ASYMMETRIC VOLATILITY IMPULSE RESPONSE FUNCTIONS

Christian Hafner, Helmut Herwartz

LIDAM Discussion Paper ISBA
2022 / 37

UCLouvain

LiDAM
ISBA | Louvain Institute of Data Analysis and Modeling in economics and statistics

ISBA
Voie du Roman Pays 20 - L1.04.01
B-1348 Louvain-la-Neuve
Email : lidam-library@uclouvain.be
https://uclouvain.be/en/research-institutes/lidam/isba/publication.html

Asymmetric volatility impulse response functions

Christian M. Hafner[*] Helmut Herwartz[†]

November 18, 2022

Abstract

Volatility impulse response functions (VIRFs) have been introduced to unravel the effects of shocks on (co-)variances for the case of classical multivariate GARCH specifications. This paper proposes generalized VIRFs for the case of asymmetric specifications which capture stylized features such as the leverage effect. In a bivariate application comprising a global equity index and gold prices, we show that generalized VIRFs can be used to reassess the role of gold as a safe-haven asset.

Keywords: Multivariate GARCH, leverage effect, volatility impulse response analysis, safe haven.
JEL Classification: C32, G15.

[*]Corresponding author, Institute of Statistics, Biostatistics and Actuarial sciences, and CORE, Université catholique de Louvain. e-mail `christian.hafner@uclouvain.be`

[†]Department of Economics, University of Göttingen, Germany. e-mail `hherwartz@uni-goettingen.de`.

1 Introduction

For the modelling of conditional volatilities (i.e. (co)variances) in multivariate systems of speculative returns, multivariate GARCH (MGARCH) models have been first considered by Bollerslev et al. (1988). As a restricted version of the so-called vec-model of these authors, Engle and Kroner (1995) have introduced the BEKK model that ensures positivity of the conditional covariance matrix and has proven useful in many empirical applications. Kroner and Ng (1998) have proposed an important generalisation of BEKK that takes account of the so-called leverage effect which describes a larger impact of negative news on volatility than positive ones of the same size. The *R* package *BEKKs*, see Fülle et al. (2022), can be used for computationally efficient estimation of MGARCH models in BEKK form, including its asymmetric version, for systems of small to medium dimension (i.e. $N \leq 6$). For a review of multivariate GARCH and BEKK models, see e.g. Bauwens et al. (2006) or Bauwens et al. (2012).

In the empirical analysis of systems of speculative returns, volatility impulse response functions (VIRFs) as defined by Hafner and Herwartz (2006) have become a popular approach to unravel nonlinear dynamic relations implied by multivariate volatility models. Generalizing VIRFs towards the case of asymmetric effects of positive and negative shocks is complicated due to the presence of nonlinear effect transmission. In this paper, we first give the definition and explicit expressions for asymmetric VIRFs (AVIRFs). We further provide feasible estimators of unknown quantities in these expressions, and suggest a bootstrap algorithm for inference. In an application to a bivariate system comprising a global equity index and gold returns, we consider four historic events (i.e. the terrorist attack on 9/11/2001, the Lehman default in 2008, the outbreak of Covid in 2020, and the Russian invasion in Ukraine in 2022). We first show that neglecting the asymmetry would imply an underestimation of subsequent volatilities after a negative shock, corresponding to the leverage effect. Second, for almost all considered cases, the effect on the covariance was negative, confirming the flight to safety tendency after negative shocks. However, this effect dissipates much faster for the asymmetric BEKK model than for the symmetric one, confirming results of Baur and McDermott (2016) that flights to safety are rather short-lived. Hence, the persistence of flights to safety would be overestimated by neglecting the asymmetric effects of shocks on volatilities and covariances.

The remainder of the paper is organized as follows. Section 2 introduces the model, defines AVIRFs and suggests a bootstrap algorithm for inference. Section 3 presents the empirical application to a global equity index and gold prices and discusses the role of gold as a safe-haven asset. Finally, Section 4 concludes.

2 The model and methodology

2.1 The asymmetric MGARCH model in BEKK form

Let r_t be an N-dimensional vector of financial returns with $t = 1, \dots, T$. The information available at time t is $\mathcal{F}_t = \{r_1, \dots, r_t\}$. To describe conditional first and second order properties of r_t, MGARCH models align with the following representation:

$$r_t = \mu_t + e_t, \tag{1}$$

$$e_t = Q_t \xi_t, \tag{2}$$

where $\mu_t = E[r_t|\mathcal{F}_{t-1}]$, and Q_t denotes a decomposition of $H_t := \mathrm{Var}[r_t|\mathcal{F}_{t-1}]$ such that $H_t = Q_t Q_t'$.[1] The shocks ξ_t are assumed i.i.d. and $\xi_t \sim (0, I_N)$, where I_N is the N-dimensional identity matrix. While empirical return processes typically show marked time variation in second order moments, there is much less information in the conditional mean, so that we set it to a constant for simplicity, $\mu_t = \mu$. Centered returns e_t are then obtained by de-meaning the return vector.

An asymmetric conditional covariance model in BEKK(1,1,1) form has been proposed by Kroner and Ng (1998) and is given by

$$H_t = CC' + Ae_{t-1}e_{t-1}'A' + GH_{t-1}G' + B\eta_{t-1}\eta_{t-1}'B'. \tag{3}$$

In (3), A, B, C, G are $N \times N$ parameter matrices, C is lower triangular, $\eta_t = \mathbb{I}(e_t < 0) \odot e_t$, where $\mathbb{I}(\cdot)$ is the elementwise indicator function and '$\odot$' denotes element-by-element multiplication. According to Grier et al. (2004), the asymmetric effects of the BEKK model might be chosen in a more flexible way. For instance, a specific combination of positive and negative returns could have strongest effects on the conditional covariance.

For covariance stationarity, it is required that all unconditional (co-)variances are finite. A necessary and sufficient condition for this is that all eigenvalues of the matrix

$$\Omega := A \otimes A + G \otimes G + (B \otimes B)\mathrm{diag}(\mathrm{vec}(W)) \tag{4}$$

are smaller than one in modulus, where the positive definite matrix W is implicitly defined by $\mathbb{E}[\eta_t\eta_t'] = \mathbb{E}[e_te_t'] \odot W = \Sigma \odot W$, see Fülle et al. (2022) who also suggest estimators for W. If covariance stationarity holds, the unconditional variance-covariance matrix Σ of e_t is given by $\mathrm{vec}(\Sigma) = (I_{N^2} - \Omega)^{-1}\mathrm{vec}(CC')$.

[1] In our application we use the spectral decomposition of H_t, which is symmetric, but one may choose, alternatively, the Cholesky decomposition or orthogonal rotations that allow, e.g., to obtain independent components as in Hafner et al. (2022).

2.2 Asymmetric volatility impulse response functions

To measure the marginal effects of shocks hitting the stochastic model components in ξ_t on the second order properties of observables e_t, Hafner and Herwartz (2006) have suggested VIRFs at horizon h that are defined as

$$\mathcal{V}_{t+h}(\xi^*, \mathcal{F}_{t-1}) = \mathbb{E}[\mathrm{vech}(H_{t+h})|\mathcal{F}_{t-1}, \xi_t = \xi^*] - \mathbb{E}[\mathrm{vech}(H_{t+h})|\mathcal{F}_{t-1}], \quad h \geq 1, \quad (5)$$

where ξ^* is a shock vector of interest, which may be obtained empirically from the estimated model and for a given covariance decomposition Q_t, i.e. $\xi^* = \xi_t = Q_t^{-1} e_t$. Hence, VIRFs quantify the volatility path expected to occur after a shock $\xi_t = \xi^*$, conditional on past observations, in comparison with a benchmark 'steady state' volatility path.

As an extension of the results of Hafner and Herwartz (2006), AVIRFs can be obtained as

$$\begin{aligned} \mathcal{V}_{t+h}(\xi^*, \mathcal{F}_{t-1}) &= \mathrm{vech}(V_{t+h}), \quad h \geq 1, && (6) \\ V_{t+1} &= A Z_t A' + B Z_t^- B' && (7) \\ V_{t+h} &= (A+G) V_{t+h-1} (A+G)' + B(\Xi_{t+h-1|t} - \Xi_{t+h-1|t-1}) B', \ h \geq 2, && (8) \end{aligned}$$

where we define

$$\begin{aligned} Z_t &:= Q_t \xi^* \xi^{*'} Q_t' - H_t, \\ Z_t^- &:= Q_t \xi^* \xi^{*'} Q_t' \odot d_t d_t' - \Xi_{t|t-1}, \quad d_t := \mathbb{I}\left(e_t < 0\right), \\ \Xi_{t+h|t} &:= \mathbb{E}[\eta_{t+h} \eta_{t+h}' | \mathcal{F}_{t-1}, \xi_t = \xi^*], \ h \geq 1. && (9) \\ \Xi_{t+h|t-1} &:= \mathbb{E}[\eta_{t+h} \eta_{t+h}' | \mathcal{F}_{t-1}], \ h \geq 1, && (10) \end{aligned}$$

Note that the conditional moments $\Xi_{t+h|t}$ and $\Xi_{t+h|t-1}$ are not available in closed form, but can be estimated, see Appendix A.

2.3 Bootstrap for AVIRFs

While asymptotic theory for the symmetric BEKK model has been developed quite extensively, see e.g. Comte and Lieberman (2003) and Hafner and Preminger (2009), we are not aware of explicit results for the asymmetric model. For a straightforward assessment of estimation uncertainty behind estimated impulse response functions, bootstrap approaches have become a conventional approach. The econometric literature on resampling parameter estimates of MGARCH models is scant. Theorem 2.4 of Boussama et al. (2011) provides mixing and ergodicity conditions that are essential for resampling MGARCH models in (symmetric) BEKK form. We determine confidence bands for AVIRFs by means of a bootstrap scheme that largely follows the lines of Ledoit et al. (2003). Specifically, it relies on the iid assumption for (estimated) model residuals $\xi_t = Q_t^{-1} e_t$. Bootstrap samples of parameter vectors $\theta := (\text{vech}(C)', \text{vec}(A)', \text{vec}(G)', \text{vec}(B)')'$ are obtained from the following steps that are adapted to the case of an asymmetric conditional (co)variance model:

1. Draw with replacement $\{\xi_t^{(r)}\}_{t=1}^T$ from data-based estimates $\{\hat{\xi}_t\}_{t=1}^T = \{Q_t^{-1} e_t\}_{t=1}^T$.

2. Determine recursively a bootstrap covariance path

$$H_t^{(r)} = \widehat{C}\widehat{C}' + \widehat{A} e_{t-1}^{(r)} e_{t-1}^{(r)\prime} \widehat{A}' + \widehat{B} \eta_{t-1}^{(r)} \eta_{t-1}^{(r)\prime} \widehat{B}' + \widehat{G} H_{t-1}^{(r)} \widehat{G}', \; t = 1, 2, \ldots, T,$$

   where $H_t^{(r)} = Q_t^{(r)} Q_t^{(r)\prime}$, $e_t^{(r)} = Q_t^{(r)} \xi_t^{(r)}$ and $\eta_t^{(r)}$ is defined in analogy to its sample counterpart η_t.[2]

[2]Unlike the suggestion of Ledoit et al. (2003) this resampling step is not conditional on data-based estimates of H_t. With conditioning on H_t it is not possible to design the bootstrap with random draws from estimates $\{\hat{\xi}_t\}_{t=1}^T$ and preserve the potential presence of an asymmetric covariance response.

3. Center the bootstrap sample such that each element of $\{e_t^{(r)}\}_{t=1}^T$ has mean zero, and re-estimate the model for each generated series $\{e_t^{(r)}\}_{t=1}^T$.

4. Replicate steps 1. to 3. for $r = 1, 2, \ldots, R$, to obtain a bootstrap sample of all estimated model parameters $\Theta^{(R)} = \{\theta^{(r)}\}_{r=1}^R$, where R is sufficiently large. In this study we use $R = 999$.

For the assessment of (A)VIRF estimation uncertainty it is important to notice that it is conditional on the estimates of covariances $H_t = \mathbb{E}[e_t e_t' | \mathcal{F}_{t-1}]$ and historical shocks $\xi^* = \xi_t$. Accordingly, $H_{t+1} = \mathbb{E}[e_{t+1} e_{t+1}' | \mathcal{F}_{t-1}, \xi_t = \xi^*]$ is not subject to resampling. Bootstrap samples of (A)VIRFs are obtained as functions of the parameter estimates in (7) and (8) for each element of $\Theta^{(R)}$. Note that also the moments in (9), for which no analytical form is available, are subject to reevaluation conditional on the bootstrap parameters.

3 An application to global equity and gold returns

As an empirical illustration, we consider a bivariate series of returns of a high-yield equity index (MSCI World Developed Markets) and a traditional safe-haven asset (LBMA gold, p.m. prices in US Dollar; see, for instance, Baur and McDermott 2016).[3] Noticing that attempts of portfolio risk reduction that rely on safe-haven

[3]The MSCI index has been taken from *https://app2.msci.com/products/index-data-search/*, with the following specifications: Developed Markets (DM), Currency: USD, Size: Standard (Large+Mid Cap), Style: None. With these options, we chose 'World' to obtain the MSCI World for Developed Markets. The index is updated daily at approximately 18:30 EDT. For the gold price, we used the R-package 'Quandl' to obtain gold prices in USD as reported by the London Bullion Market Association (LBMA), at *https://www.lbma.org.uk/*. Price quotes are set at 15:00

strategies are best complemented with a predictive analysis of emerging higher-order dependencies, AVIRFs appear as a convenient informational tool to summarize the effects of current shocks on future profiles of conditional (co)variances.

The sample period spans from January 5, 2000 until March 3, 2022 and covers $T = 5570$ daily observations. The estimation of the asymmetric MGARCH model yields a maximum of the quasi log-likelihood of -14524.4. Parameter estimates joint with bootstrap t-ratios are given in the following.

$$\widehat{C} = \begin{pmatrix} \begin{matrix} 0.112 \\ (5.10) \end{matrix} & 0 \\ \begin{matrix} -0.018 \\ (-0.56) \end{matrix} & \begin{matrix} 0.165 \\ (6.51) \end{matrix} \end{pmatrix}, \quad \widehat{A} = \begin{pmatrix} \begin{matrix} 0.192 \\ (8.88) \end{matrix} & \begin{matrix} -0.034 \\ (-2.41) \end{matrix} \\ \begin{matrix} -0.029 \\ (-1.64) \end{matrix} & \begin{matrix} 0.259 \\ (13.7) \end{matrix} \end{pmatrix},$$

$$\widehat{G} = \begin{pmatrix} \begin{matrix} 0.947 \\ (190.3) \end{matrix} & \begin{matrix} 0.011 \\ (1.72) \end{matrix} \\ \begin{matrix} 0.008 \\ (1.25) \end{matrix} & \begin{matrix} 0.954 \\ (138.1) \end{matrix} \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} \begin{matrix} 0.312 \\ (13.0) \end{matrix} & \begin{matrix} 0.008 \\ (0.38) \end{matrix} \\ \begin{matrix} -0.013 \\ (-0.62) \end{matrix} & \begin{matrix} 0.063 \\ (1.51) \end{matrix} \end{pmatrix}$$

Pointing to the prevalence of a leverage effect in multivariate (co-)variances, one out of four parameter estimates in $\widehat{B}$ is nonzero with very high significance. As a further - and more informal - criterion to favor the asymmetric model specification, the maximum of the quasi log-likelihood shrinks to -14649.2 under a symmetry restriction (i.e. imposing $B = 0$).

GMT+1 to which LBMA refers as 'p.m.' Accordingly the MSCI index is published approximately 8 hours after the gold prices.

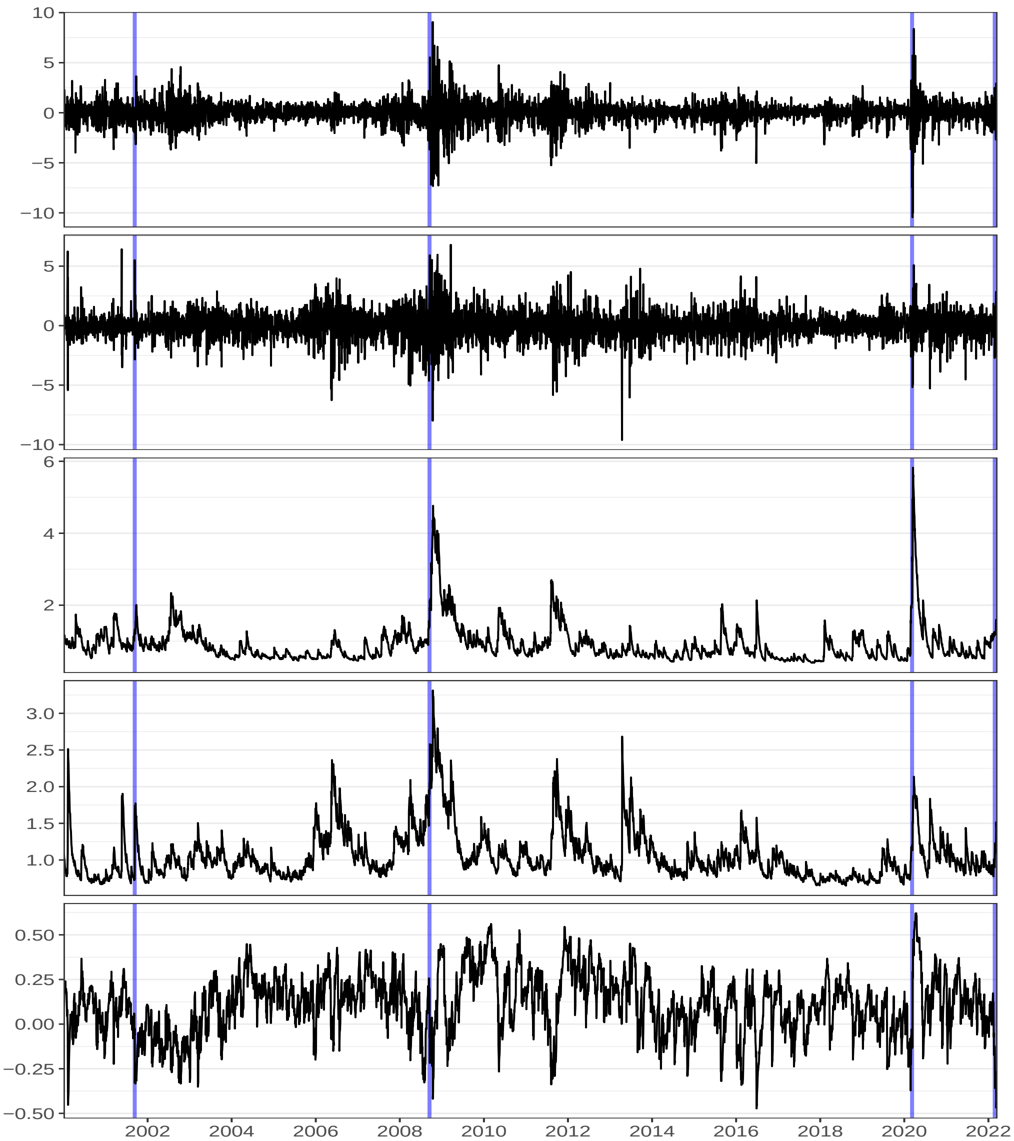

Figure 1: *Return data and second order moments as implied by the estimated asymmetric BEKK model. Equity and gold returns are shown in line 1 and 2, respectively. Rows 3,4, and 5 display the conditional standard deviations of equity returns, conditional correlations and conditional standard deviations of gold returns, respectively. Vertical lines indicate four historic events, (i) the terrorist attack on September 11, 2001, (ii) the Lehman default on September 15, 2008, (iii) the outbreak of the Covid pandemic on March 9, 2020 and the day after the Russian invasion into Ukraine (February 25, 2022).*

Bivariate log return data and model implied conditional second order moments are shown in Figure 1. Apparently, the estimated MGARCH model captures quite accurately the clustering of return variation. Although it is often considered a safe-haven asset, it is important to notice that gold returns exhibit patterns of sizeable conditional standard deviations. While they are, hence, not risk-free, gold returns are characterized by negative conditional correlations with equity for 24.5% of all time instances. On average, the conditional correlation between equity and gold returns is 10%. The following empirical analysis is mostly conditional on the asymmetric model specification.

To illustrate shock transmission by means of AVIRFs we consider four historic events, (i) the terrorist attack on September 11, 2001, (ii) the Lehman default on September 15, 2008, (iii) 'Black Monday' of March 9, 2020 as invoked by the outbreak of the Covid pandemic[4] and the day after the Russian invasion into Ukraine (February 25, 2022). These time instances are indicated in Figure 1 by means of vertical lines. Table 1 provides the exact magnitudes of log returns, shocks and conditional covariances encountered at these days. Looking at the magnitude of historical shocks $\xi^* = \xi_t$ issued by the (a)symmetric model specification, it holds that from the eight shocks, six are either below their 2.5% or above their 97.5% quantile, confirming the significant size of the considered shocks.

AVIRFs as implied by the estimated asymmetric model and historic shocks are displayed in Figure 2. For the purpose of comparison we also display VIRFs as implied by estimates of a symmetric BEKK model.[5]

[4]In the first week of March 2020, the total number of people infected by the Corona virus crossed 100,000. Abnormally adverse stock returns have been encountered on March 9, 12, and 16, 2020, of which we chose the first one, see also Mazur et al. (2021).

[5]It is worth noticing that historic shocks ξ_t^* as implied by the alternative model estimates differ slightly (see, Table 1). A comparison of respective VIRFs is, however, still meaningful.

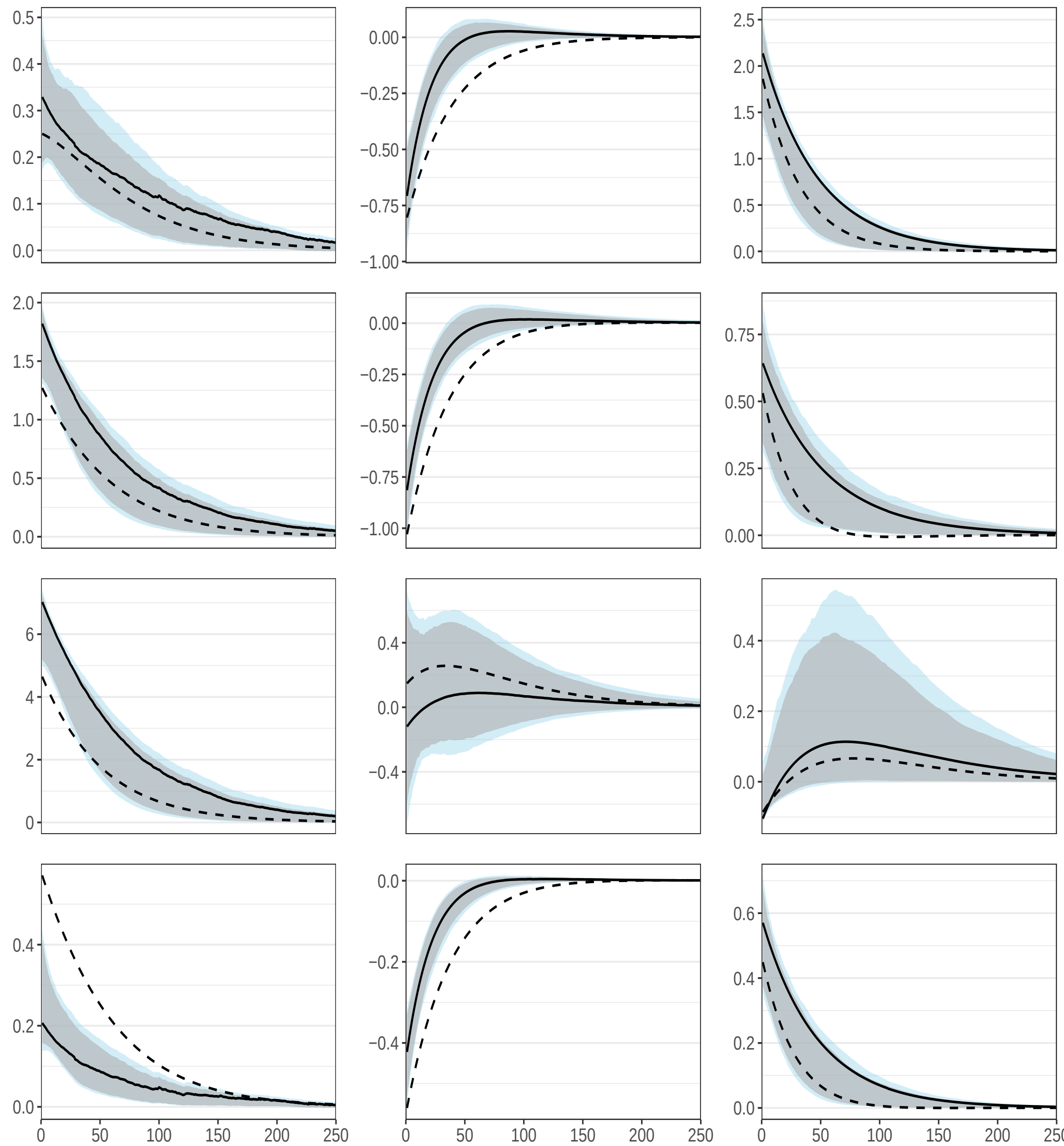

Figure 2: *AVIRFs at four occasions, terrorist attack on Sep 11, 2001 (first row), Lehman default (second), Covid outbreak (third), Russian invasion into Ukraine (fourth). For further notes on the exact dates see Figure 1. Effects on equity variance (first column), equity-gold covariance (second) and gold variance (third) are shown for $h = 1, 2, \ldots, 250$ trading days. Shaded areas correspond to 90% (grey) and 95% (blue) bootstrap confidence intervals.*

	e_t		ξ_t (sym)		ξ_t (asym)		(vech(H_t))' (asym)		
Sep. 11, 2001	-1.53	5.52	-1.31	6.98	-1.20	7.38	1.46	-.022	.556
q	.057	.999	.087	**1.00**	.104	**1.00**	.844	.203	.084
Lehman default	-3.68	3.21	-3.22	2.12	-2.90	1.95	1.92	.522	3.59
q	**.006**	**.993**	**.006**	**.983**	**.007**	**.975**	.899	.957	.968
Covid	-7.45	-.699	-3.93	-1.05	-3.68	-.898	4.22	-.335	1.45
q	**.000**	.192	**.002**	.110	**.003**	.141	.969	.037	.785
Ukraine	2.51	-2.73	2.13	-2.60	2.01	-2.69	1.36	-.140	.938
q	**.987**	**.015**	**.989**	**.010**	**.985**	**.010**	.824	.080	.503

Table 1: *Descriptive features of shock scenarios at four historic events (i.e. levels of returns, model implied shocks and (co)variances, and their empirical quantiles (q). Results indicated with 'sym' and 'asym' refer to the symmetric ($B = 0$) and the asymmetric BEKK model, respectively. To highlight empirically extreme statistics, bold entries indicate quantiles that are below 2.5% or above 97.5%. The exact dates of the considered shocks are given in the notes to Figure 1.*

With the Lehman default as an exception, the considered shocks entered the markets in states of a negative correlation between global equities and gold. The outbreak of the Covid pandemic is the only event showing both asset prices moving in the same direction. One day after the Russian invasion into the Ukraine, global equity returns have been positive while gold prices decreased, somewhat counterintuitively. This explains why the symmetric VIRF is above the AVIRF for equity volatility after the Ukraine shock. For all other shocks, AVIRF is larger than the symmetric VIRF for volatilities, meaning that neglecting the leverage effects would lead to an underestimation of subsequent volatilities.

Baur and McDermott (2016) have considered the terrorist attack and the Lehman default as typical market disruptions after which the demand for safe haven investments increased. This can be viewed as a negative shock to covariances between equities and gold, which is confirmed by the AVIRFs for covariances for all shocks, although to a less extent for the Covid shock. Comparing implications for the shock transmission in the symmetric and asymmetric BEKK model, AVIRFs exhibit a faster convergence of the conditional covariances to their unconditional level. Hence, in comparison with VIRFs, AVIRFs are more in line with the view expressed in Baur and McDermott (2016) that the quest for safe-havens is typically short lived after large shocks.

4 Conclusion

Extending the approach taken in Hafner and Herwartz (2006) we have introduced variance impulse response functions for asymmetric MGARCH models. For the purpose of inferential analysis we suggest a modification of the resampling scheme put forth by Ledoit et al. (2003). For a bivariate system of equity and gold returns we find that AVIRFs better reflect evidence that markets seek for safe havens in rather short periods after the occurrence of major news shocks.

Acknowledgements

The first author is grateful for financial support of the research grant ARC 18-23/089 of the Belgian Federal Science Policy Office. The second author gratefully acknowledges financial support by the Deutsche Forschungsgemeinschaft (HE2188/14-1).

Appendix: Estimation of $\Xi_{t+h|t}$

We propose an estimator for the conditional moments $\Xi_{t+h|t}$ and $\Xi_{t+h|t-1}$. Draw a large number S of trajectories $\xi_{t+1}^{(s)}, \ldots, \xi_{t+\nu_{max}}^{(s)}$, $s = 1, \ldots, S$, from the empirical distribution of estimated ξ_t. Then, generate $e^{(s)}_{t+1|t} := Q_{t+1}\xi^{(s)}_{t+1}$ and $\eta^{(s)}_{t+1|t} := e^{(s)}_{t+1|t} \odot I(e^{(s)}_{t+1|t} < 1)$ and

$$H^{(s)}_{t+2|t} = CC' + Ae^{(s)}_{t+1|t}e^{(s)\prime}_{t+1|t}A' + GH_{t+1}G' + B\eta^{(s)}_{t+1|t}\eta^{(s)\prime}_{t+1|t}B'$$

Then decompose $H^{(s)}_{t+2|t} = Q^{(s)}_{t+2|t}Q^{(s)\prime}_{t+2|t}$, and obtain $e^{(s)}_{t+2|t} := Q^{(s)}_{t+2|t}\xi^{(s)}_{t+2}$ and $\eta^{(s)}_{t+2|t} := e^{(s)}_{t+2|t} \odot \mathbb{I}(e^{(s)}_{t+2|t} < 1)$. Continue this recursive procedure until all trajectories $\eta^{(s)}_{t+\nu|t}$ are obtained for $h = 1, \ldots, h_{max}$, where h_{max} is some maximum horizon of interest. Finally, the estimator is

$$\widehat{\Xi}_{t+h|t} := \frac{1}{S}\sum_{s=1}^{S} \eta^{(s)}_{t+h|t}\eta^{(s)\prime}_{t+h|t},\ h = 1, \ldots, h_{max}. \tag{11}$$

Likewise, $\Xi_{t+h|t-1}$ in (10) can be estimated by drawing additionally from ξ_t, i.e., generating trajectories $\xi^{(s)}_t, \ldots, \xi^{(s)}_{t+h_{max}}$, $s = 1, \ldots, S$. Then, generate recursively $e^{(s)}_{t|t-1} := Q_t\xi^{(s)}_t$, $\eta^{(s)}_{t|t-1}$, $H^{(s)}_{t+1|t-1}$, $Q^{(s)}_{t+1|t-1}$, and so on, until all trajectories $\eta^{(s)}_{t+\nu|t-1}$ are obtained for $h = 1, \ldots, h_{max}$, and then obtain the estimator

$$\widehat{\Xi}_{t+h|t-1} := \frac{1}{S}\sum_{s=1}^{S} \eta^{(s)}_{t+h|t-1}\eta^{(s)\prime}_{t+h|t-1},\ h = 1, \ldots, h_{max}. \tag{12}$$

In our empirical applications we work with $S = 20,000$ replications to obtain very precise estimates of the conditional moments $\Xi_{t+h|t}$ and $\Xi_{t+h|t-1}$.

References

Baur, D. G. and T. K. McDermott (2016). "Why is gold a safe haven?" *Journal of Behavioral and Experimental Finance* 10, pp. 63–71.

Bauwens, L., C. Hafner, and S. Laurent (2012). *Handbook of Volatility Models*. Wiley.

Bauwens, L., S. Laurent, and J. V. K. Rombouts (2006). "Multivariate GARCH models: A survey". *Journal of Applied Econometrics* 31, pp. 79–109.

Bollerslev, T., R. F. Engle, and J. M. Wooldridge (1988). "A capital asset pricing model with time varying covariances". *Journal of Political Economy* 96, pp. 116–131.

Boussama, F., F. Fuchs, and R. Stelzer (2011). "Stationarity and geometric ergodicity of BEKK multivariate GARCH models". *Stochastic Processes and their Applications* 121.10, pp. 2331–2360.

Comte, F. and O. Lieberman (2003). "Asymptotic theory for multivariate GARCH processes". *Journal of Multivariate Analysis* 84, pp. 61–84.

Engle, R. F. and K. F. Kroner (1995). "Multivariate simultaneous generalized ARCH". *Econometric Theory* 11, pp. 122–150.

Fülle, M. J., C. Hafner, H. Herwartz, and A. Lange (2022). *BEKKs: Multivariate Conditional Volatility Modelling and Forecasting*. R package version 1.2.1.

Grier, K. B., Ólan T. Henry, N. Olekalns, and K. Shields (2004). "The asymmetric effects of uncertainty on inflation and output growth". *Journal of Applied Econometrics* 19.5, pp. 551–565.

Hafner, C. M. and H. Herwartz (2006). "Volatility impulse responses for multivariate GARCH models: An exchange rate illustration". *Journal of International Money and Finance* 25, pp. 719–740.

Hafner, C. M., H. Herwartz, and S. Maxand (2022). "Identification of structural multivariate GARCH models". *Journal of Econometrics* 227.1. Annals Issue:

Time Series Analysis of Higher Moments and Distributions of Financial Data, pp. 212–227.

Hafner, C. M. and A. Preminger (2009). "On asymptotic theory for multivariate GARCH models". *Journal of Multivariate Analysis* 100.9, pp. 2044–2054.

Kroner, K. F. and V. K. Ng (1998). "Modeling asymmetric comovements of asset returns". *The Review of Financial Studies* 11, pp. 817–844.

Ledoit, O., P. Santa-Clara, and M. Wolf (2003). "Flexible multivariate GARCH modeling with an application to international stock markets". *Review of Economics and Statistics* 85.3, pp. 735–747.

Mazur, M., M. Dang, and M. Vega (2021). "COVID-19 and the march 2020 stock market crash. Evidence from S&P1500". *Finance Research Letters* 38, p. 101690.