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The rationality of expectations formation

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Abstract

Rational expectations do not require beliefs to be consistent with history and with what agents can conclude from it. Actually, at a rational expectations equilibrium agents may hold beliefs that explain poorly the history they observe, even when restricted to only those rationalizing their choices. This paper shows that if agents hold rationally formed expectations instead —in the sense of following from beliefs that explain history better than any other beliefs justifying their choices— then additional allocations unsupported by rational expectations can be shown to be equilibrium outcomes. By means of this result, it is established too that adding common knowledge of the rationality of the formation of expectations —on top of that of rationality of choices and market clearing— does not suffice either to guarantee rational expectations. Interestingly, the rationally formed expectations equilibria produced in this paper exhibit a sunspot-like volatility that do not rely on an explicit sunspot mechanism.

Keywords: rationality, expectations, overlapping generations.

JEL classification: D84, D5, E3.

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1. Introduction

Agents make depend their decisions on their expectations about anything that may have an impact on their consequences, namely the other agents' current and future plans of action, and current and past actual events.[1] Whether expectations are rational and whether they are *rationally formed* are related but essentially distinct issues. What is a rational way of forming expectations depends on what these expectations are about. While in strategic situations common knowledge of rationality can guide an agent in the formation of his expectations about the other agents' plans,[2] when it comes to expectations about future events a competing source of information to build expectations rationally (and the only one if the relevant events are only about states of nature and not past behaviors) is empirical evidence from the past. Thus, in the absence of a strategic dimension —as in, for instance, competitive markets— only history carries weight in forming expectations rationally. Bearing this in mind, we would expect rational agents to hold expectations that follow from theories or beliefs that explain best the past they observe. Indeed, that an agent may have made a decision because of beliefs distinct from those that explain best the history he or she observes amounts to assume that that agent does not infer rationally from evidence, and must therefore be rejected on the grounds of the agent's rationality.

Does not the rational expectations hypothesis settle the issue? No, in fact the rational expectations hypothesis itself is silent about the formation of expectations:[3] (i) in sequential markets models it just forbids the agents to hold expectations that lead to systematic forecasting mistakes; (ii) in the general equilibrium literature it requires the agents' choices to be contingent only to the information available, including the information revealed by prices. At any rate, nothing prevents that in a rational expectations equilibrium an agent holds expectations following from beliefs that do not explain best the history he observes, *not even among only those that rationalize his choice*.[4] Thus, if empirical evidence is the only source of information

[1] This may include, on top of those of nature, the history of past actual actions by everyone.

[2] As a matter of fact, although a popular assumption, common knowledge of rationality is unnecessarily strong to underpin Nash equilibria: individual rationality and mutual knowledge of everybody else's strategy suffices —see Aumann and Brandemburger (1995).

[3] Lucas (1978) agues, nonetheless, that rational expectations equilibria are the asymptotic result of Bayesian learning or boundedly rational learning processes. Lucas (1986) quotes Bray (1982, 1983) and by Blume and Easley (1982) along this line. Ben Porath and Heifetz (2010) have spelled out the framework needed to state and prove such a claim.

[4] More precisely, a rational expectations equilibrium needs not impute agents, as their subjective

on which agents rationally form their beliefs about a (non-strategic) environment, then rational expectations need not be rationally formed.

This paper proposes instead an equilibrium concept that requires the agents' expectations to follow from beliefs that are not worse at explaining the available evidence than any other beliefs rationalizing his choice. Since this is a condition on what is going on inside the agents' minds when making a choice, should it make no difference outside people's minds, such condition would be irrelevant, but the fact is that (as it will be shown below) it does matter for the determination of what can be an equilibrium outcome and what cannot. Interestingly, with this equilibrium notion, new instances of sunspot-like volatility of prices and trades that cannot be supported as rational expectations equilibria happen to be, nonetheless, equilibrium outcomes.

As a matter of fact, the results of this paper contribute also to the literature seeking to provide a common knowledge foundation to competitive equilibrium outcomes. Specifically, the result above allows to establish that, in sequential market economies, adding to the common knowledge of rationality and market clearing also the common knowledge of rationality *in the way agents form their expectations* is still not enough to guarantee a rational expectations equilibrium outcome (that rational expectations equilibria outcomes need not follow from common knowledge of just rationality and market clearing has been established in Ben-Porath and Heifetz (2010) for finite exchange economies with asymmetric information; Morris (1994) had shown that common knowledge of rationality and market clearing imply rational expectations equilibria only if the agents share a common prior on the set of states of the *world* —which includes the whole system of beliefs and higher order beliefs, on top of the state of *nature*— an admittedly too demanding assumption).

More specifically, I consider a deterministic overlapping generations exchange economy of agents that hold *rationally formed expectations* in the sense that any other expectations consistent with their choices follow from beliefs implying a smaller likelihood for the history they observe.[5] I argue first that a belief that the observed

probabilities, the asymptotic value of the maximum likelihood estimate of the probabilities of the transitions (consistent with the optimality of the choices made) that they observe, which, from an admittedly frequentist viewpoint, is unsatisfactory since the maximum likelihood estimator achieves the Cramér-Rao lower bound for the asymptotic mean squared error among unbiased estimators, while any other estimator converging to a rational-expectations limit does not.

[5]Very importantly, note that this is *not* to say that each agent forms his expectations maximizing

history is Markovian can never be falsified when the agents' memory (or the history itself) is finite. Then, for Markovian beliefs I establish that when the agents' memories are finite there exist rationally formed expectations equilibria that no rational expectations equilibrium can match (Proposition 2).

It is worth noticing that limited memory and communication between agents are essential for the existence of rationally formed expectations equilibria distinct from a rational expectations equilibrium. In effect, with infinite memory the only existing rationally formed expectations equilibria would be allocationally equivalent to some rational expectations equilibrium (Proposition 1). Instead, with limited memory, different agents can hold different expectations at a rationally formed expectations equilibrium, since this diversity of beliefs follows from the fact that different generations observe different bits of the same history and therefore form their beliefs using different information —the limited memory itself captures the bounded computing abilities of actual agents. Thus, in spite of the result in Geanakoplos and Polemarchakis (1982) showing that unrestricted communication allows agents with a common prior but different information to agree in finite time on a common posterior,[6] the limited communication friction implied the demographic structure of the economy (as a sequence of overlapping generations) allows for the agents to disagree.

Specifically, a rationally formed expectations equilibrium will consist, for each agent in each generation and for each history of prices he may observe, of (1) a belief that the prices follow a particular stochastic process, and (2) consumption decisions (contingent to future prices for future consumptions), such that, for any history of prices up to any date, (i) the allocation is feasible, (ii) the agents' consumptions maximize their expected utilities given the price process they believe they face, and (iii) *the agents' beliefs about the price process are formed rationally* —i.e. their beliefs are not falsified by history and attach to the latter a likelihood not smaller than any other beliefs justifying their choices.

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the likelihood of observed history, since expectations must justify the agent's choice. As a matter of fat, rationally formed expectations typically do *not* maximize the unconstrained likelihood of observed history because of this latter condition. The actual formation of expectations itself is left un-modeled here, but the rationality condition introduced *does* restrict the expectations formation process nonetheless.

[6]The posterior needs not be the result from pooling all the information, even though this discrepancy happens only in unlikely situations with a "high degree of symmetry" (see Geanakoplos and Polemarchakis (1982)).

Note that the equilibrium concept leaves open the question of how the actual history of prices is determined. The equilibrium concept just requires that it does not falsify the agents' beliefs. Therefore, no objective process is assumed here to drive prices and, as a consequence, there is no room for agents to mistake a price process they supposedly face —which would be the ultimate rationality test under rational expectations. In case not specifying such process may be disconcerting, it is important to note that, *in the absence of shocks to the fundamentals* (which is the case considered in this paper), assuming that some objective stochastic process like the one followed by a sunspot signal drives the prices, amounts to postulate implicitly a particular price formation theory —extraneous to the equilibrium conditions and hence alien to market-clearing forces— that acts in an ad hoc way as a selecting device within the set of possible price processes.[7] Given the obvious difficulties in justifying the causation from sunspots all the way to prices and, more importantly, given that it is superfluous here, I do away with it.[8] The results in this paper establish thus the existence of equilibria akin to sunspot equilibria but without the need to make an explicit reference to sunspots —or, so to speak, to sunspot equilibria without sunspots— which shows that the introduction of sunspot mechanisms is not essential to account for pure expectations-driven fluctuations.

The remainder of the paper is organized as follows: Section 2 presents the main ideas by means of a leading example conveying what is driving the result: it produces constructively rationally formed expectations equilibria exhibiting fluctuations distinct from those of any rational expectations equilibrium. Section 3 generalizes the setup, provides a precise definition of a rationally formed expectations equilibrium for a deterministic overlapping generations economy, and establishes the existence of equilibria of this type exhibiting fluctuations that no rational expectations equilibrium can generate. Incidentally, the constructive argument used to establish this result reveals a high level of degrees of freedom to produce rationally formed expectations equilibria. Therefore, I establish next the important fact that not anything

[7]In the sunspot equilibrium literature it is customary to claim that the prices turn out to be perfectly correlated with the sunspot signal, and hence follow the same process, because the decisions made by the agents according to their belief in such a perfect correlation *causes* the *contemporaneous* prices to take the adequate values with the adequate probabilities for that perfect correlation to actually take place. But how the agents' decisions would achieve this is conspicuously left unexplained.

[8]Note that if, on the contrary, the fundamentals do follow some stochastic process, the latter will make its way through the equilibrium equations towards the equilibrium prices, in such a way that one can safely speak in that case of a stochastic process driving prices. So, the caveat above about assuming an objective price process is truly specific to the pure excess volatility case with deterministic fundamentals that we are considering in this paper.

can be a rationally formed expectations equilibrium. Section 4 embeds the notion of rationally formed expectations equilibrium within an epistemic model specifying an interactive system of beliefs and higher order beliefs in order to show that, very much like common knowledge of rationality and market clearing does not necessarily imply rational expectations in finite economies (see Ben-Porath and Heifetz (2010)), common knowledge of rationality, market clearing, *and of beliefs formation rationality* needs not lead to rational expectations equilibria either. Finally, Section 5 briefly discusses some related literature.

2. The leading example

2.1 What is a rationally formed expectations equilibrium?

Consider an overlapping generations economy with a 2-period lived representative agent. An agent born in period t decides how much to save from real income y when young (date t) in order to consume when old (date $t+1$). His decision will depend on the purchasing power he *expects* his savings to have when old. In particular, if the current price of consumption is p_t and he expects it to be p^j_{t+1} with probability π^j when old, for $j = 1, \ldots, k$, then his savings m^t would be the solution to

$$\begin{aligned} \max \sum_{j=1}^{k} \pi^j u(c^t_t, c^{tj}_{t+1}) \\ c^t_t + m^t = y \\ c^{tj}_{t+1} = \frac{p_t}{p^j_{t+1}} m^t \end{aligned} \tag{1}$$

where c^t_t is his consumption when young and c^{tj}_{t+1} is his consumption when old if the level of prices then is p^j_{t+1}. Under standard assumptions guaranteeing differentiability and the interiority of the solution, the necessary and sufficient first-order condition characterizing the solution of the problem above is, along with its budget constraints,

$$\sum_{j=1}^{k} \pi^j \Big[- u_1(y - m^t, \frac{p_t}{p^j_{t+1}} m^t) + u_2(y - m^t, \frac{p_t}{p^j_{t+1}} m^t) \frac{p_t}{p^j_{t+1}} \Big] = 0. \tag{2}$$

At equilibrium every agent must choose his consumption rationally according to his expectations about future prices, and individual consumption decisions must be compatible. Thus in an equilibrium of this economy in which the level of prices takes, at any period, one of k possible values $p^1, \ldots, p^k$, and the representative agent *believes* that the probability π^{ij} of the price being p^j when old depends only on the price p^i he faces when young, then the representative agent's savings decision *does* depend only on the level of prices p^i he faces when young, so that it can be denoted m^i and be characterized as the solution to

$$\sum_{j=1}^{k} \pi^{ij} \Big[-u_1(y - m^i, \frac{p^i}{p^j} m^i) + u_2(y - m^i, \frac{p^i}{p^j} m^i) \frac{p^i}{p^j} \Big] = 0. \tag{3}$$

Moreover, in such an equilibrium the contingent savings m^i and prices p^i, for all $i = 1, \ldots, k$, satisfy

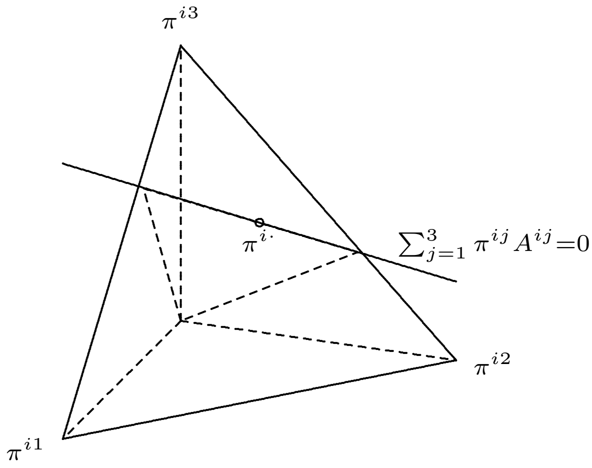
$$m^i = \frac{p^j}{p^i} m^j \tag{4}$$

for all $i, j = 1, \ldots, k$, so that not only all the agents choose their savings rationally according to their expectations, but markets clear as well (in effect, in any state i the desired saving m^i by the young agent equals then the desired consumption $\frac{p^j}{p^i} m^j$ by the contemporary old agent born in any state j). Conditions for the existence of prices p^i, savings m^i, and probabilities π^{ij}, for $i, j = 1, \ldots, k$, such that (3) and (4) hold for all $i, j = 1, \ldots, k$ are well known,[9] and such an equilibrium is known in the literature as a k-state Markovian stationary sunspot equilibrium, or k-SSE.

Note however that, as soon as $k \geq 3$ in equation (3), the same savings decision m^i can follow from different beliefs about the probabilities $\pi^{i1}, \ldots, \pi^{ik}$ of transition from a price p^i to any other p^j. In effect, at any such equilibrium each vector $(\pi^{i1}, \ldots, \pi^{ik})$ of probabilities of transition from each state $i = 1, \ldots, k$, must satisfy the two linear equations consisting of (i) being in the unit simplex in $\mathbb{R}^k$ and (ii) satisfying equation (3), so that there remains $k-2$ degrees of freedom for each row $(\pi^{i1}, \ldots, \pi^{ik})$ of the Markov matrix $(\pi^{ij})_{i,j=1}^k$ of believed probabilities of transition between prices, as illustrated in Figure 1 below for the case $k = 3$, where $\pi^{i\cdot} \equiv (\pi^{i1}, \ldots, \pi^{ik})$ and $A^{ij} \equiv -u_1(y - m^i, \frac{p^i}{p^j} m^i) + u_2(y - m^i, \frac{p^i}{p^j} m^i) \frac{p^i}{p^j}$.

[9] See the sunspot equilibrium literature, e.g. Azariadis (1981), Azariadis and Guesnerie (1986), Chiappori and Guesnerie (1988, 1989), Guesnerie (1986).

Figure 1



Thus equations (3) and (4) may hold true —i.e., (i) everyone behaves rationally *given his beliefs* and (ii) markets clear— even with different agents within and across generations *holding different beliefs* about the probabilities of transition π^{ij}, if no further requirement is made on the agents' beliefs. Of course, this possibility is excluded if the agents are supposed to hold rational expectations, since in that case all the agents must share the same "true" π^{ij}'s. Note however that no mention has been made yet of a "true" objective process from which this "true" π^{ij}'s would stem, but rather of the agents' expectations about future prices instead. That is because in a k-state stationary sunspot equilibrium the probability π^{ij} with which the agent *expects* the transition from a price p^i to a price p^j to happen is implicitly assumed to be the actual probability with which such transition does happen, because of a never falsified belief in a perfect correlation between some sunspot signal and prices.[10] Note that this amounts to assuming implicitly a price formation mechanism that is extraneous to the equilibrium notion, very much like an ad hoc choice of a particular equilibrium price out of a multiplicity of them in, for instance, an Edgeworth box. More specifically, the rational expectations hypothesis imposes the additional conditions that

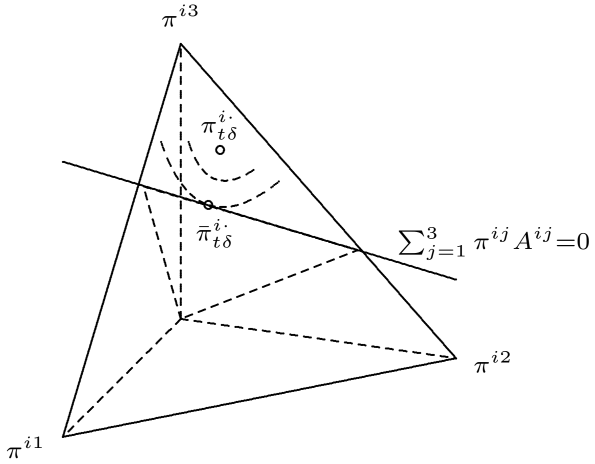
(i) *all* agents' expectations coincide, and
(ii) these common expectations correspond to those following from an objective process driving prices.

[10]No explanation of how this correlation happens to come into existence is usually provided, except for that of Woodford (1990), and in that case a small seed of uncertainty about the fundamentals is needed for an accidental correlation to get reinforced and convergence to the "sunspot theory" to obtain.

Note that, in terms of the equilibrium equations (3) and (4), condition (ii) above has no bite in the sunspot case, since it can just be dropped without any consequence for the set of solutions to the equations. It is in this sense that the assumption of an objective process driving prices is an implicitly ad hoc selection device in the absence of shocks to the fundamentals. But condition (i) without (ii) becomes arbitrary, and raises difficult questions regarding the spontaneous coordination of every agent within and across the infinity of generations on a particular belief. Accordingly, both conditions (i) and (ii) can arguably be dropped and the relevance of the rational expectations hypothesis be questioned in the absence of shocks to the fundamentals.

As a matter of fact, plenty of beliefs are compatible with the agents' behavior, and thus there is room for alternative consistency conditions at equilibrium (other than the rational expectations hypothesis) to be imposed on the agents' expectations. The most natural is to assume the agents' expectations should follow from the information available to them at the time of making their decisions. Accordingly, assuming that an agent's decision follows from expectations derived from beliefs that do not make the likelihood of the history of prices he observes as big as possible — among all the expectations that would have led to the same decision— is equivalent to assume that the agent formed his expectations using inefficiently the available information. In Figure 2 below, for the case $k = 3$, the agent's rationally formed expectations about these probabilities of transition from a price p^i (if he believes the prices follow a Markov process) would be the point $\bar{\pi}^{i\cdot}_{t\delta}$ (where t stands for the date up to which the generation t can observe a history of prices δ) attaining the highest likelihood level curve on the unit simplex *among those consistent with the first-order condition* satisfied by the agent's saving decision (represented by the plane intersecting the unit simplex in Figure 2). Note that the empirical frequencies of transitions starting from p^i (the number of observed transitions from price p^i to each price p^j over the number of times p^i has realized, depicted as $\pi^{i\cdot}_{t\delta}$ in Figure 2) would be the beliefs that best explain the observed history if no consistency with the agent's choice is required, but such expectations need not be consistent with the agent's behavior, or will be so just by chance.

Figure 2



Thus positive prices $\bar{p}^i$, savings $\bar{m}^i$, for all $i = 1, \ldots, k$, and history-dependent beliefs about a Markovian price process $(\bar{\pi}^{ij}_{t\delta})^k_{i,j=1}$, *for every history δ of prices and up to every date t*, such that

$$\sum_{j=1}^{3} \bar{\pi}^{ij}_{t\delta} \Big[-u_1(y - \bar{m}^i, \frac{\bar{p}^i}{\bar{p}^j}\bar{m}^i) + u_2(y - \bar{m}^i, \frac{\bar{p}^i}{\bar{p}^j}\bar{m}^i)\frac{\bar{p}^i}{\bar{p}^j} \Big] = 0 \tag{5}$$

and

$$\bar{m}^j = \frac{\bar{p}^i}{\bar{p}^j}\bar{m}^i, \ \forall j = 1, \ldots, k \tag{6}$$

hold for all $i = 1, \ldots, k$, and for every history δ up to every period t, constitute a Markovian rationally formed expectations equilibrium if, and only if, any other vector of probabilities $\pi^{i\cdot}$ satisfying (5) implies a lower likelihood for the realization of some history δ up to some period t than $\bar{\pi}^{i\cdot}_{t\delta}$ does, for some $i = 1, \ldots, k$.

Intuitively, as this example illustrates, at a rationally formed expectations equilibrium the expected probabilities $\bar{\pi}^{i\cdot}_{t\delta}$ will typically be different for different generations, since they will have access to histories of different length or span, and hence the observed empirical frequencies of transition $\pi^{i\cdot}_{t\delta}$ will be different for different t's even for a given history δ. Also, within generations the need for each agent's expected probabilities to be consistent with their respective different choices leaves room the agents' beliefs to differ among them as well.

2.2 Rationally formed expectations equilibria distinct from rational expectations equilibria

From equations (5) and (6) —that differ from those of a sunspot equilibrium only in that they make expectations history dependent— one could be tempted to suspect that any rationally formed expectations equilibrium should converge to a sunspot equilibrium, given that in the case in which an objective sunspot process is supposed to drive prices the empirical frequencies of transition between prices would eventually converge to the actual probabilities of transition. As a consequence, there would not be any allocational difference in the long run between the sunspot equilibrium and the rationally formed expectations equilibrium in that case. Nevertheless, this is not exactly so: there do exist rationally formed expectations equilibria whose allocations are not rational expectations equilibrium allocations.

In order to show that rationally formed expectations equilibria do not replicate rational expectations equilibria (in particular from the allocations viewpoint), I will illustrate in this framework the existence of rationally formed expectations equilibria fluctuating between a given set of states even when there is no stationary sunspot equilibrium fluctuating between those states.

The argument is constructive, starting from a k-state Markovian stationary sunspot equilibrium[11] of an overlapping generations economy with a representative agent with utility function u and endowments $e = (e_1, e_2)$. That is to say, consider, for all $i = 1, \ldots, k$, a price p^i, first and second period consumptions c_1^i and c_2^i and a Markov matrix of probabilities of transition $(\pi^{ij})_{i,j=1}^k$ such that, for all $i = 1, \ldots, k$,

$$c_1^i + c_2^i = e_1 + e_2 \tag{7}$$

and

$$(c_1^i, (c_2^j)_{j=1}^k) \in \arg\max_{p^i(\tilde{c}_1^i - e_1) + p^j(\tilde{c}_2^j - e_2) = 0, \forall j} \sum_{j=1}^k \pi^{ij} u(\tilde{c}_1^i, \tilde{c}_2^j) \tag{8}$$

Then necessarily the contingent consumptions c_1^i, c_2^i satisfy the equations, for all $i = 1, \ldots, k$

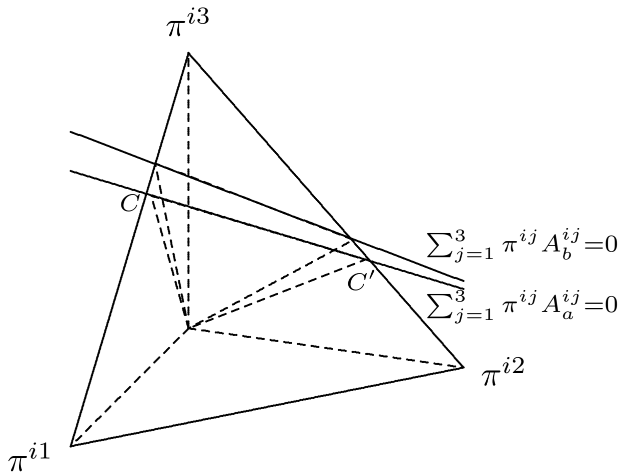
$$\sum_{j=1}^k \pi^{ij} A^{ij} = 0 \tag{9}$$

[11] See, for instance, Chappori and Guesnerie (1989).

where $A^{ij} \equiv u_1(c_1^i, c_2^j)(c_1^i - e_1) + u_2(c_1^i, c_2^j)(c_2^i - e_2)$. Figure 1 above shows for $k = 3$ the linear constraint on the simplex that the equilibrium equations impose on the probabilities of transition from any price p^i.

Now imagine this was in fact an economy of two *identical* (types of) agents a and b per generation, so that u and e are the utility and preferences u^h and e^h of both agents $h = a, b$ and, for all $i, j = 1, \ldots, k$, c_1^i and c_2^j are the equilibrium contingent consumptions c_1^{hi} and c_2^{hj} of both $h = a, b$ as well. Consider then a nearby economy in which agent b has a different utility function u^b close to u (while u^a continues to be u). Since u^b is now different from, but close enough to u (in values and, at least, first partial derivatives), then the linear constraints on each row of the Markov matrix generated by the first order conditions of agent b still intersect the simplex but differ from those of agents a. Actually, for some robust perturbations the new linear constraints on the probabilities of transition have no intersection with the old ones *on the unit simplex*, as illustrated in Figure 3 in the case $k = 3$.

Figure 3



This implies that for the economies resulting from such perturbations there is no Markov matrix that makes *both* agents a and b choose the contingent consumptions c_1^i and c_2^j whenever facing the prices p^i, p^j, for all $i, j = 1, \ldots, k$. In effect, as long as the perturbation makes the normal vector $A_b^{i\cdot}$ to the second linear subspace in (10) below —following from agent b's first-order conditions— to be distinct from the corresponding vector $A_a^{i\cdot}$ normal to a's linear subspace in (10) while in the span of $A_a^{i\cdot}$ and the normal vector to the unit simplex $(1, \ldots, 1)$, then the system in

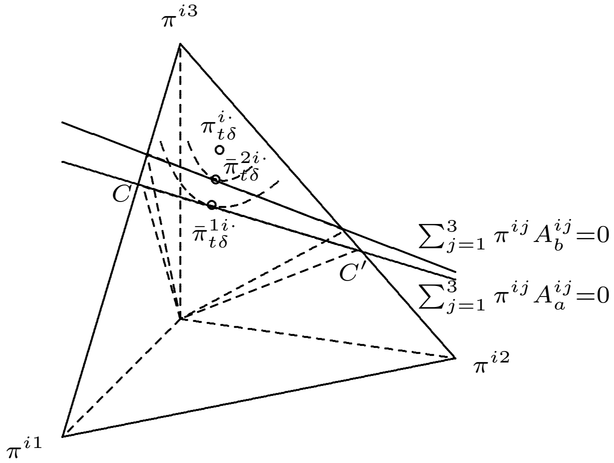
$\pi^{i1}, \ldots, \pi^{ik}$

$$\begin{aligned} \sum_{j=1}^{k} \pi^{ij} A_a^{ij} = 0 \\ \sum_{j=1}^{k} \pi^{ij} A_b^{ij} = 0 \end{aligned} \tag{10}$$

has no solution within the unit simplex.[12] Note that there is a $(k-2)$-dimensional manifold (after normalization) of possible vectors $A_b^{i\cdot}$ satisfying this condition. Of course, any other small enough perturbation of any vector on this manifold would still be such that no k-state Markovian stationary sunspot equilibrium exists with this support for the corresponding 2-agent overlapping generations economy, so that the property is robust.

Notwithstanding, there do exist rationally formed expectations equilibria over the given support for any of the 2-agent economies resulting from such robust perturbations. In effect, for small enough perturbations the unit simplex still has a nonempty intersection with the linear subspaces following from the agents' first-order conditions and hence, for all h, δ, and t, there exist probabilities $(\pi_{t\delta}^{hij})_{i,j=1}^k$, that maximize the likelihood $\prod_{i,j=1}^k (\pi^{ij})^{\sum_{\tau=1}^t \delta_{\tau-1}^i \delta_\tau^j}$ of observing the history δ up to period t, among the probabilities of transition in the unit simplex that are consistent with the agents' first-order conditions (the existence, illustrated in Figure 4 below for $k=3$, is guaranteed by the continuity of the likelihood function and the compactness of the constrained domain).

Figure 4



[12]Indeed, if $A_b^{i\cdot} = \alpha A_a^{i\cdot} + \beta \mathbf{1}$ with $\beta \neq 0$, then (10) above would imply $\sum_j \pi^{ij} = 0$!!

Thus the allocations, prices, and agent-specific, history-contingent beliefs determined by the perturbed conditions constitute a rationally formed expectations equilibrium whose allocation cannot be that of a rational expectations equilibrium. The next section shows this leading example to be general

3. Rationally formed expectations equilibria of overlapping generations economies

Consider a deterministic stationary overlapping generations 1-good exchange economy with a representative generation consisting of a number H of 2-period lived agents with utility function u^h and endowments $e^h = (e_1^h, e_2^h)$ of the good, for all $h = 1, \dots, H$. Agents have access to historical records of length m (maybe infinity), so that they know the price of the good in the last m periods. I will assume moreover, without loss of generality, that the agents *believe* that prices follow a k-state Markov chain over k prices.[13]

In what follows, histories of prices $\{p_t\}_{t \in T}$ (where T can be either the set of positive integers $\mathbb{N}$ or the set of all integers $\mathbb{Z}$)[14] taking at any period any of a finite number[15] k of possible values $p^1, \dots, p^k$, are denoted by means of a function δ_t^i indicating whether the price p^i has been realized at period t or not. Thus $\delta_t^i = 1$ whenever $p_t = p^i$, and equals 0 otherwise. Since only one price can prevail at any period t, it must hold that $\sum_{i=1}^k \delta_t^i = 1$ for all $t \in T$. Therefore, a history of realizations is a sequence $\delta = \{\delta_t\}_{t \in T}$ of k-tuples of $k-1$ zeros and one 1 at the position of the realized price at that period, that is to say, for all $t \in T$, $\delta_t \in \{0,1\}^k$ and $\sum_{i=1}^k \delta_t^i = 1$. Let Δ denote the set of such sequences.

A specific instance[16] of a *rationally formed expectations equilibrium* is defined next.

Definition. *A rationally formed expectations equilibrium of the deterministic sta-*

[13] I will argue below that this assumption is not restrictive except for the counterfactual case in which memory m is infinite and $T = \mathbb{Z}$.

[14] In the case $T = \mathbb{N}$, a special first generation of agents born old is assumed as usual.

[15] Prices can be thought of as being expressed in multiples of euro cents, and to be believed by the agents to be below some sufficiently high value with probability 0, so that prices can take indeed only a finite number (although maybe very big) of values, for all practical purposes.

[16] In particular, with Markovian beliefs over a finite number of prices.

tionary overlapping generations exchange economy with representative generation $(u^h, e^h)_{h=1}^H$ *with memory* m *consists of*

(i) *a finite number of positive prices for consumption* $p^i > 0$, $i = 1, \ldots, k$,

(ii) *nonnegative first-period consumptions and contingent plans of second-period consumptions* $(c_1^{hi}, \{c_2^{hj}\}_{j=1}^k)$ *for each agent* $h = 1, \ldots, H$ *at each possible price when young, i.e.* $i = 1, \ldots, k$

(iii) *beliefs about the probabilities of transition between prices, i.e. a Markov matrix* $(\pi_{t\delta}^{hij})_{i,j=1}^k$, *for each agent* $h = 1, \ldots, H$ *and any history of prices* $\delta \in \Delta$ *up to his date of birth* $t \in T$,[17]

such that

(c.1) *the allocation is feasible, i.e. for all* $i = 1, \ldots, k$

$$\sum_{h=1}^{H}(c_1^{hi} + c_2^{hi}) = \sum_{h=1}^{H}(e_1^h + e_2^h) \tag{11}$$

(c.2) *for any history* $\delta \in \Delta$ *and every agent* $h = 1, \ldots, H$ *born at any date* $t \in T$, *his first-period consumption and contingent plan of second-period consumptions* $(c_1^{hi}, \{c_2^{hj}\}_{j=1}^k)$ *are optimal, given his beliefs, whenever at* t *the price is* p^i, *for all* $i = 1, \ldots, k$, *so that it solves*

$$\begin{aligned} \max \quad & \sum_{j=1}^{k} \pi_{t\delta}^{hij} u^h(c_1^i, c_2^j) \\ s.t. \ & p^i(c_1^i - e_1^h) + p^j(c_2^j - e_2^h) = 0, \quad \forall j \end{aligned} \tag{12}$$

(c.3) *for any history* $\delta \in \Delta$ *and every agent* $h = 1, \ldots, H$ *born at any date* $t \in T$, *no other beliefs for which* $(c_1^{hi}, \{c_2^{hj}\}_{j=1}^k)$ *is optimal when at* t *the price is* p^i, *for all* $i = 1, \ldots, k$, *provide a higher likelihood to the history of prices he remembers, i.e. if* $\pi^{i\cdot} \in S^{k-1}$ *is such that* $(c_1^{hi}, \{c_2^{hj}\}_{j=1}^k)$ *solves*

$$\begin{aligned} \max \quad & \sum_{j=1}^{k} \pi^{ij} u^h(c_1^i, c_2^j) \\ s.t. \ & p^i(c_1^i - e_1^h) + p^j(c_2^j - e_2^h) = 0, \quad \forall j \end{aligned} \tag{13}$$

[17] Note that although with this notation every agent is supposed to hold beliefs about the probabilities of transition after *every* history (i.e. even those beyond his life-span), only the histories up to the date of his decision are relevant. If memory is finite, the number of histories relevant for the agent's decision is finite, so that he is required to hold only finitely many beliefs. In the infinite memory case this is still the case if there is a first period, but not if there is not one: in that case the number of beliefs would be countable.

then

$$\prod_{j=1}^{k}(\pi^{ij})^{\sum_{\tau=t'}^{t}\delta_\tau^i\delta_{\tau+1}^j} \leq \prod_{j=1}^{k}(\pi_{t\delta}^{hij})^{\sum_{\tau=t'}^{t}\delta_\tau^i\delta_{\tau+1}^j} \tag{14}$$

where $t' = t - m$ *if* $T = \mathbb{Z}$ *and* $t' = \max\{1, t-m\}$ *if* $T = \mathbb{N}$, *and*

(c.4) *when* $T = \mathbb{Z}$ *and* $m = \infty$, *for any history* $\delta \in \Delta$ *and every agent* $h = 1, \ldots, H$ *born at any* $t \in T$, *his beliefs are not falsified by the information available then, i.e. for all* $i, j = 1, \ldots, k$,

$$\pi_{\delta t}^{hij} = \lim_{t' \to -\infty} \frac{\sum_{\tau=t'}^{t-1}\delta_\tau^i\delta_{\tau+1}^j}{\sum_{\tau=t'}^{t-1}\delta_\tau^i} \tag{15}$$

whenever the limit exists.

Some remarks on the definition above are in order. Note first that if the beliefs are constrained to be history and agent independent (so that $\pi_{\delta t}^{hij}$ becomes π^{ij}) and the last conditions (c.3) and (c.4) are dropped, then the definition above becomes that of a stationary rational expectations (sunspot) equilibrium following a k-state Markov chain, or k-SSE.[18] Note that condition (c.4) is trivially satisfied by such a k-SSE but, crucially, (c.3) is not. As a consequence, in a rational expectations equilibrium there exist typically, for every agent, beliefs about the probabilities of transition that are consistent with his consumption choice but that make the history he observes likelier than the equilibrium beliefs do. Of course the discrepancy between the agents' beliefs and those maximizing the likelihood of history while rationalizing the choices vanishes in the limit if, as in the sunspot equilibrium interpretation, the prices are supposed to actually follow a given Markov chain. But the determination of prices by a specific stochastic process is difficult to justify in the absence of shocks to the fundamentals.

Secondly, condition (c.3) is not superfluous. If instead of condition (c.3) only the existence of subjective beliefs rationalizing the agents' choices was required (regardless of history), that would imply a set of equilibrium allocations and prices that is a strict superset of the set of rationally-formed expectations equilibria. In effect, while any rationally-formed expectations equilibrium clearly satisfies the existence of subjective beliefs rationalizing the agents' choices, there are consumption

[18]See, for instance, Azariadis (1981), Azariadis and Guesnerie (1986), Chiappori and Guesnerie (1988, 1989), Guesnerie (1986). On the notion of sunspot equilibrium see Cass and Shell (1983).

plans, prices and arbitrary, history-independent subjective beliefs rationalizing the agents' choices that are not rationally-formed expectations equilibria, since history-independent beliefs cannot solve the problem (23) below —equivalent to condition (c.3)— *for all realizations of history*.

Finally, note also that, as previously claimed, the restriction to beliefs in Markovian prices is not constraining for finite memory or $T = \mathbb{N}$. In effect, such an assumption cannot be refuted by the agents unless the data available to them is able to falsify it, but for that to be the case it must at least allow to establish that the empirical frequencies are not Cauchy.[19] That is to say, it must allow to conclude that the distance between any two empirical frequencies at dates $t < t'$ from any p^i to p^j does not become arbitrarily small, for t, t' sufficiently far away down the sequence. In other words, for the agents to be able to discard the assumption of Markovian prices they would need to have infinite histories and memories, so that no data can falsify that assumption if $T = \mathbb{N}$ or m is finite.[20]

As a matter of fact, for any two consecutive terms[21] the distance between the empirical frequencies of transitions converges to zero along the sequence, since

$$\left| \frac{\sum_{\tau=1}^{t+1} \delta_\tau^i \delta_{\tau+1}^j}{\sum_{\tau=1}^{t+1} \delta_\tau^i} - \frac{\sum_{\tau=1}^{t} \delta_\tau^i \delta_{\tau+1}^j}{\sum_{\tau=1}^{t} \delta_\tau^i} \right| = \frac{\delta_{t+1}^i}{\sum_{\tau=1}^{t+1} \delta_\tau^i} \cdot \left| \delta_{t+2}^j - \frac{\sum_{\tau=1}^{t} \delta_\tau^i \delta_{\tau+1}^j}{\sum_{\tau=1}^{t} \delta_\tau^i} \right| \tag{16}$$

and

(1) either p^i is visited finitely many times and then for some t onwards $\delta_t^i = 0$, so that the distance between the empirical frequencies of transition becomes 0 from that term on, and the empirical frequency of transition from p^i to p^j becomes constant and therefore convergent,

(2) or p^i is visited countably many times and then the first factor in the right-hand side converges to zero,[22] while the second factor between brackets is

[19] If the sequence of empirical frequencies of transitions from any p^i to p^j was Cauchy, then completeness would imply its convergence, which would support the Markovian assumption.

[20] On the contrary, when $T = \mathbb{Z}$ and m is infinite, the agents can compute the empirical frequency at any given date t of the transitions from any price p^i to p^j as the limit

$$\lim_{t' \to -\infty} \frac{\sum_{\tau=t'}^{t-1} \delta_\tau^i \delta_{\tau+1}^j}{\sum_{\tau=t'}^{t-1} \delta_\tau^i}.$$

Should this limit not exist, the Markovian assumption would then be falsified by the data in this case.

[21] Actually, for any two terms a fixed number of periods apart.

[22] The numerator is bounded and the denominator is both non-decreasing and not non-increasing.

bounded in $[0, 1]$,[23] so that the distance between empirical frequencies of transition from p^i to p^j converges to zero.

Thus, when $T = \mathbb{N}$ and agents have unrestricted memory, not only the agents do not have enough information to falsify the Markovian prices assumption, but also they will see vanish progressively any dependence of the probabilities of transition on earlier prices (as differences between subsequent empirical frequencies converge to zero), i.e. Markovian prices tend to be confirmed (although not proved), rather than falsified.

Of course, agents can all believe in Markovian prices while not necessarily agreeing on the specific probabilities of transition governing that process, since they have access to different bits of history when $T = \mathbb{N}$ or memory is finite. On the contrary, if memory is infinite and $T = \mathbb{Z}$, they all have to agree on the probabilities of transitions as well if the limit in (15) above exists for every t; while, if memory is infinite and $T = \mathbb{N}$, they all "eventually agree", meaning that discrepancies of subsequent generations tend to vanish. In the last two cases, in which agents agree (maybe asymptotically) on the probabilities of transition, the limit of the empirical frequencies would necessarily have to be in the intersection on the unit simplex of the linear subspaces determined by the agents' first-order conditions, as proclaimed in Proposition 1 below (the proof is straightforward). In other words, if memory is infinite, the only rationally formed expectations equilibria are those for which such an intersection exists, but these equilibria are allocationally equivalent to the rational expectations (sunspot) equilibrium associated with such an intersection.

Proposition 1. *If the agents' memory m is infinite, any rationally formed expectations equilibrium of the stationary deterministic overlapping generations exchange economy $(u^h, e^h)_{h=1}^H$ is allocationally equivalent*[24] *to a k-state sunspot equilibrium.*

Rationally formed expectations equilibria distinct from a rational expectations equilibrium exist in this setup, therefore, only if memory is finite. There can be many reasons why m finite is the relevant case. People tend to make forecasts based on their recent experiences, with memories of variable lengths, but certainly of finite length if only because of their actual limited recording and computing abilities. Thus the limited memory case seems to be the relevant one, while the equivalence of rationally formed expectations equilibria and rational expectations sunspot equilibria

[23] The first term is in $\{0, 1\}$ and the second is in $[0, 1]$.

[24] In the sense of sharing the same allocation as support.

in the infinite memory case rather highlights the role played by limited knowledge in making possible rationally formed expectations equilibria distinct from rational expectations equilibria.

The next proposition establishes the main result of the paper, namely that any deterministic stationary overlapping generations economy with sunspot equilibria can be perturbed robustly in order to produce rationally formed expectations equilibria that no sunspot equilibrium can match.

Proposition 2. *Arbitrarily close*[25] *to every deterministic stationary overlapping generations economy with a k-state stationary sunspot equilibrium there exists an economy with finite-memory rationally formed expectations equilibria distinct from any rational expectations equilibrium.*

Proof. Let $(u^h, e^h)_{h\in H}$ be the utility and endowments of the members of the representative generation of a stationary overlapping generations economy, and let $\left\{p^i, (\bar{c}_1^{hi}, \bar{c}_2^{hi})_{h\in H}\right\}_{i=1}^k$ be the contingent prices and consumptions of a k-state stationary sunspot equilibrium of the economy driven by a Markov chain with probabilities of transition $(\pi^{ij})_{i,j=1}^k$.

Assume, without loss of generality, that the allocation in this equilibrium to the agents of type $h_0 \in H$ is feasible with their only resources, i.e.[26]

$$\bar{c}_1^{h_0 i} + \bar{c}_2^{h_0 i} = e_1^{h_0} + e_2^{h_0}. \tag{17}$$

Consider a new economy with a representative generation $(u^h, e^h)_{h\in H\cup\{h_1\}}$ consisting of adding to H an agent h_1 with the same endowments and consumptions as agent h_0,[27] and a utility function u^{h_1} with gradients at the consumption bundles $(\bar{c}_1^{h_1 i}, \bar{c}_2^{h_1 j})_{i,j=1}^k$ such that, for some $i = 1, \ldots, k$, the vectors[28]

$$\begin{aligned} A^i_{u^{h_0}} &= (A^{i1}_{u^{h_0}}, \ldots, A^{ik}_{u^{h_0}}) \\ A^i_{u^{h_1}} &= (A^{i1}_{u^{h_1}}, \ldots, A^{ik}_{u^{h_1}}) \end{aligned} \tag{18}$$

[25] In the topology of C^1-convergence over compacta in the space of utility functions.

[26] There is always a subset of types of agents for which this is true (note that this subset needs not be proper). In general, the replication and perturbation argument to be done next would then be done on all the types of agents in the subset.

[27] The new allocation of the new economy is feasible because of the assumption in (17).

[28] Where

$$A^{ij}_{u^h} \equiv D_1 u^h(\bar{c}_1^{hi}, \bar{c}_2^{hj})(\bar{c}_1^{hi} - e_1^h) + D_2 u^h(\bar{c}_1^{hi}, \bar{c}_2^{hj})(\bar{c}_2^{hj} - e_2^h)$$

are linearly independent, while adding $\mathbf{1} = (1, \ldots, 1)$ makes them linearly dependent, i.e.

$$A^i_{u^{h_1}} = \alpha A^i_{u^{h_0}} + \beta \mathbf{1} \tag{19}$$

for some α and some $\beta \neq 0$.[29] Then the system

$$\begin{aligned} \pi^{i1} A^{i1}_{u^{h_0}} + \cdots + \pi^{ik} A^{ik}_{u^{h_0}} &= 0 \\ \pi^{i1} A^{i1}_{u^{h_1}} + \cdots + \pi^{ik} A^{ik}_{u^{h_1}} &= 0 \end{aligned} \tag{20}$$

has no solution in the probabilities $\pi^{i1}, \ldots, \pi^{ik}$. In effect, should there be one, using equation (19) above, the second equation can be written equivalently as

$$\alpha\big(\pi^{i1} A^{i1}_{u^{h_0}} + \cdots + \pi^{ik} A^{ik}_{u^{h_0}}\big) + \beta(\pi^{i1} + \cdots + \pi^{ik}) = 0 \tag{21}$$

but from the first equation in (20) and $\beta \neq 0$, then one would have to have that

$$\pi^{i1} + \cdots + \pi^{ik} = 0 \text{ !!} \tag{22}$$

This establishes that the prices and consumptions $\big\{p^i, (\bar{c}^{hi}_1, \bar{c}^{hi}_2)_{h \in H \cup \{h_1\}}\big\}^k_{i=1}$, with $(\bar{c}^{h_1 i}_1, \bar{c}^{h_1 j}_2)^k_{i,j=1} = (\bar{c}^{h_0 i}_1, \bar{c}^{h_0 j}_2)^k_{i,j=1}$, are not those of a sunspot equilibrium of the economy with representative generation $(u^h, e^h)_{h \in H \cup \{h_1\}}$.[30]

They are, nevertheless, the allocation and prices of a rationally formed expectations equilibrium of an arbitrarily close economy. In effect, if u^{h_1} is close enough to u^{h_0} in the topology of C^1-convergence over compacta, then for all $h \in H \cup \{h_1\}$, all $t \in T$, and all $\delta \in \Delta$, there exists $\big(\pi^{hij}_{t\delta}\big)^k_{i,j=1}$ solution to

$$\begin{aligned} &\max_{\pi^{ij}} \prod_{i,j=1}^{k} (\pi^{ij})^{\sum_{\tau=t'}^{t} \delta^i_{\tau-1}\delta^j_\tau} \\ &\text{s.t. } \forall i,\ \pi^{i\cdot} \in S^{k-1} \\ &\quad \forall i,\ (\bar{c}^{hi}_1, \{\bar{c}^{hj}_2\}_j) = \arg\max \textstyle\sum_j \pi^{ij} u^h(c^i_1, c^j_2) \\ &\qquad\qquad \text{s.t. } p^i(c^i_1 - e^h_1) + p^j(c^j_2 - e^h_2) = 0, \quad \forall j \end{aligned} \tag{23}$$

for any h. Note that the equilibrium conditions for a k-SSE type of sunspot equilibrium are

$$\sum_{j=1}^{k} \pi^{ij} A^{ij}_{u^h} = 0$$

for all $i = 1, \ldots, k$.

[29]Note that since $\sum_{j=1}^k \pi^{ij} A^{ij}_{u^{h_0}} = 0$, the vector $A^i_{u^{h_0}}$ cannot be collinear to $\mathbf{1}$. Moreover there is a 1-dimensional manifold of *directions* that the vector $A^i_{u^{h_1}}$ can take while satisfying these conditions.

[30]Otherwise, system (20) would have a solution in $\pi^{i1}, \ldots, \pi^{ik}$, for all i.

—where $t' = t - m$ if $T = \mathbb{Z}$, and $t' = \max\{1, t - m\}$ if $T = \mathbb{N}$— since the objective function is continuous, and the constrained set is non-empty and compact. The same is true for any robust and small enough perturbation $\tilde{u}^{h_1}$ of u^{h_1}, therefore not necessarily for $A^i_{\tilde{u}^{h_1}}$ in the span of $A^i_{\tilde{u}^{h_0}}$ and **1**.

Finally, since m is finite, the remembered empirical frequencies of the transitions do not falsify the agents' beliefs.

Q.E.D.

4. Epistemic status of rationally formed expectations equilibria

In order to compare the epistemic requirements of a rationally formed expectations equilibrium with those of rational expectations equilibria, I will discuss it here within a model specifying an interactive system of beliefs and higher order beliefs in order to compare it from an epistemic viewpoint with rational expectations equilibria. For the sake of clarity, this will be done without specifying the cardinality of agents and goods, as well as the demographics of the economy.[31] As it will be seen below, the equilibrium notions do not depend conceptually on these details, while there is a clear notational advantage in overlooking them at this stage.

A definition of a rational expectations equilibrium corresponding to Radner (1979) is next first, followed by a discussion of its epistemic implicit assumptions.

Definition 1. *Given a probability distribution π over a set of states of nature S and an economy $\{(u^h_s, e^h_s)_{s \in S}, I^h\}_{h \in H}$ (where, for all $h \in H$, I^h is a partition of S)*[32] *a* **rational expectations equilibrium** *is a set of contingent consumptions*

[31] Thus the case where the number of states, goods and agents is not finite, and agents may be endowed with, and have preferences on, only a few goods (of which the overlapping generations setup is an instance) is therefore comprised in the following discussion. Obviously, sums stand for the adequate aggregations and measures when infinities are involved.

[32] The endowments e^h_s are measurable with respect to I^h, so that agents know their endowments. In what follows I^h_s denotes the element of the partition I^h containing (state) s. Also, for all s and all h, $u^h_s \in \mathbb{R}^{\mathbb{R}^l_+}$ has the usual properties and $e^h_s \in \mathbb{R}^l_+$.

and prices $\{(c_s^h)_{h\in H}, p_s\}_{s\in S}$ *such that*

(a.1) *for all* $s \in S$

$$\sum_{h\in H}(c_s^h - e_s^h) \leq 0 \tag{24}$$

(a.2) *for all* $h \in H$

$$\begin{aligned}(c_s^h)_{s\in S} \in \arg\max_{(c_s)_{s\in S}} & \sum_{s\in S} \pi_s u_s^h(c_s) \\ & p_s(c_s - e_s^h) \leq 0, \forall s \in S \\ & c_s = c_{s'}, \forall s \in S, \forall s' \in I_s^h \text{ such that } p_s = p_{s'}\end{aligned} \tag{25}$$

The last constraint of the optimization problem in condition (a.2) in Definition 1 prevents agents that are (possibly asymmetrically) uncertain about the state of nature s from conditioning on things they cannot see, either directly through their information partition I^h or by being revealed by prices.

Implicitly in the previous definition each agent h is obviously knows at least $\{\pi_s, u_s^h, e_s^h, p_s\}_{s\in S}$ and I^h —otherwise his choice could not be modeled as in (a.2) above— other than this, the agents do not need to have any further knowledge at a rational expectations equilibrium as defined above. In particular, no common knowledge of anything is needed to sustain a rational expectations equilibrium (some common knowledge has nonetheless been required to address some strong features of the definition above, like the need of agents to know the entire price function $(p_s)_{s\in S}$ and the generic full revelation of prices —see McAllister (1990)— but as far as the epistemic requirements of rational expectations equilibria as defined above is concerned, nothing more than $\{\pi_s, u_s^h, e_s^h, p_s\}_{s\in S}$ and I^h for each agent h is required).[33]

[33] This has a parallel in the epistemic conditions for a Nash equilibrium characterized in Aumann and Brandemburger (1995). In effect, as the authors point out there, Nash equilibria —understood, as usual, as profiles of randomizations over pure strategies— require (besides the agents' rationality) only the knowledge by the agents of their own payoffs and their *mutual* knowledge of each other's strategies. Interestingly enough, again no common knowledge of anything is actually required (common knowledge is only required to make sense of the interpretation of Nash equilibria as profiles of commonly held *conjectures* about each player's action, and this only when there are at least three players). In the current context, this amounts to the agents knowledge of the elements determining (and constraining) their payoffs, i.e. $\{\pi_s, u_s^h, e_s^h, p_s\}_{s\in S}$ (the assumed price-taking behavior voiding of content in this case the requirement of mutual knowledge of each others' decisions).

Notwithstanding, at a rational expectations equilibrium each agent h knows implicitly more than just $\{\pi_s, u_s^h, e_s^h, p_s\}_{s\in S}$ and I^h. In effect, he actually knows that, *and what he can deduce from that.* In effect, firstly an agent h is able to tell, at any given state s, whether an event $E \subset S$ has happened or not, based on I^h, if, and only if, he assigns a probability to E conditional to I_s^h of either 1 or 0 respectively, i.e. if, and only if, $I_s^h \subset E$ or $I_s^h \subset E^C$ respectively, where $E^C = S \setminus E$ (more generally, he attaches at I_s^H a probability $P(E \mid I_s^h)$ to any event E). Nevertheless, the knowledge of the equilibrium prices $(p_s)_{s\in S}$ allows him to tell as well whether an event has happened or not based also on the partition $\{p^{-1}(p_s)\}_{s\in S}$ induced by prices.[34] Of course this means that at a rational expectations equilibrium h is able to tell whether E has happened or not at state s if, and only if, $I_s^h \cap p^{-1}(p_s) \subset E$ or $I_s^h \cap p^{-1}(p_s) \subset E^C$ respectively (more generally, he attaches at I_s^h and p_s a probability $P(E \mid I_s^h \cap p^{-1}(p_s))$ to event E), which allows him to notice (and hence condition on) a bigger set of events than with I^h alone.

As the definition above makes clear, a defining feature of a rational expectations equilibrium is that agents are supposed to share the same prior π over the states of nature. At a rationally formed expectations equilibrium this requirement is dropped instead, and just a rational use of the available information (I^h and p for each agent h) is required.[35] A formal definition on a rationally formed expectations equilibrium is next, after which we discuss how its implicit epistemic assumptions compare to those of the rational expectations equilibria.

Definition 2. *Given a set of states of nature S and an economy $\{(u_s^h, e_s^h)_{s\in S}, I^h\}_{h\in H}$ (where, for all $h \in H$, I^h is a partition of S)*[36] *a* **rationally formed expectations equilibrium** *is a set of contingent consumptions, beliefs and prices $\{(c_s^h, \pi_s^h)_{h\in H}, p_s\}_{s\in S}$ such that*

(a.1) *for all $s \in S$*

$$\sum_{h\in H} (c_s^h - e_s^h) \leq 0 \tag{26}$$

[34]Where $p \in (\mathbb{R}_{++}^L)^S$ stands for the function assigning p_s to s.

[35]There is thus room for different agents to hold different expectations at a rationally formed expectations equilibrium.

[36]The endowments e_s^h are measurable with respect to I^h, so that agents know their endowments.

(a'.2) *for all* $h \in H$,[37]

$$(c^h_s)_{s\in S} \in \arg\max \sum_{s\in S} \pi^h_s u^h_s(c_s) \\ p_s(c_s - e^h_s) \leq 0, \forall s \in S \\ c_s = c_{s'}, \forall s \in S, \forall s' \in I^h_s \textit{ such that } p_s = p_{s'} \tag{27}$$

(a.3) *for all* $h \in H$, *and all* $(\pi_s)_{s\in S}$ *such that*

$$(c^h_s)_{s\in S} \in \arg\max \sum_{s\in S} \pi_s u^h_s(c_s) \\ p_s(c_s - e^h_s) \leq 0, \forall s \in S \\ c_s = c_{s'}, \forall s \in S, \forall s' \in I^h_s \textit{ such that } p_s = p_{s'} \tag{28}$$

it holds that, for all $s \in S$,

$$\sum_{s'\in I^h_s \cap p^{-1}(p_s)} \pi_{s'} \leq \sum_{s'\in I^h_s \cap p^{-1}(p_s)} \pi^h_{s'}$$

As opposed to the previous definition of a rational expectations equilibrium, the agents are not required to hold the same prior anymore, but a new condition (a.3) still imposes a consistency condition that prevents agents to hold arbitrary priors: their priors $(\pi^h_s)_{s\in S}$ must satisfy that no other beliefs $(\pi_s)_{s\in S}$ rationalizing their choices attach a higher likelihood to whatever event $I^h_s \cap p^{-1}(p_s)$ they observe (either directly or through prices).

Note that from the definitions neither rational expectations implies rationally formed expectations, nor conversely. In effect, for a rationally formed expectations equilibrium to be a rational expectations equilibrium all the agents would have to hold a common prior on the state of the world, which need not be the case. Also for a rational expectations equilibrium to be a rationally formed expectations equilibrium the additional non-trivial condition (a.3) above needs to be satisfied, which again needs not be the case for any given rational expectations equilibrium.

[37]Note that this is not the same as (a.2) since beliefs can differ across agents here.

As a matter of fact, none of the two equilibrium notions is a particular case of the other but, according to the definitions above, the two share the same epistemic requirements nonetheless, since what is implied about the agents' knowledge by the definition of a rationally formed expectations equilibrium is the same as in a rational expectations equilibrium. In effect, what the agents are supposed to know at a rationally formed expectations equilibrium, as well as what they can deduce from that knowledge, is —as in the case of a rational expectations equilibrium— determined only by their information partitions I^h, for each h, and the partition $\{p^{-1}(p_s)\}_{s\in S}$ induced by prices, which is the information about the state of nature conveyed by prices. Neither the fact that in a rationally formed expectations equilibrium the agents may hold different priors about the state of nature, nor its additional condition (a.3) adds anything that is not already implicit in the knowledge by each agent h of $\{\pi_s^h, u_s^h, e_s^h, p_s\}_{s\in S}$ and I^h and what is implied by this.

In order to see that, on top of no common knowledge assumption being necessary to sustain neither a rational expectations equilibrium nor a rationally formed expectations equilibrium, the two concepts are actually more stringent than common knowledge of rationality and market clearing,[38] let us consider first the following extension of the Definition 1 above of a rational expectations equilibrium. Specifically, let us allow for each agent h to be of different types $t^h \in T^h$ that, while having no impact on the fundamentals, can nonetheless be relevant for the equilibrium, if only because the agents may believe that opponents of different types may behave differently. Making this possibility explicit calls for specifying an interactive system of beliefs and high order beliefs about the other agents beliefs. This, of course, leaves room for the agents to hold different beliefs about prices in different states of the world (which now include the profile of agents' types $(t^h)_{h\in H}$ alongside the state of nature s) thus necessarily departing from the rational expectations equilibrium notion.

Definition 3. *Given a probability distribution π over a set of states of nature S and an economy $\{(u_s^h, e_s^h)_{s\in S}, I^h\}_{h\in H}$ (where, for all $h \in H$, I^h is a partition of the set S)*[39] *a set of contingent consumptions and prices $\{(c_{st}^h)_{h\in H}, p_{st}\}_{s\in S, t\in \times_h T^h}$ (where t^h stands for a type for each agent h) is* **consistent with common knowledge**

[38] For rational expectations equilibria this has been established in Ben-Porath and Heifetz (2010) for finite exchange economies with asymmetric information.

[39] The endowments e_s^h are measurable with respect to I^h, so that agents know their endowments. Here also I_s^h denotes the element of the partition I^h containing (state) s. Also, for all s and all h, $u_s^h \in \mathbb{R}^{\mathbb{R}_+^l}$ has the usual properties and $e_s^h \in \mathbb{R}_+^l$.

of rationality and market clearing *if, and only if,*

(b.1) *for all* $s \in S$ *and all* $t \in \times_h T^h$

$$\sum_h (c^h_{st} - e^h_s) \leq 0 \tag{29}$$

(b.2) *for all* $h \in H$ *and all* $t^h \in T^h$

$$\begin{aligned} (c^h_{st})_{s,t^{-h}} \in \arg\max \sum_{s,t^{-h}} \pi^h_{st} u^h_s(c_{st}) & \\ p_{st}(c_{st} - e^h_s) \leq 0, \forall s, \forall t^{-h} & \\ c_{st} = c_{s't'}, \forall s, \forall s' \in I^h_s, \forall t, t' \text{ such that} & \\ t^h = t'^h \text{ and } p_{st} = p_{s't'} & \end{aligned} \tag{30}$$

for some belief $\pi^h_{st} \in \Delta(S \times (\times_h T^h))$ *over states of the world such that, for all* $h \in H$ *and all* $t^h \in T^h$,

$$\sum_{t^{-h}} \pi^h_{st} = \pi_s \tag{31}$$

Condition (b.1) requires the allocation to clear markets *in every state of the world*, guaranteeing thus common knowledge of market clearing. Condition (b.2), given the measurability constraint, clearly implies that, for every agent $h \in H$ and state of the world $(s, t) \in S \times (\times_h T^h)$,

$$\begin{aligned} c^h_{st} \in \arg\max u^h_s(c_{st}) & \\ p_{st}(c_{st} - e^h_s) \leq 0 & \end{aligned} \tag{32}$$

which guarantees common knowledge of rationality. Condition (b.2), finally, requires too the agents to hold beliefs that, while not necessarily in accordance with those of other agents when it comes to beliefs about the state of the *world*, they agree nonetheless of the objective probabilities about the state of *nature*.[40]

[40] Note that condition (b.2) implies also that, for all $h \in H$ and all $t^h \in T^h$, clearly

$$\sum_{s,t^{-h}} \pi^h_{st} = 1$$

so that each agent h of each type t^h knows his own type.

It is straightforward to see that any rational expectations equilibrium $((c_s^h)_h, p_s)_s$ is consistent with (although does not require) common knowledge of rationality and market clearing (letting all c_{st}^h and p_{st} be trivially c_s^h and p_s respectively, and π_{st}^h be any such that the last condition in (b.2) holds). The fact that the converse is not true is precisely what has been established for finite exchange economies with asymmetric information in Ben-Porath and Heifetz (2010).

Similarly, the Definition 2 of a rationally formed expectations equilibrium can be extended as follows to include a system of interactive beliefs guaranteeing common knowledge of rationality, market clearing, and belief formation rationality.

Definition 4. *Given an economy $\{(u_s^h, e_s^h)_{s\in S}, I^h\}_{h\in H}$ (where, for all $h \in H$, I^h is a partition of a set S of states of nature)*[41] *a set of contingent consumptions, beliefs, and prices $\{(c_{st}^h, \pi_{st}^h)_{h\in H}, p_{st}\}_{s\in S, t\in \times_h T^h}$ (where t^h stands for a type for each agent h) is* **consistent with common knowledge of rationality, market clearing, and belief formation rationality** *if, and only if,*

(b.1) *for all $s \in S$ and all $t \in \times_h T^h$*

$$\sum_h (c_{st}^h - e_s^h) \le 0 \tag{33}$$

(b'.2) *for all $h \in H$ and all $t^h \in T^h$,*

$$\begin{aligned} (c_{st}^h)_{s,t^{-h}} \in \arg\max_{(c_{st})_{s,t^{-h}}} & \sum_{s,t^{-h}} \pi_{st}^h u_s^h(c_{st}) \\ & p_{st}(c_{st} - e_s^h) \le 0, \forall s, \forall t^{-h} \\ & c_{st} = c_{s't'}, \forall s, \forall s' \in I_s^h, \forall t, t' \text{ such that} \\ & \qquad t^h = t'^h \text{ and } p_{st} = p_{s't'} \end{aligned} \tag{34}$$

(b.3) *for all $h \in H$, all $t^h \in T^h$, and all $(\pi_{st})_{s\in S, t\in\times_h T^h}$ such that*

$$\begin{aligned} (c_{st}^h)_{s,t^{-h}} \in \arg\max_{(c_{st})_{s,t_{-h}}} & \sum_{s,t_{-h}} \pi_{st} u_s^h(c_{st}) \\ & p_{st}(c_{st} - e_s^h) \le 0, \forall s, \forall t^{-h} \\ & c_{st} = c_{s't'}, \forall s, \forall s' \in I_s^h, \forall t, t' \text{ such that} \\ & \qquad t^h = t'^h \text{ and } p_{st} = p_{s't'} \end{aligned} \tag{35}$$

[41] The endowments e_s^h are measurable with respect to I^h, so that agents know their endowments. Here also I_s^h denotes the element of the partition I^h containing (state) s. Also, for all s and all h, $u_s^h \in \mathbb{R}^{\mathbb{R}^l_+}$ has the usual properties and $e_s^h \in \mathbb{R}^l_+$.

it holds that, for all $s \in S$ and all $t^h \in T^h$

$$\sum_{\substack{t' \in \times_h T^h | t'^h = t^h \\ s' \in I_s^h \cap p^{-1}(p_{st'})}} \pi_{s't'} \leq \sum_{\substack{t' \in \times_h T^h | t'^h = t^h \\ s' \in I_s^h \cap p^{-1}(p_{st'})}} \pi^h_{s't'} \tag{36}$$

Conditions (b.1) and (b'.2) guarantee, as before, common knowledge of market clearing and rationality respectively. Condition (b.3), in its turn, guarantees to be common knowledge that no one could hold other beliefs rationalizing his choice that attach a higher likelihood to the event he observes, i.e. that the beliefs held are rationally formed.

Since any rationally formed expectations equilibrium $((c_s^h, \pi_s^h)_h, p_s)_s$ is clearly consistent with (although do not require) the conditions in Definition 4 for common knowledge of rationality, market clearing, and belief formation rationality (letting all c_{st}^h, p_{st} and π_{st}^h be trivially c_s^h, p_s, and π_s^h respectively) and, according to Proposition 2, there are rationally formed expectations equilibria distinct from any rational expectations equilibrium, adding common knowledge of belief formation rationality to that of rationality and market clearing is still not enough to guarantee rational expectations equilibrium outcomes.[42]

Finally, it is important to realize also that, according to the two definitions above, deterministic environments do not necessarily imply deterministic equilibrium allocations and prices. In effect, in the definition of a rational expectations equilibrium provided above, the fundamentals u_s^h and e_s^h might actually not depend on the state of the world s and the economy might still have non-deterministic rational expectations equilbria, i.e. sunspot equilibria. This is a well-known fact that follows from the sufficient characterization by Cass and Shell (1983) of the conditions under which sunspots *do not* matter (basically those of a finite, convex, complete markets Arrow-Debreu economy), which essentially opened the path towards establishing subsequently that sunspots *do* matter in almost any other setup.[43] That the same can be said about rationally formed expectations equilibria is what this paper

[42]Ben-Porath and Heifetz (2010) had already established that common knowledge of rationality and market clearing alone is not able to guarantee a rational expectations equilibrium outcome.
[43]That is to say, with infinitely many agents or goods, or with incomplete markets, or with non convex preferences or productions sets.

establishes. Note once more that the epistemic status of both rational expectations and rationally formed expectations equilibria, being the same, has no import on this fact.

5. Discussion

Firstly, given that Proposition 2 establishes that rationally formed expectations equilibria can account for more fluctuations than rational expectations equilibria, one would like to have an idea of where do the limits of this expansion lay or, at least, whether the proposed equilibrium notion does not go too far as to be able to rationalize *any* fluctuations as an equilibrium phenomenon. In order to see that not anything can be made into a rationally formed expectations equilibrium, consider a feasible allocation of consumptions c_1^i, c_2^i, for all $i = 1, \ldots, k$, such that for some agent and some price p^i, it holds that all his trades contingent to any price p^j he may face in his second period of life imply a higher marginal rate of substitution of future for present consumption than the corresponding implicit relative price, i.e.

$$A^{ij} \equiv D_1 u(c_1^i, c_2^j)(c_1^i - e_1) + D_2 u(c_1^i, c_2^j)(c_2^i - e_2) < 0 \tag{37}$$

for all $j = 1, \ldots, k$. For this to happen, it suffices —in the case the marginal rate of substitution is smaller than 1 at the endowments point— that c_1^i is small enough whenever the solutions are guaranteed to be always interior. Then the set of expectations consistent with the agent's choice of c_1^i when facing p^i in his first period of life is empty (the fist-order conditions of (12) in the definition cannot be satisfied, since the associated hyperplane does not intersect the unit simplex, its normal direction being in the strictly positive orthant). As a consequence, no fluctuations between the feasible allocation of such consumption levels c_1^i, c_2^i, for all $i = 1, \ldots, k$, can result from a rationally formed expectations equilibrium.

Finally, the rationality condition considered here on the formation of expectations seems reminiscent of the one underlying the rational beliefs equilibrium concept of Kurz (1994a,b). Nonetheless, rationally formed expectations differ essentially from Kurz's rational beliefs. The two concepts only share the idea that the rationality of expectations or "beliefs should be defined relative to what is learnable from the data" (Kurz (1994b), p.879). Otherwise, Kurz (1994b) requires the agents to believe that prices are driven by a process whose *long term* behavior coincides with that of the true process. Leaving aside the problem posed by the ad hoc character of such a true process in the pure extrinsic uncertainty case, in order to infer such long term behavior Kurz (1994b) assumes that the agents have access to infinitely long histories of past prices, a formidable feat that the rationally formed expectations equilibrium deliberately avoids to assume.

Also Hommes (1998) and Hommes and Sorger (1998) introduce in a different setup an equilibrium notion, the consistent expectations equilibrium, which imposes as well a condition of consistency with available data, namely the zero (limit of) autocorrelations of errors made in past forecasts based on history, so that they cannot be distinguished from white noise. Note however that in a consistent expectations equilibrium agents try to forecast a relevant variable, say a price, while in the setup considered here they try to forecast the *probability distribution* of that variable. Also the consistent expectations equilibrium notion makes implicitly the counterfactual assumption, as in Kurz (1994a,b), that infinitely many records of past realizations of this variable are always available and agents have an infinite memory and computation ability allowing to process them, otherwise finite sample autocorrelations of a given number of lags will always be typically nonzero.

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