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Inequality aversion and separability in social risk evaluation

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Abstract

This paper examines how to satisfy a separability condition related to "independence of the utilities of the dead" (Blackorby et al., 1995; Bommier and Zuber, 2008) in the class of "expected equally distributed equivalent" social orderings (Fleurbaey, 2010). It also inquires into the possibility to keep some aversion to inequality in this context. It is shown that the social welfare function must either be utilitarian or take a special multiplicative form. The multiplicative form is compatible with any degree of inequality aversion, but only under some constraints on the range of individual utilities.

Keywords: risk, ex post equity, independence of the utilities of the dead.

JEL Classification: D63, D71, D81

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1 Introduction

The evaluation of social situations involving risk has been a debated topic ever since Harsanyi (1955) published his theorem on utilitarianism. He interpreted his theorem as vindicating utilitarianism. But an equivalent interpretation is that this is an impossibility theorem for those who would like to give some priority to the worst-off. If one wants to incorporate such priority in the evaluation criterion, one must relax one of Harsanyi's central postulates, social rationality or Pareto.[1]

In this paper we explore how inequality aversion can be introduced when one relaxes Pareto somewhat. We believe that the Pareto principle in risky contexts is not as compelling as in riskless contexts, because when individuals take risks, by definition they are not fully informed about the final consequences of their choices. The most striking situation is when the final distribution of welfare is known and only the identity of the well-off and worse-off individuals is unknown. Then there is no risk on social welfare. Uncertainty at the individual level is irrelevant if one wants to cater to the individuals' actual interests as they will be revealed in the final distribution.

Based on this argument, Fleurbaey (2010) has proposed to restrict the application of the Pareto principle to riskless situations and to risky situations that involve no inequalities ex post. When there are no inequalities ex post, an external observer has no better clue about the final distribution than every individual has about his own final situation. It then seems reasonable to respect their risk preferences. With such restrictions, one obtains a class of criteria that compute the expected value of the "equally distributed equivalent" (EDE) utility.[2] Any degree of inequality aversion can be put in the EDE function. In the extreme, the expected value of the lowest utility, or expected maximin, is such a criterion. This type of criterion is unfortunately highly non-separable across individuals. If Robinson wants to climb a tree, this is fine if he is worse-off than Friday in all states of nature, or better-off in all states of nature. But if he may be better-off or worse-off than Friday depending on whether he falls from

[1] On the interpretation of the implications of Harsanyi's theorem, see Weymark (1991) and Broome (1991). For a defense of Paretian ("ex ante") criteria that evaluate the distribution of individual expected utilities with some inequality aversion, see, e.g., Diamond (1967) and Epstein and Segal (1992). For a defense of rational ("ex post") criteria that compute the expected value of an inequality averse social welfare function, see, e.g., Adler and Sanchirico (2006) and Fleurbaey (2010).

[2] The equally distributed equivalent (Atkinson, 1970) of a given distribution of utility is the level of utility that, if enjoyed uniformly by all individuals, would yield the same social welfare as the contemplated distribution.

the tree or not, his adventure decreases the expected value of the lowest utility. One therefore sees that the evaluation depends on the utility level of Friday, even when Friday is on the other side of the island, totally unconcerned.

In this paper we explore the implications on permissible inequality aversion of imposing a condition of independence with respect to unconcerned individuals, which we restrict to be individuals who bear no risk and have the same utility in the two alternatives to be compared. This kind of restriction has already been studied in the context of social risk evaluation by Bommier and Zuber (2008), who showed that an egalitarian evaluator then has to adopt a multiplicative form for the social welfare function. Bommier and Zuber restricted the Pareto principle to situations in which risks are independent across individuals and assumed that the evaluation relied on expected social welfare. Here, in contrast, we restrict Pareto to riskless and to egalitarian situations,[3] and, as far as social rationality is concerned, we only assume that social evaluation satisfies statewise dominance. Even if our Pareto conditions are different and our rationality assumption is weaker, we obtain a similar result, with some additional possibilities for specific domains of individual utilities. We study the implications of this result for the degree of inequality aversion of the criterion. We show that inequality aversion is not restricted provided that one is allowed to recalibrate the utilities before applying the criterion.

The paper is organized as follows. The next section introduces the framework. The axioms and the main result are presented in Section 3. The implications for inequality are examined in Sections 4. Section 5 contains a discussion of the results.

2 The framework

The framework is the same as in Fleurbaey (2010). The population is finite and fixed, $N = \{1, \dots, n\}$. The set of states of the world is finite, $S = \{1, \dots, m\}$, and the evaluator has a fixed probability vector $\pi = (\pi_s)_{s \in S}$, with $\sum_{s \in S} \pi_s = 1$. This probability vector corresponds to the evaluator's best estimate of the likelihood of the various states of the world. We therefore abstract from the problem of aggregating beliefs. Given that what happens in null states can be disregarded, we simply assume

[3] When there are many individuals, independent risks are compatible with an almost perfect knowledge of the final distribution. In view of the argument given in the beginning of the introduction, taking account of individual risk preferences in this context is less compelling than in absence of inequalities. It turns out that the criteria we obtain do satisfy the Pareto principle in the case of independent risks, which is a noteworthy result in itself.

that $\pi_s > 0$ for all $s \in S$. Vector inequalities are denoted $\geq$, $>$ and $\gg$ as usual.

The evaluator's problem is to rank prospects $U = (U_i^s)_{i\in N, s\in S} \in \mathbb{R}^{nm}$, where U_i^s describes the utility attained by individual i in state s. Let $X \subseteq \mathbb{R}$ be an interval (not necessarily bounded) and $\mathcal{L} = X^{nm}$ denote the relevant set of prospects over which the evaluation must be made. The social ordering (i.e., a complete, transitive binary relation) over the set L is denoted R (with strict preference P and indifference I).

Let U_i denote $(U_i^s)_{s\in S}$ and U^s denote $(U_i^s)_{i\in N}$. Let $[U^s]$ denote the riskless prospect in which vector U^s occurs in all states of the world. Two subsets of $\mathcal{L}$ must be singled out: $\mathcal{L}^c$ will denote the subset of riskless prospects (i.e., $U^s = U^t$ for all $s, t \in S$); $\mathcal{L}^e$ will denote the subset of egalitarian prospects (i.e., $U_i = U_j$ for all $i, j \in N$). For two prospects $U, \widetilde{U} \in \mathcal{L}$ and a subset $Q \subset N$, let $(U_Q, \widetilde{U}_{N\setminus Q})$ denote the prospect V such that $V_i = U_i$ for all $i \in Q$ and $V_i = \widetilde{U}_i$ for all $i \in N \setminus Q$.

The utility numbers U_i^s are assumed to be fully measurable and interpersonally comparable. They may measure any subjective or objective notion of advantage that the evaluator considers relevant for social evaluation. It is assumed that, for one-person evaluations, the evaluator considers that the expected value $EU_i = \sum_{s\in S} \pi_s U_i^s$ correctly measures agent i's ex-ante interests. We also assume that for every $s \in S$, the vector U^s fully describes the relevant features of the final situation occurring in state s. Thus, the social preferences over final situations need not be state dependent. This means that U^s is deemed better than V^s in state s if and only if $[U^s] R [V^s]$. In other words, there is no need to introduce preferences over final consequences as they are equivalent to the social ordering R restricted to riskless prospects. This is a convenient and innocuous simplification.

3 Multiplicative and Additive Criteria

We now introduce some requirements that one may wish to impose on the social ordering R. First, as explained above, there are two Pareto conditions, one for riskless situations, the other for situations in which full equality prevails in all states of the world.

Axiom 1 (Strong Pareto for no risk). *For all $U, V \in \mathcal{L}^c$, if $U_i \geq V_i$ for all $i \in N$, then URV. If furthermore $U_j > V_j$ for some $j \in N$, then UPV.*

Axiom 2 (Weak Pareto for equal risk). *For all $U, V \in \mathcal{L}^e$, if $EU_i > EV_i$ for all $i \in N$, then UPV.*

Social rationality is expressed here by statewise dominance. This is a compelling requirement. Violating it would mean that one would sometimes prefer a prospect that is bound to generate worse consequences than another.[4]

Axiom 3 (Weak dominance). *For all $U, V \in \mathcal{L}$, if $[U^s]R[V^s]$ for all $s \in S$, then URV.*

The last key requirement is an independence condition, which says that the social ranking of two prospects is independent of the level of utility of individuals who bear no risk and have the same utility in the two prospects.

Axiom 4 (Independence of the utilities of the sure). *For all $U, V \in \mathcal{L}$ and $\widetilde{U}, \widetilde{V} \in \mathcal{L}^c$, and for all $Q \subset N$,*

$$\left(U_Q, \widetilde{U}_{N\backslash Q}\right) R \left(V_Q, \widetilde{U}_{N\backslash Q}\right) \Longleftrightarrow \left(U_Q, \widetilde{V}_{N\backslash Q}\right) R \left(V_Q, \widetilde{V}_{N\backslash Q}\right).$$

The restriction to individuals who bear no risk is important. Otherwise, the condition would be a strong separability property that would run afoul of the argument against Pareto developed in the introduction. They would be immediately incompatible with inequality aversion.

Consider the following prospects (rows are for individuals, columns for two equiprobable states). An egalitarian would like the social ordering R to satisfy

$$\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} P \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

because individual expected utilities are the same and less inequality ex post is obtained in the preferred prospect. If full separability were applied, this preference would imply

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} P \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix},$$

in contradiction with the same egalitarian rationale. This example shows that full separability would make it impossible to be sensitive to the correlation between indi-

[4]Certain apparent violations of dominance seem rational (Grant, 1995). If a parent would rather flip a coin to allocate a sweet between two children than give it to one child without flipping a coin, this seems to violate dominance because the final distribution of sweets is the same anyway. But this behavior is compatible with dominance if, as is natural, one incorporates the fairness of the procedure in the description of the final consequences.

vidual utilities and to the degree of information the external observer has about the final distribution of utilities.

Another issue regarding Independence of the utilities of the sure must be briefly discussed. The situation of individuals who bear no risk and are unconcerned by the options to be compared may seem rather rare in real life evaluations of public policies. However, a correct evaluation of the risky prospects of humanity should encompass all the individuals who ever lived and may live. If the population under consideration includes the previous generations, they indeed form a subgroup that is unconcerned and no longer bears any risk. Therefore independence of the utilities of the sure is not an idle axiom in practice.

Actually, if in our model individuals are successive generations and we interpret the index $i = 1, \ldots, n$ as the birth date of a generation, one may want to apply independence of the utilities of the sure to the first generations up to any particular date. This is what Blackorby et al. (1995) and Bommier and Zuber (2008) have called "Independence of the utilities of the dead". One may object that Independence of the utility of the sure is stronger because it applies to any subpopulation, whereas with Independence of the utility of the dead the unconcerned are always the past generations. Under Anonymity, however, the two axioms are equivalent. The reader can easily check that in the proof of Proposition 1, only Independence of the utilities of the dead is actually used.

Finally, we will make use of two basic axioms of anonymity and continuity.

Axiom 5 (Anonymity). *For all $U, V \in \mathcal{L}$, if there exists a bijection $\rho : N \to N$ such that $U_i = V_{\rho(i)}$ for all $i \in N$, then UIV.*

Axiom 6 (Continuity). *For all $U, V \in \mathcal{L}$, if $(U(k))_{k\in\mathbb{N}} \in \mathcal{L}^{\mathbb{N}}$ is such that $U(k) \to U$ and $U(k)RV$ for all $k \in \mathbb{N}$, then URV; if $VRU(k)$ for all $k \in \mathbb{N}$, then VRU.*

We are now able to state our main result.

Proposition 1. *The social ordering R satisfies the six axioms if and only if one of the following two statements holds:*

1. *For all $U, V \in \mathcal{L}$*

$$URV \Longleftrightarrow \sum_{s\in S} \pi_s \frac{1}{n} \sum_{i\in N} U_i^s \geq \sum_{s\in S} \pi_s \frac{1}{n} \sum_{i\in N} V_i^s. \tag{1}$$

2. *There exist* $\alpha, \beta \in \mathbb{R}$ *satisfying* $\alpha x + \beta > 0$ *for all* $x \in X$ *such that, for all* $U, V \in \mathcal{L}$,

$$URV \Longleftrightarrow \frac{1}{\alpha}\sum_{s\in S}\pi_s \prod_{i\in N}(\alpha U_i^s + \beta)^{\frac{1}{n}} \geq \frac{1}{\alpha}\sum_{s\in S}\pi_s \prod_{i\in N}(\alpha V_i^s + \beta)^{\frac{1}{n}}. \tag{2}$$

Proof. If the social ordering R satisfies (1) or (2), then it clearly satisfies the axioms.

Now assume that the social ordering R satisfies the axioms. Let $\mathbf{1}_n$ denote the n-vector $(1, \ldots, 1)$. By Strong Pareto for no risk, for every $U^s \in X^n$, there exists $a, b \in X$ such that

$$[b\mathbf{1}_n]\, R\, [U^s]\, R\, [a\mathbf{1}_n]\,.$$

By Continuity, there exists $x \in X$ such that $[x\mathbf{1}_n]\, I\, [U^s]$. By Strong Pareto for no risk, it is unique. This value of x defines the EDE function $e(U^s)$. By Anonymity, e is symmetric. By Strong Pareto for no risk, it is increasing in each argument. By definition, it satisfies $e(x, \ldots, x) = x$ for all $x \in X$.

By Weak dominance, for all $U \in \mathcal{L}$,

$$UI\left(e\left(U^1\right), \ldots, e\left(U^m\right)\right).$$

The quantity $\sum_{s\in S}\pi_s e(U^s)$ belongs to X because X is an interval. By Continuity and Weak Pareto for equal risk, one must have

$$\left(e\left(U^1\right), \ldots, e\left(U^m\right)\right) I \left[\left(\sum_{s\in S}\pi_s e(U^s)\right)\mathbf{1}_n\right].$$

Therefore, by transitivity and Strong Pareto for no risk, for all $U, V \in \mathcal{L}$,

$$URV \Longleftrightarrow \sum_{s\in S}\pi_s e(U^s) \geq \sum_{s\in S}\pi_s e(V^s). \tag{3}$$

The remainder of the proof is closely related to a similar result by Keeney and Raiffa in the case of multidimensional risks (Keeney and Raiffa, 1976, Th. 6.1, p. 289). Let u^* be an arbitrary number in X. Let $\hat{e}$ be the function defined as $\hat{e} \equiv e - u^*$, which implies $\hat{e}(u^*, \ldots, u^*) = 0$. By definition, the function $\hat{e}$ is symmetric, and for all $U, V \in \mathcal{L}$,

$$URV \Longleftrightarrow \sum_{s\in S}\pi_s \hat{e}(U^s) \geq \sum_{s\in S}\pi_s \hat{e}(V^s). \tag{4}$$

Independence of the utilities of the sure tells us that, for all $i \in \{1, \ldots, n-1\}$, for all $U \in \mathcal{L}^c$ and all $V, \widetilde{V} \in \mathcal{L}$:

$$\sum_{s \in S} \pi_s \hat{e}(U_1^s, \ldots, U_i^s, V_{i+1}^s, \ldots, V_n^s) \geq \sum_{s \in S} \pi_s \hat{e}(U_1^s, \ldots, U_i^s, \widetilde{V}_{i+1}^s, \ldots, \widetilde{V}_n^s)$$
$$\iff \sum_{s \in S} \pi_s \hat{e}(u^*, \ldots, u^*, V_{i+1}^s, \ldots, V_n^s) \geq \sum_{s \in S} \pi_s \hat{e}(u^*, \ldots, u^*, \widetilde{V}_{i+1}^s, \ldots, \widetilde{V}_n^s).$$

Because vNM utility functions are unique up to an increasing affine transformation, there must exist two functions f_i and g_i such that:

$$\hat{e}(U_1^s, \ldots, U_i^s, U_{i+1}^s, \ldots, U_n^s) = f_i(U_1^s, \ldots, U_i^s) + g_i(U_1^s, \ldots, U_i^s)\hat{e}(u^*, \ldots, u^*, U_{i+1}^s, \ldots, U_n^s), \tag{5}$$

where $g_i(U_1^s, \ldots, U_i^s) > 0$ for all $(U_1^s, \ldots, U_i^s) \in X^i$.

Define $a_1 \equiv f_1$, $b_1 \equiv g_1$, and, for all $i \in \{2, \ldots, n-1\}$, $a_i(U_i^s) = f_i(u^*, \ldots, u^*, U_i^s)$ and $b_i(U_i^s) = g_i(u^*, \ldots, u^*, U_i^s)$, and $a_n(U_n^s) = \hat{e}(u^*, \ldots, u^*, U_n^s)$. Equation (5) implies that, for all $i \in \{1, \ldots, n-1\}$:[5]

$$\hat{e}(u^*, \ldots, u^*, U_i^s, U_{i+1}^s, \ldots, U_n^s) = a_i(U_i^s) + b_i(U_i^s)\hat{e}(u^*, \ldots, u^*, U_{i+1}^s, \ldots, U_n^s). \tag{6}$$

Repeated applications of Equation (6) yield:

$$\begin{aligned} \hat{e}(U_1^s, \ldots, U_n^s) &= a_1(U_1^s) + b_1(U_1^s)\left(a_2(U_2^s) + b_2(U_2^s)(\ldots)\right) \\ &= a_1(U_1^s) + \sum_{i=2}^{n} a_i(U_i^s) \prod_{j=1}^{i-1} b_j(U_j^s). \end{aligned}$$

Using the normalization condition $\hat{e}(u^*, \ldots, u^*) = 0$ in Equation (6), we also obtain that $a_i(U_i^s) = \hat{e}(u^*, \ldots, u^*, U_i^s, u^*, \ldots, u_*)$ for all $i \in \{1, \ldots, n-1\}$ (the same is also true for a_n by definition). Therefore, by symmetry of $\hat{e}$, all the functions a_i are the same (increasing) function ϕ, such that $\phi(u^*) = 0$.

The symmetry of the function $\hat{e}$ also implies that, for all $i \in \{1, \ldots, n-1\}$:

$$\hat{e}(U_1^s, \ldots, U_i^s, U_{i+1}^s, \ldots, U_n^s) = \hat{e}(U_1^s, \ldots, U_{i+1}^s, U_i^s, \ldots, U_n^s).$$

Using Equation (6) applied to $(u^*, \ldots, u^*, U_i^s, U_{i+1}^s, u^*, \ldots, u^*)$ and $a_i \equiv \phi$, this yields,

[5]In the case $i = 1$, the equation is $\hat{e}(U_s^1, U_s^2, \ldots, U_s^n) = a_1(U_s^1) + b_1(U_s^1)\hat{e}(u^*, U_s^2, \ldots, U_s^n)$.

for all $(U_i^s, U_{i+1}^s) \in X^2$:

$$\phi(U_i^s) + b_i(U_i^s)\phi(U_{i+1}^s) = \phi(U_{i+1}^s) + b_i(U_{i+1}^s)\phi(U_i^s). \tag{7}$$

If $U_i^s = u^*$, we obtain $b_i(U_i^s) = 1$. If U_i^s and U_{i+1}^s are both different from u^*, we obtain:

$$\frac{1 - b_i(U_i^s)}{\phi(U_i^s)} = \frac{1 - b_i(U_{i+1}^s)}{\phi(U_{i+1}^s)}.$$

Therefore there exists a constant $k_i = (b_i(U_i^s) - 1)/\phi(U_i^s)$ for all U_i^s, or equivalently, $b_i(U_i^s) = 1 + k_i\phi(U_i^s)$. Note that we need $b_i(x) > 0$ for all $x \in X$ and therefore $1 + k_i\phi(x) > 0$ for all $x \in X$.

Symmetry also implies that:

$$\hat{e}(u^*, \ldots, u^*, U_i^s, U_{i+1}^s, u^*, \ldots, u^*) = \hat{e}(U_i^s, U_{i+1}^s, u^*, \ldots, u^*),$$

so that $\phi(U_i^s) + (1 + k_i\phi(U_i^s))\,\phi(U_{i+1}^s) = \phi(U_i^s) + (1 + k_1\phi(U_i^s))\,\phi(U_{i+1}^s)$ and therefore k_i is equal to a given constant k for all $i \in \{1, \ldots, n-1\}$. In the end, we obtain that:

$$\hat{e}(U_1^s, \ldots, U_n^s) = \phi(U_1^s) + \sum_{i=2}^{n} \phi(U_i^s) \prod_{j=1}^{i-1} \left(1 + k\phi(U_j^s)\right) \tag{8}$$

There are two cases.

Case 1: $k = 0$. In this case, (8) implies that $\hat{e}(U_1^s, \ldots, U_n^s) = \sum_{i\in N} \phi(U_i^s)$, so that $e(U_1^s, \ldots, U_n^s) = u^* + \sum_{i\in N} \phi(U_i^s)$. Note that the condition $1 + k\phi(x) > 0$ is always satisfied in that case. The condition $e(x, \ldots, x) = x$ implies $\phi(x) = (x - u^*)/n$, which yields (1).

Case 2: $k \neq 0$. In this case, (8) can be rewritten

$$\begin{aligned} 1 + k\hat{e}(U_1^s, \ldots, U_n^s) &= 1 + k\phi(U_1^s) + \sum_{i=2}^{n} k\phi(U_i^s) \prod_{j=1}^{i-1} \left(1 + k\phi(U_j^s)\right) \\ &= \prod_{i=1}^{n} \left(1 + k\phi(U_i^s)\right), \end{aligned}$$

so that

$$\hat{e}(U_1^s, \ldots, U_n^s) = \frac{1}{k}\left(\prod_{i=1}^{n} \left(1 + k\phi(U_i^s)\right) - 1\right),$$

and $e(U_1^s, \ldots, U_n^s) = u^* + \hat{e}(U_1^s, \ldots, U_n^s)$. The condition $e(x, \ldots, x) = x$ implies

$$\phi(x) = 1/k \left((kx + 1 - ku^*)^{1/n} - 1 \right),$$

so that

$$e(U_1^s, \ldots, U_n^s) = \frac{1}{\alpha} \prod_{i \in N} (\alpha U_i^s + \beta)^{\frac{1}{n}} + u^*.$$

where $\alpha = k$ and $\beta = 1 - ku^*$. The condition $1 + k\phi(x) > 0$ for all $x \in X$ implies that we must have $\alpha x + \beta > 0$ for all $x \in X$. This yields (2). □

The first possibility highlighted in this result is unappealing to an egalitarian because it features standard utilitarianism. The second possibility makes it possible to introduce inequality aversion, but this partly depends on the value of the parameters α, β. We study this issue in the next section.

4 Transfer principle and inequality aversion

Inequality aversion, or equivalently, priority for the worse-off, may be captured by requiring the social ordering to satisfy the Pigou-Dalton transfer principle. In our setting, individual prospects are inherently multidimensional and there are several possible adaptations of the classical transfer principle. We retain a rather natural one proposed by Fleurbaey and Trannoy (2003).[6] If i's prospect strictly dominates j's prospect in every state of the world, making a transfer of utility from i to j in every state (without reversing their relative positions) improves the social prospect.

Axiom 7 (Multidimensional transfer principle). *For all $U, V \in \mathcal{L}$, if there exist $i, j \in N$ and $\delta \in \mathbb{R}^m_{++}$ such that*

$$U_i = V_i - \delta \gg V_j + \delta = U_j,$$

and for all $k \in N \setminus \{i, j\}$, $U_k = V_k$, then

$$U P V.$$

[6]For a comparison of various multidimensional versions of the Pigou-Dalton principle, see Diez et al. (2007).

Proposition 2. *The social ordering R satisfies the seven axioms if and only if one of the three following statements holds true:*

1. *There exists a scalar $\varepsilon \in \mathbb{R}_{++}$ satisfying $\varepsilon x + 1 > 0$ for all $x \in X$ and such that for all $U, V \in \mathcal{L}$,*

$$URV \Longleftrightarrow \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon U_i^s + 1)^{\frac{1}{n}} \geq \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon V_i^s + 1)^{\frac{1}{n}}. \tag{9}$$

2. *There exists a scalar $\varepsilon \in \mathbb{R}_{++}$ satisfying $\varepsilon x - 1 > 0$ for all $x \in X$ and such that for all $U, V \in \mathcal{L}$,*

$$URV \Longleftrightarrow \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon U_i^s - 1)^{\frac{1}{n}} \geq \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon V_i^s - 1)^{\frac{1}{n}}. \tag{10}$$

3. *$X \subset \mathbb{R}_{++}$ and for all $U, V \in \mathcal{L}$,*

$$URV \Longleftrightarrow \sum_{s\in S} \pi_s \prod_{i\in N} (U_i^s)^{\frac{1}{n}} \geq \sum_{s\in S} \pi_s \prod_{i\in N} (V_i^s)^{\frac{1}{n}}. \tag{11}$$

Proof. One can easily check that the proposed social welfare functions satisfy all the axioms. For the Multidimensional transfer principle, this follows from the fact that the transfer δ^s improves the distribution in every $s \in S$.

By Proposition 1, the social ordering R satisfies the first six axioms only if either (1) or (2) holds.

Consider (1) first. When U and V are defined as in the Multidimensional transfer principle, it is clear that $\sum_{s\in S} \pi_s \frac{1}{n} \sum_{i\in N} U_i^s = \sum_{s\in S} \pi_s \frac{1}{n} \sum_{i\in N} V_i^s$, therefore the axiom cannot be satisfied.

For the case (2), if $\beta \neq 0$ we can rewrite:

$$
\begin{aligned}
URV \quad &\iff \quad \frac{1}{\alpha}\sum_{s\in S}\pi_s\prod_{i\in N}(\alpha U_i^s+\beta)^{\frac{1}{n}} \geq \frac{1}{\alpha}\sum_{s\in S}\pi_s\prod_{i\in N}(\alpha V_i^s+\beta)^{\frac{1}{n}} \\
&\iff \quad sign(\alpha)\frac{|\beta|^n}{|\alpha|}\sum_{s\in S}\pi_s\prod_{i\in N}\left(sign(\alpha)\times(|\alpha|/|\beta|)U_i^s+sign(\beta)\times 1\right)^{\frac{1}{n}} \\
&\qquad\qquad \geq sign(\alpha)\frac{|\beta|^n}{|\alpha|}\sum_{s\in S}\pi_s\prod_{i\in N}\left(sign(\alpha)\times(|\alpha|/|\beta|)V_i^s+sign(\beta)\times 1\right)^{\frac{1}{n}} \\
&\iff \quad sign(\alpha)\sum_{s\in S}\pi_s\prod_{i\in N}\left(sign(\alpha)\varepsilon U_i^s+sign(\beta)\times 1\right)^{\frac{1}{n}} \\
&\qquad\qquad\qquad \geq sign(\alpha)\sum_{s\in S}\pi_s\prod_{i\in N}\left(sign(\alpha)\varepsilon V_i^s+sign(\beta)\times 1\right)^{\frac{1}{n}},
\end{aligned}
$$

where $\varepsilon = |\alpha| \,/\, |\beta|$. There are four subcases, depending on $sign(\alpha)$ and $sign(\beta)$.

Now, considering $U, V \in \mathcal{L}^c$, we obtain that

$$URV \iff \sum_{i\in N}\phi(U_i^s) \geq \sum_{i\in N}\phi(V_i^s),$$

where $\phi(x) = sign(\alpha)\ln\left(sign(\alpha)\varepsilon x + sign(\beta)\right)$. On $\mathcal{L}^c$, Multidimensional transfer principle implies the usual Pigou-Dalton transfer principle, which is satisfied if and only if ϕ is a strictly concave function. This is the case here only when $sign(\alpha) > 0$, which leaves us with the two possibilities (9) and (10), depending on the sign of β.

If $\beta = 0$, one then has

$$URV \iff sign(\alpha)\sum_{s\in S}\pi_s\prod_{i\in N}(sign(\alpha)U_i^s)^{\frac{1}{n}} \geq sign(\alpha)\sum_{s\in S}\pi_s\prod_{i\in N}(sign(\alpha)V_i^s)^{\frac{1}{n}},$$

and here again the Multidimensional transfer principle implies $sign(\alpha) > 0$, which yields (11). □

Looking at the proof, it is worth noting that the result would not be changed if we used a weaker axiom making only simple Pigou-Dalton transfers in riskless situations. The stronger axiom has been introduced here because it is worth checking that it can be satisfied in this context.

Social welfare functions satisfying the transfer principle are said to be inequality averse. It remains to study how much inequality aversion is compatible with formulae (9) and (10). To that effect we will compare the inequality aversion of the contemplated orderings with that of benchmark orderings. It is enough to focus on riskless prospects, and we can therefore rely on standard concepts of unidimensional inequality measurement. We have the following standard method to compare inequality aversion:

Definition 1. *A social ordering R is more inequality averse than a social ordering $\tilde{R}$ if, for any $U \in \mathcal{L}^c$ and $V \in \mathcal{L}^c \cap \mathcal{L}^e$:*

$$URV \Longrightarrow U\tilde{R}V$$

In the case of social orderings represented for riskless prospects by symmetric additive social welfare functions $\sum_{i \in N} \phi(U_i^s)$, there are standard results indicating that the more concave the function ϕ, the more inequality averse the social ordering. When $\sum_{i \in N} \phi(U_i^s)$ takes the classical isoelastic form $\frac{1}{1-\alpha} \sum_{i \in N} (U_i^s)^{1-\alpha}$, it is convenient to measure its degree of inequality aversion by α.

Clearly, all the social welfare functions in the families (9) and (10) are more inequality averse than the social ordering represented by the utilitarian social welfare function (1), which has a degree of inequality aversion equal to 0.

One can also compare them with the social ordering represented by (11), which is for sure prospects ordinally equivalent to $\sum_{i \in N} \ln U_i^s$ and has a degree of inequality aversion equal to 1. We obtain the following results:

Proposition 3.

1. *Social welfare functions from family (9):*
   - *Are more inequality averse the larger ε.*
   - *Become ordinally equivalent to (1) when $\varepsilon \to 0$ and to (11) when $\varepsilon \to +\infty$.*[7]
2. *Social welfare functions from family (10):*
   - *Are less inequality averse the larger ε.*
   - *Become ordinally equivalent to (11) when $\varepsilon \to +\infty$.*
   - *Are more inequality averse than $\frac{1}{1-\alpha} \sum_{i \in N} (U_i^s)^{1-\alpha}$, for a given $\alpha > 1$, if $0 < \varepsilon x - 1 < -1/(1-\alpha)$ for all $x \in X$.*

[7] It is permissible to let $\varepsilon \to +\infty$ only if $\inf X \geq 0$.

Proof.

1.

- As indicated above, a social ordering represented by $\sum_{i\in N}\phi(U_i^s)$ is more inequality averse than a social ordering represented by $\sum_{i\in N}\tilde{\phi}(U_i^s)$ if and only if there exists a concave function ψ such that $\phi=\psi\circ\tilde{\phi}$. Let $\varphi_\varepsilon(x)=\ln(\varepsilon x+1)$. On riskless prospects, (9) is ordinally equivalent to $\sum_{i\in N}\varphi_\varepsilon(U_i^s)$. One can compute that
$$\varphi_\varepsilon(x)=\ln\left(\varepsilon\frac{\exp\left(\varphi_{\varepsilon'}(x)\right)-1}{\varepsilon'}+1\right).$$
If $\varepsilon>\varepsilon'$, the function $\psi_{\varepsilon,\varepsilon'}(z)=\ln\left((\varepsilon/\varepsilon')\exp(z)+1-\varepsilon/\varepsilon'\right)$ is strictly concave. Then the social ordering on riskless prospects represented by $\sum_{i\in N}\varphi_\varepsilon(U_i^s)$ is more inequality averse than the social ordering represented by $\sum_{i\in N}\varphi_{\varepsilon'}(U_i^s)$.

- When $\varepsilon\to 0$, $(\varepsilon x+1)^{1/n}\approx 1+\varepsilon x/n$. Therefore the function $\sum_{s\in S}\pi_s\prod_{i\in N}(\varepsilon U_i^s+1)^{\frac{1}{n}}$ becomes ordinally equivalent to $\sum_{s\in S}\pi_s\sum_{i\in N}U_i^s$.

  The function $\sum_{s\in S}\pi_s\prod_{i\in N}(\varepsilon U_i^s+1)^{\frac{1}{n}}$ is ordinally equivalent to $\sum_{s\in S}\pi_s\prod_{i\in N}(U_i^s+1/\varepsilon)^{\frac{1}{n}}$, which tends to $\sum_{s\in S}\pi_s\prod_{i\in N}(U_i^s)^{\frac{1}{n}}$ when $\varepsilon\to+\infty$.

2.

- Let $\chi_\varepsilon(x)=\ln(\varepsilon x-1)$. We have:
$$\chi_\varepsilon(x)=\ln\left(\varepsilon\frac{\exp\left(\chi_{\varepsilon'}(x)\right)+1}{\varepsilon'}-1\right).$$
If $\varepsilon<\varepsilon'$, the function $\Psi_{\varepsilon,\varepsilon'}(z)=\ln\left((\varepsilon/\varepsilon')\exp(z)+\varepsilon/\varepsilon'-1\right)$ is strictly concave. The social ordering represented by $\sum_{i\in N}\chi_\varepsilon(U_i^s)$ is more inequality averse than the social ordering represented by $\sum_{i\in N}\chi_{\varepsilon'}(U_i^s)$.

- When $\varepsilon\to+\infty$, the argument is similar as for (9).

- One has $\chi_\varepsilon(x)=\ln(\varepsilon\left[(1-\alpha)z\right]^{\frac{1}{1-\alpha}}-1)$ whenever $z=\frac{1}{1-\alpha}x^{1-\alpha}$. The function $\psi_{\varepsilon,\alpha}(z)=\ln(\varepsilon\left[(1-\alpha)z\right]^{\frac{1}{1-\alpha}}-1)$ is strictly concave if $\varepsilon\left[(1-\alpha)z\right]^{\frac{1}{1-\alpha}}-1<-1/\left(1-\alpha\right)$ for all z.

$\square$

The families (9) and (10) seemingly cover a wide range of attitudes towards inequality. However the social welfare function represented by (9) is well-defined on (subsets of) the interval $(-1/\epsilon, +\infty)$ while the social welfare function represented by (10) is well-defined on (subsets of) the interval $(1/\epsilon, +\infty)$. So the form of the set X will constrain possible degree of inequality aversions. A noteworthy configuration is the following:

Corollary 1. *If $X = \mathbb{R}_{++}$, $(0, a]$ or $(0, a)$ (where $a \in \mathbb{R}_{++}$), $\sum_{i \in N} \ln U_i^s$ is the most inequality averse social ordering satisfying the seven axioms.*

As a consequence the case of positive utility levels singles out the Nash product as the social welfare function that gives most priority to the worst-off. If a zero utility level is included, one obtains:

Corollary 2. *If $X = \mathbb{R}_+$, $[0, a]$ or $[0, a)$, (where $a \in \mathbb{R}_{++}$), the social orderings satisfying the seven axioms are less inequality averse than $\sum_{i \in N} \ln U_i^s$.*

To obtain a greater inequality aversion, a further restriction of the domain is required:

Corollary 3. *A social ordering satisfying the seven axioms is more inequality averse than $\frac{1}{1-\alpha} \sum_{i \in N} (U_i^s)^{1-\alpha}$, for a given $\alpha > 1$, if and only if $X \subset \left(\frac{1}{\varepsilon}, \frac{1}{\varepsilon} \frac{\alpha}{\alpha - 1}\right)$ and it belongs to family (10).*

Therefore, although in theory any positive degree of inequality aversion can be surpassed by social orderings satisfying the seven axioms, this may require a calibration of individual utilities which squeezes them in a tiny interval. More precisely, the greater the degree of inequality aversion one wishes to put into social evaluation, the more difficult it may be to measure utilities in a reasonable range. Whether the evaluator is free to rescale utility numbers before applying a formula like (10) is a delicate issue that depends on what utility is supposed to measure. As is well known, the issue of interpersonal comparisons is laden with difficult value judgments.

Future research should investigate whether some additional principles can provide guidance on the definition and the range of the individual welfare indices. For instance, in the context of Harsanyi's two theorems (his impartial observer theorem and his social aggregation theorem), several contributions have advocated that the cardinal indices to be used are the 0-1 normalized von Neumann Morgenstern utilities (see among other Dhillon and Mertens, 1999; Karni, 1998; Gajdos and Kandil, 2008). This is a utility range that imposes a weak degree of inequality aversion, from Corollary 1.

5 Separability versus Pareto

Another problematic consideration is that, even though the *utility* of the past generations can be ignored in the application of the social orderings highlighted in Proposition 2, the *number* of individuals in society, and therefore in the past generations, still plays a role in the computation. One might want to have independence not just of the utility of the sure, but of the existence of the sure.

If one combines independence of the existence of the sure with Weak Pareto for equal risk, one obtains the following stronger version of Weak Pareto for equal risk, that applies to the subgroup of concerned individuals independently of its size.

Axiom 8 (Weak Pareto for subgroup equal risk). *For all $U, V \in \mathcal{L}^e$ and $\widetilde{U} \in \mathcal{L}^c$, and for all $Q \subset N$, if $EU_i \geq EV_i$ for all $i \in Q$, then* $\left(U_Q, \widetilde{U}_{N \setminus Q}\right) P \left(V_Q, \widetilde{U}_{N \setminus Q}\right)$.

As shown in Fleurbaey (2010), this axiom brings us back into the grip of Harsanyi's utilitarianism. In the context of EDE criteria studied in this paper, it seems that we cannot allow more separability than permitted by independence of the utility of the sure. But this may become possible if Weak Pareto for equal risk is abandoned or modified. Consider the following weakening of Weak Pareto for subgroup equal risk where the size of the group of concerned individuals is fixed.

Axiom 9 (Weak Pareto for q group risk). *For all $U, V \in \mathcal{L}^e$ and $\widetilde{U} \in \mathcal{L}^c$, and for all $Q \subset N$ such that $|Q| = q$, if $EU_i > EV_i$ for all $i \in Q$, then* $\left(U_Q, \widetilde{U}_{N \setminus Q}\right) P \left(V_Q, \widetilde{U}_{N \setminus Q}\right)$.

We do not argue that Weak Pareto for q group risk is ethically appealing; it may appear restrictive to consider only one group size. We introduce it for analytical purposes. It encompasses cases of particular interest. When $q = n$ we are back to Weak Pareto for equal risk. The case $q = 1$ corresponds to a situation where one individual takes risks that do not affect the other members of the society. One could argue that choices for such individual risks should be respected. Considering intermediate cases will turn out to have an influence on the risk aversion of the social ordering.

If we replace Weak Pareto for equal risk by Weak Pareto for q group risk, we need to strengthen our rationality requirements to remain within the scope of the expected utility theory (see the Appendix for details). We therefore make the following assumption, which implies both Continuity and Weak Dominance.

Axiom 10 (Expected utility hypothesis). *For all $U, V \in \mathcal{L}$, there exists a continuous function F unique up to positive affine transformations such that*

$$URV \iff \sum_{s\in S} \pi_s F(U^s) \geq \sum_{s\in S} \pi_s F(V^s).$$

Using this axiom, we obtain the following characterization result.

Proposition 4. *If $n \geq q$, the social ordering R satisfies Strong Pareto for no risk, Independence of the utilities of the sure, Anonymity, Multidimensional transfer principle, Weak Pareto for q group risk, and Expected utility hypothesis in and only if one of the three following statements holds true:*

1. *There exists a scalar $\varepsilon \in \mathbb{R}_{++}$ satisfying $\varepsilon x + 1 > 0$ for all $x \in X$ and such that for all $U, V \in \mathcal{L}$,*

$$URV \iff \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon U_i^s + 1)^{\frac{1}{q}} \geq \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon V_i^s + 1)^{\frac{1}{q}}. \tag{12}$$

2. *There exists a scalar $\varepsilon \in \mathbb{R}_{++}$ satisfying $\varepsilon x - 1 > 0$ for all $x \in X$ and such that for all $U, V \in \mathcal{L}$,*

$$URV \iff \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon U_i^s - 1)^{\frac{1}{q}} \geq \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon V_i^s - 1)^{\frac{1}{q}}. \tag{13}$$

3. *$X \subset \mathbb{R}_{++}$ and for all $U, V \in \mathcal{L}$,*

$$URV \iff \sum_{s\in S} \pi_s \prod_{i\in N} (U_i^s)^{\frac{1}{q}} \geq \sum_{s\in S} \pi_s \prod_{i\in N} (V_i^s)^{\frac{1}{q}}. \tag{14}$$

Proof. By Expected utility hypothesis,

$$URV \iff \sum_{s\in S} \pi_s F(U^s) \geq \sum_{s\in S} \pi_s F(V^s) \tag{15}$$

By Strong Pareto for no risk and Anonymity, the function F must be increasing and symmetric. Thus we obtain an equivalence similar to (3) in the proof of Proposition

1. Furthermore, F can be normalized so that $F(u^*, \ldots, u^*) = 0$ like function $\hat{e}$ in the proof of Proposition 1. Using Independence of the utilities of the sure, we can proceed as in the proof of Proposition 1 to obtain that:

$$F(U_1^s, \ldots, U_n^s) = \phi(U_1^s) + \sum_{i=2}^{n} \phi(U_i^s) \prod_{j=1}^{i-1} \left(1 + k\phi(U_j^s)\right) \tag{16}$$

where ϕ is a continuous and increasing function such that $1 + k_i\phi(x) > 0$ for all $x \in X$.

There are two cases.

Case 1: $k = 0$. In this case, Weak Pareto for q group risk implies that

$$\sum_{s\in S} \pi_s q\phi(U_i^s) \geq \sum_{s\in S} \pi_s q\phi(V_i^s) \Longleftrightarrow \sum_{s\in S} \pi_s U_i^s \geq \sum_{s\in S} \pi_s V_i^s.$$

As VNM functions are unique up to an increasing affine transform, there must exist $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}$ such that $q\phi(x) = \alpha x + \beta$. Therefore

$$URV \Longleftrightarrow \sum_{s\in S} \pi_s U^s \geq \sum_{s\in S} \pi_s V^s \tag{17}$$

Case 2: $k \neq 0$. In this case, Equation (16) can be rewritten:

$$F(U_1^s, \ldots, U_n^s) = \frac{1}{k} \left(\prod_{i=1}^{n} (1 + k\phi(U_i^s)) - 1 \right).$$

Hence Weak Pareto for q group risk implies that:

$$\sum_{s\in S} \pi_s \frac{1}{k} (1 + k\phi(U_i^s))^q \geq \sum_{s\in S} \pi_s \frac{1}{k} (1 + k\phi(U_i^s))^q \Longleftrightarrow \sum_{s\in S} \pi_s U_i^s \geq \sum_{s\in S} \pi_s V_i^s.$$

Applying the same reasoning as above there must exist $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}$ such that $\frac{1}{k} (1 + k\phi(x))^q = \alpha x + \beta$.

When $\beta = 0$, it is necessary that $kx > 0$ in order to have $1 + k\phi(x) > 0$. Then $1 + k\phi(x) = (k\alpha)^{1/q} x^{1/q}$ if $k > 0$, and $(-k\alpha)^{1/q} (-x)^{1/q}$ if $k < 0$. Therefore, either $X \subset \mathbb{R}_{++}$ and

$$URV \Longleftrightarrow \sum_{s\in S} \pi_s \prod_{i\in N} (U_i^s)^{1/q} \geq \sum_{s\in S} \pi_s \prod_{i\in N} (V_i^s)^{1/q} \tag{18}$$

or $X \subset \mathbb{R}_{--}$ and

$$URV \Longleftrightarrow -\sum_{s\in S} \pi_s \prod_{i\in N} (-U_i^s)^{1/q} \geq -\sum_{s\in S} \pi_s \prod_{i\in N} (-V_i^s)^{1/q}.$$

The latter case is excluded by Multidimensional transfer principle.

When $\beta \neq 0$, it is necessary that $k(\alpha x + \beta) > 0$. One has

$$\begin{aligned} 1 + k\phi(x) &= [k(\alpha x + \beta)]^{1/q} \\ &= |\beta k|^{1/q} \left[\left(\frac{sign(k)\alpha}{|\beta|} x + sign(\beta k) \right) \right]^{1/q}. \end{aligned}$$

This gives us four possibilities, depending on $sign(k)$ and $sign(\beta k)$:
1) For some $\varepsilon > 0$, $\varepsilon x + 1 > 0$ for all $x \in X$ and

$$URV \Longleftrightarrow \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon U_i^s + 1)^{1/q} \geq \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon U_i^s + 1)^{1/q}.$$

2) For some $\varepsilon > 0$, $\varepsilon x - 1 > 0$ for all $x \in X$ and

$$URV \Longleftrightarrow \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon U_i^s - 1)^{1/q} \geq \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon U_i^s - 1)^{1/q}.$$

3) For some $\varepsilon < 0$, $\varepsilon x + 1 > 0$ for all $x \in X$ and

$$URV \Longleftrightarrow -\sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon U_i^s + 1)^{1/q} \geq -\sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon U_i^s + 1)^{1/q}.$$

4) For some $\varepsilon < 0$, $\varepsilon x - 1 > 0$ for all $x \in X$ and

$$URV \Longleftrightarrow -\sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon U_i^s - 1)^{1/q} \geq -\sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon U_i^s - 1)^{1/q}.$$

The last two cases are excluded by Multidimensional transfer principle, which ends the proof. □

The criteria highlighted in Proposition 4 are closely related to the classes of criteria (9), (10) and (11). Indeed, as far as the analysis of inequality aversion is concerned, they induce the same results as in Proposition 3.

In the case $q = 1$, Pareto for q group risk collapses to a property of Pareto for individual risk: the risk preferences of the individual are respected if all other individuals are indifferent and bear no risk. In this case, the criteria in (12) exactly correspond to the multiplicative social welfare functions satisfying "risk equity" in Bommier and Zuber (2008). It is worth noting that their multiplicative social welfare functions satisfying "catastrophe avoidance",[8] which would correspond to case 3 in the proof ($\varepsilon < 0$, $\varepsilon x + 1 > 0$ for all $x \in X$), are ruled out by the transfer principle. They did not find the social welfare functions displayed in (13) or (14) because they assumed that $X \subset \mathbb{R}_+$ and $0 \in X$, which excludes these two cases.

Interestingly the criteria in Proposition 4 all satisfy the property of independence of the *existence* of the unconcerned sure introduced in the beginning of this section. On the other hand they fail Pareto for equal risk unless $n = q$.

6 Conclusion

In this paper, we have shown that social rationality (embodied in Weak Dominance) and a reasonable dose of the Pareto principle (Pareto for no risk, Pareto for equal risk) can be reconciled with inequality aversion and some independence with respect to unconcerned individuals bearing no risk. In particular, the Nash product has been singled out as the social ordering giving the most priority to the worst-off in the relevant case where the utility possibility set is the positive real line.

In the context of the evaluation of social situations involving risks, this already constitutes some progress. Indeed, in view of the results involving Independence of the utility of the sure (or "the dead") in Bommier and Zuber (2008), or the results involving Pareto for subgroup equal risk in Fleurbaey (2010), one might have feared that the degree of inequality aversion would be severely constrained. Our results open a wider range of possibilities.

Truly enough, the tension between social rationality, Pareto, inequality aversion, and separability remains substantial. The criteria introduced in this paper satisfy a very limited form of Pareto principle. The key parameter in this respect is q, which

[8]Risk equity and catastrophe avoidance are two principles introduced in Keeney (1980). The former is the principle that, when individuals face independent risks of a specific damage (accident), inequalities in their probabilities of damage are undesirable. The latter principle seeks to minimize the risk of having a large number of fatalities. Keeney showed that the two principles are antinomic, because the best way to avoid a catastrophe is to concentrate the risk on a few (sacrificed) individuals. In an intergenerational setting with uncertain existence of future generations, Bommier and Zuber (2008) show that risk equity (resp., catastrophe avoidance) induces a low (resp., high) social discount rate.

is equal to n in Propositions 1-2 and can be less than n in Proposition 4. If $q = 1$, individual expected utility is taken into account only when one individual takes a risk. When two individuals consider a risky prospect that does not generate inequality between them (while the rest of the population is unconcerned and risk-free), the criterion (14), for instance, maximizes the expected value of U^2, introducing what looks like an artificial love for risk. Conversely, if $q > 1$ and the group considering a risk is smaller than q, the criterion is more risk averse than the members of the group. When $q = n$, the criterion is more risk averse than the members of any strict subgroup of the population.

Another way to reconcile Pareto and separability is to restrict the set of possible prospects $\mathcal{L}$ and not simply the utility possibility set X. The ex post generalized Gini criteria introduced in Fleurbaey (2010), for instance, satisfy the full Pareto principle and strong separability properties (including Independence of the existence of the dead), in an intertemporal setting, if the successive generations' utility is always increasing in all possible worlds. Sustainable development might be not only our duty toward future generations, but also the solution to this ethical dilemma.

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A.1 Appendix

In this Appendix, we discuss what would happen if we used Continuity and Weak Dominance instead of the Expected utility hypothesis in Proposition 4. It is shown that the result would still hold on a subdomain (including sure prospect). But on part of the domain, social preferences may not be expected utilities.

Proposition 5. *If $n \geq q > 1$, if the social ordering R satisfies Strong Pareto for no risk, Weak dominance, Independence of the utilities of the sure, Anonymity, Continuity, Multidimensional transfer principle, and Weak Pareto for q group risk, then there exists a subset $\bar{\mathcal{L}} \subset \mathcal{L}$ such that $\mathcal{L}^c \subset \bar{\mathcal{L}}$ and one of the three following statements holds true:*

1. *There exists a scalar $\varepsilon \in \mathbb{R}_{++}$ satisfying $\varepsilon x + 1 > 0$ for all $x \in X$ and such that for all $U, V \in \bar{\mathcal{L}}$,*

$$URV \Longleftrightarrow \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon U_i^s + 1)^{\frac{1}{q}} \geq \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon V_i^s + 1)^{\frac{1}{q}}. \tag{19}$$

2. *There exists a scalar $\varepsilon \in \mathbb{R}_{++}$ satisfying $\varepsilon x - 1 > 0$ for all $x \in X$ and such that for all $U, V \in \bar{\mathcal{L}}$,*

$$URV \Longleftrightarrow \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon U_i^s - 1)^{\frac{1}{q}} \geq \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon V_i^s - 1)^{\frac{1}{q}}. \tag{20}$$

3. *$X \subset \mathbb{R}_{++}$ and for all $U, V \in \bar{\mathcal{L}}$,*

$$URV \Longleftrightarrow \sum_{s \in S} \pi_s \prod_{i \in N} (U_i^s)^{\frac{1}{q}} \geq \sum_{s \in S} \pi_s \prod_{i \in N} (V_i^s)^{\frac{1}{q}}. \tag{21}$$

But if $q < n$, none of the statements holds true for all $U, V \in \mathcal{L}$.

Proof. Fix $x_0 \in X$. Let $Y(x_0) \subset X^n$ denote the subset such that for all $U^s \in Y(x_0)$, there exists $e_q(U^s) \in X$ such that the n-vector $\overline{U}^s(U^s)$ defined by $\overline{U}^s_1(U^s) = ... = \overline{U}^s_q(U^s) = e_q(U^s)$ and $\overline{U}^s_{q+1}(U^s) = ... = \overline{U}^s_n(U^s) = x_0$ satisfies $[U^s]\, I \left[\overline{U}^s(U^s)\right]$. By Strong Pareto for no risk and Continuity, $Y(x_0) \neq \varnothing$ for every $x_0 \in X$. By Weak Dominance, for all $U \in Y(x_0)^m$,

$$UI\left(\overline{U}^1(U^1), ..., \overline{U}^m(U^m)\right).$$

By Weak Pareto for q group risk, Anonymity, and Continuity, for all $U,V \in Y(x_0)^m$,

$$U R V \Longleftrightarrow \sum_{s\in S} \pi_s e_q(U^s) \geq \sum_{s\in S} \pi_s e_q(V^s).$$

By Strong Pareto for no risk and Anonymity, the function e_q must be increasing and symmetric. We can proceed as in the proof of Proposition 4 to obtain the same results, but limited to $U,V \in Y(x_0)^m$.

Now one can change x_0. Taking a sufficiently fine grid, the corresponding $Y(x_0)^m$ overlap and the same criterion must therefore hold on the union $\bar{\mathcal{L}}$ of the sets $Y(x_0)^m$. Note that by Strong Pareto for no risk and Continuity, one has $X^n = \bigcup_{x_0\in X} Y(x_0)$. Therefore $\mathcal{L}^c \subset \bar{\mathcal{L}} = \bigcup_{x_0\in X} Y(x_0)^m$, so that the criteria obtained in Proposition 4 are valid over sure prospects.

But one does not have $\mathcal{L} \subset \bar{\mathcal{L}}$. In order to show that the result does not hold over $\mathcal{L}$ we exhibit a counter-example for $n = m = 2$. Extending it to other values of n,m is straightforward.

Let $X = [a,b] \subset \mathbb{R}_{++}$ and $q = 1$. The ordering R is defined as follows:

$$U R V \Longleftrightarrow W(U) \geq W(U)$$

for

$$\begin{aligned} W(U) &= \sum_{s\in S} \pi_s U_1^s U_2^s \text{ if } \frac{a}{b} \leq \frac{U_1^2 U_2^2}{U_1^1 U_2^1} \leq \frac{b}{a}, \\ &= \left(\pi_1 \frac{b}{a} + \pi_2\right) U_1^2 U_2^2 \text{ if } \frac{U_1^2 U_2^2}{U_1^1 U_2^1} < \frac{a}{b}, \\ &= \left(\pi_1 + \frac{b}{a}\pi_2\right) U_1^1 U_2^1 \text{ if } \frac{U_1^2 U_2^2}{U_1^1 U_2^1} > \frac{b}{a}. \end{aligned}$$

Observe that when $\frac{U_1^2 U_2^2}{U_1^1 U_2^1} < \frac{a}{b}$, for instance, it is impossible to find $x_0, x_1^1, x_1^2 \in X$ satisfying

$$x_0 x_1^1 = U_1^1 U_2^1 \text{ and } x_0 x_1^2 = U_1^2 U_2^2,$$

because this implies

$$\frac{x_1^2}{x_1^1} = \frac{U_1^2 U_2^2}{U_1^1 U_2^1} < \frac{a}{b},$$

which is impossible for $x_1^1, x_1^2 \in [a,b]$. For this ordering, the set $\bigcup_{x_0\in X} Y(x_0)^m$ is defined by the condition $\frac{a}{b} \leq \frac{U_1^2 U_2^2}{U_1^1 U_2^1} \leq \frac{b}{a}$.

The ordering defined here satisfies all the axioms of the proposition but does not coincide with any of the criteria listed in the proposition (except on $\bigcup_{x_0 \in X} Y(x_0)^m$).

$\square$

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