TASTE CHANGE IN THE
TRUE COST-OF-LIVING INDEX

by

Eric de SOUZA

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UNITE ANALYSE ECONOMIQUE

INSTITUT DES SCIENCES ECONOMIQUES
121, Parkstraat B-3000 LOUVAIN

Belgium

INTRODUCTION

The theory of the true cost-of-living index is a natural corollary to consumer demand theory. The introduction of the linear expenditure model by Klein and Rubin (1947-48) demonstrated that it was possible to arrive at theoretically plausible demand models which could be estimated, without having direct recourse to a preference ordering, even if the class of utility functions underlying their model was soon discovered by Geary (1950-51). This developement also allowed econometric applications of the theory of the true cost-of-living index (cfr. GOLDBERGER (1967).

Recent developments in consumer demand analysis and its econometric applications have led to a dynamic version of the linear expenditure model (PHLIPS (1972)). Concurrent with these developments was the publication of an article by Franklin Fisher and Karl Shell (FISHER-SHELL (1969)) on the introduction of taste and quality change in the theory of the true cost-of-living index. However, their analysis of the effect of taste change on the index was limited in scope, excluding, for instance, the kind of change reflected in the Phlips' dynamic model. Moreover, the manner in which they allow taste change to influence the index is rather superficial and does not really endogenise taste change. The object of this paper is to present a more general framework for the endogenisation of taste change in the index.

"Asking the right question", state Fisher and Shell, "is a good part of obtaining the answer" (FISHER-SHELL (1969)p.98). However, asking a leading question, even if one is convinced of its righteousness, tends to bias the reply. On the other hand, an analysis of the answers already proposed is a good way of arriving at a correct formulation of the question: history is an excellent, if not infallible, master. We shall, therefore begin with a consideration of the manner in which the static true cost-of-living index is constructed in order to arrive at the question it is designed to answer (section one). We shall then, after a brief regard on the Fisher-Shell proposal and a definition of what we understand by taste change, propose different methods of endogenising it and examine them in turn (section two). Section three shall be devoted to how the index is affected by the introduction of taste change. We shall end by relating the Phlips' dynamic version of the linear expenditure model to the preceding discussion (section four).

I. THE STATIC TRUE COST-OF-LIVING INDEX

The true cost-of-living index is a member of the functional group of indexes, as opposed to the atomistic or purely statistical type associated with the name of Irving Fisher (FISHER (1923)). In the functional approach, prices and quantities are considered as being linked by certain typical relations. An arbitrary movement of prices entails a determined variation of quantities. Consider a set of situations, each characterised by a price vector. We suppose that the consumer has access potentially to a continuum of quantities and that his actual choice in each situation depends on the given price vector and the income allotted to him. We select two situations from the set, index them 0 and 1, respectively, and seek a criterion to determine objectively whether a pair of incomes, one allotted to him in situation 0 and the other in situation 1, would render him indifferent between the two situations. If so, the two incomes are said to be equivalent to each other ; and the functional price index is the ratio of the income allotted in situation 1, and called comparison income, to that allotted in situation 0, called base income. One can also consider base income as determined by a prior decision and seek a comparison income equivalent to it (cfr. FRISCH (1936)).

The criterion that the true cost-of-living index uses derives from the theory of consumer demand with its underlying preference theory. We assume throughout that the consumer's preferences are complete (or total), reflexive, transitive, continuous, monotonic (or increasing) and strictly convex. Moreover, any utility function representing it must be twice differentiable. Two systems of prices and income are said to be equivalent if they allow the consumer to attain, in an optimal fashion, the same indifference surface. Unfortunately, the ratio of the incomes respectively needed to reach an indifference surface under two different price systems is not in general independent of the surface retained [1]. For this reason, a reference indifference class must be specified beforehand.

The determination of the indifference class is subordinated to the prior choice of a reference vector of prices and income. The manner in which the latter are in practice chosen shall be considered further on. However, whatever the manner of selection, there remains a certain degree of arbitrariness ; therefore, the vector of prices and income serving as reference shall in general be taken as arbitrary but fixed. The reference indifference class is the one to which the reference vector of prices and income belongs when one considers the space of prices and income (on which the indirect utility function is defined) ; it is the one optimally attained in the commodity space given the budget constraint that the reference vector of prices and income imposes.

The true cost-of-living index is easily calculated once the reference vector of prices and income is chosen. In the commodity space : maximisation of a direct utility function representing the consumer's preferences, subject to the budget constraint determined by the reference vector of prices and income, leads to the relevant indifference curve (or, more precisely, to the utility level indicating the indifference class). The minimum costs of attaining the so-determined "reference" utility level given comparison-period prices, i.e., those associated with situation 1, provides the comparison income, or numerator of the index [2] ; a similar minimum cost, given base-period prices, i.e., those of situation 0, is the base income or denominator of the index. In the space of prices and income : the indirect utility function will directly yield the relevant utility level by mere substitution of the reference vector of prices and income therein. Inserting the base-period price regime into the indirect utility function and equating it with the reference utility level will yield base income on resolution. A similar operation using comparison-period prices will give comparison income. Whichever the technique employed, the resulting index will not only be identical (which is evident) but also ordinal, that is, independent of the choice of the utility function as long as it represents the same preferences. It should be noted that the index involves the use of only one indifference, or preference, map.

The true cost-of-living index such as it has been developed above can be indifferently considered as designed to answer any of the three following problems :

(1) given base-period prices and comparison-period prices of goods, to find two incomes : one which renders the consumer indifferent between base-period prices and a reference vector of prices and income, the other rendering

him indifferent between comparison-period prices and the same reference vector ;

(2) given base-period prices and comparison-period prices of goods, to find two incomes : one which allows the consumer to optimally acquire a bundle of goods, under base-period prices, lying in the same indifference class as that bundle which he optimally obtained given reference prices and income ; the other allowing him to optimally acquire, under comparison-period prices, a commodity vector lying the same indifference class as defined above ;

(3) given base-period prices and comparison-period prices, to find two incomes : one which when combined with base-period prices results in the same utility level as that obtained with a reference vector of prices and income ; the other which together with comparison-period prices yields exactly the same utility level.

The logic behind the three different formulations is the following. Corresponding to the reference vector of prices and income there is a budget constraint. The latter determines a budget set in the commodity space. For any given preference ordering, this set has an optimal element, defined as that point which is strictly preferred to all others in the budget set. We shall call this point the reference commodity vector. Once a utility function is chosen, the indifference class to which the reference vector of prices and income belongs (in the space of prices and income) and that to which the reference commodity vector belongs (in the commodity space) are indicated by the same utility level, which we denominate the reference utility level. From the economic-theoretical point of view, it is the second formulation of the problem which seems to us to be the most important, for the consumer's preference ordering is defined on the commodity space. The preference ordering defined on the space of prices and income is, in fact, an induced relation. Therefore, in our opinion, the reference commodity vector plays a central role, the reference vector of prices and income serving essentially to identify it.

The set of situations have until now been considered as being characterised uniquely by price vectors. In fact, for each situation data is available not only concerning the ruling prices but also on total consumer expenditure, i.e., the income the (average) consumer actually had at his disposition in that situation. This leads us to two natural, or obvious, choices for the reference vector of prices and income. The first possible choice is base-period income and prices. Their use for reference purposes results in Laspeyres-type indexes : those which have base-period income as base income. The second possibility is comparison-period prices and income, resulting in Paasche-type indexes, i.e., those having comparison-period income as comparison income. As will be shown further on, the index proposed by Fisher and Shell is of the Laspeyres type ; and it is essentially this kind of index that will retain our attention.

II. TASTE CHANGE IN THE TRUE COST-OF-LIVING INDEX

The theory of the true cost-of-living index as presented above is based on the hypothesis of invariant consumer preferences : his indifference, or preference, map does not change over time. In this section we shall introduce the existence of taste change. However, we shall limit our scope to changes in taste that can be translated in the utility function in the form of parameters that are functions of continuous time. This implies that the dynamic preference ordering incorporating taste change, whatever the cause be of it, can be represented by the following general type of function : $u(x_1,\ldots, x_n, b_1(t),\ldots, b_m(t))$, or $u(x, b(t))$ in short. We shall make a further assumption concerning, this time, the consumer's maximisation process : at each moment of time, the consumer takes his current preferences and maximises them subject to the budget constraint that the current price system and the income available to him imposes. There is no taking account of the future, or intertemporal maximisation. The past, however, may be reflected in the manner his taste parameters vary, e.g., a sort of stock adjustment process, as described by Houthakker and Taylor (1970). This manner of maximisation is often known as the dynamic myopic case.

What influence should the introduction of taste change have on the true cost-of-living index ? F. Fisher and K. Shell contend that the only reasonable manner of effecting its introduction is in answer to the following question that, according to them, the index is supposed to answer : "Given base-period prices $\hat{p}_1, \hat{p}_2,\ldots, \hat{p}_n$, base-period income $\hat{y}$, current prices of goods $p_1, p_2,\ldots, p_n$, the problem is to find that income y such that the representative consumer is <u>currently</u> indifferent between facing current prices with income y and facing base-period prices with base-period income. The true cost-of-living index is then $(y/\hat{y})$". (FISHER-SHELL (1969) p. 102).

In what way does the Fisher-Shell index, as we shall name the above, reflect the existence of taste change ? Solely, in the fact that it uses the current, or comparison-period, preference map of the consumer. As a consequence of taste change, this preference map is different from that of the base-period ; and, therefore, the comparison income y resulting from the use of the former is not the same as that which would have resulted from the employ of the latter. The two maps, and consequently comparison incomes, would coincide in the absence of taste change. The index implicitly assumes that base period income and base income coincide. The authors, therefore, have only the Laspeyres type of index in mind.

Yet, in what seems to us an important sense, the Fisher-Shell index does not really endogenise taste change. The crucial test for us is the following : suppose that prices do not change between the base and comparison period whereas tastes continue to evolve. Comparison income shall equal base income, and the index will stand at unity (or hundred, if scaled to percentages).

Are there methods of constructing a true index of the cost-of-living which endogenises taste change and withstands the ordinalist critique ? Since Fisher and Shell maintain the contrary, one possible manner of finding out is by searching for a fault in their exposé. This seems to lie in their apparent failure to distinguish between the role of base-period prices as characteristic of a situation (and to be compared with comparison-period prices, characteristic of another situation) ; and the use of the very same prices, together with base-period income, as reference units. The authors appear to be aware of the distinction in footnote 7 of their article ; however, it plays no role in the main body of the text. What is in fact a Laspeyres type index defined on the space of prices and income is presented as a general answer to the problem of the true cost-of-living index. Their failure to differentiate between the two vitiates their approach, and closes their eyes to other possibilities [3]. From the logical point of view, base-period prices and comparison-period prices are not compared directly to one another in the index ; but to a common standard. This standard can be chosen in three different manners as the first section pointed out. The fact that in general it is the reference vector of prices and income which is used to determine the other standards is chiefly on operational grounds (its choice delimits a budget set). As already remarked, from an economic-theoretical viewpoint, the reference commodity vector has priority.

We shall now propose three methods, based on the three standards : a reference vector of prices and income, a reference commodity vector and a reference utility level, of endogenising taste change in the index :

1. given base-period prices and comparison-period prices, to find two incomes : one such that the consumer is indifferent between it and base-period prices and a reference vector of prices and income, under base-period tastes ; the other such that he is indifferent between it and comparison-period prices and the same reference vector, under comparison-period tastes ; the index is the ratio of the latter income to the former ;

2. given base-period prices and comparison-period prices of goods, to find two incomes : one which allows the consumer, under base-period prices and base-period tastes, to optimally acquire a bundle of goods lying in the same indifference class as the reference commodity vector ; another which allows him to optimally acquire a bundle in the same indifference class as the reference vector, under comparison-period prices and tastes ; the index is the ratio of the second income to the first one ;

3. given base-period prices and comparison-period prices of goods, to find two incomes : one which when combined with base-period prices results in a given reference utility level, subject to base period tastes ; the other which, together with comparison-period prices results in the same level, subject this time to comparison-period tastes.

The first two formulations above, those based on a reference vector of prices and income and on a reference commodity vector, are irreproachably ordinal in our view. There is no comparison of utilities - only indifference surfaces are involved - and no direct inter-period comparisons. However, the third formulation using a reference utility level creates a problem in the dynamic context. Consider any utility function representing the changing preferences, say, $u(x,b(t))$. Any transformation of the type, $v(x,b(t)) = a(t) + u(x,b(t))$ causes trouble. The reason why is evident : such a transformation does not affect the indifference surfaces at each period. Hence the dynamic myopic demand functions obtained by a constrained maximisation of either u or v are identical. In other words, the transformation is ordinal in the classical sense. However, such a transformation re-numbers the indifference surfaces in a different manner at each period. Thus if preferences are not homothetic at every period, the index becomes arbitrary. In general, any neutral (in the sense of Hicks neutral technological change of production theory) change creates problems. One can, on the other hand, object to this kind of transformation. The introduction of time-variant parameters is intended to reflect some kind of information. Now the only kind of information reflected in a neutral change is a change in "man's efficiency as a pleasure machine" (FISHER-SHELL (1969), p. 99). It is not easy to be more cardinalist than that ! The exclusion of neutral change leads to an index that appears to be perfectly invariant under monotonic increasing transformations of the utility function, and reference utility level, used.

The above indexes aim at a comparison of prices reigning at two different moments while taking into account the fact that tastes have changed in between. For this reason we shall refer to them as dynamic temporal indexes. A different class of indexes are those which wish to realise a contemporaneous comparison of two price vectors, such as the Fisher-Shell index. It, and its like, shall be named a dynamic contemporaneous index.

In the above formulations no link has been established between the reference vector of prices and income, the reference commodity vector and the reference utility level. In order to establish a relationship between them one is obliged to choose a reference period in order to "fix" the values of the taste change parameters. The preference ordering so fossilised shall be called the reference ordering and any utility function representing it a reference utility function. Maximising this function subject to the constraint implied in the reference vector of prices and income will determine a reference commodity vector and utility level. When we are dealing with a dynamic contemporaneous index it is natural to choose as reference period the moment when the two price vectors are compared. This shall be our policy, and shall be implicitly assumed in what follows. In this case, this kind of index can be likened to a static index. If the reference period and moment of comparison were different this would not be so. There is no such unique manner of choosing a reference year for the dynamic temporal indexes ; hence, they will normally yield three different results.

The next step lies in choosing a reference vector of prices and income for the dynamic temporal and contemporaneous indexes, as well as a reference period for the former group. Since we limit our scope in this paper to Laspeyres-type indexes, our choices will be effected accordingly. In the case of the dynamic contemporaneous indexes, a sufficient condition to ensure having base-period income as base income is the selection of base-period prices and income for reference purposes. The Fisher-Shell index is of this kind, where, moreover, the comparison of base-period and comparison-period prices is effected during the comparison period. Since this index resembles the static kind, there is no need of specifying the standard used : all three lead to the same result.

In order that the dynamic temporal indexes be of the Laspeyres type, it is sufficient that the reference vector of prices and income as well as the reference period be of the base period. Under this hypothesis, we shall examine what becomes of the three indexes belonging to this group. We shall begin with the constant utility index. This method supposes the prior selection of a utility function, say, $u(x, b(t))$. The reference utility level is obtained by maximising $u(x, b(t_0))$, where t_0 is the base period, subject to the budget constraint imposed by (p_0, y_0), the base-period prices and income. Let x_0 be the value of x maximising the function. Then the reference level $u_0 = u(x_0, b(t_0))$. Base income is, of course, y_0 ; and comparison income $\overline{y}$ can be found by minimising $p_1' x$ subject to $u(x, b(t_1)) = u_0$, where t_1 is the comparison period.

This index has already been proposed in a paper co-authored by L. Phlips and R. Sanz-Ferrer (1972). Its relative merit from the ordinalist viewpoint has already been discussed above.

Next in line is the dynamic temporal index based on a reference commodity vector. The first step in its calculation consists in selecting a commodity vector x_0 in the same way as for the constant utility index. Comparison income $\overline{y}$ is the minimum cost of attaining the indifference class in which

x_o lies, given comparison-period prices and tastes. When we subjected this index to the endogenisation test we discovered a serious handicap or bias or what you will in it. The test lies in supposing that comparison-period prices are the same as base-period prices, whereas tastes have changed ; and then observing whether the index is different from unity or not. In this case, the index is in general different from unity, but always inferior to it ! In other words, comparison income $\overline{y}$ is smaller than base income y_o, no matter how tastes change. The reasons for this are clarified in the statement and proof of the following proposition :

Let $\underset{o}{\precsim}$ and $\underset{1}{\precsim}$ be two preference orderings defined on the same commodity space ; (p_o,y_o) the base-period vector of prices and income ; and x_o the optimal element, according to the ordering $\underset{o}{\precsim}$, in the budget set $B(p_o,y_o)$. $B(p_o,y_o) \overset{\Delta}{=} \{x : x \geqq 0 ; p_o'x \leq y_o\}$. Let $\overline{y}$ be such that there exists a commodity vector $\overline{x}$ which is (1) the optimal element, according to $\underset{1}{\precsim}$, in $B(p_o,\overline{y})$; and (2) indifferent to x_o, also according to $\underset{1}{\precsim}$. Then $\overline{y} \leq y_o$.

We shall prove this proposition by showing the absurdity of the contrary. Suppose that $\overline{y} > y_o$. This is equivalent to saying that the budget set $B(p_o,y_o)$ is strictly contained in $B(p_o,\overline{y})$. Now, in virtue of the axioms of monotonicity and strict convexity relative to preferences (cfr. section one), the optimal point $\overline{x}$ is strictly preferred, according to $\underset{1}{\precsim}$, to all other points in $B(p_o,\overline{y})$. Therefore x_o, being a member of $B(p_o,y_o)$ which is strictly contained in $B(p_o,\overline{y})$, cannot be indifferent to $\overline{x}$!

A couple of remarks are in place here. First of all, if we had defined a Paasche kind of index, the equalisation of comparison-period prices to base-period prices would result in an index always superior to unity. Secondly, in cases other than Laspeyres and Paasche, an index based on the reference commodity vector seems to be perfectly acceptable. However, this case has not been very deeply examined. In the Laspeyres context which concerns us here, we are obliged to conclude that a dynamic temporal index based on a reference commodity vector is unacceptable.

Finally, there is the dynamic temporal index based on a reference vector of prices and income. Comparison income is obtained by solving the equation :

$u^{+}(p_1, \bar{y}, b(t_1)) = u^{+}(p_0, y_0, b(t_1))$ for $\bar{y}$, where u^{+} is the indirect utility function, (p_0, y_0) the base-period vector of prices and income serving here as reference, p_1 the comparison period price vector, and t_1 the comparison period. Base income is, of course, the same as base-period income since it is obtained by solving

$u^{+}(p_0, y, b(t_0)) = u^{+}(p_0, y_0, b(t_0))$ for y.

Although the approach of the index just described is radically different from that of the Fisher-Shell index, the results are the same for the simple reason that comparison income is calculated in exactly the same manner for the two indexes ; although base income is differently calculated, both result in base-period income, being of the Laspeyres type.

Consequently, when we limit ourselves to Laspeyres type indexes and after all the logical dissection and sifting is over and done with, we are for practical purposes left with two indexes : the Fisher-Shell index and the dynamic temporal constant utility index (subject to the limitation expressed above). How these two indexes are related to the static true cost-of-living index shall be the theme of the following section.

III. THE EFFECT OF TASTE CHANGE ON THE INDEX

If we are to compare the Fisher-Shell index, and at the same time, the dynamic temporal index based on the base-period vector of prices and income, with the static index ; and likewise the dynamic temporal constant utility index with the static index - all this in a world of changing preferences which can be characterised by a member of the general class of utility functions $u(x, b(t))$, where x is an n-vector and b an m-dimensional vector-valued function of time -, we need to define what we mean by a static index in this case. First of all, we shall only consider the Laspeyres type true cost-of-living index. Secondly, the index will be based on the tastes of a base period that shall be fixed once and for all for the three indexes. Our model is more general than that of Fisher and Shell who study the special case of one good's being influenced by an own-augmenting taste change. This generalisation was necessitated by the inapplicability of the Fisher-Shell case [4] to the dynamic version of the Linear Expenditure Model proposed by L. Phlips (1972).

Since all three indexes are of the Laspeyres type and given that the base period is fixed and common to all three, we need only consider the effect of taste change on comparison income. In order to realise this aim we shall describe how comparison income is determined in the commodity space for the static index. We shall then see how this derivation must be changed in order to calculate comparison income in the other two cases. This is basically the logic followed by Fisher and Shell, and will allow us to profit from their article.

Let $(p_o, y_o) = (p_{o1}, p_{o2}, \ldots, p_{on}, y_o)$ be the base-period (and reference) vector of prices and income ; $p_1 = (p_{11}, p_{12}, \ldots, p_{1n})$ the comparison-period price vector ; and $u(x, b(t_o)) = u(x_1, \ldots, x_n ; b_1(t_o), \ldots, b_m(t_o))$ a base-period utility function.

Comparison income is derived in two steps :

Step I : derivation of the reference utility level

Define $L^{\circ} = u(x, b(t_o)) - \lambda(p_o' x - y_o)$ and maximise it with respect to the variables x and λ. The solution (x_o, λ_o) is found by solving the first-order conditions :

$$\begin{aligned} p_o' x - y_o &= 0 \\ \nabla_x u - \lambda p_o &= 0 \end{aligned} \tag{1}$$

where ∇ is the gradient sign. The reference utility level is equal to to $u_o \triangleq u(x_o, b(t_o))$.

Step II : determination of comparison income $\bar{y}$

Define $L^1 = p_1' x - (\lambda^{-1})(u(x, b(t_o)) - u_o)$ and minimise it with respect to x and λ^{-1}. The vector $(\bar{x}, \bar{\lambda})$ satisfying the system

$$\begin{aligned} u(x, b(t_o)) - u_o &= 0 \\ \nabla_x u - \lambda p_1 &= 0 \end{aligned} \tag{2}$$

is the sought-for solution. Comparison income is given by

$$\bar{y} \triangleq p_1' \bar{x} \tag{3}$$

Comparison income for the dynamic temporal constant utility index is obtained by re-defining the lagrangian L^1 in step II as equal to : $p_1' x - \lambda^{-1}(u(x, b(t_1)) - u_o)$, and minimising it with respect to x and λ^{-1}. Step I remains untouched. In other terms, one uses the comparison-period utility function in step II, while maintaining the base-period utility function in step I. In order to calculate comparison income for the Fisher-Shell index, one must, in addition to the change effected above, also introduce the comparison-period utility function in the first step. The lagrangian L° then becomes : $L^{\circ} = u(x, b(t_1)) - \lambda(p_o' x - y_o)$. Re-calculating comparison income for the two indexes incorporating taste change consists, therefore, in replacing the taste parameters $b(t_o)$ by $b(t_1)$ in step II only for the constant utility index, in both steps for the Fisher-Shell index.

The variation of comparison income $d\bar{y}$ resulting from these adaptations is given by the expression

$$d\bar{y} = \nabla_b \bar{y} \Big|'_{u=u_o \text{ constant}} \cdot db$$

for the dynamic temporal constant utility index ; and by

$$d\bar{y} = \frac{\partial \bar{y}}{\partial u_o} \cdot \nabla_b u'_o \cdot db + \frac{\partial \bar{y}}{\partial u_o} \cdot \nabla_{x_o} u'_o \cdot J_b(x_o) \cdot db$$

$$+ \nabla_b \bar{y} \Big|'_{u=u_o \text{ constant}} \cdot db$$

for the Fisher-Shell index where J is the jacobian of x_o with respect to b. The three terms composing the latter expression represent, respectively, the change in comparison income resulting from :

(i) a change in the reference utility level

(a) on account of the direct effect of a variation in the taste change parameters intervening as such in the utility function ;

(b) on account of the effect of a variation in the taste change parameters on the indifference surfaces and hence on the optimal vector x_o ;

(ii) a change in the optimal vector $\bar{x}$ in step II even if u_o were to remain constant because of the presence of b in the second step.

The three terms can be considerably simplified with the help of three lemmas analagous to those derived by Fisher and Shell (1969, pp. 106-110)

Lemma 1. $$\nabla_b \bar{y} \Big|_{u=u_o \text{ constant}} = - \frac{1}{\bar{\lambda}} \nabla_b u(\bar{x}, b(t_o))$$

Proof : Total differentiation of the system (2) in step II yields :

$$\begin{bmatrix} 0 & \nabla_x' u & \nabla_b' u \\ p_1 & H_{xx}(u) & H_{xb}(u) \end{bmatrix} \begin{bmatrix} -\,d\lambda \\ dx \\ db \end{bmatrix} = 0,$$

from which we obtain

$$\begin{bmatrix} -\,d\lambda \\ dx \end{bmatrix} = - \begin{bmatrix} 0 & \nabla_x' u \\ p_1 & H_{xx}(u) \end{bmatrix}^{-1} \begin{bmatrix} \nabla_b' u \\ H_{xb}(u) \end{bmatrix} db,$$

where H is a matrix of second derivatives with respect to the subscripts.

$$\text{Now } d\bar{y} = p_1' \, d\bar{x} = (0 \quad p_1') \begin{bmatrix} -d\lambda \\ dx \end{bmatrix}$$

$$= -(0 \quad p_1') \begin{bmatrix} 0 & \nabla_x' u \\ p_1 & H_{xx}(u) \end{bmatrix}^{-1} \begin{bmatrix} \nabla_b' u \\ H_{xb}(u) \end{bmatrix} db$$

$$= -(0 \quad p_1') . \frac{\text{adjoint } A}{\text{det. } A} . \begin{bmatrix} \nabla_b' u \\ H_{xb}(u) \end{bmatrix} db$$

where A is the matrix to be inverted. Since, by (2), the first row of A is equal to $\bar{\lambda}$ times $(0 \quad p_1)$, we obtain, on account of the rule of expansion by alien cofactors, that

$$d\bar{y} = - \frac{1}{\bar{\lambda}} \nabla_b' u \, . \, db$$

$$\text{It follows that } \nabla_b \bar{y} \Big|_{u=u_o \text{ constant}} = - \frac{1}{\bar{\lambda}} \nabla_b' u \; (\bar{x}, \, b(t_o))$$

Q.E.D.

<u>Lemma 2</u> $\nabla'_{x_o} u_o \cdot J_b(x_o) = 0$

<u>Proof</u> Consider any column of J, say the $i^{\underline{th}}$ one :

$$(\frac{\partial x_{o1}}{\partial b_i}, \ldots\ldots\ldots, \frac{\partial x_{on}}{\partial b_i})' \,.$$

This vector indicates a movement from x_o along the budget surface. The vector $\nabla_{x_o} u_o$, on the contrary, is orthogonal to the budget surface on account of the optimality status of x_o. The inner product of the two is, therefore, equal to zero. Q.E.D.

<u>Lemma 3</u> $\frac{\partial \bar{y}}{\partial u_o} = \frac{1}{\lambda}$

<u>Proof</u> Total differentiation of (2) with respect to u_o results in

$$\begin{bmatrix} 0 & \nabla'_x u & \nabla'_b u \\ & & \\ p_1 & H_{xx}(u) & H_{xb}(u) \end{bmatrix} \begin{bmatrix} -\,d\lambda/du_o \\ dx/du_o \\ db/du_o \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

However, $db/du_o = 0$; hence

$$\begin{bmatrix} 0 & \nabla'_x u \\ & \\ p_1 & H_{xx}(u) \end{bmatrix} \begin{bmatrix} -\,d\lambda/du_o \\ \\ dx/du_o \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Therefore, $\frac{d\bar{y}}{du_o} = p'_1 \frac{d\bar{x}}{du_o}$

$$= (0 \quad p'_1) \begin{bmatrix} 0 & \nabla_x u \\ p_1 & H_{xx}(u) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\lambda} \,.$$

Q.E.D.

As a consequence of the above lemmas, the difference between the dynamic temporal constant utility index and the static index becomes :

$$d\bar{y} = -\frac{1}{\bar{\lambda}} \nabla_b u(\bar{x}, b(t_o))' \cdot db \;;$$

whereas the gap between the Fisher-Shell index and the static index is equal to :

$$d\bar{y} = \left(+ \frac{1}{\bar{\lambda}} \nabla_b u(x_o, b(t_o)) - \frac{1}{\bar{\lambda}} \nabla_b u(\bar{x}, b(t_o)) \right)' db.$$

IV. TASTE CHANGE IN THE LINEAR EXPENDITURE MODEL AND THE COST-OF-LIVING INDEXES

In a recent article, L. Phlips (1972) presented a dynamic version of the linear expenditure model. It was based on a "dynamisation" of the Geary utility function :

$$u(x) = \Sigma_{i=1}^{n} \beta_i \log (x_i - \gamma_i) \text{ , where } \Sigma_{i=1}^{n} \beta_i = 1.$$

A temporal movement is introduced in the utility function by relating the constants γ_i to certain state variables s_i (Houthakker-Taylor (1970), pp. 13-14). Let s_i be a state variable reflecting the effects of the past, be it a stock effect or habit , on the minimum required quantities, γ_i, the interpretation by Samuelson (1947-48) of these constants. The relation is presumed to be linear : $\gamma_i (t) = 0_i + \alpha_i s_i(t)$. γ_i "moves" on account of the rule of motion affecting s_i which is defined by a differential equation : $\dot{s}_i = x_i - \delta_i s_i$. The model is now complete :

$$u(x, \gamma(t)) = \Sigma_{i=1}^{n} \beta_i \log \left[x_i - \gamma_i(t)\right]$$

$$\Sigma \beta_i = 1 \; ; \quad \gamma_i(t) = \theta_i + \alpha_i s_i(t) \quad ; \dot{s}_i = x_i - \delta_i s_i$$

On account of the dynamic myopic hypothesis, the dynamic demand functions can be immediately derived from the static linear expenditure model. The former are :

$$x_i = \gamma_i(t) + \frac{\beta_i}{p_i} \left[y - \Sigma_{j=1}^{n} p_j \gamma_j(t)\right] \qquad i=1, \ldots, n$$

The minimum cost of attaining an indifference curve, indicated by a utility level u_o , is equal to

$$\bar{y} = \Sigma_{i=1}^{n} p_i \gamma_i(t) + (\exp. u_o). \pi_{i=1}^{n} \left[\frac{p_i}{\beta_i}\right]^{\beta_i}$$

With this information at our disposal we are now ready to write down the formules for the comparison incomes of the three indexes discussed in the previous section. We obtain as comparison income for

(1) the static index :

$$\bar{y} = \Sigma_{i=1}^{n} p_{1i}\, \gamma_i(t_o) + (\exp.\, u_o).\, \pi_{i=1}^{n} \left[\frac{p_{1i}}{\beta_i}\right]^{\beta_i}$$

$$\text{where } u_o = \Sigma_{i=1}^{n} \beta_i \log\left[\frac{\beta_i}{p_{oi}} (y_o - \Sigma_{j=1}^{n} p_{oj}\, \gamma_j(t_o))\right]$$

(2) the dynamic temporal constant utility index

$$\bar{y} = \Sigma_{i=1}^{n} p_{1i}\, \gamma_i(t_1) + (\exp.\, u_o).\, \pi_{i=1}^{n} \left[\frac{p_{1i}}{\beta_i}\right]^{\beta_i}$$

$$\text{where } u_o = \Sigma_{i=1}^{n} \beta_i \log\left[\frac{\beta_i}{p_{oi}} (y_o - \Sigma_{j=1}^{n} p_{oj}\, \gamma_j(t_o))\right]$$

(3) the Fisher-Shell index

$$\bar{y} = \Sigma_{i=1}^{n} p_{1i}\, \gamma_i(t_1) + (\exp.\, u_o).\, \pi_{i=1}^{n} \left[\frac{p_{1i}}{\beta_i}\right]^{\beta_i}$$

$$\text{where } u_o = \Sigma_{i=1}^{n} \beta_i \log\left[\frac{\beta_i}{p_{oi}} (y_o - \Sigma_{i=1}^{n} p_{oj}\, \gamma_j\, (t_1))\right]$$

There are three possible ways of analysing the effect of taste change on the index (more precisely, on comparison income) in the case of the dynamic linear expenditure model. The first lies in simple subtraction , it is tedious; the second in differentiation of the above equations; and the third in an application of the preceding section. We need a further bit of information in order to apply the last method, viz., the formula of λ :

$$\lambda^{-1} = y - \Sigma_{i=1}^{n} p_i\, \gamma_i(t) .$$

The three approaches should, and do, lead to the same conclusions.

The difference between the comparison income of the dynamic temporal constant utility index and that of the static index is obtained by means of the partial derivatives

$$\left.\frac{\partial \bar{y}}{\partial \gamma_i}\right|_{u = u_o \text{ constant}} = -\frac{1}{\bar{\lambda}} \cdot \frac{\partial u(\bar{x}, \gamma(t_o))}{\partial \gamma_i} = p_{1i} \quad i=1,\ldots,n.$$

The use of the Fisher-Shell index instead of the static index means that comparison income differs according to

$$\frac{\partial \bar{y}}{\partial \gamma_i} = \frac{1}{\bar{\lambda}} \left(-\frac{\partial u(\bar{x}, \gamma(t_o))}{\partial \gamma_i} + \frac{\partial u(x_o, \gamma(t_o))}{\partial \gamma_i} \right)$$

$$= p_{1i} - p_{oi} \cdot \frac{\bar{y} - \Sigma_{i=1}^{n} p_{1i}\, \gamma_i(t_o)}{y_o - \Sigma_{i=1}^{n} p_{oi}\, \gamma_i(t_o)}$$

$$= p_{1i} - p_{oi} \cdot \pi_{i=1}^{n} \left(\frac{p_{1i}}{p_{oi}}\right)^{\beta_i}$$

(For the last equality see Goldberger (1967) or Phlips (1974)).
Consequently, the differential $d\bar{y}$ in the two cases is respectively :

$$d\bar{y} = \Sigma_{i=1}^{n} p_{1i}\, d\gamma_i = \Sigma_{i=1}^{n} p_{1i} \cdot (\gamma_i(t_1) - \gamma_i(t_o))$$

and $$d\bar{y} = \Sigma_{i=1}^{n} \left[p_{1i} - p_{oi} \cdot \pi_{i=1}^{n} \left(\frac{p_{1i}}{p_{oi}}\right)^{\beta_i} \right] d\gamma_i$$

$$= \Sigma_{i=1}^{n} \left[p_{1i} - p_{oi} \cdot \pi_{i=1}^{n} \frac{p_{1i}}{p_{oi}}^{\beta_i} \right] \cdot (\gamma_i(t_1) - \gamma_i(t_o)).$$

CONCLUSION

The intention of the above paper was to elaborate a rather general framework for the analysis of true cost-of-living indexes under the two-fold hypotheses of taste change and dynamic myopic maximisation. Within this framework, the special case of Laspeyres' type indexes were studied and the resulting conclusions were applied to the dynamic version of the linear expenditure system that has been proposed by L. Phlips (1972). An algorithm for calculating the Fisher-Shell index, the dynamic temporal constant utility index and the static index has been expounded by L. Phlips and R. Sanz-Ferrer (1972). Moreover, once the dynamic LES has been estimated it is possible to estimate the values taken by the state variables [5] at each period. This should allow a concrete consideration of the taste change forces at work in the true index with respect to this particular model.

° ° °

N O T E S

1 The ratio of the two incomes is independent of the choice of the indifference surface in the case of homothetic preferences, i.e., preferences such that if a commodity vector x_0 is indifferent to another vector x_1, then automatically ax_0 is indifferent to ax_1, for any strictly positive scalar a.

2 The reader is asked to continuously distinguish between base-period prices, base-period income, comparison-period prices and comparison period income, on the one hand, and base income and comparison income, on the other. The former set of concepts are the data available; the latter form, respectively, the denominator and numerator of the index, and are calculated values.

3 Why the authors chose their approach and whether they are aware of the other cases is not for us to say. All we can criticise is their systematisation in the article, or, rather, in the main body of the text. There is a certain ambiguity in their article: the footnotes often give the impression of opening perspectives that the main text closes.

4 When treating the problem of quality change the authors partially treat a more general case : they demonstrate lemma one (see further on) for a scalar function of time, b(t).

5 The estimation procedure proposed by L. Phlips (1972) calculates simultaneously for each period the optimal quantities of goods, $\hat{x}_{it}$, as well as the corresponding marginal utility of money, $\hat{\lambda}_t$, given the prices and income of that period. Now since

$$\hat{x}_{it} = \gamma_{it} + \frac{\beta_i}{p_{it}}(y_t - \Sigma_{j=1}^{n} p_{jt}\, \gamma_{jt})$$

and $$\hat{\lambda}_t^{-1} = y_t - \Sigma_{j=1}^{n} p_{jt}\, \gamma_{jt}$$

we obtain $$\hat{\gamma}_{it} = \hat{x}_{it} - \frac{\beta_i}{\hat{\lambda}_t\, p_{it}}$$

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