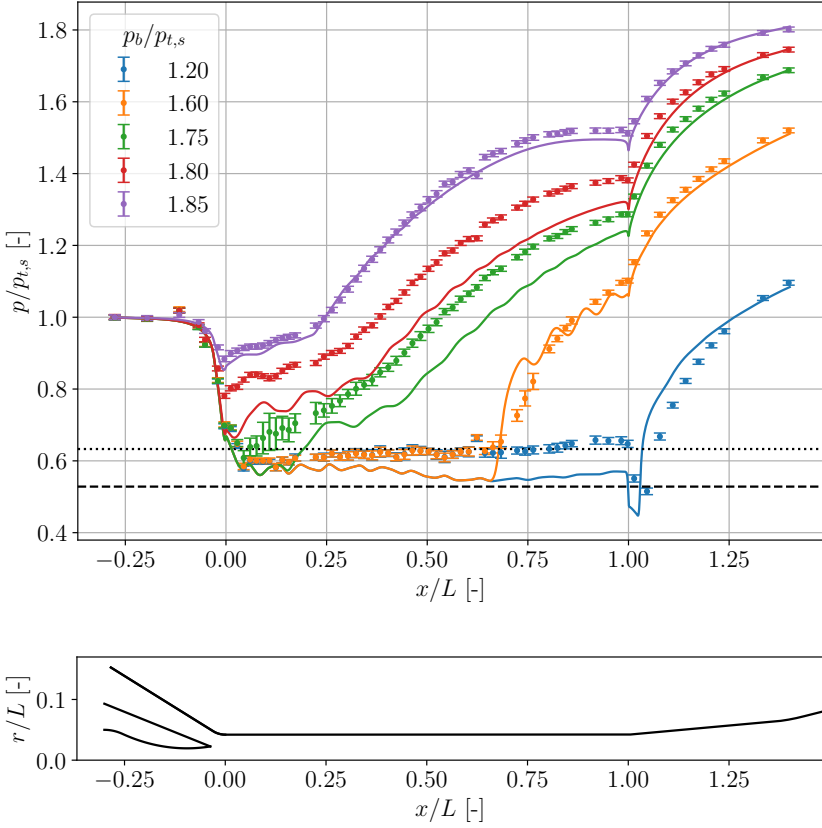


Fig. 2.9 Wall pressure distributions from experiments and RANS simulations on the ejector at UCLouvain, for $p_{t,p}/p_{t,s} = 5$. The dotted and dashed lines indicate the isentropic compound- and Fabri-sonic pressure ratios. The compound theory seems most adequate here since the minimum of the pressure distribution at the critical point $p_b/p_{t,s} = 1.75$ roughly corresponds to the compound line (see chapter 7).

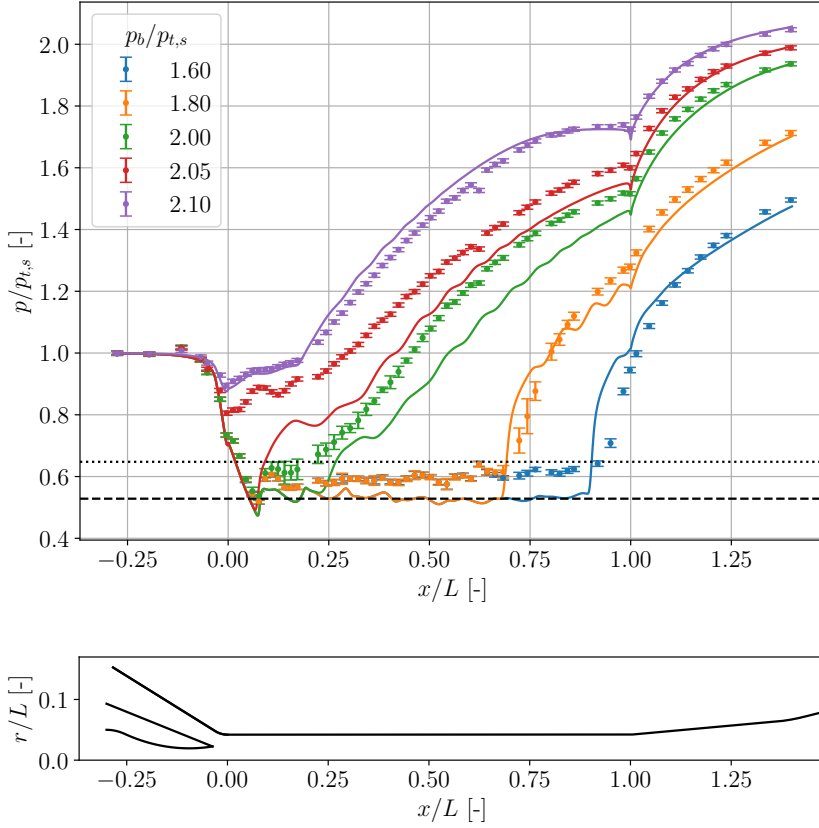


Fig. 2.10 Wall pressure distributions from experiments and RANS simulations on the ejector at UCLouvain, for $p_{t,p}/p_{t,s} = 6$. The dotted and dashed lines indicate the isentropic compound- and Fabri-sonic pressure ratios. A Fabri-sonic pocket is present in the secondary stream, which appears in the two-dimensional flow fields as a Y-shock (see for example Lamberts et al. [17]).

3

Data-driven methods for calibration

3.1 Introduction

This chapter presents a general framework for machine learning-based calibration of systems governed by differential equations. The objective is to establish a relation between closure parameters (e.g., friction coefficients) and the system state (e.g., Reynolds number).

Two training methods are considered: a *physics-separated* and a *physics-integrated* approach, following the classification by Chang and Dinh [67]. In the first case, the model is not involved during calibration and the closure relations are identified in a classic supervised fashion; this requires to construct a database with relevant physical quantities as inputs and reference closure coefficients as outputs. While the setup is relatively straightforward, extracting reliable reference coefficients is often non-trivial, especially from sparse experimental data. Moreover, the decoupling of the model during training does not guarantee physically consistent or numerically stable predictions once deployed.

In contrast, the physics-integrated approach includes the model during training, thereby overcoming the aforementioned limitations, though at the cost of increased complexity. In particular, the gradients of the model

predictions with respect to the closure coefficients are generally difficult to compute, especially in dynamical systems governed by ODEs or PDEs.

The adjoint method is an efficient tool to avoid computing these derivatives explicitly (see for example [68] in reinforcement twinning or [69,70] among many others for state-of-the-art turbulence modelling). Presented here for IVPs, this method provides a relatively simple expression for the gradient of the cost function with respect to the closure parameters through the backward integration of an adjoint system of equations. This allows gradient-based regression tools to exploit the sensitivity of model predictions with respect to closure parameters. A key advantage is that the computational cost is independent of the number of parameters (unlike approaches based on finite difference or forward sensitivities), making the method well-suited for parametric regression methods, such as Radial Basis Function (RBF) regression or Artificial Neural Networks (ANNs). To this end, a primer on these methods and on uncertainty quantification is also included in this chapter. In particular, it is shown that ANNs have strong potential as a convenient and efficient means of constructing surrogate models for ejectors.

Finally, the developed tools are demonstrated on two test cases. The first is an academic study in which a non-linear closure relation is learned for the advection speed in a one-dimensional Burgers-like setting. The second is a practical application on closure modelling for ejectors, where the friction forces in the mixing pipe are calibrated by sequentially applying the physics-separated and the physics-integrated approach [71].

3.2 General framework

We consider a physical problem stated as a Boundary Value Problem (BVP) in the state variables $\mathbf{u} \in \mathbb{R}^{n_u}$ and a set of parameters $\mathbf{p} \in \mathbb{R}^{n_p}$ to be linked to the state variables. In this work, we focus on boundary value problems in $x \in [0, L]$, and denote the sought-after closure relation as $\mathbf{p} = g(\mathbf{u}, x)$. Thus the closure problem can be written as

$$f\left(\mathbf{u}, \frac{d\mathbf{u}}{dx}, \frac{d^2\mathbf{u}}{dx^2}, \dots, x, \mathbf{p} = g(\mathbf{u}, x)\right) = 0, \quad (3.1)$$

with boundary conditions $\mathbf{u}(0) = \mathbf{u}_0$ and $\mathbf{u}(L) = \mathbf{u}_L$. This framework is general enough to encompass many physical systems, from turbulence

modelling [69, 70, 72, 73] to general inverse modelling [74]. The PDEs encountered in chapter 8 on quasi-two dimensional modelling also fit in this framework after projection on a basis along the radial coordinate r . In that case, the ODEs govern the axial dynamics of the coefficients related to the basis functions. This thesis presents the first implementation of this framework for the closure of ejector models. The problem of finding the unknown function is generally a difficult variational problem. Machine learning offers an alternative approach by approximating the unknown function using a parametric model which depends on a finite set of weights $\mathbf{w} \in \mathbb{R}^{n_w}$:

$$\mathbf{p} = g(\mathbf{u}, x; \mathbf{w}). \quad (3.2)$$

This could range from a simple linear relation to ANNs, and the range of possible functions that the given model can represent defines the ‘hypothesis set’ in the machine learning formalism [75]. Following [67], the identification of the weights $\mathbf{w}$ can be carried out using physics-separated and physics-integrated approaches (see also [76], and [69, 70, 72–74]).

The physics-separated approach is a classic supervised learning formulation which assumes that a set of instances $(\tilde{\mathbf{u}}_i, x_i)$ and the associated parameters $\tilde{\mathbf{p}}_i$ are available. Therefore, the optimal set of weights is the one that minimises a cost function $J(\mathbf{w})$ such as, for example:

$$J_1(\mathbf{w}) = \sum_i \sum_r \frac{(\tilde{\mathbf{p}}_{i,r} - g_r(\mathbf{u}_i, x_i; \mathbf{w}))^2}{\tilde{\mathbf{p}}_{i,r}^2}, \quad (3.3)$$

where i is the index spanning the instances $\tilde{\mathbf{p}}_i$ at coordinates x_i and r is the index spanning the entries in each vector of parameters (i.e. the closure coefficients that make up $\mathbf{p}$). The relative error is preferred here due to the composite nature of common state vectors (e.g., coefficients for friction or heat exchange). Note that the chosen formulation is but a single option; different normalisations or weighted sums are also viable options.

This regression problem is shown schematically in block II of figure 3.1 and starts with an initial guess $\mathbf{w}_0$. The notation $d_b a$ denotes the derivative, gradient or Jacobian of a with respect to b , where either can be a scalar or a vector (see also table 3.1). The gradient of the cost function $d_{\mathbf{w}} J$ only requires the gradient of the parametric function $d_{\mathbf{w}} g$; this is easily available for usual machine learning models (e.g. using backpropagation in ANNs), hence classic gradient-base optimisers such as the BFGS algorithm can be implemented efficiently. However, this approach requires that both the pa-

rameters $\tilde{p}_i$ and the full states $\tilde{u}_i$ can be extracted from a sufficiently rich dataset with sufficient accuracy (cf. block I in figure 3.1). This is rarely possible from experimental data. Moreover, the trained model g is unaware of the underlying physical problem in (3.1) since the model is never called during training.

The physics-integrated approach includes the physical problem in (3.1) during the learning process (cf. block III in figure 3.1). Note that both approaches are self-standing, but can be applied successively. The extraction of the model parameters from data is not required, and the training is based on some observation of the states and the prediction that the model f can achieve for a given set of weights w . Considering the simplest case of full observation of the states $\tilde{u}_i$ and denoting the solution of (3.1) as $u_i(g(u_i, x_i; w))$ when the closure is achieved with the weights w , the optimal set of weights minimises a cost function of the form

$$J_2(w) = \sum_i \sum_r \frac{(\tilde{u}_{i,r} - u_{i,r}(g(u_i, x_i; w)))^2}{\tilde{u}_{i,r}^2} \quad (3.4)$$

where i is the index spanning the instances $\tilde{u}_i$ at coordinates x_i and r is the index spanning the entries in each state vector (i.e. the variables considered for the cost function definition).

Using the chain rule, it is easy to see that in this formulation, the gradient of the cost function $d_w J$ requires the gradient of the problem solution with respect to the parameters and the gradient of the parameters with respect to the weights, i.e. $d_w J(u(g(w))) = d_u J d_p u d_w g$. Computing this gradient symbolically is challenging and generally requires adjoint differentiation (cf. [69, 70, 72–74]). Besides avoiding the need for extracting the parameters $\tilde{p}$ from data, this approach has the main merit of pairing the closure with the solver with which it is later deployed. On the other hand, the model sensitivity to the parameters ($d_p u$) strongly influences the cost function gradient and may suffer from vanishing gradients in regions where the sensitivity is small [77].

A combination of the two approaches is feasible if the dataset is sufficiently rich, whereby the solution of a physics-separated approach is used as a starting point for a physics-integrated approach (hence successively following blocks I to III in figure 3.1). This offers an excellent balance between accuracy, robustness and computational cost.

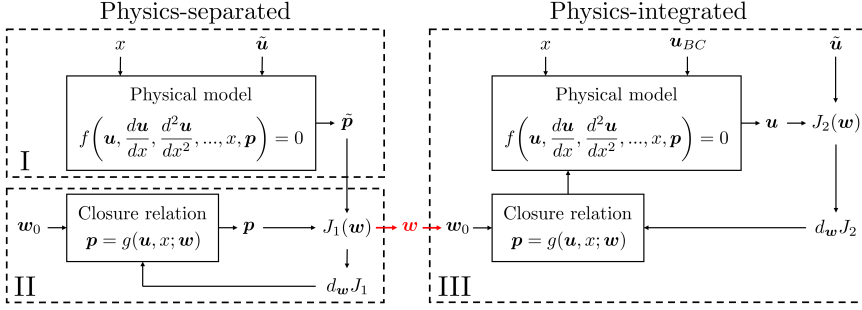


Fig. 3.1 Schematic overview of the physics-separated and the physics-integrated approach for calibrating a physical model f through a closure relation g , given data $\tilde{u}$. The physics-separated approach consists of (I) extracting the reference parameters $\tilde{p}$ from data and (II) applying a classic supervised machine learning technique for regressing p . The physics-integrated approach (III) includes the physical model in the training process and thus requires the sensitivity of the model with respect to its parameters $d_p u$. Both methods are self-standing, but can be applied successively (I-III) as indicated in red.

3.3 The adjoint method

3.3.1 Definitions

In this section, the adjoint equations are derived for first order IVPs (the interested reader is referred to [78] for a complete tutorial). The practical equations in chapters 6, 7 and 8 of the general BVP formulation (3.1) are solved with a shooting method from a guessed state u_0 at a given position to the outlet of the domain, delimited here without loss of generality between $x = 0$ and $x = L$. The adjoint equations are therefore derived here as an IVP. Note that the shooting method is not required for the adjoint solution, only the final solution is required. The system dynamics are governed by a function f , which returns the first derivative of the state in the continuous setting or the state at the next grid point in the discrete setting:

$$\begin{cases} \frac{du}{dx} = f_c(u, x, p) \\ p = g_c(u, x; w) \end{cases} \quad \text{or} \quad \begin{cases} u^{(n+1)} = f_d(u^{(n)}, x^{(n)}, p^{(n)}) \\ p^{(n)} = g_d(u^{(n)}, x^{(n)}; w) \end{cases}, \quad (3.5)$$

where $u \in \mathbb{R}^{n_u}$ denotes the state vector, x denotes the axial coordinate, $p \in \mathbb{R}^{n_p}$ is a vector of closure parameters that depends on the state, the

coordinate x and tunable weights $\mathbf{w} \in \mathbb{R}^{n_w}$ through the function g . This function is for example embodied by an artificial neural network or radial basis functions as defined in section 3.4.

The goal is to minimise the discrepancy between the predicted state $\mathbf{u}$ and observations $\tilde{\mathbf{z}} = h(\tilde{\mathbf{u}})$ of the reference state. These are potentially different quantities or the state may not be fully observable. The error is quantified through a cost function J that returns a scalar value as in equation (3.4). Vectors are denoted here as rows. Table 3.1 lists the dimensions of all the required derivatives in the sections below.

Table 3.1 Dimensions of the gradients and the Jacobians. Gradients are defined as row vectors. Rows in the Jacobians represent the gradient of a single component.

gradient	sensitivity of the ...	with respect to the ...	dimensions
$d_{\mathbf{w}}J$	cost	weights	n_w
$d_{\mathbf{w}}\mathbf{u}$	state	weights	$n_u \times n_w$
$d_{\mathbf{w}}f$	dynamics	weights	$n_u \times n_w$
$d_{\mathbf{w}}\mathbf{p}$	parameters	weights	$n_p \times n_w$
$d_{\mathbf{w}}g$	parameters	weights	$n_p \times n_w$
$d_{\mathbf{z}}J$	cost	observations	n_z
$d_{\mathbf{u}}\mathbf{z}$	observations	state	$n_z \times n_u$
$d_{\mathbf{u}}f$	dynamics	state	$n_u \times n_u$
$d_{\mathbf{u}}\mathbf{p}$	parameters	state	$n_p \times n_u$
$d_{\mathbf{p}}f$	dynamics	parameters	$n_u \times n_p$

3.3.2 The continuous adjoint method for IVPs

We consider the continuous system of equations in (3.5). The cost function J penalises the prediction error between the ‘numerical’ observation $\mathbf{z} = h(\mathbf{u})$ and the reference $\tilde{\mathbf{z}}$ through a function e . A classic choice consists of the summed squared error, where the indicator function indicates at which points the system is observed (e.g. the positions of sensors):

$$J(\mathbf{w}) = \int_0^L e(\mathbf{z}, \tilde{\mathbf{z}}) dx = \int_0^L \mathbb{1}_Z(x) \|\mathbf{h}(\mathbf{u}(x)) - \tilde{\mathbf{z}}(x)\|_2^2 dx. \quad (3.6)$$

The gradient with respect to the weights is given by the following equation:

$$\frac{dJ}{d\mathbf{w}} = \int_0^L \frac{de}{dz} \frac{dz}{d\mathbf{u}} \frac{d\mathbf{u}}{d\mathbf{w}} dx = 2 \int_0^L \mathbb{1}_Z(\mathbf{z} - \tilde{\mathbf{z}}) \frac{dz}{d\mathbf{u}} \frac{d\mathbf{u}}{d\mathbf{w}} dx, \quad (3.7)$$

omitting the explicit dependence on x . The dimensions of each term can be found in table 3.1. Their product matches the dimensions of the gradient, which is a vector with n_w components. The gradient of the state $\mathbf{u}$ with respect to the weights $\mathbf{w}$ is difficult to compute, since the state carries a history which was also affected by the weights. For example, the state at $x = L$ has been affected by the weights at all points during the integration from $x = 0$. We therefore introduce the adjoint variable $\boldsymbol{\lambda}(x) \in \mathbb{R}^{n_u}$ in the form of a Lagrangian multiplier:

$$\mathcal{L}(\mathbf{w}) = J(\mathbf{w}) + \int_0^L \boldsymbol{\lambda}(x) \left[\frac{d\mathbf{u}}{dx} - f(\mathbf{u}, x, \mathbf{p}) \right] dx. \quad (3.8)$$

The added term always equals zero by definition. This implies that the Lagrangian in the equation above, and its gradient (!), are identical to the original cost function for *any* adjoint variable $\boldsymbol{\lambda}(x)$. A clever choice of $\boldsymbol{\lambda}$ allows to eliminate the problematic derivative of the state with respect to the weights, which forms the backbone of the adjoint method. We first derive the Lagrangian above with respect to the weights and omit the arguments of the functions for conciseness:

$$\frac{d\mathcal{L}}{d\mathbf{w}} = \frac{dJ}{d\mathbf{w}} + \int_0^L \boldsymbol{\lambda} \frac{d^2\mathbf{u}}{dx d\mathbf{w}} dx - \int_0^L \boldsymbol{\lambda} \left[\left(\frac{\partial f}{\partial \mathbf{u}} + \frac{\partial f}{\partial \mathbf{p}} \frac{\partial \mathbf{p}}{\partial \mathbf{u}} \right) \frac{d\mathbf{u}}{d\mathbf{w}} + \frac{\partial f}{\partial \mathbf{p}} \frac{\partial \mathbf{p}}{\partial \mathbf{w}} \right] dx. \quad (3.9)$$

Integrating by parts and reshuffling leads to the following equation:

$$\begin{aligned} \frac{d\mathcal{L}}{d\mathbf{w}} = & \int_0^L \left[\frac{d\boldsymbol{\lambda}}{dx} \frac{d\mathbf{u}}{dx} - \boldsymbol{\lambda} \left(\frac{\partial f}{\partial \mathbf{u}} + \frac{\partial f}{\partial \mathbf{p}} \frac{\partial \mathbf{p}}{\partial \mathbf{u}} \right) \right] \frac{d\mathbf{u}}{d\mathbf{w}} dx \\ & + \boldsymbol{\lambda}(L) \frac{d\mathbf{u}}{d\mathbf{w}}(L) - \boldsymbol{\lambda}(0) \frac{d\mathbf{u}}{d\mathbf{w}}(0) - \int_0^L \boldsymbol{\lambda} \frac{\partial f}{\partial \mathbf{p}} \frac{\partial \mathbf{p}}{\partial \mathbf{w}} dx. \end{aligned} \quad (3.10)$$

The terms including the problematic derivative have been grouped and are eliminated if the adjoint variable $\boldsymbol{\lambda}$ respects the following equation:

$$\frac{d\boldsymbol{\lambda}}{dx} = \frac{d\boldsymbol{\lambda}}{dz} \frac{dz}{d\mathbf{u}} - \boldsymbol{\lambda} \left(\frac{\partial f}{\partial \mathbf{u}} + \frac{\partial f}{\partial \mathbf{p}} \frac{\partial \mathbf{p}}{\partial \mathbf{u}} \right). \quad (3.11)$$

with a boundary condition at the point $x = L$:

$$\boldsymbol{\lambda}(L) = 0. \quad (3.12)$$

Note that the derivative du/dw at $x = 0$ equals zero in this setting since the initial condition is specified in the physical problem. The final expression for the gradient becomes:

$$\frac{dJ}{dw} = \frac{d\mathcal{L}}{dw} = - \int_0^L \lambda \frac{\partial f}{\partial \mathbf{p}} \frac{\partial \mathbf{p}}{\partial w} dx. \quad (3.13)$$

Generally, observations are available at distinct points and not as continuous functions, as illustrated in the proposed cost function with the use of the indicator function. This function leaves a set of dirac pulses at the observation points and generates a sudden jump in the adjoint variable λ during the integration. It is important to note that the integration is carried out backwards in space, so the integral of the dirac pulse equals -1. Practically, one should subtract $dze \, d_u z$ from the adjoint variables λ at each observation point. The main advantage of the continuous adjoint method lies in its independence of the numerical method used to integrate the system of equations.

3.3.3 The discrete adjoint method for IVPs

The derivation of adjoint system in the discrete setting of (3.5) follows the same steps. The integral in the cost function is replaced by a summation:

$$\mathcal{L}(w) = \frac{1}{2} \sum_{n=0}^{n_x} \mathbb{1}_Z(n) \|z^n - \tilde{z}^n\|_2^2 + \sum_{n=0}^{n_x-1} \lambda^n (u^{n+1} - f_d(u^n, x^n, \mathbf{p}^n)), \quad (3.14)$$

where $z^n = h(u^n)$ denotes the numerical observation at the discrete grid point n and the sum is carried over $n_x + 1$ grid points. The gradient reads:

$$\frac{d\mathcal{L}}{dw} = \sum_{n=0}^{n_x} \mathbb{1}_Z(n) (z^n - \tilde{z}^n) \frac{dz^n}{du^n} \frac{du^n}{dw} + \sum_{n=0}^{n_x-1} \lambda^n \frac{du^{n+1}}{dw} \quad (3.15)$$

$$- \sum_{n=0}^{n_x-1} \lambda^n \left(\left(\frac{\partial f_d}{\partial u^n} + \frac{\partial f_d}{\partial \mathbf{p}^n} \frac{\partial \mathbf{p}^n}{\partial u^n} \right) \frac{du^n}{dw} + \frac{\partial f_d}{\partial \mathbf{p}^n} \frac{\partial \mathbf{p}^n}{\partial w} \right) \quad (3.16)$$

Similarly as before, the term du^n/dw is burdensome since it carries the history of the numerical integration. It is eliminate here as well by grouping these terms and by formulating an adjoint system of equations. The integration by parts is replaced by a change of indices in the second term,

eventually revealing the following adjoint system:

$$\lambda^{n-1} = \lambda^n \left(\frac{\partial f_d}{\partial \mathbf{u}^n} + \frac{\partial f_d}{\partial \mathbf{p}^n} \frac{\partial \mathbf{p}^n}{\partial \mathbf{u}^n} \right) - \mathbb{1}_Z(n)(\mathbf{z}^n - \tilde{\mathbf{z}}^n) \frac{d\mathbf{z}^n}{d\mathbf{u}^n}, \quad (3.17)$$

with the following boundary condition:

$$\lambda^{n_x-1} = -\mathbb{1}_Z(n_x)(\mathbf{z}^{n_x} - \tilde{\mathbf{z}}^{n_x}) \frac{d\mathbf{z}^{n_x}}{d\mathbf{u}^{n_x}}. \quad (3.18)$$

Note that the condition above corresponds to the source term from the full equation at the final grid point. The remaining equation for the gradient is given by:

$$d_w J = - \sum_{n=0}^{n_x-1} \lambda^n \frac{\partial f_d}{\partial \mathbf{p}^n} \frac{\partial \mathbf{p}^n}{\partial \mathbf{w}}. \quad (3.19)$$

3.4 Regression tools

3.4.1 Radial Basis Functions

Regression with Radial Basis Functions (RBFs) is a linear method [79], the closure relation g being linear with respect to the parameters $\mathbf{w}$. Considering the case of a single input and single output, the prediction $\mathbf{y} \in \mathbb{R}^{N_p}$ for N_p inputs $\mathbf{x} \in \mathbb{R}^{N_p}$ consists of a weighted sum of N_w basis functions, which can be written as the following matrix multiplication:

$$\mathbf{y} = \tilde{f}(\mathbf{x}; \mathbf{w}) = \begin{bmatrix} \phi_1(\mathbf{x}) & \phi_2(\mathbf{x}) & \dots & \phi_N(\mathbf{x}) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} = \Phi(\mathbf{x})\mathbf{w} \quad (3.20)$$

where $\Phi(\mathbf{x}) \in \mathbb{R}^{N_p \times N_w}$ is the feature matrix for the basis functions ϕ_i evaluated at the points $\mathbf{x}$. The prediction is thus a linear combination of the columns of $\Phi(\mathbf{x}_{**})$, which acts as a basis. In the case of RBF regressions, the basis functions are *radial*, in the sense that they solely depend on the distance from the points in which they are collocated. For example, Gaussian RBFs are defined as

$$\phi_k(\mathbf{x}; \mathbf{x}_{c,k}, \sigma_k) = \exp \left(-\sigma_k^2 \|\mathbf{x} - \mathbf{x}_{c,k}\|^2 \right), \quad (3.21)$$

where $\mathbf{x} \in \mathbb{R}^{N_x}$ is the coordinate vector in an N_x -dimensional input space, $\mathbf{x}_{c,k}$ is the k -th collocation point and σ_k the shape parameter of the basis, defining its width. An example of a RBF with compact support is the following C^4 -continuous function:

$$\phi_k(\mathbf{x}; \mathbf{x}_{c,k}, \sigma_k) = \begin{cases} \left(1 + \frac{d_k}{\sigma_k}\right)^5 \left(1 - \frac{d_k}{\sigma_k}\right)^5 & \text{if } d_k \leq \sigma_k, \\ 0 & \text{otherwise,} \end{cases} \quad (3.22)$$

where d_k denotes the distance with respect to the k^{th} collocation point:

$$d_k = \|\mathbf{x} - \mathbf{x}_{c,k}\|. \quad (3.23)$$

The definition of the basis therefore requires defining the matrix $\mathbf{X}_c \in \mathbb{R}^{N_w \times N_x}$ collecting the collocation points of all bases and the vector $\boldsymbol{\sigma} \in \mathbb{R}^{N_w}$ collecting their shape factors.

Linearity makes the training of these models particularly simple. In the case of classic quadratic cost functions such as the mean squared error or its l_2 penalised version (Ridge regression), an analytic solution for the optimal set of parameters can be derived. Otherwise, numerical optimisation of the weights $\mathbf{w}$ takes the form of a gradient-based approach since the gradient of the model prediction with respect to the parameters is simply the feature matrix itself $d_{\mathbf{w}} \tilde{f}(\mathbf{x}) = \boldsymbol{\Phi}(\mathbf{x})$. These methods should always be prioritised in relatively low dimensional problems and when a careful craft of the basis feasible. Nonlinear methods, such as neural networks, can have a larger model capacity (i.e. be able to represent more complex functions) while requiring less engineering in the model formulation, but are generally far more difficult to train.

3.4.2 Artificial Neural Networks

Definition

Artificial Neural Networks (ANNs) are tools to encode a mapping $\mathbf{y} = f(\mathbf{x})$, with $\mathbf{x} \in \mathbb{R}^{n_x}$ and $\mathbf{y} \in \mathbb{R}^{n_y}$ using a dataset $\{\mathbf{x}_*, \mathbf{y}_*\}$ for training and a dataset $\{\mathbf{x}_{**}, \mathbf{y}_{**}\}$ for validation.

The encoding is done in the form of a parametric function which we denote as $\tilde{f}(\mathbf{x}; \mathbf{w})$, where the parameters $\mathbf{w}$ must be tuned to find an optimal mapping between inputs and outputs. This parametric function is built as a distributed architecture consisting of many simply connected units

(called neurons), organised in layers [80]. More specifically, the mapping from input to output takes the following recursive form:

$$\mathbf{y} = \tilde{f}(\mathbf{x}; \mathbf{w}) = \mathbf{a}^{(L)}, \quad \text{with} \quad \begin{cases} \mathbf{a}^{(l)} = \sigma^{(l)}(\mathbf{z}^{(l)}), \\ \mathbf{z}^{(l)} = \mathbf{W}^{(l-1)}\mathbf{a}^{(l-1)} + \mathbf{b}^{(l)}, \end{cases} \quad (l = 2, \dots, L)$$

and $\mathbf{a}^{(1)} = \mathbf{x}.$ (3.24)

Here $\mathbf{a}^{(l)}, \mathbf{z}^{(l)}, \mathbf{b}^{(l)} \in \mathbb{R}^{n_l}$ are the *output vector*, *activation vector* and the *bias vector* of layer l , composed of n_l neurons, $\sigma^{(l)}$ is the activation function in each layer, $\mathbf{W}^{(l-1)} \in \mathbb{R}^{n_l \times n_{l-1}}$ is the matrix containing the weights connecting layer $l-1$ with layer l , and L is the number of layers. The input x is considered as the output $\mathbf{a}^{(1)}$ of the first layer, which therefore has no corresponding bias vector. The parameter vector $\mathbf{w} \in \mathbb{R}^{n_w}$ collects all the weights and biases across the network, hence $n_w = (n_L n_{L-1} + n_{L-2} n_{L-3} + \dots n_2 n_1) + (n_L + n_{L-1} + \dots n_2)$. The activation functions are nonlinear functions such as hyperbolic tangents. The zoology of activation functions counts dozens of possibilities and the reader is referred to [80] for a concise introduction.

Uncertainty calculation via Ensemble Networks

The uncertainty of any surrogate model can be split into two categories: the aleatory uncertainty and the model (epistemic) uncertainty [81]. The aleatory is due to unpredictable (random) noise in the data and can not be reduced by introducing more data or increasing the model complexity. On the other hand, the epistemic uncertainty is systematic: it exists because the model is not capturing some physical process or conditions.

For neural networks, an essential source of model uncertainty is due to the stochastic computation of its parameters. These are uncertain because multiple minima exist [82] in the minimisation of the cost function and because of the stochastic training algorithm. As a result, training the same network twice leads to different final weights and predictions.

In this work we use an ensemble learning approach (see also [83] and [84]) to compute the epistemic uncertainties due to the parameters. This approach consist in training an ensemble of n_E versions of the same ANN. Then, interrogating the full ensemble gives, for each input, a distribution of outputs with a mean and a standard deviation:

$$\mu_E(\mathbf{x}) = \mathbb{E}_E \{f_i(\mathbf{x}; \mathbf{w}_i)\} , \quad (3.25)$$

$$\sigma_e^2(\mathbf{x}) = \mathbb{E}_E \left\{ (f_i(\mathbf{x}; \mathbf{w}_i) - \mu_E(\mathbf{x}))^2 \right\} . \quad (3.26)$$

where $\mathbb{E}_E$ denotes the expectation operator on the ensemble of networks, and $f_i(\mathbf{x}; \mathbf{w}_i)$ denotes the prediction of the i^{th} ANN, receiving an input vector $\mathbf{x}$ and depending on its own set of weights $\mathbf{w}_i$. The aleatory uncertainty is assumed to be uniformly distributed over the inputs and is estimated from the ensemble mean of the Root Mean Square Error (RMSE):

$$\sigma_a = \mathbb{E}_E \{RMSE_i\} . \quad (3.27)$$

Finally, assuming the aleatory and epistemic uncertainties are independent, the total uncertainty can be estimated as follows:

$$\sigma_E(\mathbf{x}) = \sqrt{\sigma_e^2(\mathbf{x}) + \sigma_a^2} . \quad (3.28)$$

3.5 ANNs as surrogate models for ejectors

In the context of the Ejemod project¹, a robust and computationally inexpensive global ejector model was required for system-scale simulations. Zero-dimensional ejector models [32, 46, 47] satisfy these constraints, but their validity range is limited and they rely on the calibration of multiple engineering parameters (e.g., isentropic efficiencies). These challenges have driven recent developments in surrogate modelling, including polynomial regression [85] and Gaussian processes [24].

In this work, the following regression task is addressed using artificial neural networks (ANNs):

$$\hat{m}_s = f_s \left(\frac{p_b}{p_{t,p}}, \frac{p_{t,s}}{p_{t,p}}, \frac{A_{th}}{A_m} \right) , \quad (3.29)$$

i.e., the ANN predicts the normalised secondary mass flow rate (see equation (2.7)) as a function of two pressure ratios and the area ratio between the primary throat and the mixing pipe. Although this is a low-dimensional problem that could in principle be approximated without ANNs, the data

¹<https://cleansky-ejemod.eu/>

exhibit distinct operating regimes (cf. figure 1.2 from chapter 1): a constant entrainment at low back pressure and a sharp corner at the critical point. ANN activation functions capture such piecewise behaviour efficiently, whereas for example RBF models may struggle to represent constant regions due to their local nature. The aim is not to outperform physics-based formulations in accuracy, but to build a more robust surrogate. ANN evaluation requires only evaluating a differentiable function: no numerical integration or regime-dependent branching (if/else logic) is required.

This work focuses on providing a proof of concept for the ANN ejector model, and the training dataset was generated using the 0D model from Metsue et al. [46]. A total of 10,000 operating points were computed, fixing the mixing-pipe diameter to $D_m = 52$ mm and varying the primary-nozzle diameter between $0.5 D_m$ and $0.8 D_m$. Operating conditions were sampled from uniform distributions within the ranges defined in the Ejemod proposal. The following efficiencies were assumed in all simulations: $\eta_p = 0.97$, $\eta_{py} = 0.93$, $\eta_s = 0.85$, $\eta_m = 0.95$ and $\eta_d = 0.97$. Although these values may vary across ejector designs and operating regimes, their effect is implicitly captured during training. This does not limit the generality of the approach, as the surrogate model is intended to represent a specific ejector configuration.

The ANN consists of three hidden layers with 16 neurons each. These hyperparameters were selected to balance prediction accuracy and network complexity [65]. The network is implemented in Tensorflow [86], employing SELU activations and LeCun-normal weight initialisation. The loss function is the root mean squared error, optimised using Nadam with a batch size of 32. The learning rate starts at 0.001 and is reduced by a factor of two when improvements stagnate (patience of 100 epochs, minimum delta of 0.001). The dataset is split into 80 % for training and 20 % for validation, and early stopping is applied with a patience of 100 epochs.

An ensemble of 65 independently trained networks was constructed following Section 3.4.2. The resulting ensemble mean prediction and 95 % confidence interval are shown in figure 3.2. Owing to the ensemble averaging, the predictive spread is low, with a mean uncertainty of approximately 3.5%, suggesting that the regression problem is well-posed. The function has indeed been reduced and normalised to a maximum extent, including as much prior physical knowledge as possible. This considerably reduces the workload for the ANN.

The relatively small network ($n_L = 3$ layers, $n_l = 16$ neurons each) predicts the normalised secondary mass flow rate with an average uncer-

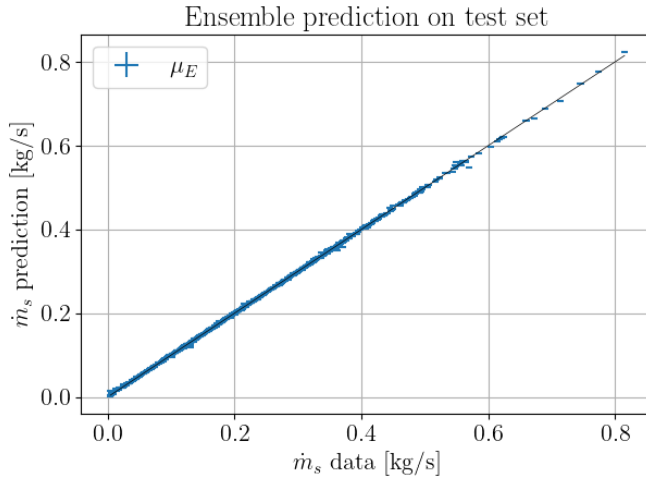


Fig. 3.2 Ensemble prediction of the secondary mass flow rate with 95 % confidence interval.

tainty of only 3.5 % over the operating domain relevant to Ejemod. Given its accuracy, robustness, and differentiability, the ANN surrogate provides a valuable tool for design optimisation and system-level integration.

Although the present work focuses on conventional operating regimes (where physics-based models are available), the same methodology could be extended to more complex scenarios (e.g., reverse flow or two-phase regimes) where traditional models face difficulties. In such cases, data-driven surrogates may provide an attractive alternative.

3.6 Practical application of the methods

The tools from the sections above are applied on two test cases: a one-dimensional advection equation where the closure relation for the advection speed is learnt using the adjoint method and a neural network; and the calibration of the friction forces in the mixing pipe of an ejector using the framework of section 3.2.

3.6.1 A non-linear advection equation

The test case consists of the following non-linear advection equation:

$$\begin{cases} \partial_t u + a \partial_x u = 0, \\ a = g(u; \mathbf{w}), \end{cases} \quad (3.30)$$

on the domain $(x, t) \in [0, 1] \times [0, 0.5]$, with periodic boundary conditions and with the following initial state:

$$u_0(x) = 0.6 + 0.3 \sin(2\pi x). \quad (3.31)$$

The space derivative is discretised using a first order upwind scheme on a uniform grid with 50 points. The PDE above has thus been semi-discretised into a system of 50 ODEs. As such, it fits in the general framework presented in section 3.3. The advection speed a is prescribed by a neural network with input u and weights $\mathbf{w}$. The neural network contains two hidden layers with ten nodes and ELU activation functions [87]. The output node has a sigmoid activation function to normalise the output between zero and one, hence avoiding stability issues during training with a reference speed of 1 and a CFL number of 0.8.

The reference solution for the advection speed is given by the following equation:

$$a_{ref} = 0.5 + 0.4 \sin(2\pi u). \quad (3.32)$$

This is deliberately bounded between 0.1 and 0.9, allowing the neural network to approximate this function with its normalised output. This entails no loss of generalisation, since different problems can be normalised with appropriate scaling. The reference solution $\tilde{u} \in \mathbb{R}^{50}$ in the discrete grid points is found by integrating the system of equations above with the chosen discretisation, from time $t = 0$ to $t = 0.5$ using an explicit first order scheme and a CFL number of 0.8. The cost function penalises the error on the solution u at the final time $t = 0.5$:

$$J(\mathbf{w}) = \frac{1}{2} \|\mathbf{u} - \tilde{\mathbf{u}}\|_2^2. \quad (3.33)$$

The neural network is therefore not trained in the classic supervised fashion with a database of inputs u and outputs a , as in the physics-separated approach (cf. section 3.2), but from observations of the physical variable u linked to the network through a system of ODEs. The modest model

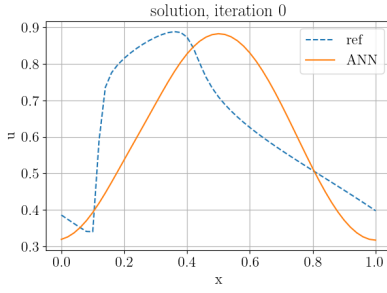
describe above already contains 141 weights, illustrating why the adjoint method is the tool of choice for this type of problem. The network is initialised randomly and trained with the BFGS algorithm [88], available in the SciPy minimize package [89]. In particular, the gradient is computed using the discrete adjoint method from section 3.3.3 and provided to the optimiser through the ‘jac’ argument of SciPy’s minimize function.

Figure 3.3 shows the solution at the final time step (left) and the closure relation predicted by the neural network (right) as the training progresses. After ten iterations the neural network learnt to lower the advection speeds at high values of u . The main contribution to the cost at the initial time is indeed an overshoot of the peak between $x = 0.1$ and $x = 0.5$. The algorithm converged after 141 iterations, yielding a solution that is indistinguishable from the reference. This outcome is consistent with expectations, given the relatively simple test case and the absence of noise. Nevertheless, it is worth emphasising that the neural network was not provided with the reference closure relation during training; instead, it successfully inferred a suitable functional representation solely by observing the physical state during the integration of the system of ODEs. Similar to standard neural networks, it performs poorly when extrapolating: since values of u below 0.3 never occurred during training, the network did not learn any behaviour in this region.

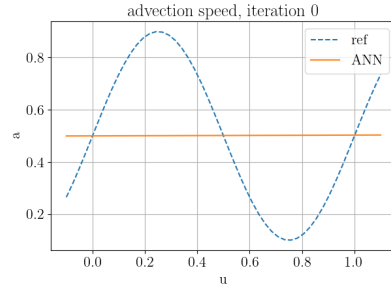
This simple test case provided a first proof of concept of the adjoint-based regression tool to infer closure relations in IVPs from observations of the system’s physical state. This scenario is particularly relevant for the one-dimensional and quasi-two dimensional modelling of ejectors (see chapters 6, 7 and 8), which are mathematically similar and rely on closure relations for friction coefficients or for the eddy viscosity. For the one-dimensional models, existing correlations significantly reduced the need for data-driven closure, which was initially not foreseen at the start of this thesis. Nevertheless, these tools play a crucial role in the quasi-two dimensional model discussed in chapter 8.

3.6.2 Friction in the mixing pipe

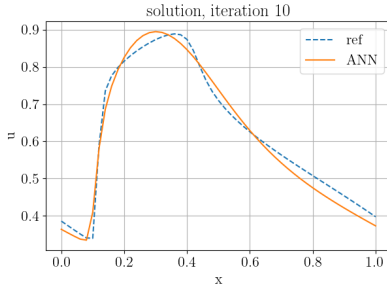
The model calibration framework introduced in section 3.2 is applied to the modelling of a supersonic ejector using a 1D formulation with two streams (cf. figure 1.4c). The calibration is based on data from a 2D axisymmetric RANS simulation of the VKI ejector (see section 2.3) with the convergent primary nozzle [27,51]. These results were presented in [71] and stem from



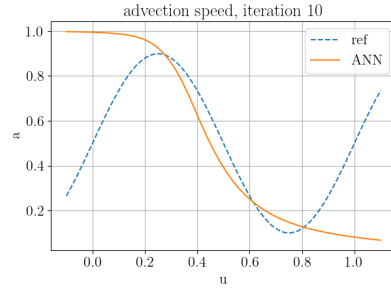
(a) Solution at iteration 0



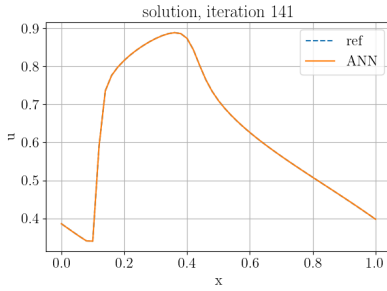
(b) Advection speed at iteration 0



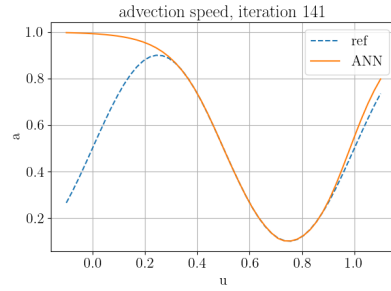
(c) Solution at iteration 10



(d) Advection speed at iteration 10



(e) Solution at iteration 141



(f) Advection speed at iteration 141

Fig. 3.3 Training history (top-bottom) of a non-linear advection equation where the speed (right) is prescribed by a neural network (cf. equation (3.30)). The error on the solution at the final time step (left) is penalised through a cost function, which is minimised with an adjoint method for the enhanced physical system. The network manages to learn a sinusoidal function indirectly through the physical system.

an early version of the ejector model presented in chapter 7.

The CFD Dataset

We consider a single operating point with inlet pressures $p_{t,p} = 5$ bar, $p_{t,s} = 1$ bar and back pressure $p_b = 2.2$ bar. The total temperatures equal $T_{t,p} = T_{t,s} = 293$ K. The numerical domain and a contour of the Mach number field are shown in figure 3.4. The primary jet is under-expanded due to the large pressure difference between the inlets, which leads to a shock train in the mixing pipe. This is evidenced by the dividing streamline as indicated by the full red line in figure 3.4. The ejector operates in off-design conditions, so the maximal flow rate is not reached, and the mixed flow remains subsonic. Consequently, no shock train is present in the diffuser as would be the case in on-design conditions [9]. Interested readers are referred to [27, 51] for further numerical details and the validation against experiments.

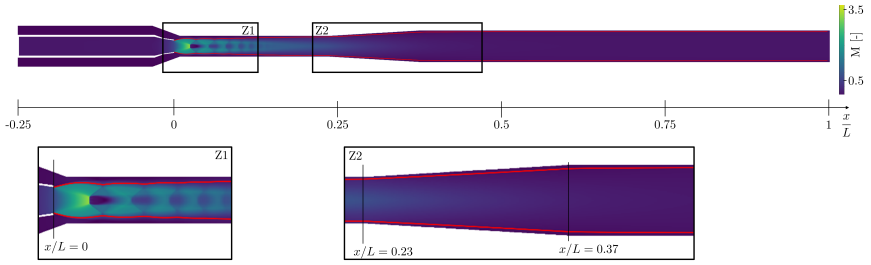


Fig. 3.4 Mirrored Mach number field of an ejector operating in off-design conditions, obtained through an axisymmetric RANS simulation [27, 51]. The red line indicates the dividing streamline.

The data assimilation in this work focuses on the mixing pipe since the primary and secondary inlets can be accurately described with classic quasi-1D flow theory. The mixing pipe features complex flow phenomena, such as the shear layer and shock trains. This portion of the ejector is the one where closure relations are required the most.

The data are averaged over the two streams as reported in detail in section 2.2.2. The final dataset consists of the cross-sections A_i , and the cross-stream averaged static pressures p_i , total pressures $p_{t,i}$ and total temperatures $T_{t,i}$ at 1530 spatial coordinates x of the single CFD simulation shown in figure 3.4. Hence, the available data is locally rich, but limited to a single geometry and a single operating condition. Therefore, the re-

sulting calibration can not be expected to generalise to different operating conditions.

Model definition

The mixing pipe is modelled as a 1D domain with two streams that exchange momentum through shear and with the wall. The division of the complete section between the two streams introduces a degree of freedom, e.g., the same primary mass flow rate can pass through a narrow or a wide flow passage, with the secondary cross-section adapting accordingly. Therefore, the static pressure is assumed to be equal in both streams as an additional constraint. This set of equations is known as the compound flow theory, originally proposed by Bernstein [45] and adopted later in ejector modelling [19,23,35,46]. The set of ODEs consists of the following governing equations:

$$\begin{aligned} \frac{1}{p_{t,i}} \frac{dp_{t,i}}{dx} &= \frac{F_i}{A_i p}, \quad \text{and} \quad \frac{1}{T_{t,i}} \frac{dT_{t,i}}{dx} = 0, \quad \text{for } i \in [p, s], \\ \frac{d_x p}{p} &= \frac{1}{\beta} \left(\frac{dA}{dx} + \sum_{i \in [p, s]} \left[\frac{1 + (\gamma - 1) \text{Ma}_i^2}{\gamma \text{Ma}_i^2} \right] \frac{F_i}{p} \right), \end{aligned} \quad (3.34)$$

where

$$\beta = A_p \frac{1 - \text{Ma}_p^2}{\gamma \text{Ma}_p^2} + A_s \frac{1 - \text{Ma}_s^2}{\gamma \text{Ma}_s^2}. \quad (3.35)$$

The forces F_i consist of inter-stream friction F_{ps} and wall friction F_w :

$$F_p = -F_{ps}, \quad \text{and} \quad F_s = F_{ps} - F_w, \quad (3.36)$$

which are defined as follows in this section:

$$F_w = \frac{1}{2} f_w \gamma p \text{Ma}_s^2 l_w, \quad \text{and} \quad F_{ps} = \frac{1}{2} f_{ps} \gamma p (\text{Ma}_p^2 - \text{Ma}_s^2) l_{ps}, \quad (3.37)$$

where l_w and l_{ps} respectively denote the perimeters of the wall and the primary cross-section. The single equation for the static pressure implies that it is equal on the primary and secondary sides, i.e. $p_p = p_s = p$. Note that the inter-stream friction was defined in this early version of the model [71] through the difference of squared Mach numbers, instead of velocities in the final model presented in chapters 6 and 7.

The two closure parameters to be identified in a data-driven model cal-

ibration are the inter-stream friction coefficient f_{ps} and the wall friction coefficient f_w . Therefore, in the formalism of section 3.2, these variables constitute the model parameters $\mathbf{p} = [f_{ps}, f_w]$ to be provided by the closure function g , while the state variables are $\mathbf{u} = [p, p_{tp}, p_{ts}, T_{tp}, T_{ts}, A_p, A_s]$.

It is worth stressing that the assumption of equal static pressure is problematic at the inlet of the mixing pipe because the primary flow is generally under-expanded (see figure 3.4). Therefore, the proposed model cannot be used from the exit of the primary nozzle ($x = 0$) unless a pressure equalisation mechanism is introduced in the model (see chapter 7 for a detailed analysis). Alternatively, the areas for the primary and secondary flows must be provided at the inlet of the mixing channel: in this case, the pressures naturally equalise within a short distance from the inlet; this is akin to what happens through the shock cells in the CFD simulation. The first approach is developed in chapter 7. In this section, the dividing streamline identified from the CFD is imposed up to the point where the pressures equalise, and downstream, we use the compound equations (3.34). Practically, the governing equations are integrated with space marching from the known ‘initial’ condition at the inlet of the mixing pipe. This is possible in the current setting due to the available reference data. A shooting method is applied in the predictive setting of chapters 6 and 7 to enforce the static back pressure as a boundary condition. The procedure in this section thus allows for errors on the back pressure as the error accumulates progressively during the integration.

The research question addressed in this section is the feasibility of deriving the spatial distribution of friction coefficients that makes the 1D model comply with the post-processed CFD data. We do not link the closure parameters to the state, which is a more complex task and should be studied across a wide range of operating conditions. A broader analysis is carried out in chapter 7. We focus on the parameters’ spatial distributions and their derivation by implementing the physics-separated and physics-integrated approaches.

Calibration

Following the framework in figure 3.1, the physics-separated approach consists of calculating the closure coefficients from the processed CFD data. To this end, the equations for the total pressure were used to compute the friction coefficients f_{ps} and f_w by extracting all the other terms from data via appropriate processing. These two equations give:

$$f_{ps} = -\frac{2A_p}{\gamma \left(\text{Ma}_p^2 - \text{Ma}_s^2 \right) l_{ps}} \frac{d_x p_{tp}}{p_{tp}}, \quad (3.38)$$

$$f_w = \frac{2}{\gamma p \text{Ma}_s^2 l_w} \left(F_{ps} - p A_s \frac{d_x p_{ts}}{p_{ts}} \right). \quad (3.39)$$

Both definitions rely on the differentiation of the total pressure evolution, computed using finite differences on a smoothed version of the signal. The derivation was carried out with a second-order centred scheme, while the smoothing was carried out with a Savitzky-Golay filter with a second-order polynomial. The resulting closure coefficients are, therefore, functions of the spatial coordinate x and could be linked to the local value of the state variables $\mathbf{u}$, or local values of dimensionless numbers such as Reynolds and Mach numbers in each stream, or pressure and temperature ratios. However, this regression has a high risk of overfitting because the flow field (and thus the closure coefficients) can change drastically with the operating conditions. This section focuses on a single off-design operating point, with regression as a function of the axial coordinate x . Practically, we minimise the cost function (3.3) separately for the shear and the wall friction coefficient with a parametric function g heuristically constructed after analysing the data. The specific choice is provided further below. Note that we can split the regression in two smaller problems because the predicted closure coefficients $\mathbf{p}$ can be evaluated independently against their references $\tilde{\mathbf{p}}$.

The closure coefficients derived in the previous section are those that minimise the error in the definitions (3.38) and (3.39), but this does not guarantee that the model prediction is the most accurate. The complex interplay of these parameters with the governing equations in the model introduces additional sensitivities to the model prediction. The physics-integrated approach seeks to account for all of these using the cost function (3.4) to penalise model prediction (that indirectly depends *also* on the parameters). In the investigated calibration problem, the observed quantities included in the cost function are the pressures p_{tp} and p_{ts} and the cross-sections A_p and A_s . The static pressure p is not included because the shock trains make p_p and p_s oscillate. Since the 1D model can only reproduce the mean trends due to the assumption of equal static pressures, this oscillation could unfairly penalise the model prediction and produce local minima in the cost function landscape. Moreover, the total temperature is

excluded because it is assumed to be constant in the mixing pipe and thus does not contribute to accentuating the cost function gradient distribution along the spatial coordinate. Finally, an additional term is added at the exit ($x = L$) to penalise the potential error on the back pressure. This penalisation helps enforce the boundary condition of the model. Therefore, the cost function (3.4) for this approach becomes

$$J_2(\mathbf{w}) = \sum_i \left[\frac{(\tilde{p}_{tp,i} - p_{tp,i}(g))^2}{(\tilde{p}_{tp,i})^2} + \frac{(\tilde{p}_{ts,i} - p_{ts,i}(g))^2}{(\tilde{p}_{ts,i})^2} + \frac{(\tilde{A}_{p,i} - A_{p,i}(g))^2}{(\tilde{A}_{p,i})^2} + \frac{(\tilde{A}_{s,i} - A_{s,i}(g))^2}{(\tilde{A}_{s,i})^2} \right] + \frac{(\tilde{p}(L) - p(g, L))^2}{(\tilde{p}(L))^2}. \quad (3.40)$$

where the summation is carried out over the available instances (e.g. grid points in x), the functional dependency on g denotes the model prediction based on the closure g and the summation over the index r is made explicit over the four variables involved. We recall that the closure function depends on weights $\mathbf{w}$, i.e. $g := g(x; \mathbf{w})$. For a given guess of the weights, hence a given closure law $\mathbf{p} = g(x; \mathbf{w})$, the cost function $J_2(\mathbf{w})$ in (3.40) is computed by first solving numerically the set of governing equations.

Results

Figure 3.5 shows the original and filtered 1D distributions of the total pressure in both streams from the post-processed CFD, near the exit of the primary nozzle at $x/L = 0$. The primary total pressure undergoes a sharp drop at $x/L = 0.025$, which corresponds to the Mach disk in figure 3.4. This is a 2D effect which can not be reproduced by the 1D model and is thus better filtered out before the data is used for calibration. The filtered signal thus ‘averages’ the shock train for the computation of the closure coefficients. The window length of the Savitzky-Golay filter has been tuned to this end.

The friction coefficients computed with equations (3.38) and (3.39) are shown in figure 3.6. The unfiltered coefficients show a sharp peak at the position of the Mach disk, as the primary total pressure changes suddenly at this point. This artefact is avoided through filtering. Still, both results indicate large coefficients at the inlet of the mixing pipe, especially in the wall friction f_w , which reaches an extreme value of 25 (with values expected to be of the order 0.01). A possible explanation could be the separation region

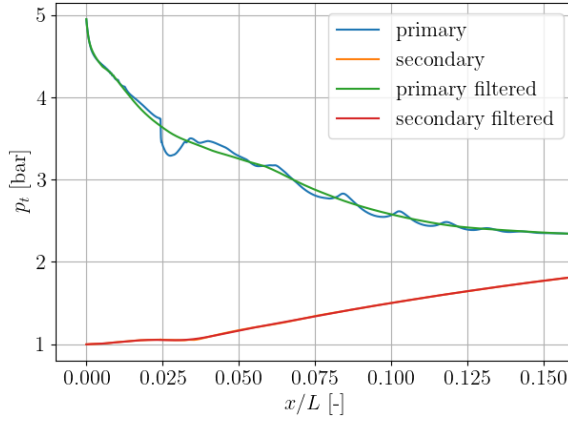


Fig. 3.5 Filtering of the 1D total pressure field obtained through post-processing the CFD simulation in figure 3.4. The total pressures tend towards a common value as the primary and secondary streams exchange momentum. The shock train at the inlet of the mixing pipe strongly affects the primary total pressure, which is therefore filtered with a Savitzky-Golay filter before being differentiated (cf. equations (3.38) and (3.39)). The filtered secondary total pressure coincides with its raw counterpart since it is already quite smooth.

behind the wall between the exit of the primary nozzle and the secondary inlet. This trailing edge is not sharp, so it is followed by a region of separated flow which slows down both the primary and the secondary stream through shear. This effect is locally quite strong for the primary flow due to a momentum deficit between the sonic flow and the stagnant flow in the bubble, hence a large value of the shear coefficient f_{ps} . However, this force is applied equally but with an opposite sign on the secondary flow, which therefore tends to *accelerate* rather than *decelerate*. In turn, a high wall friction coefficient is thus needed to overcome this numerical acceleration and still represent the actual friction forces on the secondary stream.

The peak at the inlet of the mixing domain shows an exponential decay, so an exponential function is chosen in the parametric closure function g for the calibration. Downstream, the evolution is rather flat, except for the oscillations induced by the shock train. Therefore, the closure function g is parametrised with a linear trend downstream. This leads to the following parametric function with 5 weights:

$$\begin{cases} y = w_1 + w_2 \exp(-w_3 x) & \text{if } x \leq w_0 \\ y = w_1 + w_2 \exp(-w_3 w_0) + w_4 x & \text{if } x > w_0 \end{cases} \quad (3.41)$$

This function is used for both closure coefficients, bringing the number of weights to 10. The mismatch with the post-processed coefficients is minimised using the summed squared error as a cost function and with the BFGS algorithm available in SciPy [89]. The gradient computation is performed with finite differences, which is affordable due to the low cost of the function to be called and the low number of weights. The resulting regression is shown alongside the raw and the filtered signals in figure 3.6.

Next, we use the regressed closure relation to predict the flow field of the ejector with the 1D model. The resulting flow field compares well to the post-processed CFD in figure 3.7. Upstream, the cross-section is imposed by the CFD, leading to a close match. Downstream, the unique pressure deviates from the values in the CFD due to the shock train (the primary stream is particularly affected). Nevertheless, the 1D model captures the global rising trend. This pressure recovery arises from the momentum exchange between the two streams of the compound flow, as predicted by the static pressure equation if $\beta > 0$. The pressure rises more strongly in the diffuser from $x/L = 0.23$ and reaches a constant value where the streams reach the same total pressure. The constant total temperature in the model matches the post-processed CFD within 3 K. The deviation is attributed to the low enthalpy in the separation bubble at the exit of the primary nozzle, which influences the total temperature field downstream near the dividing streamline. The constant total temperature remains a good approximation for the bulk of the streams.

Despite the indirect penalisation on the closure coefficients, an excellent match is obtained. The assumption of equal static pressure in both streams proves to be an acceptable assumption to close the system of equations (see chapter 7 for a detailed analysis). The match of the cross-sections between the 1D model and the CFD best validates the model. However, the 1D model shows a mismatch in back pressure (2.3 bar in the model and 2.2 bar in the CFD). Therefore, we apply the physics-integrated method below to improve the prediction in the flow quantities both in the internal domain and the boundary condition for the static pressure.

The closure relation from the section above serves as an initial guess for the physics-integrated approach, where the prediction error on the physical quantities is minimised (including the back pressure as a boundary con-

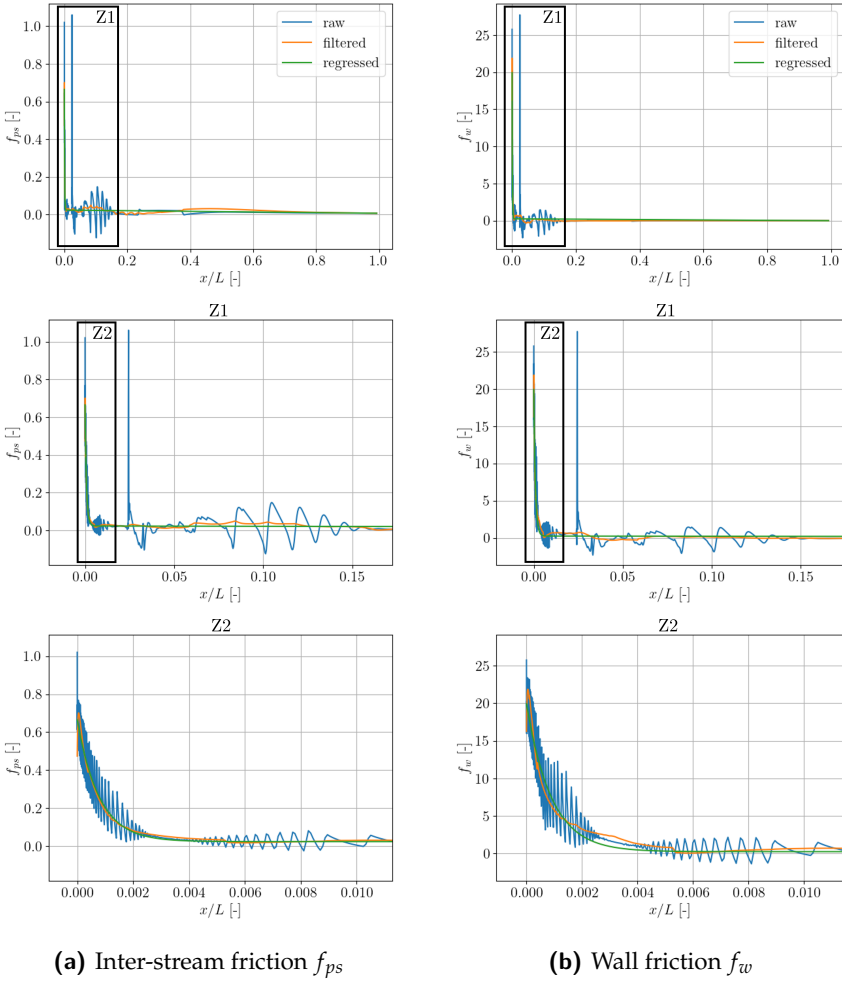


Fig. 3.6 Inter-stream and wall friction coefficient calculated from the raw and filtered total pressure field from figure 3.5 with equations (3.38) and (3.39). The exponential decay at the inlet is attributed to a separation bubble at the exit of the primary nozzle. The calculation on the raw signal suffers from sharp gradients in the shock train and the Mach disk ($x/L = 0.025$). The physics-separated regression is carried out with equation (3.41) on the filtered coefficients.

dition, cf. equation (3.40)). The resulting evolution of the shear and friction coefficients is shown in figure 3.8. The wall friction coefficient increased

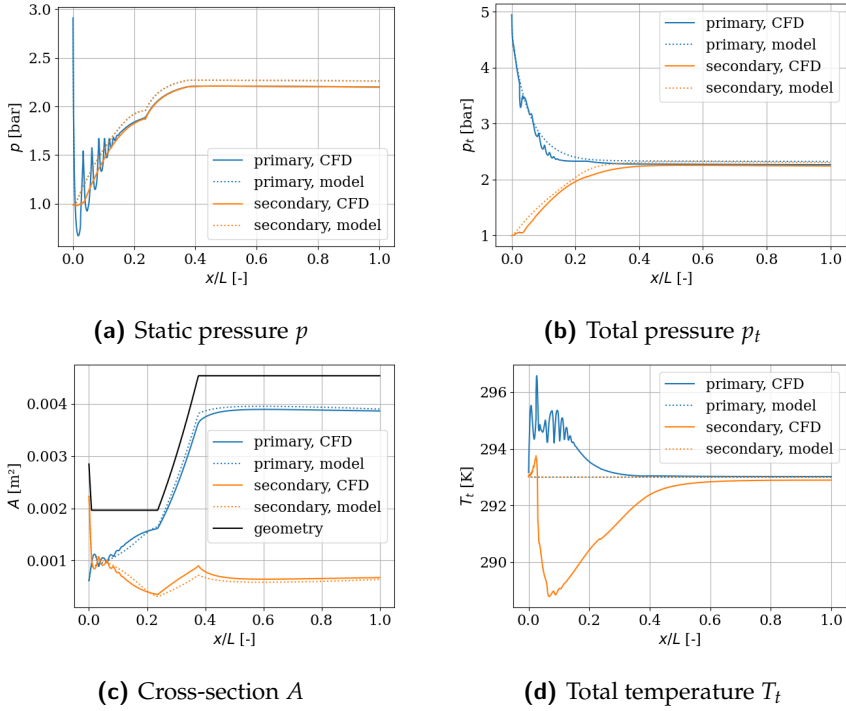


Fig. 3.7 Model predictions with the regressed closure relation from the physics-separated approach (cf. figure 3.6). All variables match the post-processed 1D data quite well (the cost function (3.40) equals 0.0044). However, the boundary condition of the static pressure at the outlet is not met due to the indirect regression on the post-processed closure coefficients instead of on the physical quantities (cf. equations (3.3) and (3.40)).

significantly, resulting in lower static pressure and a satisfied boundary condition (cf. figure 3.9a). An increased shear coefficient compensates for the corresponding loss in secondary total pressure. Finally, we note that the exponential decay between the inlet and $x/L = 0.01$ from figures 3.6a and 3.6b has sharpened, which results in a flat plateau in the coefficients in figure 3.8. This indicates the extreme values found with the physics-separated approach were not necessary.

The physics-integrated approach has decreased the value of the cost function (equation (3.40)) from 0.0044 with the initial guess from the physics-separated approach to 0.0036. This significant improvement can be appre-

ciated visually by comparing figures 3.7 and 3.9. However, the integrated approach is less straightforward a priori; one does not know which type of parametric function allows a close match. Additionally, initialising the weights is also non-trivial. For example, an excessive wall friction coefficient could decrease the static pressure enough to choke the flow and lead to problems in the integration of the system of ODEs (which becomes singular if $\beta = 0$, see chapter 6). Therefore, the initialisation through the physics-separated approach followed by fine-tuning with the physics-integrated proved an effective strategy for the model calibration, profiting from their complementary advantages.

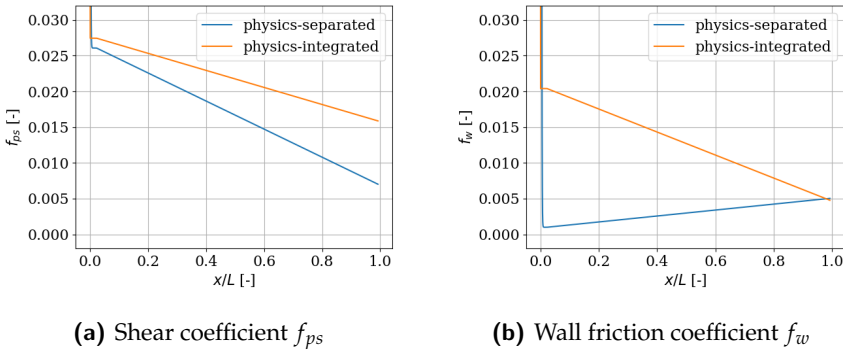


Fig. 3.8 Comparison of the regressed closure relations of the physics-separated and the physics-integrated approach. The first serves as initial guess for the latter method and is optimised to minimise cost function (3.40) (final value of 0.0036). A significant increase of the wall friction coefficient has allowed to decrease the static pressure in the mixing duct, leading to a satisfied boundary condition (cf. figure 3.9a).

3.7 Conclusion

This chapter proposed a general machine learning framework for calibrating physical models governed by differential equations through two approaches. The physics-separated approach consists of calculating the closure coefficients from higher resolution data (CFD) and then regressing the post-processed coefficients; alternatively, the physics-integrated approach calls the model during the optimisation phase to minimise the prediction error on the observed physical quantities. Both approaches have been applied successfully on a 1D ejector model to find the shear and wall fric-

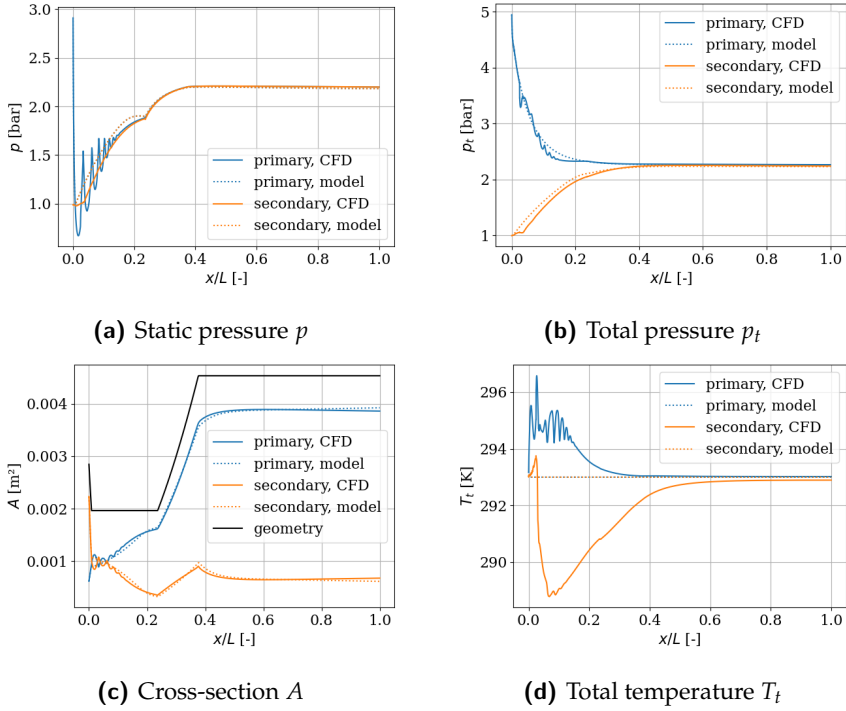


Fig. 3.9 Model predictions with the regressed closure relation from the physics-integrated approach (cf. figure 3.8). The agreement improved compared to the physics-separated approach in figure 3.7 (the cost function (3.40) equals 0.0036). Moreover, the boundary condition of the static pressure at the outlet is now respected.

tion coefficients as a function of the spatial coordinate. In particular, the physics-integrated approach can naturally be combined with the adjoint method and parametric regression techniques such as RBFs or ANNs. A first test case proved the successful combination of these tools.

The results on the ejector model above show that a successive application of both approaches proves to be a convenient and robust method to calibrate the 1D model. The physics-separated approach guides the choice of the parametric function and provides a valid initial guess for the physics-integrated approach,. This then acts as a refinement tool to further minimise the prediction error by *directly* penalising mispredictions in the state variables.

The proposed methodology is robust and flexible: it is compatible with state-of-the-art optimisers and leaves a free choice of the regressor. Any parametric tool fits the framework, ranging from linear regression to artificial neural networks. Furthermore, the method brings physical insight through closure relations for low-order but highly interpretable models. Finally, the framework can be categorised as *physics-constrained*, as opposed to *physics-informed*, machine learning since the conservation equations are always respected. From a more global perspective, splitting the problem in a set of physical equations f and closure relations g allows the continued use of dedicated solvers for the physical problem f , which are conservative and efficient, and simplifies the machine learning task by restricting its scope to the closure problem g . In this sense, the framework is an extension of neural ODEs, which solve the combined problem of f and g with neural networks without including prior physical knowledge. Consequently, less complex regressors may suffice for solving the problem, reducing the amount of required data or improving the performance on a fixed dataset.

The adjoint-based regression tool enabled efficient data-driven calibration of the quasi-two dimensional surrogate model for the mixing pipe presented in section 8.2. Although this model is still in an early stage of development, the numerical optimisation tool demonstrates clear potential for calibrating more complex versions in the near future.

4

Governing equations

4.1 Introduction

This chapter derives the governing equations of the 1D and quasi 2D models in the following chapters. These stem from the compressible RANS equations, simplified through a leading-order analysis in section 4.2 and then integrated over the relevant cross-sections in section 4.4. The reduced two-dimensional equations correspond to the compressible boundary layer equations, which are shown to provide an accurate first order approximation in section 4.3 by computing the respective residuals from a RANS simulation on the UCLouvain ejector (see chapter 2).

4.2 Scaling analysis

4.2.1 Scaling of the conservation equations

The starting point of the present work consists of the steady, compressible, Reynolds-Averaged Navier-Stokes (RANS) equations in axisymmetric coordinates without swirl. In the context of high-speed or high-Reynolds-number flows, the density field may exhibit significant spatial variations and thus the classical Reynolds decomposition is replaced by the Favre (density-weighted) averaging procedure. For any generic flow quantity ϕ ,

the Reynolds and Favre decompositions read

$$\phi = \bar{\phi} + \phi', \quad \phi = \tilde{\phi} + \phi'', \quad \tilde{\phi} = \frac{\bar{\rho}\phi}{\bar{\rho}}, \quad (4.1)$$

where $\bar{\phi}$ represents the Reynolds averaged value, $\tilde{\phi}$ the Favre averaged value, and ϕ' and ϕ'' their corresponding fluctuations. Throughout this work, the overbar ($\bar{\cdot}$) denotes a Reynolds average and the tilde ($\tilde{\cdot}$) a Favre average. The Reynolds stresses and turbulent heat fluxes are modelled using eddy-viscosity and gradient-diffusion approaches. Under these assumptions, the Favre-averaged compressible RANS equations in axisymmetric coordinates without swirl are recalled below for completeness:

$$\frac{\partial(\bar{\rho}\tilde{u})}{\partial x} + \frac{1}{r} \frac{\partial(r\bar{\rho}\tilde{v})}{\partial r} = 0, \quad (4.2)$$

$$\bar{\rho}\tilde{u} \frac{\partial\tilde{u}}{\partial x} + \bar{\rho}\tilde{v} \frac{\partial\tilde{u}}{\partial r} = -\frac{\partial\bar{p}}{\partial x} + \frac{\partial\tau_{xx}}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r}(r\tau_{rx}), \quad (4.3)$$

$$\bar{\rho}\tilde{u} \frac{\partial\tilde{v}}{\partial x} + \bar{\rho}\tilde{v} \frac{\partial\tilde{v}}{\partial r} = -\frac{\partial\bar{p}}{\partial r} + \frac{\partial\tau_{xr}}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r}(r\tau_{rr}) - \frac{\tau_{\theta\theta}}{r}, \quad (4.4)$$

$$\bar{\rho}\tilde{u} \frac{\partial\tilde{h}_t}{\partial x} + \bar{\rho}\tilde{v} \frac{\partial\tilde{h}_t}{\partial r} = \frac{\partial}{\partial x}(\tau_{xx}\tilde{u} + \tau_{xr}\tilde{v} - q_x) + \frac{1}{r} \frac{\partial}{\partial r}(r(\tau_{rx}\tilde{u} + \tau_{rr}\tilde{v} - q_r)), \quad (4.5)$$

where $\tilde{u}$ and $\tilde{v}$ denote the Favre averaged velocity components. The total enthalpy is defined as

$$\tilde{h}_t = c_v \tilde{T} + \frac{1}{2}(\tilde{u}^2 + \tilde{v}^2) + \frac{\bar{p}}{\bar{\rho}} = c_p \tilde{T}_t. \quad (4.6)$$

The viscous stresses follow the Stokes hypothesis and read

$$\tau_{xx} = 2\mu_{\text{eff}} \frac{\partial\tilde{u}}{\partial x} - \frac{2}{3}\mu_{\text{eff}} \left(\frac{\partial\tilde{u}}{\partial x} + \frac{1}{r} \frac{\partial r\tilde{v}}{\partial r} \right), \quad (4.7)$$

$$\tau_{rr} = 2\mu_{\text{eff}} \frac{\partial\tilde{v}}{\partial r} - \frac{2}{3}\mu_{\text{eff}} \left(\frac{\partial\tilde{u}}{\partial x} + \frac{1}{r} \frac{\partial r\tilde{v}}{\partial r} \right), \quad (4.8)$$

$$\tau_{\theta\theta} = 2\mu_{\text{eff}} \frac{\tilde{v}}{r} - \frac{2}{3}\mu_{\text{eff}} \left(\frac{\partial\tilde{u}}{\partial x} + \frac{1}{r} \frac{\partial r\tilde{v}}{\partial r} \right), \quad (4.9)$$

$$\tau_{xr} = \tau_{rx} = \mu_{\text{eff}} \left(\frac{\partial\tilde{u}}{\partial r} + \frac{\partial\tilde{v}}{\partial x} \right), \quad (4.10)$$

and Fourier's law provides the heat fluxes

$$q_x = -k_{\text{eff}} \frac{\partial \tilde{T}}{\partial x}, \quad q_r = -k_{\text{eff}} \frac{\partial \tilde{T}}{\partial r}. \quad (4.11)$$

Both the viscosity μ_{eff} and thermal conductivity k_{eff} include molecular and turbulent contributions through an eddy-viscosity closure.

Finally, assuming a calorically perfect gas, the equation of state is

$$\bar{p} = \bar{\rho} R_{\text{gas}} \tilde{T}, \quad \tilde{h}_t = c_p \tilde{T}_t. \quad (4.12)$$

The averaging operation has no direct impact on the scaling analysis, so the explicit overbars are omitted in the remainder of this section. The coordinates are normalised with the length L of the constant-area section and its radius R , which differ at least by an order of magnitude:

$$\hat{x} = \frac{x}{L} = O(1), \quad \hat{r} = \frac{r}{R} = O(1), \quad \Rightarrow \quad \varepsilon = \frac{R}{L} \ll 1. \quad (4.13)$$

In the ejectors considered in this work, the ratio ε is of the order of 0.04. The slenderness plays a crucial role in the derivation of the leading-order equations further below. The non-dimensional counterparts to the physical variables appearing in the equations above are denoted with a hat:

$$\begin{aligned} \hat{\rho} &= \frac{\rho}{[\rho]}, \quad \hat{u} = \frac{u}{[u]}, \quad \hat{v} = \frac{v}{[v]}, \quad \hat{p} = \frac{p}{[p]}, \quad \hat{T} = \frac{T}{[T]}, \quad \hat{T}_t = \frac{T_t}{[T]}, \\ \hat{\tau}_{ij} &= \frac{\tau_{ij}}{[\tau]}, \quad \hat{h}_t = \frac{h_t}{[h_t]}, \quad \hat{q}_i = \frac{q_i}{[q]}, \quad \hat{\mu}_{\text{eff}} = \frac{\mu_{\text{eff}}}{[\mu]}, \quad \hat{k}_{\text{eff}} = \frac{k}{[k]}. \end{aligned} \quad (4.14)$$

Scaling the continuity equation reveals two dimensional terms which should be of the same order of magnitude:

$$\frac{[\rho][u]}{L} \frac{\partial(\hat{\rho}\hat{u})}{\partial \hat{x}} + \frac{[\rho][v]}{R} \frac{1}{\hat{r}} \frac{\partial(\hat{r}\hat{\rho}\hat{v})}{\partial \hat{r}} = 0. \quad (4.15)$$

Consequently, the scale of the radial velocity is of the order ε relative to the axial velocity:

$$\frac{[v]}{[u]} = \frac{R}{L} = \varepsilon. \quad (4.16)$$

The scales for the density, the velocity, the viscosity and the thermal con-

ductivity are left independent. The other scales are defined as follows:

$$[p] = [\rho] R_{gas} [T], \quad [T] = \frac{[u]^2}{c_p}, \quad [\tau] = \frac{[\mu][u]}{L}, \quad [h_t] = [u]^2, \quad [q] = \frac{[k][T]}{L}. \quad (4.17)$$

The normalised ideal gas law and total temperature become:

$$\hat{p} = \rho \hat{T} \quad \text{and} \quad \hat{T}_t = \hat{T} + \frac{1}{2}(\hat{u})^2. \quad (4.18)$$

The ratio ε appears in the non-dimensional shear stresses and heat conduction terms due to the combination of axial and radial derivatives:

$$\hat{\tau}_{xx} = 2\hat{\mu}_{\text{eff}} \frac{\partial \hat{u}}{\partial \hat{x}} - \frac{2}{3}\hat{\mu}_{\text{eff}} \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{1}{\hat{r}} \frac{\partial \hat{r} \hat{v}}{\partial \hat{r}} \right), \quad (4.19)$$

$$\hat{\tau}_{rr} = 2\hat{\mu}_{\text{eff}} \frac{\partial \hat{v}}{\partial \hat{r}} - \frac{2}{3}\hat{\mu}_{\text{eff}} \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{1}{\hat{r}} \frac{\partial \hat{r} \hat{v}}{\partial \hat{r}} \right), \quad (4.20)$$

$$\hat{\tau}_{\theta\theta} = 2\hat{\mu}_{\text{eff}} \frac{\hat{v}}{\hat{r}} - \frac{2}{3}\hat{\mu}_{\text{eff}} \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{1}{\hat{r}} \frac{\partial \hat{r} \hat{v}}{\partial \hat{r}} \right), \quad (4.21)$$

$$\hat{\tau}_{xr} = \hat{\tau}_{rx} = \hat{\mu}_{\text{eff}} \left(\frac{1}{\varepsilon} \frac{\partial \hat{u}}{\partial \hat{r}} + \varepsilon \frac{\partial \hat{v}}{\partial \hat{x}} \right), \quad (4.22)$$

$$\hat{q}_x = -\hat{k}_{\text{eff}} \frac{\partial \hat{T}}{\partial \hat{x}}, \quad \hat{q}_r = -\frac{1}{\varepsilon} \hat{k}_{\text{eff}} \frac{\partial \hat{T}}{\partial \hat{r}}. \quad (4.23)$$

The scaled conservation equations become:

$$\frac{\partial(\hat{\rho}\hat{u})}{\partial \hat{x}} + \frac{1}{\hat{r}} \frac{\partial(\hat{r}\hat{\rho}\hat{v})}{\partial \hat{r}} = 0, \quad (4.24)$$

$$\hat{\rho}\hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{\rho}\hat{v} \frac{\partial \hat{u}}{\partial \hat{r}} = -\frac{1}{\Pi} \frac{\partial \hat{p}}{\partial \hat{x}} + \frac{1}{Re} \frac{\partial \hat{\tau}_{xx}}{\partial \hat{x}} + \frac{1}{\varepsilon Re} \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} (\hat{r} \hat{\tau}_{rx}), \quad (4.25)$$

$$\varepsilon \left(\hat{\rho}\hat{u} \frac{\partial \hat{v}}{\partial \hat{x}} + \hat{\rho}\hat{v} \frac{\partial \hat{v}}{\partial \hat{r}} \right) = -\frac{1}{\Pi} \frac{1}{\varepsilon} \frac{\partial \hat{p}}{\partial \hat{r}} + \frac{1}{Re} \frac{\partial \hat{\tau}_{xr}}{\partial \hat{x}} + \frac{1}{\varepsilon Re} \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} (\hat{r} \hat{\tau}_{rr}) - \frac{1}{\varepsilon Re} \frac{\hat{\tau}_{\theta\theta}}{\hat{r}}, \quad (4.26)$$

$$\begin{aligned} \hat{\rho}\hat{u} \frac{\partial \hat{h}_t}{\partial \hat{x}} + \hat{\rho}\hat{v} \frac{\partial \hat{h}_t}{\partial \hat{r}} &= \frac{1}{Re} \frac{\partial}{\partial \hat{x}} \left(\hat{\tau}_{xx} \hat{u} + \varepsilon \hat{\tau}_{xr} \hat{v} \right) + \frac{1}{\varepsilon Re} \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r} (\hat{\tau}_{rx} \hat{u} + \varepsilon \hat{\tau}_{rr} \hat{v}) \right) \\ &\quad - \frac{1}{Re} \frac{1}{Pr} \left(\frac{\partial \hat{q}_x}{\partial \hat{x}} + \frac{1}{\varepsilon} \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} (\hat{r} \hat{q}_r) \right), \end{aligned} \quad (4.27)$$

where the Reynolds number, Prandtl number and Euler number appear:

$$Re = \frac{[\rho]UL}{[\mu]}, \quad Pr = \frac{[\mu]C_p}{[k]}, \quad \Pi = \frac{[\rho]U^2}{[p]} = \frac{\gamma}{\gamma - 1}. \quad (4.28)$$

The Euler number is fixed by an expression of the ratio of specific heats γ due to the thermodynamic scaling of the static pressure through the ideal gas law.

4.2.2 Leading-order analysis

The non-dimensional equations derived above are reduced here to their leading-order form. Taking the averaged flow variables at the outlet of the primary nozzle as reference values, the Reynolds number is of the order of 10^7 , the Prandtl number equals 0.71 (air) and $\Pi = 3.5$ since the working fluid is air, and the aspect ratio ε equals 0.04. The coordinates are of order unity through their separate scaling, and the variations of the physical quantities are assumed to be of the same order. This implies that variables vary more sharply over r than over x in physical coordinates (which makes sense in ejector flows), and that they evolve at a similar rate in the non-dimensional coordinates. Practically, this implies that the spatial derivatives in non-dimensional space do not alter the order of the derived term.

The continuity equation has already yielded the scale difference of ε in the reference velocities. The viscous stress $\hat{\tau}_{xr}$ can be simplified at leading order:

$$\hat{\tau}_{xr} = \hat{\tau}_{rx} = \hat{\mu}_{\text{eff}} \left(\frac{1}{\varepsilon} \frac{\partial \hat{u}}{\partial \hat{r}} + \varepsilon \frac{\partial \hat{v}}{\partial \hat{x}} \right) \approx \frac{1}{\varepsilon} \hat{\mu}_{\text{eff}} \frac{\partial \hat{u}}{\partial \hat{r}}. \quad (4.29)$$

Consequently, the off-diagonal component $\hat{\tau}_{xr}$ appears at the order $1/\varepsilon$ with respect to the diagonal terms in the stress tensor (see equations (4.19)-(4.22)).

The advection terms in the axial momentum equation (4.25) and the axial pressure gradient appear as first order terms. The axial diffusion is of order ε^2 compared to the radial diffusion, so is assumed to be negligible. Despite the division by the large Reynolds number, the diffusion term can not be neglected due to strong gradients in the boundary and shear layers. The boundary layer thickness is indeed small with respect to the radius R . The leading-order axial momentum equation becomes:

$$\hat{\rho} \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{\rho} \hat{v} \frac{\partial \hat{u}}{\partial \hat{r}} = -\frac{1}{\Pi} \frac{\partial \hat{p}}{\partial \hat{x}} + \frac{1}{\varepsilon^2 Re} \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r} \hat{\mu}_{\text{eff}} \frac{\partial \hat{u}}{\partial \hat{r}} \right). \quad (4.30)$$

The leading-order term in the radial momentum equation (4.26) is the radial pressure gradient, which appears at $O(1/\varepsilon)$, whereas the advection terms are of order $O(\varepsilon)$ and the viscous terms are divided by the Reynolds number. The leading-order approximation therefore yields a uniform static pressure across $\hat{r}$, which complies with the compound flow theory from chapter 6 and is also a clone result of boundary layer theory.

The advection terms in the energy equation are of the order unity. The leading-order viscous work term is the one containing $\hat{\tau}_{rx}\hat{u}$, which appears at $O(1/(\varepsilon^2 Re))$. The other viscous terms are smaller by ε^2 . Similarly, the radial heat conduction is dominant over the axial conduction, leading to the following equation for the total energy:

$$\begin{aligned} \hat{\rho}\hat{u}\frac{\partial\hat{h}_t}{\partial\hat{x}} + \hat{\rho}\hat{v}\frac{\partial\hat{h}_t}{\partial\hat{r}} &= \frac{1}{\varepsilon^2 Re}\hat{\mu}_{\text{eff}}\left(\frac{\partial\hat{u}}{\partial\hat{r}}\right)^2 + \frac{1}{\varepsilon^2 Re}\frac{\hat{u}}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left(\hat{r}\hat{\mu}_{\text{eff}}\frac{\partial\hat{u}}{\partial\hat{r}}\right) \\ &+ \left(\frac{1}{\varepsilon^2 Re Pr}\right)\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left(\hat{r}\hat{k}_{\text{eff}}\frac{\partial\hat{T}}{\partial\hat{r}}\right). \end{aligned} \quad (4.31)$$

The viscous dissipation is expected to be important in the boundary layer, where the velocity varies over a local length scale $\delta \ll R$. The shear work and heat transfer terms are active at the boundaries of the shear and boundary layer, where the second derivatives of the velocity and temperature reach their maximum. In the present configurations, these two effects are expected to partially compensate each other due to the identical total temperatures imposed at the inlets: the primary stream exhibits a higher velocity but a lower static temperature than the secondary stream, leading to radial gradients of opposite sign. With this consideration, the energy equation can be expressed in an alternative form as follows:

$$\hat{\rho}\hat{u}\frac{\partial\hat{h}_t}{\partial\hat{x}} + \hat{\rho}\hat{v}\frac{\partial\hat{h}_t}{\partial\hat{r}} = \left(\frac{1}{\varepsilon^2 Re}\right)\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left[\hat{r}\left(\frac{\hat{k}_{\text{eff}}}{Pr}\frac{\partial\hat{T}}{\partial\hat{r}} + \hat{\mu}_{\text{eff}}\frac{\partial}{\partial\hat{r}}\left(\frac{1}{2}(\hat{u})^2\right)\right)\right]. \quad (4.32)$$

After introducing the total temperature, the equation becomes:

$$\hat{\rho}\hat{u}\frac{\partial\hat{h}_t}{\partial\hat{x}} + \hat{\rho}\hat{v}\frac{\partial\hat{h}_t}{\partial\hat{r}} = \left(\frac{1}{\varepsilon^2 Re}\right)\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left(\hat{r}\hat{k}_{\text{eff}}\frac{\partial\hat{T}_t}{\partial\hat{r}}\right). \quad (4.33)$$

At leading order, the Reynolds analogy implies that the effective thermal and momentum diffusivities are of the same order, that is: $O(\hat{k}_{\text{eff}}) = O(\hat{\mu}_{\text{eff}})$. Using this simplification, the radial diffusion term in the energy

equation can be compactly expressed through the total temperature:

$$\hat{\rho} \hat{u} \frac{\partial \hat{h}_t}{\partial \hat{x}} + \hat{\rho} \hat{v} \frac{\partial \hat{h}_t}{\partial \hat{r}} = \frac{1}{\varepsilon^2 Re} \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r} \hat{k}_{\text{eff}} \frac{\partial \hat{T}_t}{\partial \hat{r}} \right). \quad (4.34)$$

As a stronger simplification, the right-hand side is therefore negligible considering the uniform total temperature profile at the inlet and the term $\varepsilon^2 Re \gg 1$. A pure advection equation remains for the total enthalpy, which from a uniform inlet profile amounts to a uniform total enthalpy field:

$$\hat{h}_t(\hat{x}, \hat{r}) = \hat{h}_{t,0}. \quad (4.35)$$

Two reduced energy equations ((4.31) and (4.35)) have thus been obtained, with the viscous dissipation in the boundary layer as the main expected difference. The final set of equations corresponds to the axisymmetric compressible boundary layer equations [15]:

$$\frac{\partial(\hat{\rho}\hat{u})}{\partial \hat{x}} + \frac{1}{\hat{r}} \frac{\partial(\hat{r}\hat{\rho}\hat{v})}{\partial \hat{r}} = 0, \quad (4.36)$$

$$\hat{\rho}\hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{\rho}\hat{v} \frac{\partial \hat{u}}{\partial \hat{r}} = -\frac{1}{\Pi} \frac{\partial \hat{p}}{\partial \hat{x}} + \frac{1}{\varepsilon^2 Re} \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r} \hat{\mu}_{\text{eff}} \frac{\partial \hat{u}}{\partial \hat{r}} \right), \quad (4.37)$$

$$\frac{\partial \hat{p}}{\partial \hat{r}} = 0, \quad (4.38)$$

$$\begin{aligned} \hat{\rho}\hat{u} \frac{\partial \hat{h}_t}{\partial \hat{x}} + \hat{\rho}\hat{v} \frac{\partial \hat{h}_t}{\partial \hat{r}} &= \frac{1}{\varepsilon^2 Re} \hat{\mu}_{\text{eff}} \left(\frac{\partial \hat{u}}{\partial \hat{r}} \right)^2 + \frac{1}{\varepsilon^2 Re} \frac{\hat{u}}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r} \hat{\mu}_{\text{eff}} \frac{\partial \hat{u}}{\partial \hat{r}} \right) \\ &+ \left(\frac{1}{\varepsilon^2 Re Pr} \right) \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r} \hat{k}_{\text{eff}} \frac{\partial \hat{T}_t}{\partial \hat{r}} \right). \end{aligned} \quad (4.39)$$

In the next section, the terms in the equation above are evaluated using a RANS simulation of the UCLouvain ejector to verify whether the terms retained in the asymptotic analysis are indeed dominant.

4.2.3 An explicit equation for the axial velocity

Akin to the explicit one-dimensional equations derived below in section 4.4 (see also appendix A), we seek an explicit differential equation for $\partial \hat{u} / \partial \hat{x}$ in this section. Here, the sharp variation in the radial direction motivates introducing a concise notation for the terms containing radial derivatives, which dominate over the smooth variation along the axial direction $\hat{x}$. Assuming a uniform total temperature, these terms are collected as $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ for the contributions of radial advection and diffusion in the continuity

equation (4.36) and axial momentum equation (4.37). They act as source terms in the resulting system for the long-wave variation along the axial direction $\hat{x}$:

$$\hat{\rho} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{u} \frac{\partial \hat{\rho}}{\partial \hat{x}} + \hat{a}_1 = 0, \quad (4.40)$$

$$\hat{\rho} \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{a}_2 = -\frac{1}{\Pi} \frac{\partial \hat{p}}{\partial \hat{x}} + \hat{a}_3, \quad (4.41)$$

$$\frac{\partial \hat{T}_t}{\partial \hat{x}} = 0 = \frac{\partial \hat{T}}{\partial \hat{x}} + \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}}, \quad (4.42)$$

$$\frac{\partial \hat{p}}{\partial \hat{x}} = \hat{T} \frac{\partial \hat{\rho}}{\partial \hat{x}} + \hat{\rho} \frac{\partial \hat{T}}{\partial \hat{x}}, \quad (4.43)$$

The strong simplification of uniform total energy is adapted here (equation (4.35)), and the contribution of the squared radial velocity to the total temperature has been neglected in equation (4.42), as it scales with ε^2 with respect to the axial component. The axial derivative of the static temperature is eliminated from the system by combining the ideal gas law (4.43) and the energy equation (4.42):

$$\frac{\partial \hat{p}}{\partial \hat{x}} = \hat{T} \frac{\partial \hat{\rho}}{\partial \hat{x}} - \hat{\rho} \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}}. \quad (4.44)$$

Introducing the continuity equation, this becomes:

$$\frac{\partial \hat{p}}{\partial \hat{x}} = -\frac{\hat{T}}{\hat{u}} \left(\hat{a}_1 + \hat{\rho} \frac{\partial \hat{u}}{\partial \hat{x}} \right) - \hat{\rho} \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}}. \quad (4.45)$$

Substituting the pressure gradient in the axial momentum equation (4.41) leads to the following explicit equation for the axial velocity:

$$\frac{\hat{p}}{\Pi} \left(\frac{(\hat{u})^2}{(\gamma - 1)\hat{T}} - 1 \right) \frac{\partial \hat{u}}{\partial \hat{x}} = \frac{\hat{T}}{\Pi} \hat{a}_1 + \hat{u} \hat{a}_3 - \hat{u} \hat{a}_2. \quad (4.46)$$

The term between the brackets gives rise to the Mach number, which is defined as follows in terms of the non-dimensional quantities:

$$\text{Ma}_x^2 = \frac{u^2}{\gamma R_{gas} T} = \frac{(\hat{u})^2}{\hat{T}} \frac{[u]^2}{\gamma R_{gas}[T]} = \frac{(\hat{u})^2}{(\gamma - 1)\hat{T}}, \quad (4.47)$$

where the subscript x highlights its derivation from the axial velocity. Reintroducing the terms $\hat{a}_i$, the full equation reads:

$$\left(\text{Ma}_x^2 - 1\right) \frac{\partial \hat{u}}{\partial \hat{x}} = \frac{1}{\hat{r}\hat{\rho}} \frac{\partial(\hat{r}\hat{\rho}\hat{v})}{\partial \hat{r}} + \frac{\Pi}{\varepsilon^2 Re} \frac{\hat{u}}{\hat{p}} \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r}\hat{\mu}_{\text{eff}} \frac{\partial \hat{u}}{\partial \hat{r}} \right) - \gamma \text{Ma}_x \text{Ma}_r \frac{\partial \hat{u}}{\partial \hat{r}}. \quad (4.48)$$

The term with the Mach number is reminiscent of the single-stream one-dimensional equations (see for example chapter 8 of Shapiro [57]), and could eventually become the denominator of the right-hand side for a practical quasi-2D model. This would introduce a singularity for the axial derivative of the velocity, which is local since the velocity still depends on the radial coordinate. The implications on the integration of the equations and on potential choking presently remain open questions for future research (see chapter 8).

Explicit expressions can be derived for the static pressure from equation (4.45) and for the static temperature from the conservation of energy, equation (4.42). It is worth emphasizing that the assumption of uniform pressure is not inherently enforced by equation (4.45) for the static pressure since it depends on the radial profiles of the velocity and the temperature. This practical challenge will likely arise in quasi-2D models and depends on the parametrisation of the radial profiles.

The system of equations (4.42), (4.45) and (4.48) still presents significant challenges for practical ejector modelling.

Mainly, the equations are still PDEs. Two classical pathways exist to obtain a practical low-cost quasi-2D model: (a) impose tailored parametric radial profiles of the relevant quantities or (b) represent the radial profiles as an expansion of prescribed basis functions, for example Chebyshev polynomials or a data-driven basis such as POD modes. The first is a more ad-hoc solution and likely the most parameter-efficient (e.g. 5 coefficients, including 2 free-stream velocities and three radial coordinates to fix the self-similar profiles in space). The equations governing the axial dynamics of the coefficients that determine the radial profiles are obtained by integrating the two-dimensional equations in the radial direction, for example over the primary and secondary stream (see section 4.4). The latter is potentially more general, as the functional form is less restrictive. This quasi-2D approach is explored in chapter 8. If the assumption of uniform total temperature is deemed sufficiently accurate, only the radial profile of the axial velocity must be prescribed, since the static temperature follows from

the uniform total temperature through equation (4.42). The density follows from the uniform static pressure profile and the static temperature.

Furthermore, the radial velocity poses a challenge as its functional form is a priori less clear. It follows from integration of the continuity equation along $\hat{r}$, which involves the axial derivatives of the density and the velocity, which in turn contain the radial velocity (see equation (4.48)). An iterative procedure might be required at the inlet of the domain, for example initialising the radial velocity at zero, computing the axial derivatives of $\hat{u}$ and $\hat{\rho}$, integrating the continuity to update the radial velocity and iterating.

4.3 Off-line analysis using RANS simulations

The chosen test case is the on-design operating point of the UCLouvain ejector, with $p_{t,p}/p_{t,s} = 5$ and $p_b/p_{t,s} = 1.2$. The normalised static pressure, axial velocity and total temperature fields are displayed in figure 4.1 (scaled through equations (4.14) and (4.17)). The static pressure field features structures resulting from the shock train in the primary stream. Aside from these shocks, the uniform static pressure in $\hat{r}$ seems a reasonable approximation. The axial velocity shows the growing shear layer and boundary layer, which start interacting approximately halfway in the mixing pipe ($\hat{x} \approx 0.4$). It reaches values larger than unity, indicating that the primary stream has expanded in the mixing pipe from the reference velocity at the outlet of the primary nozzle. The total temperature is not uniform in the shear layer and boundary layer. The uniform regions mark the potential cores of the primary and secondary streams, which penetrate beyond the constant-area section and halfway the constant-area section respectively. The lowest value of total temperature is reached in the boundary layer. The assumption of uniform total temperature is overly strong in the boundary layer (reaching a minimum $\hat{T}_t = 1.41$), yet, the values in the shear layer are compromised between 1.47 and 1.50. It may therefore still serve as a simplifying assumption for a low order quasi-2D model, in the same spirit as the one-dimensional double stream model.

The continuity equation has not been reduced with respect to the original RANS equations, so its residual is not analysed here. The residual on the radial momentum is shown in figure 4.2 and equals the radial pressure gradient, therefore highlighting the shock train (compressed in $\hat{x}$ due to the different length scales). These are suppressed in the leading-order equations, which therefore implies that a pressure equalisation mechanism will equally be required in an eventual quasi-2D model.

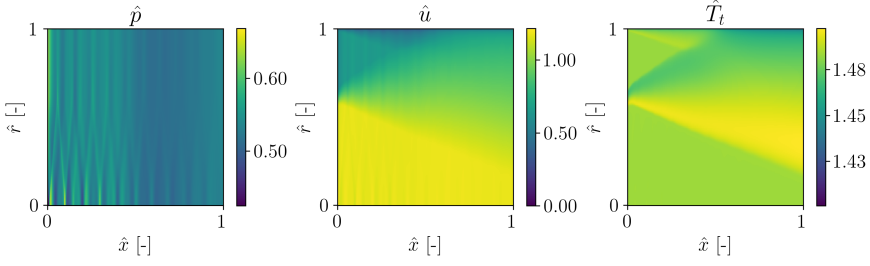


Fig. 4.1 Non-dimensional fields of the static pressure, the axial velocity and the total temperature (cf. equations (4.14) and (4.17)). The static pressure gradient is mainly oriented along $\hat{x}$, except in the shock train. The velocity and the total temperature outline the shear layer and the boundary layer, across which both variables vary, albeit on a smaller range for $\hat{T}_t$.

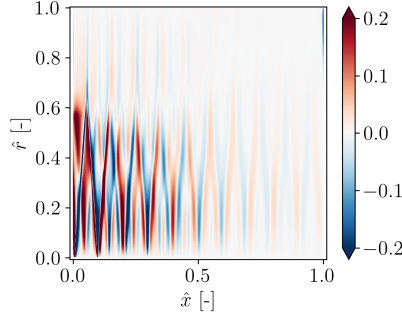


Fig. 4.2 The residual on the radial momentum equation (4.38) corresponds to the radial pressure gradient, which is strong in the shock train.

The leading-order equations (4.37) and (4.39) for the axial momentum and the total energy contain viscous terms, which in the context of RANS are defined through the eddy viscosity. However, neither of the turbulent quantities (ν_t , k or ω) were stored from the simulations, so a priori only the laminar viscosity was available through Sutherland’s law and the static temperature.

Therefore, the eddy viscosity was induced from the density, pressure and velocity fields through field inversion; treating the unknown eddy viscosity as the correction parameter [76]. The advection terms and the static pressure gradient of the axial momentum equation (4.37) were computed with second order finite differences and are displayed in figure 4.3. The

pressure gradient induced by the shock train is compensated by the axial advection, as the shocks also induce a strong local disturbance of the velocity. The axial advection term is locally strong in the top left corner, indicating an accelerating secondary flow, which occurs due to the converging secondary channel and the entrainment by the primary stream.

The residual $\mathcal{R}_{excl,diff}$ considering only the advection terms and the pressure gradient is shown at the top left of figure 4.4. This field is at leading order attributed purely to the radial diffusion, which is plausible considering it is most active in regions with large second derivatives of the velocity, hence at the edges of the shear layer and the boundary layer. The objective here is to identify the eddy viscosity $\hat{\mu}_{data}$ that reproduces the required diffusion term. The unknown term is governed by the following equation:

$$\frac{\partial}{\partial \hat{r}} \left(\hat{r} \hat{\mu}_{data} \frac{\partial \hat{u}}{\partial \hat{r}} \right) = \hat{r} \epsilon^2 Re \mathcal{R}_{excl,diff}. \quad (4.49)$$

Identifying $\hat{\mu}_{data}$ is mathematically equivalent to the field inversion step of the procedure proposed by Parish and Duraisamy [76], by considering it as a ‘corrective’ term in the original equations. The inversion procedure presently involves integration the equation above along $\hat{r}$. The field $\hat{r} \hat{\mu}_{data} \partial \hat{u} / \partial \hat{r}$ has been obtained by integration from $\hat{r} = 0$ to each radial coordinate $\hat{r}$, considering that it equals zero at $\hat{r} = 0$, and is displayed at the top center of figure 4.4. The negative residual $\mathcal{R}_{excl,diff}$ in the bottom half of the domain induces negative values of the integrated field, which is consistent with the negative velocity gradient. It is compensated by the positive portion of the residual to reach near-zero in the core of the secondary stream. Finally, the integrated field is divided by the radial coordinate and the radial derivative of the axial velocity to obtain the eddy viscosity $\hat{\mu}_{data}$. Numerical issues were avoided in this procedure by masking the points where the derivative $\partial \hat{u} / \partial \hat{r}$ was smaller than 0.1, a value that was empirically found to mask the cores of the primary and secondary streams.

The resulting eddy viscosity is shown in figure 4.4, which clearly corresponds to the shear layer. Interestingly, it reduces to low values in the laminar sublayer due to the strong velocity gradient. This is expected from boundary layer theory, but it is not trivial that the current procedure has captured this effect by integrating numerically differentiated flow fields. The radial diffusion term and the total residual were recomputed using the identified eddy viscosity, yielding the fields at the bottom of figure 4.4. Provided with the correct eddy viscosity, the leading-order equation (4.37)

therefore accurately describes the axial momentum balance in the mixing pipe of axisymmetric ejectors with respect to RANS simulations, although these naturally have their own intrinsic shortcomings [28].

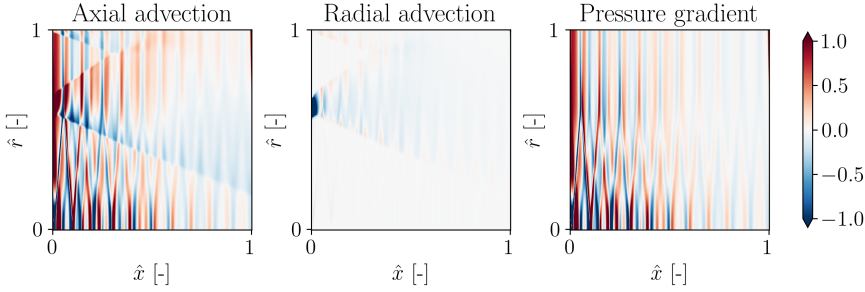


Fig. 4.3 Terms of the axial momentum equation computed from the RANS simulation with finite differences, excluding radial diffusion. The strong pressure gradient in the shocks is compensated by the axial advection and the radial advection plays a smaller role in the shear layer.

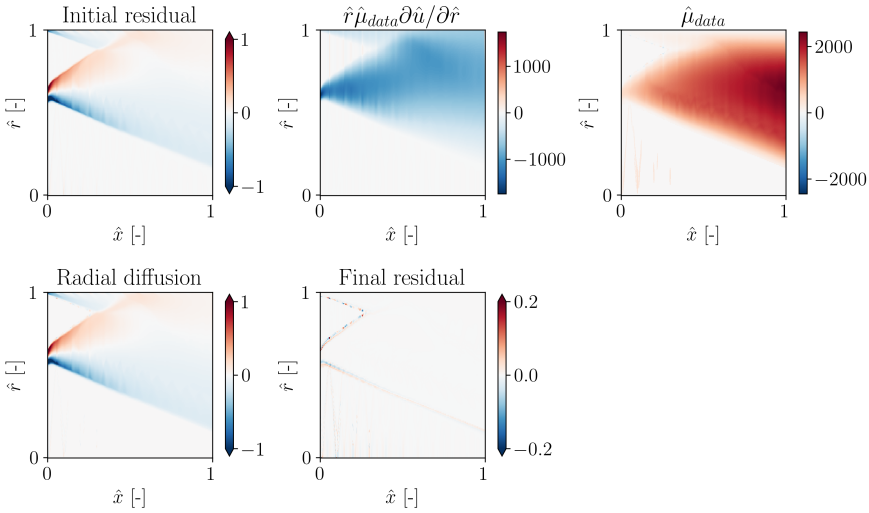


Fig. 4.4 Field inversion for the unknown eddy viscosity $\hat{\mu}_{data}$. The initial residual excludes radial diffusion (summing the three fields in figure 4.3) and is integrated along $\hat{r}$ (top center) to ultimately obtain the eddy viscosity. The resulting radial diffusion accurately matches the initial residual and thus minimises the final residual on the leading-order equation (4.37).

The identified eddy viscosity in figure 4.4 allows to compute the viscous terms in the energy equation taking into account the turbulence. The resulting terms and the residual are shown in figure 4.5. The radial conduction, viscous dissipation and the radial viscous work are most apparent (with the highest dissipation in the boundary layer). These three terms almost cancel each other, supporting the simplified form of the energy equation in section 4.2.2, where they were combined in a single radial derivative of the total temperature under strong assumptions. The assumption of uniform total temperature thus seems further justified, provided the temperatures at the inlets are identical. In any case, the full leading-order equation (4.39) is an accurate simplification of the full RANS equation. The analysis in this section leads to the fascinating conclusion that the flow in the mixing pipe of ejector can be quite accurately represented by, in essence, the boundary layer equations. This observation anchors the one-dimensional formulation of the ejector models in chapters 5-7, which assume a cross-stream uniform static pressure profile as in the compound flow formulation. The two-dimensional boundary layer equations derived here further provide a natural middle ground between these cross-stream averaged representations and RANS simulations (see chapter 8).

4.4 Cross-stream averaged equations

Derivation

Integrating the reduced equations (4.36)-(4.39) from section 4.2 in the radial direction yields the conservation equations required for the one-dimensional models introduced in figure 1.4 and developed in chapters 5, 6 and 7. These averaged equations are the interface between high-fidelity physics and low-order models. All subsequent models in this thesis are closures of this system. The equations are first integrated over arbitrary surfaces delimited by $\hat{r}_i(\hat{x})$ and $\hat{r}_o(\hat{x})$. The axisymmetry is taken into account by multiplying the integrand by $2\pi\hat{r}$. The continuity equation yields:

$$\int_{\hat{r}_i}^{\hat{r}_o} \frac{\partial(\hat{\rho}\hat{u})}{\partial\hat{x}} 2\pi\hat{r}d\hat{r} + \int_{\hat{r}_i}^{\hat{r}_o} \frac{\partial(\hat{r}\hat{\rho}\hat{v})}{\partial\hat{r}} 2\pi d\hat{r} = 0. \quad (4.50)$$

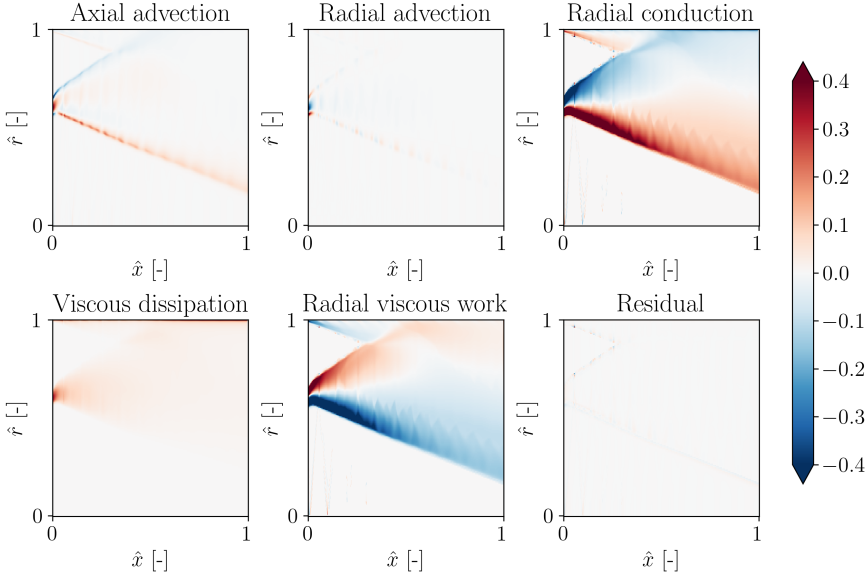


Fig. 4.5 Terms and residual of the leading-order total energy equation (4.39), using the identified turbulent viscosity from figure 4.4. The heat produced by viscous dissipation and the work done by radial shear stress is radially conducted. The advection terms are relatively small due to the relatively uniform total temperature (cf. figure 4.1).

Leibniz rule for integration yields:

$$2\pi \frac{\partial}{\partial \hat{x}} \left(\int_{\hat{r}_i}^{\hat{r}_o} \hat{\rho} \hat{u} \hat{r} d\hat{r} \right) + 2\pi \hat{r}_o \hat{\rho}_o \left(\hat{v}_o - \hat{u}_o \frac{d\hat{r}_o}{d\hat{x}} \right) - 2\pi \hat{r}_i \hat{\rho}_i \left(\hat{v}_i - \hat{u}_i \frac{d\hat{r}_i}{d\hat{x}} \right) = 0, \quad (4.51)$$

where the subscripts i and o indicate that the respective variables are evaluated at the corresponding boundary. If these are streamlines, only the first term remains, since the flow is parallel to the boundary:

$$\frac{\partial}{\partial \hat{u}} = \frac{1}{\varepsilon} \frac{v}{u} = \frac{1}{\varepsilon} \frac{dr}{dx} = \frac{d\hat{r}}{d\hat{x}}. \quad (4.52)$$

In practice, this holds for all radial boundaries in the proposed models, as the flow is parallel to the wall, the centerline (by axisymmetry), and the dividing streamline between the primary and secondary domains (see

chapter 6). The averaged flow quantities are denoted as follows:

$$\langle \hat{q} \rangle = \frac{\int_{\hat{r}_i}^{\hat{r}_o} \hat{q} 2\pi \hat{r} d\hat{r}}{\int_{\hat{r}_i}^{\hat{r}_o} 2\pi \hat{r} d\hat{r}} = \frac{1}{\hat{A}} \int_{\hat{r}_i}^{\hat{r}_o} \hat{q} 2\pi \hat{r} d\hat{r}. \quad (4.53)$$

The conservation equations reduce to:

$$\frac{d}{d\hat{x}} (\langle \hat{\rho} \hat{u} \rangle \hat{A}) = 0, \quad (4.54)$$

$$\frac{d}{d\hat{x}} (\langle \hat{\rho} (\hat{u})^2 \rangle \hat{A}) = -\frac{1}{\Pi} \hat{A} \frac{d\hat{p}}{d\hat{x}} + \hat{F}_o - \hat{F}_i, \quad (4.55)$$

$$\frac{d}{d\hat{x}} (\langle \hat{\rho} \hat{u} \hat{h}_t \rangle \hat{A}) = \hat{Q}_o - \hat{Q}_i, \quad (4.56)$$

taking into account the uniform static pressure along $\hat{r}$ at leading order, and where the momentum and energy exchange terms were introduced:

$$\hat{F}_{o,i} = \frac{1}{\varepsilon^2 Re} 2\pi \hat{r}_{o,i} \hat{\mu}_{\text{eff},o,i} \left. \frac{\partial \hat{u}}{\partial \hat{r}} \right|_{o,i}, \quad (4.57)$$

$$\hat{Q}_{o,i} = \frac{1}{\varepsilon^2 Re} 2\pi \hat{r}_{o,i} \left(\frac{1}{Pr} \hat{k}_{\text{eff},o,i} \left. \frac{\partial \hat{T}}{\partial \hat{r}} \right|_{o,i} + \hat{u}_{o,i} \hat{\mu}_{\text{eff},o,i} \left. \frac{\partial \hat{u}}{\partial \hat{r}} \right|_{o,i} \right). \quad (4.58)$$

The terms $\hat{F}_{o,i}$ and $\hat{Q}_{o,i}$ represent friction, heat transfer, and mechanical work. They vanish at the centerline by axisymmetry; at the wall, only friction remains owing to the assumption of adiabatic walls and the no-slip condition.

For the single-stream configuration (chapter 5, figure 1.4b), integration over the full cross-section leaves only wall friction and no energy exchange. In the double-stream configurations (chapters 6 and 7, figure 1.4c), all exchange terms are in principle present at the internal interface. Energy exchange is nevertheless neglected, as inlet total temperatures are identical in the considered RANS dataset (section 2.4) and the associated contributions through Q_i are two orders of magnitude smaller than the cross-sectional gradient and force terms. This is consistent with the LES results of [8], which show only a short, localised exchange of total enthalpy in the converging section of the mixing pipe. The net effect is further reduced by compensation between kinetic-energy transfer and opposite

heat transfer due to the low temperature of the expanded primary stream (cf. (4.32)). Accordingly, $Q_i = 0$ is assumed, implying uniform total temperature from (4.35). Energy-exchange closures for unequal inlet total temperatures are left for future work.

The ODEs (4.54) are not yet suitable for 1D modelling, as they involve averaging operators acting on multiple variables and exchange terms defined through radial derivatives. Closure relations based on averaged flow quantities are therefore introduced in the following chapters for the specific models. Equations expressed in terms of separately averaged variables are obtained by prescribing parametrised radial profiles. In this work, these profiles are assumed piece-wise uniform (and thus uniform in the single-stream configuration), thereby avoiding the introduction of shape factors accounting for nonlinear interactions within the averaging operator, such as the parameters α and β proposed by Grazzini et al. [15]:

$$\langle \hat{\rho} \hat{u} \rangle = \langle \hat{\rho} \rangle \langle \hat{u} \rangle, \quad \beta = \frac{\langle (\hat{u})^2 \rangle}{\langle \hat{u} \rangle^2}, \quad \text{and} \quad \alpha = \frac{\langle (\hat{u})^3 \rangle}{\langle \hat{u} \rangle^3}, \quad (4.59)$$

leading to the following equations:

$$\frac{d\hat{m}}{d\hat{x}} = 0, \quad (4.60)$$

$$\hat{m} \frac{d}{d\hat{x}} \left(\alpha \langle \hat{u} \rangle \right) = -\frac{1}{\Pi} \hat{A} \frac{d\hat{p}}{d\hat{x}} + \hat{F}_o - \hat{F}_i, \quad (4.61)$$

$$\hat{m} \frac{d}{d\hat{x}} \left(\frac{1}{2} \beta \langle \hat{u} \rangle^2 + \langle \hat{T} \rangle \right) = \hat{Q}_o - \hat{Q}_i, \quad (4.62)$$

where $\hat{m} = \langle \hat{\rho} \rangle \langle \hat{u} \rangle \hat{A}$ is the non-dimensional mass flow rate. Piece-wise uniform profiles imply $\alpha = \beta = 1$ for both the single- and double-stream configurations presented in the following chapters.

Final dimensional equations

The one-dimensional models in the following chapters have been constructed with dimensional equations. Omitting the explicit averaging operators and returning to dimensional quantities, the unsteady, quasi-1D conservation equations read:

$$A \frac{\partial}{\partial t} (\rho) + \frac{\partial}{\partial x} (\rho A u) = 0, \quad (4.63)$$

$$A \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho A u^2) = -A \frac{\partial p}{\partial x} + F, \quad (4.64)$$

$$A \frac{\partial}{\partial t}(\rho e_t) + \frac{\partial}{\partial x}(\rho A u h_t) = Q, \quad (4.65)$$

where F and Q collect all forces and energy exchanges at the boundaries of the domain. The equations above can be applied over the entire cross-section of the mixing pipe as in chapter 5, or over the separate streams as in chapters 6 and 7. In the latter case, the temporal derivatives are not considered since the developed model is based on the dividing streamline between both streams, which can only be defined in steady state. More details are presented in the derivation of the corresponding equations in chapter 6. In steady state, The conservation equations above can be converted to explicit ODEs for any flow variable [57]. The equation for the static pressure reads as follows:

$$\frac{1}{p} \frac{dp}{dx} = \frac{\gamma \text{Ma}^2}{1 - \text{Ma}^2} \frac{1}{A} \frac{dA}{dx} + \frac{1 + (\gamma - 1) \text{Ma}^2}{1 - \text{Ma}^2} \frac{F}{Ap} - \frac{\gamma \text{Ma}^2}{1 - \text{Ma}^2} \frac{Q}{\rho A u c_p T}, \quad (4.66)$$

having introduced the ratio of specific heats γ and the Mach numbers Ma :

$$\text{Ma} = u/a, \quad \text{and} \quad a = \sqrt{\gamma R_{\text{gas}} T}, \quad (4.67)$$

where R_{gas} denotes the specific gas constant. Notably, the equation (4.66) becomes singular at the sonic point $\text{Ma} = 1$. This observation is crucial, since the equation above is the first point in the derivation from the full RANS equations where choking naturally appears. The series of simplifications above has preserved a formal description of choking and has allowed to discern it from other flow phenomena at play in ejectors, for example shock trains and turbulent mixing. The one-dimensional description of choking is discussed in more details in chapters 6 and 7. The singularity at the sonic point is paired with a compatibility constraint on the geometry and the exchange terms, which appears by transferring the common denominator $1 - \text{Ma}^2$ to the left hand side and considering $\text{Ma} = 1$:

$$\gamma \text{Ma}^2 \frac{1}{A} \frac{dA}{dx} + \left[1 + (\gamma - 1) \text{Ma}^2 \right] \frac{F}{Ap} - \gamma \text{Ma}^2 \frac{Q}{\rho A u c_p T} = 0. \quad (4.68)$$

The sum of these terms should equal zero to obtain a finite (and physical) pressure gradient. Choking is discussed in further details in chapters 6

and 7. The following equations for the stagnation pressure and stagnation temperature close the system, since three variables suffice to define the full state:

$$\frac{1}{p_t} \frac{dp_t}{dx} = \frac{F}{Ap} - \frac{\gamma \text{Ma}^2}{2 \left(1 + \frac{\gamma-1}{2} \text{Ma}^2\right)} \frac{Q}{\rho A u c_p T}, \quad (4.69)$$

$$\frac{1}{T_t} \frac{dT_t}{dx} = \frac{1}{1 + \frac{\gamma-1}{2} \text{Ma}^2} \frac{Q}{\rho A u c_p T}, \quad (4.70)$$

These equations never become singular and are further particularly practical since the stagnation conditions are typical boundary conditions at the inlets (see chapter 1). A comprehensive list of the explicit ODEs for all relevant quantities is presented in appendix A (in line with chapter 8 of Shapiro [57]).

5

Single stream one-dimensional model

5.1 Introduction

The model developed in this chapter is based on the single-stream framework introduced in chapter 1 (see figure 1.4b). In this formulation, the flow variables in the mixing section are averaged across the entire cross-section, and the ejector is represented as a pipeline network with two inlets and one outlet. The governing equations consist of integral conservation laws in both space and time, following classic unsteady pipeline-network formulations [90–92]. The domains are coupled by a junction model inspired by Lang and Mindt [92].

Several ejector applications involve inherently unsteady operating conditions. Examples include pulsed solar refrigeration [93], pulsed thrust augmentation [94–98], pulsed mixing for combustion or compactness [99, 100], and pulse detonation engines [101]. The characteristic time scales in these systems range from a few seconds down to milliseconds, giving rise either to sequences of quasi-steady states or to traveling pressure waves [99, 100, 102, 103]. These disturbances primarily propagate in the streamwise direction, owing to the slender geometry of ejectors, and their dynamics are analogous to those studied in pipeline networks [90–92].

Against this backdrop, the single-stream formulation provides a compact yet flexible description of ejector operation that extends naturally to unsteady conditions. It allows the imposition of time-varying boundary inputs, such as sudden pressure rises or valve closures, and can track the resulting pressure waves and flow reversals. In this way, the model preserves the low computational cost of reduced-order methods while avoiding the quasi-steady assumptions of earlier approaches.

We first validate the model against steady experimental data for entrainment ratios from Besagni *et al.* [12] and Mazzelli *et al.* [13]. We then use the 1D framework to investigate the choking mechanism predicted by the integral formulation, and finally benchmark the model against 2D RANS simulations under both steady and transient conditions.

The chapter is structured as follows: section 5.2 introduces the model derivation. Section 5.3 details the numerical implementation, while section 5.4 outlines calibration procedures based on local and global quantities. Section 5.5 reviews the CFD setup. Section 5.6 presents experimental and numerical validation, and section 5.7 concludes with key findings and future directions. These sections are largely based on the journal paper in which the model has been presented [104].

5.2 Definition

In what follows, we assume that the operating fluid is an ideal gas and the flow is adiabatic. We restrict our model to conditions at which the primary flow is choked and the two streams are parallel when entering the mixing region. We then picture the ejector as a pipeline network consisting of two inlets (for the primary and the secondary flows) and one outlet. Each of these pipes is modeled as a 1D domain, as shown in Figure 5.1.

Naturally, the physics of the mixing process in a supersonic ejector differ significantly from that occurring in the junction of a pipe network. Nevertheless, considering that we focus on integral quantities and not on the detailed flow field, this is not a limit. On the contrary, the results in this work show that the approximation is appropriate.

We denote the stream-wise coordinate as x in the three domains. Within each of these, we describe the flow using the quasi-1D Euler equations developed in chapter 4, considering the cross-sectional area variation $A(x)$, adiabatic walls and friction losses:

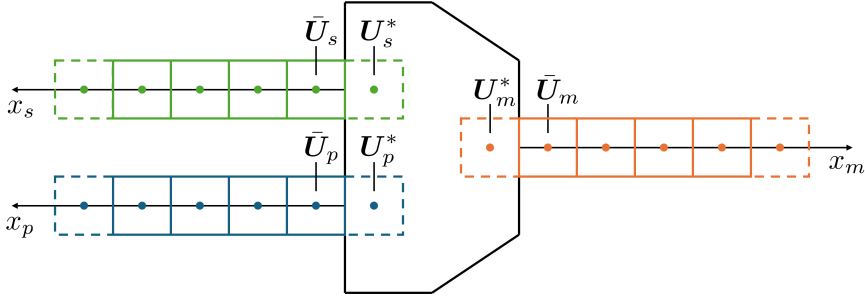


Fig. 5.1 Definitions of ‘initial’ and ‘star’ states in the 1D ejector model pictured as three 1D domains joined through a junction model.

$$\partial_t \begin{pmatrix} \rho \\ \rho u \\ \rho e_t \end{pmatrix} + \partial_x \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u h_t \end{pmatrix} = -\frac{\partial_x A}{A} \begin{pmatrix} \rho u \\ \rho u^2 \\ \rho u h_t \end{pmatrix} - \frac{1}{2} f \frac{l_w}{A} \begin{pmatrix} 0 \\ \rho |u| u \\ 0 \end{pmatrix}$$

where ρ is the density, u the (average) velocity, $e_t = C_V T + 0.5u^2$ the total specific internal energy, p the pressure, $h_t = e_t + p/\rho$ the total specific enthalpy, f a friction factor, l_w is the cross-section’s perimeter and A is the cross-section’s area.

These equations are typically used in nozzle flows [105]. All variables depend on the stream-wise coordinate x , except for the friction factor f which is computed with a correlation inspired by Blasius’ formula [106]:

$$f = c (Re_D)^{-0.25} , \quad (5.1)$$

with the Reynolds number Re_D based on the local hydraulic diameter. The value for c is 0.0791 for turbulent flow in smooth pipes, but in this work this coefficient is considered as a model parameter that needs to be calibrated (see chapter 3). We allow this coefficient to differ between the primary, secondary and mixing sections and use the subscripts p, s, m respectively (thus we have c_p, c_s and c_m). This allows for tuning the losses in each line separately, hence accounting for different local losses such as, e.g., bends or valves. Although the current model can not localize these losses, this term allows to distribute their impact over the whole domain.

The quasi 1-D equation can be more concisely written in terms of state vector $\mathbf{U} = [\rho, \rho u, \rho e_t]^T$ and appropriate flux function $\mathbf{F}(\mathbf{U})$ and source

function $S(U)$:

$$\partial_t U + \partial_x F(U) = S(U) . \quad (5.2)$$

All the relevant variables can be computed from the state vector.

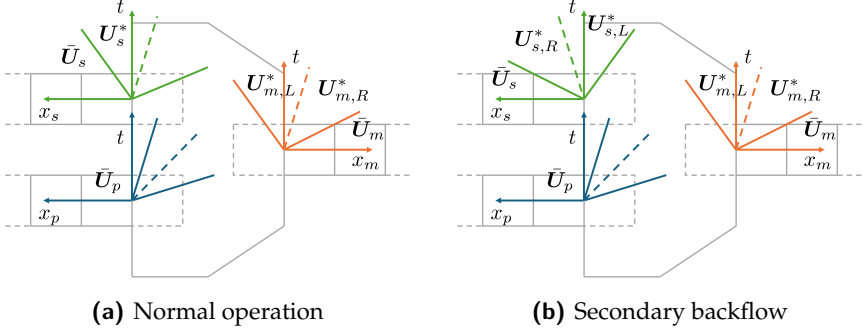


Fig. 5.2 Schematic of wave structures at the junction in normal (top) and backflow (bottom) operation with a supersonic primary flow and subsonic secondary and mixed flow. Solid lines represent pressure waves, dashed lines represent contact lines. The secondary and mixing boundary lie inside a star region.

The solution of (5.2) is coupled with a mixing model inspired by the junction model by Lang and Mindt [92]. This treats the junction as a generalized Riemann problem, which reduces to the classic shock tube problem [107] if two pipes are considered. The extension to an arbitrary number of pipes is carried out by imposing a pressure wave in each pipe and a contact discontinuity in the outlet pipes. Figure 5.2a illustrates the wave structure in the normal operating conditions of an ejector: the characteristic lines of the primary supersonic flow are directed downstream; the subsonic secondary and mixed flow admit one characteristic line directed upstream. Figure 5.2b shows the same schematic in the case of backflow in the secondary inlet.

The definition of the (time dependent) boundary conditions for the three domains is computed by solving a Riemann problem for each pipe. These are characterized by ‘star’ states U_i^* linked to the initial states $\bar{U}_i$ by the Rankine-Hugoniot relations [107]. In both cases, we have $i = p, s, m$.

Following [92], the jump relations are

$$u^*(p^*, \bar{U}) = \bar{u} - (p^* - \bar{p}) \sqrt{\frac{1 - \mu^2}{\bar{\rho}(p^* + \mu^2 \bar{p})}} \quad (5.3)$$

$$\bar{\rho}^*(p^*, \bar{U}) = \bar{\rho} \left(\frac{p^* + \mu^2 \bar{p}}{\mu^2 p^* + \bar{p}} \right) \quad (5.4)$$

in case of a compression, i.e. $p^* > \bar{p}$, and

$$u^*(p^*, \bar{U}) = \bar{u} - \frac{2\bar{a}}{\gamma - 1} \left(\left(\frac{p^*}{\bar{p}} \right)^{\frac{\gamma-1}{2\gamma}} - 1 \right) \quad (5.5)$$

$$\bar{\rho}^*(p^*, \bar{U}) = \bar{\rho} \left(\frac{p^*}{\bar{p}} \right)^{\frac{1}{\gamma}} \quad (5.6)$$

in case of an expansion, i.e. $p^* < \bar{p}$. In these relations, $\mu = (\gamma - 1)/(\gamma + 1)$. A contact discontinuity splits the star region of the outlet pipes (cf. figure 5.2) into left (U_L^*) and right (U_R^*) states. These have equal velocity and pressure but differ in the density by an (unknown) jump $\Delta\rho$, i.e. $\rho_L^* = \rho_R^* + \Delta\rho$.

These conditions must be combined with coupling conditions ensuring conservation of mass, momentum and energy. To simplify the problem, the formulation in [92] assumes equal total enthalpy for all inlets and outlets. In this work, we release this assumption and consider the complete balance equations. In normal operating conditions, these read

$$0 = \dot{m}_p + \dot{m}_s + \dot{m}_m \quad (5.7)$$

$$0 = \eta_m (\dot{m}_p u_p + A_p p_p + \dot{m}_s u_s + A_s p_s) - \dot{m}_m u_m - A_m p_m \quad (5.8)$$

$$0 = \dot{m}_p h_{t,p} + \dot{m}_s h_{t,s} + \dot{m}_m h_{t,m} \quad (5.9)$$

where all variables refer to 'star' conditions but the stars are omitted for conciseness. In the momentum balance, the mixing efficiency η_m has been introduced to account for momentum losses due to, e.g., force exchanges with solid walls. In case of backflow, the energy equation is simplified as $h_{t,p} = h_{t,s} = h_{t,m}$ since the junction is supplied by one single line.

The lack of an unsteady term in the conservation of mass implies that the mixing is inertia-less and no mass accumulation occurs in the junction.

Moreover, variables in the outlet pipe are to be interpreted as ‘averaged’ quantities, not implying the complete mixing between the two streams. Finally, the momentum equation focuses on the stream-wise coordinate because of the parallel streams assumption while the total enthalpy conservation stems from the adiabatic assumption.

5.3 Numerical methods

5.3.1 0D initialisation

The 1D model described in the previous section can be easily reduced to a 0D model in steady and frictionless conditions. In this case, the flow field is fully determined by the coupling equations and the boundary conditions. The 1D steady Euler equations in each pipe ensure the conservation of total pressure and total enthalpy. Moreover, in steady conditions, the Mach number can be computed in the whole domain using the classic area-Mach relation [105] if it is known in only one section:

$$\left(\frac{A_1}{A_2}\right)^2 = \frac{\text{Ma}_2^2}{\text{Ma}_1^2} \left[\frac{\left(1 + \frac{\gamma-1}{2} \text{Ma}_1^2\right)}{\left(1 + \frac{\gamma-1}{2} \text{Ma}_2^2\right)} \right]^{\frac{\gamma+1}{\gamma-1}} \quad (5.10)$$

where the indices 1, 2 indicate two arbitrary cross-sections between which the flow is isentropic.

The resulting 0D model allows for initializing the 1D (unsteady) computations with a solution that respects the junction conditions, hence avoiding numerical instabilities and speeding up convergence to steady state conditions.

All variables related to the primary flow can be easily computed from the choked throat using equation (5.10) and the inlet conditions. The computation in the other domains depends on the operating conditions of the ejector.

For a given ejector ($\mathbf{A} = [A_p, A_s, A_m]^T$) and operating conditions ($\mathbf{c} = [p_{t,p}, T_{t,p}, p_{t,s}, T_{t,s}, p_b]^T$), the identification of the operating regime can be carried out given the critical point (at back pressure p_b^*) and the breakdown point (at back pressure p_b^0). Therefore, these conditions are calculated first.

The critical operating point is characterized by the maximum pressure for which the mixing pipe is choked. This choking is to be interpreted in the compound choking formalism (see chapter 6). Therefore, the Mach num-

ber $\text{Ma}_m(x)$ can be calculated in the same way as for the primary nozzle. In the critical point, the flow reaches $\text{Ma}_m(x) = 1$ in the mixing pipe and is subsonic in the diffuser. The associated secondary Mach number Ma_s can be found with an iterative procedure, as the solution of an equation of the kind $\mathcal{R}(\text{Ma}_s) = 0$. For given Ma_s and stagnation states at the secondary inlet, one can compute all flow variables related to the secondary flow at the junction. Conservation of mass (5.7) and energy (5.9) then provide $\dot{m}_m$ and $h_{t,m}$ since both inlet states are now known. Then, these variables can be introduced in the momentum equation (5.8): the function $\mathcal{R}(\text{Ma}_s)$ is the residual in this equation. It is worth noting that the mixing efficiency η_m influences the solution of the 0D model.

The breakdown point is characterized by no secondary flow, so the secondary pipe has zero velocity and uniform total pressure and total temperature. The mass flow rate and total enthalpy are identical in the primary and mixing pipes, leaving the Mach number Ma_m as the only unknown. This is found with a similar iterative procedure as for the critical point, leading to the breakdown pressure p_b^0 .

Given p_b^* and p_b^0 , the computation of the remaining variables depends on the operating regime and is described in what follows.

Critical regime: $p_b < p_b^*$

In the critical regime, the flow is choked but the static pressure is lower than the one in isentropic conditions. Consequently, the flow is supersonic in the diffuser and expands further: shocks must occur in the diffuser to let the pressure reach the prescribed back pressure. These appear as a sequence of oblique shocks in a real ejector [9], but the proposed 1D formulation pictures them as a single normal shock. This is a classic assumption in the LPM of ejectors [30, 32, 46].

In these conditions, all variables of interest depend on the position of this fictitious normal shock. We herein denote as x_{sh} its distance from the diffuser inlet. For a given x_{sh} , the geometry of the diffuser prescribes the cross-sectional area A_{sh} at which this occurs, while the area-Mach equation (5.10) prescribes the Mach number Ma_{up} upstream of the shock. The normal shock equations then provide the Mach number Ma_{down} and pressure p_{down} downstream of the shock:

$$\text{Ma}_{down}^2 = \frac{1 + \frac{\gamma-1}{2}\text{Ma}_{up}^2}{\gamma\text{Ma}_{up}^2 - \frac{\gamma-1}{2}} \quad (5.11)$$

$$\frac{p_{down}}{p_{up}} = 1 + \frac{2\gamma}{\gamma + 1} \left(\text{Ma}_{up}^2 - 1 \right). \quad (5.12)$$

Downstream the shock, the subsonic flow decelerates isentropically to a Mach number Ma_b at the diffuser exit (cf. equation 5.10). The static pressure p_b follows from the conservation of total pressure after the shock. One can thus iterate on x_{sh} until the resulting back pressure is equal to the imposed one.

Sub-critical regime: $p_b^* < p_b < p_b^0$

In the sub-critical regime the flow is subsonic in both the secondary and the mixing pipes. Therefore, one must iterate on both Mach numbers Ma_s and Ma_m . These are identified by ensuring the compliance with the momentum equation *and* the matching with the back pressure.

Backflow regime: $p_b^0 < p_b$

In this work, we restrict the modeling to the backflow in the secondary pipe (cf. the characteristic lines in figure 5.2). In principle, the flow can reverse in any section at sufficiently high back pressure, but this might conflict with the assumption of choked primary flow. The (unknown) Mach numbers Ma_s and Ma_m are below unity, so they are iterated upon as in the normal sub-critical regime. The reversed secondary flow has two consequences on the calculation procedure: (1) the energy conservation equation in the junction changes to equality of total enthalpy in all pipes (cf. section 5.2) and (2) the stagnation pressure $p_{t,s}$ at the inlet is now the static discharge pressure of the reversed secondary flow. The mass flow rate $\dot{m}_s$ can be calculated from the guessed Mach number Ma_s , the total temperature $T_{t,s} = T_{t,p}$ and the static discharge pressure $p_{t,s}$ with gas dynamic relations. The remaining calculation is similar to the sub-critical regime: the conservation of mass provides $\dot{m}_m$, which allows calculating the residual on the momentum equation, and the static back pressure from the guessed Mach number Ma_m and the known total temperature $T_{t,m} = T_{t,p}$.

5.3.2 Unsteady computations

In the numerical implementation of the model, every time step requires the solution of equations (5.3)-(5.9) for the unknown p_s^* , p_m^* and $\Delta\rho_m$. An additional jump in density $\Delta\rho_s$ for the secondary line appears in the case of backflow, bringing the total number of unknowns to 4. In these conditions, we simplify the energy conservation equation by imposing more restric-

tive equality of total enthalpy as in [92]. The resulting star states U^* are then provided as boundary conditions to advance the numerical integration of (5.2). This system of equations is solved using a classic finite volume approach. The source term is treated explicitly and the time marching is carried out with a first order Euler scheme (cf. chapter 15 of [107]):

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} \left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n \right) + \Delta t S_i^n, \quad (5.13)$$

where the subscript i denotes the cell index in space and n denotes the index in time. The fluxes are calculated with a first order Roe scheme:

$$F_{i+\frac{1}{2}} = \frac{1}{2} (F_i + F_{i+1}) - \frac{1}{2} \sum_{k=1}^3 \tilde{\alpha}_k |\tilde{\lambda}_k| \tilde{K}_k, \quad (5.14)$$

where $\tilde{\alpha}_k$ denotes the wave strength and $\tilde{\lambda}_k$ and $\tilde{K}_k$ denote the eigenvalues and eigenvectors of the system. The interested reader is referred to chapter 11 of [107] for more details. Boundary conditions are handled with ghost cells as proposed in chapter 7 of [108] (cf. figure 5.1). The boundary fluxes can thus be calculated with the internal scheme, which requires the states at both sides of the interface.

The junction model only provides boundary conditions and does not impose any requirements on the numerical scheme. The first order Roe scheme was chosen in this work for its low diffusion and its relative simplicity.

5.4 Calibration

The current model can be tuned with 4 coefficients (cf. section 5.2) which we group into a single vector for conciseness: $w = [\eta_m, c_p, c_s, c_m]^T$. These coefficients must be adjusted to match the model prediction with experimental or numerical data. Depending on the available data, the calibration might be based on global quantities (e.g. the entrainment ratio) or locally averaged quantities (e.g. the downstream evolution of the momentum).

To the authors' knowledge, no experimental data is available in the literature concerning locally averaged quantities. Most of the available experimental works provide data on entrainment ratio ϕ as a function of various operating conditions, while only a few authors give both mass flow rates. On the other hand, 2D or 3D simulations can easily provide data for the local validation of the proposed model.

Regardless of the approach, the calibration procedure is a regression problem in which a cost function $J(\mathbf{w})$ must be minimized. In a global approach, this cost function might take the form

$$J(\mathbf{w}) = \begin{cases} \sum_{j=1}^{n_p} \left(\dot{m}_{p,j} - \dot{m}_{p,j}^d \right)^2 + \left(\dot{m}_{s,j} - \dot{m}_{s,j}^d \right)^2 & \text{if available} \\ \sum_{j=1}^{n_p} \left(\phi_j - \phi_j^d \right)^2 & \text{otherwise} \end{cases} \quad (5.15)$$

where n_p is the number of available data points, the superscript d refers to available data and the j -th prediction is provided by the model at all the conditions at which the data was measured:

$$\dot{m}_{p,j}, \dot{m}_{s,j} = f(\mathbf{A}_j, \mathbf{c}_j; \mathbf{w}) . \quad (5.16)$$

Likewise, in a local approach the cost function could be

$$J(\mathbf{w}) = \sum_{i=p,s,m} \sum_{n=1}^{n_t} \left(\alpha_1 \left\| \dot{\mathbf{m}}_{i,n} - \dot{\mathbf{m}}_{i,n}^d \right\|_2^2 + \alpha_2 \left\| \mathcal{M}_{i,n} - \mathcal{M}_{i,n}^d \right\|_2^2 + \alpha_3 \left\| (\dot{\mathbf{m}}\mathbf{h}_t)_{i,n} - (\dot{\mathbf{m}}\mathbf{h}_t)_{i,n}^d \right\|_2^2 \right) \quad (5.17)$$

where $\|\bullet\|_2$ denotes the l_2 norm of a vector, the coefficients $\alpha_1, \alpha_2, \alpha_3$ can be used to define different weights and the variables $\dot{\mathbf{m}}_{i,n}$, $\mathcal{M}_{i,n}$ and $(\dot{\mathbf{m}}\mathbf{h}_t)_{i,n}$ denote the vectors of mass, momentum and enthalpy flow rates in all cells of domain i at time step n . These quantities are defined by integrating over the cross stream section as follows

$$\dot{m}(x) = \int_A \rho u dA \quad (5.18)$$

$$\mathcal{M}(x) = \int_A (\rho u^2 + p) dA \quad (5.19)$$

$$\dot{m}h_t(x) = \int_A \rho u h_t dA \quad (5.20)$$

Of course many other cost functionals can be defined and one might not necessarily restrict to single objective optimization. Similarly, one might set up the closure problem in terms of more complex friction losses (e.g. nonuniform friction coefficients or closure parameters depending on the

operating conditions) but none of these extensions fall within the scope of this work.

In both global and local calibrations, the minimization of the cost functions was carried out using a the L-BFGS-B algorithm [109], as implemented in the Python package SCIPY, with the following bounds for the closure parameters: $c_p, c_s, c_m \in [0, 0.5]$ and $\eta_m \in [0.85, 1.05]$. It is worth noting that a value $\eta_m > 1$ can occur if $A_m > A_p + A_s$ due to, e.g., the finite thickness of the surfaces separating the primary from the secondary flows.

5.5 2D CFD benchmark

2D URANS computations were performed in both steady and unsteady conditions, to benchmark the proposed 1D model. The ejector geometry and the computational domain is adapted from Lamberts et al. [16] and is shown in figure 5.3. The mesh is mostly structured, except near the exit of the primary nozzle. It is resolving the boundary layer with $y^+ < 1$. The simulations were carried out with the settings from section 2.2.1. The numerical values of the pressures and temperatures imposed at the boundaries are reported in section 5.6. For all simulations, the mixing length for computing the turbulence dissipation ω and the turbulence intensity are the same as in Lamberts [16], that is $l_{mix} = 0.0044$ m and $TI = 5\%$. The wave transmissive condition is used with a relaxation length scale of 10^{-4} m. The mesh is resolving the boundary layer with $y^+ < 1$.

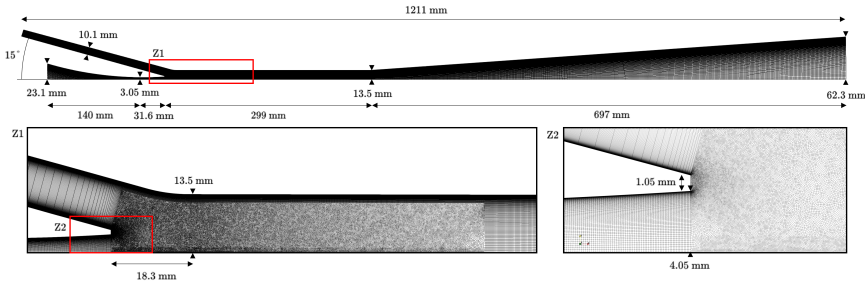


Fig. 5.3 Geometry and mesh used in the axisymmetric RANS simulations to verify the unsteady 1D model. The geometry is planar and has a width of 48.8 mm. The 2D mesh is mostly structured and contains 338164 cells. A symmetry boundary condition is imposed on the bottom plane.

5.6 Results

The model is validated globally against experimental data from Mazzelli [13] and Besagni [12] in section 5.6.1. The results from this 0D validation are further analyzed to reveal the choking mechanism according to the proposed model in section 5.6.2. Sections 5.6.3 and 5.6.4 report on the benchmarking in steady and transient conditions.

5.6.1 Global validation

We begin with the model calibration and testing using the experimental data by Mazzelli et al. [13] concerning a planar and supersonic ejector. The dimensions of the key sections are reported in Table 5.1; the reader is referred to the original publication for more details.

domain	inlet	outlet	length
primary	110.25	8.00	422
secondary	2×62.14	2×12.72	430
mixing	27.06	130.14	1019

Table 5.1 Height and length of the inlet and outlet sections for the experiments reported by Mazzelli [13]. The geometry is planar with a uniform depth of 48.8 mm. The primary throat is 5.98 mm high. The secondary section is counted twice to account for both inlets. All dimensions are in mm.

The experiments were conducted keeping the secondary port open to the atmosphere and inlet pressure for the primary nozzle varying in the range 2-5 bar. The back pressure varies to cover a primary-to-back pressure ratio $p_b/p_{t,p}$ in the range 0.3 – 0.7. Figure 5.4 compares experimental data and model for various pressures in the primary flow. The global calibration (see chapter 3) was carried out including the entire data set (that is the four operating curves) and the resulting coefficients are $\eta_m = 0.85$, $c_p = 0$, $c_s = 0.5$ and $c_m = 0$.

The large friction factor in the secondary inlet indicates high losses in the secondary flow. This is in line with the comparatively low isentropic efficiency ($\eta_s = 0.75$, versus $\eta_p = 0.977$ in the primary) used by Metsue et al. [46] in the calibration of their model in the same data. The different losses in the two lines might be justified by the fact that the experimental set-up features a 90-degree bend downstream of the secondary pressure measurement.

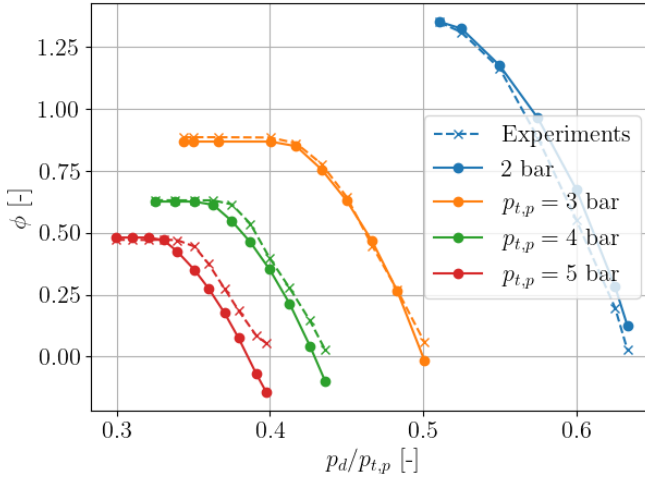


Fig. 5.4 Comparison of experimental data (in [13], Fig. 4) with the converged unsteady solver for different primary pressures. The different curves are obtained keeping the same pressure in the primary nozzle. Stars are used for the experimental data and circular markers for the model prediction. The model calibration yields $\eta_m = 0.85$, $c_p = 0$, $c_s = 0.50$ and $c_m = 0$ (cf. section 5.2 and chapter 3).

After calibration, the matching with experimental data is satisfactory. Better matching could be obtained by independently repeating the calibration for each operating curve. This allows for different coefficients for each operating curve and the derivation of a link between coefficients and operating conditions. This extension of the model calibration will be addressed in a future contribution.

As a second test case, we consider the experimental data by Besagni et al. [12] on a planar and sonic ejector. The relevant geometrical parameters are collected in Table 5.3. The comparison between model and experimental data on the four available points is summarized in Table 5.3. The relative errors are below 6 % with the model coefficients set to $\eta_m = 1$, $c_p = c_s = c_m = 0$.

The higher mixing efficiency in this test case might be due to a better design of the mixing area and the different locations chosen to define the mixing section. This was taken at the start of the constant area section in the geometry by Mazzelli et al. [13] and right after the primary nozzle exit for

the geometry by Besagni et al. [12]. The much larger hydraulic diameters in all sections of the ejector could also justify the negligible friction losses in the geometry by Besagni et al. [12].

domain	inlet	outlet	length
primary	21.7	3.1	75.0
secondary	44.5	44.5	75.0
mixing	47.6	47.6	304.8

Table 5.2 Height and length of the inlet and outlet sections for the experiments reported by Besagni [12]. The geometry is planar with a width of 203.2 mm. The primary throat is convergent, hence its area corresponds to the outlet section. All dimensions are in mm.

p_p [bar]	p_s [bar]	T_p [K]	T_s [K]	p_m [bar]	ϕ_{exp} [-]	ϕ_{mod} [-]	error [%]
2.47	1.02	361	304	0.997	3.91	3.87	1.1
2.47	1.02	359	302	0.981	4.17	4.11	1.4
2.46	1.02	358	306	0.927	4.66	4.69	0.7
2.46	1.01	367	304	0.927	5.01	4.73	5.5

Table 5.3 Operating conditions and experimental and numerical entrainment ratio on the sonic ejector by Besagni et al. [12]. The mixing efficiency and the friction factors were calibrated as constants on all four points: $\eta_m = 1$, $c_p = 0$, $c_s = 0$ and $c_m = 0$.

5.6.2 A note on the choking mechanism

We now consider the operating points at $p_{t,p} = 3$ bar from the dataset by Mazzelli et al. [13] (see Figure 5.4) and we analyze the 1D distribution of the (local) Mach number for all the available back pressures. The results are shown in Figure 5.5 for the primary, the secondary and the mixing lines, together with the corresponding area distributions. These were obtained by a quadratic fit of the originally piece-wise linear contours to avoid differentiability problems.

Since the primary flow is choked, the local Mach distribution is independent of the back pressure and reaches $M = 1.61$ as expected by the isentropic condition (given that $c_p = 0$). The range of back pressures covers the entire set of regimes. At $p_b = 1.5$ bar, the model is in the back-flow

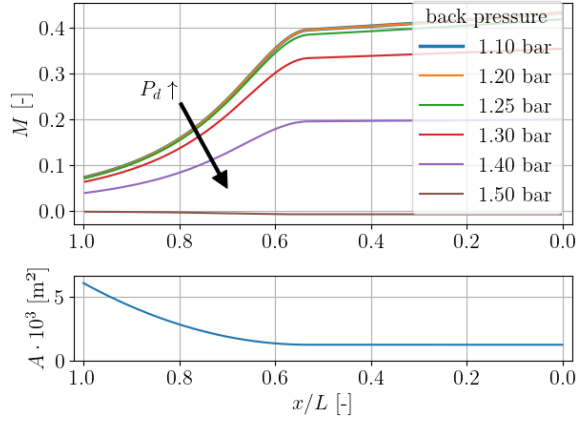
regime, and the Mach number is slightly negative in the secondary line. As visible from figure 5.4, this is due to a slight miss-match between the model and the experimental data. The assumption of constant $c_s = 0.5$ over the entire range of investigated conditions is too restrictive for this operating point, which is close to the break-down point (hence no secondary flow). Nevertheless, the matching in all other conditions is excellent, and Figure 5.5 shows that the secondary flow is far from being choked even when the ejector reaches the critical regime and the mixed flow is choked. This is the essence of the compound choking theory: the mixture of two streams of compressible flows can choke in a channel even if one of the two is subsonic. This mechanism was recently proposed by Lamberts *et al* [19] and explicitly embedded in a 0D model by Metsue *et al.* [46].

Finally, the flow reaches supersonic conditions in the mixing line for the cases with back pressure of 1.1 bar and 1.2 bar. As also described in the model by Banasiak and Hafner [37], this results in an initial expansion in the diffuser (in this case, starting at $x/L = 0.2$), followed by a normal shock to match the back pressure. The shock moves downstream as the back pressure decreases further. It is worth stressing that this normal shock is a common modeling simplification of the actual pattern of oblique shocks encountered in a supersonic ejector [9,10], and the present model naturally adjusts its location and strength.

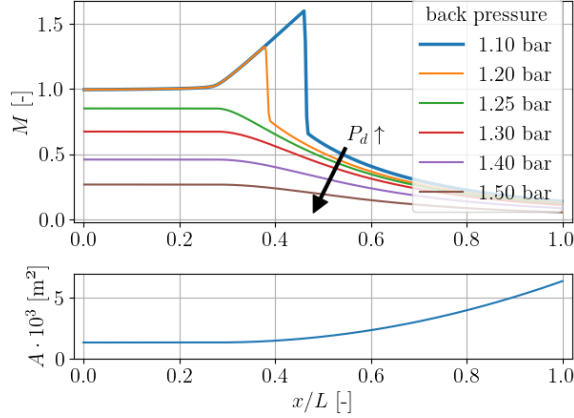
5.6.3 Steady state benchmark

We compare the performances of the proposed 1D model with the 2D CFD simulations in one of the operating points in the experiments by Mazzelli [13]. The use of a 2D formulation (see section 5.5) for the selected geometry is questionable because it does not allow for capturing important 3D effects (e.g. friction on the side walls). This results in over prediction of the secondary flow and an erroneous computation of the critical point (see Figure 9 in [13]). Nevertheless, our interest in this benchmark was to compare the prediction of stream-wise evolution of important variables such as mass, momentum and energy fluxes with the prediction from CFD. The proposed 1D model can easily compensate for the difference with respect to the experiments by tuning the model coefficients (as proven by the global validation in section 5.6.1).

We thus consider an operating condition with $p_{t,p} = 5.00$ bar, $p_{t,s} = 0.974$ bar, $p_b = 1.20$ bar and $T_{t,p} = T_{t,s} = 300$ K. This is taken from Table A.1 in [13] and corresponds to an on-design condition with large $p_{t,p}/p_b$



(a) Secondary line



(b) Mixing line

Fig. 5.5 Mach number in the secondary inlet and in the mixing pipe for different back pressures, as in figure 5.4 ($p_{t,p} = 3$ bar). The primary nozzle is choked and therefore not shown. In the critical regime, the secondary flow is subsonic, while a shock is formed in the diffuser.

ratio. Strong shocks are thus expected in the diffuser, as confirmed by the CFD investigation in [13].

Figure 5.6 shows a contour of the Mach number obtained in our CFD simulation, together with a comparison of the 1D distribution of mass flow

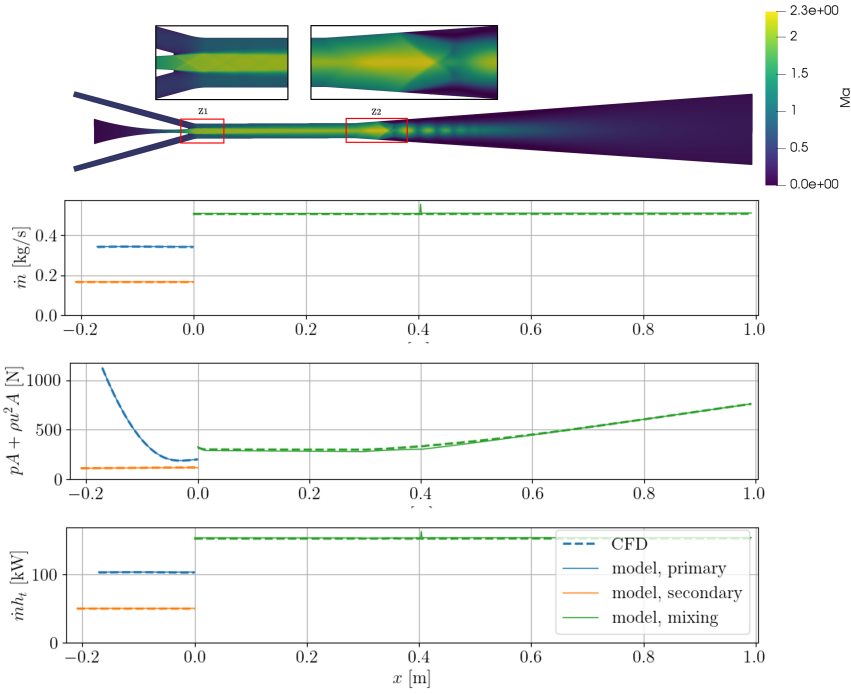


Fig. 5.6 Local Mach number from the steady CFD simulation (top) and the comparison of the mass, momentum and energy flow rates with the 1D model (bottom). The 2D simulation reveals expansion waves in the mixing duct (Z1) and a strong shock train in the diffuser (Z2). The coupling equations (5.7)-(5.9) impose the conservation of the shown quantities at the junction ($x = 0$). The computational domain has been mirrored for visualisation purposes.

rates, momentum and total enthalpy flow rates according to the proposed 1D model.

The well-known oblique shock pattern appears at the nozzle exit (zoom region Z1) and at the diffuser's inlet (zoom region Z2). These are strong enough to trigger boundary layer separation (also reported in [9] in similar conditions). The proposed model naturally places vertical shocks in these locations, and the comparison in the integral quantities is in excellent agreement. The model calibration was here carried out locally (see chapter 3) on the CFD data and yields $\eta_m = 1.04$, $c_p = 0.062$, $c_s = 0.391$ and $c_m = 0.148$.

Finally, it is worth stressing that the integral quantities in the mixing line are to be considered as ‘compound’ averages. The thermodynamic state of the mixture is defined regardless of whether the two streams have actually mixed (as opposed to the 1D models in [37,41]). While the integration/averaging of CFD results works well for extensive quantities such as the illustrated mass, momentum and energy rates, no similar averaging is naturally possible for intensive quantities such as pressure or Mach numbers. While the link between the local intensive quantities and the compound averages in the proposed 1D model is out of the scope of this work, the accuracy in the prediction of valuable integral quantities is proven.

5.6.4 Transient benchmark

We here consider the transient between two operating conditions for the same ejector as in the previous section. The initial operating point is the previously investigated one ($p_{t,p} = 5.00$ bar, $p_{t,s} = 0.974$ bar, $p_b = 1.20$ and $T_{t,p} = T_{t,s} = 300$ K) while the final one is obtained by suddenly increasing the back pressure to $p_b = 1.8$ bar while keeping everything else equal. The previous section’s solution serves as the initial condition for both the CFD and 1D models.

Figure 5.7 shows the static pressure contours from the 2D CFD simulation at six time steps, together with the mass flow rate predictions in both the CFD and the 1D model. The sudden increase in the back pressure triggers a strong pressure wave travelling upstream. In the 1D model, this takes the form of a normal shock while in the 2D CFD this has a curved wavefront. This is primarily due to the lower velocity near the walls. Interestingly, the velocity in the 1D model is close to the average wave velocity, and the dynamic of the back-ward travelling wave is well captured (see snapshot 5.7b at time $t = 1$ ms). This result is remarkable if one considers the compound averaging mechanism underpinning the definition of the mixture flow quantities in the 1D model: the wave propagation speed depends on the local ‘global’ Mach number, whose interpretation is obscure as the two streams are not completely mixed.

When the pressure wave reaches the junction, the coupling conditions require a change in the secondary flow but not in the primary one, which is assumed to be choked. Figure 5.7d shows that the 1D model correctly captures the wave passage while, at the same time, the flow in the mixing pipe recovers from the sudden change and returns positive. The wave then reflects at the constant total pressure inlet and re-enters the domain as

a much faster upstream travelling wave (cf. Figure 5.7e at time 7ms). This results in several oscillations before the ejector settles into the new steady state (cf. Figure 5.7e at time 32.4ms).

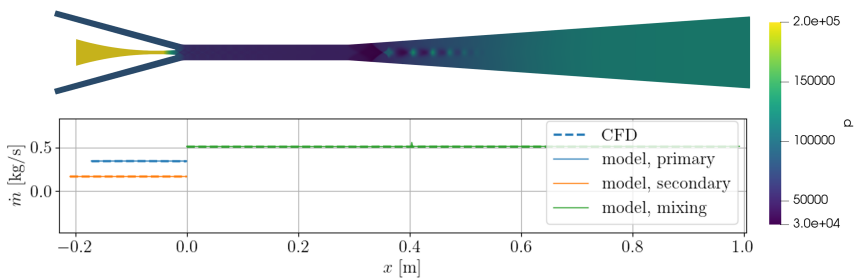
This test case proves that the proposed model can handle transitions between operating regimes, correctly describing traveling waves and the characteristic response time of the ejectors. Furthermore, the numerical implementation and the junction model proved sufficiently robust to handle sudden variations in the flow variables and flow reversal.

5.7 Conclusion

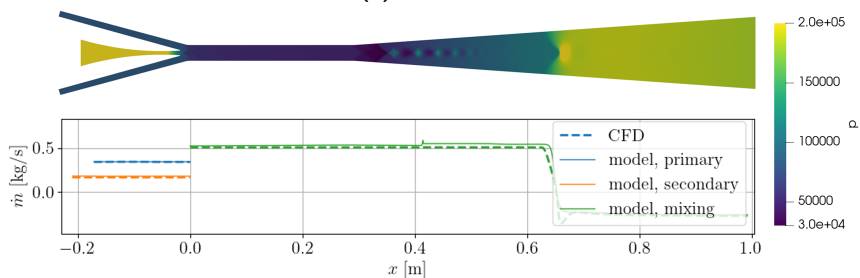
We have presented a new one-dimensional transient model for ejectors, formulated through a pipeline-network analogy. The model combines the unsteady 1D Euler equations with a junction model, enabling the study of ejector performance at a fraction of the computational cost of CFD. In the mixing section, the gas dynamics are expressed in terms of compound-averaged states, describing the streamwise evolution of integral quantities (such as mean velocity and momentum flux) together with the thermodynamic conditions of the mixture (pressure, Mach number). This formulation overcomes the traditional quasi-steady assumption.

To the best of the authors' knowledge, this is the first reduced-order model capable of simulating transient ejector operation without resorting to CFD. The model has been validated and benchmarked under both steady and transient conditions. In steady operation, it reproduces entrainment ratios with good accuracy across the range of experimental data available in the literature. In transients, it captures traveling pressure waves in the mixing and secondary sections, as well as flow reversal triggered by sudden changes in boundary conditions.

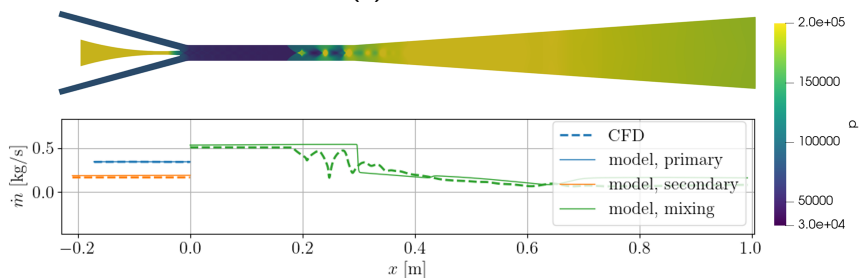
It should also be noted that the averaging operator in the mixing pipe conceals a complex interplay of flow phenomena. These are resolved in the models presented in the following two chapters, which assign separate numerical domains to the primary and secondary streams. This raises fundamental questions on how to partition the mixing pipe into parallel domains, how to model stream interaction, and how to capture choking effects. Chapter 6 introduces an extension of compound theory with frictional effects and focuses specifically on the mixing section, while chapter 7 presents the complete ejector model, including a detailed analysis of choking mechanisms and a comparison of model predictions with the experimental and numerical data of chapter 2.



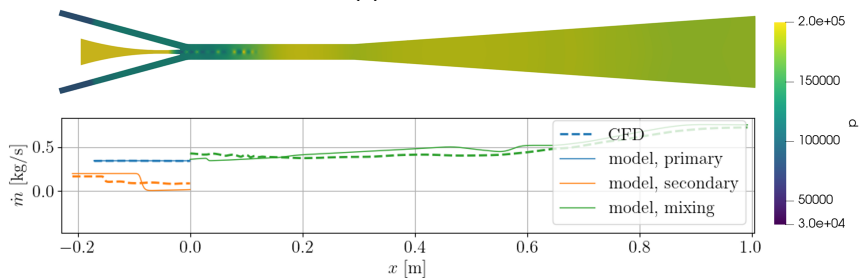
(a) $t = 0$ ms



(b) $t = 1.0$ ms



(c) $t = 2.4$ ms



(d) $t = 4.4$ ms

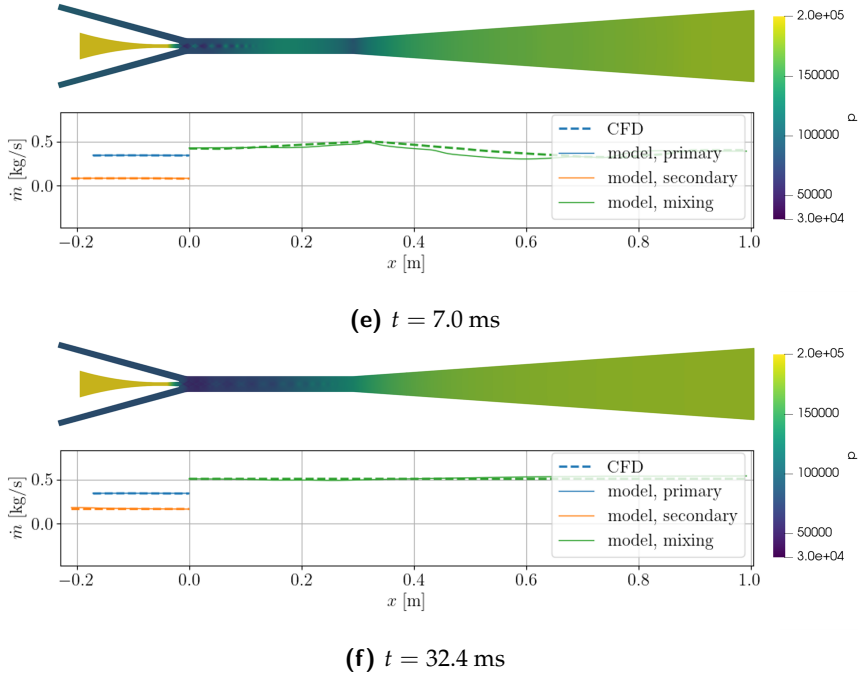


Fig. 5.7 Sequence of snapshots in the transient test case, showing the evolution of the pressure field in the CFD simulation together with the mass flow rates distribution from the CFD and the proposed 1D model. A sudden increase in back pressure from the steady state in Figure 5.6 creates a shock wave traveling upstream. This makes the temporarily reverse. The shock wave is reflected at the secondary inlet, leading to oscillations that die out in approximately 30 ms, after which the ejector reaches the new steady state. The 1D model captures the transient in mass flow rate and smoothly switches between the ejector's operating modes. The scale of the static pressure is limited to 2 bar for better visualisation. A similar match is obtained for the momentum and total enthalpy flow rates (not shown for conciseness). An animation of these snapshots is included in the supplementary material of the original publication [104].

6

Extension of the compound flow theory

6.1 Introduction

The previous chapter concluded that distinguishing between the primary and the secondary streams in a one-dimensional model raises fundamental questions regarding (1) how to partition the mixing pipe into parallel domains, (2) how to capture choking, and (3) how to model stream interaction. The existing isentropic compound theory [43,45] provides a practical framework that addresses the first two of these challenges; an extension was required here to incorporate friction forces between the streams and at the wall. This relatively straightforward modification not only improved predictive accuracy but, more importantly, offered new insights into the displacement of the sonic section—an effect that had previously remained only partially understood.

The first clear observation of such a displacement was made by Lamberts et al. [19], who reported that compound flows could choke *upstream* of the geometrical throat, in contrast to the classic *downstream* shift of the sonic section associated with friction. Kracik and Dvorak [26] later demonstrated numerically that increased wall roughness can produce the same upstream displacement. While Lamberts et al. [19] attributed the effect to non-isentropic phenomena, its fundamental cause remained unresolved.

Both works approached the problem through axisymmetric RANS simulations, but, to the best of the authors' knowledge, no theoretical analysis of the sonic section displacement has yet been provided.

The first theoretical treatment of compound flows was proposed by Pearson et al. [43]. These authors introduced the idea of splitting the domain into parallel 1D streams bounded by dividing streamlines and walls. The position of these streamlines can be identified by solving mass, momentum and energy conservation in each stream, together with a force balance at the dividing streamlines. The authors assumed adiabatic and isentropic conditions for the compound flow. Hence, only a pressure force is exerted between the streams, and the force balance requires equal static pressure for the entire cross-section. This formulation leads to a closed system of equations linking the evolution of mass, momentum and energy in each stream to the evolution of the respective cross-sections and stream-wise pressure distribution. Pearson et al. [43] state that the assumption of equal pressure is reasonable if the curvature of the streamlines remains limited.

Through some manipulations of the conservation equations, Bernstein, Heiser & Hevenor [45] showed that a single equation can be obtained for the uniform static pressure. Similarly to a single 1D stream (see, for example, [57]), this equation is singular with a division of zero by zero if the flow is choked. The major result of the study by Pearson et al. [43] lies in the fact that a compound flow can be choked with a combination of subsonic and supersonic streams. They showed that information cannot travel upstream in the subsonic streams due to the assumption of uniform static pressure. The derivation by Bernstein et al. [45] confirms the findings of Pearson et al. [43] identifying the compound choking from the singularity of the pressure equation. The same compound choking condition was found by Hoge & Segars [44] by minimising the impulse function defined in chapter 4 of Shapiro [57]. An alternative approach by Agnone [110] identifies the compound choking from the singularity of the governing equations, written in a matrix form, while Fage [111] defines it through the so-called 'elasticity of the fluid boundary' between the streams. Bernstein et al. [45] gave the name 'compound flow theory', which we also adopt. These works did not consider exchanges between the streams; hence, the flow remained isentropic.

Later work investigated the effect of forces on the compound flow. Hoge & Segars [44] explored the effects of friction between the streams and the walls, assuming isentropic and adiabatic conditions but without deriving

an explicit model to account for their impact on the flow. These authors postulated that the exchange of momentum between the streams would have a minimal impact on the compound choking mechanism. A more explicit approach is presented by Clark [35], Papamoschou [36] and Grazzini, Mazzelli & Milazzo [15], who used the correlation proposed by Papamoschou [112] to compute the shear stress between the streams. The authors used the resulting compound flow equations *with* momentum exchange in a 1D ejector model but did not investigate the choking mechanism. Earlier work by Otis [113] treats the mixing and choking of a compound flow for constant-area ducts. An important conclusion of that work is that additional effects such as forces or heat exchange do not affect the choking condition as derived in the original compound theory. To the best of the author's knowledge, a general treatment of compound choking with momentum exchange in variable area ducts has not yet been proposed in the literature.

Such an extension is presented in this chapter, specifically by investigating the effect of friction between the streams and the walls on the choking mechanism. It is analytically shown that the sonic section moves upstream or downstream depending on the relative magnitude of the inter-stream and wall friction. The analysis is similar to 'generalised 1D flow' [57, chapter 8], which describes quasi-1D compressible flow of a *single* stream with any combination of area change, friction, heat exchange and/or mass injection. Similarly, the isentropic *compound* flow theory, which only includes area change, is generalised in this work by including momentum exchanges. Our contribution is both theoretical and numerical: (1) we propose a simple theoretical model that captures the position of the sonic section, and (2) we propose a quasi-third-order approximation of the pressure gradient in the vicinity of the sonic section to circumvent the numerical issues near the singular sonic point.

The governing equations are derived in section 6.2 and converted into an explicit system of Ordinary Differential Equations (ODEs). Section 6.3 goes into more detail on the singularity of the equation for the static pressure at the sonic section. The numerical solution of the model is discussed in section 6.4. Numerical test cases are presented in section 6.5, to compare the results with the 1D predictions in section 6.6. Section 6.7 closes the paper with conclusions and perspectives. This chapter is largely based on the journal paper in which the extension has been presented [64].

6.2 Model definition

The governing equations are derived in their general form in section 6.2.1. Section 6.2.2 introduces the treatment of the friction forces in the equation governing the streamwise pressure gradient. The corresponding choking mechanism is analysed in section 6.2.3.

6.2.1 Governing equations

Consider an internal flow consisting of two parallel streams (cf. figure 6.1) of the same ideal gas. The theory can be extended to any number of streams, but this generalisation is left for future work. The streams are modelled as uniform 1D streams, i.e., in terms of cross-stream averaged quantities. The main assumption for compound-compressible flows is that the static pressure is uniform in all cross-sections, hence:

$$p_p(x) = p_s(x) = p(x), \quad (6.1)$$

where p denotes the static pressure, x is the axial coordinate and the subscripts p and s refer to the primary and secondary stream. The distribution of the cross-sections $A_i(x)$, with $i \in [p, s]$, is initially unknown and should be calculated. These are subject to a geometrical constraint imposed by the cross-section of the channel $A(x)$:

$$\sum_{i \in [p, s]} A_i(x) = A(x). \quad (6.2)$$

The governing equations consist of the conservation of mass, momentum and energy in each stream. For a steady, quasi-1D flow without energy exchange, these read as follows (cf. chapter 4):

$$\frac{d}{dx}(\rho_i A_i u_i) = 0, \quad (6.3)$$

$$\frac{d}{dx}(\rho_i A_i u_i^2) = -A_i \frac{dp_i}{dx} + F_i, \quad (6.4)$$

$$\frac{d}{dx}(\rho_i A_i u_i h_{t,i}) = 0, \quad (6.5)$$

where ρ denotes the density, u denotes the axial velocity, F_i is the net force per unit length exerted on stream i and h_t denotes the total enthalpy. The force F_i accounts for the friction between the streams and at the wall, and

is defined in section 6.2.2. The equations imply that the streams do not exchange mass and that the separating line between the streams is the *dividing streamline*. Heat exchange is not considered as discussed in chapter 4. This system of equations can be converted to an explicit system of ODEs for numerical integration, see also appendix A. The derivation can be found in classic textbooks; for example in chapter 8 of Shapiro [57]. The equations in terms of the static pressure p , the total pressure p_t and the total temperature T_t read as follows (cf. chapter 4):

$$\frac{1}{p_i} \frac{dp_i}{dx} = \left[\frac{1 + (\gamma - 1) \text{Ma}_i^2}{1 - \text{Ma}_i^2} \right] \frac{F_i}{A_i p_i} + \left[\frac{\gamma \text{Ma}_i^2}{1 - \text{Ma}_i^2} \right] \frac{1}{A_i} \frac{dA_i}{dx}, \quad (6.6)$$

$$\frac{1}{p_{t,i}} \frac{dp_{t,i}}{dx} = \frac{F_i}{A_i p_i}, \quad (6.7)$$

$$\frac{1}{T_{t,i}} \frac{dT_{t,i}}{dx} = 0, \quad (6.8)$$

having introduced the ratio of specific heats γ and the Mach numbers Ma_i :

$$\text{Ma}_i = u_i / a_i, \quad \text{and} \quad a_i = \sqrt{\gamma R_{gas} T_i}, \quad (6.9)$$

where R_{gas} denotes the specific gas constant. Equations (6.6)-(6.8) do not form a closed system of equations because the cross-sections $A_i(x)$ are unknown (as opposed to a single stream where $A_i = A$). Nonetheless, their sum should equal the known channel's cross-section $A(x)$ (6.2). The derivative dA/dx can be computed from the channel's profile, so the unknown terms dA_i/dx can be eliminated from the system by summing over the streams i . The gradient of the cross-section A_i is first isolated from equation (6.6):

$$\frac{dA_i}{dx} = \left[A_i \frac{1 - \text{Ma}_i^2}{\gamma \text{Ma}_i^2} \right] \frac{1}{p} \frac{dp}{dx} - \left[\frac{1 + (\gamma - 1) \text{Ma}_i^2}{\gamma \text{Ma}_i^2} \right] \frac{F_i}{p}, \quad (6.10)$$

where the index for the pressure is omitted because of (6.1). Note that (6.10) can be integrated to compute the cross-section of a single stream if the pressure gradient is known. Summing over the streams i gives:

$$\frac{dA}{dx} = \sum_{i \in [p,s]} \left[A_i \frac{1 - \text{Ma}_i^2}{\gamma \text{Ma}_i^2} \right] \frac{1}{p} \frac{dp}{dx} - \sum_{i \in [p,s]} \left[\frac{1 + (\gamma - 1) \text{Ma}_i^2}{\gamma \text{Ma}_i^2} \right] \frac{F_i}{p}. \quad (6.11)$$

The pressure gradient is the only remaining unknown, for which an explicit equation is obtained from (6.11):

$$\frac{1}{p} \frac{dp}{dx} = \frac{1}{\beta} \left(\frac{dA}{dx} + \sum_{i \in [p,s]} \left[\frac{1 + (\gamma - 1) \text{Ma}_i^2}{\gamma \text{Ma}_i^2} \right] \frac{F_i}{p} \right), \quad (6.12)$$

where β denotes the compound choking indicator:

$$\beta = \sum_{i \in [p,s]} A_i \frac{1 - \text{Ma}_i^2}{\gamma \text{Ma}_i^2}. \quad (6.13)$$

Bernstein et al. [45] showed that $\beta > 0$ corresponds to a compound-subsonic flow, $\beta = 0$ to a compound-sonic flow and $\beta < 0$ to a compound-supersonic flow. As a result, the pressure gradient and the area gradient of a compound-subsonic flow share the same sign in the absence of forces, leading to an expansion in a convergent channel and a compression in a divergent channel. Recently, Metsue et al. [46] extended the compound choking condition to real gases and showed that the condition $\beta = 0$ corresponds to the maximal combined mass flow rate.

A convenient variable to identify the compound flow regime is the so-called equivalent Mach number Ma_{eq} , introduced by Hedges and Hill [114]:

$$\text{Ma}_{eq} = \left(\gamma \frac{\beta}{A} + 1 \right)^{-\frac{1}{2}}. \quad (6.14)$$

This equals one for a compound choked flow. Section 6.2.3 analyses choking in more detail. It is important to note that the forces do not affect β and hence the compound choking criterion, as pointed out by Otis [113]. Moreover, as shown by Bernstein et al. [45], compound waves can no longer travel upstream a location x if $\beta(x) \leq 0$, which constitutes the fundamental principle of choking as in the case of a single stream.

Equations (6.7), (6.8), (6.10) and (6.12) form a closed system for the (uniform) static pressure p and the total pressure $p_{t,i}$, the total temperature $T_{t,i}$ and the cross-section A_i in each stream. These variables fully define the state of the flow and other variables can easily be calculated from them using classic gas dynamic relations. The explicit nature of the equations facilitates numerical integration with a shooting method. More details on the numerical solution are provided in section 6.4.

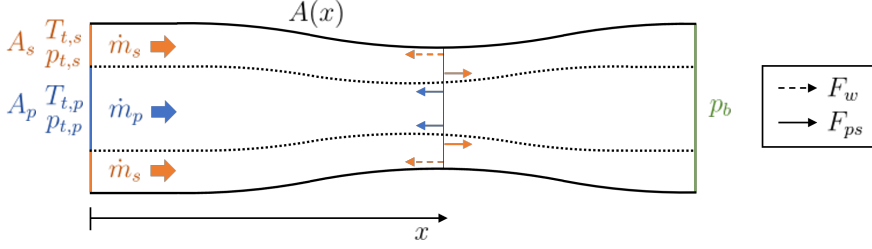


Fig. 6.1 The considered axisymmetric nozzle has a contour $A(x)$ and contains two streams, which have the same total temperature T_t , but can have a different total pressure p_t . They exchange momentum through a friction force F_{ps} at the dividing streamline and a wall friction force F_w . The static pressure p is assumed to be uniform in each cross-section. The solution consists of the streamwise distribution of the flow variables p , $p_{t,p}$, $p_{t,s}$, $T_{t,p}$, $T_{t,s}$ and the individual cross-sections A_p and A_s .

6.2.2 Model closure

The compound flow is assumed to be axisymmetric, with the primary stream surrounded by the secondary stream. Planar flows are left as technical extensions for future studies. Hence, wall friction acts only on the secondary stream. A friction coefficient f_w is defined such that the friction force per unit length F_w equals:

$$F_w = \frac{1}{2} f_w \rho_s u_s^2 l_w = \frac{1}{2} f_w \gamma p \text{Ma}_s^2 l_w, \quad (6.15)$$

where l_w denotes the wetted perimeter of the cross-section. In a circular duct, l_w is related to the cross-section A as follows:

$$l_w = 2\pi r_w = 2\sqrt{\pi A}, \quad (6.16)$$

where r denotes the radius of the total cross-section A . Similarly, the friction between the streams is defined as in the model of Papamoschou [112]:

$$F_{ps} = \frac{1}{2} f_{ps} \frac{\rho_p + \rho_s}{2} (u_p - u_s) |u_p - u_s| l_{ps}, \quad (6.17)$$

where f_{ps} denotes a friction coefficient and l_{ps} denotes the perimeter of the primary stream:

$$l_{ps} = 2\pi r_p = 2\sqrt{\pi A_p}. \quad (6.18)$$

The friction force between the streams opposes the stream with the highest velocity and is defined to be positive by (6.17) if $u_p > u_s$. Therefore, F_{ps} opposes the primary stream, while the secondary stream is entrained by the inter-stream friction and decelerated by the wall friction:

$$F_p = -F_{ps}, \quad \text{and} \quad F_s = F_{ps} - F_w. \quad (6.19)$$

Two approaches were used in this work for both coefficients f_w and f_{ps} and tested to the RANS simulations in Section 6.6.1.

The first approach was to leverage known correlations for similar flow configurations. Van Driest's [115] formula for the local wall friction coefficient f_w under a turbulent boundary layer in a compressible flow gives:

$$\frac{0.242}{\sqrt{f_w}} \sqrt{1 - \lambda^2} \frac{\arcsin(\lambda)}{\lambda} = 0.41 + \log_{10}(f_w Re_x) + \log_{10} \left((1 - \lambda^2) \left(1 - \frac{\theta \lambda^2}{1 + \theta} \right) \right), \quad (6.20)$$

where

$$1 - \lambda^2 = \left(1 + \frac{\gamma - 1}{2} Ma_s^2 \right)^{-1}, \quad Re_x = \frac{\rho_s u_s x}{\mu_s}, \quad \text{and} \quad \theta = \frac{S}{T_s}, \quad (6.21)$$

with the dynamic viscosity μ computed with Sutherland's [63] law, a reference viscosity μ_{ref} of $1.716 \cdot 10^{-5}$ Pa s at $T_{ref} = 273.2$ K, and $S = 110.4$ K. For the inter-stream friction coefficient f_{ps} , Papamoschou's [112] formula gives

$$f_{ps} = 0.013 \frac{(1 + u_s/u_p)(1 + \sqrt{\rho_s/\rho_p})}{1 + (u_s/u_p)\sqrt{\rho_s/\rho_p}} \left(0.25 + 0.75 \exp(-3Ma_c^2) \right), \quad (6.22)$$

with all quantities evaluated locally and the local convective Mach number Ma_c is defined as follows:

$$Ma_c = \frac{u_p - u_s}{a_p + a_s}. \quad (6.23)$$

The second approach was to infer these coefficients from the RANS results using an inverse method. In this setting, both coefficients f_w and f_{ps} are assumed to be constant and are determined by an optimizer which seeks to minimize the l_2 norm of the discrepancy between the proposed

model and the results of RANS simulations. This approach is expected to be more accurate due to the calibration, provided that the assumption of constant coefficient is adequate. On the other hand, these coefficients are not expected to generalize well outside the reference data, while the correlations (6.20) and (6.22) make the model self-sufficient and closed.

6.2.3 Choking mechanism

Substituting the forces F_p and F_s as defined in the section above in equation (6.12) leads to a rational equation for the static pressure gradient:

$$\frac{1}{p} \frac{dp}{dx} = \frac{N}{\beta}, \quad (6.24)$$

with β defined in (6.13) and the numerator:

$$N = \frac{dA}{dx} + N_{ps} + N_w, \quad (6.25)$$

where

$$N_{ps} = \frac{f_{ps} l_{ps}}{2} \frac{\text{Ma}_p^2 - \text{Ma}_s^2}{\gamma \text{Ma}_p^2 \text{Ma}_s^2} \frac{\rho_p + \rho_s}{2} (u_p - u_s) |u_p - u_s| \frac{1}{p}, \quad (6.26)$$

$$N_w = -\frac{f_w l_w}{2} (1 + (\gamma - 1) \text{Ma}_s^2). \quad (6.27)$$

The term N_w stems from wall friction (f_w) and is negative by definition. The term N_{ps} is due to friction between the streams and it is positive unless the terms $\text{Ma}_p^2 - \text{Ma}_s^2$ and $u_p - u_s$ have a different sign. In the natural case $u_p > u_s$, this is possible only if $T_p/T_s \geq u_p^2/u_s^2$. This scenario is not encountered in the works of Lamberts et al. [19] and of Kracik and Dvorak [26], and thus not discussed further. Therefore, the inter-stream friction has a positive contribution to the numerator N in (6.24) in the considered conditions and the wall friction has a negative contribution. This leads to some fundamental implications.

A compound-subsonic flow ($\beta > 0$) expands in a convergent section ($dA/dx < 0$) and under the action of wall friction (as in a classic Fanno flow). Conversely, it recovers pressure in a divergent section and under the action of inter-stream friction. This force opposes the stream with the highest velocity, seeking to equalize velocities between streams. Consequently, momentum exchange between the streams promotes flow unifor-

mity and results in a net increase in static pressure. The opposite is true for a compound-supersonic flow.

Equation (6.24) is singular in the sonic section ($\beta = 0$), similarly to a single stream where the denominator equals $1 - \text{Ma}^2$ (cf. [57, 116]). Nevertheless, classic nozzle flows and compound flows expand continuously through the sonic section, meaning that their pressure gradient is finite at $\beta = 0$. This is possible according to equation (6.24) only if both the numerator N and the denominator β tend to zero, leading to an indeterminate form with a finite value. This observation is corroborated by (6.11), which simplifies to $N = 0$ if $\beta = 0$. In isentropic conditions ($f_{ps} = f_w = 0$), a compound flow becomes sonic at the throat ($dA/dx = 0$), as shown by Bernstein et al. [45]. In non-isentropic conditions, the relative magnitudes of N_{ps} and N_w in (6.25) determine the position of the sonic section.

Wall friction has a negative contribution to the numerator. Therefore, in the absence of inter-stream friction ($f_{ps} = 0$), one must have $dA/dx > 0$ to respect $N = 0$. This implies that the sonic section is located in a divergent section, hence downstream of the throat. The wall friction has the same effect on a nozzle flow with a single stream [57, 116, 117]. Likewise, in the absence of wall friction ($f_w = 0$), one must have $dA/dx < 0$ to respect $N = 0$. This implies that the inter-stream friction pushes the sonic section upstream of the throat. This analysis offers a plausible explanation for the observed displacement of the sonic section by [19] and [26]. The interplay between both forces is investigated further in section 6.6.

The indeterminate pressure gradient can be computed by applying de l'Hôpital's rule on equation (6.24) (cf. [45, 116]):

$$\left(\frac{1}{p} \frac{dp}{dx}\right)^* = \frac{N^*}{\beta^*} = \frac{0}{0} = \frac{(dN/dx)^*}{(d\beta/dx)^*}, \quad (6.28)$$

where the superscript $*$ refers to the sonic section. The derivatives dN/dx and $d\beta/dx$ give rise to the derivatives of Mach numbers Ma_i , cross-sections A_i , forces F_i and the static pressure p (see (6.25) and (6.13)), and these can be all related to the local pressure gradient. The relevant equations are presented in appendix B.

For example, the gradient of β is given by:

$$\frac{d\beta}{dx} = \sum_{i \in [p, s]} \left(\frac{dA_i}{dx} \frac{1 - \text{Ma}_i^2}{\gamma \text{Ma}_i^2} - \frac{A_i}{\gamma \text{Ma}_i^4} \frac{d\text{Ma}_i^2}{dx} \right). \quad (6.29)$$

Writing the Mach numbers Ma_i as a function of the total and the static pressure:

$$\text{Ma}_i^2 = \frac{2}{\gamma - 1} \left(\left(\frac{p_{t,i}}{p} \right)^{\frac{\gamma-1}{\gamma}} - 1 \right), \quad (6.30)$$

the derivatives $d\text{Ma}_i^2/dx$ in (6.29) can be substituted by the gradients of the static and total pressures. The resulting expression for the gradient of β thus contains first derivatives of the static pressure, the cross-sections A_i and the total pressures $p_{t,i}$. The latter two can be substituted using (6.7) and (6.10), which contain the static pressure gradient. The resulting expression for the gradient of β is linear with respect to the static pressure gradient, with non-linear coefficients depending on the Mach numbers, forces and static pressure:

$$\frac{d\beta}{dx} = c_{\beta 0}(\text{Ma}_p, \text{Ma}_s) \left(\frac{1}{p} \frac{dp}{dx} \right) + c_{\beta 1}(\text{Ma}_p, \text{Ma}_s, F_p, F_s, p). \quad (6.31)$$

The analytical expressions of $c_{\beta 0}$ and $c_{\beta 1}$ are given by (B.102) and (B.103). The same approach leads to a similar linear equation for the derivative of the numerator (see (B.98)). These equations are derived for constant friction coefficients, i.e., the correlations (6.20) and (6.22) are not explicitly differentiated for simplicity. The result is an implicit equation for the static pressure gradient in the sonic section:

$$\left(\frac{1}{p} \frac{dp}{dx} \right)^* = \frac{c_{N0}^* \left(\frac{1}{p} \frac{dp}{dx} \right)^* + c_{N1}^*}{c_{\beta 0}^* \left(\frac{1}{p} \frac{dp}{dx} \right)^* + c_{\beta 1}^*}. \quad (6.32)$$

Rearranging the equation above leads to a quadratic equation for the static pressure gradient (see appendix B). The two roots correspond to the subsonic and the supersonic solution downstream of the sonic point (cf. [57]). For an isentropic compound flow, the derivative of the numerator corresponds to the second derivative of the cross-section d^2A/dx^2 ($f_{ps} = f_w = 0$), leading to the tractable expression reported by Bernstein et al. [45]. The additional terms introduced by the friction forces preclude such a concise expression, which is why the coefficients and their analytical expressions are reported in the appendix B. Note, however, that these equations are exact and analytical.

6.3 Circumventing the singularity at the sonic section

The pressure gradient can be computed in any section using (6.32) if the flow is compound-sonic or by using (6.24) otherwise. However, equation (6.24) is numerically ill-conditioned near the sonic section. Therefore, approximations of the pressure gradient near the sonic section help stabilising the numerical integration. In this section, such approximations are developed analytically.

De l'Hôpital's rule used in section 6.2.3 arises from a Taylor series expansion of the numerator and the denominator around the sonic section x^* , which is then evaluated at the sonic section x^* itself. More generally, these expansions lead to the following equation in x , which is still exact:

$$\frac{1}{p} \frac{dp}{dx} = \frac{N}{\beta} = \frac{N^* + (x - x^*) \frac{dN^*}{dx} + \frac{1}{2}(x - x^*)^2 \frac{d^2N^*}{dx^2} + \frac{1}{6}(x - x^*)^3 \frac{d^3N^*}{dx^3} + \dots}{\beta^* + (x - x^*) \frac{d\beta^*}{dx} + \frac{1}{2}(x - x^*)^2 \frac{d^2\beta^*}{dx^2} + \frac{1}{6}(x - x^*)^3 \frac{d^3\beta^*}{dx^3} + \dots} \quad (6.33)$$

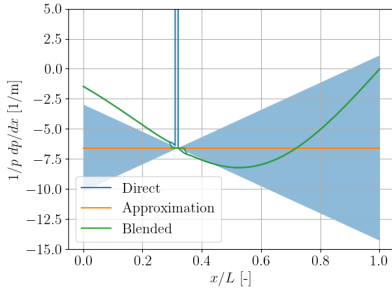
Since $N^* = \beta^* = 0$, the equation above simplifies to the following expression:

$$\frac{1}{p} \frac{dp}{dx} = \frac{N}{\beta} = \frac{\frac{dN^*}{dx} + \frac{1}{2}(x - x^*) \frac{d^2N^*}{dx^2} + \frac{1}{6}(x - x^*)^2 \frac{d^3N^*}{dx^3} + \dots}{\frac{d\beta^*}{dx} + \frac{1}{2}(x - x^*) \frac{d^2\beta^*}{dx^2} + \frac{1}{6}(x - x^*)^2 \frac{d^3\beta^*}{dx^3} + \dots} \quad (6.34)$$

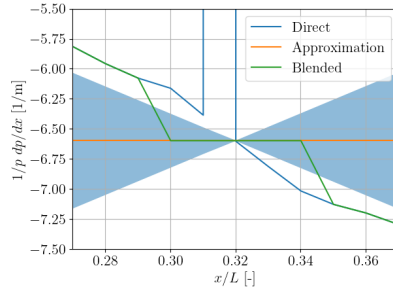
Truncating the Taylor series leads to approximations of arbitrary order in space. The first order approximation only retains the first derivatives, which are constant since they are evaluated at the sonic section. Hence, the pressure gradient in the vicinity of the sonic section is approximated as a constant, equal to its value in the sonic section. Note that the resulting expression is identical to (6.28) and thus leads to the quadratic (6.32) in section 6.2.3.

Higher accuracy is obtained by retaining higher order derivatives. The analytical expressions for the second derivatives are given by (B.127) and (B.134). The derivation could continue up to the third derivative, but to avoid overly cumbersome expressions, this work proceeds with a third order approximation by numerically differentiating the analytical second derivatives.

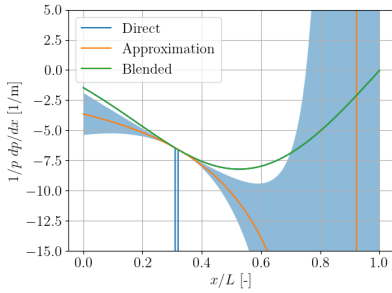
Numerically, one has to decide when to switch from the direct equation (6.24) to the approximation (6.34). This can be achieved by imposing a tolerance on β (or Ma_{eq}), e.g., using the approximation when $-\epsilon < \beta < \epsilon$,



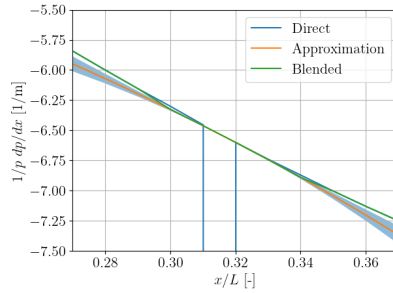
(a) First order



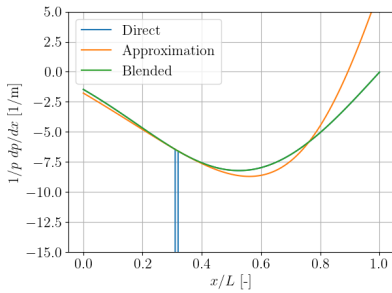
(b) First order (zoom)



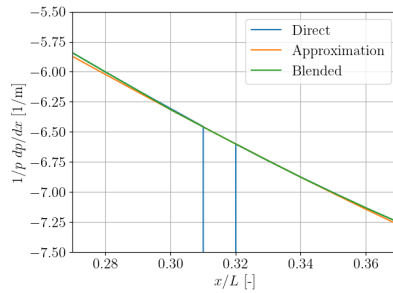
(c) Second order



(d) Second order (zoom)



(e) Quasi-third order



(f) Quasi-third order (zoom)

Fig. 6.2 The pressure gradient computed directly with (6.24), with the approximation (6.34) (truncated after the first, second and third derivatives) and with the blended function (6.36). The shaded region indicates the truncation error estimated by (6.35), which is unavailable for the quasi-third order approximation.

with ϵ an arbitrary small positive number (cf. [116]). An alternative condition is based on an estimate of the truncation error in the Taylor series, which can be estimated as the magnitude of the first term after the truncation. The resulting truncation errors on the numerator and the denominator, denoted by δN and $\delta \beta$, can be propagated to the pressure gradient as follows:

$$\delta \left(\frac{1}{p} \frac{dp}{dx} \right) = \sqrt{\left(\frac{1}{\beta} \right)^2 (\delta N)^2 + \left(-\frac{N}{\beta^2} \right)^2 (\delta \beta)^2}. \quad (6.35)$$

Equation (6.35) is analytically available for the first order approximation using the second derivatives for δN and $\delta \beta$ (cf. appendix B). The second order approximation requires third derivatives to estimate its error, which can be computed with finite differences (as in the quasi-third order approximation). The switch from the direct equation (6.24) to the approximation (6.34) is based on the relative error on the static pressure gradient:

$$\frac{1}{p} \frac{dp}{dx} = \begin{cases} \text{equation (6.34)} & \text{if } \frac{\delta \left(\frac{1}{p} \frac{dp}{dx} \right)}{\left(\frac{1}{p} \frac{dp}{dx} \right)} \leq 5\%, \\ \text{equation (6.24)} & \text{otherwise.} \end{cases} \quad (6.36)$$

Examples of the distribution of the pressure gradient using the approximations of increasing order are shown in figure 6.2. The higher order approximations are more accurate in a wider region around the sonic section. The estimated truncation error for the quasi-third order approximation is not shown. The compound equations have been integrated using the blended equation (6.36) for the pressure. More details on the numerical integration are provided in section 6.4.

The blended function (6.36) matches the direct equation (6.24) at the inlet and switches to the approximation (6.34) as the truncation error, computed with (6.35), drops below 5 %. The flow then passes through the sonic section using the approximation (6.34), effectively avoiding the indeterminate form (6.28) of the direct equation. Downstream, the direct equation is integrated when the threshold on the truncation error is exceeded again.

In this study, the quasi-third order approximation is used with a backward second order finite difference formula, with a threshold of 5 % on the second order approximation, since the fourth derivatives needed for the truncation error are not available.

6.4 Numerical solution

The governing equations (6.7), (6.8), (6.10), (6.24), with the approximation (6.34) near the sonic point, form a system of ODEs in function of the axial coordinate x , which can thus be integrated in space. The fourth order Runge-Kutta method is used in this study, which is readily available in Python with the 'solve_ivp' function in SciPy¹. In this study, the mesh consists of 100 equally distributed points, which has been found sufficient to accurately respect the conservation equations.

The boundary conditions consist of the total pressures $p_{t,p}, p_{t,s}$, total temperatures $T_{t,p}, T_{t,s}$ and the cross-sections A_p, A_s at the inlet ($x = 0$) and of the static pressure p_b at the outlet ($x = L$), as indicated in figure 6.1. The distribution of the channel's cross-section $A(x)$ is prescribed. Integrating the system of ODEs requires an 'initial' condition at the inlet or at any point from which the integration proceeds forward in x . In a single stream computation, the sonic condition $N^* = 0$ is used to identify the sonic section and split the integration for the Mach number from the inlet to the sonic section and from the sonic section to the outlet (as proposed by [117] and [116]).

This is not a viable option for compound flows since the sonic state is not uniquely defined (as $Ma = 1$), but can consist of various combinations of cross-sections A_i and Mach numbers Ma_i . The system state is available at the inlet except for the static pressure $p(0)$. This was found with a shooting method until the boundary condition at the outlet was respected. Firstly, bounds for the inlet pressure are established, considering that $p(0)$ can not exceed the total pressure of any stream by definition, that is the upper bound of $p(0)$ is

$$p_{max} = \min(p_{t,p}(0), p_{t,s}(0)) . \quad (6.37)$$

The compound flow is assumed to be subsonic at the inlet (as in a classic nozzle flow), so the second limiting condition is a sonic flow at the inlet ($\beta(0) = 0$). Using (6.13) and (6.30), the sonic condition at the inlet leads to the lower bound p_{min} for the static pressure at the inlet:

$$\beta(0) = 0 = \sum_{i \in [p,s]} \frac{A_i}{\gamma} \left(\left[\frac{2}{\gamma - 1} \left(\left(\frac{p_{t,i}(0)}{p_{min}} \right)^{\frac{\gamma-1}{\gamma}} - 1 \right) \right]^{-1} - 1 \right) , \quad (6.38)$$

¹https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.solve_ivp.html

where p_{min} can be found iteratively. The bounds $[p_{min}, p_{max}]$ allow to find the correct inlet pressure $p(0)$ with a bisection method, which iteratively restricts the bounds of the interval as depicted in figure 6.3. In each iteration, the governing equations are integrated using the mean of the current interval, after which either its bottom or top half is used in the next iteration. This update depends on whether the computed flow is choked.

In the subsonic case, the inlet pressure $p(0)$ and the computed outlet pressure $p(L)$ are positively correlated: a lower inlet pressure leads to a lower outlet pressure. If the computed back pressure is too low with respect to the boundary condition ($p(L) < p_b$), the inlet pressure should increase and the average pressure $p(0)$ becomes the new lower bound of the bracket in the next iteration. The opposite is true if $p(L) > p_b$. If the pressures $p(L)$ and p_b match, the solution has been found and the flow is compound-subsonic.

If the computed flow becomes sonic ($\beta(x) = 0$), the integration stops locally at the sonic point and the numerator N is computed with (6.25). If the condition $N = 0$ is not respected (see (6.28)), the inlet pressure $p(0)$ must be updated. A negative numerator indicates that a subsonic flow tends to expand. If this occurs in the sonic section, the sonic point is reached too far upstream, and hence the inlet pressure should be increased. Considering, for example, an isentropic compound flow ($f_{ps} = f_w = 0$), where the numerator N is reduced to the gradient dA/dx , it is known that the flow becomes sonic at the throat ($dA/dx = 0$). The case $N = dA/dx < 0$ at the sonic section implies that the sonic point is reached in a convergent section; hence, the inlet pressure should be increased to avoid having a non-physical mass flow rate exceeding the choked one. The same argument holds for a non-isentropic flow.

If $N = 0$, the flow is sonic in the correct section, and the approximation (6.36) can be computed with the desired order of accuracy. This slightly modifies the governing equations, so the choked solution also changes. Therefore, the procedure to find the inlet pressure $p(0)$ is repeated, replacing (6.24) with (6.36).

The resulting choked solution is available from the inlet to the sonic section ($\beta(x) = 0$). The governing equations are then integrated up to the outlet, using the positive root $(dp/dx)^*$ of (6.32) to obtain the subsonic solution. The complete solution is thus subsonic throughout, except at the sonic point. This limiting case corresponds to the highest back pressure which chokes the flow, which is commonly referred to as the critical back pressure p_b^* in ejector modelling. The final step checks whether the choked

solution respects the boundary condition p_b . The flow remains choked if the imposed back pressure p_b is lower than the critical value p_b^* . Otherwise, the static pressure at the inlet should increase, leading to a subsonic solution.

The supersonic solution can be obtained with the negative root $(dp/dx)^*$ of (6.32). If the back pressure p_b lies between the pressures corresponding to the two continuous choked solutions, a shock appears in the diffuser as in the case of a nozzle flow with a single stream. The shock position can be found iteratively by imposing the computed back pressure $p(L)$ to match p_b (see, for example, [116]). However, to the authors' knowledge, a compound shock model has not been presented in the literature.

6.5 Validation test cases with axisymmetric simulations

The predictions of the 1D theoretical model are compared to post-processed axisymmetric RANS simulations of a compound nozzle in section 6.6. The numerical set-up of these 2D simulations is detailed in section 6.5.1, and the test cases are described in section 6.5.2.

6.5.1 Numerical set-up for the RANS validation

The computational domain is shown in figure 6.4. It consists of an axisymmetric convergent-divergent nozzle with two distinct inlets. The simulation was carried out with a wedge-shaped domain of 5 degrees and a single cell in the tangential direction. The contour of the upper wall was defined as a cosine between $\pi/2$ and 2π with appropriate scaling to reach the dimensions shown in figure 6.4:

$$r(x) = \frac{R_o + R_{th}}{2} + \frac{R_o - R_{th}}{2} \cos\left(\frac{3\pi x}{2L} + \frac{\pi}{2}\right) \quad (6.39)$$

Hence, the radius at the inlet ($x = 0$) follows from the outlet radius R_o , the throat radius R_{th} and the length L . The throat is located at one-third of the domain. The analysis of the sonic section displacement was focused on a test case with nozzle lengths of 187.5 mm, outlet radius $R_o = 10$ mm, inlet radius $R_i = 9.5$ mm, throat radius $R_{th} = 9$ mm and $R_p = 0.5R_i$. For the model validation, several other geometries were considered as reported in more details in section 6.5.2. The simulations have been carried out and post-processed with the tools presented in section 2.2.

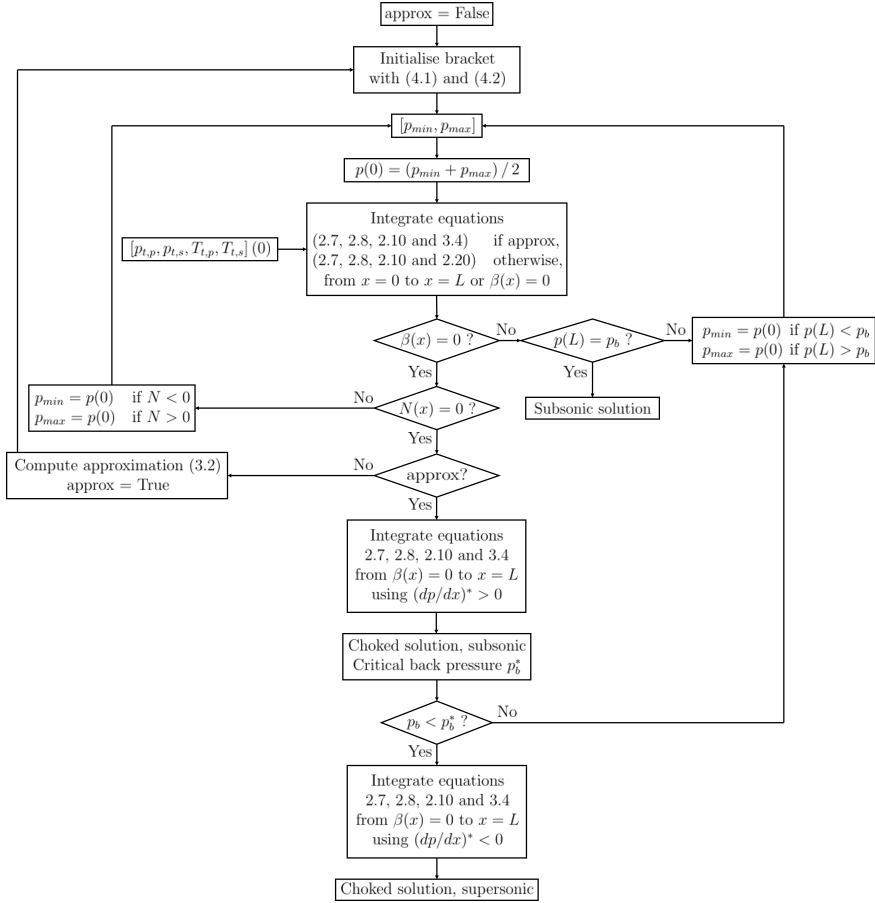


Fig. 6.3 Flowchart of the numerical procedure to solve the compound nozzle flow. The static pressure $p(0)$ at the inlet is found with a bisection method. A subsonic solution is accepted if the computed back pressure $p(L)$ matches the boundary condition p_b . A choked flow requires both β and the numerator N to equal zero (see (6.28)). The approximation (6.34) is computed after convergence of the choked procedure, which is then repeated, replacing (6.24) by (6.36).

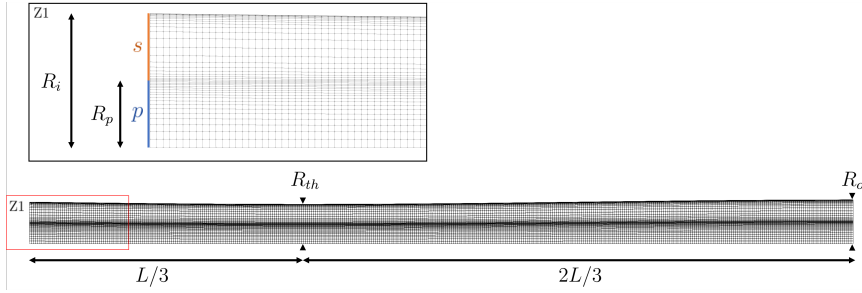


Fig. 6.4 The computational domain consists of an axisymmetric convergent-divergent nozzle with two distinct inlets. The primary stream (index p) is surrounded by the secondary stream (index s). The reference geometry is defined by $L = 187.5$ mm, $R_i = 9.5$ mm, $R_{th} = 9.0$ mm, $R_o = 10$ mm and $R_p = 0.5R_i$. The model validation considered various throat R_{th} and the primary inlet R_p radii, as described in section 6.5.2, while keeping the same resolution at the wall. The coarse mesh is displayed for visibility.

6.5.2 Selected Test Cases

Three simulations were carried out where the wall and inter-stream friction forces were numerically (de)-activated. The boundary conditions are listed in table 6.1. At both inlets, the turbulent mixing length was set equal to 7 % of the hydraulic diameter, as recommended by [60]. The turbulence intensity was set to 1 %, as this value proved to best match the friction coefficients predicted by (6.22).

The first simulation is inviscid and thus serves as a reference for the original compound flow theory without momentum exchange. The second simulation is viscous with a slip condition at the wall: only friction forces between the streams are thus active in this case. Finally, wall friction is included in the third case by imposing a no-slip condition along the upper boundary. A mesh convergence study was carried out on the viscous simulation with the no-slip condition. The number of cells was doubled in each direction, leading to meshes with 400×25 , 800×52 and 1600×106 cells. The resulting distributions of y^+ near the wall and the Mach number at the centerline are shown for the three meshes in figure 6.5. The medium and fine meshes show good agreement and are both wall-resolved. The wall friction Reynolds number $Re_\tau = \rho u_\tau D_h / \mu$ equals $1.9 \cdot 10^4$ at the inlet and $5.3 \cdot 10^3$ at the outlet in case 3. The results on the fine mesh are used in the next section to profit from its higher resolution.

Table 6.1 Boundary conditions of the simulations presented in section 6.6. The inlet conditions are identical in the three cases: $p_{t,p} = 3$ bar, $p_p = 1.21$ bar, $p_{t,s} = 1.5$ bar and $T_{t,p} = T_{t,s} = 300$ K. The primary inlet is supersonic with $Ma_p = 1.22$. They respectively cover (1) an isentropic case, (2) a case with friction between the streams but without wall friction and (3) a case with both forces.

Case	TI [%]	l_{mix} [m]	Wall	Comments
1	n/a	n/a	slip	inviscid
2	1	$6.65 \cdot 10^{-4}$	slip	viscous
3	1	$6.65 \cdot 10^{-4}$	no-slip	viscous

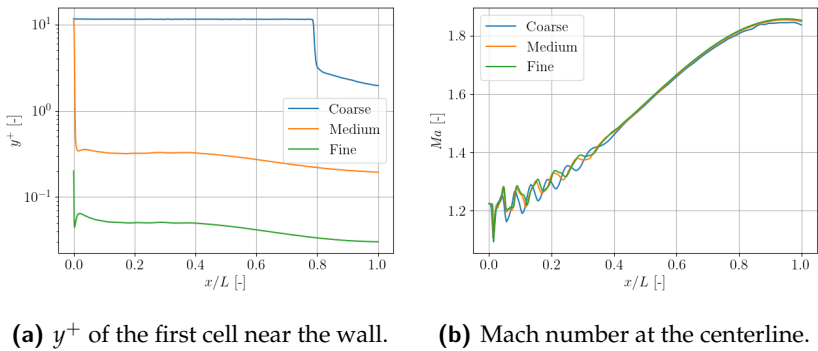


Fig. 6.5 The coarse, medium and fine mesh contain 400×25 , 800×52 and 1600×106 elements respectively. The medium and fine meshes lead to a wall-resolved simulation and to good agreement for the Mach number at the centerline. The fine mesh is retained for the other simulations and the post-processing.

A parametric study was carried out to validate the model in a wider range of conditions. The reference case is the viscous simulation with the no-slip condition at the wall (case 3 in table 6.1), as it is the most representative for practical applications. Three parameters were varied independently: (1) the total pressure at the secondary inlet between $p_{t,s} = 0.5$ and 3 bar, (2) the radius of the throat between $R_{th}/R_o = 0.6$ and 0.9 and (3) the radius of the primary inlet between $R_p/R_i = 0.3$ and 0.7 (see figure 6.4). The correlations (6.20) and (6.22) were used in all cases, so the model was not tuned further.

The resulting distributions of the static pressure and the Mach numbers depend on the geometry and the inlet conditions, so the static pressure imposed at the supersonic primary inlet changes too. However, the static pressure at the inlet corresponding to a choked flow without a shock train is initially unknown. The 1D model proved to be a useful tool to compute this pressure, as it provides a cross-stream averaged description of a choked flow with a *uniform* static pressure profile. The 1D model was thus run *first*, to provide the boundary conditions for the RANS simulations *afterwards* (see table 6.2). The primary stream is supersonic in most cases, except if $p_{t,s} \geq 2$ bar. In that case, only the total pressure and total temperature were imposed. The other boundary conditions remained identical to those in table 6.1.

6.6 Results

The two approaches to close the model from section 6.2.2 are compared in section 6.6.1, followed by a validation of the model in various operating conditions and geometries in section 6.6.2. The effect of the friction forces on compound choking is illustrated in section 6.6.3. Compound choking is shown to be a plausibly more general flow blockage mechanism than individual streams reaching a unitary Mach number in section 6.6.4. This is demonstrated through a choked compound flow with a fully subsonic secondary stream, both with the 1D model and with an axisymmetric RANS simulation.

6.6.1 Model closure

Before proceeding with the analysis of the investigated test cases, figure 6.6 showcases the impact of the friction coefficients at the wall and at the dividing streamline for the test case 3. Figures 6.6a and 6.6b show the stream-

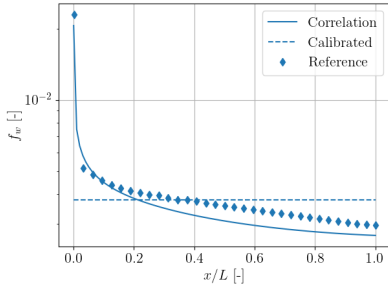
$p_{t,s}$ [bar]	R_{th}/R_o [-]	R_p/R_i [-]	p_p [bar]	Comments
1.5	0.9	0.5	1.21	reference case
0.5	0.9	0.5	0.40	
1.0	0.9	0.5	0.81	
2.0	0.9	0.5	n/a	subsonic primary inlet
2.5	0.9	0.5	n/a	subsonic primary inlet
3.0	0.9	0.5	n/a	subsonic primary inlet
1.5	0.6	0.5	1.45	
1.5	0.7	0.5	1.40	
1.5	0.8	0.5	1.32	
1.5	0.9	0.3	1.16	
1.5	0.9	0.4	1.18	
1.5	0.9	0.6	1.25	
1.5	0.9	0.7	1.30	

Table 6.2 Boundary conditions for the parametric study. The primary static pressure p_p was computed a priori with the 1D model and served a posteriori as a boundary condition for the RANS simulations. The primary stream is subsonic at the inlet for $p_{t,s} \geq 2$ bar, so the static pressure is not imposed.

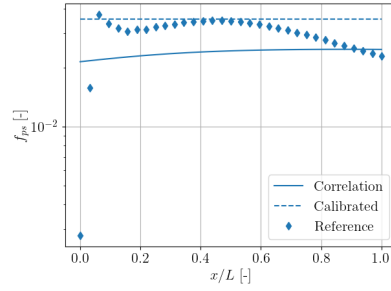
wise evolution of the friction coefficients (1) obtained with the correlations (6.20) and (6.22), (2) identified as constants by the inverse method as described in section 6.2.2 and (3) extracted from the viscous RANS simulations as reference. The correlation (6.20) for the wall friction is adequate and follows the correct trend. The calibrated constant matches the reference on average. The friction between the streams is mostly underpredicted by correlation (6.22), as also observed by [15]. This was found to be sensitive to the turbulent intensity at the inlets, which directly impacts the turbulent viscosity and hence the friction between the streams. Analysing this effect in further details is beyond the scope of the current work.

More importantly, the distribution of the static pressure is predicted accurately by the 1D model in both cases, as shown in figures 6.6c and 6.6d. The averaged static pressure is computed from the RANS simulations with equations (2.2) - (2.4) and is displayed with diamonds as the reference. For plotting purposes, the spacing in the markers is much larger than the mesh resolution.

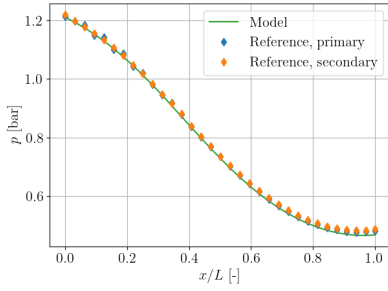
It is worth noticing that the static pressure in the two streams oscillates near the inlet due to a weak shock train. This effect is not captured in the 1D model because of the assumption of uniform static pressure. The agree-



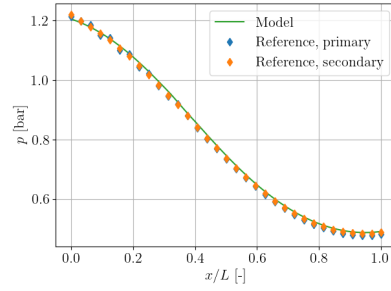
(a) Wall friction coefficient f_w .



(b) Inter-stream friction coefficient f_{ps} .



(c) Prediction of the static pressure with the correlations.



(d) Prediction of the static pressure with calibrated constants.

Fig. 6.6 Comparison of the two approaches for determining the friction coefficients (see section 6.2.2). The model predicts the static pressure accurately with the correlations (6.20) and (6.22), and with the calibrated constants ($f_w = 0.00377$, $f_{ps} = 0.0355$).

ment between the model and the RANS simulations shows that the correlations (6.20) and (6.22) are adequate for the purposes of this work. Moreover, the agreement between predictions with both constant and varying coefficients indicates a weak sensitivity to these parameters, justifying the exclusion of their derivatives in the approximation in section 6.3. All the results presented for the 1D model in the remaining of this section were obtained with the correlations. This approach has the advantage of not requiring calibration since the model is fully closed when using (6.20) and (6.22).

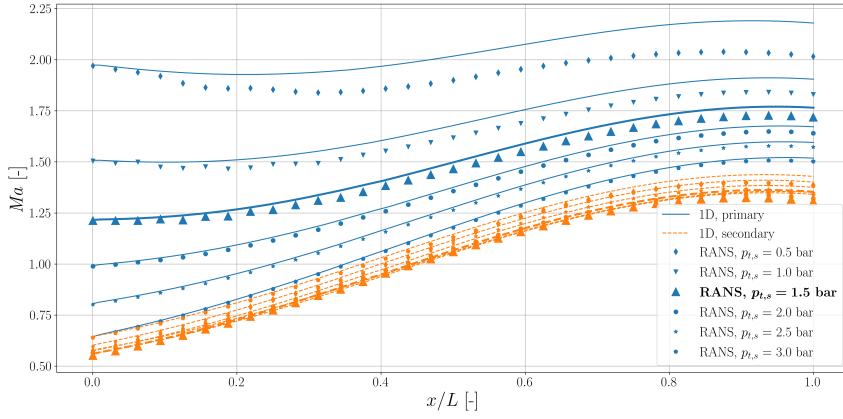
6.6.2 Model validation

Figure 6.7 illustrates the distribution of the Mach numbers and the dividing streamlines for the different secondary inlet pressures (see table 6.2). The primary Mach number decreases significantly with increasing total pressure $p_{t,s}$, due to an increased static pressure at the inlet. Similarly, the secondary Mach number increases with increasing $p_{t,s}$, but is less sensitive. When the total pressures are equal (uniform flow), the Mach numbers are equal in each stream at the inlet. Downstream, the secondary Mach number decreases relative to the primary due to wall friction, which is not fully counteracted by the weak shear force from the quasi-uniform flow.

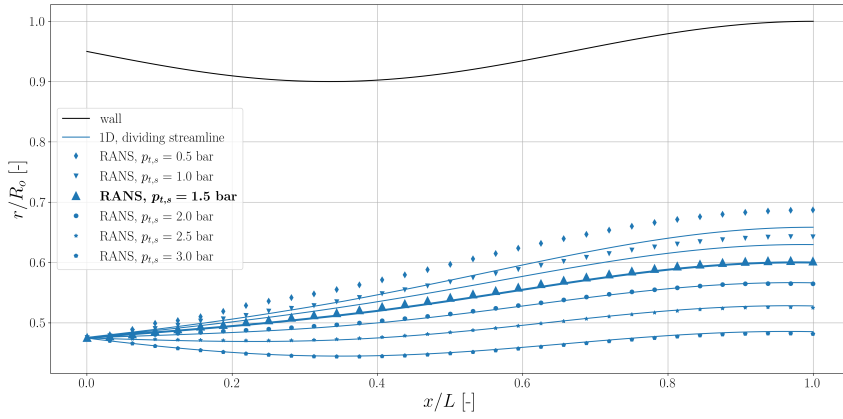
The model agreement with RANS simulations worsens as the inlet pressure difference grows. Notably, in these cases, the primary Mach number and dividing streamline oscillate near the inlet, due to the formation of a mild shock train. The resulting oblique shocks introduce pressure losses that are not captured by the 1D model, which assumes a uniform static pressure across the streams. Additionally, the maximal relative error of the model amounts to 10 %, which is comparable to the scatter of the data on which the correlation (6.22) is based (see [112]). Despite this, the overall agreement is satisfactory, especially considering that the model was not specifically calibrated for these conditions.

The throat radius R_{th} was varied between 60 % and 90 % of the outlet radius R_o , while maintaining the same total pressures as in case of table 6.1. A narrower throat induces a stronger expansion, resulting in lower Mach numbers upstream and higher Mach numbers downstream of the sonic section (see figure 6.8). The agreement between the model and the RANS simulations is excellent, with discrepancies below 10 %. The overpredicted Mach numbers at the outlet are attributed to underpredicted friction coefficients by (6.20) and (6.22) with respect to the RANS simulations, as also observed in figures 6.6a and 6.6b. It is worth noticing that the radius of the wall at the inlet changes with the throat due to the cosine-shaped profile between $\pi/2$ and 2π , as defined by (6.39). The primary inlet changes accordingly and extends halfway to the total inlet in each configuration.

Finally, the radius of the primary inlet is varied between 30 and 70 % of the total radius at the inlet, while keeping the same profile of the wall and the same total pressures as the reference case. Figure 6.9 shows that both Mach numbers decrease with an increasing radius of the primary inlet. This is due to a higher average total pressure at the inlet (a larger portion with $p_{t,p} = 3$ bar instead of $p_{t,s} = 1.5$ bar), which in turn leads to a

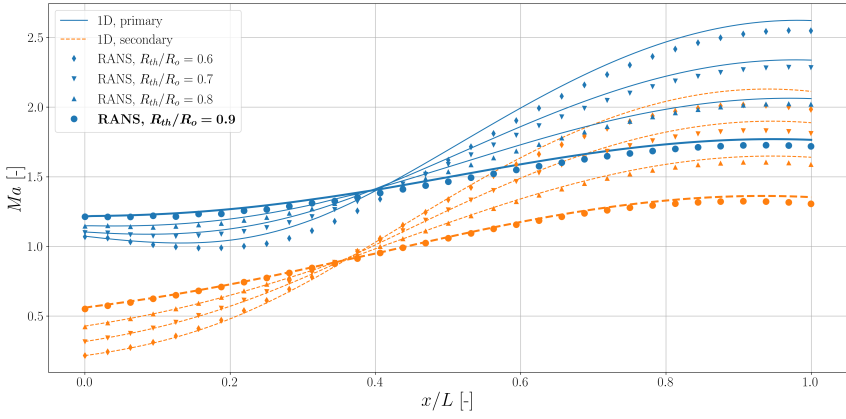


(a) Mach numbers.

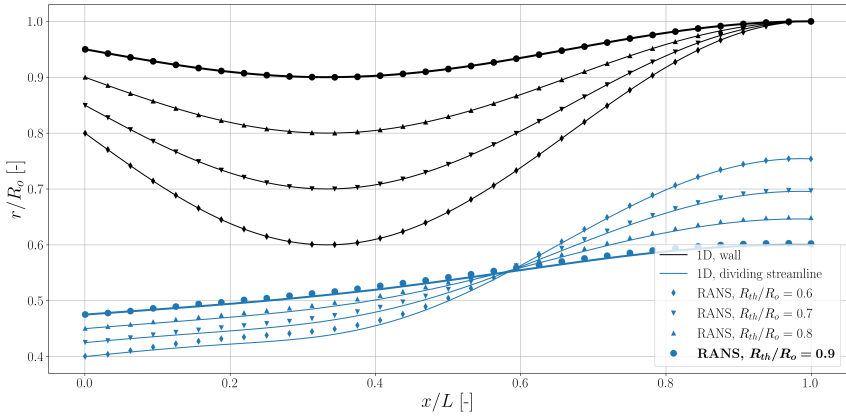


(b) Dividing streamlines.

Fig. 6.7 Comparison of the model predictions and the post-processed RANS simulations with varying total pressure $p_{t,s}$ at the secondary inlet and constant total pressure $p_{t,p} = 3$ bar at the primary inlet. The reference case is highlighted in bold. The geometry is identical to the reference case in section 6.5.1. The primary Mach number decreases with increasing $p_{t,s}$ due to a higher static pressure at the inlet and eventually becomes sub-sonic. The agreement degrades with a larger difference in inlet pressures.



(a) Mach numbers.



(b) Dividing streamlines.

Fig. 6.8 Comparison of the model predictions and the post-processed RANS simulations with varying radius of the throat. The reference case is highlighted in bold. The total pressures at the inlet are identical to the reference case 3 in table 6.1. The Mach numbers decrease at the inlet and increase at the outlet with a narrowing throat. The curvature becomes important for the smallest throat, inducing 2D effects and reducing the accuracy of a 1D formulation.

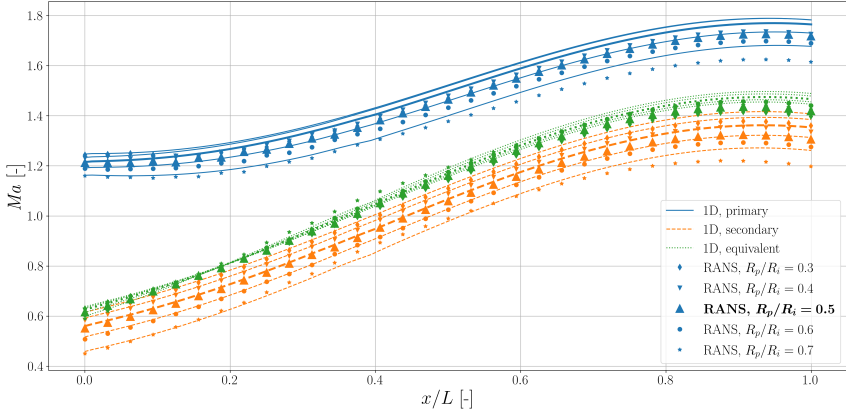
higher static pressure of the choked flow. Despite the lower Mach numbers in both streams, the equivalent Mach number is comparable. This surprising effect is possible through the non-linear relation between the Mach numbers and the cross-sections in the terms of β in (6.13). Clearly, the Mach numbers alone do not suffice to conclude on the state of the flow and should be complemented with information on the cross-sections. The 1D model captures these trends accurately, although the Mach numbers are overpredicted near the outlet as observed earlier in figure 6.8.

6.6.3 The effect of friction on compound choking

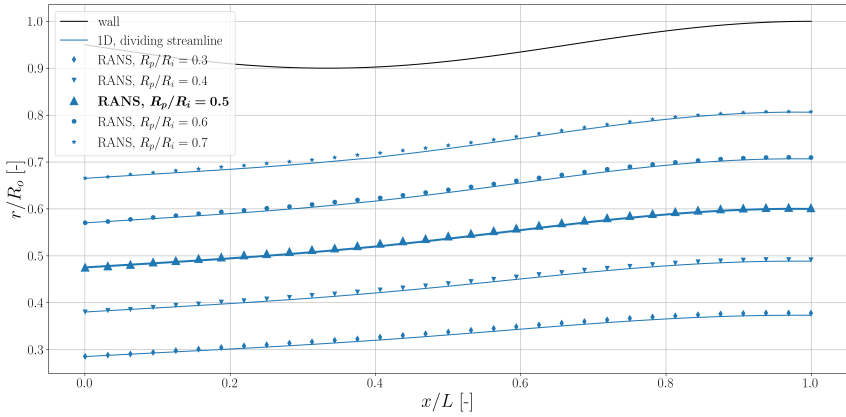
Figure 6.10 shows the Mach number fields of the three simulations in table 6.1, along with the dividing streamline in white and the sonic line in black. The shear layer in the inviscid case is only due to numerical diffusion, so the two streams remain clearly distinct across the dividing streamline. The sonic line ends on the wall downstream of the geometrical throat (indicated by the arrows). The primary stream is thus supersonic at the throat and the secondary stream is still subsonic. The secondary stream is accelerated through shear in the viscous simulations, causing the sonic line to move radially outwards. The sonic line no longer reaches the wall when a no-slip condition is imposed. The development of the boundary layer tends to slow down the flow, leading to a lower maximal Mach number.

The averaged Mach numbers extracted from the RANS simulations are displayed with diamonds in figure 6.11 as the reference. The predictions of the proposed model are shown in full lines. It is worth noticing that the static pressure at the inlet for the 1D model prediction is not imposed to be equal to the one from the RANS simulations, but rather results from the iterative procedure laid out in section 6.3. The only difference in the three cases is the friction coefficients, which are used to obtain the 1D predictions. In the inviscid case, both naturally equal 0. The friction coefficients are computed with (6.20) and (6.22) for the viscous simulations.

The key finding in these results is the displacement of the sonic section, which is highlighted by the vertical dashed line and commented below. The sonic section is located in the geometrical throat (indicated by the red triangle) in the inviscid case, as predicted by the original compound flow theory (see, for example, [45]). The viscous simulation, without wall friction but with shear forces between the streams, shows an upstream shift towards the convergent section, as expected in the analysis of section 6.2.3. Conversely, wall friction is dominant in the final simulation, overcoming



(a) Mach numbers.



(b) Dividing streamlines.

Fig. 6.9 Comparison of the model predictions and the post-processed RANS simulations with varying radius of the primary inlet. The reference case is highlighted in bold. The total pressures at the inlet are identical to the reference case 3 in table 6.1. A larger primary cross-section at the inlet increases the average total pressure at the inlet and hence increases the static pressure required for choking. Consequently, both Mach numbers decrease with an increasing radius of the primary inlet.

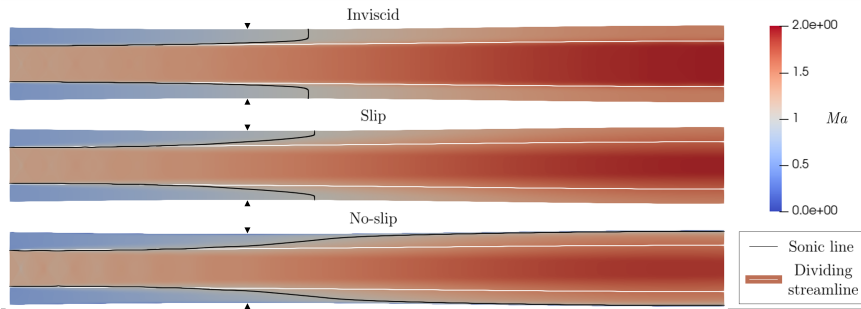
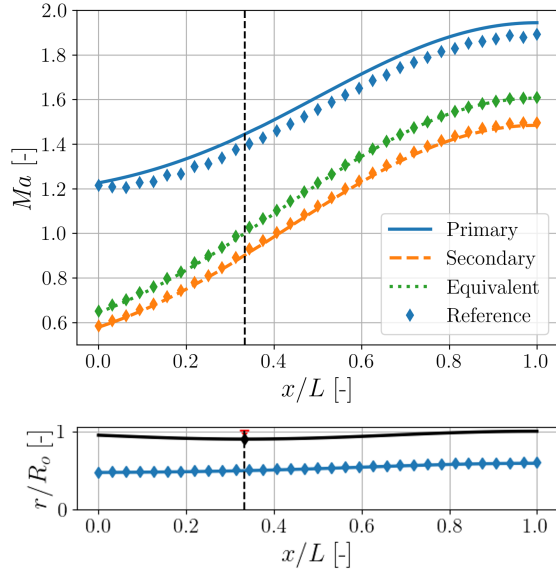


Fig. 6.10 Mach number contours of the three simulations (cf. table 6.1). The black line indicates the sonic line and tends towards the wall as the compound flow becomes supersonic. The dividing streamline is indicated in white. The arrows indicate the geometrical throat. The numerical domain depicted in figure 6.4 has been mirrored for visibility.

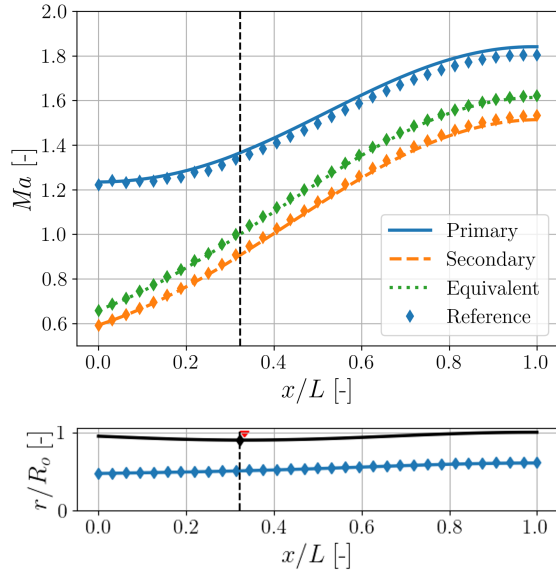
the effects of the shear force and ultimately pushing the sonic section in the divergent portion of the nozzle.

Note that the primary Mach number is slightly overestimated in all cases. This is attributed to the weak shock train at the inlet (see figures 6.6c, 6.6d and 6.10), which introduces a pressure loss in the primary stream.

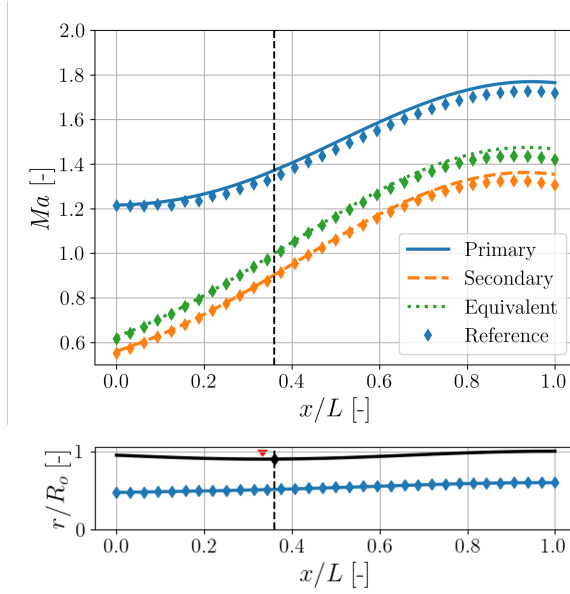
The 1D predictions allow for computing each term in the equation for the static pressure gradient (cf. equation (6.24)) separately. The evolution of the components in the numerator is shown in figure 6.12 for the slip and no-slip cases. The inviscid case is trivial since the gradient of the cross-section is the only remaining term. The terms are normalised with the maximal gradient of the cross-section, as this is a fixed quantity for the three simulations. The evolution of the two friction terms supports the theoretical considerations in section 6.2.3, the first (inter-stream friction) always being positive and the second (wall friction) being negative. As a result, the zero-crossing of the numerator moves up or downstream. The gradient of the cross-section balances the effect of the forces in either direction by moving the sonic section in the convergent or the divergent section. Note that the inter-stream friction tends to decrease from left to right as the difference in Mach number decreases, while friction tends to increase as the flow accelerates through the nozzle.



(a) Case 1, sonic at $x/L = 0.333$.

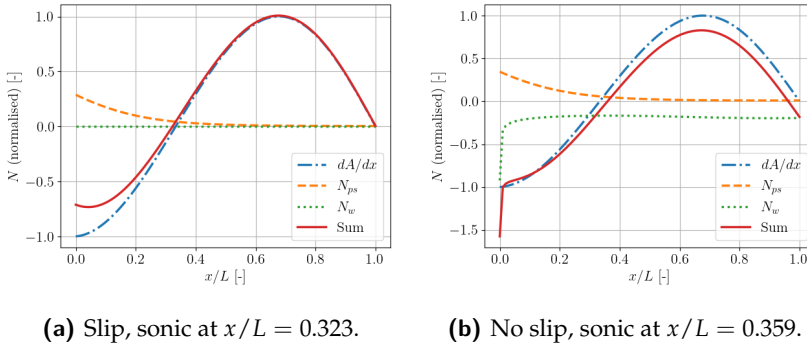


(b) Case 2, sonic at $x/L = 0.323$ (continued below).



(c) Case 3, sonic at $x/L = 0.359$.

Fig. 6.11 Mach numbers computed with the 1D model and from the RANS simulations. The sonic section (dotted line) coincides with the throat (red triangle) in the inviscid case 1. Inter-stream friction pushes the sonic section upstream (case 2), while wall friction pushes it downstream in case 3 (see (6.25)). The used friction coefficients are given by (6.20) and (6.22).



(a) Slip, sonic at $x/L = 0.323$.

(b) No slip, sonic at $x/L = 0.359$.

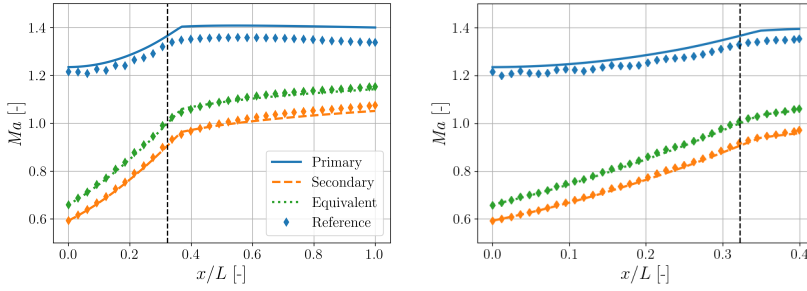
Fig. 6.12 Normalised components of the numerator in equation (6.24). The positive inter-stream friction term pushes the zero-crossing of the sum upstream, while the negative wall friction term pushes it downstream.

6.6.4 Flow blockage mechanism

As observed in figure 6.11, the secondary Mach number is below unity in the compound-sonic section if the primary stream is supersonic. It reaches unity further downstream, where the compound flow continues to expand in the diverging section of the nozzle. It can thus be stated that the *compound* flow is sonic in the first section and that the *secondary* stream is sonic, separately, in a section further downstream. Both sonic sections are plausible explanations for the blockage of the secondary mass flow rate, e.g., in ejectors, these choking mechanisms are referred to as ‘compound choking’ and ‘Fabri choking’ (see [19]). There is currently no consensus on which of these mechanisms limits the secondary mass flow rate.

The excellent match between the RANS simulations and the 1D predictions suggests that the compound choking condition is the more general mechanism. Furthermore, the secondary Mach number reaches unity necessarily *downstream* of the compound-sonic section since a combination of a sonic and a supersonic stream is necessarily compound-supersonic (cf. equation (6.13)). This suggests that cases identified as ‘Fabri choking’ also feature compound choking further upstream (see [44]). Moreover, [19] and [26] showed that the static pressure required to reach compound choking is higher than the one required to reach $Ma_s = 1$. This implies that the compound condition is less restrictive (the flow needs to expand less to reach the compound condition), but still sufficient to choke the flow. As a definitive proof of the generality of the compound choking condition to explain the mass flow limitation (compared to Fabri choking), a compound choked flow is demonstrated with a fully subsonic secondary stream.

To lower the secondary Mach number below unity, the divergent section of the nozzle presented in section 6.5 is replaced by a constant cross-section having the same diameter as the throat. The first simulation was carried out on a nozzle of the same length, while the second is cut shorter at $x/L = 0.4$ (so downstream of the throat at $x/L = 0.333$). The settings for the mesh, the boundary conditions and the numerical solver remain identical to the previous simulations. The absence of a divergent section would severely impact the viscous case with the no-slip condition at the wall since the compound sonic section is located in the divergent section. The resulting flow field forms a Fanno flow, which becomes nearly sonic at the outlet of the constant-area pipe. Therefore, the viscous case with a slip condition at the wall is retained, which becomes compound-sonic upstream of the throat regardless of the geometry downstream.



(a) Case 2, convergent-constant configuration.

(b) Case 2, cut configuration.

Fig. 6.13 Evolution of the Mach number for the convergent-constant and cut nozzle with friction between the streams and a slip condition at the wall (see figure 6.11b). The results coincide in the convergent section as the flow is choked in all cases. The secondary Mach number does not reach unity in the shorter nozzle.

Figure 6.13 shows the evolution of the post-processed Mach number of the convergent-constant and cut (still convergent-constant) configurations. The compound choking condition is reached at $x/L = 0.323$ in both cases, and the secondary mass flow rates are identical to 0.0622 kg/s (since the geometry and the boundary conditions at the inlet are also identical). The flow expands less in the constant area section compared to the convergent-divergent case, but the secondary Mach number still reaches unity at $x/L = 0.56$. However, the secondary cross-section does not reach a minimum in this section, suggesting that the secondary stream does not choke separately. This is confirmed by the simulation with the shorter nozzle, which coincides with the results of the complete nozzle. Despite the low pressure imposed as a boundary condition at the outlet, the secondary flow does not accelerate beyond $Ma_s = 0.97$ (both in the RANS simulation and in the 1D model). Hence, no larger secondary mass flow rate can be obtained, and the compound flow is choked with a fully subsonic secondary stream. These results suggest that compound choking is the more general choking condition in ejectors and, more generally, for parallel compressible streams with different stagnation pressures.

6.7 Conclusion

The isentropic compound flow theory has been extended with friction forces between the streams and at the wall. The resulting 1D model captures the upstream and downstream displacement of the sonic section observed by [19] and [26]. Moreover, the 1D predictions have been validated numerically to 2D axisymmetric RANS simulations with appropriate boundary conditions to include or eliminate the effect of friction and a parametric study has demonstrated that the model generalises well across different operating conditions and different geometries. An extension of this work to the modelling of ejectors is foreseen in the future.

To the authors' knowledge, our work is the first to propose an explanation for choking of a compound flow *upstream* of the geometrical throat. The Taylor expansion of the equation for the static pressure gradient allows for circumventing its singularity, avoiding numerical instabilities and increasing the accuracy of the numerical predictions. Analytical expressions have been presented for all required derivatives for a second-order approximation, with a quasi-third-order approximation obtained with finite differences. These expressions and the governing equations have first been derived in a general way in terms of a force F_i , facilitating future developments such as a compound flow with more than two streams or a planar geometry instead of the axisymmetrical one considered in this work. Another open path includes heat exchange and different total temperatures at the inlet, which may affect the shifting mechanism due to friction between the streams, as discussed in section 6.2.3. In particular, a high primary inlet temperature may invert the effect of inter-stream friction.

The developed model not only provides more insight into the choking of compound flows but also opens perspectives in the one-dimensional modelling of ejectors, as presented in the following chapter.

7

Double stream one-dimensional model

7.1 Introduction

The previous chapter introduced the extension of the compound theory with friction forces and their effect on the sonic section. The analysis isolated the mixing pipe to simplify the analysis of choking and to comply with the constraint of uniform static pressure. This chapter builds upon this earlier work, extending the analysis and the modelling to the full geometry of the ejector, thereby confronting under- and over-expanded primary streams in the mixing pipe. Furthermore, both the compound- and the Fabri-sonic conditions are implemented and evaluated on a common one-dimensional framework. This allows to evaluate the impact of the choking model on the one-dimensional predictions.

This work presents four configurations of a one-dimensional model, aimed at assessing the range of validity of both choking mechanisms. The first two configurations are based on the recent extension of the compound choking theory by Van den Berghe et al. [64], which accounts for wall friction effects at the sonic section and provides a practical framework for integrating the governing equations through the singular sonic point. In the first case, the friction forces are prescribed using calibrated correlations, yielding the only fully predictive model considered here. In the second,

the friction forces are directly imposed from axisymmetric RANS simulations, thereby removing uncertainties associated with empirical closures and isolating the influence of the choking mechanism itself. The remaining two configurations implement the Fabri choking condition, which requires an external specification of the dividing streamline. This input is provided either from RANS data or from the compound model. The latter case establishes a direct comparison between the two theories under otherwise identical conditions, allowing us to explore possible overlap or convergence between the Fabri and compound frameworks.

The common base of the model and both choking mechanisms are formally introduced in section 7.2. Practical considerations for implementing the model are presented in section 7.3. Section 7.4 explains the solution procedure of the governing equations. The calibration of the first model is discussed in section 7.5. The resulting predictions of the four models in on-design operation are compared and analysed in section 7.6. Finally, the predictive model is evaluated on a broader range of operating conditions on the UCLouvain ejector in section 7.7, including off-design operation.

7.2 Definition

Section 7.2.1 introduces the general one-dimensional governing equations. These leave one degree of freedom associated with the dividing streamline in the mixing pipe. Closure proceeds along two alternative routes: (i) enforcing equal static pressure across the streams, yielding the compound-choking formulation; or (ii) prescribing the streamwise evolution of the cross-sectional partition, yielding the Fabri-choking formulation. Further closure is required to account for the exchange terms between the two streams. This is addressed either by (a) employing friction coefficients and predictive correlations; or (b) imposing the exchanges extracted from an external source—specifically, the RANS simulations presented in section 2.4. These choices yield four distinct model configurations, which are introduced in section 7.2.2.

7.2.1 General equations

The 1D modelling approach considered in this work treats the primary and secondary streams as two parallel quasi-1D domains along the axial coordinate x , in an axisymmetric geometry with radial coordinate as shown schematically in figure 7.1. The domains span the full ejector: single-stream

inlets merge at $x = 0$ and continue as a coupled, double quasi-1D mixing pipe.

Cross-sections $A_i(x)$, with $i \in [p, s]$, vary smoothly along x and reveal a local discontinuity for the secondary at $x = 0$, where inlet inclination and finite primary-nozzle thickness are treated as in section 7.3.2. The governing equations are the conservation laws of mass, momentum, and energy in variable-area ducts, with ideal-gas behavior and no inter-stream mass transfer—hence the interface in figure 7.1 is a dividing streamline. The explicit relations (6.6)–(6.8) from the previous chapter apply both in the separate inlets and in the mixing pipe with the practical difference that the definitions of the momentum exchanges F_i depend on the domain. For example, wall friction acts where a stream is in contact with a wall, whereas inter-stream shear appears only in the mixing pipe; details are given in section 7.3. Energy-exchange terms are set to zero in the present study as discussed in chapter 4. Energy-exchange closures may be required when inlet total temperatures differ significantly but this is left for future work.

The radial position $r_d(x)$ of the dividing streamline determines the local cross-sections of both domains and is defined implicitly from the primary mass flow rate (see equation (2.2)) so that, at the exit of the primary nozzle, the streamline originates at the nozzle lip (figure 7.1). The cross-sectional areas A_i and their axial gradients are prescribed by geometry in the inlet sections but remain unknown in the mixing region, where the position of the dividing streamline must be determined. These unknown areas satisfy the geometric constraint $A_p(x) + A_s(x) = A(x)$, with $A(x)$ the total cross-section of the mixing duct, leaving a single degree of freedom. A unique solution can then be obtained by closing the system in one of two ways: by enforcing equal static pressure in both streams—yielding the compound-choking-based models (see chapter 6)—or by prescribing the dividing streamline from an external source, which defines the Fabri-choking-based models.

In this work, both approaches were implemented and evaluated, highlighting their respective advantages and limitations, as well as their potential common ground. Furthermore, two alternatives are considered for the exchange terms in Equations (6.6)–(6.8): they were either defined through correlations or extracted from the axisymmetric RANS simulations presented in section 2.4. The test cases in section 7.6 are organised according to these two aspects and are introduced below.

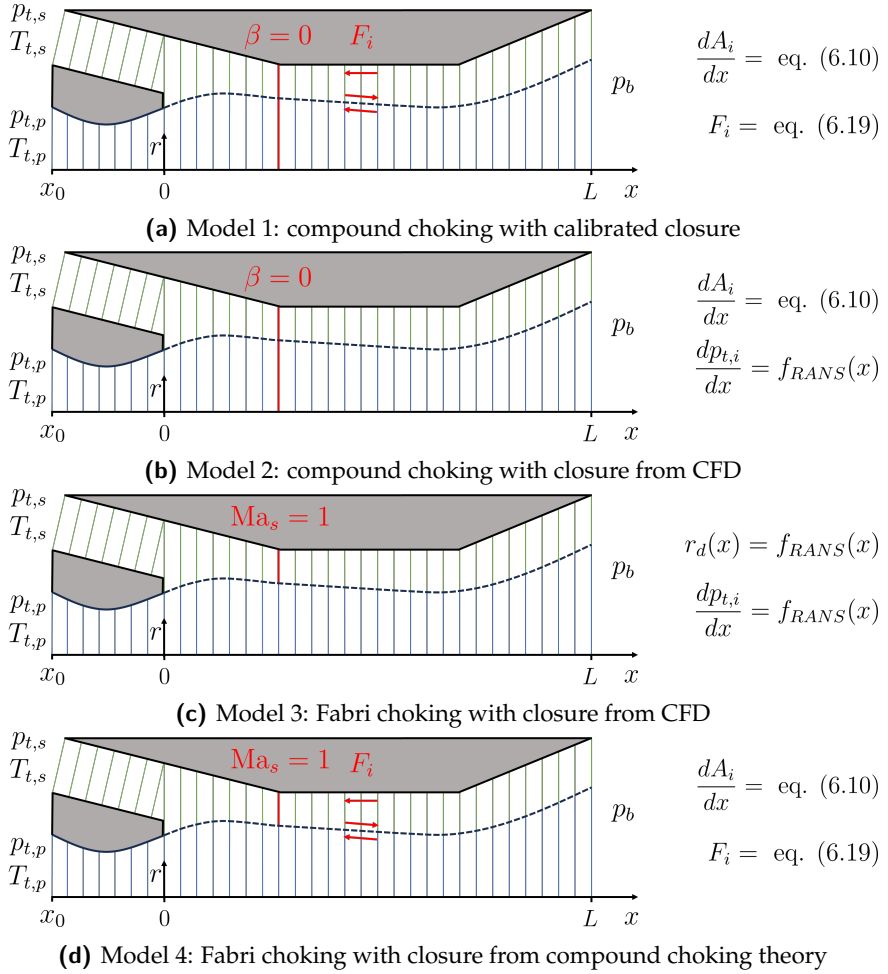


Fig. 7.1 The ejector is modelled with two 1D domains that are coupled in the mixing pipe. The geometry is assumed to be axisymmetric. The boundary conditions consist of the total pressures $p_{t,i}$ and total temperatures $T_{t,i}$ at the inlets and the static back pressure p_b . The dividing streamline, indicated by the dotted line, is computed with the equation (6.10) from the compound choking theory or imposed from the RANS simulations. Momentum exchange is accounted for through friction forces (see (6.19)) or through imposed total pressure gradients from RANS.

7.2.2 Selected models

The four model configurations investigated in this work combine two treatments of momentum exchange with two choking mechanisms, as illustrated in figure 7.1. The compound-based models include a fully predictive case using empirical friction correlations (Model 1, section 7.2.2) and a non-predictive one where total pressure gradients are imposed from the RANS simulations (Model 2, section 7.2.2). Comparing these two isolates the role of the closure terms and reveals the intrinsic limitations of the compound formulation.

The Fabri-based models follow the same logic: the dividing streamline and forces are either imposed from RANS data (Model 3, section 7.2.2) or taken from the compound model while retaining its closure correlations (Model 4, section 7.2.2). These comparisons highlight how the choking mechanism itself influences the predicted flow evolution and help identify conditions where the Fabri and compound frameworks may converge.

Model 1: compound choking with calibrated closures

In the first model, the streams in mixing pipe are governed by the compound equations developed in chapter 6. The friction forces are defined through equations (6.19), using the correlations (6.20) and (6.22), yielding a predictive model. The main difference with the test cases in chapter 6 lies in the addition of the inlet domains and consequently, the need for a mechanism that handles non-uniform static pressure at the inlet of the mixing pipe, which arises from a non-perfectly expanded primary stream. These practical considerations are presented in section 7.3. This model is also further analysed in off-design operation in section 7.7.

Model 2: compound choking with closure from CFD

In this configuration, the compound model from above is used without the friction forces defined in equation (6.19). Instead, the forces are prescribed from the reference RANS simulations. The model therefore serves as a diagnostic rather than a predictive tool. Rather than computing the forces F_i directly from local shear stresses within the RANS data, the total pressure gradients were extracted and imposed in the model, effectively replacing equation (6.7) and being incorporated into equation (6.6) through the force term. These gradients were obtained by filtering and differentiating the axial distributions of the cross-stream averaged total pressures. The pressure distributions themselves were obtained using the post-processing tools described in section 2.2.2. Practical details of the filtering and numerical dif-

ferentiation procedures are provided in section 7.3.5.

Model 3: Fabri choking with closures from CFD

In the Fabri choking framework, the two streams are not constrained to share the same static pressure, and the position of the dividing streamline must be prescribed externally. Consequently, equation (6.6) is integrated independently in each stream, $i \in [p, s]$. Choking occurs when the secondary stream reaches unit Mach number—the singular point of equation (6.6)—while the primary stream, having expanded through its converging–diverging nozzle, is already supersonic. Beyond this point, no information can propagate upstream in either stream, marking the onset of the Fabri choking regime.

In this third model, the dividing streamline is taken from the axisymmetric RANS simulations described in section 2.4 and imposed on the one-dimensional formulation. This configuration is therefore non-predictive and serves instead to analyse the choking mechanism and assess the impact of the simplifying assumptions inherent to one-dimensional models. In particular, it isolates the effect of cross-stream averaging while avoiding any assumptions on the pressure distribution, allowing a direct comparison with the compound-choking model that enforces equal pressures. The implications for choking behavior are discussed in section 7.6.

On the other hand, the Fabri-based models proved highly sensitive to the pressure evolution. Small deviations in the friction forces from the RANS data disrupt the balance in equation (6.6), especially near the sonic point, where the equation becomes stiff. Consequently, even minor inaccuracies in the closure terms can produce large discrepancies in the pressure distribution or cause numerical failure. This sensitivity stems from the coupled dependence of equation (6.6) on the dividing streamline and the force term, through dA_i/dx and F_i . As the secondary flow approaches Mach 1, the denominator of equation (6.6) tends to zero, and maintaining a finite pressure gradient requires the combined effect of the area gradient and the force term to vanish simultaneously. Hence, a consistent relation between the force closure and the dividing streamline is essential for numerical stability.

This limitation does not occur in compound-based models, where the dividing streamline evolves under the equal-pressure constraint. Since equation (6.12) already includes the closure terms, the streamline naturally adapts to them, and the area and force gradients act coherently. In contrast, the Fabri formulation retains an additional degree of freedom,

which can lead to instability when the two effects are misaligned near the sonic point. Practically, this means that the imposed forces must be consistent with those determining the dividing streamline, as obtained from the RANS simulations in this model configuration.

Model 4: Fabri choking with closure from compound choking theory

In this model, the predictive compound model from section 7.2.2 serves as the external source for the dividing streamline, while keeping the Fabri-sonic condition as the criterion for choking. The sensitivity discussed in the previous section remains, so the correlations for the friction forces are used in this configuration to be consistent with the dividing streamline model. This approach implies that the governing equations are identical to those of the predictive compound model. The only remaining difference lies in the sonic condition, which determines the solution for the choked flow (see section 7.4). Consequently, the corresponding analysis in section 7.6.4 is a direct comparison of the choking mechanisms on a common basis, albeit on an artificial dividing streamline to comply with the constraint of uniform pressure.

7.3 Practical implementation

7.3.1 Treatment of the inlets

For all the models discussed in the previous section, the friction coefficients in the inlets are taken using the empirical correlations in (6.19).

The boundary conditions specify the total pressure $p_{t,i}$ and total temperature $T_{t,i}$ at each inlet. The governing equations (6.6)–(6.8) form an explicit set of ODEs that are integrated axially from the inlets, where the total quantities are known. A shooting method is used to adjust the inlet static pressure until the target sonic condition is met (see section 7.4).

The same formulation applies to both streams. However, since the primary jet is choked in its converging–diverging nozzle, it can be solved independently. The secondary flow, on the other hand, depends on the back pressure p_b at the outlet of the diffuser and must therefore be solved in conjunction with the mixing section, as discussed below.

7.3.2 Jump relations at the inlet of the mixing pipe

The full flow states are known at the outlets of the separate domains ($x = 0$) after integrating the respective ODEs in the inlets. The dividing streamline

originates at the bottom of the primary nozzle lip, so the primary cross-sections and variables match between the inlet, denoted with the subscript L , and in the mixing pipe, with subscript R (see figure 7.2). This is not the case for the secondary stream due to its inclined inlet and a potential finite thickness A_w of the nozzle. The secondary flow state on the right is found through conservation of mass, momentum and energy. The control volume is infinitesimally thin ($\Delta x \rightarrow 0$ in figure 7.2). Since $A_{s,R} + A_{p,R} = A(x=0)$, three unknowns remain (e.g., density, velocity and temperature). Two forces intervene in the axial momentum balance: the force on the bottom wall $\bar{F}_{w,b}$ and the force on the nozzle lip $\bar{F}_{w,ps}$. It is assumed that friction acting on these surfaces is negligible compared to the pressure force, which has been confirmed during the post-processing of the RANS simulations in this work. These forces are therefore orthogonal to the surfaces on which they act. The inclinations of the bottom and top wall are generally different and are denoted here with $\alpha_b > 0$ and $\alpha_t > 0$. The boundary of the secondary inlet is defined as the line originating from the top wall at the exit of the primary nozzle ($x = 0$), with a central angle $\alpha_c = 0.5(\alpha_b + \alpha_t)$. Projecting the momentum fluxes and the forces on the x-axis with their respective angles leads to the following balance:

$$(\dot{m}_s u_{s,L} + p_{s,L} A_{s,L}) \cos \alpha_c + F_{w,b} \sin \alpha_b + F_{w,ps} = \dot{m}_s u_{s,R} + p_{s,R} A_{s,R}, \quad (7.1)$$

where u denotes the average velocity as described in the 1D equations. The magnitude of the forces $F_{w,b}$ and $F_{w,ps}$ need to be estimated from the averaged quantities of the left and right states. The expansion of the secondary stream from the inclined section on the left to $x = 0$ is generally small, hence the force on the bottom wall is approximated as $F_{w,b} = p_{s,L} A_{w,b}$, where the area can be computed from the geometry. Similarly, the force on the nozzle lip is defined as $F_{w,ps} = p_{s,R} A_{w,ps}$, where the (unknown) static pressure on the right side is applied since it is in direct contact with this wall.

Conservation of energy implies that the total temperature is conserved, since there is no source term and the walls are assumed to be adiabatic. The remaining unknowns in equation (7.1) are the velocity $u_{s,R}$ and the static pressure $p_{s,R}$. These are linked through the known mass flow rate $\dot{m}_s$, the known total temperature $T_{t,s}$ and the ideal gas law:

$$\dot{m}_s = \frac{p_{s,R}}{RT_{s,R}} A_{s,R} u_{s,R}, \quad \text{and} \quad T_{s,R} = T_{t,s,R} - \frac{u_{s,R}^2}{2c_p}. \quad (7.2)$$

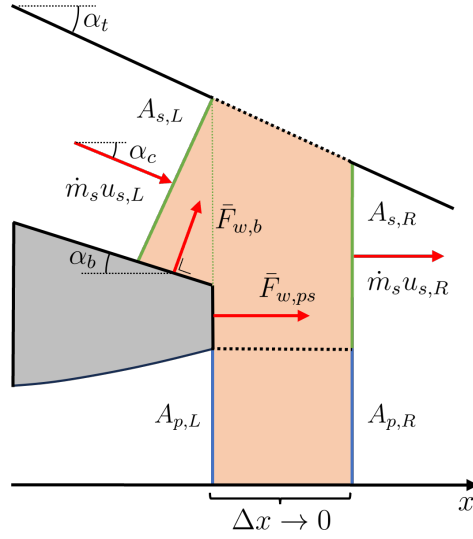


Fig. 7.2 The jump conditions from the inlets to the mixing pipe. The primary cross-sections and flow states are identical on both sides. The secondary stream requires a force balance, where the momentum fluxes and pressure forces are projected on the x-axis.

The two equations above allow to eliminate $p_{s,R}$ from the momentum balance (7.1), leading to a non-linear equation in $u_{s,R}$, which can be solved iteratively. The velocity $u_{s,L}$ from the secondary inlet is a good initial guess. The secondary flow state can then be computed from $\dot{m}_s$, $T_{t,s,R}$ and $u_{s,R}$.

The resulting states generally have a different static pressure because the primary stream is over- or under-expanded. A uniform static pressure $p(x) = p_p(x) = p_s(x)$ is required for the compound model (see section 7.2.2), so first a pressure equalisation mechanism is defined below. This is not required when using the Fabri-based model, where the static pressures are left free in each stream.

7.3.3 Pressure equalisation mechanism for the compound theory

At the inlet of the mixing pipe, the primary stream is typically either under- or over-expanded relative to the surrounding secondary stream, as evidenced by the shock-train structures observed in numerical studies [9, 11]. This behavior directly conflicts with the core assumption of equal static pressures in the compound flow theory and thus with model 1 (sec-

tion 7.2.2) and model 2 (section 7.2.2). To address this limitation, the present work introduces a *pressure equalisation mechanism* that brings the two streams to a uniform static pressure over a finite distance downstream of the primary nozzle outlet, as illustrated schematically in figure 7.3a.

The proposed mechanism is inspired by the first cell of the shock train and reproduces the local pressure-adjustment process up to the point where the average static pressures of the two streams first become equal. The analysis is local in nature and provides the orientation of the dividing streamline as a function of the local static-pressure ratio, as shown in figure 7.3. Separate formulations are adopted for under-expanded and over-expanded primary streams, though the underlying principle remains the same.

For an under-expanded primary stream, further expansion is modelled through a diverging cross-section computed using Prandtl–Meyer expansion theory, following the approach of [41]. The process is assumed to be isentropic, ensuring conservation of total pressure from the current state (p_p, Ma_p) to a virtual state $(p_s, \widetilde{\text{Ma}}_p)$ characterised by the same static pressure as the secondary stream:

$$p_{t,p} = p_s \left(1 + \frac{\gamma - 1}{2} \widetilde{\text{Ma}}_p^2 \right)^{\frac{\gamma}{\gamma - 1}}, \quad (7.3)$$

from which the expanded Mach number $\widetilde{\text{Ma}}_p$ is determined. The deviation angle θ is then defined as the difference between the Prandtl–Meyer angles ν and $\tilde{\nu}$ corresponding to the local and virtual primary Mach numbers Ma_p and $\widetilde{\text{Ma}}_p$, respectively. For completeness, the Prandtl–Meyer angle is given by [57]

$$\nu(\text{Ma}) = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \arctan \left(\sqrt{\frac{\gamma - 1}{\gamma + 1}} (\text{Ma}^2 - 1) \right) - \arctan \left(\sqrt{\text{Ma}^2 - 1} \right). \quad (7.4)$$

If the primary and secondary static pressures were equal, the dividing streamline would be tangent to the wall at the outlet of the primary nozzle (without sudden change of angle). The computed deviation angle $\tilde{\nu} - \nu$ should thus be added to the angle θ_p of the primary nozzle at its outlet to obtain the angle of the dividing streamline:

$$\theta = \theta_p + (\tilde{\nu} - \nu). \quad (7.5)$$

An oblique shock is computed for the pressure ratio p_s/p_p and a primary Mach number Ma_p in case of an over-expanded primary stream. The shock angle α can be computed from the following equation ([57]):

$$\frac{p_s}{p_p} = 1 + \frac{2\gamma}{\gamma + 1} \left(Ma_p^2 \sin^2(\alpha) - 1 \right). \quad (7.6)$$

The deviation angle θ is directly related to the shock angle α and the Mach number Ma_p :

$$\tan(\theta) = -\frac{2 \cot(\alpha) \left(Ma_p^2 \sin^2(\alpha) - 1 \right)}{Ma_p^2 (\gamma + \cos(2\alpha)) + 2}. \quad (7.7)$$

Note the minus sign for the inward deviation of the primary flow. In the axisymmetric geometry, the axial derivative of the primary cross-section can be decomposed in terms of the deviation angle θ :

$$\frac{dA_p}{dx} = 2\pi r_p \frac{dr_p}{dx} = 2\sqrt{\pi A_p} \tan(\theta). \quad (7.8)$$

The axial derivative of the secondary cross-section follows from the geometry $A(x)$ of the mixing pipe. These local gradients of the cross-sections allow to integrate the governing Equations (6.6)-(6.8) (which remain valid in the mixing pipe) to the next grid point, using the closure relations (6.19). The overall process is not isentropic due to the friction forces; the isentropic pressure equalisation mechanism is used only to determine the local angle of the dividing streamline. The procedure above is repeated until the static pressure difference drops below an arbitrary threshold (10 Pa in this work).

This approach reproduces only the first cell of the shock train and cannot capture downstream behavior, as reflections of Mach lines on the centerline and dividing streamline are inherently two-dimensional and thus excluded from the present one-dimensional model. Nevertheless, the proposed pressure equalisation mechanism offers two main advantages: (1) its local formulation makes it directly compatible with standard ODE integrators, and (2) the pressure equalisation occurs over a short distance, enabling the compound flow theory to be applied as far upstream as possible to accurately locate the sonic point.

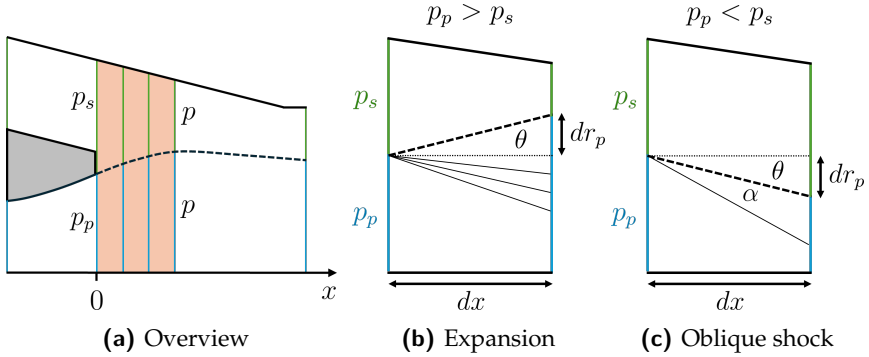


Fig. 7.3 An under- or over-expanded primary stream is brought to a uniform static pressure over a finite distance using Prandtl-Meyer expansion theory or oblique shocks respectively. The procedure results in the angle θ of the dividing streamline, which suffices to complement the conservation Equations (6.6)-(6.8).

7.3.4 Normal compound shocks

An oblique shock train typically forms in the diffuser under on-design conditions, decelerating the flow to subsonic speed and reestablishing the back pressure imposed by the boundary conditions [9, 11]. In simplified models, this system is often represented as a normal shock in 0D formulations [32,46,47], and it also emerges naturally in the unsteady single-stream model of [104].

In the compound-flow framework of model 1 and model 2, however, a normal-shock representation cannot be directly applied because the compound supersonic region may include a subsonic component, rendering standard shock relations invalid. Furthermore, shocks of unequal strength would develop in the two streams due to their differing velocities, producing a downstream pressure imbalance and violating the assumption of uniform static pressure. The inherently two-dimensional nature of the oblique shock train and its interaction with boundary layers further preclude an accurate one-dimensional description under the constraint of pressure uniformity.

Since the 1D formulation cannot resolve the flow structure downstream of the shock, the mixing between the two streams also becomes undefined. To apply normal-shock relations in this context, it is therefore necessary to define an *equivalent single-stream state* that represents the combined effect

of the primary and secondary streams at the shock location. This equivalent, or “fully mixed,” state is a notional construct used solely for the computation of the normal shock and is defined to have the same total mass flow rate, momentum, and energy as the actual double-stream configuration. This approach leads to an artificial increase of the secondary total pressure to the uniform value downstream (see figures 7.11-7.13), due to the assumption of instantaneous mixing inherent to the definition of the equivalent state. The normal-shock model is thus employed solely to recover the correct back pressure, without physical interpretation of the local shock train or mixing processes. The normal-shock relations are then applied between this equivalent upstream state, denoted by the subscript L , and the downstream state, denoted by R . The properties of the equivalent state upstream of the shock are obtained from the conservation of mass and energy as

$$\dot{m}_L = \dot{m}_p + \dot{m}_s, \quad \text{and} \quad T_{t,L} = \frac{\dot{m}_p T_{t,p} + \dot{m}_s T_{t,s}}{\dot{m}_p + \dot{m}_s}. \quad (7.9)$$

The mass flow rate and the momentum flux can be defined in terms of the static pressure p , the total temperature T_t , the cross-section A and the Mach number Ma :

$$\dot{m} = pA Ma \sqrt{\frac{\gamma}{R_{gas} T_t}} \sqrt{1 + \frac{\gamma-1}{2} Ma^2}, \quad \text{and} \quad \dot{m}u + pA = pA \left(1 + \gamma Ma^2\right). \quad (7.10)$$

The static pressure and the cross-section can be eliminated by taking the ratio of the momentum flux and the mass flow rate, which becomes a function of the Mach number and the total temperature:

$$\frac{\dot{m}u + pA}{\dot{m}} = \frac{1 + \gamma Ma^2}{Ma \sqrt{1 + \frac{\gamma-1}{2} Ma^2}} \sqrt{\frac{R_{gas} T_t}{\gamma}}. \quad (7.11)$$

Hence, conservation of momentum between the double state and the equivalent state L implies:

$$\frac{\sum_{i \in [p,s]} (\dot{m}_i u_i + p_i A_i)}{\sum_{i \in [p,s]} \dot{m}_i} = \left[\frac{1 + \gamma Ma_L^2}{Ma_L \sqrt{1 + \frac{\gamma-1}{2} Ma_L^2}} \right] \sqrt{\frac{R_{gas} T_{t,L}}{\gamma}}, \quad (7.12)$$

where the total temperature $T_{t,L}$ is known from the equation (7.9). The

term between the brackets is nonlinear in Ma_L and admits a subsonic and a supersonic solution for a given ratio of momentum to mass flow rate. The minimum corresponds to a sonic flow, indicating a minimal momentum for a given mass flow rate (or a maximal mass flow rate for a given amount of momentum). The supersonic solution is retained, followed by the normal shock relations to compute the right state R (see for example [57]):

$$\text{Ma}_R^2 = \frac{(\gamma - 1)\text{Ma}_L^2 + 2}{2\gamma\text{Ma}_L^2 - (\gamma - 1)},$$

$$\frac{p_R}{p_L} = \left(1 + \frac{2\gamma}{\gamma + 1} (\text{Ma}_L^2 - 1)\right), \quad \text{and} \quad \frac{T_{t,R}}{T_{t,L}} = 1. \quad (7.13)$$

The flow downstream is computed as an isentropic single stream, which has an analytical solution considering that the total pressure and total temperature are conserved and that the mass flow rate is known. The Mach number Ma_m of the single stream respects the following (non-linear) equation in any point downstream of the shock:

$$\dot{m}_L = p_{t,R} A(x) \sqrt{\frac{\gamma}{R_{gas} T_{t,R}}} \text{Ma}_m(x) \left[1 + \frac{\gamma - 1}{2} \text{Ma}_m^2(x)\right]^{-\frac{1}{2} \frac{\gamma + 1}{\gamma - 1}}. \quad (7.14)$$

7.3.5 Filtering, differentiation and interpolation

Imposing cross-section or total-pressure gradients from the RANS simulations requires computing and interpolating these quantities so they can be evaluated by the solver at any axial position x . The primary and secondary cross-sections and total pressures are obtained directly from the geometry, the dividing streamline, and the flow field. However, their derivatives in equation (6.6), obtained by numerical differentiation, are sensitive to noise due to interpolation and cross-stream averaging.

The distributions of the total pressures and the primary cross-section are therefore first filtered before applying finite differences with Numpy's *gradient* function. The cross-section of the secondary stream and its derivatives follow from the known geometry and the primary stream. The filter used in this work is based on the filter of Chambolle [118] available in the '*denoise to chambolle*' function from Scikit-image [119]. The total variation denoising formulation avoids spurious overshoot near sharp transitions, but it is subject to boundary effects, which are problematic in the current work because the dividing streamline originates from the lip of the pri-

mary nozzle. Therefore, the signal used in the model is a combination of the original and the filtered signal :

$$\tilde{u}[j] = w[j]u[j] + \hat{w}[j]\hat{u}[j], \quad (7.15)$$

where $\tilde{u}[j]$ denotes the j^{th} entry of the final discrete signal, composed of the original and the filtered signals u and $\hat{u}$. These are blended via a partition of unity that balances the unfiltered and the filtered signal such that the sum of the weights equals one at each point:

$$w[j] = \begin{cases} 0.5 + 0.5 \cos\left(\pi \frac{j}{60}\right) & \text{if } j \leq 60, \\ 0.5 - 0.5 \cos\left(\pi \frac{j-(N-61)}{60}\right) & \text{if } j > N - 60, \\ 0 & \text{otherwise.} \end{cases} \quad (7.16)$$

and

$$\hat{w}[j] = \begin{cases} 0.5 - 0.5 \cos\left(\pi \frac{j}{60}\right) & \text{if } j \leq 60, \\ 0.5 + 0.5 \cos\left(\pi \frac{j-(N-61)}{60}\right) & \text{if } j > N - 60, \\ 1 & \text{otherwise.} \end{cases} \quad (7.17)$$

The original signal is thus kept near the boundaries, transitioning over 60 entries to the filtered signal which is kept elsewhere. The width of 60 entries has been chosen arbitrarily as a compromise between filter intrusiveness near the boundary and noise tolerance.

Finally, since the full solver requires the evaluation of the governing Equations (6.6)-(6.8) at any axial position x , an interpolation within the filtered grid was applied. This is a cubic interpolation using SciPy's 'interp1d' function ([89]).

7.4 Numerical solution

The governing equations presented in the section above yields a system of ODEs in space, integrated with a shooting method to match the boundary condition at the outlet. The integration was carried out with the explicit Runge-Kutta 4 scheme using the `SciPy.integrate` library. The choked flow in the primary nozzle is computed first, since this is independent from the flow in the rest of the ejector. The flows in the secondary inlet and in the mixing pipe are coupled and solved afterwards. This section presents the solution procedure on the level of the complete ejector model and of its

components.

Figure 7.4 shows the high-level procedure, starting with the choked flow in the primary nozzle which can be solved independently. Next, the critical point of the ejector is computed, which marks the division between on- and off-design operation. It occurs at the maximal back pressure that chokes the flow, i.e., the flow becomes compound sonic and becomes subsonic immediately downstream. The supersonic solution is computed if the computed critical back pressure p_{crit} is higher than the imposed boundary condition p_b , indicating on-design operation. A normal shock brings the flow back to subsonic conditions as specified in the section above. Its position is computed to match the back pressure p_b from the boundary condition. In the off-design case, the secondary mass flow rate is found iteratively to match the back pressure. The model predictions therefore respect the boundary conditions at the inlets and at the outlet in all operating conditions. The following sections go in more detail on the procedures inside each block in figure 7.4.

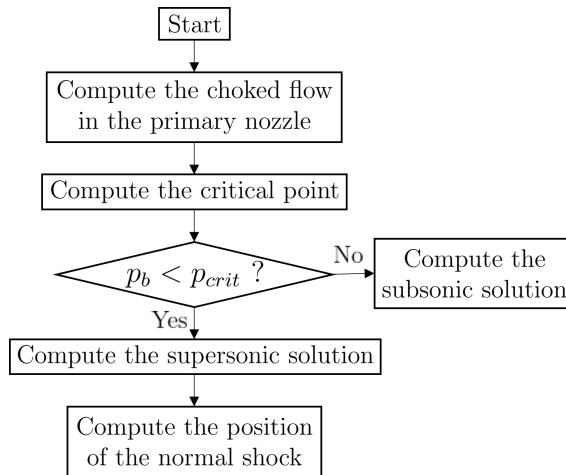


Fig. 7.4 High-level flow chart of the numerical solution. The primary stream is assumed to be choked, so can be computed independently. The critical point is calculated first, as in 0D models [46, 47], to decide on the operating regime depending on the back pressure p_b . A normal shock is computed in the on-design operation to match the back pressure.

Primary inlet

Given the inlet totals $(p_{t,p}, T_{t,p})$, the primary mass flow is obtained by a *bracketed shooting* on the inlet static pressure $p_p(x_0)$ to enforce $\text{Ma}_p = 1$ (and $N_p = 0$ in equation (7.20)). By definition the upper bound is $p_p(x_0) \leq p_{t,p}$ while the lower bound is the isentropic sonic pressure

$$p_{\text{sonic}} = p_{t,p} \left(\frac{\gamma + 1}{2} \right)^{-\gamma/(\gamma-1)}, \quad (7.18)$$

since the flow is subsonic in the converging section. The search therefore brackets $p_p(x_0) \in [p_{\text{sonic}}, p_{t,p}]$ and proceeds by bisection: at each iteration a trial $p_p(x_0)$ (the bracket midpoint) is used to integrate (6.6)–(6.8) from x_0 to either the point where $\text{Ma}_p = 1$ or the nozzle exit. Reaching the exit while still subsonic indicates $p_p(x_0)$ is too high and hence the upper bound is set to $p_p(x_0)$. Otherwise the bracket is tightened around the $\text{Ma}_p = 1$ solution. This monotone update exploits the fact that decreasing $p_p(x_0)$ increases the mass flow toward choking (see the flowchart in figure 7.5).

As the sonic condition is approached, the denominator in (6.6) tends to zero. To obtain a finite ratio and allow the flow to continuously expand to supersonic conditions, also the numerator must tend to zero (see [57]). Isolating the common denominator and defining the numerator N_p :

$$\frac{1}{p_p} \frac{dp_p}{dx} = \frac{N_p}{1 - \text{Ma}_p^2}, \quad (7.19)$$

where

$$N_p = \gamma \text{Ma}_p^2 \frac{1}{A_p} \frac{dA_p}{dx} + \left(1 + (\gamma - 1) \text{Ma}_p^2 \right) \frac{F_p}{A_p p_p}, \quad (7.20)$$

a finite pressure gradient is obtained if:

$$\lim_{(p, \text{Ma}_p) \rightarrow (p^*, 1)} \frac{1}{p_p} \frac{dp_p}{dx} = \frac{dN_p/dx}{-d\text{Ma}_p^2/dx} \quad (7.21)$$

following de l'Hôpital's rule. The derivative of the Mach number can be analytically evaluated from the conservation equation (see equation (B.5)), which introduces the derivative of the static pressure in the denominator. This leads to the quadratic equation (B.24) for the static pressure gradient in the sonic point, which is derived in appendix B (see also [57, 64]). The static pressure gradient can thus be calculated in all admissible conditions.

If the procedure in figure 7.5 runs into the sonic point, the numerator N_p is computed with the equation (7.20). If its absolute value drops below an arbitrary threshold (10^{-6} in this work), the correct sonic condition is found and the calculation proceeds. Otherwise, the bracket is adjusted. A negative value of the numerator indicates that the flow would expand if it is subsonic ($dp/dx < 0$ in (7.19)). This occurs for example in the convergent section for an isentropic flow (see (7.20) with $F = 0$). The fact that the flow already reached $Ma_p = 1$ indicates that the flow rate is too high: the Mach number should be lower, allowing the flow to expand further without running into the singularity of equation (7.19). In the isentropic example, the sonic point should be reached further downstream at the throat. The guessed static pressure is thus too low if $N_p(x) < 0$ and $Ma_p(x) = 1$. Therefore, the used average pressure $p(x_0)$ becomes the new lower bound p_{min} . The inverse is true if $N_p > 0$. The friction force $F_p < 0$ interferes in the numerator N_p , but it does not alter the mechanism discussed above. Note that dA/dx must be positive to compensate the negative term containing the force F_p in equation (7.20): the wall friction force pushes the sonic section downstream in a divergent section of the nozzle. The algorithm iterates until the tolerance on the numerator is reached or until the bracket reaches the following relative tolerance:

$$\frac{p_{max} - p_{min}}{0.5(p_{min} + p_{max})} \leq 10^{-6}. \quad (7.22)$$

The procedure above provides the solution up to the sonic point. At this point, the static pressure gradient is undetermined as (6.6) approaches the limit $0/0$, so a second order approximation of the pressure gradient near the sonic point is computed using Taylor expansions (see equation (B.26) in appendix B if $Ma_p \leq 1.01$). Once the primary stream expands beyond $Ma_p = 1.01$, the pressure is computed from equation (7.19). The threshold of 1.01 was found to provide a good compromise between accuracy and smoothness of the transition from the approximation (B.26) to the governing equation (7.19). In practice, the approximation (B.26) is blended with equation (7.19) in a single function that automatically switches between both depending on the primary Mach number Ma_p , as indicated in figure 7.5. The interested reader is referred to [64] for a more detailed description of this approach.

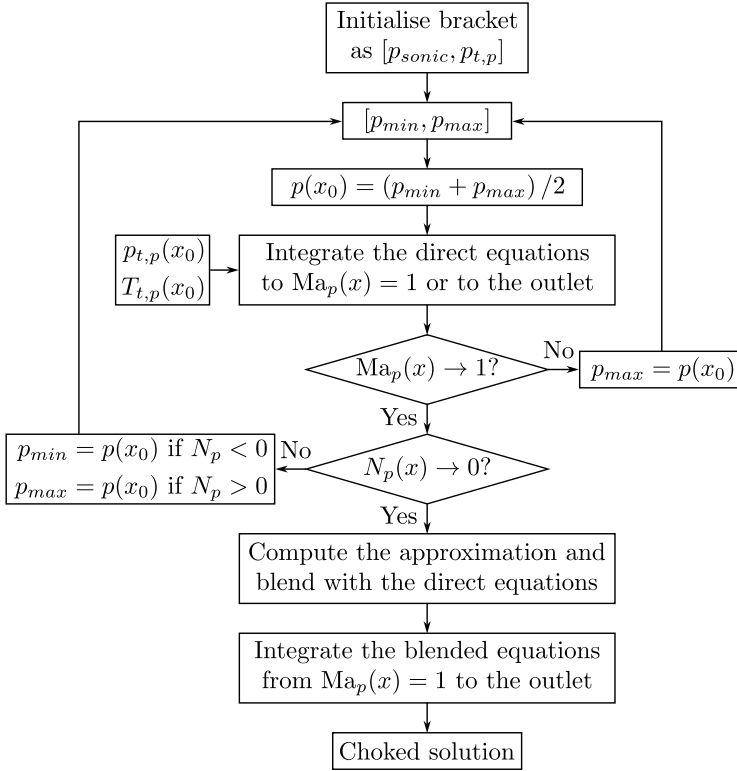


Fig. 7.5 Flow chart for computing the choked flow in the primary nozzle. The static pressure at the inlet is found iteratively to satisfy the sonic condition $Ma_p \rightarrow 1, N_p \rightarrow 0$ in equation (7.19). The approximations in appendix B allow to integrate the governing equations through the sonic point without numerical issues related to division by zero (see also [64, 116]).

Critical point and on-design operation

The procedure for the critical operating point is similar to the one of the primary nozzle, as shown in figure 7.6: the static pressure is guessed at the secondary inlet, the governing equations are integrated to the compound- or Fabri-sonic point ($\beta = 0$ in (6.12) or $Ma_s = 1$ in (6.6)) and the solution is accepted if the numerator of equation (6.12) or (6.6) equals zero.

An approximation of the static pressure gradient is then computed near the sonic point, either taking the positive or the negative pressure gradient at the sonic point to compress the flow to subsonic conditions (critical point) or to expand to supersonic conditions. The required derivatives are listed in the appendix B. The blended equations are integrated to the outlet, switching back to the direct equations once $|Ma_{eq} - 1| \geq 0.01$ or $|Ma_s - 1| \geq 0.01$. The axial position of the normal shock is found iteratively to respect the outlet condition in case the back pressure is lower than the computed critical back pressure (on-design operation in figure 7.4).

The flow might become sonic in the secondary inlet, in which case the guessed static pressure is increased to reduce the mass flow rate and reach a sonic condition in the mixing pipe instead. For the geometry analysed in this work, the sonic point is always located in the mixing pipe, which is the most typical scenario. The analysis of the sign of the numerator in the computed sonic point is identical to the one for the primary nozzle above due to the similarity of Equations (6.6) and (6.12).

The critical point has already been computed in the previous step, so the on-design solution has already been found up to the sonic point. To match the back pressure, the fully supersonic solution is computed first. This allows to apply the normal shock procedure from section 7.3.4 at any position between the sonic point and the outlet. Its axial position is found iteratively to respect the outlet condition.

Off-design operation

The main steps in the off-design regime are similar to the one in on-design (see figure 7.7). In this case, the back pressure influences the secondary mass flow rate since the flow remains compound-subsonic in the mixing pipe. The static pressure $p_{s,crit}$ at the secondary inlet corresponding to the critical point is already known at this stage (cf. figure 7.4), and serves as a lower limit for the starting bracket. The upper limit consists of zero mass flow rate ($p = p_{t,s}$), as the model breaks down in case of back flow. The governing equations are integrated to the outlet of the mixing pipe, pro-

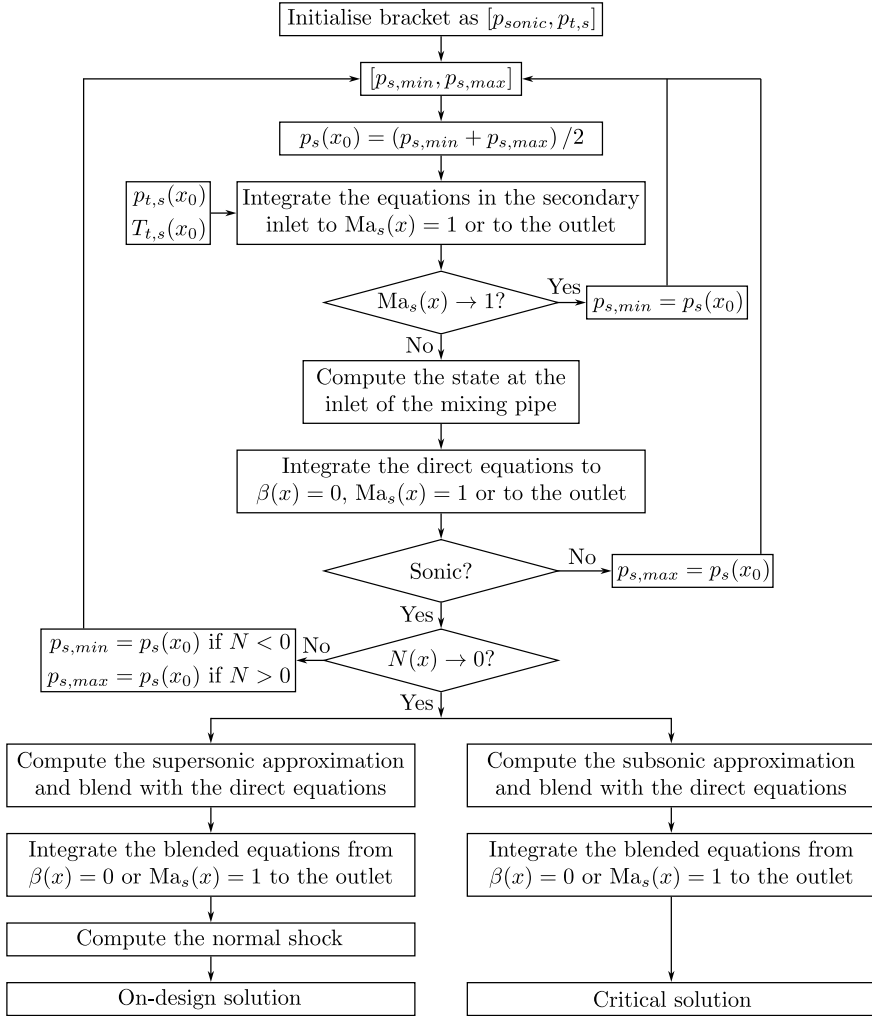


Fig. 7.6 The solution procedure iterates on the static pressure at the secondary inlet to find the correct compound- or Fabri-sonic point. A sonic flow in the secondary inlet is not accepted as a valid solution since it is atypical for ejectors. The procedure differs in the compound and Fabri case through the sonic condition and the corresponding approximations of the pressure gradient.

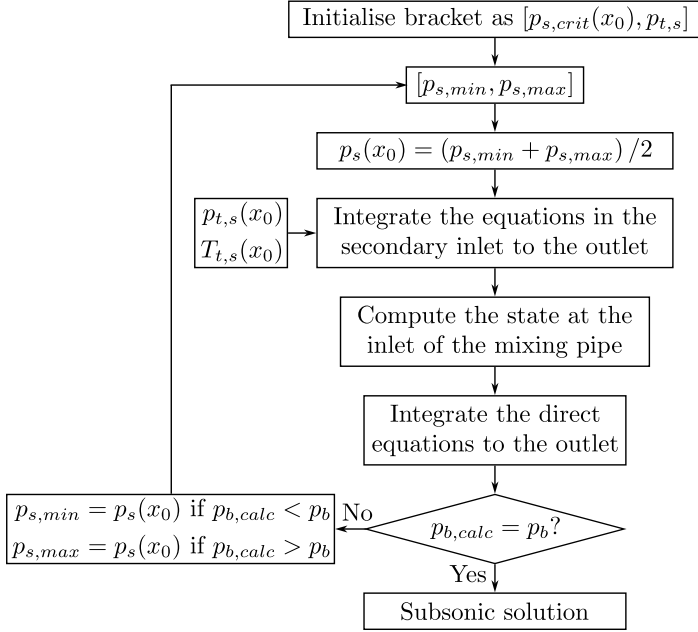


Fig. 7.7 The numerical procedure for the off-design operation iterates on the static pressure at the secondary inlet until the back pressure is matched. The previously found critical point provides an upper limit for the secondary mass flow rate.

viding a computed back pressure. If the computed value is lower than the boundary condition, the guessed static pressure at the secondary inlet should increase and vice versa. The procedure stops if the error on the back pressure drops below a predefined threshold (1 Pa in this work) or if the relative tolerance on the bracket is reached (see equation (7.22)).

7.5 Calibration of model 1

7.5.1 Definition of the calibration parameters

The closure equations for model 1 in section 7.2.2 include two friction forces, each represented by a friction coefficient. These have to be provided by empirical correlations or by a model calibration. For the wall friction, reliable correlations exist and this work used the correlation of [115]. For inter-stream friction, however, the only available correlation in the liter-

ature is that of [112]. This correlation is rooted in experimental studies of compressible shear layers and expresses the friction in terms of free-stream quantities. However, such free stream quantities are not readily available from the present 1D model (as opposed to quasi-2D models such as [25]), which relies on cross-stream averaged values. Moreover, the flow mixing in the mixing pipe eventually leads to the vanishing of the free streams once the shear layer reaches both the centerline and the wall boundary layers. Therefore, an empirical law for the inter-stream coefficient was proposed by correcting the (6.22) as follows

$$f_{ps}^* = w(x)f_{ps} = \left(\frac{w_1}{2} \tanh(30(x - w_2)) + \frac{w_1}{2} + 1 \right) f_{ps}, \quad (7.23)$$

where w_1 and w_2 are calibration parameters that control the amplitude and axial location of the correction. By construction, the factor tends to unity near the pipe inlet, so the original correlation is retained there, while the friction is increased further downstream (for $w_1 > 0$). The constant 30 was chosen to give a smooth step over the length of the constant-area section ($L = 0.5$ m); it could also be included in the calibration but behaves similarly to w_2 and was therefore fixed.

7.5.2 Sensitivity analysis

Figure 7.8 shows the effect of the calibration parameters w_1 and w_2 on the distributions of the weight $w(x)$, on the total pressures, and on the operating curve. The boundary conditions correspond to the case with $p_{t,p}/p_{t,s} = 5$, $p_b/p_{t,s} = 1.2$ of the UCLouvain ejector (see section 2.4). The total pressure is the most direct indicator for changes in the closure, since it is solely affected by the force terms in equation (6.7). The step induces a stronger inter-stream friction, which in turn leads to stronger gradients of the total pressure. Consequently, the total pressures converge more quickly towards a common value, indicating mixing. The discontinuity near $x/L = 1.2$ is caused by the normal shock occurring for the leftmost on-design point of the operating curves in figure 7.8c.

The global effect of w_1 consists of increasing the critical back pressure due to larger momentum exchange between the streams. The entrainment in on-design operation is unaffected, since the sonic section is typically located near the inlet of the mixing pipe at $x/L = 0$. The shown distributions of $w(x)$ are all approximately equal to unity, so the portion upstream of the sonic section is not affected by the calibration. Lower values of w_2

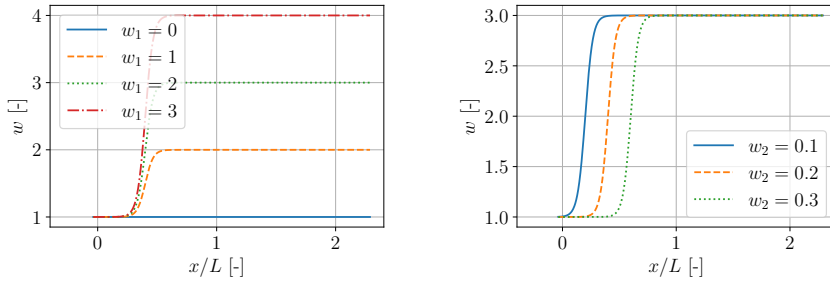
allow to adapt the inter-stream friction at the inlet of the mixing pipe, but the short distance between the outlet of the primary nozzle and the sonic section limits the amount of momentum that can be exchanged between the streams, and hence its effect on the entrainment is small. This leads to the important observation that choking and thus on-design performance is mainly pressure-driven and that the inter-stream friction is mostly responsible for the pressure recovery downstream. The effects of w_1 and w_2 on the predicted critical back pressure are opposite: increasing w_1 and/or decreasing w_2 tends to increase the total exchanged momentum and hence to increase the critical back pressure. The effect of w_2 is more pronounced in the local distributions of the total pressure, shifting the inflection point in the total pressure upstream or downstream. The combination of both parameters gives the model the flexibility to predict a wide range of total pressure distributions. It is noteworthy to mention that the two last points of the operating curve (the two highest back pressures) can not be reached by the model when using the original correlation ($w_1 = 0$), as it would lead to back flow. This illustrates the need for calibration, since no back flow is encountered in the considered RANS simulations.

The horizontal shift of the operating curve implies that the model is capable to match the critical back pressure of reference data through calibration. This ensures that the model predicts the correct operating regime. It is equally possible to calibrate the model in off-design operation based on the entrainment: the predicted mass flow rate varies with w_1 and w_2 at for example $p_b/p_{t,s} = 1.8$ in figure 7.8c. The insensitivity of the on-design entrainment with respect to w_1 and w_2 precludes calibration to match this quantity.

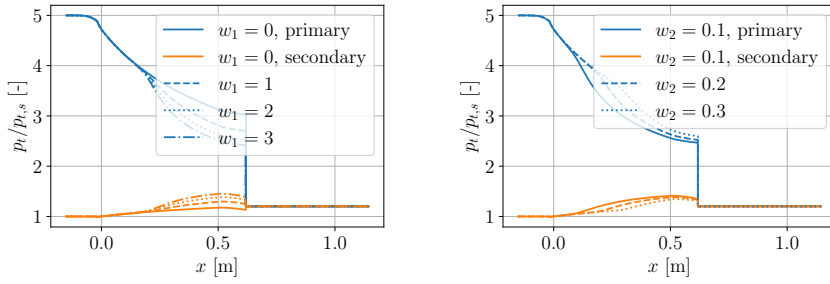
Note that the calibration of the inter-stream friction does not affect the primary mass flow rate, since the primary stream is assumed to be choked independently. This is effectively the case in all conditions considered in this work, leading to errors on the primary mass flow rate of the order of one percent.

7.5.3 Calibration strategy

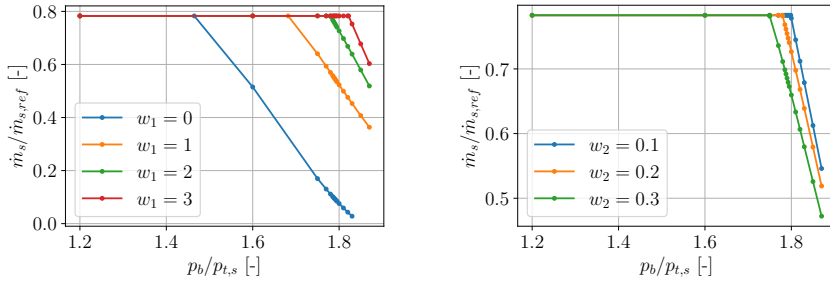
The post-processed RANS simulations provide global reference data in the form of mass flow rates and critical back pressures, and local data in terms of cross-stream averaged quantities, e.g. $p_{t,i}(x)$. Experiments provide similar global data and local wall pressure measurements. The internal flow field changes considerably across the operating curve: from a supersonic



(a) $w(x)$



(b) Total pressure



(c) Operating curve

Fig. 7.8 Effect of the calibration parameters w_1 (left) and w_2 (right) on the weight function $w(x)$ for the inter-stream friction (cf. (7.23)), on the total pressure distribution of the on-design point $p_{t,p}/p_{t,s} = 5$, $p_b/p_{t,s} = 1.2$ and on the operating curve of the ejector. The parameter w_1 regulates the magnitude of the inter-stream friction downstream ($w_2 = 0.2$). This is reflected in the steeper gradients of the total pressure for higher values of w_1 and in the higher predicted critical back pressure. The parameter w_2 displaces the inflection point in the total pressure distributions ($w_1 = 2$). The entrainment in on-design operation remains unaffected.

flow with a shock in the diffuser to a compound-subsonic flow throughout. The corresponding inter-stream friction changes accordingly, so the points are calibrated separately.

The normal shock with instantaneous mixing adopted in the model is a strong simplification which was introduced to match the back pressure, but without the intention of representing the complex shock train (which moreover often entails separation of the boundary layer). The prediction downstream of the shock should therefore not be used for calibration. The portion upstream is common to the fully supersonic solution (cf. section 7.4) and becomes smaller as the shock moves upstream with higher back pressure. The maximal amount of information is therefore available in the leftmost operating point with the lowest back pressure. The remaining on-design points do not provide additional information since their flow fields are identical up to the onset of the shock train. Therefore, the calibration in on-design conditions is limited to the points with the lowest back pressure for each available inlet pressure ratio. The cost function to be minimised is the relative difference in the secondary total pressure with respect to the RANS simulations up to the inlet of the diffuser (since the shock train is downstream):

$$J_{on}(w_1, w_2) = \sqrt{\frac{\sum_i \left(p_{t,s}(x_i) - \tilde{p}_{t,s}(x_i) \right)^2}{\sum_i \left(\tilde{p}_{t,s}(x_i) \right)^2}}, \quad (7.24)$$

where the $p_{t,s}$ is an implicit function of the calibration parameters through the closure relation (7.23) and the tilde indicates the averaged RANS data. The summation is carried out over the grid points of the mesh up to the inlet of the diffuser. The mass flow rates are not included since they are insensitive to w_1 and w_2 . The total pressure is chosen as it is solely affected by the friction forces. The primary total pressure is excluded, as it is dominated by inlet shock-train losses not captured by the model, leading to overly optimistic predictions. The secondary stream, governed mainly by wall and inter-stream friction, provides the most reliable diagnostic.

The total pressure is unavailable from experimental data, so for on-design points, the model is calibrated based on the critical back pressure. The resulting weights are kept identical for the operating points at lower

back pressure. The corresponding cost function is given by:

$$J_{crit}(w_1, w_2) = \frac{|p_{b,crit} - \tilde{p}_{b,crit}|}{\tilde{p}_{b,crit}}, \quad (7.25)$$

where $p_{b,crit}$ stems from the procedure for the critical point (see section 7.4 and figure 7.6). The off-design points are calibrated similarly, but including the secondary mass flow rate:

$$J_{off}(w_1, w_2) = \frac{|\dot{m}_s - \tilde{\dot{m}}_s|}{\tilde{\dot{m}}_s} + \sqrt{\frac{\sum_i (p_{t,s}(x_i) - \tilde{p}_{t,s}(x_i))^2}{\sum_i (\tilde{p}_{t,s}(x_i))^2}}, \quad (7.26)$$

if the reference data stems from RANS simulations, and

$$J_{off}(w_1, w_2) = \frac{|\dot{m}_s - \tilde{\dot{m}}_s|}{\tilde{\dot{m}}_s} + \sqrt{\frac{\sum_i (p_s(x_i) - \tilde{p}_{wall}(x_i))^2}{\sum_i (\tilde{p}_{wall}(x_i))^2}}, \quad (7.27)$$

for experimental data with the static pressure from the wall taps. The two parameters provide enough flexibility to reduce the error on the mass flow rate below 1 % in the off-design cases considered in this work, with relatively small errors on the total pressure.

The cost functions outlined above all consider a single operating point. A first calibration strategy therefore considers each point separately, leading to an erroneous operating curve as shown in figure 7.9a, with a non-physical corner near the critical point. The mismatch stems from an overestimated secondary flow rate in on-design operation, on which the calibration parameters have a minimal effect (see figure 7.8). The cause of this overestimation is discussed in detail further below in section 7.6.1.

A second strategy instead aims to provide a physically consistent operating curve. The points near the critical point in figure 7.9a should thus shift towards the top right, introducing limited errors on the on-design entrainment and on the critical back pressure, as shown in figure 7.9b. This trend is similar to the overestimated entrainment and critical back pressure by RANS simulations with respect to experiments (see figure 2.7).

To obtain the operating curve in figure 7.9b, the points near the critical point were no longer individually calibrated, since minimising the error on the secondary flow rate yields the non-physical corner in figure 7.9a.

Instead, the target for the critical back pressure in equation (7.25) is arbitrarily shifted towards the right with respect to the one from the reference to obtain a satisfactory operating curve. The corresponding weights are used for the predictions of all on-design points, except the left-most one, and the points near the critical point. Most of the separately calibrated off-design points are kept, presently the four points with the highest back pressure in figure 7.9b.

The leftmost on-design point is an exception, since it has been calibrated separately using equation 7.24 to maximise the local accuracy on the total pressure distributions, disregarding the critical point. It is noteworthy to mention that this is a minor issue; the more important limitation of the model is the overestimation of the flow rate in on-design operation as discussed in section 7.6.1.

Each prediction of model 1 presented in sections 7.6 and 7.7 therefore relies on a specific set of calibration weights, which limits the model's extrapolation capability outside the calibration envelope. This limitation is not specific to the present framework: zero-dimensional models are subject to the same constraint, while typically relying on a larger number of closure coefficients (four to five) and incorporating less detailed physical mechanisms. The explicit inclusion of additional physics in the present model is expected to improve robustness with respect to extrapolation; nevertheless, predictions outside the calibrated range should be interpreted with caution.

7.6 Influence of the choking model

In this section, the four models introduced in section 7.2.2 are evaluated and compared in on-design operation with inlet pressure ratios $p_{t,p}/p_{t,s} = 4, 5$ and 6 , on the UCLouvain ejector (see chapter 2). The model 1 is presented first, as it constitutes the only predictive framework. The remaining configurations serve as diagnostic tools to interpret the RANS simulations and to explore potential pathways for improving the predictive model. Models 1 and 2 are based on compound-choking, while models 3 and 4 are based on Fabri-choking. However, their predictions need not be interpreted strictly as such: all cases presented below are compound-sonic ($\beta = 0$) but not necessarily Fabri-sonic ($Ma_s < 1$).

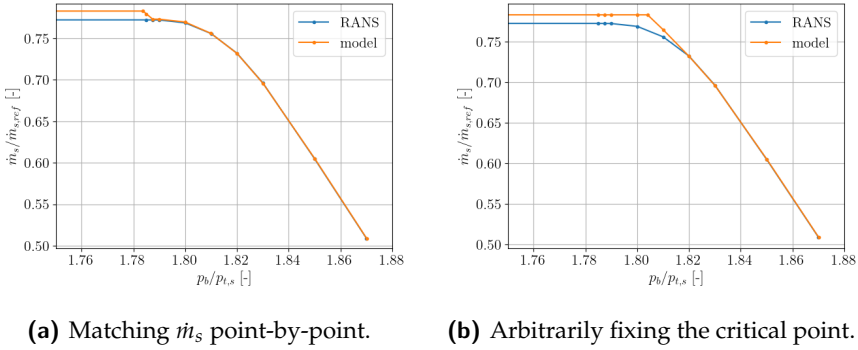


Fig. 7.9 Calibration strategy ($p_{t,p}/p_{t,s} = 5$): (a) one-by-one calibration of all off-design points based on the secondary total pressure and the secondary mass flow rate; (b) the resulting non-physical corner in the curve is avoided by arbitrarily displacing the predicted critical back pressure ($p_b/p_{t,s} = 1.805$ instead of 1.785 from RANS). Introducing this small error yields a physically consistent operating curve, in which the separately calibrated points at higher back pressures are retained (the four right-most points in the shown case).

7.6.1 Model 1: Compound choking with calibrated closures

Predictions on the selected operating points

Table 7.1 compares the mass flow rates from the RANS simulations with the predictions of the compound-based model, for the three considered pressure ratios $p_{t,p}/p_{t,s}$. The weights w_1 and w_2 for the calibration of the inter-stream coefficient (7.23) are also listed for completeness.

The predicted normalised primary mass flow rate equals 0.993 in all three cases, overestimating the reference from the RANS simulations by 0.5 %. This indicates a slight underestimation of the wall friction. The normalised secondary mass flow rate in on-design operation is further below unity, since the used cross-section for the reference mass flow rate $\dot{m}_{s,ref}$ in equation (2.6) is generally larger than the actual cross-section where the secondary stream reaches Mach one.

The compound-based model overestimates the secondary mass flow rate, with the error increasing with the inlet pressure ratio $p_{t,p}/p_{t,s}$ and reaching up to 5.2 %. This overprediction adds to that of the RANS simulations, which already overestimate the entrainment measured in the experiments of [56] by 3–5 %. The origin of the discrepancy between RANS

Table 7.1 Overview of the mass flow rates predicted by the compound-based model, the used calibration parameters w_1 and w_2 , and the errors with respect to the RANS simulations. The error increases with the inlet pressure ratio $p_{t,p}/p_{t,s}$ due to an underexpanded primary stream (cf. figure 7.14). Finally, it should be noted that the model error aggravates the overestimation of the RANS simulations with respect to the experiments of [55].

$p_{t,p}/p_{t,s}$ [-]	$p_b/p_{t,s}$ [-]	w_1 [-]	w_2 [-]	$\dot{m}_s/\dot{m}_{s,ref}$ (model) [-]	$\dot{m}_s/\dot{m}_{s,ref}$ (RANS) [-]	Error [%]
4	1.10	1.287	0.188	0.830	0.827	0.4
5	1.20	1.098	0.180	0.783	0.772	1.4
6	1.55	0.616	0.108	0.739	0.703	5.2

and experiments remains unclear and is not further investigated here. The cause of the model–simulation mismatch is discussed further below based on the local axial distributions.

Figure 7.10 shows the distributions of the dividing streamline and the cross-stream averaged static pressures near the inlet of the mixing pipe. The predicted streamline consists of two regions: between the vertical dotted lines, the pressure equalisation mechanism from section 7.3.3 is active; beyond this region, equation (6.10) is integrated to obtain the cross-sections and hence the dividing streamline.

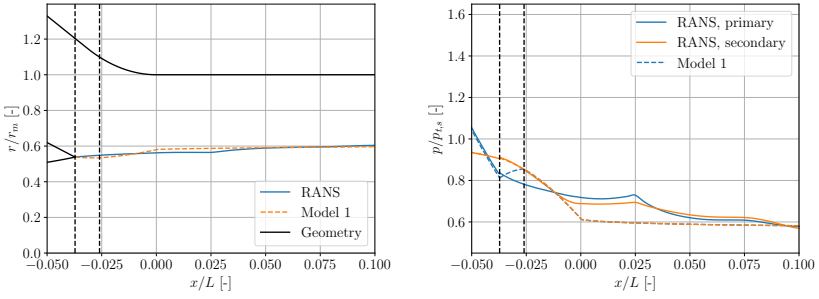
The angle of the dividing streamline at the outlet of the primary nozzle reflects whether the primary stream is under- or over-expanded. It is deflected inward in the over-expanded case (figure 7.10a), aligns with the nozzle lip in the nearly perfectly expanded case (figure 7.10b), and expands outward in the under-expanded case (figure 7.10c). The predicted streamlines match the reference well for under-expanded primary streams. However, the pressure equalisation mechanism becomes less accurate in the over-expanded regime. It enforces a contraction of the primary stream (see equation (7.7)), whereas the RANS simulation still shows an outward spread. In the RANS solution, this compression occurs through an oblique shock, which cannot be captured by the 1D model. In the model, the oblique shock is only used to compute the inward deflection angle and to compress the supersonic primary stream through a converging cross-section. While a similar inward deflection is present in the RANS solution, it does not overcome the outward inclination imposed by the nozzle lip (figure 7.10a).

In reality, the pressures equalise over a much longer distance, as also visible in figure 7.10. The proposed mechanism only mimics the first portion of the shock train. This limitation of the compound-based model is necessary to address pressure equalisation and choking separately. A generalised compound choking theory with non-uniform static pressure would render this mechanism redundant, but this has not been presented in literature. The error on the dividing streamline has a direct consequence on the static pressure distributions on the right of figure 7.10, which show a superior agreement in the region of pressure equalisation (between the vertical lines) for under-expanded primary streams. Note that the distance between the nozzle exit and the predicted point of equal pressure is minimal in the near-perfectly expanded case in figure 7.10b.

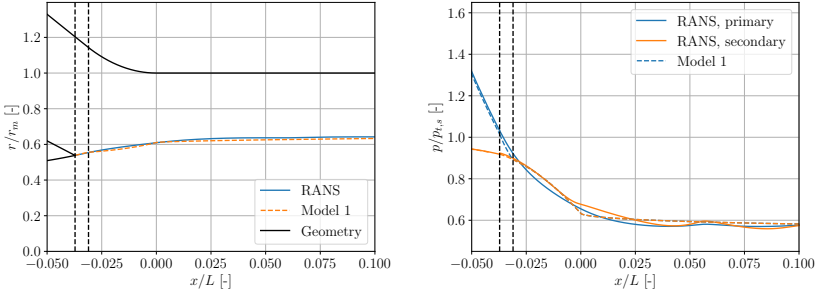
A sharp corner appears in the predicted dividing streamlines at the point of equal static pressure, deviating from the reference in the under-expanded cases. In reality, inertia carries the primary stream further outward, causing it to overexpand and triggering a shock wave that sustains the shock train. A different gradient of the dividing streamline is required to enforce identical pressures. This local constraint makes the compound theory oblivious to inertial effects from upstream, and thus permits sharp corners. This limitation is an inherent consequence of capturing choking within the compound framework, as no generalised compound theory for non-uniform static pressure is currently available.

The predicted uniform static pressure is a reasonable ‘average’ of the primary and secondary pressures in the almost perfectly expanded case $p_{t,p}/p_{t,s} = 5$, as the identical static pressure is reached far upstream due to the small initial difference. The flow in the RANS simulations expands further than in the compound prediction in the stronger under-expanded case $p_{t,p}/p_{t,s} = 6$, as evidenced by the stronger divergence of the dividing streamline and the lower static pressures in figure 7.10c. A second sharp corner appears in the compound predictions at the inlet of the constant-area section ($x/L = 0$). This arises because the gradient dA/dx of the total cross-section disappears from the numerator in equation (6.12) and only the force terms remain. Upstream, the convergent geometry of the mixing pipe (considered to start at the outlet of the primary nozzle at $x/L = -0.0372$) thus dominates.

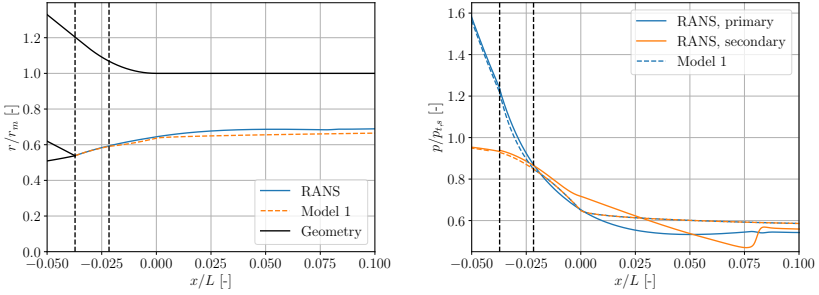
Figures 7.11-7.13 show the distributions of the Mach number, the static pressure and the total pressure for the three investigated inlet pressure ratios (see table 7.1). The total temperature remains uniform at $T_{t,p} = T_{t,s} = 300$ K with deviations below 1 % relative to the RANS simulations. All



(a) $p_{t,p}/p_{t,s} = 4$, over-expanded primary stream.



(b) $p_{t,p}/p_{t,s} = 5$, slightly under-expanded primary stream.



(c) $p_{t,p}/p_{t,s} = 6$, under-expanded primary stream.

Fig. 7.10 The dividing streamline (left) and the cross-stream averaged static pressure (right) from the RANS simulations, and from the compound-based model for three inlet pressure ratios. The pressure equalisation acts between the vertical dotted lines and the compound theory predicts the uniform static pressure downstream. The model is most accurate in the near-perfectly expanded case $p_{t,p}/p_{t,s} = 5$. However, it constrains the dividing streamline regardless of inertial effects in the primary stream, leading to sharp and non-physical corners.

cases are characterised by a choked flow, as evidenced by a unitary equivalent Mach number.

The model shows generally good agreement with the RANS simulations. The main differences arise from substituting the two-dimensional shock trains in the diffuser with normal shocks. These serve only to match the outlet pressure; the instantly mixed state downstream does not reflect the actual flow in the diffuser.

The compound-sonic point is predicted slightly upstream of the constant-area inlet ($x/L = 0$). However, the sonic points in the RANS simulations are consistently located downstream in the constant-area section. The origin of this discrepancy is unclear. It may arise from inaccuracies in the closure modelling, altering the balance of the force terms in equation (6.12), or from the strong assumption of uniform static pressure in the compound flow theory. Indeed, the static pressures at the compound-sonic points generally differ, as seen near $x/L = 0$ in figure 7.10.

The primary and the secondary total pressures converge from their boundary conditions toward a unique value at the outlet of the diffuser. This indicates mixing, since a uniform flow cannot be distinguished as originating from a single reservoir or from two reservoirs as in ejectors. This is also apparent from the modelled normal shock which involves instant mixing. The inter-stream friction is strongest at the inlet of the mixing pipe, where the velocity difference is largest. Consequently, the primary total pressure decreases while the secondary total pressure increases, consistent with the signs of the forces in equation (6.19). As the secondary stream accelerates, the wall friction gains importance, ultimately leading to a reduction in the secondary total pressure.

A small mismatch in primary total pressure appears between the RANS and the model at the outlet of the primary nozzle ($x/L = -0.0372$) due to underestimated friction. The overestimated primary total pressure is likely due to losses in the shock train, which are not included in the momentum balance and mainly affect the primary stream. The secondary total pressure shows excellent agreement since its error has been minimised by calibrating the inter-stream friction (see equation (7.24)). In the present model, wall friction constitutes the only net sink of momentum. In reality, additional mechanical losses arise from turbulent dissipation and from the presence of shock trains, which are not accounted for in the current formulation.

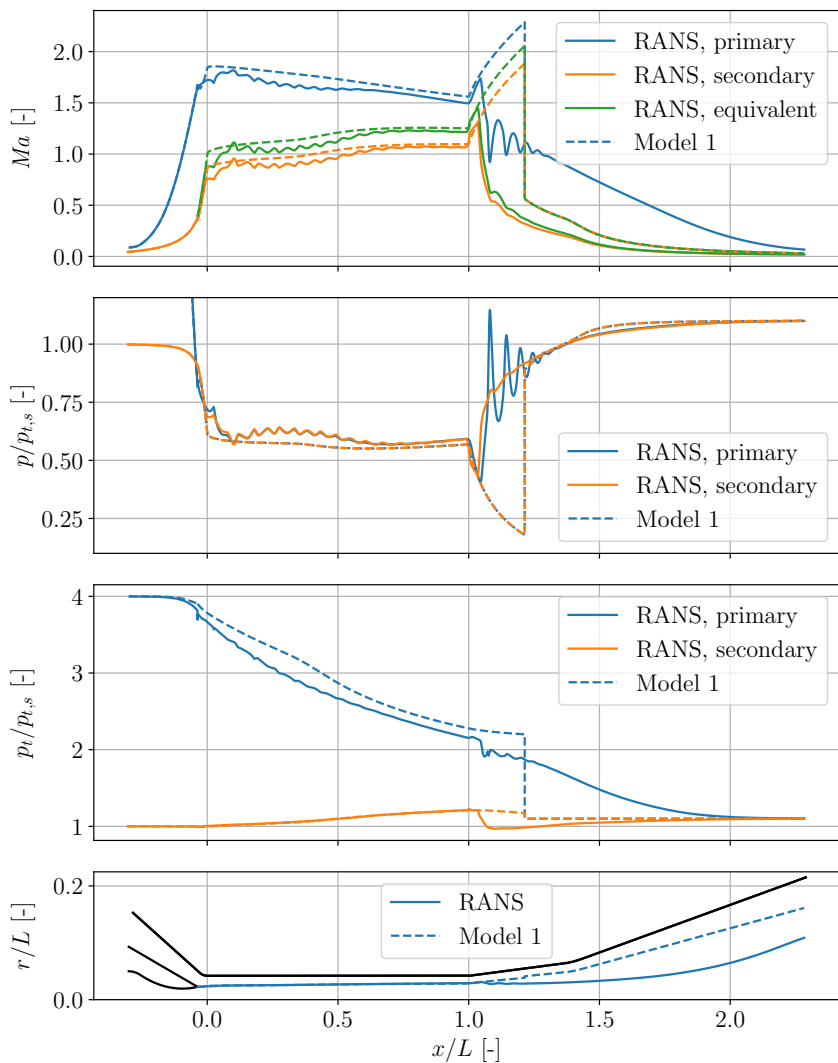


Fig. 7.11 Axial distributions of the Mach number, the static and the total pressure, and the dividing streamline from the RANS simulations and from the compound-based model for the case $p_{t,p}/p_{t,s} = 4$. The shock resulting from the over-expanded primary stream (see figure 7.10a) induces a sharp decrease of the Mach number at the outlet of the primary nozzle. The other distributions are similar to those in figure 7.12.

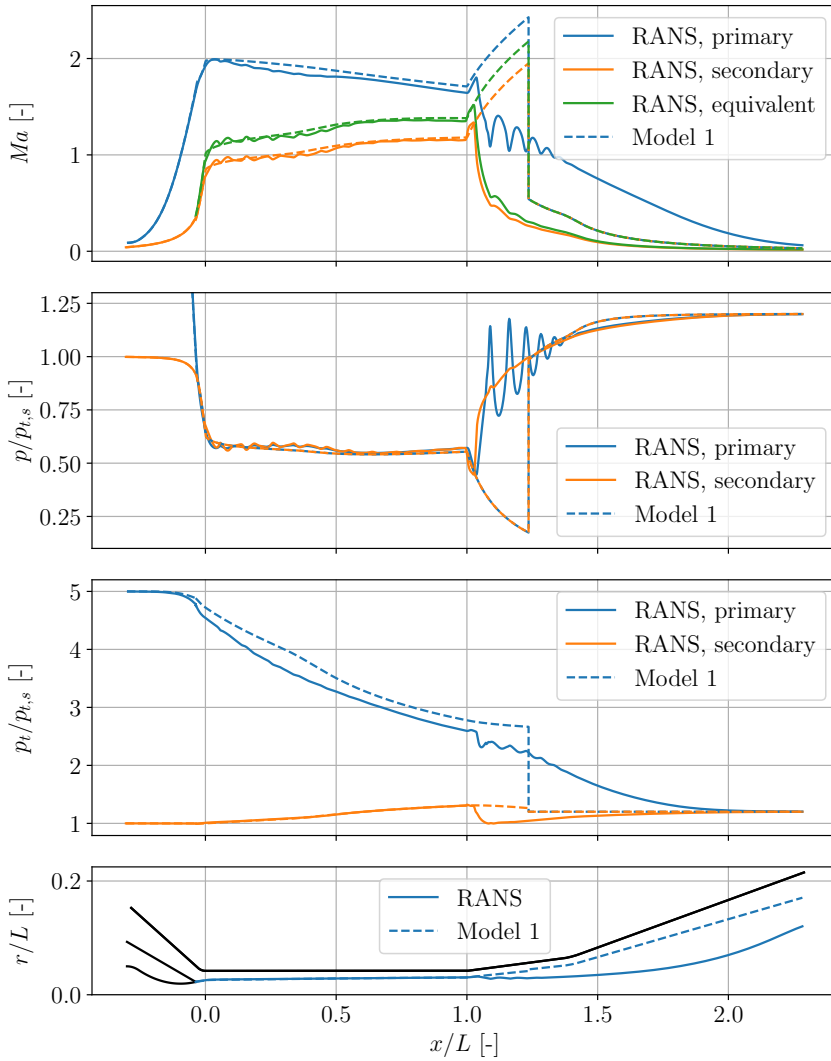


Fig. 7.12 Axial distributions of the Mach number, the static and the total pressure, and the dividing streamline from the RANS simulations and from the compound-based model for the case $p_{t,p}/p_{t,s} = 5$. Model 1 is in excellent agreement with the reference data, which is partly due to the near-perfectly expanded primary stream. The primary total pressure is overestimated due to the losses in the shock train that are not included in the model. The shock train in the diffuser is too complex to be captured by a 1D model (see also appendix C).

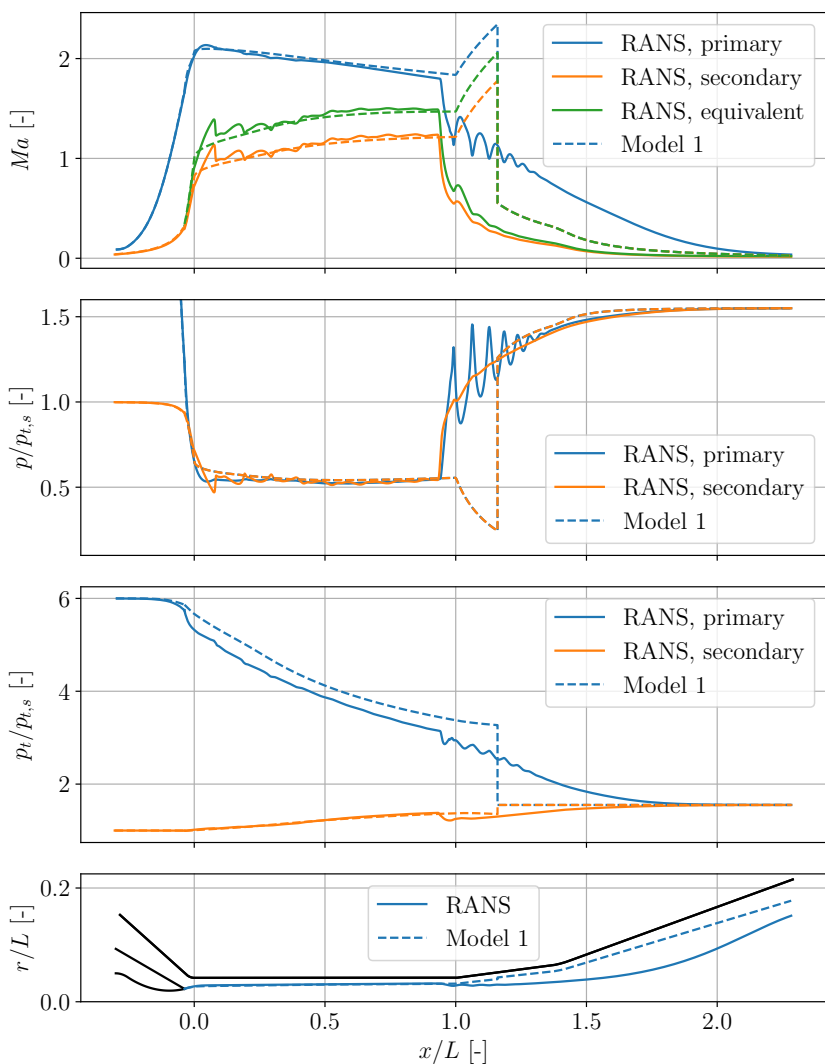


Fig. 7.13 Axial distributions of the Mach number, the static and the total pressure, and the dividing streamline from the RANS simulations and from the compound-based model for the case $p_{t,p}/p_{t,s} = 6$. Model 1 underestimates the expansion in the constant-area section, as the secondary stream reaches its Fabri-sonic point farther upstream in the RANS simulations ($x/L \approx 0.1$) than in the model ($x/L \approx 0.2$). The other distributions are similar to those in figure 7.12.

Overestimated entrainment using the compound theory

The compound model overestimates the entrainment in the under-expanded case $p_{t,p}/p_{t,s} = 6$ (see table 7.1). To reach the same secondary mass flow rate as in the RANS simulation, the model therefore predicts off-design operation. This prediction is obtained here by imposing the averaged RANS flow state as a boundary condition at the inlet of the mixing pipe and integrating the governing equations downstream.

Figure 7.14 shows the resulting distributions of the Mach number and the dividing streamline. The flow remains compound-subsonic and stops expanding at the inlet of the constant area section ($x/L = 0$). The predicted dividing streamline departs from the reference curve with a relatively sharp corner as in figure 7.10c due to the constraint of uniform static pressure. Consequently, both streams miss the stronger expansion in the constant area section, and the iterative procedure from section 7.4 incorrectly detects off-design operation. The secondary flow rate is then increased until the sonic condition is reached, as shown in figure 7.13.

The overestimated flow rate in on-design operation is thus inherent to the assumption of uniform static pressure in the mixing pipe, challenging the foundations of the compound flow theory. This limitation is particularly noteworthy, since uniform static pressure in the mixing pipe is a common assumption in 0D ([46,47], among others) and 1D ejector models (see [15,35,37]).

7.6.2 Model 2: Compound choking with closure from CFD

The closure model incorporating friction forces performs reasonably well but tends to overestimate the primary total pressure due to unmodelled pressure losses in the shock train. To assess potential improvements in accuracy, the total pressure gradient extracted from the RANS simulations is imposed within the mixing pipe (see section 7.2.2).

Globally, the error on the flow rate decreased only slightly (from 5.2 to 4.8 % in the case $p_{t,p}/p_{t,s} = 6$), indicating that further refinement of the closure model yields negligible improvement in entrainment prediction. This limited effect arises from the short distance between the outlet of the primary nozzle and the sonic section, across which friction forces can influence the mass flow rate. Variations downstream affect the mixing, the associated pressure recovery and ultimately the critical back pressure, but not the flow rate itself.

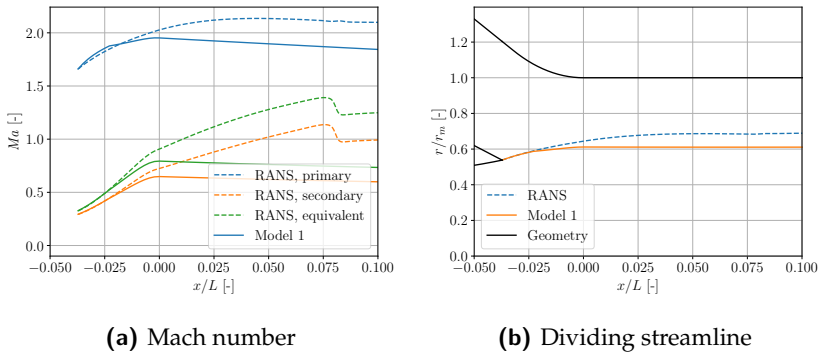


Fig. 7.14 The model predicts a compound-subsonic solution when using the averaged states from the RANS simulations as boundary conditions at the inlet of the mixing pipe ($p_{t,p}/p_{t,s} = 6$). The expansion of the primary stream in the constant area section is not captured by the compound theory since the gradients of the primary and the total cross-sections are directly related through Equations (6.10) and (6.12). The missed expansion is a consequence of the assumption of uniform static pressure and causes the overestimated entrainment.

Figure 7.15 compares the distributions of the Mach number and the total pressure with their RANS references and to the original predictions using the friction forces. The improvement in total pressure is evident, though unsurprising, since the model is no longer predictive but rather used to interpret and diagnose the RANS results. The constant offset in primary total pressure originates from the primary nozzle, where the wall friction remained modelled through the correlation of [115]. An error persists in the secondary Mach number due to the overestimated flow rate, as the missed expansion mechanism remains active (see figure 7.14). The static pressure continues to be equalised and constrained to uniformity, as in the original compound model.

This analysis shows that the prediction error on the entrainment is fundamentally governed by the dividing streamline, and ultimately linked to the constraint of uniform pressure. It can therefore not be substantially reduced through improvements of closures modelling. In the following section, the dividing streamlines from the RANS simulations are imposed in the frame of the Fabri approach.

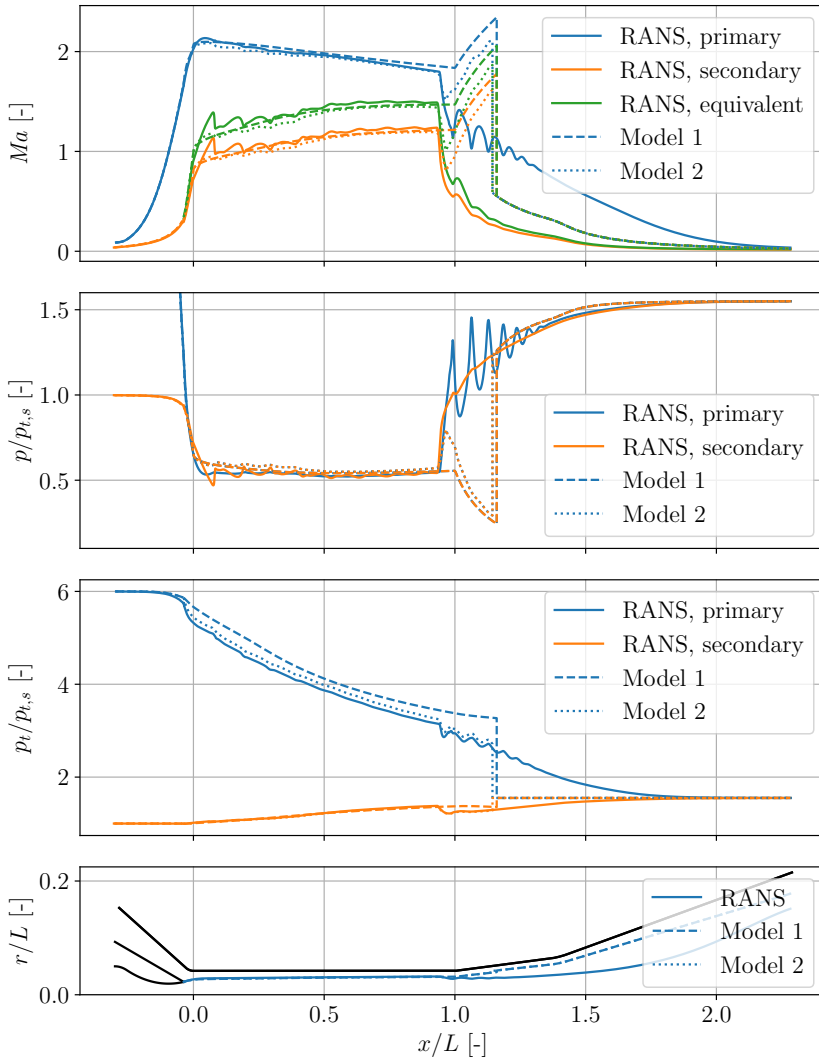


Fig. 7.15 Axial distributions of the Mach number, the static and the total pressure, and the dividing streamline from the RANS simulations, and from the compound-based model using the correlations (model 1) or the imposed total pressure gradient (model 2) for the case $p_{t,p}/p_{t,s} = 6$. Imposing the total pressure gradient of the RANS simulation in the mixing pipe yields higher accuracy on the local distributions, but marginally reduces the error on the secondary flow rate. This error is therefore linked to the dividing streamline (which is practically identical for both models) and not to the closure modelling of the exchanges.

7.6.3 Model 3: Fabri choking with closures from CFD

The dividing streamline and total pressure gradients obtained from the RANS simulations are now imposed within the model. This gives rise to the Fabri-sonic condition, as the constraint of uniform static pressure is relaxed, leading to a singularity in the one-dimensional equations if $Ma_s = 1$. The Van Driest correlation continues to be used for the wall friction in the inlets (section 7.2.2). We recall that the model is highly sensitive to the gradients of the dividing streamline and the total pressures near the sonic point, where the sum of the corresponding terms in equation (6.6) should tend to zero. This sensitivity prevents the use of predictive correlations in conjunction with the dividing streamline extracted from the RANS simulations.

Predictions on the selected operating points

Table 7.2 summarises the predicted mass flow rates in the three previously studied operating points (see also table 7.1). The primary mass flow rates are identical to those predicted by the compound model, since the primary stream is assumed to be choked and the same wall friction is applied. The error on the secondary flow rate remains below 1 % in all cases. This low discrepancy is also obvious from the local distributions, shown in figures 7.16 and 7.17 for the inlet pressure ratios $p_{t,p}/p_{t,s} = 4$ and 6 respectively.

The predicted distributions are nearly coincident with those from the RANS simulations, except near the sonic point in figure 7.17, which lies in the second shock cell of the RANS simulation. The secondary Mach number reaches 0.96 near the reference sonic point at $x/L = 0.05$. This slight discrepancy is likely sufficient to explain the underestimation of 0.9 % on the secondary flow rate, given the strong sensitivity of equation (6.6) close to unity Mach number. The total pressure distributions match those from section 7.6.2 (compound model with extracted closures), where the same total pressure gradients were imposed, they are therefore not shown. The corresponding 4.8 % error in the secondary mass flow rate in the under-expanded case ($p_{t,p}/p_{t,s} = 6$) can therefore be attributed to the dividing streamline predicted by the compound model, since all imposed exchange terms are otherwise identical in both approaches.

Note that the predicted static pressures do not fully equalise. For instance, a difference of about 0.05 between the normalised static pressures remains in figure 7.16 near the diffuser inlet ($x/L = 1$), whereas the refer-

Table 7.2 Overview of the mass flow rates predicted by the Fabri-based model and the errors with respect to the RANS simulations. The error on both mass flow rates remains below 1 % in all conditions.

$p_{t,p}/p_{t,s}$ [-]	$p_b/p_{t,s}$ [-]	$\dot{m}_s/\dot{m}_{s,ref}$ (model) [-]	$\dot{m}_s/\dot{m}_{s,ref}$ (RANS) [-]	Error [%]
4	1.10	0.821	0.827	-0.7
5	1.20	0.767	0.772	-0.7
6	1.55	0.696	0.703	-0.9

ence distributions intersect at several points and eventually equalise. Unlike in the compound model, no pressure equalisation mechanism is active here. The discrepancy remains small because the dividing streamline and total pressures are imposed from the RANS simulations. However, in a fully predictive setting, small errors in the dividing streamline or in the momentum exchange could lead to significant, non-physical pressure gaps. This underlines the need for strong coupling between the closure models for momentum exchange and dividing streamline location, ensuring that the primary and secondary static pressures converge toward a common value.

The accurate performance of the Fabri-based model demonstrates that the cross-stream averaged framework (cf. figure 7.1) can yield reliable predictions when the correct dividing streamline and corresponding closure relations are provided. Currently, the compound-based model remains the only formulation capable of predicting the dividing streamline. As discussed in sections 7.6.1 and 7.6.2, its assumption of uniform static pressure proves overly restrictive, leading to inaccuracies in the dividing streamline and, consequently, in the predicted mass flow rates. The compound model provided the most accurate axial distributions for the case nearest to a perfectly expanded primary stream (see figure 7.12).

A choked case with a subsonic secondary stream

[19,55,64] have reported compound-choked ejector flows with a subsonic secondary stream. The Fabri-sonic condition is therefore not universal, despite the accurate results obtained in the previous section. An intuitive reason for the broader validity of the compound theory lies in the fact that Fabri-sonic parallel streams are compound-supersonic by definition ($Ma_i \geq 1$ in equation (6.13)), and therefore compound-sonic farther upstream. If the shock train in the diffuser migrates upstream between

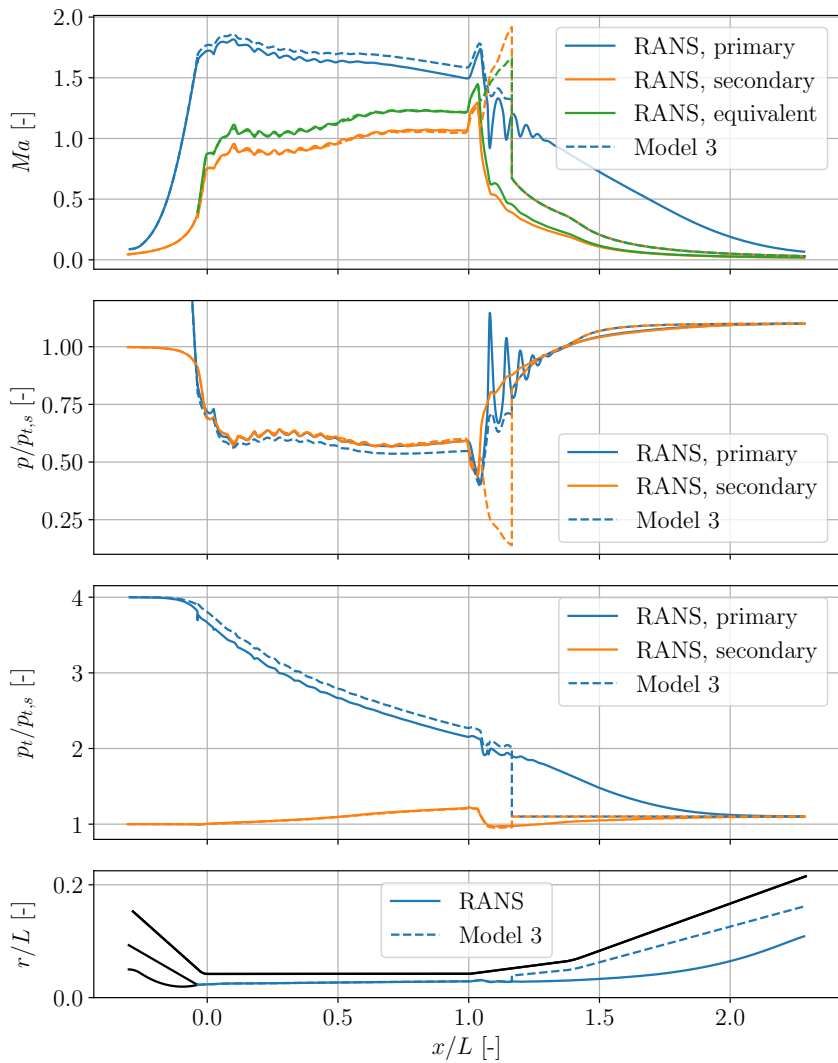


Fig. 7.16 Axial distributions of the Mach number, the static and the total pressure, and the dividing streamline from the RANS simulations, and from the Fabri-based model using the imposed total pressure gradient ($p_{t,p}/p_{t,s} = 4$). The distributions are in excellent agreement, but a small pressure difference persists at the inlet of the diffuser ($x/L = 1$).

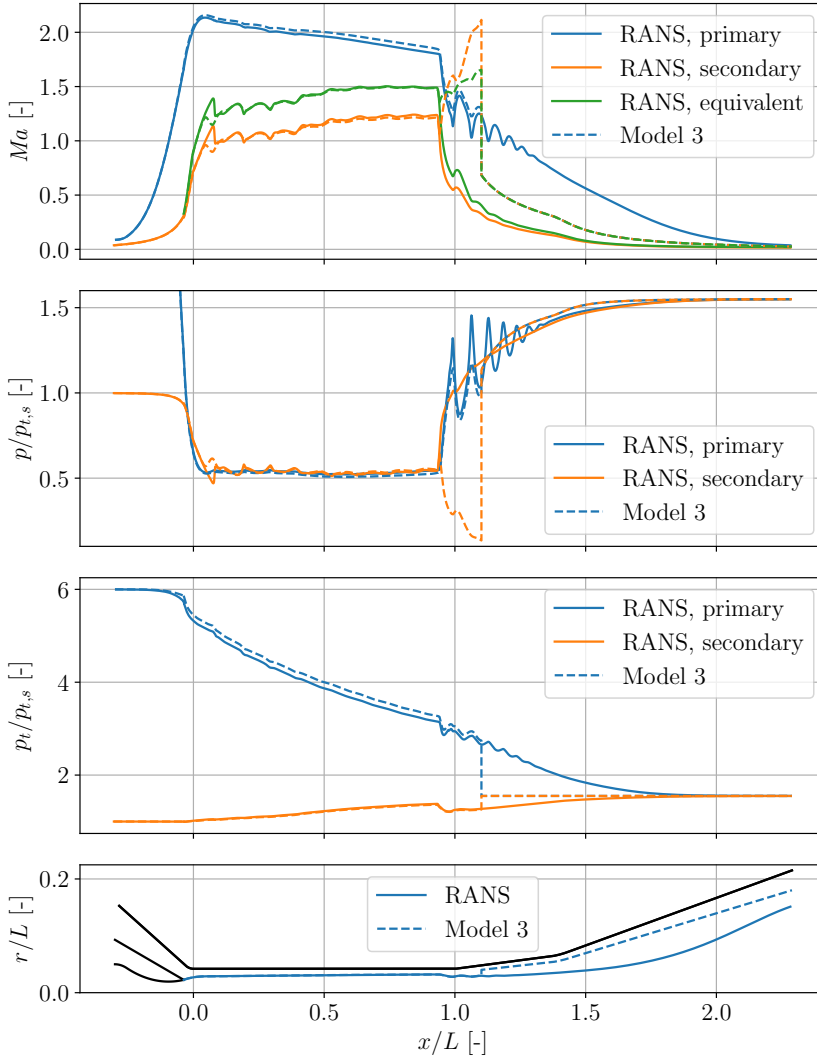


Fig. 7.17 Axial distributions of the Mach number, the static and the total pressure, and the dividing streamline from the RANS simulations, and from the Fabri-based model using the imposed total pressure gradient ($p_{t,p}/p_{t,s} = 6$). The Fabri-sonic condition is reached in the second shock cell, but overall, the Fabri-based model predictions show excellent agreement with the reference distributions.

these two sonic locations due to increasing back pressure, the flow remains choked in the compound sense, even though the secondary stream does not reach a unitary Mach number.

Such a case is presented in figure 7.18 for the case $p_{t,p}/p_{t,s} = 4$, with an increased back pressure from $p_b/p_{t,s} = 1.10$ in figure 7.16 to $p_b/p_{t,s} = 1.48$. The shock train in the simulation with the lower back pressure has moved from $x/L = 1.1$ (in the diffuser) upstream to $x/L = 0.1$ in figure 7.18, which lies beyond the Fabri-sonic from figure 7.16 point near $x/L = 0.5$. Consequently, the secondary stream no longer reaches a unitary Mach number in figure 7.18. This also becomes apparent from the sonic line, which does not penetrate the bulk of the secondary stream in the two-dimensional flow field. Nevertheless, the mass flow rates in the RANS simulations with both back pressures are identical, confirming that compound choking is the more general condition governing flow blockage (see also [64]).

The choked case with a subsonic secondary stream gives rise to the question how model 3, based on the Fabri-sonic condition, behaves in this case. Two predictions are shown in figure 7.18; one following the iterative procedure from section 7.4 to find the choked solution, and one where the secondary mass flow rate is imposed to match the RANS simulations.

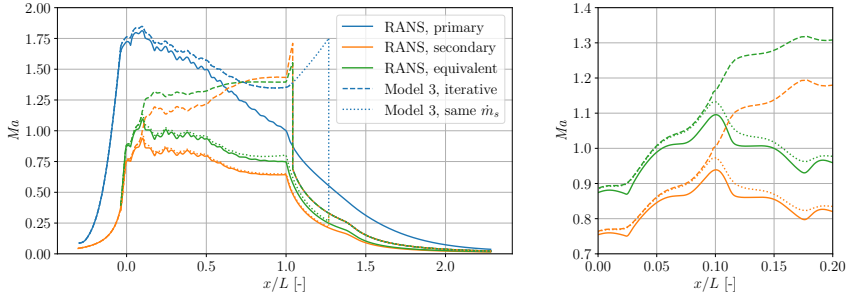
The Fabri-sonic point in the iterative case is reached at $x/L = 0.1$, which coincides with the position where the secondary Mach number is maximal in the RANS simulations due to a local minimum of the cross-section. The sonic point has thus moved upstream with respect to the case with the lower back pressure and the same inlet conditions in figure 7.16. This shift is accompanied by an increase of the predicted secondary flow rate by 1 %. The Fabri-based model 3 thus predicts a higher flow rate near the critical point than at lower back pressures when imposing the corresponding dividing streamlines and exchanges from RANS simulations. This non-physical behavior has been described earlier in 0D modelling by [46], who observed a bump in the operating curves near the critical point. The cause of the different choked solutions in this work lies in the different dividing streamlines (and consequently, the flow variables) in the RANS simulations for $p_b/p_{t,s} = 1.10$ and $p_b/p_{t,s} = 1.48$, which start differing from the compound-sonic point at $x/L = 0.1$. The prediction with matching flow rate remains Fabri-subsonic, and is hence classified as off-design by the iterative procedure in section 7.4. These observations further indicate the close link between the dividing streamline and the accuracy on the secondary flow rate.

The results presented above indicate that developing a predictive model based on the Fabri-sonic condition without assuming uniform static pressure is inherently challenging. This difficulty arises from the strong coupling required between the dividing streamline and the closure relations, as well as from the sensitivity of equation (6.6) near the sonic point. An additional complication occurs when the primary stream decelerates to subsonic speeds. This situation occurs in the RANS simulation in figure 7.18 at $x/L = 1$, but not in the model, which instead strays away from the reference and stays supersonic. This discrepancy gradually grows from $x/L = 0$ and substantially increases near the second sonic point of the primary stream at $x/L = 1$, introducing a similar sensitivity in equation (6.6) as in the Fabri-sonic point of the secondary stream. This situation inevitably manifests itself also under off-design conditions, where the static pressure increases monotonically along the mixing duct (see for example [8]). In such cases, the primary stream passes through its sonic point between the outlet of the primary nozzle (where it is supersonic) and the diffuser exit (where it becomes subsonic). This transition poses a fundamental problem within the proposed framework, since the numerator of the pressure equation (6.6) may not vanish at that point, leading to an infinite pressure gradient.

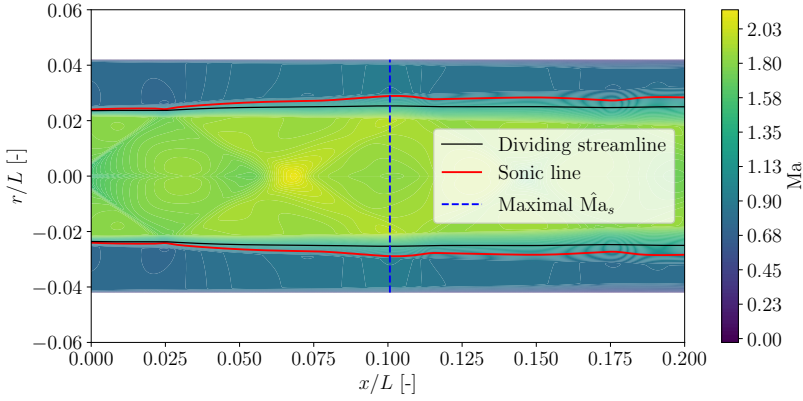
In contrast, the compound framework does not suffer from this limitation: the corresponding equation (6.12) remains well-behaved when $Ma_p = 1$ and $Ma_s < 1$ ($\beta > 0$). The secondary stream does not exhibit this issue either; encountering a singularity instead signifies an incorrect estimate of the secondary mass flow rate in the iterative procedure described in section 7.4. To the authors' knowledge, no existing one-dimensional model in the literature can operate in off-design regimes without assuming uniform static pressure. For instance, the Fabri-based model of [41] is explicitly limited to on-design operation.

7.6.4 Model 4: Fabri choking with closure from compound choking theory

The final model seeks to reconcile both choking conditions by enforcing the dividing streamline and momentum exchange predicted by the compound model within the Fabri-based framework. The governing equations and closures are identical, but the stopping criteria differ: the compound model converges to ($N \rightarrow 0, \beta \rightarrow 0$) in equation (6.12), while the Fabri-based model seeks ($N \rightarrow 0, Ma_s \rightarrow 1$) in equation (6.6). Consequently, the choked solutions are not necessarily identical.



(a) Axial distributions of the averaged Mach number



(b) Mach number field from the RANS simulation (mirrored about $r = 0$).

Fig. 7.18 The Mach number from the RANS simulations at the critical operating point ($p_{t,p}/p_{t,s} = 4$, $p_b/p_{t,s} = 1.48$), from model 3 iterating to the choked solution, and from model 3 imposing the same mass flow rate. The reference flow becomes compound-sonic, but not Fabri-sonic, as as evidenced by the averaged distributions and the sonic line, which penetrates only a small portion of the secondary stream. This indicates the generality of the compound theory, since the flow rate is unchanged compared to figure 7.16.

Two situations were examined: the fully supersonic and the critical solutions predicted by the compound model for $p_{t,p}/p_{t,s} = 6$ (see table 7.1). The critical solution mimics a case with higher back pressure compared to the case shown in figure 7.13. These were obtained using the positive and negative roots of the quadratic equation for the static pressure gradient at the sonic point, analogous to the two isentropic solutions in a choked nozzle flow. The supersonic solution corresponds to the compound prediction in figure 7.13 prior to applying the normal shock procedure described in section 7.3.4.

The Fabri-based model was then initialised using the gradients of the cross-sections from equation (6.10), evaluated by interpolating the distributions predicted by the compound model. The exchange terms were obtained from the closure relations in section 7.2.2. This interpolation ensures that the geometric gradients remain independent of the flow field computed in the Fabri-based framework.

Figure 7.19 presents the resulting Mach number distributions. In the supersonic case, both model predictions coincide, reaching the Fabri-sonic condition near $x/L = 0.2$, consistent with figure 7.13. The compound solution thus also satisfies the vanishing-numerator condition in the Fabri-sonic point. This equivalence becomes evident when substituting the compound relation for the cross-section, equation (6.10), in the static pressure equation (6.6): The $(1 - Ma_s^2)$ term in the numerator of equation (6.10) effectively regularises the singularity that would otherwise occur at the Fabri-sonic point. As a result, both sonic conditions yield an identical solution, if the same dividing streamline and closure relations are imposed.

The agreement confirms that the 5.2 % flow-rate overprediction of the compound model (cf. table 7.1) stems from inaccuracies in the dividing streamline rather than from the choking condition itself. The matching distributions in figure 7.19a confirm that the Fabri-based model predicts the same overestimated flow rate as the compound model. This discrepancy is therefore unrelated to the choking condition, which is the only formal distinction between the two predictions shown in figure 7.19a.

In contrast, for the critical case, the Fabri-sonic point is not reached in the prediction of model 1, consistent with the RANS results in figure 7.18. The iterative process then drives $Ma_s \rightarrow 1$, leading to a 9 % overestimation of the secondary flow rate relative to the compound model. The likely cause of this discrepancy is the artificial nature of the dividing streamline predicted by the compound theory at the critical condition. The abrupt reversal of the static pressure gradient required to restore subsonic condi-

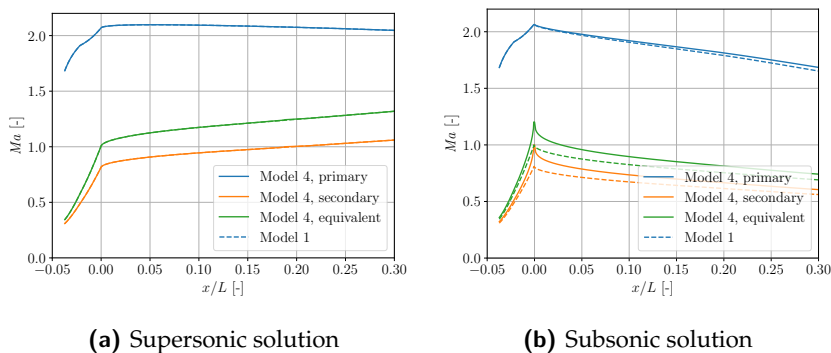


Fig. 7.19 Predicted distributions of the Mach number based on the compound model, and the Fabri model using the same dividing streamline and the same closure relations. Both theories agree if the Fabri-sonic point is reached. The Fabri-based model overestimates the flow rate with respect to the critical operating point predicted by the compound model.

tions in the compound model does not accurately represent the complex un-choking process in reality involving a two-dimensional shock train (cf. figure 7.18). These findings underline the limitations of the one-dimensional framework defined by equations (6.6)–(6.8), especially near the critical operating point.

7.7 Validation on a broader range of operating conditions

This section analyses the predictive capabilities of the compound model in a broader range of operating conditions on the UCLouvain ejector (see chapter 2). More specifically, the model was calibrated and validated also in off-design operating conditions (following the strategy in section 7.5), thereby extending the analysis of the section above. The discrepancy between the experimental and numerical data (see chapter 2) induces different weights in the model when calibrated on either of the data types. Therefore, the model was calibrated on both and the results are compared separately in the sections below. The global performance is evaluated in terms of the secondary flow rate; the local analysis consists of the cross-stream averaged quantities from the RANS simulations or of the experimental pressure taps.

7.7.1 Validation against numerical data

Global results

The predicted operating curves are shown in figure 7.20 for inlet pressure ratios $p_{t,p}/p_{t,s} = 4, 5, 6$. The numerical values are listed in table 7.3, along with the used calibration parameters w_1 and w_2 . Most on-design points are omitted from the table to avoid redundancy. The entries in bold correspond to the critical point of the RANS simulations. Below, the secondary mass flow rate starts to decrease with increasing back pressure p_b (although at first only in the fifth decimal due to the gentle transition).

The points with the lowest back pressure of the three inlet conditions have already been extensively analysed in section 7.6.1, including the cause of the overestimated flow rate in on-design operation. The error disappears in off-design operation due the calibration procedure: the entrainment becomes sensitive to the calibrated friction forces in off-design operation, since the momentum exchange affects the pressure recovery in the mixing pipe and consequently the flow rate through the back pressure. For instance, stronger inter-stream friction induces a higher back pressure for a given flow rate, allowing to displace the predicted critical back pressure, and inversely, the inter-stream friction affects the flow rate at a given back pressure, allowing to minimise the error on the secondary flow rate (see section 7.5). Besides the small error on the entrainment in off-design operation, the predicted critical back pressures also closely match those of the RANS simulations, within 0.02 bar.

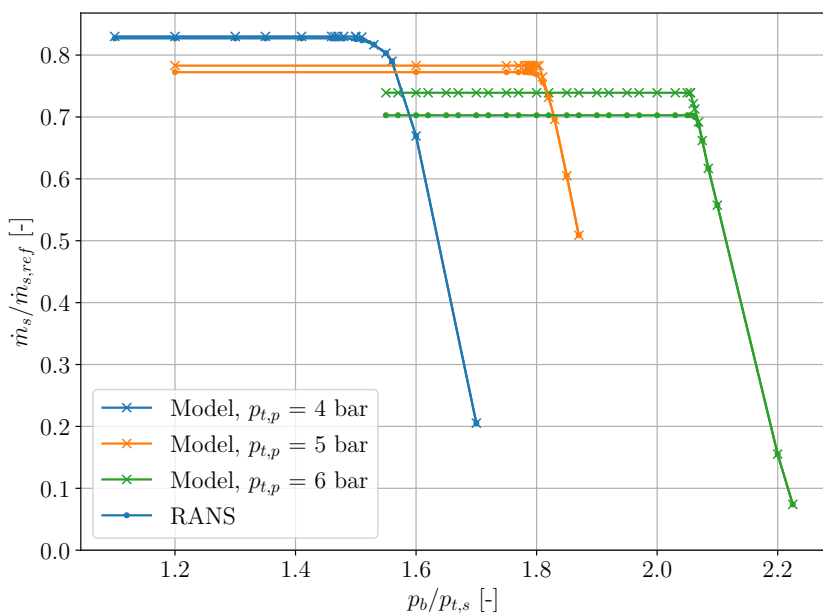


Fig. 7.20 Comparison of the operating curves of the ejector at UCLouvain, obtained from the RANS simulations and as predicted by the model. The model matches the reference in off-design operation through calibration (see table 7.3), but it overestimates the on-design entrainment by 5 % in the case $p_{t,p} = 6$ bar.

Table 7.3 Overview of the boundary conditions, the calibration parameters w_1 and w_2 , the resulting secondary mass flow rates and the errors with respect to the RANS simulations. The secondary reference mass flow rate $\dot{m}_{s,ref}$ equals 0.2536 kg/s. Most on-design points are omitted for conciseness since the predictions remain identical. The critical point from the RANS simulations is indicated in bold.

$p_{t,p}/p_{t,s}$ [-]	$p_b/p_{t,s}$ [-]	w_1 [-]	w_2 [-]	$\dot{m}_s/\dot{m}_{s,ref}$ (model) [-]	$\dot{m}_s/\dot{m}_{s,ref}$ (RANS) [-]	Relative error [%]
4	1.10	1.287	0.188	0.830	0.827	0.4
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
	1.48	1.287	0.188	0.830	0.827	0.4
	1.50	2.070	0.331	0.830	0.826	0.5
	1.51	2.151	0.314	0.829	0.824	0.6
	1.53	2.364	0.302	0.817	0.817	0.0
	1.55	2.559	0.273	0.803	0.803	0.0
	1.56	2.605	0.254	0.790	0.790	0.0
	1.60	2.284	0.158	0.669	0.669	0.0
	1.70	0.693	0.000	0.205	0.205	0.0
5	1.20	1.098	0.180	0.783	0.772	1.4
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
	1.785	2.729	0.250	0.783	0.772	1.4
	1.79	2.729	0.250	0.783	0.772	1.4
	1.795	2.729	0.250	0.783	0.771	1.5
	1.80	2.729	0.250	0.783	0.769	1.8
	1.81	2.729	0.250	0.765	0.756	1.2
	1.82	2.613	0.229	0.732	0.732	0.0
	1.83	2.439	0.195	0.696	0.696	0.0
	1.85	2.037	0.136	0.605	0.605	0.0
	1.87	1.650	0.082	0.509	0.509	0.0
6	1.55	0.616	0.108	0.739	0.703	5.2
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
	2.055	1.943	0.115	0.739	0.703	5.2
	2.06	1.943	0.115	0.721	0.702	2.8
	2.0625	1.943	0.115	0.713	0.700	1.9
	2.0688	1.943	0.115	0.692	0.687	0.7
	2.075	1.995	0.153	0.662	0.662	0.0
	2.085	1.815	0.121	0.617	0.617	0.0
	2.10	1.653	0.092	0.557	0.558	0.0
	2.20	0.726	0.231	0.155	0.155	0.1
	2.225	0.348	0.139	0.074	0.074	0.0

Local distributions

Figures 7.21, 7.22 and 7.23 show the axial distributions of the Mach number, the total pressure and the dividing streamline for three off-design operating points ($p_{t,p}/p_{t,s} = 4, p_b/p_{t,s} = 1.60$), ($p_{t,p}/p_{t,s} = 5, p_b/p_{t,s} = 1.87$) and ($p_{t,p}/p_{t,s} = 6, p_b/p_{t,s} = 2.10$) respectively. They agree well with the references from the RANS simulations, especially on the secondary total pressure which was included in the cost function (7.26). The integration of the governing equations is more straight-forward in these cases due to the absence of the singularity at the sonic point. However, the shock train at the outlet of the primary nozzle is more pronounced in these off-design operating points compared to the on-design cases, for example visible by comparison of figures 7.13 (on-design) and 7.23a (off-design).

The accurate predictions of model 1 in these off-design conditions encourages to continue using the constraint on the static pressure. Despite the uncaptured oscillations due to the shock train, the calibrated model manages to match the reference flow rate *and* obtain reasonable accuracy on the local distributions. Developing a ‘Fabri-based’ model without constraining the static pressure in off-design conditions is therefore unlikely worth the effort, given the requirement to develop a new model for the dividing streamline. The constraint of uniform static pressure proves an excellent compromise, obtaining a fully predictive model at the cost of missing two-dimensional effects (which is admissible in the current low-order one-dimensional setting). Finally, it is noteworthy to mention that these distributions were obtained in a few seconds on a standard laptop, while presenting significantly more information than conventional 0D models and relying on less calibration parameters (2 vs. 4 or 5). The main area of improvement seems to include a model for the losses in the shock train, which could potentially decrease the overestimation of the primary total pressure. Constructing such a loss term is however not straight-forward without access to the underlying two-dimensional shock structure.

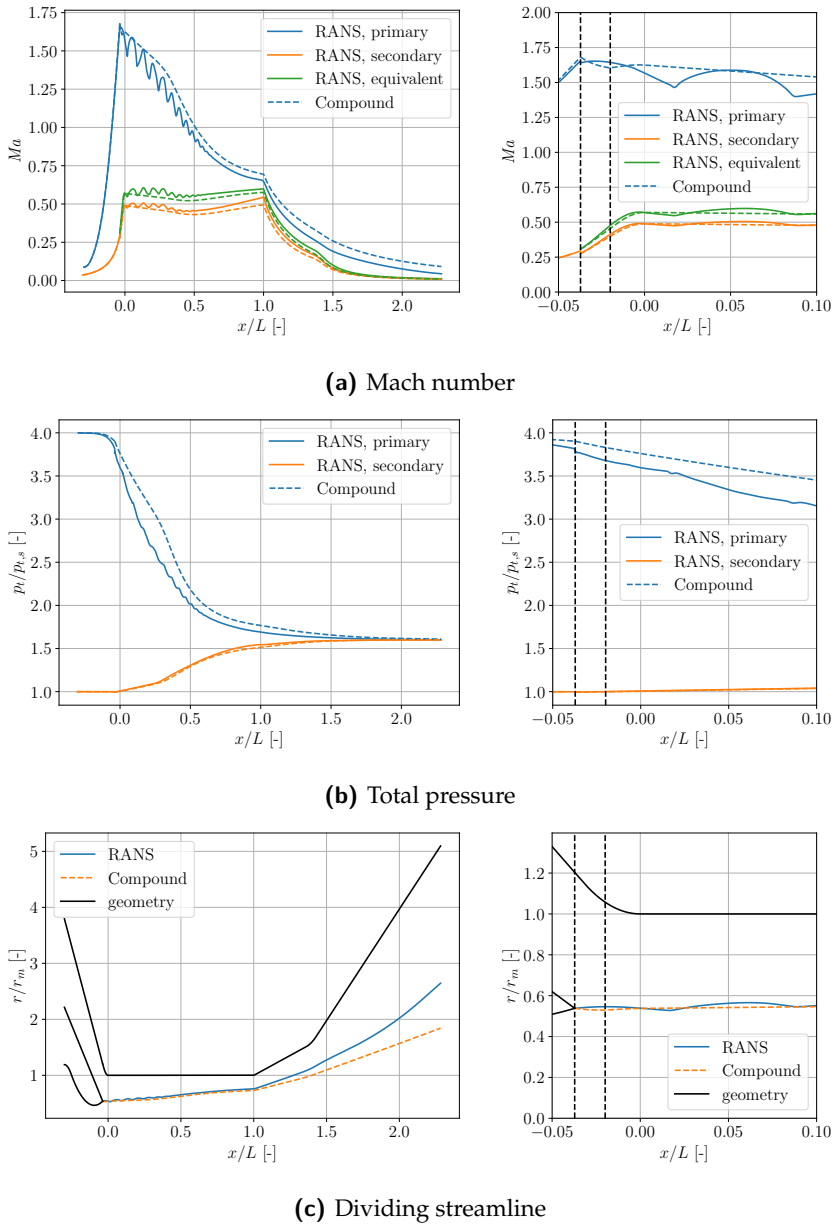
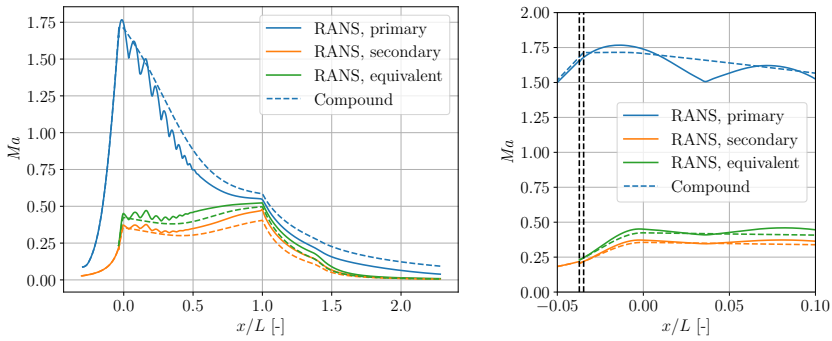
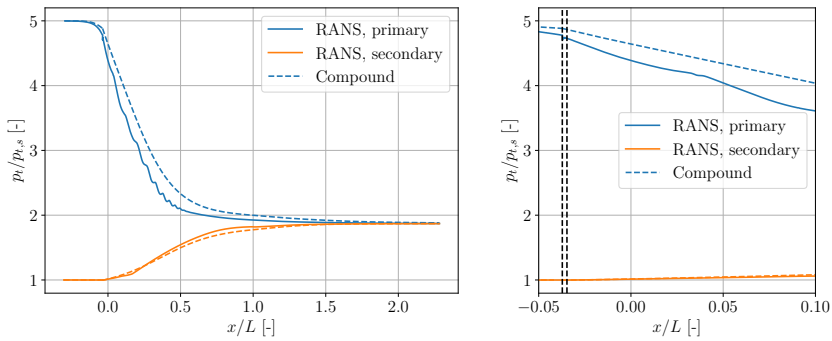


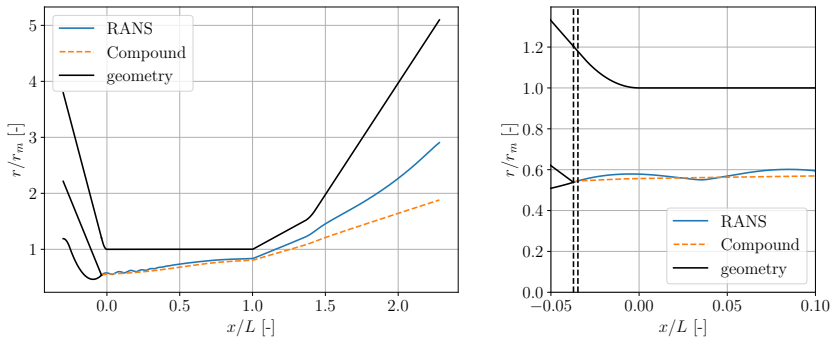
Fig. 7.21 Axial distributions of the Mach numbers, total pressures and the dividing streamline for the case $p_{t,p}/p_{t,s} = 4$ and $p_b/p_{t,s} = 1.60$. The ejector is operating in off-design conditions, remaining compound-subsonic throughout as indicated by the equivalent Mach.



(a) Mach number



(b) Total pressure



(c) Dividing streamline

Fig. 7.22 Axial distributions of the Mach numbers, total pressures and the dividing streamline for the case $p_{t,p}/p_{t,s} = 5$ and $p_b/p_{t,s} = 1.87$. The ejector is operating in off-design conditions, remaining compound-subsonic throughout as indicated by the equivalent Mach.

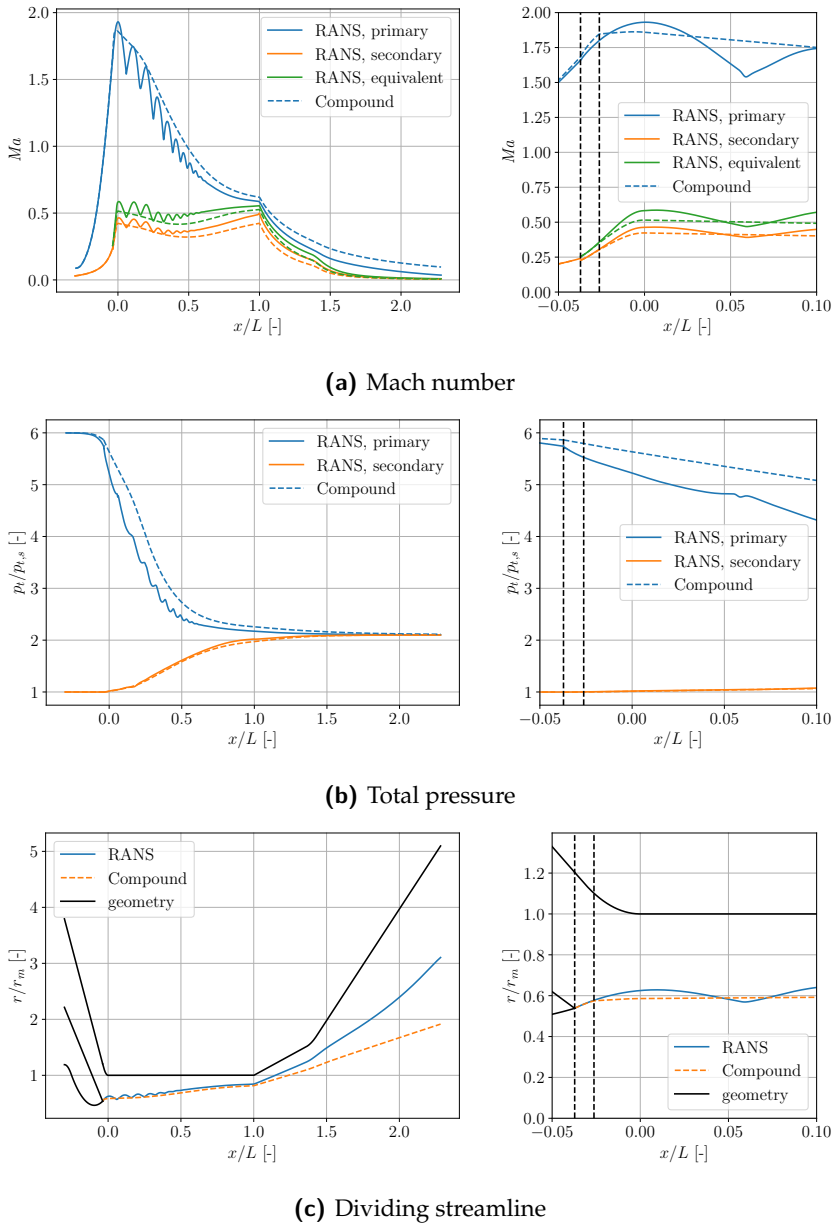


Fig. 7.23 Axial distributions of the Mach numbers, total pressures and the dividing streamline for the case $p_{t,p}/p_{t,s} = 6$ and $p_b/p_{t,s} = 2.10$. The ejector is operating in off-design conditions, remaining compound-subsonic throughout as indicated by the equivalent Mach.

7.7.2 Validation against experimental data

Global results

Figure 7.24 and table 7.4 display the normalised secondary flow rate from the experiments by Brosteaux and doucet [56] and from model 1 calibrated on those experiments. The calibration procedure was identical to the one in the section above, but applied on the experimental data instead. The model proves to be flexible enough in its calibration parameters to obtain similar accuracy with respect to the experimental and numerical data. The calibrated weights w_1 and w_2 are significantly smaller in table 7.4 compared to those in table 7.3, which is consistent with the lower critical back pressure and lower entrainment measured experimentally. The flow rate in on-design operation is insensitive to inter-stream friction as discussed in section 7.5, so the previous error of 1-5 % with respect to the RANS simulations has increased here to 3-10 %.

The calibrated weights are more uniform across the inlet pressure ratios, which is encouraging for finding a more universal closure law for the inter-stream friction. The fact that the original correlation (6.22) is rooted in experimental data likely favours this trend, pushing the calibration weights closer to zero and hence closure to the original correlation. Clearly, many open questions remain regarding the momentum exchange in RANS simulations of ejectors, as also pointed out by Debroeyer et al. [28].

Local distributions

The wall pressure distributions from the experiments are compared to the averaged secondary static pressure from the model in figure 7.25. The results are in excellent agreement in off-design operation, due to the matching flow rate and the calibration procedure that includes the wall pressure in the cost function (7.27) if available. The agreement degrades for lower entrainment ratios, such as the case ($p_{t,p}/p_{t,s} = 4$, $p_b/p_{t,s} = 1.70$), where the static pressure distribution remains flat at the inlet of the mixing pipe up to $x/L \approx 0.1$ and subsequently rises (see also figures 2.8-2.10). This effect is reproduced in RANS simulations [55], but its origin currently stays unclear. Nevertheless, the agreement in the other off-design points is striking. The model underestimates the static pressure in on-design operation, similarly as RANS simulations with respect to the experiments, so the discrepancy is most likely related to the overestimated flow rate in both cases. The mismatch is maximal in the over-expanded case $p_{t,p}/p_{t,s} = 4$, where

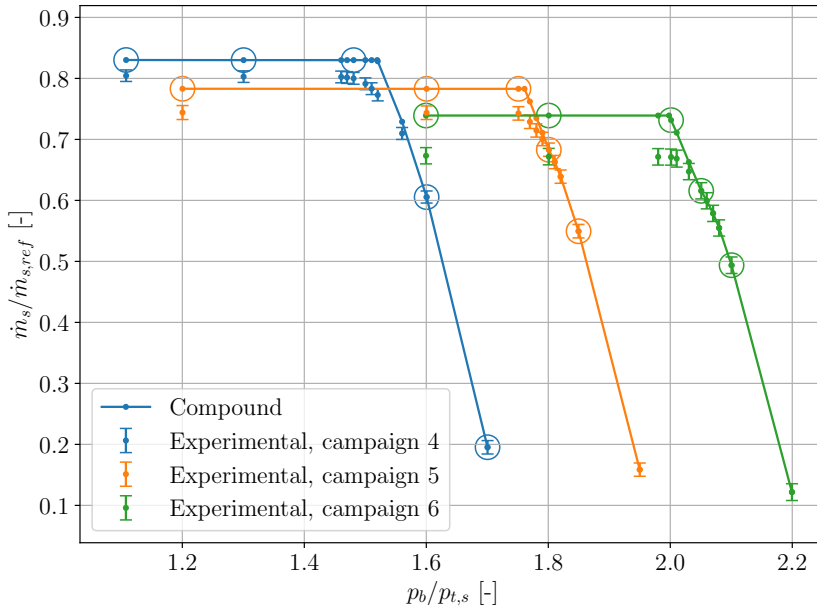
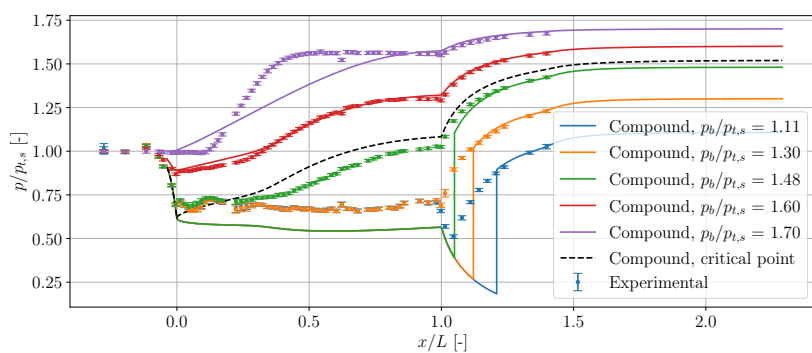


Fig. 7.24 Comparison of the experimental and predicted operating curves of the ejector at UCLouvain (see table 7.4). The prediction error in on-design conditions increased from 5 to 10 % in the case $p_{t,p} = 6$ bar compared to the RANS simulations in figure 7.20 and table 7.3. The wall pressure distributions for the circled points are displayed in figure 7.25.

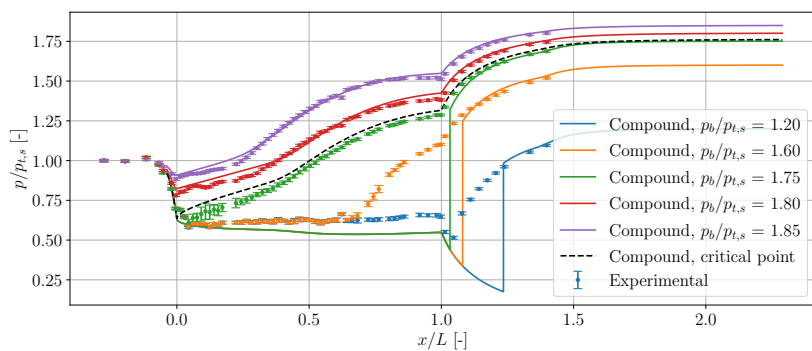
the primary stream is directly compressed at the outlet of the primary nozzle by the shock train. The flow expands further in the under-expanded case $p_{t,p}/p_{t,s} = 6$ and overshoots due to the inertial effect of the primary stream. This might explain the lower pressure ratio $p/p_{t,s}$ in those cases. The model in turn overestimates the flow rate more in the under-expanded cases, therefore also reaching a lower static pressure. The relatively good match in figure 7.25c is therefore more likely to be coincidental given the large discrepancy in flow rate (10 %). Furthermore, the losses in the shock train likely contribute to the observed differences in static pressure, but it is not straight-forward to draw conclusions on partial data where the full velocity field is not available. Particle Image Velocimetry (PIV) could shed more light on these questions and is therefore a promising avenue for future investigations (also regarding compound- and Fabri-choking).

Table 7.4 Overview of the boundary conditions, the calibration parameters w_1 and w_2 , the resulting mass flow rates and the errors with respect to the experiments on the UCLouvain ejector. Most on-design points are omitted for conciseness since the predictions remain identical. The critical point from the experiments is indicated in bold.

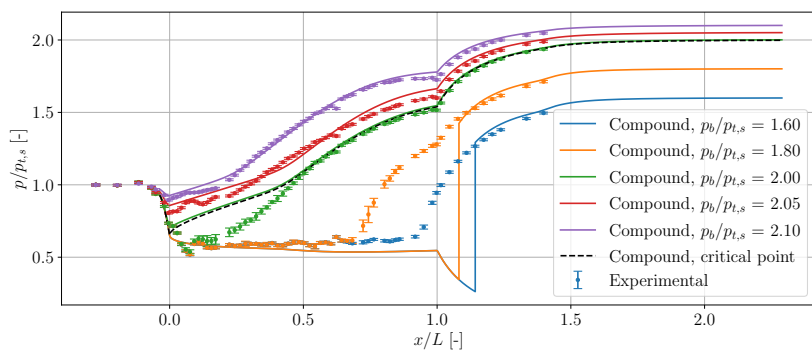
$p_{t,p}/p_{t,s}$ [-]	$p_b/p_{t,s}$ [-]	w_1 [-]	w_2 [-]	$\dot{m}_s/\dot{m}_{s,ref}$ (model) [-]	$\dot{m}_s/\dot{m}_{s,ref}$ (exp) [-]	Relative error [%]
4.00	1.11	0.602	0.168	0.830	0.804	3.2
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	1.47	0.602	0.168	0.830	0.801	3.6
	1.48	0.602	0.168	0.830	0.800	3.7
	1.50	0.602	0.168	0.830	0.791	4.9
	1.51	0.602	0.168	0.830	0.783	6.0
	1.52	0.602	0.168	0.828	0.773	7.2
	1.56	0.602	0.168	0.729	0.710	2.7
	1.60	0.602	0.168	0.605	0.606	0.0
	1.70	0.219	0.003	0.195	0.195	0.1
5.00	1.20	0.606	0.206	0.783	0.744	5.2
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	1.75	0.606	0.206	0.783	0.743	5.4
	1.77	0.606	0.206	0.762	0.729	4.6
	1.78	0.606	0.206	0.735	0.715	2.8
	1.79	0.606	0.206	0.711	0.701	1.4
	1.80	0.606	0.206	0.683	0.683	0.0
	1.81	0.556	0.140	0.663	0.663	0.0
	1.82	0.559	0.134	0.639	0.639	0.0
	1.85	0.545	0.133	0.549	0.550	0.0
	1.95	0.236	0.200	0.159	0.159	0.1
6.00	1.60	0.538	0.215	0.739	0.673	9.8
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	2.00	0.538	0.215	0.732	0.671	9.0
	2.01	0.538	0.215	0.711	0.669	6.4
	2.03	0.538	0.215	0.663	0.647	2.4
	2.05	0.538	0.215	0.616	0.616	0.0
	2.06	0.468	0.126	0.599	0.599	0.0
	2.07	0.471	0.122	0.579	0.579	0.0
	2.08	0.471	0.120	0.555	0.555	0.0
	2.10	0.455	0.123	0.494	0.494	0.0
	2.20	0.173	0.191	0.122	0.122	0.0



(a) $p_{t,p}/p_{t,s} = 4$



(b) $p_{t,p}/p_{t,s} = 5$



(c) $p_{t,p}/p_{t,s} = 6$

Fig. 7.25 Experimental and predicted wall pressure profiles for the UCLouvain ejector. The excellent agreement in off-design operation is in accordance with the low error on the flow rate.

7.8 Conclusion

This chapter presents a one-dimensional analysis of choking in supersonic ejectors, examining the interaction between compound and Fabri formulations within a unified two-stream framework. This approach isolates their assumptions and effects on predicted flow evolution. The results show that compound and Fabri choking can coexist in the same operating map. However, all Fabri-choked cases are also compound-choked, but not vice versa, indicating that compound choking represents the more general flow-limiting mechanism.

The compound model predicts the secondary mass flow rate within approximately 5 % of CFD results (10 % relative to experiments). The overestimation of entrainment under on-design conditions arises from the uniform static-pressure assumption, which neglects the primary stream's inertial overshoot. The iterative scheme compensates by converging to a higher secondary flow rate. Imposing the momentum exchange from the RANS simulations did not improve this error, showing that it can not be reduced further through calibration of the forces. Although the accuracy of the local flow distributions can be slightly improved, the correlation-based predictions remain remarkably accurate—particularly considering that the model executes within seconds on a standard laptop, compared to several hours required for equivalent RANS simulations. Off-design operation benefits from the absence of a compound-sonic point, simplifying the model and yielding accurate global and local predictions. Even with pronounced shock trains, the uniform-pressure assumption ultimately yields an accurate dividing streamline and the model predicts a decent 'average' of the oscillating static pressure.

In contrast, the Fabri formulation allows static pressures to evolve freely but requires the dividing streamline to be prescribed by an external model or data source. Using the filtered dividing streamline extracted from RANS simulations reduced the error in secondary mass flow rate to below 1 %. However, the integration procedure proved extremely sensitive to both the dividing streamline and the closure relations small perturbations in the cross-sections induced large fluctuations in the Mach number and in some cases to infinite pressure gradients. This sensitivity is expected to persist if the dividing streamline is obtained from a predictive model. Notably, even when RANS-based streamlines were applied, complete static pressure equalisation was not achieved within the Fabri framework. This limitation is absent from the compound model, whose cross-sectional for-

mulation inherently accounts for the effects of forces, regardless of their precise definition.

Both choking formulations yield identical results when employing the same dividing streamline and closure relations, provided the Fabri-sonic condition is reached. In that case, the Fabri-based model reproduces the same prediction error as the compound model, confirming that discrepancies in entrainment arise from inaccuracies in the dividing streamline rather than from the choking condition itself. Nonetheless, distinct solutions appear when using the dividing streamline from the critical point as predicted by the compound model. Such cases—featuring a subsonic secondary stream—have been presented here and in literature. Although it should be kept in mind that these flow fields near the critical point remain strongly two-dimensional, thereby testing the limits of the present one-dimensional framework, the compound sonic condition emerges as the more general criterion governing flow rate blockage.

However, the investigated test cases exhibit nonuniform static pressure profiles, which conflict with the fundamental assumption of uniform static pressure in compound models. This suggests the existence of a *generalised* compound condition, where the sonic condition depends on both streams without requiring uniform static pressure. Such an extension could significantly enhance one-dimensional ejector modelling, though at present, no straightforward research path exists for developing a compound formulation without this simplifying assumption.

The proposed double-stream model provides a more comprehensive representation than the single-stream formulation discussed in chapter 5, addressing the requirements outlined in the hierarchy of figure 1.3. The compound-based model's accuracy has been maximised locally through calibration. However, the observed overestimation of the global flow rate in on-design conditions is directly linked to the fundamental constraint of uniform static pressure. Relaxing this constraint leads to Fabri choking, which in turn represents a less general mechanism for flow blockage. These modelling limitations are therefore intrinsic to the one-dimensional double-stream framework and can only be marginally mitigated through calibration. The following chapter continues the trend established in chapters 5–7, enhancing the radial resolution from a single to a double state, and begins exploring quasi-two-dimensional formulations to recover full velocity profiles without resorting to RANS simulations—thus achieving comparable insight at a fraction of the computational cost.

8

Quasi-two-dimensional modelling: early developments and perspectives

8.1 Introduction

The discrepancies identified in the one-dimensional models from the previous chapters motivate an explicit representation of the radial structures in the flow field. This chapter explores two main avenues for quasi-2D modeling.

The first avenue is that of reduced-order modeling, in which the solution of the full governing PDE is approximated as a linear combination of a small number of basis functions. This leads to a significantly lower-dimensional ODE systems, and hence to a far more computationally tractable framework. The resulting reduction in cost makes it feasible to perform data-driven closure modelling for the unresolved terms, leveraging the physics-integrated methodology introduced in section 3.2.

The second avenue concerns improved one-dimensional, cross-stream-averaged models to gain insight into the evolution of the radial velocity,

pressure and density profiles. This strategy underpins classical approaches such as the Kármán–Pohlhausen integral models for boundary layers [120–122], as well as the Kapitza–Shkadov and weighted-residual models for falling liquid films [123–126]. In all these cases, suitable assumptions about the radial profiles enable the reduction to low-dimensional systems—either ODEs for boundary-layer thickness or PDEs for film thickness and flow rate.

This chapter investigates both avenues using deliberately simplified models and assumptions.

Firstly, the leading-order equation for the axial velocity identified in section 4.2 is simplified to a linear diffusion equation in section 8.2. This extreme simplification shifts a greater burden onto the data-driven closure law, which is identified using spatial radial basis functions and an adjoint-based optimisation method. As such, this framework provides the first building blocks for a quasi-two dimensional reduced-order model. Finally, section 8.3 outlines possible directions for the development of integral models. The chapter concludes with perspectives and future research directions for both avenues in section 8.4.

8.2 Simplified linear model

This section develops the numerical tools required to construct and calibrate the envisaged quasi-2D model. The approach taken here expands the radial profiles of the axial velocity on a basis of tailored Chebyshev functions that satisfy the boundary conditions by construction. This work deliberately prioritizes a physics-informed basis over a data-driven POD approach in order to infer a two-dimensional description of choking without embedding configuration-specific information into the model structure. The POD remains attractive for configuration-specific optimisation, where it may further reduce dimensionality or improve accuracy.

Equation (4.48) is here strongly simplified to a linear model for the axial velocity alone. The resulting framework is therefore a deliberately minimal proof-of-concept, serving as a first building block toward a conservative quasi-two-dimensional model that respects continuity and the full leading-order equations.

The following sections outline the definition of the model, the construction of the radial basis functions, the regression tools involving the continuous adjoint method, and the resulting behaviour of the system.

8.2.1 Definition

The model is based on a non-uniform eddy viscosity distribution that distorts a velocity profile at the inlet of the mixing pipe. The coordinates and the axial velocity are normalised as before in section 4.2. The governing equation for the velocity is given below:

$$\frac{\partial \hat{u}}{\partial \hat{x}} = \frac{\partial}{\partial \hat{r}} \left(\hat{\mu}_{\text{eff}} \frac{\partial \hat{u}}{\partial \hat{r}} \right), \quad (8.1)$$

with the following boundary conditions:

$$\hat{u}(0, \hat{r}) = \hat{u}_0(\hat{r}), \quad \frac{\partial \hat{u}}{\partial \hat{r}}(\hat{x}, 0) = 0, \quad \hat{u}(\hat{x}, 1) = 0. \quad (8.2)$$

and where

$$\hat{\mu}_{\text{eff}} = g(\hat{x}, \hat{r}; \mathbf{w}), \quad (8.3)$$

is the eddy viscosity for which a parametric closure relation is sought. The dimensional counterpart of equation (8.1) would entail inconsistent dimensions of the viscosity, which is not an issue in the current data-driven context, since the closure relation (8.3) for $\hat{\mu}_{\text{eff}}$ is to be inferred through the physics-integrated approach from section 3.2. Its distribution in space is parametrised through the closure weights $\mathbf{w}$. The specific regression tools to identify this relation are described in section 8.2.3. The boundary conditions stem from the no-slip condition at the wall ($\hat{r} = 1$), the axisymmetry and the velocity profile at the inlet which is assumed to be known. In a predictive quasi-2D model, this profile would be provided from the modelled inlets upstream, for example as in the double stream 1D model of chapter 7.

Equation (8.1) is obviously a strong simplification with respect to the compressible axisymmetric RANS equations, which themselves are also insufficient to accurately capture the on-design entrainment and the pressure distribution at the wall with respect to experiments (see chapter 2 and [11, 28]). The central question in this section is therefore to assess to what extent the proposed model can assimilate the velocity field with data-driven closure. This analysis serves as a base for the future development of Reduced Order Models (ROM) for ejectors.

The cost function to be minimised is a classic quadratic function, penalising the discrepancy between the velocity fields $\hat{u}$ and $\tilde{u}$ of the model and

of the reference respectively:

$$J(\mathbf{w}) = \frac{1}{2} \int_0^1 \int_0^1 \mathbb{1}_Z(\hat{x}) (\hat{u} - \tilde{u})^2 d\hat{r} d\hat{x}, \quad (8.4)$$

where the identity function $\mathbb{1}_Z(x)$ equals unity at positions where the reference is available (following section 3.3.2). Practically, these consist of the grid points of the mesh. The continuous adjoint method is used in this chapter to compute the gradient of the cost function (8.4). The adjoint equations are given by:

$$\frac{\partial \hat{\lambda}}{\partial \hat{x}} = -\frac{\partial}{\partial \hat{r}} \left(\hat{\mu}_{\text{eff}} \frac{\partial \hat{\lambda}}{\partial \hat{r}} \right) - \mathbb{1}_Z(\hat{x}) (\hat{u} - \tilde{u}), \quad (8.5)$$

with the following boundary conditions:

$$\hat{\lambda}(1, \hat{r}) = 0, \quad \frac{\partial \hat{\lambda}}{\partial \hat{r}}(\hat{x}, 0) = 0, \quad \hat{\lambda}(\hat{x}, 1) = 0. \quad (8.6)$$

The boundary conditions at the centerline and at the wall are hence identical to those of the original system, which is beneficial for the numerical method applied in the section below. The gradient of the cost function is given by the following equation:

$$\frac{dJ}{d\mathbf{w}} = \int_0^1 \int_0^1 \hat{\lambda} \left(\frac{\partial \hat{\mu}_{\text{eff}}}{\partial \mathbf{w}} \frac{\partial^2 \hat{u}}{\partial \hat{r}^2} + \frac{\partial^2 \hat{\mu}_{\text{eff}}}{\partial \hat{r} \partial \mathbf{w}} \frac{\partial \hat{u}}{\partial \hat{r}} \right) d\hat{r} d\hat{x}. \quad (8.7)$$

8.2.2 Spectral collocation with tailored Chebyshev basis

The equations are solved using a spectral collocation method based on an expansion in tailored Chebyshev polynomials that inherently satisfy the boundary conditions. The normalisation coordinates are well suited for these polynomials, which are supported on the $[-1, 1]$ interval. They are defined through cosine functions and a change of variable (see for example [127]):

$$T_n(\hat{r}) = \cos(n\theta), \quad \text{where } \hat{r} = \cos(\theta). \quad (8.8)$$

The coordinate $\hat{r}$ is therefore bounded between $[-1, 1]$, with $\theta \in [0, \pi]$. They form a series of polynomials of increasing order (through trigonometric sum formulas), of which the first $n - 1$ components are discretely

orthogonal when evaluated in the zeros of the n^{th} polynomial, given by:

$$\hat{r}_{n,k} = \cos \left(\pi \frac{2k+1}{2n} \right), \quad k \in [0, \dots, n-1]. \quad (8.9)$$

Furthermore, the even functions $[T_0, T_2, \dots]$ are symmetric around $\hat{r} = 0$, thereby respecting the symmetry condition at the bottom of the numerical domain. These functions equal unity at the boundary $\hat{r} = 1$, so the following basis functions also comply with the no slip condition at $\hat{r} = 1$:

$$\mathcal{B} = \{T_2(\hat{r}) - 1, T_4(\hat{r}) - 1, \dots, T_{2N}(\hat{r}) - 1\}. \quad (8.10)$$

The polynomial $T_0(\hat{r})$ was excluded here, since it equals unity. A discrete orthogonal basis was finally obtained through linear combinations of the basis functions in $\mathcal{B}$, numerically found through a QR-decomposition of the following matrix:

$$\mathbf{T}_{2N} - \mathbf{1} = \begin{bmatrix} | & & | \\ \mathbf{T}_2 - 1 & \dots & \mathbf{T}_{2N} - 1 \\ | & & | \end{bmatrix} = \mathbf{Q}_N \mathbf{R} = \begin{bmatrix} | & & | \\ \hat{\mathbf{q}}_1 & \dots & \hat{\mathbf{q}}_N \\ | & & | \end{bmatrix} \mathbf{R}, \quad (8.11)$$

where the vectors $\hat{\mathbf{q}}_n$ and $\mathbf{T}_{2n}$ denote the respective basis functions evaluated at the zeros of the $N + 1^{\text{th}}$ even polynomial. Since the numerical domain is bounded between $\hat{r} \in [0, 1]$ and the basis functions are even, only the positive roots are used to construct the basis, which is therefore evaluated in $N + 1$ root points (see equation (8.9)):

$$\hat{\mathbf{r}}_{2(N+1)} = \begin{bmatrix} \hat{r}_{2(N+1),0} & \hat{r}_{2(N+1),1} & \dots & \hat{r}_{2(N+1),N} \end{bmatrix}^T. \quad (8.12)$$

The basis matrix $\mathbf{Q}_N$ thus has the dimensions $(N + 1 \times N)$. The fundamental property of discrete orthonormality of the basis yields:

$$\mathbf{Q}_N^T \mathbf{Q}_N = \mathbf{I}. \quad (8.13)$$

The derivatives and the integrals of the original Chebyshev polynomials are known analytically:

$$\frac{dT_n}{d\hat{r}}(\hat{r}) = nU_{n-1}(\hat{r}), \quad \frac{d^2T_n}{d\hat{r}^2}(\hat{r}) = n \frac{nT_n(\hat{r}) - \hat{r}U_{n-1}(\hat{r})}{\hat{r}^2 - 1}, \quad (8.14)$$

$$\int_{-1}^1 T_n(\hat{r}) d\hat{r} = \begin{cases} \frac{(-1)^n + 1}{1 - n^2} & \text{if } n \neq 1 \\ 0 & \text{if } n = 1 \end{cases}, \quad (8.15)$$

where U_n denote Chebyshev polynomials of the second kind:

$$U_n(\hat{r}) = \frac{\sin((n+1)\theta)}{\sin(\theta)}. \quad (8.16)$$

The derivatives of the basis functions in $\mathcal{B}$ (see equation (8.10)) are identical to those above. Finally, the derivatives of the base functions $\hat{q}_n$, evaluated at the zeros $\hat{r}_{2(N+1)}$, follow from the linear QR-decomposition:

$$\frac{dT_{2N}}{d\hat{r}} = \frac{dQ_N}{d\hat{r}} \mathbf{R}, \quad \text{and} \quad \frac{d^2 T_{2N}}{d\hat{r}^2} = \frac{d^2 Q_N}{d\hat{r}^2} \mathbf{R}. \quad (8.17)$$

The integrals follow in a similar fashion:

$$\begin{bmatrix} \int_0^1 T_2(\hat{r}) d\hat{r} - 1 \\ \vdots \\ \int_0^1 T_{2N}(\hat{r}) d\hat{r} - 1 \end{bmatrix}^T = \begin{bmatrix} \int_0^1 \hat{q}_1(\hat{r}) d\hat{r} \\ \vdots \\ \int_0^1 \hat{q}_N(\hat{r}) d\hat{r} \end{bmatrix}^T \mathbf{R} = \mathbf{i}_N^T \mathbf{R}. \quad (8.18)$$

Evaluating the integral on the positive half of the domain is straightforward since the base functions are even, hence dividing the result in (8.15) by two. The derivatives $dQ_N/d\hat{r}$ and $d^2 Q_N/d\hat{r}^2$ of the basis functions in the zeros $\hat{r}_{2(N+1)}$ and the integrals $\mathbf{i}_N$ are computed once, by solving the linear systems in (8.17) and (8.18), and are then stored for later use. For illustration, the first eight functions of a basis with 32 components are shown in figure 8.1. The dots indicate the zeros from equation (8.9) at which the basis is evaluated numerically and in which its components are discretely orthogonal. The functions all individually respect the boundary conditions (8.2).

The velocity field is approximated by a truncated expansion of the modified Chebyshev polynomials $\hat{q}$:

$$\hat{u}(\hat{x}, \hat{r}) = \sum_{i=1}^N c_i(\hat{x}) \hat{q}_i(\hat{r}). \quad (8.19)$$

The numerical algorithm thus solves for the coefficients c_i , which vary along the axial coordinate $\hat{x}$. Introducing this decomposition in the gov-

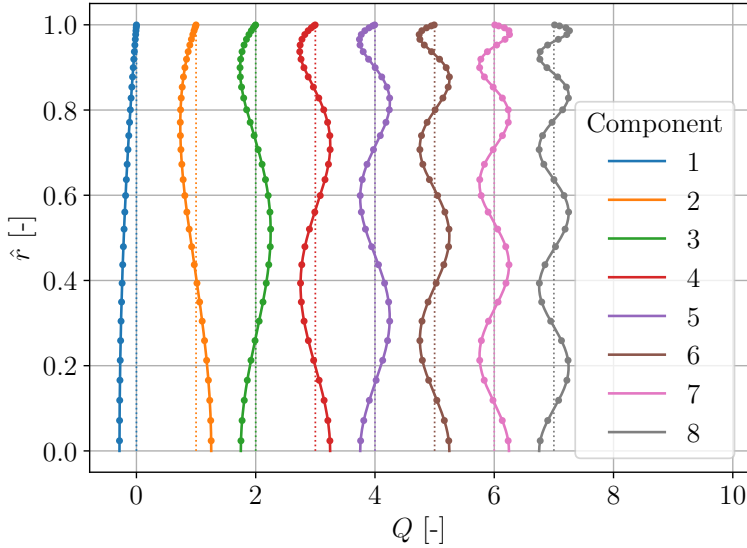


Fig. 8.1 First eight functions $\hat{q}$ of the basis $\mathbf{Q}_N$ with $N = 32$ components. They are all homogeneous at $\hat{r} = 1$ and have zero derivative at $\hat{r} = 0$.

erning equation (8.1) results in the following equation:

$$\sum_{i=1}^N \frac{\partial c_i}{\partial \hat{x}} \hat{q}_i = \frac{\partial}{\partial \hat{r}} \left(\hat{\mu}_{\text{eff}} \sum_{i=1}^N c_i \frac{\partial \hat{q}_i}{\partial \hat{r}} \right). \quad (8.20)$$

Evaluated at the zeros $\hat{r}_{2(N+1)}$ of the orthonormal basis, this equation becomes:

$$\mathbf{Q}_N \frac{\partial \mathbf{c}}{\partial \hat{x}} = \frac{\partial \hat{\mu}_{\text{eff}}}{\partial \hat{r}} \odot \left(\frac{d\mathbf{Q}_N}{d\hat{r}} \mathbf{c} \right) + \hat{\mu}_{\text{eff}} \odot \left(\frac{d^2 \mathbf{Q}_N}{d\hat{r}^2} \mathbf{c} \right), \quad (8.21)$$

where $\mathbf{Q}_N \in \mathbb{R}^{N+1 \times N}$ is the previously constructed basis, $\mathbf{c} \in \mathbb{R}^N$ collects the coefficients of the N basis functions in (8.19) at the current axial coordinate $\hat{x}$, $\hat{\mu}_{\text{eff}} \in \mathbb{R}^{N+1}$ is the eddy viscosity evaluated at the zeros of the basis through closure relation (8.3) and $\odot$ denotes an element-wise multiplication. The orthonormality of the basis (see (8.13)) allows to construct the following equation for the dynamics of the coefficients $\mathbf{c}$:

$$\frac{\partial \mathbf{c}}{\partial \hat{x}} = \mathbf{Q}_N^T \left[\frac{\partial \hat{\mu}_{\text{eff}}}{\partial \hat{r}} \odot \left(\frac{d\mathbf{Q}_N}{d\hat{r}} \mathbf{c} \right) \right] + \mathbf{Q}_N^T \left[\hat{\mu}_{\text{eff}} \odot \left(\frac{d^2 \mathbf{Q}_N}{d\hat{r}^2} \mathbf{c} \right) \right], \quad (8.22)$$

The computational cost of these operations is relatively limited, since the most expensive operation consists of a matrix-vector product of size $N + 1$. The numerical integration of these equations are therefore performed in a matter of seconds on an ordinary laptop. It is noteworthy to mention that tailoring the Chebyshev basis has allowed to simplify the original system of PDEs (8.1) to the system of ODEs (8.22) above, which respect the boundary conditions by construction. The adjoint system is solved in a similar way using the same basis functions.

Evaluating the cost function and its gradient with the adjoint method requires integrating terms not expressed in terms of the basis functions (for example, equation (8.7) contains the closure relation (8.3)). Rather than computing these integrals analytically, Clenshaw-Curtis quadrature was applied. This method consists of approximating the integrand through Chebyshev polynomials, for which the integrals are known analytically (see equation (8.18)). For example, the integral of $\hat{u}$ along $\hat{r}$ then follows from an inner product between the coefficients c of the expansion (8.19) and the integrals i_N of the basis functions from equation (8.18):

$$\int_0^1 \hat{u} d\hat{r} = \sum_{i=1}^N c_i \int_0^1 \hat{q}_i d\hat{r} = c \cdot i_N. \quad (8.23)$$

The integral of a general function f (e.g., the squared error on $\hat{u}$) follows from projection on the basis. Practically, the function f is evaluated at the zeros $\hat{r}_{2(N+1)}$ of the orthonormal basis, which are readily available, and projected on coefficients c_f through the basis matrix Q_N :

$$\int_0^1 f(\hat{r}) d\hat{r} \approx c_f \cdot i_N = \left(Q_N^T f \left(\hat{r}_{2(N+1)} \right) \right) \cdot i_N, \quad (8.24)$$

where $Q_N \in \mathbb{R}^{N+1 \times N}$, $\hat{r}_{2(N+1)} \in \mathbb{R}^{N+1}$ and $i_N \in \mathbb{R}^N$. This formulation allows to efficiently integrate any function f using the precomputed basis functions and their integrals. The projection is required to enable the computation of the gradient of the cost function in equation (8.7), which involves the generic closure relation g .

To choose the number of basis functions, a representative normalised velocity profile is projected onto bases with increasing size, as shown in figure 8.2. The reference profile stems from the RANS simulation on the UCLouvain ejector with conditions $p_{t,p}/p_{t,s} = 5$, $p_b/p_{t,s} = 1.2$ and was taken at $\hat{x} = 0.2$. At that axial position, the primary and secondary streams

still exhibit an unperturbed core. Visually, the approximation with $N_b = 32$ basis functions eliminates most of the oscillations due to Gibbs phenomenon near the sharp corners in the profile. The corresponding error is generally below 2 %, and the accuracy in the boundary layer is deemed sufficient. The number of basis functions was fixed to 32 to obtain the results discussed below.

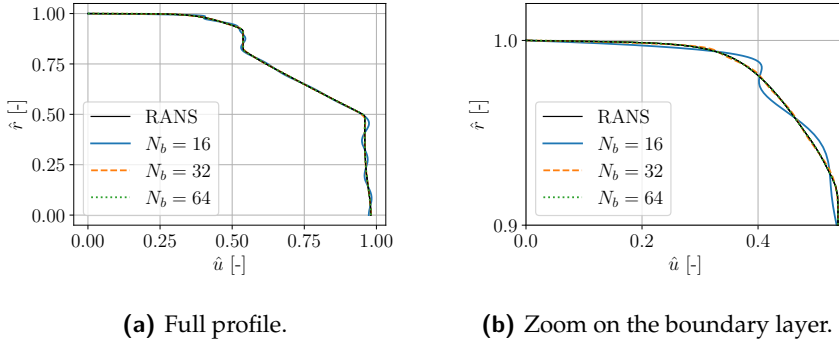


Fig. 8.2 Projection of the normalised velocity profile at $\hat{x} = 0.2$ (UCLouvain ejector, $p_{t,p}/p_{t,s} = 5$, $p_b/p_{t,s} = 1.2$) onto the tailored Chebyshev bases. The profile is accurately captured with 32 basis functions.

8.2.3 Regression tools

The objective of this section is to identify the weights of the parametric closure law g for the eddy viscosity $\hat{\mu}_{\text{eff}}$, following the physics-integrated calibration framework introduced in chapter 3. In this exploratory phase, the regression is deliberately restricted to a single operating point (UCLouvain ejector, $p_{t,p}/p_{t,s} = 5$, $p_b/p_{t,s} = 1.2$), with the viscosity parametrised directly as a function of space (see equation (8.3)) in order to decouple the reduced-order formulation from additional transport equations. More elaborate turbulence closures, such as $k-\omega$ SST-type models embedded in the conservative governing equations, constitute a natural extension and are left for future work.

The closure is parametrised using the same C^4 -continuous Radial Basis Functions (RBFs) introduced earlier (see equation (3.22)). The closure relation g is expressed as

$$\hat{\mu}_{\text{eff}} = g(\hat{x}, \hat{r}; \mathbf{w}) = \phi(\hat{\mathbf{d}}) \cdot \mathbf{w}, \quad (8.25)$$

where $\phi \in \mathbb{R}^{N_w}$ is the vector of RBFs evaluated at $(\hat{x}, \hat{r})$ and $\mathbf{w} \in \mathbb{R}^{N_w}$ contains the corresponding weights.

Following the philosophy of physics-separated and physics-integrated approaches from chapter 3, the identified field of the eddy viscosity in section 4.3 provides a suitable initial guess for the linear model. Since the governing equations differ, the absolute values of $\hat{\mu}_{data}$ must be rescaled. By comparing equations (4.48) and (8.1), neglecting additional terms and assuming static pressure of order unity, the scale factor is given by the non-dimensional group $\Pi/(\varepsilon^2 Re) \approx 10^{-4}$. Consequently, the target to initialise the viscosity field is thus given by

$$\hat{\mu}_{\text{eff}, \text{targ}} = 10^{-4} \hat{\mu}_{data}. \quad (8.26)$$

The field above was regressed with RBFs on a uniform 32×32 grid over the computational domain with a uniform shape factor of $\sigma = 0.1$, which were empirically found to provide an accurate representation of the identified turbulent viscosity field. The gradient of the cost function (see equations (8.4) and (8.7)) is computed using the adjoint method, making this relatively high-dimensional parametrization computationally feasible. The basis in matrix form is:

$$\Phi_g(\hat{\mathbf{x}}, \hat{\mathbf{r}}) = \begin{bmatrix} \phi(\hat{\mathbf{d}}_1; \hat{x}_{c1}, \hat{r}_{c1}, \sigma) & \dots & \phi(\hat{\mathbf{d}}_{N_w}; \hat{x}_{cN_w}, \hat{r}_{cN_w}, \sigma) \end{bmatrix}, \quad (8.27)$$

where $\Phi_g \in \mathbb{R}^{N_p \times N_w}$ is evaluated at N_p points with coordinates $\hat{\mathbf{x}}$ and $\hat{\mathbf{r}}$. The distances to each center $(\hat{x}_{ci}, \hat{r}_{ci})$ are denoted by $\hat{\mathbf{d}}_i$. To reduce computational cost, the basis was evaluated on a subset of the original RANS grid, leaving out two-thirds of points in each direction. Initial weights were obtained through Ridge regression with regularisation $\alpha = 0.2$:

$$J_{\text{init}}(\mathbf{w}_0) = \left\| \Phi_g \mathbf{w}_0 - \hat{\mu}_{\text{lin}, \text{targ}} \right\|_2^2 + \alpha \|\mathbf{w}_0\|_2^2, \quad (8.28)$$

where $\|\bullet\|_2$ denotes the L2 norm over the grid points (first term) and over the weights (second term). The regression problem was solved with the L-BFGS-B algorithm under the constraint $\mathbf{w}_0 > 0$ to enforce positive diffusion coefficients, with the gradient of the cost function given by:

$$\frac{dJ_{\text{init}}}{d\mathbf{w}_0} = 2\Phi_g^T \left(\Phi_g \mathbf{w}_0 - \hat{\mu}_{\text{lin}, \text{targ}} \right) + 2\alpha \mathbf{w}_0. \quad (8.29)$$

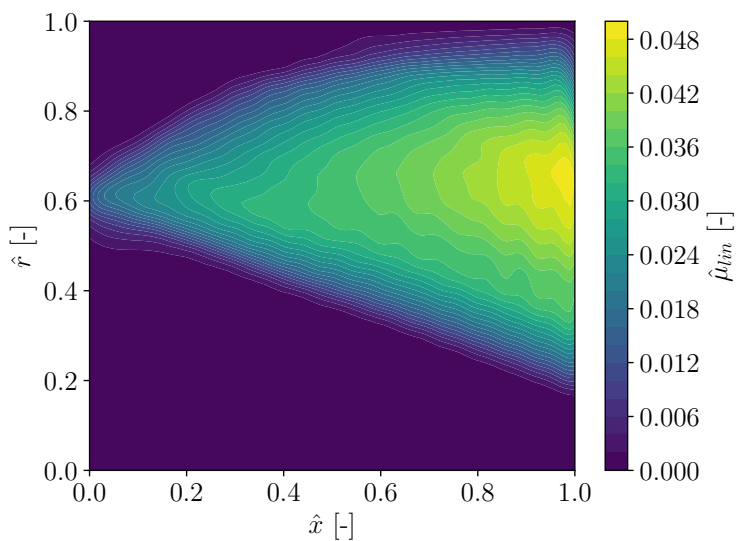
8.2.4 Results

The optimised linear viscosity field is shown in figure 8.3a, along with the corresponding fields of the axial velocity $\hat{u}$ and the adjoint variable $\hat{\lambda}$. The viscosity field looks qualitatively identical to the eddy viscosity field identified from the RANS simulations, with a scale difference of $4 \cdot 10^4$, indicating that the initial guess was relatively accurate. Nevertheless, the velocity profiles differ considerably from the reference, and tend to be underestimated. The likely reason for this discrepancy is the violation of conservation laws, which force the unperturbed secondary flow to accelerate due to boundary layer growth. The radial velocity probably also plays an important role near the inlet of the domain.

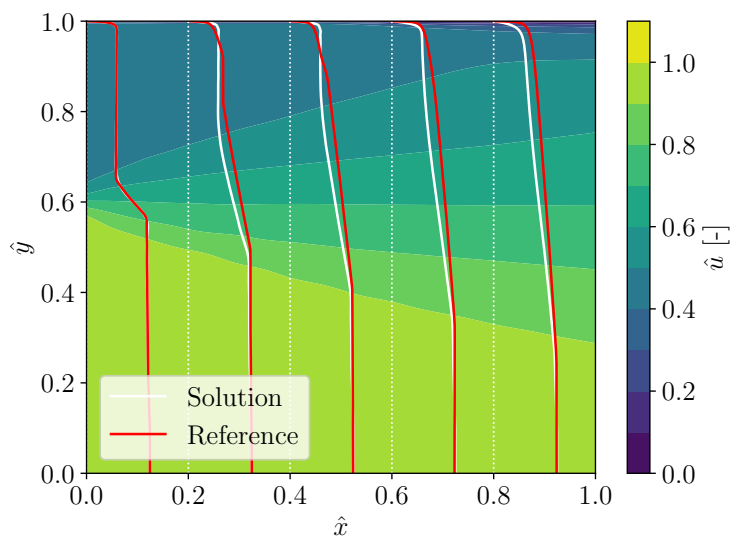
The solution $\hat{\lambda}$ of the adjoint equations (8.5) and (8.6) corresponding to the final physical solution is shown in figure 8.3c. It is dominated by the integration of the discrepancy between the numerical solution $\hat{u}$ and the reference $\tilde{u}$ through the source term in equation (8.5). Consequently, it highlights regions where the model is most sensitive to changes in the closure relation, as the integrand in equation (8.7) to compute the gradient of the cost function involves a multiplication by $\hat{\lambda}$. Not surprisingly, the adjoint variable reaches its maximum near the region of highest shear at the inlet, which is indeed where the solution in the full domain would be affected by changes in the closure relation.

Equation (8.7) provides the gradient of the cost function by integrating N_w fields (for each component of the gradient). Investigating these fields allows to evaluate the local contribution of a particular weight on its component in the gradient. In the case of the RBFs, these contributions are local by definition since the derivatives of $\hat{\mu}_{\text{eff}}$ and $\partial \hat{\mu}_{\text{eff}} / \partial \hat{r}$ with respect to a single weight w_i is only non-zero in the neighbourhood of the RBF center. Figure 8.4 displays the components of the gradient at the end of the optimisation, sorted by magnitude and normalised by its largest component. A sharp drop occurs after 64 components, indicated by the dashed line. The fields integrated in equation (8.7) to obtain the gradient of the cost function are summed and normalised for these 64 components, hence highlighting which RBFs are associated with the highest components of the gradient. These are clearly localised near the top wall, suggesting a finer grid with smaller shape factors could potentially improve the accuracy of the linear model. This is a logical consequence of the small internal scale of the boundary layer.

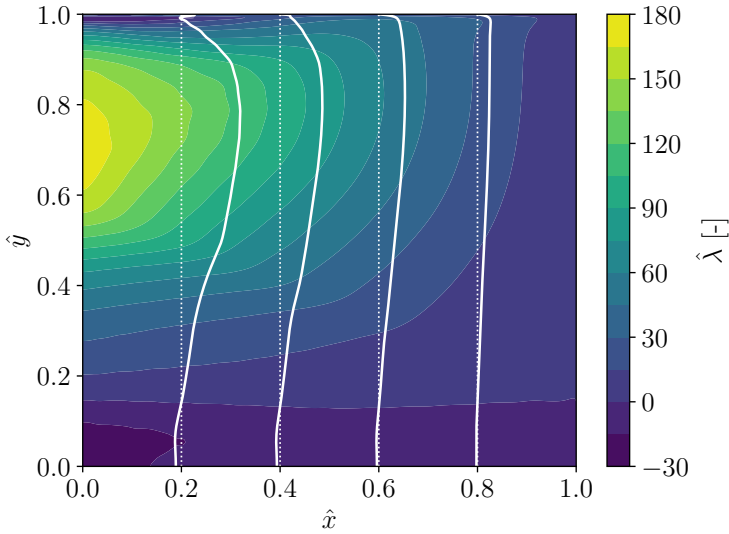
Despite its simplicity, the linear model reproduces the growth of the



(a) Linear viscosity $\hat{\mu}_{eff}$.



(b) Solution $\hat{u}$ and reference $\hat{u}$ (continued below).



(c) Adjoint solution $\hat{\lambda}$.

Fig. 8.3 Results of the optimised linear model. The linear viscosity follows the same qualitative trend as the eddy viscosity identified from RANS. The model underestimates the axial velocity due to the absence of conservation equations. The adjoint variables highlight the region of highest sensitivity of the model relative to its closure relation.

shear layer with remarkable accuracy. Its numerical integration requires only a few seconds on a standard laptop, which makes it an attractive candidate for reduced-order modelling. Future developments should primarily aim at enforcing the underlying conservation laws, thereby improving predictive accuracy and reducing the burden on the regression procedure. This would allow the regression tools to focus more effectively on identifying the closure relations rather than compensating for missing physics, all while preserving the low computational cost of the model.

8.3 Perspectives on integral modelling

As discussed in Section 8.1, integral one-dimensional models are constructed by prescribing analytical radial profiles for the relevant flow quantities and subsequently integrating a reduced form of the full two-dimensional equa-

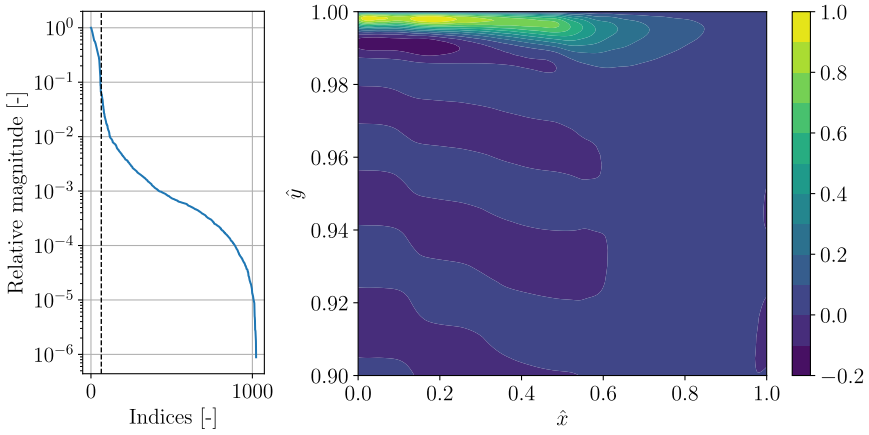


Fig. 8.4 Relative magnitude of the components in the cost gradient (left) and the integrand in equation (8.7) for computing the gradient, summed and normalised for the 64 largest components (right). The boundary layer is thus primarily responsible for the remaining discrepancy in figure 8.3, which is not surprising given the relatively coarse grid for the RBFs in the boundary layer.

tions. In contrast to the expansion over a generic family of basis functions considered in the previous section, these models rely on more tailored parametric profiles. Such profiles are not explicitly derived in this thesis and are therefore left for future work; symbolic regression may offer a promising avenue, with polynomial and hyperbolic tangent functions being a priori suitable candidates for representing the boundary layer and the shear layer, respectively. The primary objective of integral models is not to accurately reconstruct the radial profiles themselves, but rather to predict the axial evolution of integral quantities, such as the widths of the boundary and shear layers.

In practice, the velocity profile in the mixing pipe could be described by five parameters evolving along the axial direction x : the free-stream velocities in the primary and secondary streams, the widths of the boundary layer and the shear layer, and the radial position of the shear-layer centre. Given these parameters, the velocity profile is fully specified through the chosen polynomial and hyperbolic functions. The profiles of other quantities are a priori less straightforward to prescribe; however, a uniform

total temperature field may be assumed as a first approximation, in line with the one-dimensional models developed in previous chapters. The local static pressure is uniform in the radial direction and can therefore be treated as a sixth describing parameter. Together, the fixed total temperature and the parametrised axial velocity allow the static temperature to be computed (neglecting the contribution of the squared radial velocity to the total energy), with the density then obtained from the ideal gas law. Finally, the radial velocity may be inferred from conservation of mass. Different parametrisations may be required throughout the mixing pipe, for instance downstream of the point where the shear layer merges with the boundary layer, as proposed in the models of Chen et al. [39] and Hickman et al. [38].

The leading-order equations can then be integrated in the radial direction over the primary and secondary cross-sections, as well as across the boundary layer. This procedure should yield a closed set of equations governing the axial evolution of the describing parameters, in the spirit of the Kármán–Pohlhausen approach [120–122]. A more detailed analysis, including careful matching of the number of unknowns and governing equations, is left for future work.

Despite this reduction, the resulting one-dimensional equations still involve two-dimensional descriptions of interfacial exchanges, such that closure relations remain necessary for the eddy viscosity and turbulent heat transfer, or directly for the shear stresses. A potential advantage of the present framework is the availability of an underlying velocity profile, which may allow the free-stream values required by the correlation of Papamoschou [112] to be consistently sampled. However, as the shear layer and boundary layer grow and eventually merge at the centreline, the unperturbed cores of the two streams cease to exist, raising the question of which characteristic values should then be used in the correlation.

8.4 Conclusion

This chapter has shown that quasi-two-dimensional modelling constitutes a computationally viable middle ground between one-dimensional ejector models—whose structural limitations were identified in chapters 5–7—and full RANS simulations. The results demonstrate that the physics-integrated calibration framework from chapter 3 extends naturally to governing PDEs when combined with an efficient spectral collocation discretisation. The resulting reduced-order formulation remains computationally

lightweight while already reproducing the correct qualitative behaviour, despite the deliberately simplified physics. This establishes the feasibility of learning closures directly at the PDE level and confirms that quasi-2D modelling is compatible with the hierarchical framework developed throughout this thesis.

Future developments should prioritise stricter enforcement of conservation constraints within the reduced-order framework. Beyond improving accuracy, this would shift part of the modelling burden from regression toward physical structure, allowing the calibration tools to focus on genuinely unresolved flow features at comparable computational cost. Extending the framework to broader operating conditions will require replacing purely spatial regression with closure laws expressed in terms of physical state variables, in a manner analogous to the friction closures introduced in chapters 6 and 7. Turbulence closures inspired by $k-\omega$ -type models embedded in the conservative equations represent a natural direction. Such developments would transform the present proof-of-concept into a predictive quasi-two-dimensional modelling framework. These steps follow directly from the modelling hierarchy established in this thesis.

The quasi-two-dimensional formulation also exposes open questions regarding choking. In its current form, choking does not emerge as clearly as in the cross-stream averaged theory. This raises the possibility that either the local singular structure at $\text{Ma}_x = 1$ requires further analysis, or that choking in quasi-2D systems must be interpreted through an averaged compatibility constraint similar to the compound-flow theory. Extending the present framework to variable-area ducts is expected to clarify this issue, since convergent–divergent geometry remains central to compound choking (see chapter 6). In particular, the compound-nozzle benchmark from that chapter provides a natural test case and would enable direct comparison with the extended compound-flow theory [64].

Collectively, these directions define a coherent roadmap toward quasi-2D ejector models that reconcile physical consistency, computational efficiency, and data-driven closure. The chapter therefore marks the first step beyond purely one-dimensional modelling within the unified framework developed in this thesis.

9

Conclusions and perspectives

Lumped-parameter ejector models constitute essential tools for the design and optimisation of ejectors at the system scale, whether integrated within thermodynamic cycles (e.g., refrigeration systems) or embedded in large fluidic networks (e.g., aeronautics, chemical or process industries). Given the high computational cost of CFD simulations, full-scale analyses of such applications are often impractical. It is therefore crucial to understand the intrinsic limitations of lumped ejector models and to assess the extent of their predictive capabilities.

The work presented in this thesis has aimed to identify and quantify these limitations across the hierarchy of ejector models, ranging from integral zero-dimensional (0D) formulations to one-dimensional (1D) single- and double-stream models, and further to quasi-two-dimensional (quasi-2D) representations. By examining alternative modelling assumptions and applying systematic calibration, the boundaries of these models have been explored and extended, with the ultimate objective of reconciling their predictions with experimental measurements and CFD simulations.

The main contributions of this thesis are summarised in the following section, with emphasis on the modelling capabilities and the limitations identified throughout this work. The chapter concludes by outlining perspectives for future developments and research directions.

Summary and main achievements

Chapter 2 presented a database of experimental and numerical data for two ejector geometries. These include RANS simulations, from which any flow quantity can be computed, such as cross-stream averaged distributions. The experimental data consist of flow rates and local wall pressure measurements. Together, these datasets provide a broad range of operating conditions on which to assess the accuracy of the developed, and future, ejector models.

In parallel, practical tools for physics-separated and physics-integrated calibration were developed in chapter 3. These tools were ultimately not required to calibrate the 1D models, due to their low computational cost and the physics-based closure used for the momentum exchange. The definition of friction forces through coefficients prescribed by correlations allowed to reduce the number of closure parameters to two or three in the one-dimensional models. In contrast, the quasi-two-dimensional model greatly benefits from adjoint-based optimisation due to its higher dimensionality and larger parameter space.

Zero-dimensional models have been briefly discussed in this thesis, as they are extensively documented in the literature and their capabilities and limitations are well-established. They provide solely global information in terms of flow rates, but remain flexible through calibration of four to five efficiency coefficients associated with subcomponents. Consequently, an important goal in this thesis was to develop one-dimensional models capable of providing local information at a fraction of the cost of RANS simulations.

The first one-dimensional model (chapter 5) predicts axial distributions of flow variables averaged over the entire cross-sections; effectively behaving as a network of three pipes connected through a junction model. Its main strength lies in its capability to capture travelling pressure waves and, therefore, the characteristic response time of ejectors. However, the complex dynamics between the primary and the secondary stream in the mixing pipe remain hidden under the averaging operator. The averaged intensive variables, such as the Mach number, are generally not representative of the states within each stream, and the results must therefore be interpreted in terms of global quantities, such as the mass flow rate, over space and time. In this sense, the model behaves as a 0D formulation in steady state conditions, as the steady solution is governed primarily by the junction, while the length of the mixing pipe only has a minor influ-

ence through wall friction. On the other hand, the simplification to a single stream in the mixing pipe allows choking to emerge naturally in the unsteady solver, without raising the question whether choking occurs according to the Fabri- or the compound-theory.

The compound flow theory was extended with momentum exchange terms in chapter 6, forming the foundation for the double-stream model in chapter 7. The balance between wall and inter-stream friction forces proved responsible for the upstream and downstream displacement of the compound-sonic section. Counterintuitively, a sonic flow can occur in a convergent cross-section when inter-stream friction is dominant—an effect that had not previously been explained. This analysis highlights an important link between choking and friction forces, which also becomes apparent when comparing Fabri- and compound-choking mechanisms.

The double-stream model offers a more physically representative framework for ejector flows by assigning distinct numerical domains for the primary and secondary streams, thereby improving the radial resolution relative to the single-stream formulation. The added complexity enables the inclusion of additional physical mechanisms, such as local exchanges between the streams, but also raises several key questions: which choking mechanism dominates, how should the inter-stream exchanges be modelled, and how can the non-uniform pressure and shock trains be handled? These aspects have been systematically analysed in chapter 7.

Firstly, the compound and the Fabri choking mechanism were compared on a common basis within the same one-dimensional framework. Although the Fabri-sonic condition yielded the highest accuracy (1 %), it heavily relied on RANS data by imposing the dividing streamline and the total pressure gradients. The compound model employing the correlations for the friction coefficients is the only predictive double-stream model presented here, but it overestimates the secondary flow rate by up to 5 % with respect to the RANS simulations. It is more accurate in off-design operation, both globally and locally, despite the presence of shock trains. The assumption of uniform static pressure therefore appears acceptable to obtain axial distributions where the oscillations induced by the shock train have been ‘filtered out’. Furthermore, both choking mechanism were shown to coexist and to yield the same flow rate when imposing the same dividing streamline and exchanges terms, provided that the secondary stream reaches Mach one. Otherwise, the predictions differ as choked flows can exist with a subsonic secondary stream. The compound theory is therefore more general, although this situation also occurs under non-uniform static

pressure. Consequently, a generalised compound choking condition exists in nature, yet it remains challenging to formulate the observed behaviour mathematically without resorting to the constraint of uniform pressure.

The potential impact of calibration of the friction forces was analysed by imposing the total pressure gradients from the reference RANS simulations into the 1D model; hence emulating an idealised closure. The effect on the flow rate is minimal, owing to the short distance between the primary nozzle outlet and the sonic section. Minor improvements could further improve the accuracy of the axial distributions, particularly by introducing a loss term in the primary stream related to the shock train. However, assimilating this two-dimensional phenomenon within a strictly one-dimensional framework is not straightforward. Moreover, the calibrated correlations already achieve satisfactory accuracy, especially under off-design conditions.

The two-dimensional shock trains represent the most significant obstacle for the double-stream formulation, primarily due to the absence of oblique shocks in 1D. The pressure equalisation mechanism reproduces the first portion of the shock train accurately if the primary stream is under-expanded. In that case, an isentropic expansion to reach the same static pressure as the secondary stream is a physically meaningful representation. However, inertia of the primary stream causes an overshoot of the pressure difference, leading to further expansion. This effect is absent in the compound model, which locally provides the dividing streamline that complies with the static pressure constraint, neglecting inertia. This omission may induce sharp corners in the dividing streamline and consequently, errors in the secondary flow rate. Similar difficulties arise in the second shock train within the diffuser, where the boundary layer typically separates from the wall. Even without boundary layer separation, a compound shock model would be too restrictive in enforcing uniform static pressure downstream. Therefore, the normal shock procedure in the double-stream model first imposes instantaneous mixing to a uniform state, followed by a single-stream solution up to the outlet of the diffuser.

Overall, the double-stream model showcases that the cross-stream averaged representation of ejector flows allows constructing reduced order models that achieve a balance between accuracy and simplicity, at negligible computational cost.

The exploratory work on the quasi-two dimensional model in chapter 8 yielded a set of leading-order RANS equations which form the base for two approaches: (1) integral modelling which results in ODEs for in-

tegral quantities such as the boundary layer thickness and the mass flow rate; and (2) Reduced-Order Models (ROMs) based on an expansion of the solution on a carefully constructed basis for the radial profiles. Such a basis was constructed for the axial velocity and was applied on a first linear test case. The closure relation was inferred from the data using the sequential physics-separated and physics-integrated calibration procedures from chapter 3, including the continuous adjoint method. The results are encouraging, yet display the requirement to strictly enforce conservation.

The ultimate question in this thesis is whether the engineering models have been successfully reconciled with experiments and CFD. This objective has been partially achieved. The proposed models provide low-order representations of the underlying flow physics, such as the transient behaviour through the single-stream model and the axial distributions of cross-stream averaged quantities through the double-stream model. The latter, in particular, has been calibrated and validated against both numerical and experimental data. The identification of their intrinsic limits has revealed several open questions that remain partially unresolved. Particularly, compound choking with non-uniform static pressure remains elusive from a one-dimensional perspective, and the models presented here do not fully explain the discrepancies observed between RANS simulations and experimental measurements. Quasi-two-dimensional modelling emerges as a natural next step, increasing the radial resolution from one or two averaged streams to a continuous radial distribution, and may further help bridge the gap between reduced order models and high-fidelity data.

Limitations and perspectives

The main limitation of the double-stream model remains the constraint of uniform static pressure. It neglects the inertia of the primary stream and allows for sharp corners in the dividing streamline, resulting in a missed expansion in the mixing pipe that is ultimately compensated by an over-estimated secondary flow rate. Relaxing this constraint, however, induces Fabri choking, which has been shown to be less general than compound choking. This dilemma thus provides a trade-off between the generality of the choking condition and the accuracy of the dividing streamline, and consequently, of the flow rate. An extension of the pressure equalisation mechanism could be envisaged, incorporating the inertia of the primary stream and delaying the enforcement of uniform pressure to a subsequent shock cell farther downstream. Care must be taken not to overshoot the

sonic point before allowing the pressures equalise. Two practical questions arise:

How can the inertia of the primary stream be incorporated into the one-dimensional pressure equalisation mechanism?

At what point downstream of the primary nozzle are the pressures sufficiently equal to begin enforcing uniformity without compromising the compound-sonic point?

One could envisage a second order differential equation for the primary cross-section—thereby conserving a continuous axial derivative and avoiding sharp corners. However, this introduces unclosed terms for which no clear closure presently exists.

An alternative approach lies in quasi-two-dimensional modelling, which has been briefly introduced in this thesis. Employing continuous radial profiles for the velocity while maintaining uniform static pressure might suffice to overcome the limitations of the double-stream model. Analysing the resulting model in terms of singularities along the axial direction could reveal a continuous analogue to compound choking of double streams. The original research questions in the introduction 1.2 still stand for the quasi-2D model, which deserves to be developed further. In any case, bridging quasi-2D and double-stream 1D models seems a promising avenue for future research, motivated by the following central question:

Can a quasi-2D model complement the double-stream 1D model to overcome its limitations in on-design operation?

Moreover, choking in the quasi-2D setting was not yet considered in the early exploratory work of this thesis. Extending the analysis to variable-area ducts will be essential to address this question:

How does choking manifest in a quasi-two dimensional setting considering variable-area ducts?

Furthermore, the ideal gas law has been employed throughout this work. Extending the proposed one-dimensional models to real gases and two-phase flows will be essential for modelling practical ejector operation in refrigeration cycles. Analysing the interaction between these phenomena and choking will likely be complex, yet it offers substantial potential for accurate low-order modelling of ejectors in more diverse applications.

Finally, this work has considered equal total temperatures at the inlets. Exchange of energy between the streams becomes important in case of a large temperature difference (and phase change), leading to convergence in the total temperature distributions similar to that observed for total pressures in this study.

In summary, this thesis has systematically explored the hierarchy of lumped-parameter ejector models, clarifying their capabilities and intrinsic limitations. The development of one-dimensional single- and double-stream models has enabled the accurate prediction of transient behaviour and axial distributions at a fraction of the computational cost of RANS simulations, while highlighting the key role of inter-stream friction and choking mechanisms. The findings provide a rigorous foundation for low-order modelling, bridging the gap between integral models and complex CFD data. Looking forward, quasi-two-dimensional extensions and the inclusion of real-gas and phase-change effects offer promising avenues to further enhance predictive fidelity and expand the applicability of the proposed reduced-order models in practical engineering systems.



Explicit Ordinary Differential Equations

The table below summarises the required Ordinary Differential Equations (ODEs) for the relevant flow variables. These equations can be directly derived from the quasi-one-dimensional conservation equations without injection of mass, on which a force F (N/m) acts in the direction of the flow and receiving energy Q (W/m) [57]. Note that the equations for the total pressure and the total temperature are not singular if $Ma = 1$. The table should be read horizontally, e.g.:

$$\frac{d_x p}{p} = \frac{\gamma Ma^2}{1 - Ma^2} \frac{d_x A}{A} + \frac{1 + (\gamma - 1) Ma^2}{1 - Ma^2} \frac{F}{Ap} - \frac{\gamma Ma^2}{1 - Ma^2} \frac{Q}{\rho A u c_p T} \quad (A.1)$$

Another useful relation is the link between the Mach number and the static pressure as written below. It can be retrieved from the table below by eliminating the gradient of the cross-section A from the lines of the static pressure and the Mach number.

$$\frac{d_x Ma^2}{Ma^2} = \left[-2 \frac{1 + \frac{\gamma-1}{2} Ma^2}{\gamma Ma^2} \right] \frac{d_x p}{p} + \left[2 \frac{1 + \frac{\gamma-1}{2} Ma^2}{\gamma Ma^2} \right] \frac{F}{Ap} - \frac{Q}{\rho A u c_p T} \quad (A.2)$$

Table A.1 Summary of the 1D equations.

	$\frac{d_x A}{A}$	$\frac{F}{Ap}$	$\frac{Q}{\rho A u c_p T}$
$\frac{d_x p}{p}$	$\frac{\gamma \text{Ma}^2}{1 - \text{Ma}^2}$	$\frac{1 + (\gamma - 1) \text{Ma}^2}{1 - \text{Ma}^2}$	$\frac{-\gamma \text{Ma}^2}{1 - \text{Ma}^2}$
$\frac{d_x u}{u}$	$\frac{-1}{1 - \text{Ma}^2}$	$-\frac{1}{1 - \text{Ma}^2}$	$\frac{1}{1 - \text{Ma}^2}$
$\frac{d_x T}{T}$	$\frac{(\gamma - 1) \text{Ma}^2}{1 - \text{Ma}^2}$	$\frac{(\gamma - 1) \text{Ma}^2}{1 - \text{Ma}^2}$	$\frac{1 - \gamma \text{Ma}^2}{1 - \text{Ma}^2}$
$\frac{d_x \text{Ma}^2}{\text{Ma}^2}$	$-2 \frac{1 + \frac{\gamma - 1}{2} \text{Ma}^2}{1 - \text{Ma}^2}$	$-2 \frac{1 + \frac{\gamma - 1}{2} \text{Ma}^2}{1 - \text{Ma}^2}$	$\frac{1 + \gamma \text{Ma}^2}{1 - \text{Ma}^2}$
$\frac{d_x \rho}{\rho}$	$\frac{\text{Ma}^2}{1 - \text{Ma}^2}$	$\frac{1}{1 - \text{Ma}^2}$	$\frac{-1}{1 - \text{Ma}^2}$
$\frac{d_x p_t}{p_t}$	0	1	$\frac{-\gamma \text{Ma}^2}{2 \left(1 + \frac{\gamma - 1}{2} \text{Ma}^2 \right)}$
$\frac{d_x T_t}{T_t}$	0	0	$\frac{1}{1 + \frac{\gamma - 1}{2} \text{Ma}^2}$

B

Approximations of the static pressure gradient

2.1 Single stream

In this section, the first and second order approximations are computed for the pressure gradient of a single stream, using Taylor expansions of the numerator and denominator of equation (7.19). The notation and the governing equations are provided in section 2.1.1, which are required for computing the first and second derivatives in the Taylor expansions in sections 2.1.2 and 2.1.3.

2.1.1 Definitions

We reformulate the definition (7.19) of the static pressure gradient as follows:

$$\frac{d \ln(p)}{dx} = \frac{N_A + N_F}{D}, \quad (\text{B.1})$$

where

$$N_A = \gamma \text{Ma}^2 \frac{d \ln(A)}{dx}, \quad (\text{B.2})$$

$$N_F = \left(1 + (\gamma - 1) \text{Ma}^2\right) \frac{F}{Ap}, \quad (\text{B.3})$$

$$D = 1 - \text{Ma}^2. \quad (\text{B.4})$$

The indices p have been omitted since the analysis that follows is valid for any single stream in a channel $A(x)$. The following equation for the Mach number can be derived from the governing equations (6.6)-(6.8) [57] and will be required below:

$$\frac{d\text{Ma}^2}{dx} = c_{M0} \frac{d \ln(p)}{dx} + c_{M^2 1} F, \quad (\text{B.5})$$

where:

$$c_{M^2 0} = -\frac{2}{\gamma} \left(1 + \frac{\gamma-1}{2} \text{Ma}^2 \right), \quad (\text{B.6})$$

$$c_{M^2 1} = \frac{2}{\gamma} \left[1 + \frac{\gamma-1}{2} \text{Ma}^2 \right] \frac{1}{Ap}. \quad (\text{B.7})$$

For conciseness, we also define:

$$c_1 = 1 + (\gamma - 1) \text{Ma}^2. \quad (\text{B.8})$$

2.1.2 First order approximation

The first order approximation consists of using the pressure gradient in the sonic section as a constant in the neighbourhood. It is computed with de l'Hôpital's rule as in (7.21):

$$\left(\frac{d \ln(p)}{dx} \right)^* = \frac{N^*}{D^*} = \frac{0}{0} = \frac{(dN/dx)^*}{(dD/dx)^*}. \quad (\text{B.9})$$

We therefore need the derivatives of the numerator and the denominator, as computed below. The derivative of N_A reads:

$$\frac{dN_A}{dx} = c_{NA0} \frac{d \ln(p)}{dx} + c_{NA1}, \quad (\text{B.10})$$

where:

$$c_{NA0} = \gamma \frac{d \ln(A)}{dx} c_{M^2 0}, \quad (\text{B.11})$$

$$c_{NA1} = \gamma \frac{d \ln(A)}{dx} c_{M^2 1} F + \gamma \text{Ma}^2 \frac{d^2 \ln(A)}{dx^2}. \quad (\text{B.12})$$

The derivative of N_F reads:

$$\frac{dN_F}{dx} = c_{NFp} \frac{d \ln(p)}{dx} + c_{NFF} \frac{dF}{dx} + c_{NF2}, \quad (\text{B.13})$$

where

$$c_{NFp} = [(\gamma - 1) c_{M^2 0} - c_1] \frac{F}{Ap}, \quad (\text{B.14})$$

$$c_{NFF} = \left(1 + (\gamma - 1) \text{Ma}^2\right) \frac{1}{Ap}, \quad (\text{B.15})$$

$$c_{NF2} = (\gamma - 1) c_{M^2 1} \frac{F^2}{Ap} - c_1 \frac{F}{Ap} \left(\frac{d \ln(A)}{dx} \right). \quad (\text{B.16})$$

We introduce the following decomposition for the derivative of the force F :

$$\frac{dF}{dx} = c_{F0} \frac{d \ln(p)}{dx} + c_{F1}. \quad (\text{B.17})$$

This operation is general: any definition of the force F can be reduced to this form using the governing equations. The definition of these coefficients depends on the closure of the conservation equations; the definitions used in this work are given in section 2.1.4. Introducing this decomposition in (B.13) leads to:

$$\frac{dN}{dx} = c_{N0} \frac{d \ln(p)}{dx} + c_{N1}, \quad (\text{B.18})$$

where

$$c_{N0} = c_{NA0} + c_{NFp} + c_{NFF} c_{F0}, \quad (\text{B.19})$$

$$c_{N1} = c_{NA1} + c_{NFF} c_{F1} + c_{NF2}. \quad (\text{B.20})$$

The derivative of the denominator equals:

$$\frac{dD}{dx} = -\frac{d \text{Ma}^2}{dx} = c_{D0} \frac{d \ln(p)}{dx} + c_{D1}, \quad (\text{B.21})$$

where:

$$c_{D0} = -c_{M^2 0}, \quad (\text{B.22})$$

$$c_{D1} = -c_{M^2 1} F. \quad (\text{B.23})$$

Introducing the computed derivatives of the numerator and the de-

nominator in equation (B.9) leads to a quadratic equation for the pressure gradient with a positive root (compressing to subsonic speed) and a negative root (expanding to supersonic speed):

$$c_{D0}^* \left(\left(\frac{d \ln(p)}{dx} \right)^* \right)^2 + (c_{D1}^* - c_{N0}^*) \left(\frac{d \ln(p)}{dx} \right)^* - c_{N1}^* = 0. \quad (\text{B.24})$$

2.1.3 Second order approximation

The full Taylor expansions of the numerator and denominator provide the following exact equation (recall that $N^* = D^* = 0$):

$$\frac{d \ln(p)}{dx} = \frac{N^* + (x - x^*) \left(\frac{dN}{dx} \right)^* + \frac{(x - x^*)^2}{2} \left(\frac{d^2 N}{dx^2} \right)^* + \dots}{D^* + (x - x^*) \left(\frac{dD}{dx} \right)^* + \frac{(x - x^*)^2}{2} \left(\frac{d^2 D}{dx^2} \right)^* + \dots}. \quad (\text{B.25})$$

Truncating after second derivatives allows to build a second order approximation near the sonic point:

$$\frac{d \ln(p)}{dx} \approx \frac{\left(\frac{dN}{dx} \right)^* + \frac{(x - x^*)}{2} \left(\frac{d^2 N}{dx^2} \right)^*}{\left(\frac{dD}{dx} \right)^* + \frac{(x - x^*)}{2} \left(\frac{d^2 D}{dx^2} \right)^*}. \quad (\text{B.26})$$

The second derivatives of the numerator and the denominator are computed below. The derivatives of the coefficients $c_{M^2 0}$ and $c_{M^2 1}$ are given by:

$$\frac{dc_{M^2 0}}{dx} = -\frac{\gamma - 1}{\gamma} \frac{d \text{Ma}^2}{dx}, \quad (\text{B.27})$$

$$\frac{dc_{M^2 1}}{dx} = \frac{\gamma - 1}{\gamma} \frac{d \text{Ma}^2}{dx} \frac{1}{Ap} - c_{M^2 1} \left(\frac{d \ln(A)}{dx} + \frac{d \ln(p)}{dx} \right). \quad (\text{B.28})$$

These first derivatives are directly available from the equations in the section above, so are not expanded further for conciseness. We differentiate the coefficients related to N_A :

$$\frac{dc_{NA0}}{dx} = \gamma \frac{d^2 \ln(A)}{dx^2} c_{M^2 0} + \gamma \frac{d \ln(A)}{dx} \frac{dc_{M^2 0}}{dx}. \quad (\text{B.29})$$

$$\begin{aligned} \frac{dc_{NA1}}{dx} = & \gamma \text{Ma}^2 \frac{d^3 \ln(A)}{dx^3} + \gamma \frac{d^2 \ln(A)}{dx^2} \left[\frac{d\text{Ma}^2}{dx} + c_{M^21} F \right] \\ & + \gamma \frac{d \ln(A)}{dx} \left[F \frac{dc_{M^21}}{dx} + c_{M^21} \frac{dF}{dx} \right]. \end{aligned} \quad (\text{B.30})$$

The second derivative of N_A becomes:

$$\frac{d^2 N_A}{dx^2} = c_{NA0} \frac{d^2 \ln(p)}{dx^2} + c_{NA21}, \quad (\text{B.31})$$

where:

$$c_{NA21} = \frac{dc_{NA0}}{dx} \frac{d \ln(p)}{dx} + \frac{dc_{NA1}}{dx}. \quad (\text{B.32})$$

We differentiate the coefficients related to N_F :

$$\begin{aligned} \frac{dc_{NFp}}{dx} = & (\gamma - 1) \frac{F}{Ap} \left(\frac{dc_{M^20}}{dx} - \frac{d\text{Ma}^2}{dx} \right) \\ & + [(\gamma - 1) c_{M^20} - c_1] \frac{d}{dx} \left(\frac{F}{Ap} \right). \end{aligned} \quad (\text{B.33})$$

$$\frac{dc_{NFF}}{dx} = (\gamma - 1) \frac{d\text{Ma}^2}{dx} \frac{1}{Ap} + c_1 \frac{d}{dx} \left(\frac{1}{Ap} \right). \quad (\text{B.34})$$

$$\begin{aligned} \frac{dc_{NF2}}{dx} = & (\gamma - 1) \left[\frac{F^2}{Ap} \frac{dc_{M^21}}{dx} + c_{M^21} \frac{d}{dx} \left(\frac{F^2}{Ap} \right) \right] \\ & - (\gamma - 1) \frac{d\text{Ma}^2}{dx} \frac{F}{Ap} \frac{d \ln(A)}{dx} - c_1 \frac{F}{Ap} \frac{d^2 \ln(A)}{dx^2} \\ & - c_1 \left(\frac{d \ln(A)}{dx} \right) \frac{d}{dx} \left(\frac{F}{Ap} \right). \end{aligned} \quad (\text{B.35})$$

Despite these lengthy definitions, they can be evaluated directly. Differentiating N_F twice leads to:

$$\frac{d^2 N_F}{dx^2} = c_{NFp} \frac{d^2 \ln(p)}{dx^2} + c_{NFF} \frac{d^2 F}{dx^2} + c_{NF22}, \quad (\text{B.36})$$

where

$$c_{NF22} = \frac{dc_{NFp}}{dx} \frac{d \ln(p)}{dx} + \frac{dc_{NFF}}{dx} \frac{dF}{dx} + \frac{dc_{NF2}}{dx}. \quad (\text{B.37})$$

Differentiating the decomposition (B.17) gives:

$$\frac{d^2 F}{dx^2} = c_{F0} \frac{d^2 \ln(p)}{dx^2} + c_{F21}, \quad (\text{B.38})$$

where

$$c_{F21} = \frac{dc_{F0}}{dx} \frac{d \ln(p)}{dx} + \frac{dc_{F1}}{dx} \quad (\text{B.39})$$

The second derivative of N_F becomes:

$$\frac{d^2 N_F}{dx^2} = c_{NF20} \frac{d^2 \ln(p)}{dx^2} + c_{NF21}, \quad (\text{B.40})$$

where

$$c_{NF20} = c_{NFp} + c_{NFF} c_{F0}, \quad (\text{B.41})$$

$$c_{NF21} = c_{NFF} c_{F21} + c_{NF22}. \quad (\text{B.42})$$

The second derivative of the numerator equals:

$$\frac{d^2 N}{dx^2} = c_{N20} \frac{d^2 \ln(p)}{dx^2} + c_{N21}, \quad (\text{B.43})$$

where

$$c_{N20} = c_{NA0} + c_{NF20}, \quad (\text{B.44})$$

$$c_{N21} = c_{NA21} + c_{NF21}. \quad (\text{B.45})$$

The second derivative of the denominator equals:

$$\frac{d^2 D}{dx^2} = c_{D0} \frac{d^2 \ln(p)}{dx^2} + c_{D21}, \quad (\text{B.46})$$

where:

$$c_{D21} = \frac{dc_{D0}}{dx} \frac{d \ln(p)}{dx} + \frac{dc_{D1}}{dx}, \quad (\text{B.47})$$

and the derivatives of the coefficients are given by:

$$\frac{dc_{D0}}{dx} = -\frac{dc_{M^2 0}}{dx}, \quad (\text{B.48})$$

$$\frac{dc_{D1}}{dx} = -\frac{dc_{M^2 1}}{dx} F - c_{M^2 1} \frac{dF}{dx}. \quad (\text{B.49})$$

The second derivative of the static pressure is given by:

$$\frac{d^2 \ln(p)}{dx^2} = \frac{d}{dx} \left(\frac{N}{D} \right) = \frac{D(dN/dx) - N(dD/dx)}{D^2}. \quad (\text{B.50})$$

The equation above can be evaluated directly if $D \neq 0$. Otherwise, de l'Hôpital's rule yields the following equation:

$$\left(\frac{d^2 \ln(p)}{dx^2} \right)^* = \frac{0}{0} = \frac{(d^2 N/dx^2)^*}{2(dD/dx)^*} - \frac{N^*}{D^*} \frac{(d^2 D/dx^2)^*}{2(dD/dx)^*}, \quad (\text{B.51})$$

where N^*/D^* equals the static pressure gradient in the sonic section. Substituting the second derivatives $d^2 N/dx^2$ and $d^2 D/dx^2$ eventually leads to:

$$\left(\frac{d^2 \ln(p)}{dx^2} \right)^* = \frac{-c_{N21}^* + \left(\frac{d \ln(p)}{dx} \right)^* c_{D21}^*}{c_{N20}^* - \left(\frac{d \ln(p)}{dx} \right)^* c_{D0}^* - 2 \left(\frac{dD}{dx} \right)^*}. \quad (\text{B.52})$$

The terms in (B.26) can be computed with equations (B.18), (B.21), (B.43), (B.46), which can be evaluated in the sonic point using the derivatives of the static pressure given by (B.24) and (B.52). The result is a the second order approximation of the static pressure gradient near the sonic point.

2.1.4 Derivatives of the friction force

The decompositions (B.17) and (B.38) introduce the coefficients c_{F0} and c_{F1} and their derivatives, which are given by the equations below for the definition in equation (6.15):

$$c_{F0} = F \left(1 + \frac{c_{M0}}{\text{Ma}^2} \right), \quad (\text{B.53})$$

$$c_{F1} = c_{M1} \frac{F^2}{\text{Ma}^2} + \frac{F}{l_w} \frac{dl_w}{dx}. \quad (\text{B.54})$$

$$\frac{dc_{F0}}{dx} = \left(1 + \frac{c_{M0}}{\text{Ma}^2} \right) \frac{dF}{dx} + \frac{F}{\text{Ma}^2} \frac{dc_{M0}}{dx} - \frac{Fc_{M0}}{\text{Ma}^4} \frac{d\text{Ma}^2}{dx}, \quad (\text{B.55})$$

$$\frac{dc_{F1}}{dx} = c_{F1} \left(\frac{1}{c_{M1}} \frac{dc_{M1}}{dx} + 2 \frac{1}{F} \frac{dF}{dx} - \frac{1}{\text{Ma}^2} \frac{d\text{Ma}^2}{dx} \right) \quad (\text{B.56})$$

$$+ \frac{1}{l_w} \frac{dl_w}{dx} \frac{dF}{dx} - \frac{F}{l_w^2} \left(\frac{dl_w}{dx} \right)^2 + \frac{F}{l_w} \frac{d^2 l_w}{dx^2} \quad (\text{B.57})$$

2.2 Compound flow equations

2.2.1 Definitions

We reformulate the definition (6.12) of the static pressure gradient as follows:

$$\frac{d \ln(p)}{dx} = \frac{1}{\beta} \left(\frac{dA}{dx} + \sum_{i \in [p,s]} Z_i + \sum_{i \in [p,s]} Y_i \right) = \frac{N}{\beta}, \quad (\text{B.58})$$

where:

$$Z_i = \left[\frac{1 + (\gamma - 1) \text{Ma}_i^2}{\gamma \text{Ma}_i^2} \right] \frac{F_i}{p}, \quad (\text{B.59})$$

$$Y_i = -\frac{\gamma - 1}{\gamma} \frac{Q_i}{p u_i}. \quad (\text{B.60})$$

Differentiating the equation above gives rise to derivatives of quantities such as the velocity, which are linearly related to the static pressure gradient through the conservation equations (see for example chapter 8 of Shapiro [57]). For a general quantity ϕ_i , this relation reads:

$$\frac{d\phi_i}{dx} = c_{\phi i 0} \frac{d \ln(p)}{dx} + c_{\phi i 1}. \quad (\text{B.61})$$

The coefficients for the velocities u_i , the cross-sections A_i , the squared Mach numbers Ma_i^2 , the densities ρ_i and the static temperatures T_i are given below:

$$c_{ui0} = -\frac{u_i}{\gamma \text{Ma}_i^2}, \quad (\text{B.62})$$

$$c_{ui1} = \frac{u_i}{\gamma \text{Ma}_i^2} \frac{F_i}{A_i p}, \quad (\text{B.63})$$

$$c_{Ai0} = \frac{A_i}{\gamma} \left(\frac{1}{\text{Ma}_i^2} - 1 \right), \quad (\text{B.64})$$

$$c_{Ai1} = -\left(\frac{1 + (\gamma - 1) \text{Ma}_i^2}{\gamma \text{Ma}_i^2} \right) \frac{F_i}{p} + \frac{\gamma - 1}{\gamma} \frac{Q_i}{p u_i}, \quad (\text{B.65})$$

$$c_{M^2i0} = -\frac{2}{\gamma} \left(1 + \frac{\gamma-1}{2} \text{Ma}_i^2 \right), \quad (\text{B.66})$$

$$c_{M^2i1} = \frac{2}{\gamma} \left(1 + \frac{\gamma-1}{2} \text{Ma}_i^2 \right) \frac{F_i}{A_i p} - \frac{\gamma-1}{\gamma} \text{Ma}_i^2 \frac{Q_i}{p A_i u_i}, \quad (\text{B.67})$$

$$c_{\rho i0} = \frac{\rho_i}{\gamma}, \quad (\text{B.68})$$

$$c_{\rho i1} = \rho_i \frac{\gamma-1}{\gamma} \frac{F_i}{A_i p} - \rho_i \frac{\gamma-1}{\gamma} \frac{Q_i}{p A_i u_i}, \quad (\text{B.69})$$

$$c_{Ti0} = T_i \frac{\gamma-1}{\gamma}, \quad (\text{B.70})$$

$$c_{Ti1} = T_i \frac{\gamma-1}{\gamma} \frac{F_i}{A_i p} - T_i \frac{\gamma-1}{\gamma} \frac{Q_i}{p A_i u_i}. \quad (\text{B.71})$$

Their second derivatives share the following form:

$$\frac{d^2 \phi_i}{dx^2} = c_{\phi i0} \frac{d^2 \ln(p)}{dx^2} + \frac{dc_{\phi i0}}{dx} \frac{d \ln(p)}{dx} + \frac{dc_{\phi i1}}{dx}, \quad (\text{B.72})$$

where

$$\frac{dc_{ui0}}{dx} = -\frac{1}{\gamma \text{Ma}_i^2} \frac{du_i}{dx} + \frac{u_i}{\gamma \text{Ma}_i^4} \frac{d\text{Ma}_i^2}{dx}, \quad (\text{B.73})$$

$$\frac{dc_{ui1}}{dx} = \frac{c_{ui1}}{u_i} \frac{du_i}{dx} - \frac{c_{ui1}}{\text{Ma}_i^2} \frac{d\text{Ma}_i^2}{dx} + \frac{u_i}{\gamma \text{Ma}_i^2} \frac{d}{dx} \left(\frac{F_i}{A_i p} \right), \quad (\text{B.74})$$

$$\frac{dc_{Ai0}}{dx} = \frac{1}{\gamma} \left(\frac{1}{\text{Ma}_i^2} - 1 \right) \frac{dA_i}{dx} - \frac{A_i}{\gamma \text{Ma}_i^4} \frac{d\text{Ma}_i^2}{dx}, \quad (\text{B.75})$$

$$\begin{aligned} \frac{dc_{Ai1}}{dx} = & - \left(\frac{1 + (\gamma-1) \text{Ma}_i^2}{p \gamma \text{Ma}_i^2} \right) \left(\frac{dF_i}{dx} - F_i \frac{d \ln(p)}{dx} \right) \\ & + \frac{F_i}{p \gamma \text{Ma}_i^4} \frac{d\text{Ma}_i^2}{dx} + \frac{\gamma-1}{\gamma} \frac{d}{dx} \left(\frac{Q_i}{p u_i} \right), \end{aligned} \quad (\text{B.76})$$

$$\frac{dc_{M^2i0}}{dx} = -\frac{\gamma-1}{\gamma} \frac{d\text{Ma}_i^2}{dx}, \quad (\text{B.77})$$

$$\begin{aligned} \frac{dc_{M^2i1}}{dx} = & \frac{\gamma-1}{\gamma} \left(\frac{F_i}{p A_i} - \frac{Q_i}{p A_i u_i} \right) \frac{d\text{Ma}_i^2}{dx} + \frac{2}{\gamma} \frac{d}{dx} \left(\frac{F_i}{A_i p} \right) \\ & + \frac{\gamma-1}{\gamma} \text{Ma}_i^2 \left[\frac{d}{dx} \left(\frac{F_i}{A_i p} \right) - \frac{d}{dx} \left(\frac{Q_i}{p A_i u_i} \right) \right], \end{aligned} \quad (\text{B.78})$$

$$\frac{dc_{\rho i0}}{dx} = \frac{1}{\gamma} \frac{d\rho_i}{dx}, \quad (\text{B.79})$$

$$\begin{aligned} \frac{dc_{\rho i1}}{dx} = & \frac{\gamma-1}{\gamma} \left(\frac{F_i}{A_i p} - \frac{Q_i}{p A_i u_i} \right) \frac{d\rho_i}{dx} + \rho_i \frac{\gamma-1}{\gamma} \frac{d}{dx} \left(\frac{F_i}{A_i p} \right) \\ & - \rho_i \frac{\gamma-1}{\gamma} \frac{d}{dx} \left(\frac{Q_i}{p A_i u_i} \right), \end{aligned} \quad (\text{B.80})$$

$$\frac{dc_{T i0}}{dx} = \frac{\gamma-1}{\gamma} \frac{dT_i}{dx}, \quad (\text{B.81})$$

$$\begin{aligned} \frac{dc_{T i1}}{dx} = & - \frac{\gamma-1}{\gamma} \left(\frac{F_i}{A_i p} - \frac{Q_i}{p A_i u_i} \right) \frac{dT_i}{dx} - T_i \frac{\gamma-1}{\gamma} \frac{d}{dx} \left(\frac{F_i}{A_i p} \right) \\ & + T_i \frac{\gamma-1}{\gamma} \frac{d}{dx} \left(\frac{Q_i}{p A_i u_i} \right). \end{aligned} \quad (\text{B.82})$$

2.2.2 First order approximation

As in 2.1.2, the first order approximation consists of the pressure gradient in the sonic section, requiring the first derivatives of the numerator and the denominator in equation (B.58). The derivative of the force term Z_i reads:

$$\frac{dZ_i}{dx} = c_{Zi0} \frac{dF_i}{dx} + c_{Zi1} \frac{d \ln(p)}{dx} + c_{Zi2}, \quad (\text{B.83})$$

where

$$c_{Zi0} = \frac{1 + (\gamma-1) \text{Ma}_i^2}{\gamma p \text{Ma}_i^2}, \quad (\text{B.84})$$

$$c_{Zi1} = -Z_i + c_{ZiM} c_{M^2 i0}, \quad (\text{B.85})$$

$$c_{Zi2} = c_{ZiM} c_{M^2 i1}, \quad (\text{B.86})$$

$$c_{ZiM} = -\frac{1}{\gamma \text{Ma}_i^4} \frac{F_i}{p}. \quad (\text{B.87})$$

The first derivative of the energy term Y_i is given by:

$$\frac{dY_i}{dx} = c_{Yi0} \frac{dQ_i}{dx} + c_{Yi1} \frac{d \ln(p)}{dx} + c_{Yi2}, \quad (\text{B.88})$$

where

$$c_{Yi0} = -\frac{\gamma-1}{\gamma} \frac{1}{p u_i}, \quad (\text{B.89})$$

$$c_{Yi1} = -Y_i + c_{Yiu}c_{ui0} , \quad (\text{B.90})$$

$$c_{Yi2} = c_{Yiu}c_{ui1} , \quad (\text{B.91})$$

$$c_{Yiu} = \frac{\gamma - 1}{\gamma} \frac{Q_i}{pu_i^2} . \quad (\text{B.92})$$

The first derivative of the numerator becomes:

$$\frac{dN}{dx} = c_{Np} \frac{d \ln(p)}{dx} + \sum_{i \in [p,s]} c_{Zi0} \frac{dF_i}{dx} + \sum_{i \in [p,s]} c_{Yi0} \frac{dQ_i}{dx} + c_{N2} , \quad (\text{B.93})$$

where

$$c_{Np} = \sum_{i \in [p,s]} c_{Zi1} + \sum_{i \in [p,s]} c_{Yi1} , \quad (\text{B.94})$$

$$c_{N2} = \frac{d^2 A}{dx^2} + \sum_{i \in [p,s]} c_{Zi2} + \sum_{i \in [p,s]} c_{Yi2} . \quad (\text{B.95})$$

As in 2.1, we can decompose the gradients of the closure terms as follows:

$$\frac{dF_i}{dx} = c_{Fi0} \frac{d \ln(p)}{dx} + c_{Fi1} , \quad (\text{B.96})$$

$$\frac{dQ_i}{dx} = c_{Qi0} \frac{d \ln(p)}{dx} + c_{Qi1} . \quad (\text{B.97})$$

Introducing these decompositions into the derivative of the numerator gives:

$$\frac{dN}{dx} = c_{N0} \frac{d \ln(p)}{dx} + c_{N1} , \quad (\text{B.98})$$

where

$$c_{N0} = c_{Np} + \sum_{i \in [p,s]} (c_{Zi0}c_{Fi0}) + \sum_{i \in [p,s]} (c_{Yi0}c_{Qi0}) , \quad (\text{B.99})$$

$$c_{N1} = c_{N2} + \sum_{i \in [p,s]} (c_{Zi0}c_{Fi1}) + \sum_{i \in [p,s]} (c_{Yi0}c_{Qi1}) . \quad (\text{B.100})$$

The first derivative of β is given by the following equation:

$$\frac{d\beta}{dx} = c_{\beta 0} \frac{d \ln(p)}{dx} + c_{\beta 1} , \quad (\text{B.101})$$

where

$$c_{\beta 0} = \sum_{i \in [p,s]} c_{\beta i A} c_{Ai0} + \sum_{i \in [p,s]} c_{\beta i M} c_{M^2 i 0}, \quad (\text{B.102})$$

$$c_{\beta 1} = \sum_{i \in [p,s]} c_{\beta i A} c_{Ai1} + \sum_{i \in [p,s]} c_{\beta i M} c_{M^2 i 1}, \quad (\text{B.103})$$

$$c_{\beta i A} = \frac{1}{\gamma} \left(\frac{1}{\text{Ma}_i^2} - 1 \right), \quad (\text{B.104})$$

$$c_{\beta i M} = -\frac{A_i}{\gamma \text{Ma}_i^4}. \quad (\text{B.105})$$

The pressure gradient in the sonic section follows from the following quadratic equation:

$$c_{\beta 0}^* \left(\left(\frac{d \ln(p)}{dx} \right)^* \right)^2 + (c_{\beta 1}^* - c_{N0}^*) \left(\frac{d \ln(p)}{dx} \right)^* - c_{N1}^* = 0. \quad (\text{B.106})$$

2.2.3 Second order approximation

The second order approximation for the compound equations shares the same form as equation (B.26). The second derivatives of the numerator and the denominator of equation (B.58) are computed in this section. We first calculate the derivatives of the coefficients related to the force terms.

$$\frac{dc_{Zi0}}{dx} = -c_{Zi0} \frac{d \ln(p)}{dx} - \frac{1}{\gamma p \text{Ma}_i^4} \frac{d \text{Ma}_i^2}{dx}, \quad (\text{B.107})$$

$$\frac{dc_{Zi1}}{dx} = -\frac{dZ_i}{dx} + c_{M^2 i 0} \frac{dc_{ZiM}}{dx} + c_{ZiM} \frac{dc_{M^2 i 0}}{dx}, \quad (\text{B.108})$$

$$\frac{dc_{Zi2}}{dx} = c_{M^2 i 1} \frac{dc_{ZiM}}{dx} + c_{ZiM} \frac{dc_{M^2 i 1}}{dx}, \quad (\text{B.109})$$

$$\frac{dc_{ZiM}}{dx} = \frac{c_{ZiM}}{F_i} \frac{dF_i}{dx} - \frac{c_{ZiM}}{\text{Ma}_i^2} \frac{d \text{Ma}_i^2}{dx} - c_{ZiM} \frac{d \ln(p)}{dx}. \quad (\text{B.110})$$

The second derivative of the force term is given by:

$$\frac{d^2 Z_i}{dx^2} = c_{Zi0} \frac{d^2 F_i}{dx^2} + c_{Zi1} \frac{d^2 \ln(p)}{dx^2} + c_{Zi22}, \quad (\text{B.111})$$

where

$$c_{Zi22} = \frac{dc_{Zi0}}{dx} \frac{dF_i}{dx} + \frac{dc_{Zi1}}{dx} \frac{d \ln(p)}{dx} + \frac{dc_{Zi2}}{dx}. \quad (\text{B.112})$$

The derivatives of the coefficients related to the energy term read:

$$\frac{dc_{Yi0}}{dx} = -c_{Yi0} \left(\frac{d \ln(p)}{dx} + \frac{1}{u_i} \frac{du_i}{dx} \right), \quad (\text{B.113})$$

$$\frac{dc_{Yi1}}{dx} = -\frac{dY_i}{dx} + c_{ui0} \frac{dc_{Yiu}}{dx} + c_{Yiu} \frac{dc_{ui0}}{dx}, \quad (\text{B.114})$$

$$\frac{dc_{Yi2}}{dx} = c_{ui1} \frac{dc_{Yiu}}{dx} + c_{Yiu} \frac{dc_{ui1}}{dx}, \quad (\text{B.115})$$

$$\frac{dc_{Yiu}}{dx} = \frac{c_{Yiu}}{Q_i} \frac{dQ_i}{dx} - \frac{2c_{Yiu}}{u_i} \frac{du_i}{dx} - c_{Yiu} \frac{d \ln(p)}{dx}. \quad (\text{B.116})$$

The second derivative of the energy term is given by:

$$\frac{d^2 Y_i}{dx^2} = c_{Yi0} \frac{d^2 Q_i}{dx^2} + c_{Yi1} \frac{d^2 \ln(p)}{dx^2} + c_{Yi22}, \quad (\text{B.117})$$

where

$$c_{Yi22} = \frac{dc_{Yi0}}{dx} \frac{dQ_i}{dx} + \frac{dc_{Yi1}}{dx} \frac{d \ln(p)}{dx} + \frac{dc_{Yi2}}{dx}. \quad (\text{B.118})$$

The derivatives of the coefficients related to the numerator are given by:

$$\frac{dc_{Np}}{dx} = \sum_{i \in [p,s]} \frac{dc_{Zi1}}{dx} + \sum_{i \in [p,s]} \frac{dc_{Yi1}}{dx}, \quad (\text{B.119})$$

$$\frac{dc_{N2}}{dx} = \frac{d^3 A}{dx^3} + \sum_{i \in [p,s]} \frac{dc_{Zi2}}{dx} + \sum_{i \in [p,s]} \frac{dc_{Yi2}}{dx}. \quad (\text{B.120})$$

The second derivative of the numerator becomes:

$$\frac{d^2 N}{dx^2} = c_{Np} \frac{d^2 \ln(p)}{dx^2} + \sum_{i \in [p,s]} c_{Zi0} \frac{d^2 F_i}{dx^2} + \sum_{i \in [p,s]} c_{Yi0} \frac{d^2 Q_i}{dx^2} + c_{N22}, \quad (\text{B.121})$$

where

$$c_{N22} = \frac{dc_{Np}}{dx} \frac{d \ln(p)}{dx} + \sum_{i \in [p,s]} \frac{dc_{Zi0}}{dx} \frac{dF_i}{dx} + \sum_{i \in [p,s]} \frac{dc_{Yi0}}{dx} \frac{dQ_i}{dx} + \frac{dc_{N2}}{dx}. \quad (\text{B.122})$$

Differentiating the decompositions (B.96) and (B.97):

$$\frac{d^2 F_i}{dx^2} = c_{Fi0} \frac{d}{dx} \frac{d \ln(p)}{dx} + c_{Fi21}, \quad (\text{B.123})$$

$$\frac{d^2 Q_i}{dx^2} = c_{Qi0} \frac{d}{dx} \frac{d \ln(p)}{dx} + c_{Qi21}, \quad (\text{B.124})$$

where

$$c_{Fi21} = \frac{dc_{Fi0}}{dx} \frac{d \ln(p)}{dx} + \frac{dc_{Fi1}}{dx}, \quad (\text{B.125})$$

$$c_{Qi21} = \frac{dc_{Qi0}}{dx} \frac{d \ln(p)}{dx} + \frac{dc_{Qi1}}{dx}. \quad (\text{B.126})$$

Introducing these decompositions into (B.121) gives:

$$\frac{d^2 N}{dx^2} = c_{N20} \frac{d^2 \ln(p)}{dx^2} + c_{N21}, \quad (\text{B.127})$$

where

$$c_{N20} = c_{Np} + \sum_{i \in [p,s]} c_{Zi0} c_{Fi20} + \sum_{i \in [p,s]} c_{Yi0} c_{Qi20}, \quad (\text{B.128})$$

$$c_{N21} = c_{N22} + \sum_{i \in [p,s]} c_{Zi0} c_{Fi21} + \sum_{i \in [p,s]} c_{Yi0} c_{Qi21}. \quad (\text{B.129})$$

The derivatives of the coefficients related to β are given by:

$$\begin{aligned} \frac{dc_{\beta 0}}{dx} = \sum_{i \in [p,s]} & \left(c_{Ai0} \frac{dc_{\beta iA}}{dx} + c_{\beta iA} \frac{dc_{Ai0}}{dx} \right. \\ & \left. + c_{M^2 i0} \frac{dc_{\beta iM}}{dx} + c_{\beta iM} \frac{dc_{M^2 i0}}{dx} \right), \end{aligned} \quad (\text{B.130})$$

$$\begin{aligned} \frac{dc_{\beta 1}}{dx} = \sum_{i \in [p,s]} & \left(c_{Ai1} \frac{dc_{\beta iA}}{dx} + c_{\beta iA} \frac{dc_{Ai1}}{dx} \right. \\ & \left. + c_{M^2 i1} \frac{dc_{\beta iM}}{dx} + c_{\beta iM} \frac{dc_{M^2 i1}}{dx} \right), \end{aligned} \quad (\text{B.131})$$

where

$$\frac{dc_{\beta iA}}{dx} = \frac{-1}{\gamma \text{Ma}_i^4} \frac{d\text{Ma}_i^2}{dx}, \quad (\text{B.132})$$

$$\frac{dc_{\beta iM}}{dx} = -\frac{1}{\gamma \text{Ma}_i^4} \frac{dA_i}{dx} + \frac{2A_i}{\gamma \text{Ma}_i^6} \frac{d\text{Ma}_i^2}{dx}. \quad (\text{B.133})$$

The second derivative of the compound choking indicator reads:

$$\frac{d^2\beta}{dx^2} = c_{\beta 0} \frac{d^2 \ln(p)}{dx^2} + c_{\beta 21}, \quad (\text{B.134})$$

where

$$c_{\beta 21} = \frac{dc_{\beta 0}}{dx} \left(\frac{d \ln(p)}{dx} \right) + \frac{dc_{\beta 1}}{dx}. \quad (\text{B.135})$$

The second derivative of the static pressure in the sonic section is given by:

$$\left(\frac{d^2 \ln(p)}{dx^2} \right)^* = \frac{-c_{N21}^* + \left(\frac{d \ln(p)}{dx} \right)^* c_{\beta 21}^*}{c_{N20}^* - \left(\frac{d \ln(p)}{dx} \right)^* c_{\beta 0}^* - 2 \left(\frac{d\beta}{dx} \right)^*}, \quad (\text{B.136})$$

2.2.4 Derivatives of the friction forces

The derivatives of the wall friction have already been described in section 2.1.4 and are also valid for the compound equations, using the flow variables from the secondary stream and the geometry of the mixing pipe. Below, we provide the derivatives of the inter-stream friction (see 6.17).

The inter-stream friction force explicitly depends on densities, velocities and the perimeter l_{ps} of the primary stream. The vector $\mathbf{q} = [\rho_p, \rho_s, u_p, u_s]$ is introduced to ease the notation. The friction coefficient is not differentiated for simplicity. The first derivative of the friction between the streams is given by the following equation:

$$\frac{dF_{ps}}{dx} = \frac{\partial F_{ps}}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dx} + \frac{\partial F_{ps}}{\partial l_{ps}} \frac{dl_{ps}}{dx}, \quad (\text{B.137})$$

where the dot operator indicates an inner product. The equations below are derived for the case where $u_p > u_s$ to avoid the absolute value operator. The derivatives in the case $u_p < u_s$ have the same magnitude, but only differ in the sign. It thus suffices to add a minus sign in the equation above.

The partial derivatives of F_{ps} with respect to $\mathbf{q}$ are given by:

$$\frac{\partial F_{ps}}{\partial \mathbf{q}} = \begin{pmatrix} \frac{f_{ps} l_{ps}}{4} (u_p - u_s)^2 \\ \frac{f_{ps} l_{ps}}{4} (u_p - u_s)^2 \\ \frac{f_{ps} l_{ps}}{2} (\rho_p + \rho_s) (u_p - u_s) \\ -\frac{f_{ps} l_{ps}}{2} (\rho_p + \rho_s) (u_p - u_s) \end{pmatrix}. \quad (\text{B.138})$$

The derivative $d\mathbf{q}/dx$ follows from the governing equations which depend directly on the static pressure:

$$\frac{d\mathbf{q}}{dx} = \mathbf{c}_{q0} \frac{d \ln(p)}{dx} + \mathbf{c}_{q1}, \quad (\text{B.139})$$

where $\mathbf{c}_{q0}$ is a vector of the same length as $\mathbf{q}$, and itself a function of $\mathbf{q}$:

$$\mathbf{c}_{q0} = \begin{pmatrix} c_{\rho p0} \\ c_{\rho s0} \\ c_{u p0} \\ c_{u s0} \end{pmatrix} \quad \text{and} \quad \mathbf{c}_{q1} = \begin{pmatrix} c_{\rho p1} \\ c_{\rho s1} \\ c_{u p1} \\ c_{u s1} \end{pmatrix}. \quad (\text{B.140})$$

These coefficients have been defined in equations (B.62)-(B.71). The partial derivative of the inter-stream friction with respect to l_{ps} equals:

$$c_{F_{ps} l_{ps}} = \frac{\partial F_{ps}}{\partial l_{ps}} = \frac{1}{2} f_{ps} \frac{\rho_p + \rho_s}{2} (u_p - u_s)^2, \quad (\text{B.141})$$

The treatment of l_{ps} depends on whether the geometry is axisymmetric or planar. In the planar case, l_{ps} corresponds to the depth of the channel, which is constant, so its spatial derivative is zero. Otherwise:

$$l_{ps} = 2\pi r_p = 2\sqrt{\pi} \sqrt{\pi r_p^2} = 2\sqrt{\pi} \sqrt{A_p}. \quad (\text{B.142})$$

Its derivative equals:

$$\frac{dl_{ps}}{dx} = c_{l_{ps}0} \frac{d \ln(p)}{dx} + c_{l_{ps}1}, \quad (\text{B.143})$$

where

$$c_{lps0} = \sqrt{\frac{\pi}{A_p}} c_{Ap0}, \quad \text{and} \quad c_{lps1} = \sqrt{\frac{\pi}{A_p}} c_{Ap1}. \quad (\text{B.144})$$

This leads to the final equation for the gradient of the inter-stream friction:

$$\frac{dF_{ps}}{dx} = c_{Fps0} \frac{d \ln(p)}{dx} + c_{Fps1}. \quad (\text{B.145})$$

where

$$c_{Fps0} = \frac{\partial F_{ps}}{\partial \mathbf{q}} \cdot \mathbf{c}_{q0} + c_{Fpslps} c_{lps0}, \quad (\text{B.146})$$

$$c_{Fps1} = \frac{\partial F_{ps}}{\partial \mathbf{q}} \cdot \mathbf{c}_{q1} + c_{Fpslps} c_{lps1} \quad (\text{B.147})$$

The second derivative of the inter-stream friction force is given by:

$$\frac{d^2 F_{ps}}{dx^2} = c_{Fps0} \frac{d^2 \ln(p)}{dx^2} + \frac{dc_{Fps0}}{dx} \frac{d \ln(p)}{dx} + \frac{dc_{Fps1}}{dx}. \quad (\text{B.148})$$

We compute the derivatives of the coefficients c_{Fps0} and c_{Fps1} .

$$\frac{dc_{Fps0}}{dx} = \left(\frac{\partial^2 F_{ps}}{\partial \mathbf{q}^2} \frac{d\mathbf{q}}{dx} \right) \cdot \mathbf{c}_{q0} + \frac{\partial F_{ps}}{\partial \mathbf{q}} \cdot \frac{d\mathbf{c}_{q0}}{dx} \quad (\text{B.149})$$

$$+ c_{lps0} \frac{dc_{Fpslps}}{dx} + c_{Fpslps} \frac{dc_{lps0}}{dx}. \quad (\text{B.150})$$

The second derivative of F_{ps} with respect to $\mathbf{q}$ is a symmetrical matrix:

$$\frac{\partial^2 F_{ps}}{\partial \mathbf{q}^2} = \frac{f_{pslps}}{2} \begin{pmatrix} 0 & 0 & \Delta u & -\Delta u \\ 0 & 0 & \Delta u & -\Delta u \\ \Delta u & \Delta u & \bar{\rho} & -\bar{\rho} \\ -\Delta u & -\Delta u & -\bar{\rho} & \bar{\rho} \end{pmatrix}, \quad (\text{B.151})$$

where $\Delta u = u_p - u_s$ and $\bar{\rho} = \rho_p + \rho_s$. The derivatives of c_{q0} and c_{q1} read:

$$\frac{dc_{q0}}{dx} = \begin{pmatrix} \frac{dc_{pp0}}{dx} \\ \frac{dc_{ps0}}{dx} \\ \frac{dc_{up0}}{dx} \\ \frac{dc_{us0}}{dx} \end{pmatrix}, \quad \text{and} \quad c_{q1} = \begin{pmatrix} \frac{dc_{pp1}}{dx} \\ \frac{dc_{ps1}}{dx} \\ \frac{dc_{up1}}{dx} \\ \frac{dc_{us1}}{dx} \end{pmatrix}, \quad (\text{B.152})$$

for which the definitions are given by equations (B.73)-(B.82). The derivative of c_{Fpslps} equals the derivative of F_{ps}/l_{ps} :

$$\frac{dc_{Fpslps}}{dx} = \frac{\partial F_{ps}}{\partial \mathbf{q}} \cdot c_{q0} \frac{d \ln(p)}{dx} \frac{1}{l_{ps}}. \quad (\text{B.153})$$

The derivatives of the coefficients c_{lps0} and c_{lps1} equal:

$$\frac{dc_{lps0}}{dx} = c_{lps0} \left(\frac{1}{c_{Ap0}} \frac{dc_{Ap0}}{dx} - \frac{1}{2} \frac{1}{A_p} \frac{dA_p}{dx} \right), \quad (\text{B.154})$$

$$\frac{dc_{lps1}}{dx} = c_{lps1} \left(\frac{1}{c_{Ap1}} \frac{dc_{Ap1}}{dx} - \frac{1}{2} \frac{1}{A_p} \frac{dA_p}{dx} \right). \quad (\text{B.155})$$

The derivative of c_{Fps1} equals:

$$\frac{dc_{Fps1}}{dx} = \left(\frac{\partial^2 F_{ps}}{\partial \mathbf{q}^2} \frac{d\mathbf{q}}{dx} \right) \cdot c_{q1} + \frac{\partial F_{ps}}{\partial \mathbf{q}} \cdot \frac{dc_{q1}}{dx} \quad (\text{B.156})$$

$$+ c_{lps1} \frac{dc_{Fpslps}}{dx} + c_{Fpslps} \frac{dc_{lps1}}{dx}, \quad (\text{B.157})$$

for which we already computed all terms. In summary:

$$\frac{d^2 F_{ps}}{dx^2} = c_{Fps0} \frac{d^2 \ln(p)}{dx^2} + c_{Fps21}, \quad (\text{B.158})$$

where

$$c_{Fps21} = \frac{dc_{Fps0}}{dx} \frac{d \ln(p)}{dx} + \frac{dc_{Fps1}}{dx}. \quad (\text{B.159})$$

Given the definition of the primary and the secondary forces in (6.19), the related coefficients are defined as follows:

$$c_{Fp0} = -c_{Fps0}, \quad (\text{B.160})$$

$$c_{Fp1} = -c_{Fps1} , \quad (\text{B.161})$$

$$c_{Fs0} = c_{Fps0} - c_{Fw0} , \quad (\text{B.162})$$

$$c_{Fs1} = c_{Fps1} - c_{Fw1} . \quad (\text{B.163})$$

Their derivatives follow the same structure:

$$\frac{dc_{Fp0}}{dx} = -\frac{dc_{Fps0}}{dx} , \quad (\text{B.164})$$

$$\frac{dc_{Fp1}}{dx} = -\frac{dc_{Fps1}}{dx} , \quad (\text{B.165})$$

$$\frac{dc_{Fs0}}{dx} = \frac{dc_{Fps0}}{dx} - \frac{dc_{Fw0}}{dx} , \quad (\text{B.166})$$

$$\frac{dc_{Fs1}}{dx} = \frac{dc_{Fps1}}{dx} - \frac{dc_{Fw1}}{dx} . \quad (\text{B.167})$$

All terms related to energy exchange are zero in this work as discussed in section 7.2.



Compound shocks

General remarks

In this appendix, we develop a model for a compound shock, i.e., a shock across two parallel streams which are compound-supersonic, with the constraint of uniform static pressure up- and downstream of the shock. If the primary and secondary states are equal, the solution should match the classic 1D shock equations; otherwise, the solution is less evident since an oblique shock train usually appears. Assimilating this 2D phenomenon into a 1D model is the main challenge tackled here.

The classic solution is not a valid option for the case with different Mach numbers due to the static pressure constraint. It forms the base of the compound equations (chapter 6), so this condition should be respected across the shock in order to use those equations downstream. Separate normal shocks in each stream would lead to a pressure difference downstream due to the different shock strengths. Moreover, a flow can be compound supersonic with a subsonic component. An alternative solution is sought after in the following section.

Derivation of the model

Firstly, the unknowns are defined. The state of a stream consists of 4 quantities: its cross-section and three independent flow quantities, e.g., a static pressure p , a total temperature T_t and a Mach number M . The state upstream is considered to be known, as the shock model should work for any

compound-supersonic state. The state downstream of the shock is fully unknown, leaving also the cross-sections free since imposing the same cross-sections as the upstream state may prove overly restrictive. A normal compound shock is by definition an artificial construct, as the flow compresses through a complex 2D shock train in reality. The following 8 unknowns are identified:

$$[A_{pR}, A_{sR}, p_{pR}, p_{sR}, T_{t,pR}, T_{t,sR}, Ma_{pR}, Ma_{sR}] . \quad (C.1)$$

These unknowns are now reduced to the following two non-dimensional quantities:

$$\Pi_p = \frac{p_R}{p_L}, \quad \text{and} \quad \Pi_A = \frac{A_{pR}}{A}, \quad (C.2)$$

where $A = A_{sL} + A_{sR}$ denotes the total cross-section. The other unknowns follow from the constraints and the conservation equations. The static pressures and cross-sections follow directly from the constraints of the compound theory:

$$p_{pR} = p_{sR} = p_R = p_L \Pi_p . \quad (C.3)$$

$$A_{pR} = A \Pi_A, \quad \text{and} \quad A_{sR} = A - A_{pR} . \quad (C.4)$$

The total temperatures of the streams are conserved:

$$T_{t,pR} = T_{t,pL} \quad \text{and} \quad T_{t,sR} = T_{t,sL} . \quad (C.5)$$

The Mach numbers in each stream are the only remaining unknowns, and can be found from the conservation of mass in each stream $i \in [p, s]$:

$$\dot{m}_i = p_{Ri} A_{Ri} Ma_{Ri} \sqrt{\frac{\gamma}{R_{gas} T_{t,Ri}}} \sqrt{1 + \frac{\gamma - 1}{2} Ma_{Ri}^2} . \quad (C.6)$$

Two equations other than the constraints, conservation of mass and conservation of energy thus remain to be defined to solve for the two fundamental unknowns $[\Pi_p, \Pi_A]$.

Conservation of momentum of the compound stream provides a first equation. Applying the conservation law on the separate streams proved to be constrictive and did not yield a solution. The impulse function $\mathcal{M}$ is introduced first to ease the notation:

$$\mathcal{M} = \dot{m}u + pA . \quad (C.7)$$

The momentum balance across the shock reads:

$$\mathcal{M}_{pL} + \mathcal{M}_{sL} = \mathcal{M}_{pR} + \mathcal{M}_{sR}. \quad (\text{C.8})$$

The equations above a priori admit multiple solutions, but not all are physically admissible. From the state up- and downstream of the shock, we can calculate the entropy difference as follows:

$$\Delta s_i = s_{iR} - s_{iL} = c_v \ln \left(\frac{p_{iR}/\rho_{iR}^\gamma}{p_{iL}/\rho_{iL}^\gamma} \right) > 0. \quad (\text{C.9})$$

However, these constraints do not further restrict the problem to a unique solution. Therefore, the classic normal shock relation is applied on the equivalent Mach number:

$$\tilde{M}_{eq,R}^2 = \frac{(\gamma - 1)M_{eq,L}^2 + 2}{2\gamma M_{eq,L}^2 - (\gamma - 1)}, \quad (\text{C.10})$$

where the induces L and R denote the state upstream ('left') and downstream ('right') of the shock. The tilde highlights the fact that the Mach number is computed through the shock equation and not from its definition in equation (6.14).

Example

For illustration, we look at the following case: $M_{pL} = 2$, $M_{sL} = 1.5$, $p_L = 1$ bar, $T_{t,pL} = T_{t,sL} = 300$ K and $A_{pL} = A_{sL} = 0.005$ m². The residuals on the equations are computed as relative squared errors:

$$\mathcal{R}_{mom} = \frac{\left(\mathcal{M}_{pR} + \mathcal{M}_{sR} - \mathcal{M}_{pL} - \mathcal{M}_{sL} \right)^2}{\left(\mathcal{M}_{pL} + \mathcal{M}_{sL} \right)^2}, \quad (\text{C.11})$$

$$\mathcal{R}_{Meq} = \frac{\left(M_{eq,R} - \tilde{M}_{eq,R} \right)^2}{\tilde{M}_{eq,R}^2}. \quad (\text{C.12})$$

Figure C.1a displays the sum of the residuals $\mathcal{R}_{mom}$ and $\mathcal{R}_{Meq}$ in function of the pressure ratio Π_p and area ratio Π_A . The solid lines indicate the loci at which $\Delta s_i = 0$, hence the boundaries of the admissible solutions.

Figures C.1b and C.1c display the numerical values of these entropy differences. Positive values are found in the region enclosed by the two solid lines in figure C.1a. Another criterion could be to look at the global entropy balance as weighed by the mass flow rates:

$$\dot{m}_p s_{pR} + \dot{m}_s s_{sR} \geq \dot{m}_p s_{pL} + \dot{m}_s s_{sL} \quad (\text{C.13})$$

This global constraint is shown on figure C.1d. The global balance is less restrictive than the union of the two individual constraints. Although the global formulation looks attractive, it has been found to allow an increase in total pressure in one of the streams, with a corresponding decrease in entropy. Therefore, we propose to keep the more strict separate constraints as indicated on figure C.1a.

The residuals are minimised from an initial guess $[\Pi_p, \Pi_A] = [3, 0.4]$, as shown on figure C.1a, which turns out to be close to the minimum indicated with the cross. The summed residuals are however non-zero and correspond to a relative error of 2.5 %.

Figure C.2 shows the contributions of both residuals on the total cost in figure C.1a. It turns out that the equivalent Mach number dominates the cost function by one order of magnitude with respect to the momentum (which is also reflected in the final residuals). Using the classic shock relation with the equivalent Mach number is therefore too restrictive. In fact, the minimum of the cost related to the equivalent Mach number is located outside of the admissible region based on the entropy.

Instead, using only the global momentum conservation equation (C.8) allows for multiple solutions, located on a circular shape in figure C.2a. This shape crosses the admissible region of solutions (including the one previously found). This implies multiple solutions exist, with different pressure and area ratios. Note that the original area ratio 0.5 lies outside of this region, so constraining the cross-sections to remain identical is also too restrictive.

Closer inspection of the solutions revealed that the admissible solutions present a different distribution of the pressure losses between both streams. In particular, the solutions at the extremities ($\Delta s_i = 0$) place the loss in one of the streams and the total pressure is conserved in the other. This observation has finally led to the modelling choice in section 7.3.4, where the same total pressure is imposed downstream to emulate perfect mixing.

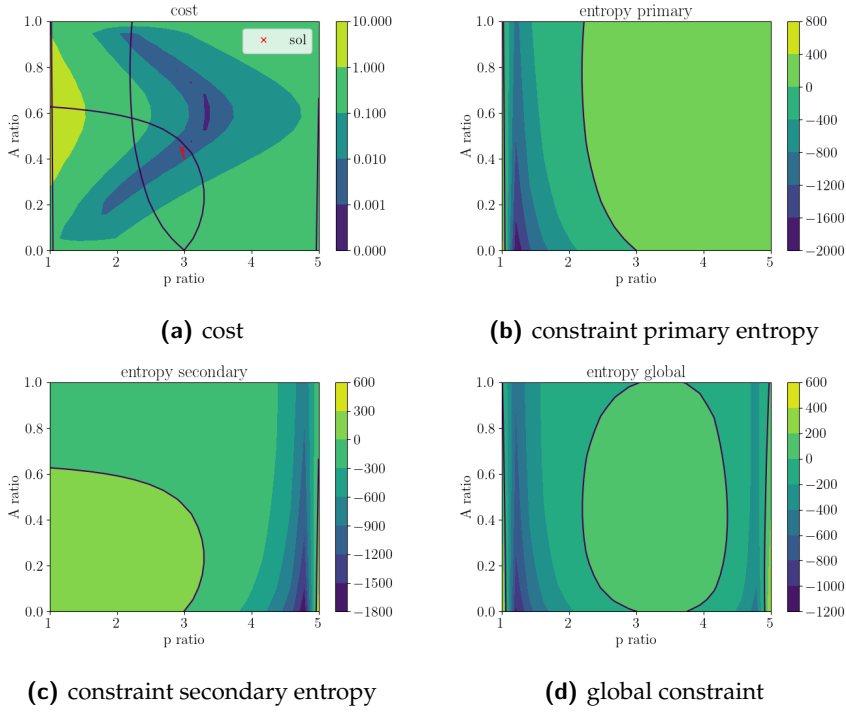


Fig. C.1 Residuals and entropy constraints

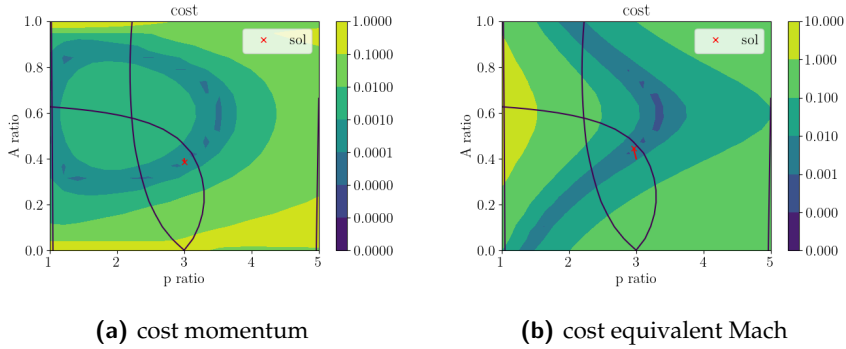


Fig. C.2 Separate contributions to the cost in figure C.1a by the residuals on the momentum (left) and the equivalent Mach number (right).



Results on the VKI ejector

This appendix presents the uncertainty analysis of the experiments on the VKI ejector (see chapter 2) and the corresponding predictions of the double stream model from chapter 7.

Uncertainties

The absolute uncertainty on the static pressures at the inlets and outlets equals $1.4 \cdot 10^3$ Pa (0.2 % of the full range of the Validyne sensor). The thermocouples are deemed accurate up to 1 K and the uncertainty on the mass flow rates equals 1.5 % of the full range, which means 0.026 kg/s for the flow meter with the largest range (used in most of the experiments and always for the primary flow) and 0.007 kg/s for the most sensitive flow meter. Finally, the pressure taps have an uncertainty of $3.5 \cdot 10^3$ Pa (0.5 % of the full range).

Uncertainties are denoted here by δ followed by the quantity of interest. The uncertainty on a quantity $q_{dep} \pm \delta q_{dep}$ that is computed from other quantities $q_i \pm \delta q_i$ is defined as follows:

$$\delta q_{dep} = \sqrt{\sum_i \left(\frac{\partial q_{dep}}{\partial q_i} \delta q_i \right)^2}. \quad (D.1)$$

The equations for the uncertainty on the Mach numbers, the total temper-

atures, the total pressures are given below for completeness.

$$\delta \text{Ma}_i = \sqrt{\left(\frac{\text{Ma}_i}{\dot{m}_i} \delta \dot{m}_i\right)^2 + \left(\frac{\text{Ma}_i}{p_i} \delta p_i\right)^2 + \left(\frac{\text{Ma}_i}{2T_i} \delta T_i\right)^2}, \quad (\text{D.2})$$

$$\delta T_{t,i} = \sqrt{\left(\frac{T_{t,i}}{T_i} \delta T_i\right)^2 + (T_i (\gamma - 1) \text{Ma}_i \delta \text{Ma}_i)^2}, \quad (\text{D.3})$$

$$\delta p_{t,i} = \sqrt{\left(\frac{p_{t,i}}{p_i} \delta p_i\right)^2 + \left(p \left(1 + \frac{\gamma - 1}{2} \text{Ma}_i^2\right)^{\frac{-1}{\gamma - 1}} \gamma \text{Ma}_i \delta \text{Ma}_i\right)^2}. \quad (\text{D.4})$$

The uncertainty on the reference mass flow rates in equations (2.5) and (2.6) are given by:

$$\delta \dot{m}_{i,ref} = \sqrt{\left(\frac{\dot{m}_{i,ref}}{p_{t,i}} \delta p_{t,i}\right)^2 + \left(\frac{\dot{m}_{i,ref}}{2T_{t,i}} \delta T_{t,i}\right)^2}. \quad (\text{D.5})$$

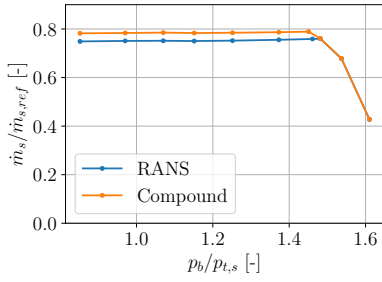
Finally, the uncertainty on the non-dimensional mass flow rates is given by the following equation:

$$\delta \hat{m}_i = \sqrt{\left(\frac{\delta \dot{m}_i}{\dot{m}_{i,ref}}\right)^2 + \left(\frac{\dot{m}_i}{\dot{m}_{i,ref}^2} \delta \dot{m}_{i,ref}\right)^2}. \quad (\text{D.6})$$

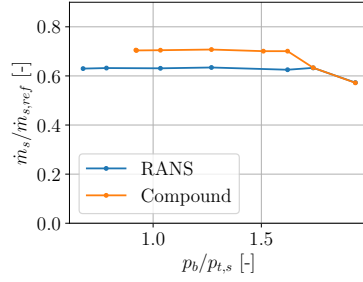
The shown uncertainties on the experimental data from the VKI ejector correspond directly to the equations above, i.e., we do not multiply by the canonical value of 1.96 to obtain the 95 % confidence interval. The uncertainties given on the measured base quantities stem directly from the specifications of the manufacturer.

Results

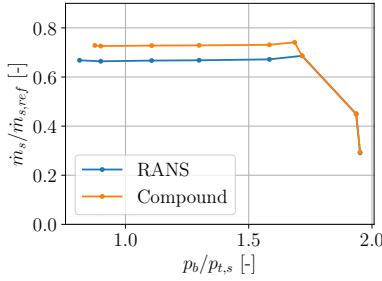
The qualitative behaviour of the model is similar to the results in sections 7.6 and 7.7, with an overestimated entrainment in on-design operation and a good match on global and local data in off-design operation through calibration. The predictions therefore do not provide new insights other than showcasing the capability of the model to operate on a different geometry. The results below are therefore not discussed in full detail.



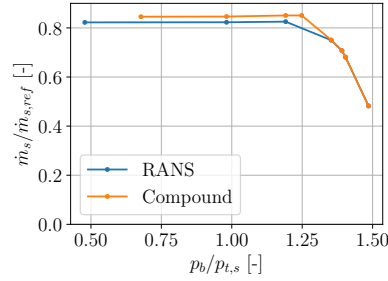
(a) Campaign 3, $p_{t,p}/p_{t,s} = 3.46$.



(b) Campaign 8, $p_{t,p}/p_{t,s} = 4.53$.



(c) Campaign 9, $p_{t,p}/p_{t,s} = 4.19$.



(d) Campaign 14, $p_{t,p}/p_{t,s} = 2.78$.

Fig. D.1 Comparison of the non-dimensional secondary mass flow rate from the RANS simulations on the VKI ejector and from the double-stream model. The left-most predictions do not match the back-pressure. In these cases, the normal shock procedure is applied at the outlet of the diffuser and the back pressure can not be matched (the fully supersonic solution would induce lower pressures).

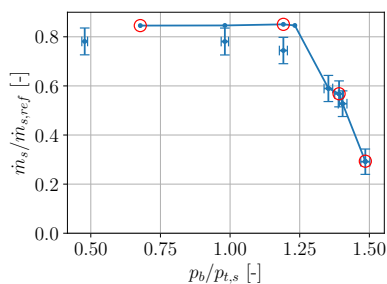
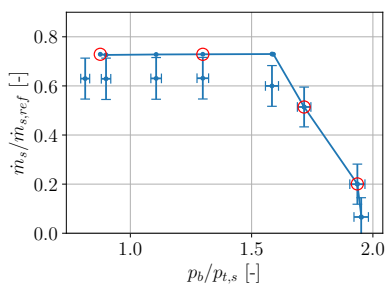
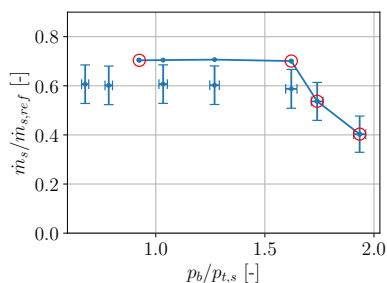
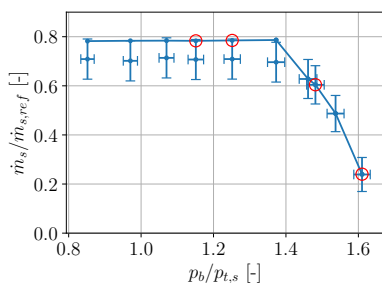


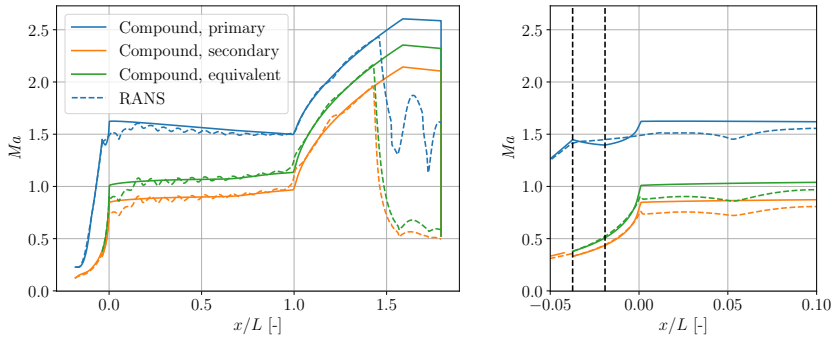
Fig. D.2 Comparison of the non-dimensional secondary mass flow rate from the experiments on the VKI ejector and from the double-stream model. The wall pressure distributions of the circled points are shown in figure D.7.

Table D.1 Overview of the calibration parameters w_1 and w_2 , the resulting secondary mass flow rates and the errors with respect to the RANS simulations on the VKI ejector. The critical point from the RANS simulations is indicated in bold.

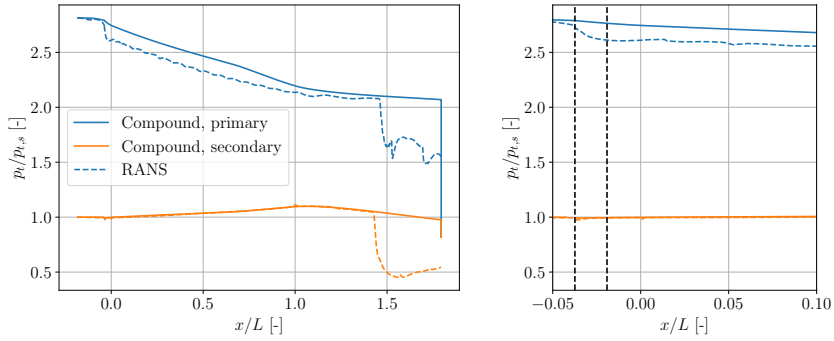
Campaign [-]	Test [-]	w_1 [-]	w_2 [-]	$\dot{m}_s/\dot{m}_{s,ref}$ (model) [-]	$\dot{m}_s/\dot{m}_{s,ref}$ (RANS) [-]	Relative error [%]
3	1	0.332	0.257	0.782	0.749	4.5
	4	0.332	0.257	0.783	0.751	4.3
	5	0.332	0.257	0.785	0.751	4.5
	6	0.332	0.257	0.783	0.750	4.4
	7	0.332	0.257	0.784	0.752	4.3
	8	0.332	0.257	0.787	0.755	4.1
	10	0.450	0.286	0.761	0.760	0.1
	12	0.534	0.198	0.678	0.678	0.1
	13	0.534	0.105	0.428	0.428	0.1
8	1	0.197	0.147	0.704	0.630	11.7
	2	0.197	0.147	0.705	0.632	11.6
	3	0.197	0.147	0.705	0.631	11.7
	4	0.197	0.147	0.707	0.634	11.5
	5	0.361	0.345	0.701	0.625	12.1
	6	0.275	0.285	0.633	0.633	0.1
	7	0.564	0.208	0.572	0.573	-0.2
9	1	0.119	0.120	0.729	0.668	9.1
	2	0.439	0.226	0.726	0.664	9.3
	3	0.439	0.226	0.728	0.667	9.2
	4	0.439	0.226	0.729	0.668	9.1
	5	0.439	0.226	0.731	0.672	8.8
	6	0.455	0.243	0.686	0.686	0.0
	7	0.388	0.000	0.449	0.450	-0.1
	8	0.346	0.027	0.294	0.292	0.8
14	1	0.314	0.281	0.846	0.822	2.8
	4	0.314	0.281	0.846	0.823	2.8
	5	0.314	0.281	0.851	0.826	3.1
	6	0.449	0.252	0.751	0.750	0.1
	8	0.517	0.227	0.707	0.707	0.0
	9	0.549	0.215	0.681	0.681	0.0
	14	0.603	0.139	0.482	0.482	0.1

Table D.2 Overview of the calibration parameters w_1 and w_2 , the resulting secondary mass flow rates and the errors with respect to the experiments on the VKI ejector.

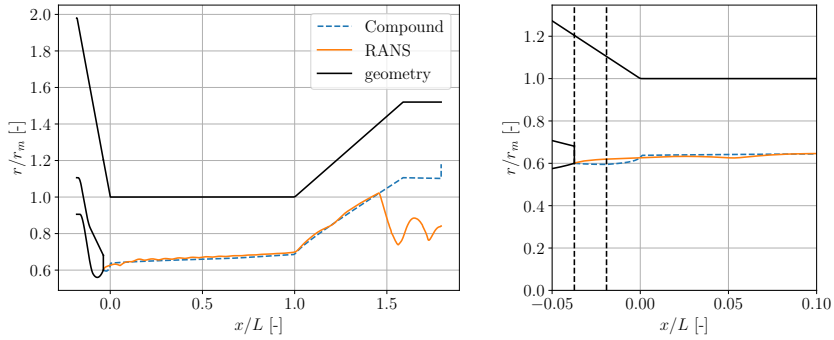
Campaign [-]	Test [-]	w_1 [-]	w_2 [-]	$\dot{m}_s / \dot{m}_{s,ref}$ (model) [-]	$\dot{m}_s / \dot{m}_{s,ref}$ (exp) [-]	Relative error [%]
3	1	0.176	0.319	0.782	0.709	10.3
	4	0.176	0.319	0.783	0.702	11.6
	5	0.176	0.319	0.784	0.714	9.8
	6	0.176	0.319	0.783	0.707	10.8
	7	0.176	0.319	0.785	0.709	10.7
	8	0.176	0.319	0.787	0.696	13.0
	9	0.151	0.256	0.628	0.628	0.1
	10	0.176	0.209	0.604	0.604	0.0
	12	0.230	0.139	0.487	0.487	0.1
	13	0.188	0.021	0.240	0.239	0.5
8	1	0.161	0.312	0.704	0.607	16.0
	2	0.161	0.312	0.705	0.602	17.2
	3	0.161	0.312	0.705	0.607	16.1
	4	0.161	0.312	0.707	0.602	17.3
	5	0.161	0.312	0.701	0.588	19.3
	6	0.139	0.176	0.537	0.537	0.1
	7	0.258	0.077	0.403	0.403	0.1
9	1	0.241	0.289	0.729	0.630	15.7
	2	0.241	0.289	0.726	0.629	15.4
	3	0.241	0.289	0.728	0.631	15.4
	4	0.241	0.289	0.729	0.631	15.4
	5	0.241	0.289	0.730	0.600	21.7
	6	0.213	0.162	0.514	0.514	0.0
	7	0.205	0.105	0.201	0.200	0.5
	8	0.197	0.181	0.066	0.066	-0.8
14	1	0.165	0.201	0.846	0.781	8.2
	4	0.165	0.201	0.846	0.781	8.4
	5	0.165	0.201	0.851	0.744	14.3
	6	0.165	0.201	0.590	0.589	0.1
	8	0.221	0.139	0.568	0.568	0.1
	9	0.212	0.103	0.528	0.528	0.0
	14	0.188	0.029	0.293	0.291	0.7



(a) Mach number

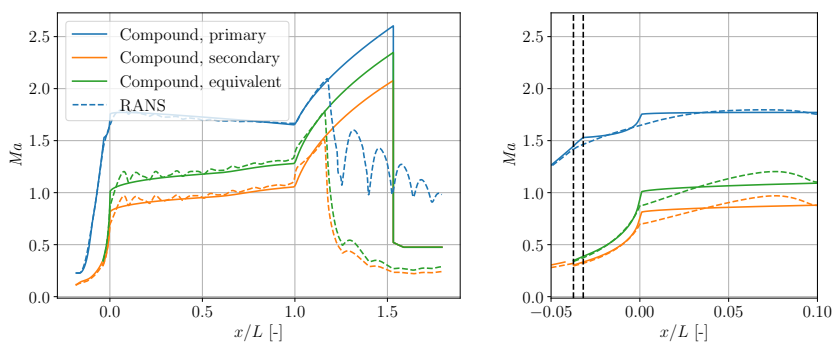


(b) Total pressure

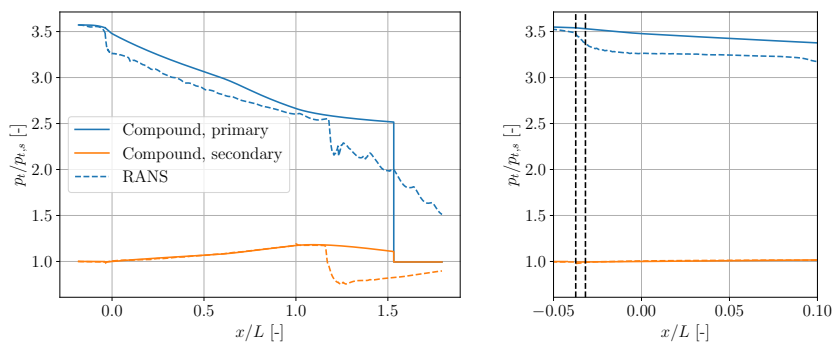


(c) Dividing streamline

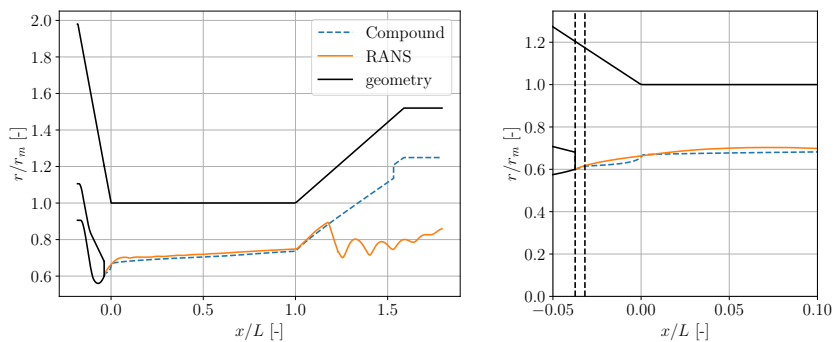
Fig. D.3 Axial distributions of the Mach numbers, total pressures and the dividing streamline for test 1 of campaign 14 on the VKI ejector.



(a) Mach number

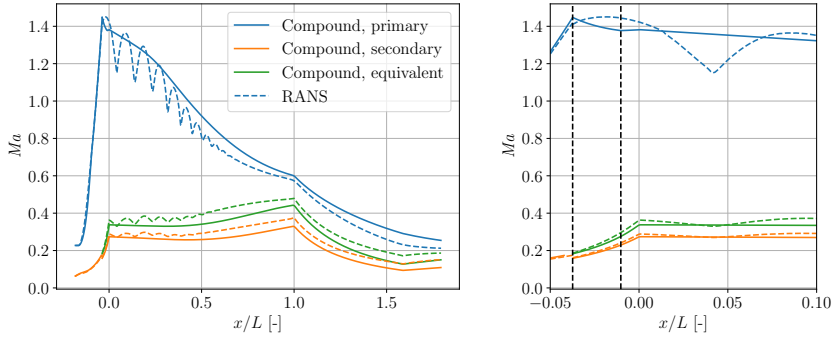


(b) Total pressure

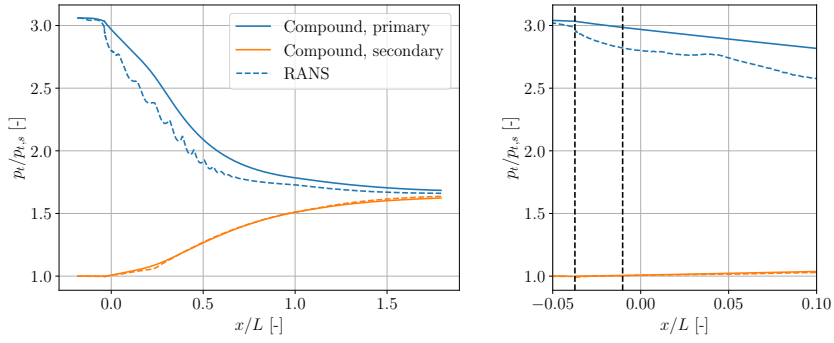


(c) Dividing streamline

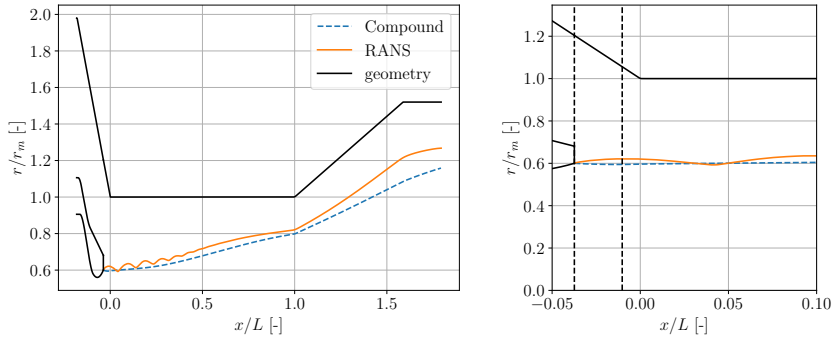
Fig. D.4 Axial distributions of the Mach numbers, total pressures and the dividing streamline for test 1 of campaign 3 on the VKI ejector.



(a) Mach number

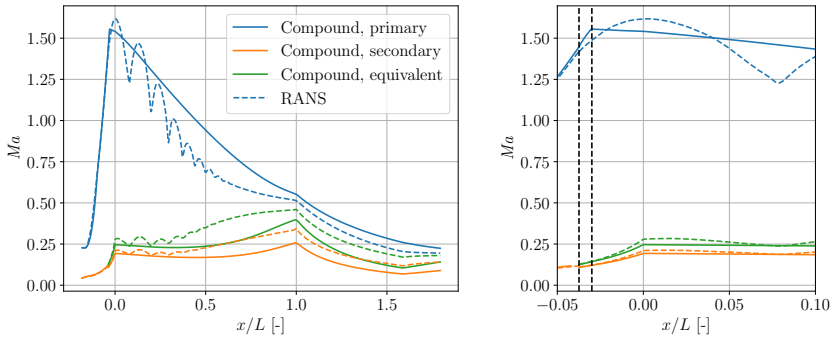


(b) Total pressure

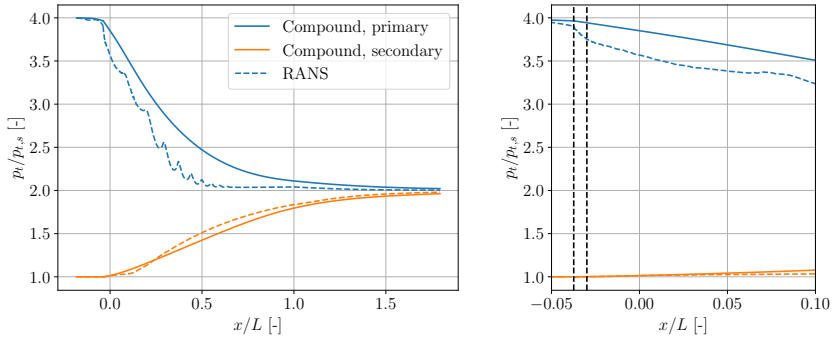


(c) Dividing streamline

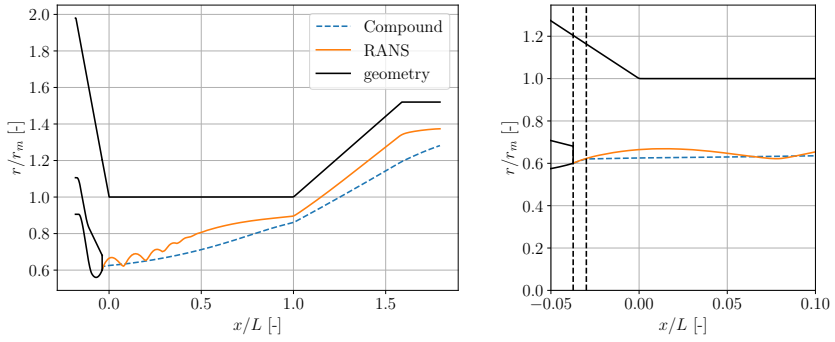
Fig. D.5 Axial distributions of the Mach numbers, total pressures and the dividing streamline for test 13 of campaign 3 on the VKI ejector.



(a) Mach number

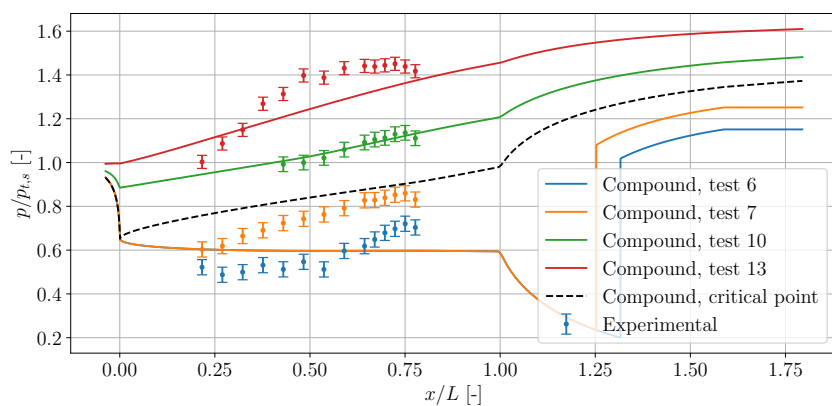


(b) Total pressure

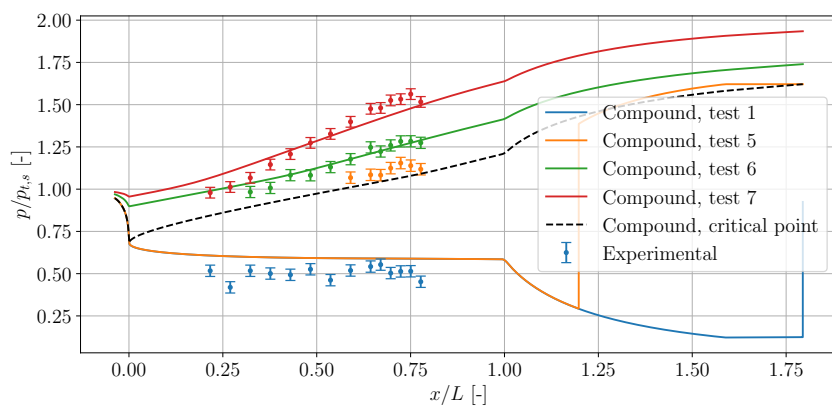


(c) Dividing streamline

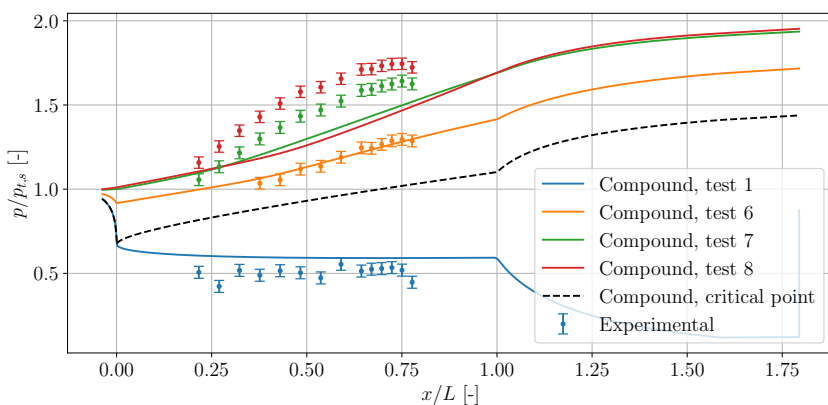
Fig. D.6 Axial distributions of the Mach numbers, total pressures and the dividing streamline for test 8 of campaign 9 on the VKI ejector.



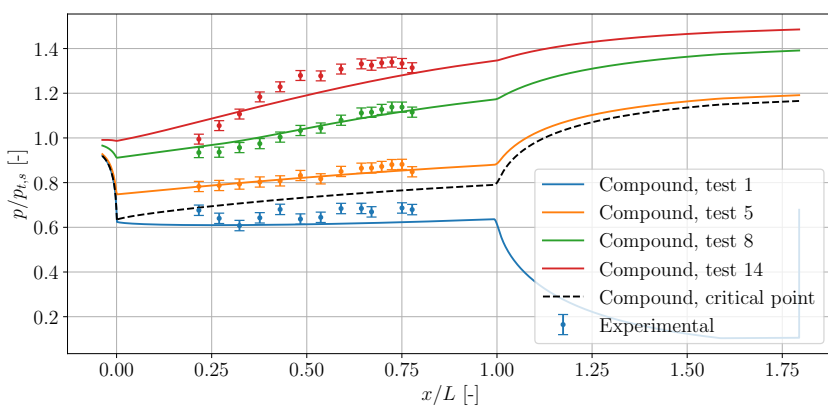
(a) Campaign 3.



(b) Campaign 8.



(c) Campaign 9.



(d) Campaign 14.

Fig. D.7 Experimental and predicted wall pressure profiles for the VKI ejector.

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