Optimal Control in Infinite Horizon Problems: A Sobolev space approach*

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Abstract

In this paper, we make use of the Sobolev space $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$ to derive at once the Pontryagin conditions for the standard optimal growth model in continuous time, including a necessary and sufficient transversality condition. An application to the Ramsey model is given. We use an order ideal argument to solve the problem inherent to the fact that L^1 spaces have natural positive cones with no interior points.

Keywords: Optimal control, Sobolev spaces, Transversality conditions, Order ideal

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1 Introduction

Typically, the first-order necessary conditions of optimization problems in continuous time, the so-called Pontryagin conditions, are established using variational methods. This kind of methods is for example used throughout the textbooks of Hadley and Kemp (1973) and Kamien and Schwartz (1991), but it is indeed at the basis of optimal control theory as initially designed by Pontryagin et al (1962). For a finite time horizon, the set of Pontryagin conditions include optimality conditions with respect to the control, state and co-state variables, plus the corresponding transversality conditions which depend on the assumptions on the time horizon and the terminal state. All these conditions can be identified using standard variational methods.

When the optimization time horizon goes to infinity, things become much more complicated. In particular, it turns out that while the usual Pontryagin conditions obtained for finite horizons with respect to the control, state and co-state variables are preserved, the transversality conditions cannot be safely extrapolated. As the horizon gets to infinity, it is quite easy to show (see for example Halkin, 1974) that taking the limits of the transversality conditions obtained for finite time horizons is highly misleading. In particular, the traditional "economic" condition according to which the value of the stock variables should go to zero as the time horizon goes to infinity was shown to be clearly erroneous in the case of non-discounted problems.

This has lead to a kind of split in the optimal control treatment under infinite horizons: while the Pontryagin conditions can still be obtained by variational methods, the transversality condition is obtained using another type of argument. This is for example true in the seminal paper of Michel (1982), who concentrates on the necessary transversality condition part. Michel provides a fairly general inspection into this issue in the case of discounted problems (without *a priori* sign or concavity assumptions on the objective and state functions). In such a framework, he proves that the right necessary transversality condition when time tends to infinity is the limit of the maximum of the Hamiltonian going to zero. This extends the property valid in a finite horizon problem with free terminal time to the infinite horizon case. On the other hand, he shows that the this necessary condition implies the traditional "economic" transversality condition, mentioned above, provided the objective function is non-negative and if "enough possibilities of changing the state's speed exist indefinitely". Ye (1993) extends this analysis by allowing for the non-differentiability of the problem data and obtains the maximum principle in terms of differential inclusions in analogy to the finite horizon problem.

Unfortunately the resulting characterization of the cases where the "economic" transversality condition holds reveals unpractical. Alternative duality-based theories for discounted problems were developed starting with Benveniste and Scheinkman (1982). Under some concavity conditions (needed to apply an envelop condition), Benveniste and Scheinkman (1982) establishes the necessity of the transversality condition, $\lim_{t\to\infty}\left[-v_2\left(x\left(t\right),\dot{x}\left(t\right),t\right)\right]x = 0$ for the continuous time reduced form model:

$$\begin{array}{ll} \max & \int\limits_0^\infty v\left(x\left(t\right),\dot{x}\left(t\right),t\right)dt \\ \text{subject to} & \\ & x\left(0\right)=x_0, \left(x\left(t\right),\dot{x}\left(t\right)\right)\subset\left(\mathbb{R}^n\right)^2. \end{array}$$

when the assumptions of non-negativity and integrability of v for all feasible paths are verified. Kamihigashi (2001) generalizes this analysis by allowing for unbounded v with the assumptions of local boundedness of v_1 and v_2 and the existence of an open set that the optimal pair $\left(x^*\left(t\right),\dot{x}^*\left(t\right)\right)$ belongs to and under which $v\left(.,.,t\right)$ is continuously differentiable. Long and Shimomura (2003) prove the necessity of a transversality condition of the form $\lim_{t\to\infty} [x^*(t) - x_0] [v_2(x(t), \dot{x}(t), t)] = 0$ under the assumption that v is twice differentiable and the optimal pair belongs to the interior of a set under $(\mathbb{R}^n)^2$.

This paper provides a simple and unified functional analysis argument to derive **at once** the Pontryagin conditions, including the transversality condition in infinite horizon problems. More specifically, we make use of the Sobolev space $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$, which appears to be quite natural to derive not only the convenient transversality conditions, but also the whole set of Pontryagin conditions. Our choice of the Sobolev space $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$ is relevant for many optimal growth models, e.g. the Ramsey model, in which the feasible capital paths are proved to belong to this space and the feasible consumption paths belong to L^1 (see Askenazy and Le Van, 1999, page 42). We don't use variational methods. Rather our setting is based on an assumption (Assumption 4 in the text), which is close to the concept of g-supported control trajectories used in Carlson, Haurie and Leizarowitz (1991), chapter 7, for example. Combining this concept with the Sobolev space $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$ turns out to be a powerful tool to get through the problem.

To our knowledge, the first analysis that uses Sobolev spaces in economics was Chichilnisky (1977). She studies the problem of existence and the characterization of the solutions of optimal growth models in many sector economies. In this context, the prices are continuous linear functionals defined on the space of consumption paths. Mathematically, the question turns out to be the existence of an appropriate continuous linear functional separating the set of feasible paths from a translation of the positive cone with vertex on the feasible element which optimizes the discounted social utility of the stream of consumption. Chichilnisky makes use of the Hilbert space with L^2 norms as the spaces of consumption paths on which the optimization is performed. Accordingly, the space of admissible capital accumulation paths is given a certain Hilbert space structure called a Sobolev space:

$$W^{1,2}(\mathbb{R}_+,\mathbb{R}^n) \equiv \left\{u \in L^2(\mathbb{R}_+,\mathbb{R}^n) : D^\alpha u \in L^2(\mathbb{R}_+,\mathbb{R}^n) \text{ for } 0 \le |\alpha| \le 1\right\}$$

where α is a n-tuple of nonnegative integers with $|\alpha| = \sum_{i=1}^{n} \alpha_i$ and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n}$ with $D_j = \frac{\partial}{\partial x_j}$.

The basic tool needed to prove the existence of competitive prices for optimal programs, the Hahn-Banach theorem, requires one of the convex sets being separated to have an interior or an internal point. However, all L^p spaces with $1 \le p < \infty$ have natural positive cones with no interior or internal points. To overcome this problem, the objective function being maximized is shown to be continuous in weaker L^2 topology. Another inconvenient feature of L^2 spaces is related to the fact that their topology is weaker. It creates a difficulty in having conditions on the utility function which yield L^2-*continuity* of nonlinear objective functional, the discounted social utility of the stream of consumption.

As mentioned above, in contrast to the previous studies, we shall use Sobolev space $W^{1,1}(\mathbb{R}_+,\mathbb{R}^n)$. Nonetheless, as in the alternative approaches listed above, we still face the problem that the involved L^1 spaces have natural positive cones with no interior or internal points. In order to overcome this problem, we shall use the concepts of properness and order ideal. The notion of properness is proved to be very useful in analyzing the existence of equilibrium in Banach lattices or Riesz spaces (see the excellent survey of Aliprantis, Cornet and Tourky, 2002, and its references). The properness is a notion weaker than continuity. A complete characterization for strictly increasing separable concave functions in L_+^p is given in Araujo and Monteiro (1989). Le Van (1996) characterizes properness for separable concave functions in L_+^p without assuming monotonicity. Dana, Le Van and Magnien (1997) provides an existence theorem when the consumptions sets being the positive orthant of a locally convex solid Riesz space has an empty interior. They use the approach of Mas-Colell and Zame (1991) by considering an economy restricted to the order ideal generated by the total resource, which is dense in the initial consumption space. This suffices to obtain a quasi-equilibrium price which can be extended to a linear form in the initial topology by the properness of the every utility function.

The paper is organized as follows. Section 2 presents the considered optimization problem, and gives some preliminary definitions and assumptions needed to derive our necessary and sufficient transversality condition. Section 3 proves the latter condition in the described mathematical framework, yielding the main result in Theorem 1. Section 4 is an application to the Ramsey model.

2 Preliminaries

Let $C_c^1(\mathbb{R}_{++}, \mathbb{R}^n)$ denote the set of continuously differentiable functions from $\mathbb{R}_{++}$ to $\mathbb{R}^n$ with compact support. We have the following general definitions and notations.

Definition 1 *The space $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$ is the space of functions, $x \in L^1(\mathbb{R}_+, \mathbb{R}^n)$ such that there exists a function $x' \in L^1(\mathbb{R}_+, \mathbb{R}^n)$ that satisfies*

$$\int_0^\infty x\phi' \, dt = -\int_0^\infty x'\phi \, dt, \ \forall \phi \in C_c^1(\mathbb{R}_{++}, \mathbb{R}^n).$$

In this case, x' is called the derivative of x in the sense of distributions.

We recall some results that will be useful in our analysis (see Brezis, 1983, for the proofs, pp 119-148) about Sobolev space $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$:

- $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$ is a Banach space for the norm: $\|x\|_{W^{1,1}} = \|x\|_{L^1} + \|x'\|_{L^1}$.
- If $x \in W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$, there exists a unique continuous mapping $\widetilde{x}$ on $\mathbb{R}_+$ such that $x = \widetilde{x}$ almost everywhere.
- For all τ, $\tau' \in \mathbb{R}_+$, $\widetilde{x}(\tau) - \widetilde{x}(\tau') = \int_{\tau'}^{\tau} x'(t)\, dt$ and $\lim_{t\to\infty} \widetilde{x}(t) = 0$.

We consider a standard optimal control problem with an infinite horizon arising in dynamic models in continuous time:

$$\begin{array}{ll} \max & \int_0^{\infty} u(x(t), c(t))\, e^{-rt} dt \\ \text{subject to} & \\ & \dot{x}(t) = f(x(t), c(t)) \\ & x(0) = x_0 \end{array}$$

where $x(t) \in \mathbb{R}_+^n$ and $c(t) \in \mathbb{R}_+^n$.

We denote by E the space of functions from $\mathbb{R}_+$ to $\mathbb{R}^n$ such that $xe^{-rt} \in W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$. Let $\|x\|_E = \int_0^{\infty} \|x\| e^{-rt} dt + \int_0^{\infty} \|x'\| e^{-rt} dt$. By $L^1(e^{-rt})$, we define the set of functions such that $xe^{-rt} \in L^1$, for a given $r > 0$. Observe that $x \in E$ implies $\|x(t)\| e^{-rt} \to 0$ when $t \to \infty$.

Next we make the following assumptions.

Assumption 1 *$x \in E$ and $c \in L^1(e^{-rt})$.*

Assumption 2 *f and u are continuous and continuously differentiable with respect to x.*

Assumption 3 *$\forall x \in L^1(e^{-rt})$ and $\forall c \in L^1(e^{-rt})$, we have $f_x(x, c) \in L^1(e^{-rt})$ and $u_x(x, c) \in L^1(e^{-rt})$.*

The condition $x(0) = x_0$ must be understood in the sense that the unique continuous function $\widetilde{x}$ which is almost everywhere equal to x satisfies $\widetilde{x}(0) = x_0$.

Definition 2 *A trajectory* $(x(t), c(t))$, $t \in [0, \infty)$ *is admissible if* $x(t)$ *satisfies the constraint qualifications with control* $c(t)$, $t \in [0, \infty)$ *and if the integral in the objective function converges. A trajectory* $(x^*(t), c^*(t))$ *is an optimal solution of the problem if it is admissible and if the value of the objective function corresponding to any admissible trajectory is not greater than that of* $(x^*(t), c^*(t))$.

Lemma 1 *Let* $L : E \to \mathbb{R}_+$ *be defined by* $L(x) = x(0)$. *The mapping* L *is Lipschitzian.*

Proof. See Bonnisseau and Le Van (1996). ■

Lemma 2 *Let* $D : x(t) \to Dx(t) = \dot{x}(t)$. D *is continuous from* E *into* $L^1(e^{-rt})$.

Proof. It is easy. ■

The optimization problem under consideration can be recast in the following form ($\mathcal{P}$):

$$\begin{array}{ll} \max & U(x,c) = \int_0^\infty u(x(t), c(t)) e^{-rt} dt \\ \text{subject to} & \\ & Dx = f(x,c) \\ & Lx = x_0 \end{array}$$

where $U : \left(E \cap L^1_+(e^{-rt})\right) \times L^1(e^{-rt}) \to \mathbb{R} \cup \{-\infty\}$.

We now set an assumption, which is most crucial to our analysis:

Assumption 4 *Let* $(x^*(t), c^*(t))$ *be a solution. Assume that there exist multipliers* $(a, q, \lambda) \in \mathbb{R}_+ \times L^\infty \times \mathbb{R}^n$ *such that:* $\forall x \in \left(E \cap L^1_+\left(e^{-rt}\right)\right)$, $\forall c \in L^1_+\left(e^{-rt}\right)$,

$$aU(x^*, c^*) - q(Dx^* - f(x^*, c^*)) - \lambda(Lx^* - x_0) \geq$$
$$aU(x, c) - q(Dx - f(x, c)) - \lambda(Lx - x_0). \quad (1)$$

Assumption 4 is close in spirit to the definition of *g-supported control trajectories* used in Carlson, Haurie and Leizarowitz (1991), chapter 7.[1] This definition is indeed quite common in the optimization literature. For example, Carlson and Angell (1998) have the same definition of supported controls (Definition 5, page 76) and easily prove for a class of undiscounted optimization problems with finite horizon that a control-trajectory which is feasible and supported is necessarily optimal (Theorem 6, page 76).[2] However, there are two main differences between this approach and ours. First of all, the literature mentioned just above follows different optimality criteria, namely overtaking optimality,[3] which has a lot to do with the undiscounted nature of the problems under consideration. Moreover, we know by the Halkin's counter-example that these undiscounted problems may not satisfy the usual transversality conditions. Therefore our framework and the associated optimality criterion (see Definition 2 above) are much better suited to the study of transversality conditions in economic problems.

Second, and more importantly, the treatment of the supporting function, q in our case, is far from similar, and it can, by no way, be the same because the involved functional spaces are completely different. Our application section

[1] see for example Definition 7.1.

[2] In this class of paper, the optimal steady states are similarly characterized, namely by using the concept of supported steady state. For example, Definition 8 in Carlson and Angell, 1998.

[3] see for example, Carlson and Angell, 1998, Definition 19, page 85.

provides an insightful constructive method to get the supporting function q, using the concept of order ideal in L^1 topology.
The next section gives the main results of the paper.

3 Main results

In this section, we shall show how our approach allows to derive properly and easily the Pontryagin conditions, and more importantly, it will be shown how it settles in a simple and natural way the problem of the necessity and sufficiency of the transversality condition for infinite horizon problems.

The next proposition can be viewed as a more accurate characterization of the supporting function q under assumptions 1 to 4.

Proposition 1 *Let Assumptions 1-4 be satisfied. Assume that $x^*(t) > 0$, $\forall t$. Then $\exists\, p \in L^1$ such that:*

$$a u_x(x^*, c^*) e^{-rt} + \dot{p}(t) + p(t) f_x(x^*, c^*) = 0, \tag{2}$$

in the sense of distributions.

Proof. It is clear from (1) that one can write: $\forall x \in \left(E \cap L^1_+\left(e^{-rt}\right)\right),$

$$a \int_0^\infty [u(x^*, c^*) - u(x, c^*)] e^{-rt} dt - \int_0^\infty q(t) [Dx^* - Dx] e^{-rt} dt$$
$$+ \int_0^\infty q(t) [f(x^*, c^*) - f(x, c^*)] e^{-rt} dt - \lambda [x^*(0) - x(0)] \geq 0, \tag{3}$$

Let $h(t) \in C^1_c(\mathbb{R}_+, \mathbb{R}^n)$. If $x^*(t) > 0$, $\forall t$, as $x^*(t)$ can be assumed to be continuous [recall that every element of the Sobolev space $W^{1,1}$ can be identified with a continuous function], we can choose μ sufficiently small such that $x(t) = x^*(t) + \mu h(t) \in E$. We obtain:

$$\int_0^\infty a u_x(x^*, c^*) e^{-rt} h(t)\, dt - \int_0^\infty q(t) e^{-rt} \dot{h}(t)\, dt + \int_0^\infty q(t) e^{-rt} f_x(x^*, c^*) h(t)\, dt = 0$$

and hence, with $p(t) = q(t) e^{-rt} \in L^1$,

$$a u_x(x^*, c^*) e^{-rt} + \dot{p}(t) + p(t) f_x(x^*, c^*) = 0,$$

in the sense of distributions. ∎

It is easy to see that equation (2) is indeed the Pontryagin condition with respect to the state variable. Notice that the derivation of such a result is done in a very elementary way within our functional framework. A finer characterization of $p(t)$ (or $q(t)$) is indeed allowed.

Corollary 1 *If $c^*(t)$ is piecewise continuous then $\dot{p}(t)$ is piecewise continuous.*

Proof. Since $p(t) \in L^1$, it follows from (2) that $\dot{p}(t) \in L^1$ and hence, $p(t)$ is continuous. This implies that $\dot{p}(t)$ is piecewise continuous. ∎

The next proposition gives the Pontryagin condition with respect to the control. Again, our approach allows for an almost immediate proof.

Proposition 2 *Let Assumptions 1-4 be satisfied. Assume that $c^*(t)$ is continuous and hence, $\dot{x}^*(t)$ is continuous. Then we have,* $\forall c \geq 0$,

$$a u(x^*(t), c^*(t)) e^{-rt} + p(t) f(x^*(t), c^*(t)) \geq a u(x^*(t), c) e^{-rt} + p(t) f(x^*(t), c). \quad (4)$$

Proof. ¿From (1), it can be noted that, $\forall c \in L^1_+(e^{-rt})$,

$$a \int_0^\infty [u(x^*, c^*) - u(x^*, c)] e^{-rt} dt + \int_0^\infty p(t) [f(x^*, c^*) - f(x^*, c)]\, dt \geq 0. \quad (5)$$

Now consider a point t with $c^*(t) > 0$ and where c is continuous. Assume on the contrary, by continuity, $au(x^*(t), c^*(t))e^{-rt} + p(t) f(x^*(t), c^*(t)) < au(x^*(t), c) e^{-rt} + p(t) f(x^*(t), c)$ in some interval I around t with some $c \geq 0$. Let $c'(t) = c^*(t)$, $t \notin I$ and $c'(t) = c$ when $t \in I$. Note that $c'(t) \in L^1_+(e^{-rt})$. However, (5) is not satisfied leading to a contradiction. ■

Let us move now to the transversality conditions. Notice that actually, $\lim_{t\to\infty} p(t) = 0$. Furthermore, knowing that $p(t) x^*(t) = q(t) e^{-rt} x^*(t) \to 0$ and $p(t) \to 0$ as t$\to \infty$, we can easily revisit the result of Long and Shimomura:

$$\lim_{t\to\infty} p(t)(x^*(t) - x_0) = 0.$$

These establish the following result.

Proposition 3 *Under Assumptions 1-4, the following transversality condition is a necessary condition for the optimality of* $(x^*(t), c^*(t))$*:*

$$p(t) e^{rt} \in L^\infty. \tag{6}$$

We now move to the sufficiency part and prove among others that (6) is sufficient under some conditions which are known in the optimization literature (see for example, Carlson, Haurie and Leizarowitz, 1991). To this end, we use the Hamiltonian concept.

Assumption 5 *Define the Hamiltonian*

$$H(x, c, p, t) = u(x(t), c(t)) e^{-rt} + p(t) f(x(t), c(t)).$$

Suppose that , $\max_{c\geq 0} H(x, c, p, t)$ *is concave in* x *and* $H(x^*, c^*, p, t) \geq H(x^*, c, p, t)$, $\forall c \geq 0$.

The next proposition shows the sufficiency of the transversality condition when Assumption 5 is added to our assumptions set.

Proposition 4 *Under Assumptions 1-5, a sufficient condition for* $(x^*(t), c^*(t))$ *to be optimal is*

$$p(t)\, e^{rt} \in L^{\infty}.$$

Proof. By Assumption 5, the following holds [see Carlson, Haurie and Leizarowitz (1991)] for every $T > 0$:

$$\int_0^T u(x^*(t), c^*(t))\, e^{-rt} dt - \int_0^T u(x(t), c(t))\, e^{-rt} dt \geq p(T)\,(x(T) - x^*(T)).$$

By assumption, $p(t)\, e^{rt} \in L^{\infty}$. We then have:

$$\begin{aligned} |\, p(T)\,(x(T) - x^*(T))| &\leq \|p(T)\|\, e^{rT}\ \left[\|x(T)\|\, e^{-rT} + \|x^*(T)\|\, e^{-rT}\right] \\ &\leq K\ \left[\|x(T)\|\, e^{-rT} + \|x^*(T)\|\, e^{-rT}\right]. \end{aligned}$$

Since $x \in E$ and $x^* \in E$, we get $K\ \left[\|x(T)\|\, e^{-rT} + \|x^*(T)\|\, e^{-rT}\right] \to 0$ as $T \to \infty$. That ends the proof. ■

This establishes the transversality condition part of the paper.

Proposition 5 *Let Assumptions 1-5 be satisfied. Then,* $p(t)\, e^{rt} \in L^{\infty}$ *is a necessary and a sufficient optimality condition for* $(x^*(t), c^*(t))$ *on the set of all admissible trajectories of problem* $(\mathcal{P})$.

As mentioned before, the crucial part of our analysis is Assumption 4. We show in the next section how this assumption is checked in the Ramsey-like models.

4 Application to the Ramsey Model

We consider the following usual type of Ramsey model:

$$\begin{aligned}
&\max && \int_0^\infty u(c(t))\,e^{-rt}dt \\
&\text{subject to} && \\
& && c(t) + \dot{x}(t) \le f(x(t)) - \delta x(t) \\
& && c(t) \ge 0, \forall t \\
& && x(t) \ge 0, \forall t \\
& && x(0) = x_0 > 0, \textit{ is given.}
\end{aligned}$$

under the following assumptions.

Assumption 6 *u is C^1, strictly concave, increasing with $u'(0) \le +\infty$.*

Assumption 7 *f is C^1, strictly concave, increasing with $f'(0) > \delta$, $f'(0) < \infty$, $f'(\infty) = 0$.*

Proposition 6 *The optimal solution $(x^*(t), c^*(t))$ satisfy $x^* \in W^{1,1} \cap L^1_+$ and $c^* \in L^1_+(e^{-rt})$.*

Proof. See Askenazy and Le Van (1999). ■

In accordance with this proposition, we can use $W^{1,1} \cap L^1_+$ as the state space and L^1_+ as the control space. Let $X = \left(W^{1,1} \cap L^1_+\right) \times L^1_+$. The problem becomes:

$$\begin{aligned}
&\max && U(x,c) = \int_0^\infty u(c(t))\,e^{-rt}dt \\
&\text{subject to} && \\
& && g(x,c) \ge 0 \\
& && Lx = x_0
\end{aligned}$$

where $g(x,c) = f(x) - \delta x - c - Dx$ and $Lx = x(0)$. Note that g takes values in L^1_+ and L^1_+ has an empty interior. Hence, the direct application of the theorem V.3.1 of Hurwicz (1958) is not possible for proving the existence of the multipliers $(a, q, \lambda) \in \mathbb{R}_+ \times L^\infty \times \mathbb{R}^n$ associated with this problem. We then use the same approach as Mas-Colell and Zame (1991) and Dana, Le Van and Magnien (1997). We consider an order ideal which is dense in the original space. There we have the positive orthant of the order ideal with a nonempty interior for its lattice norm.

It is well known (see for example, Askenazy and Le Van, 1999, Proposition 5) that there exists $\alpha > 0$ such that the optimal consumption path satisfies: $c^*(t) \geq \alpha$, $\forall t \geq 0$. Let $\bar{c} = c^*$ and $I(\bar{c}) = \{y \in L^1 : \exists \mu > 0 \text{ s.t. } |y| \leq \mu \bar{c}\}$.

The ideal $I(\bar{c})$ is dense for both the L^1 topology and for the weak topology (see Aliprantis, Brown and Burkinshaw, 1989, pages 103,104). We define on $I(\bar{c})$ the norm $\|.\|_{\bar{c}}$:

$$\|y\|_{\bar{c}} = \inf \{\mu > 0 : \ |y| \leq \mu \bar{c}\}.$$

One can verify that the positive orthant of $I(\bar{c})$, $I_+(\bar{c})$ has nonempty interior for the topology defined by $\|.\|_{\bar{c}}$. More precisely, $\bar{c} \in int\ I(\bar{c})$.

As u is increasing by Assumption 6, along an optimal path, we have $g(x^*, c^*) = 0$, i.e. $g(x^*, c^*) \in I_+(\bar{c})$. Consider the problem:

$$\begin{array}{ll} \max & U(x,c) \\ \text{subject to} & \\ & g(x,c) \in I_+(\bar{c}) \\ & Lx = x_0. \end{array}$$

where one can now apply the Theorem V.3.1 of Hurwicz (1958) and obtain:

$$\exists (a, q, \lambda) \in \mathbb{R}_+ \times I(\bar{c})' \times \mathbb{R} \text{ s.t.}$$

$$aU(x^*, c^*) + qg(x^*, c^*) + \lambda(Lx^* - x_0) \geq aU(x,c) + qg(x,c) + \lambda(Lx - x_0), \ \forall x, \forall c. \tag{7}$$

Now we shall prove that q is a continuous linear form on $I(\bar{c})$ for the L^1-norm topology. Since $I(\bar{c})$ is dense in L^1, it extends to a continuous form on L^1 for the L^1-norm topology. To this end, we follow Dana, Le Van and Magnien (1997) and utilize the notion of properness.

Since $c_t^* \geq \alpha > 0$, $\forall t$, it is clear that $u^{'}(c^*(t)) \in L^1(e^{-rt})$. From Le Van (1996), U is proper at c^*. Hence, there exists an open neighborhood of 0, denoted by A and a vector $v \in L^1_+$ such that

$$\forall \mu \in]0,1[\,,\ U(x,\ c^* + \mu(v+z)) > U(x,\ c^*)\ \ if\ z \in A\ and\ if\ c^* + \mu(v+z) \in L^1_+.$$

Let $y \in A \cap I_+(\bar{c})$. There exists $\mu > 1$ such that $0 \leq y \leq \mu\ \bar{c} = \mu c^*$. Define $z = (1/\mu)\, y$. We have $c^* + \frac{1}{\mu}(v-y) \geq 0$. By applying the inequality (7):

$$aU(x^*, c^*) + qg(x^*, c^*) + \lambda(Lx^* - x_0) \geq$$
$$aU\left(x^*, c^* + \frac{1}{\mu}(v-y)\right) + q\left(g(x^*, c^*) + \frac{1}{\mu}(v-y)\right) + \lambda(Lx^* - x_0)$$

together with the properness condition lead us to obtain that

$$qy \geq qv.$$

On the other hand, since $c^* + \frac{1}{\mu}(v+y) \geq 0$, we have also

$$-qy \geq qv.$$

Now, let $y \in A \cap I(\bar{c})$. y^+ and y^- belong to $A \cap I_+(\bar{c})$. We have $qy^+ \geq qv$ and $-qy^- \geq qv$ so that $qy \geq 2qv$. We have proved that the linear form q is bounded from below in an open neighborhood of 0. Therefore, q is continuous on $I(\bar{c})$ with the initial topology. Since $I(\bar{c})$ is dense in L^1, q has a unique extension in $(L^1)^{'} = L^\infty$.

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