# Using Symbolic Dynamics to Compare Path-Complete Lyapunov Functions

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***Abstract*— In this paper, we study discrete-time linear switched systems by leveraging tools from symbolic dynamics and language theory. More specifically, viewing path-complete Lyapunov functions (PCLF) associated with the switched system as finite automata, we investigate the coverings of bi-infinite words generated via a PCLF and develop a framework for comparing two different PCLFs via these coverings. However, in most of the cases PCLFs are not comparable with regards to one corresponding to a better stability criterion than the other. For this purpose, we utilize the notion of support sets, which is a subset of paths in a PCLF that is sufficient to obtain the same performance index as that of the entire set of bi-infinite words generated by the PCLF, to obtain partial relations between two coverings. We also illustrate a numerical example to justify the study of support sets via the covering framework.**

***Index Terms*— finite automata, graph theory, switched systems, multiple Lyapunov functions.**

## I. Introduction

Switched systems form an important class of hybrid systems providing suitable complexity for numerous mathematical models like consensus in social networks [3], smart sustainable buildings [19], digital circuits [9], robotics [22] etc. Pertaining to the involved structure of hybrid systems, Lyapunov methods have proved to be crucial in providing tools for understanding the stability of switched systems [1], [17], [21]. One such method involving multiple Lyapunov functions was introduced in [2] comprising of a directed labelled graph with each edge representing an inequality between two Lyapunov functions and the labels on the edges denoting different modes of the switched system. By construction these graphs are strongly connected and are commonly referred to as path-complete graphs (see Section II for formal definition and formulation of Lyapunov criteria via these graphs). The graph theoretic structure allows for construction of Lyapunov criteria with added versatility. Efforts have been made in the recent past to systematically characterize these path-complete graphs based on the performance of the corresponding candidate Lyapunov criteria provided. For example, in [16], combinatorial techniques are used to obtain an ordering between path-complete graphs based on the ability of the corresponding criteria to show stability for a large class of switched systems. However, in practice we need to define a template of Lyapunov functions along with the labelled directed graph setup to comment on the conservativeness of a criterion. For this purpose, template dependent ordering of path-complete graphs has been studied in [24] and [25] for polytopic templates and templates closed under addition respectively.

In this paper, we propose a language theory based framework for comparison of path-complete graphs for the purpose of understanding the stability properties of a linear discrete-time switched system. More specifically, the strong connectivity in path-complete graphs allocates them to finite automata generating regular languages (we refer the readers to [7] for a comprehensive study on regular languages), which allows us to associate a subset of bi-infinite words from a regular language to a particular node of the path-complete graph. A collection of such subsets is denoted by "a covering" and is formally defined in Section II. This in turn enables us to study ordering between path-complete graphs from a set-theoretic perspective. However, in most scenarios, the path-complete graphs are not comparable in the sense that for some cases, a graph $\mathcal{G}_1$ provides a better stability criterion than another graph $\mathcal{G}_2$, while for some other cases $\mathcal{G}_2$ provides a better stability criterion than $\mathcal{G}_1$. Taking this into account, in this paper, we establish a weaker form of ordering between graphs which relies on the construction of suitable support sets (described in Section IV).

This paper is organised as follows: In Section II, we provide the necessary preliminaries and definitions required. In Section III, we establish a relationship between coverings of path-complete graphs obeying simulation relation. We define the notion of support set on these graphs in Section IV and furthermore analyze the structural properties of graphs constructed via these support sets. In Section V, we use the concept of support sets to establish a more relaxed notion of simulation between graphs and obtain a relation between corresponding coverings. To provide a rationale for the study of support sets and coverings of path-complete graphs, we give a numerical example for the same in Section VI. Finally, conclusions and future work are stated in Section VII.

## II. Preliminaries and Definitions

Throughout the paper, we consider the following formulation for a linear discrete-time switched system:

$$x(t+1) = A_{\phi(t)}x(t) \quad \text{for all} \quad t \in \mathbb{Z}, \tag{1}$$

where the state $x(t) \in \mathbb{R}^n$ for all time instances $t \in \mathbb{Z}$, $\phi(t) : \mathbb{Z} \to \{1, \cdots, M\}$ is the switching signal with $M$ modes and $A_i$ are fixed $n \times n$ matrices for $i \in \{1, \cdots, M\}$. We focus on the following stability problem for switched system (1):

*Definition 1:* (Stability of a linear discrete-time switched system). The system (1) is said to be *globally asymptotically stable* if for any initial state $x(0) \in \mathbb{R}^n$, and any switching signal $\phi(t)$, the following holds:

$$\lim_{t\to\infty} x(t) = 0.$$

The stability problem of the system (1) has been widely studied via the *joint spectral radius* which was introduced in paper [8] and is defined as follows:

*Definition 2:* (Joint Spectral Radius). Given a finite set of square matrices $\mathcal{A} := \{A_1, \cdots, A_M\}$ having same size, the *joint spectral radius* (JSR) of the set $\mathcal{A}$ is given by:

$$\rho(\mathcal{A}) := \lim_{k\to\infty} \max_{\substack{\phi(i)\in\{1,\cdots,M\}\\ 1\le i\le k}} ||A_{\phi(k)} \cdots A_{\phi(2)} A_{\phi(1)}||^{\frac{1}{k}}, \quad (2)$$

where, $||.||$ is a matrix norm and $\rho(\mathcal{A})$ does not depend on the norm.

It is well-known that the system (1) is globally asymptotically stable if and only if the joint spectral radius of the set $\mathcal{A} := \{A_1, A_2, \cdots, A_M\}$ is strictly less than one. However, the problem of verification of JSR being less than one given a set of matrices $\mathcal{A}$ is undecidable. Several Lyapunov methods have been proposed to study the stability property of system (1). We refer the readers to [6] for an extensive survey of various Lyapunov criteria developed for stability analysis of switched systems. In this paper, we focus on analyzing the performance of different path-complete Lyapunov functions which serve as stability criteria for system (1). To this end, we state all the graph theoretic preliminaries required to formally define path-complete Lyapunov criteria for switched systems.

*Definition 3:* (Labeled Graphs on $\Sigma$). Given a set $\Sigma$, a labeled graph on $\Sigma$ is defined by a directed graph $\mathcal{G} = (S, E)$, where $S$ is a finite set of nodes and $E \subseteq S \times S \times \Sigma$ is the set of labeled edges. Given $e \in (s, q, i) \in E$, $s$ and $q$ are the *starting* and *arrival* nodes of $e$, respectively.

*Definition 4:* (Paths on $\mathcal{G}$). Given $\mathcal{G} = (S, E)$ and a tuple $\hat{\imath} = (i_0, \cdots, i_{K-1}) \in \Sigma^K$, a *path* on $\mathcal{G}$ labeled by $\hat{\imath}$ is a sequence of consecutive edges $\bar{e} = e_1, \cdots, e_K = (s_0, s_1, i_0), (s_1, s_2, i_1), \cdots, (s_{K-1}, s_K, i_{K-1}) \in E^K$ labeled by $\hat{\imath}$. $s_0$ and $s_K$ are the starting and arrival node of this path respectively. Given a path $\bar{e}$, we denote its label sequence by $L(\bar{e}) := i_0 i_1 \cdots i_{K-1}$.

*Definition 5:* (Path-Complete Graph) A directed and labelled graph $\mathcal{G} = (S, E)$ is said to be *path-complete* if and only if for any given sequence of labels $(i_0, i_1 \cdots i_{K-1}) \in \Sigma^K$, there exists a path in $\mathcal{G}$ with label sequence $i_0 i_1 \cdots i_{K-1}$.

We now state the formulation of path-complete Lyapunov functions as given in [2].

*Definition 6:* (Path-Complete Lyapunov Function) Given a linear discrete-time switched system with modes $\mathcal{A} = \{A_1, A_2, \cdots, A_M\}$, a *path-complete Lyapunov function* (PCLF) for $\mathcal{A}$ on a directed graph $\mathcal{G} = (S, E)$ labelled by $\Sigma = \{1, 2, \cdots, M\}$ is given by $\mathcal{V} := \{V_s | s \in S\}$ such that each Lyapunov function $V_s : \mathbb{R}^n \to \mathbb{R}$ is continuous, positive, homogeneous and the following inequalities are satisfied:

$$V_b(A_i x) \le V_a(x) \quad \text{for all} \quad (a, b, i) \in E, x \in \mathbb{R}^n. \quad (3)$$

The set $\mathcal{V}$ is said to be *admissible* for graph $\mathcal{G}$ and set of matrices $\mathcal{A}$.

In order for inequalities in (3) to be tight, we can presume that the graph $\mathcal{G}$ is strongly connected. Furthermore, for the convenience of notation, we denote a PCLF criterion $\mathcal{V}$ by the underlying path-complete graph $\mathcal{G}$ hereafter. As shown in paper [2], the existence of a PCLF on $\mathcal{A}$ guarantees the global asymptotic stability of its associated switched system. However, a given PCLF criteria can happen to be too stringent or conservative depending on the set of matrices $\mathcal{A}$. In this paper, we want to analyze the quality of different PCLFs by exploiting the underlying graph structure. To this end, we state the definition of performance index for a PCLF as given in [20].

*Definition 7:* (Performance index of path-complete graph) Given a set of matrices $\mathcal{A}$ and a path-complete graph $\mathcal{G} = (S, E)$ in which the associated Lyapunov functions in the set $\mathcal{V} = \{V_s | V_s \in S\}$ follow a template $\mathcal{T}$, we define the *performance index* of $\mathcal{G}$ (denoted by $\rho_{\mathcal{G},\mathcal{T}}(\mathcal{A})$) as the infimum over all non-negative real $\rho$ such that $\mathcal{G}$ has an admissible solution for set $\mathcal{A}/\rho := \{A_i/\rho | A_i \in \mathcal{A}\}$.

It is shown in paper [2, Theorem 2.4], that the performance index provides an upper bound on the trajectories of system (1) and thus smaller the performance index $\rho_{\mathcal{G},\mathcal{T}}(\mathcal{A})$, better is the stability criteria provided by the path-complete graph $\mathcal{G}$ for the set of matrices $\mathcal{A}$ and template $\mathcal{T}$.

Furthermore, from language theory perspective, given a set of finite modes of the switched system $\Sigma := \{1, \cdots, M\}$, we want to study the bi-infinite sequences $(\phi(t))_{t\in\mathbb{Z}}$ as elements of a *full-*$\Sigma$ *shift space* which is formally defined as:

$$\Sigma^{\mathbb{Z}} := \{(z_i)_{i\in\mathbb{Z}} \mid z_i \in \Sigma, \forall i \in \mathbb{Z}\}. \quad (4)$$

We refer to the set $\Sigma$ as the *alphabet* of a language. A $K$-length word of $\Sigma$ is a finite sequence of elements from $\Sigma$ of length $K > 0$ and is given by $\hat{\imath} := (i_0, \cdots, i_{K-1}) \in \Sigma^\star := \bigcup_{K\in\mathbb{N}} \Sigma^K$(the *Kleene closure* of $\Sigma$). We denote the length of word $\hat{\imath}$ by $|\hat{\imath}|$, i.e., $|\hat{\imath}| = K$. We further define the *shift function* $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ as follows:

$$\sigma(z) = \omega \quad \text{such that} \quad \omega_k = z_{k+1} \quad \text{for all} \quad k \in \mathbb{Z} \quad (5)$$

*Definition 8:* (Bi-infinite walks on $\mathcal{G}$). Given a bi-infinite sequence $\bar{z} = (\cdots, z_{-1}, z_0, z_1, \cdots) \in \Sigma^{\mathbb{Z}}$, a *bi-infinite walk* labeled by $\bar{z}$ is a bi-infinite sequence of consecutive edges $\bar{\pi} = (\cdots, e_{-1}, e_0, e_1, \cdots) \in E^{\mathbb{Z}}$, such that $L(e_k) = z_k$ for all $k \in \mathbb{Z}$ (where, $L : E \to \Sigma$ is the labeling function defined in Definition 4).

Given a labeled graph $\mathcal{G} = (S, E)$ on an alphabet $\Sigma$, we define the following two subsets of the full-$\Sigma$ shift space:

$$\mathcal{Z}(\mathcal{G}) := \{\bar{z} \in \Sigma^{\mathbb{Z}} \mid \exists \text{ a bi-infinite walk } \bar{\pi} \text{ in } \mathcal{G} \text{ labeled by } \bar{z}\}, \quad (6)$$

$$\mathcal{Z}(\mathcal{G}, s) := \{\bar{z} \in \Sigma^{\mathbb{Z}} \mid \exists \text{ a bi-infinite walk } \bar{\pi} \text{ in } \mathcal{G} \text{ labeled by } \bar{z} \text{ starting at } s\}, \quad (7)$$

where $s \in S$. We denote $\mathcal{Z}(\mathcal{G})$ to be the *graph presentation* of the labelled graph $\mathcal{G}$. It is easy to see by Definition 5, that for a path-complete graph $\mathcal{G}$ we have $\mathcal{Z}(\mathcal{G}) = \Sigma^{\mathbb{Z}}$.

*Definition 9:* (Cylinder sets of bi-infinite sequences) Given a finite word $\omega \in \Sigma^l$ of length $l \in \mathbb{Z}^{>0}$, we define a cylinder set of bi-infinite sequences ($[\omega]_{n,n+l-1}$ for $n \in \mathbb{Z}$) as follows:

$$[\omega]_{n,n+l-1} = \{z \in \Sigma^{\mathbb{Z}} \mid z_n = \omega_n, z_{n+1} = \omega_{n+1}, \cdots$$

$$\cdots, z_{n+l-1} = \omega_{n+l-1}\}. \quad (8)$$

*Definition 10:* (g-covering of a path-complete graph). Given a path-complete graph $\mathcal{G}$ with a graph presentation $\mathcal{Z}(\mathcal{G})$, a collection of sets $\mathcal{C} = \{C_1, \cdots, C_K\} \subseteq \mathcal{P}(\Sigma^{\mathbb{Z}})$ (where $\mathcal{P}(Z)$ denotes the power set of $Z$) is said to be *graph-induced covering* (g-covering) of $\mathcal{G}$ if

$$C_j = \mathcal{Z}(\mathcal{G}, s_j), \quad \text{for all} \quad s_j \in S,$$
$$\bigcup_{\substack{C_i \in \mathcal{C} \\ i \in \{1,\cdots,K\}}} C = \Sigma^{\mathbb{Z}}.$$

Note that, by definition there is a bijection between the elements of the set of nodes $S$ of the graph presentation and elements of the covering induced by this graph.

We say that a covering $\mathcal{C}$ is *non-redundant* when $C_i \cap C_j = \varnothing$ for all elements $C_i, C_j \in \mathcal{C}$. For a non-redundant covering, there always exists a unique graph presentation (we refer the readers to [12], Lemma 4 for proof). Going forward in this paper, we only study path-complete graphs which induce non-redundant coverings. The investigation of graphs that generate redundant coverings is left to future work.

## III. Covering based Simulation Relations between Path-Complete Graphs

In this section, we present the definition of simulation between graphs and derive a relationship between non-redundant coverings of two graph presentations when a simulation relation exists between them. The following definition of simulation between two directed labelled graphs is derived from the notion of simulation between two automata as defined in [4, pp. 91-92].

*Definition 11:* (Simulation between graphs) Consider two path-complete graphs $\mathcal{G}_1 = (S^{(1)}, E^{(1)})$ and $\mathcal{G}_2 = (S^{(2)}, E^{(2)})$ defined on the same alphabet $\Sigma$. We say that $\mathcal{G}_1$ *simulates* $\mathcal{G}_2$ if there exists a function $R: S^{(2)} \to S^{(1)}$ such that for any edge $(s, d, \sigma) \in E^{(2)}$, there exists an edge $(R(s), R(d), \sigma) \in E^{(1)}$.

It is easy to see that if $\mathcal{G}_1 = (S^{(1)}, E^{(1)})$ simulates $\mathcal{G}_2 = (S^{(2)}, E^{(2)})$, then given an admissible solution $\mathcal{V} = \{V_s | s \in S^{(1)}\}$ of $\mathcal{G}_1$, the set of Lyapunov functions $\mathcal{U} = \{V_{R(s)} | s \in S^{(1)}\}$ is an admissible solution for $\mathcal{G}_2$ (where $R$ is the simulation relation). Therefore, a solution for $\mathcal{G}_1$ implies a solution for $\mathcal{G}_2$ which implies that the performance index of $\mathcal{G}_2$ is at most as large as the performance index of $\mathcal{G}_1$ indicating that $\mathcal{G}_2$ is a better criteria than $\mathcal{G}_1$. We next show that in the case of graph presentations of non-redundant coverings, one graph simulates another if and only if the g-covering of the latter is finer than the former. Before stating the main theorem, we present the formal definition of one covering being finer than the other and a prerequisite lemma required in the proof of the main result.

*Definition 12:* Given two graph-induced coverings (g-coverings) $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\Sigma^{\mathbb{Z}}$, the covering $\mathcal{C}_2$ is said to be *finer* than $\mathcal{C}_1$ when for all $C_2 \in \mathcal{C}_2$, there exists an element $C_1 \in \mathcal{C}_1$ such that $C_2 \subseteq C_1$.

*Lemma 1:* Given a non-redundant g-covering $\mathcal{C}$ with graph presentation $\mathcal{G}$, the set of all bi-infinite sequences $\mathcal{Z}(\mathcal{G})$ is in bijection with the set of all bi-infinite walks on $\mathcal{G}$.

*Proof 1:* We first note that by Definition 8, the function which associates, to a given bi-infinite walk on $\mathcal{G}$, its bi-infinite sequence of labels, is surjective on $\mathcal{Z}(\mathcal{G})$. Now, suppose to the contrary that there exists a bi-infinite sequence $\bar{z} \in \mathcal{Z}(\mathcal{G})$ such that there are two bi-infinite walks on $\mathcal{G}$ denoted by $\bar{\pi}_1 = (\cdots, e_{-1}^{(1)}, e_0^{(1)}, e_1^{(1)}, \cdots)$ and $\bar{\pi}_2 = (\cdots, e_{-1}^{(2)}, e_0^{(2)}, e_1^{(2)}, \cdots)$ such that $\bar{\pi}_1 \neq \bar{\pi}_2$. Let $\bar{z} \in C$, where $C \in \mathcal{C}$. Since $\mathcal{C}$ is a g-covering, $C = \mathcal{Z}(\mathcal{G}, s)$ for some node/state $s$ of the graph presentation $\mathcal{G}$. Due to non-redundancy of the covering $\mathcal{C}$, $\bar{z} \notin \mathcal{Z}(\mathcal{G}, r)$ for all nodes $r \neq s$, i.e., the bi-infinite walk $\bar{z}$ can only start at node/state $s$. Hence, if we denote $e_1^{(1)} := (s(1), q(1), i)$ and $e_1^{(2)} := (s(2), q(2), i^{'})$, then $s(1) = s(2)$. Moreover, since $\bar{\pi}_1$ and $\bar{\pi}_2$ are labelled by the same bi-infinite word $\bar{z}$, we have $i = i^{'} = z_0$. The existence of edges $e_1^{(1)}$ and $e_2^{(2)}$ implies that $\sigma(z) \in C_{q(1)}$ and $\sigma(z) \in C_{q(2)}$, which is a contradiction because $C_{q(1)} \cap C_{q(2)} = \varnothing$ (where, $C_{q(1)}$ and $C_{q(2)}$ are the covering elements corresponding to nodes $q(1)$ and $q(2)$). Hence, $q(1) = q(2)$ which implies $e_1^{(1)} = e_1^{(2)}$. We can now use the same argument recursively to show that $e_j^{(1)} = e_j^{(2)}$ for all $j \in \mathbb{Z}$.

We now show the main result for non-redundant g-coverings and simulation relation.

*Theorem 1:* Given two graphs $\mathcal{G}_1 = (S^{(1)}, E^{(1)})$ and $\mathcal{G}_2 = (S^{(2)}, E^{(2)})$ inducing non-redundant coverings $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\Sigma^{\mathbb{Z}}$ respectively, $\mathcal{G}_1$ simulates $\mathcal{G}_2$ if and only if $\mathcal{C}_2$ is finer than $\mathcal{C}_1$. Moreover, in this case the simulation relation $R: S^{(2)} \to S^{(1)}$ is given by

$$R(C_k^{(2)}) = \{C_l^{(1)}\} \quad \text{for} \quad C_k^{(2)} \subseteq C_l^{(1)}. \quad (9)$$

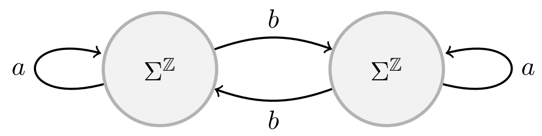
*Proof 2:* Given that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are non-redundant, and $\mathcal{C}_2$ is finer than $\mathcal{C}_1$, the function $R$ in (9) is well defined. In order to show that $\mathcal{G}_1$ simulates $\mathcal{G}_2$, we need to show that definition 11 hold true for the function $R$ in (9). Consider an edge $(C_k^{(2)}, C_{k'}^{(2)}, i) \in E^{(2)}$. There exist unique elements $C_l^{(1)}, C_{l'}^{(1)} \in \mathcal{C}_1$ such that $C_k^{(2)} \subseteq C_l^{(1)}$ and $C_{k'}^{(2)} \subseteq C_{l'}^{(1)}$ i.e., $R(C_k^{(2)}) = C_l^{(1)}$ and $R(C_{k'}^{(2)}) = C_{l'}^{(1)}$.

We now show that $(C_l^{(1)}, C_{l'}^{(1)}, i) \in E^{(1)}$ by contradiction. Suppose that $(C_l^{(1)}, C_{l'}^{(1)}, i) \notin E^{(1)}$, then $\forall z \in C_l^{(1)}$ such that $z_0 = i$, $\sigma(z) \notin C_{l'}^{(1)}$ which implies that $\sigma(z) \notin C_{k'}^{(2)}$. But since this holds for all $z \in C_l^{(1)}$ such that $z_0 = i$, it holds for all $z \in C_k^{(2)}$ such that $z_0 = i$. Hence, for all $z \in C_k^{(2)}$ such that $z_0 = i$, we have that $\sigma(z) \notin C_{k'}^{(2)}$ implying that $(C_k^{(2)}, C_{k'}^{(2)}, i) \notin E^{(2)}$, which is a contradiction. Therefore,

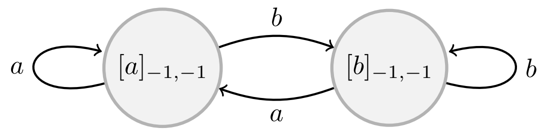
$$(C_k^{(2)}, C_{k'}^{(2)}, i) \in E^{(2)} \Rightarrow (R(C_k^{(2)}), R(C_{k'}^{(2)}), i) \in E^{(1)},$$

and hence $\mathcal{G}_1$ simulates $\mathcal{G}_2$.

To show the converse statement, let us consider a simulation relation $R: S^{(2)} \to S^{(1)}$. We will show that for all $s \in S^{(2)}$, we have $C_s^{(2)} \subseteq C_{R(s)}^{(1)}$. Choose a bi-infinite sequence $\bar{z} \in C_s^{(2)}$, where $C_s^{(2)} \in \mathcal{C}_2$. By Lemma 1, there exists a unique bi-infinite walk $\bar{\pi} = (\cdots, e_{-1}, e_0, e_1, \cdots)$ labelled by $\bar{z}$ starting at the node $s$ (where the node $s$ in $\mathcal{G}_2$ corresponds to the element $C_s^{(2)}$ of the covering $\mathcal{C}_2$). Since




(a) Graph Presentation $\mathcal{G}_1$




(b) Graph Presentation $\mathcal{G}_2$

Fig. 1: The covering $\mathcal{C}_2 = \{[a]_{-1,-1}, [b]_{-1,-1}\}$ is finer than the redundant covering $\mathcal{C}_1 = \{\Sigma^{\mathbb{Z}}, \Sigma^{\mathbb{Z}}\}$ but $\mathcal{G}_1$ does not simulate $\mathcal{G}_2$. Therefore, Theorem 1 cannot be extended to redundant coverings.

$\mathcal{G}_1$ simulates $\mathcal{G}_2$, there exists a bi-infinite walk labelled by $\bar{z}$ in $\mathcal{G}_1$ given by $R(\bar{\pi}) := (\cdots, R(e_{-1}), R(e_0), R(e_1), \cdots)$ such that $R(e_j) = (R(s_j), R(q_j), i)$ for $e_j = (s_j, q_j, i)$ for all $j \in \mathbb{Z}$. Therefore, $\bar{z} \in R(s)$ which implies $C_s^{(2)} \subseteq C_{R(s)}^{(1)}$. Hence $\mathcal{C}_2$ is finer than $\mathcal{C}_1$.

*Remark 1:* Theorem 1 does not hold when even one of the coverings $\mathcal{C}_1$ and $\mathcal{C}_2$ is redundant. In this remark, we illustrate a counterexample for the same. Consider an alphabet $\Sigma = \{a, b\}$ and two g-coverings $\mathcal{C}_1 = \{\Sigma^{\mathbb{Z}}, \Sigma^{\mathbb{Z}}\}$ and $\mathcal{C}_2 = \{[a]_{-1,-1}, [b]_{-1,-1}\}$. We represent graph presentations of these coverings in Figure 1. Since, $\mathcal{C}_1$ is redundant, the graph presentation $\mathcal{G}_1$ is not unique. On the other hand $\mathcal{C}_2$ is non-redundant and hence $\mathcal{G}_2$ is a unique graph presentation. $\mathcal{C}_2$ is finer than $\mathcal{C}_1$ but $\mathcal{G}_1$ does not simulate $\mathcal{G}_2$ i.e., one cannot construct a mapping $R : S^{(2)} \to S^{(1)}$ as defined in Definition 11.

## IV. On Support Sets of Path-Complete Graphs

The notion of identifying an appropriate support set has been extensively utilized in convex optimization. We refer the readers to [10], [13], [26] for a few examples. In this section, we employ the same support set structure for path-complete graphs, i.e., for all the constraints given by (3), we consider a subset of constraints which is sufficient to achieve the same performance. The motivation behind finding support sets for path-complete graphs is to obtain a less stringent simulation relation between graphs as compared to Definition 11. To this end, we give the formal definition of support set (as stated in [20]) of a path-complete graph $\mathcal{G}$.

*Definition 13:* Given a set of matrices $\mathcal{A}$, a path-complete graph $\mathcal{G} = (S, E)$ and a template $\mathcal{T}$ associated with the Lyapunov functions, we say that a set $\Pi$ of paths in $\mathcal{G}$ is a *support set* with respect to $\mathcal{A}$ and $\mathcal{T}$ if the graph

$$\mathcal{G}_\Pi := (S, \{(v, v', L(\pi)) \,|\, \pi \in \Pi \text{ is a path from } v \text{ to } v'\}), \tag{10}$$

has the same performance index as $\mathcal{G}$ for a set of matrices $\mathcal{A}$ and template $\mathcal{T}$. Furthermore, we say that a support set $\Pi$ is *strict* if any proper subset of $\Pi$ is not a support set. Note that given a path-complete $\mathcal{G}$, its support set is not unique.

The usefulness of the concept of support set is that it removes edges that are irrelevant for the specific set of matrices at stake. Thus, it allows to characterize an ordering relation that would not hold in general between two graphs, but that holds for a specific case. In the theorem below, we establish some structural properties of these support sets.

*Theorem 2:* Given a path-complete graph $\mathcal{G}$ with strictly positive performance index $\rho$ and a support set $\Pi$ of $\mathcal{G}$, the graph $\mathcal{G}_\Pi$ must contain a cycle.

*Proof 3:* Suppose to the contrary that the graph $\mathcal{G}_\Pi$ is a directed acyclic graph (DAG), then there exists at least one *topological ordering* on $\mathcal{G}_\Pi$ such that for every directed edge $u \to v$, we have $u > v$ in the ordering (we refer the readers to [5], section 2.2.3 for a detailed description on topological ordering/sorting of DAGs). Choose a vertex $s_1$ in this ordering such that there is no other vertex $v$ in the graph such that $v > s_1$. We choose an edge $(s_1, s_2, L(\pi))$, where $L(\pi) := i_1 i_2 \cdots i_n$ is the label on the edge and $\pi \in \Pi$ (note that without loss of generality, we can assume that $s_2$ exists, because otherwise $s_1$ would be an isolated vertex which cannot have a self loop due to $\mathcal{G}_\Pi$ being acyclic and hence can be removed from $\mathcal{G}_\Pi$).

By definition of a support set, $\mathcal{G}$ and $\mathcal{G}_\Pi$ have the same performance index which we denote by $\rho$, and hence the constraint corresponding to the edge $(s_1, s_2, L(\pi))$ is given by:

$$V_{s_2}(A_{i_n} \cdots A_{i_2} A_{i_1} x) \leq \rho^n V_{s_1}(x) \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{11}$$

If we define $\widetilde{V}_{s_1}(x) := V_{s_1}(2^P x) = 2^{Pd} V_{s_1}(x)$ for all $x \in \mathbb{R}^n$ (where $d$ is the degree of homogeneity of $V_{s_1}$ and $P$ is a positive integer greater than the number of edges in $\mathcal{G}_\Pi$), $\widetilde{\rho} := \rho/2$ and $\widetilde{V}_{s_2}(x) := V_{s_2}(2^{P-1} x) = 2^{(P-1)d} V_{s_2}(x)$ for all $x \in \mathbb{R}^n$, then using (11), we have

$$\widetilde{V}_{s_2}(A_{i_n} \cdots A_{i_2} A_{i_1} x) \leq \widetilde{\rho}^n \widetilde{V}_{s_1}(x) \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{12}$$

We now choose a vertex $s_3$ such that $s_2 > s_3$. The constraint corresponding to the directed edge from $s_2$ to $s_3$ with label $j_1, j_2, \cdots, j_m$ is as follows:

$$V_{s_3}(A_{j_m} \cdots A_{j_2} A_{j_1} x) \leq \rho^m V_{s_2}(x) \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{13}$$

Similar to the construction in (12), if we define, $\widetilde{V}_{s_3}(x) := V_{s_3}(2^{P-2} x) = 2^{(P-2)d} V_{s_3}(x)$ for all $x \in \mathbb{R}^n$, then from (13):

$$\widetilde{V}_{s_3}(A_{j_m} \cdots A_{j_2} A_{j_1} x) \leq \widetilde{\rho}^m \widetilde{V}_{s_2}(x) \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{14}$$

We can iterate this construction of Lyapunov functions on every path with strict ordering of vertices $s_1 > s_2 > \cdots > s_N$. Since $\mathcal{G}_\Pi$ is a finite acyclic graph with no isolated vertices, any path with strict ordering is of finite length and union of all the paths with strict ordering contains the entire vertex set. Therefore, $(\widetilde{V}_{s_1}, \widetilde{V}_{s_2}, \cdots, \widetilde{V}_{s_N})$ is an admissible solution for $\mathcal{G}_\Pi$ with performance index $\widetilde{\rho} < \rho$ which is a contradiction because by definition $\mathcal{G}$ and $\mathcal{G}_\Pi$ have the same performance index. Hence, $\mathcal{G}_\Pi$ must contain a cycle.

*Remark 2:* Using a similar reasoning as in the proof of Theorem 2, it is easy to see that for a strict support set $\Pi$, the graph $\mathcal{G}_\Pi$ does not contain any degree one node with an incoming edge (i.e., a node with indegree one and outdegree zero) because if such a node (say $s_1$) would exist with the incoming edge $(s_2, s_1, j_1 j_2 \cdots j_K)$, then we would have the following Lyapunov inequality:

$$V_{s_2}(A_{j_K} \cdots A_{j_2} A_{j_1} x) \leq \rho^K V_{s_1}(x) \quad \text{for all} \quad x \in \mathbb{R}^n,$$

and since there is no Lyapunov function in the admissible solution set which bounds $V_{s_1}$, by the same construction as presented in the proof of Theorem 2, we can replace $(V_{s_1}, V_{s_2}, \cdots, V_{s_N})$ by another admissible solution $(\widetilde{V}_{s_1}, \widetilde{V}_{s_2}, \cdots, \widetilde{V}_{s_N})$ and $\rho$ by $\widetilde{\rho}$ such that $\widetilde{\rho} < \rho$ which is a contradiction.

Note that, we did not address the case when $\rho = 0$ in the proof of Theorem 2, because within the framework of comparing graphs based on their performances, a graph $\mathcal{G}$ with performance index 0 always provides a better criterion than any other graph and hence for the sake of brevity, we can set any support set of $\mathcal{G}$ to be an empty set in this case.

## V. Covering based Partial Simulation Relations between Path-Complete Graphs

The construction of the graph $\mathcal{G}_\Pi$ for a support set $\Pi$ along with Theorem 2 in Section IV enables us to examine the covering generated by the paths in $\Pi$ which in turn allows us to establish a more relaxed version of Theorem 1. For this purpose, we state the definition of Partial Simulation between graphs, as given in [20].

*Definition 14:* (Partial Simulation between Graphs) We say that $\mathcal{G}_1 = (S^{(1)}, E^{(1)})$ partially simulates $\mathcal{G}_2 = (S^{(2)}, E^{(2)})$ with respect to set of matrices $\mathcal{A}$ and template $\mathcal{T}$ if there exists a support set $\Pi$ for $\mathcal{G}_2$ and simulation function $F_\Pi : S^{(2)} \to S^{(1)}$ such that for any $\pi \in \Pi$,

$$\pi = (s_0, s_1, i_0), (s_1, s_2, i_1), \cdots, (s_{K-1}, s_K, i_{K-1}) \in E^{(2)K},$$

there exists a path $F_\Pi(\pi) \in E^{(1)K}$ given by:

$$\begin{aligned} F_\Pi(\pi) =& (F_\Pi(s_0), F_\Pi(s_1), i_0), (F_\Pi(s_1), F_\Pi(s_2), i_1), \\ & \cdots, (F_\Pi(s_{K-1}), F_\Pi(s_K), i_{K-1}). \end{aligned}$$

We denote this partial simulation ordering by $\mathcal{G}_1 <_{(\mathcal{A},\mathcal{T})} \mathcal{G}_2$. As shown in [20], a partial simulation allows to establish a performance ordering between two graphs for a specific set of matrices $\mathcal{A}$ , even if this ordering does not hold in general:

*Theorem 3:* (Theorem 14 in [20]) Consider two graphs $\mathcal{G}_1 = (S^{(1)}, E^{(1)})$ and $\mathcal{G}_2 = (S^{(2)}, E^{(2)})$, a template $\mathcal{T}$ and a set of matrices $\mathcal{A}$. If there exists a support set $\Pi$ of $\mathcal{G}_2$ with respect to $\mathcal{A}$,$\mathcal{T}$ and a partial simulation $F_\Pi$ as defined in Definition 14, then $\rho_{\mathcal{G}_2,\mathcal{T}}(\mathcal{A}) \leq \rho_{\mathcal{G}_1,\mathcal{T}}(\mathcal{A})$.

We now establish the formulation of partial simulation between graphs through coverings.

*Definition 15:* (covering restricted to a set of paths) Given a graph $\mathcal{G} = (S, E)$ and a set of paths $\Pi$ on $\mathcal{G}$, we denote the covering restricted to $\Pi$ by $\mathcal{C}_\Pi$ and it is defined as follows:

$$\mathcal{C}_\Pi := \{z \in \mathcal{C} | \text{there exists a bi-infinite walk labelled by } z \text{ in } \mathcal{G}_\Pi\}. \quad (15)$$

Theorem 2 ensures that the set $\mathcal{C}_\Pi$ as defined in (15) is non-empty. Similar to Theorem 1, we now obtain a relationship between the coverings of two non-redundant graph presentations when a partial simulation relation exists between them.

*Theorem 4:* Given a set of matrices $\mathcal{A}$, a template $\mathcal{T}$ and two path-complete graphs $\mathcal{G}_1 = (S^{(1)}, E^{(1)})$ and $\mathcal{G}_2 = (S^{(2)}, E^{(2)})$ inducing non-redundant coverings $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, $\mathcal{G}_1 <_{(\mathcal{A},\mathcal{T})} \mathcal{G}_2$ if and only if there exists a support set $\Pi$ of $\mathcal{G}_2$ such that $\mathcal{C}_{2,\Pi}$ (the covering $\mathcal{C}_2$ restricted to the elements of the language generated by $\Pi$) is finer than $\mathcal{C}_1$.

*Proof 4:* The proof is almost analogous to the proof of Theorem 1. Given that $\mathcal{C}_{2,\Pi}$ is finer than $\mathcal{C}_1$, we claim that the partial simulation function $F_\Pi : S^{(2)} \to S^{(1)}$ (as defined in Definition 14) is uniquely given by $F(C_k^{(2,\Pi)}) = C_l^{(1)}$, $C_k^{(2,\Pi)} \subseteq C_l^{(1)}$. Consider an edge $(C_k^{(2,\Pi)}, C_{k'}^{(2,\Pi)}, L(\pi)) \in E^{(2)}(\Pi)$ for some $\pi \in \Pi$. There exists unique elements $C_l^{(1)}, C_{l'}^{(1)} \in \mathcal{C}_1$ such that $C_k^{(2,\Pi)} \subseteq C_l^{(1)}$ and $C_{k'}^{(2,\Pi)} \subseteq C_{l'}^{(1)}$. Suppose that $(C_l^{(1)}, C_{l'}^{(1)}, L(\pi))$ is not a path in $\mathcal{G}_1$, then $\forall z \in C_l^{(1)}$ such that $z_{-(K-1)} = i_{K-1}, \cdots, z_{-1} = i_1, z_0 = i_0$ (where $L(\pi) = i_0 i_1 \cdots i_{K-1}$), $\sigma^K(z) \notin C_{l'}^{(1)}$ which implies $\sigma^K(z) \notin C_{k'}^{(2,\Pi)}$ such that $z_{-(K-1)} = i_{K-1}, \cdots, z_{-1} = i_1, z_0 = i_0$. Hence for all $z \in C_k^{(2,\Pi)}$ such that $z_{-(K-1)} = i_{K-1}, \cdots, z_{-1} = i_1, z_0 = i_0$, we have that $\sigma^K(z) \notin C_{k'}^{(2,\Pi)}$ implying that $(C_k^{(2,\Pi)}, C_{k'}^{(2,\Pi)}, L(\pi)) \notin E^{(2)}(\Pi)$ which is a contradiction.

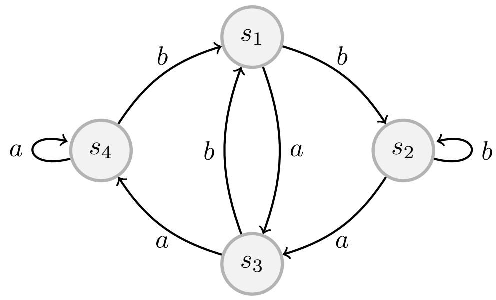
To show the converse statement, choose a bi-infinite sequence $\bar{z} \in C_s^{(2,\Pi)}$, where $C_s^{(2,\Pi)} \in \mathcal{C}_{2,\Pi}$. By Lemma 1, there exists a unique bi-infinite walk $\bar{\pi} = (\cdots, e_{-1}, e_0, e_1, \cdots)$ labelled by $\bar{z}$ starting at the node $s$ (where the node $s$ in $\mathcal{G}_{2,\Pi}$ corresponds to the element $C_s^{(2,\Pi)}$ of the covering $\mathcal{C}_{2,\Pi}$). Since $\mathcal{G}_1$ partially simulates $\mathcal{G}_2$ on the support set $\Pi$, there exists a bi-infinite walk labelled by $\bar{z}$ in $\mathcal{G}_1$ given by $F(\bar{\pi}) := (\cdots, F(e_{-1}), F(e_0), F(e_1), \cdots)$ such that for all edges $e_j = (s_j, q_j, i_0 i_1 \cdots, i_{K-1})$ (where the path $i_0 i_1 \cdots i_{K-1} \in \Pi$), $F(e_j)$ is a path in $\mathcal{G}_1$ starting at $F(s_j)$ and ending at $F(q_j)$. Therefore, $\bar{z} \in F(s)$ which implies $C_s^{(2,\Pi)} \subseteq C_{F(s)}^{(1)}$, where $C_{F(s)}^{(1)}$ is the covering element in $\mathcal{C}_1$ corresponding to node $F(s)$ in $\mathcal{G}_1$ which implies that every element of $\mathcal{C}^{(2,\Pi)}$ is contained in some element of $\mathcal{C}^{(1)}$. Hence $\mathcal{C}_{2,\Pi}$ is finer than $\mathcal{C}_1$.

Using Theorem 3 and Theorem 4, we obtain a relationship between performances of graphs which are induced by non-redundant coverings as follows:

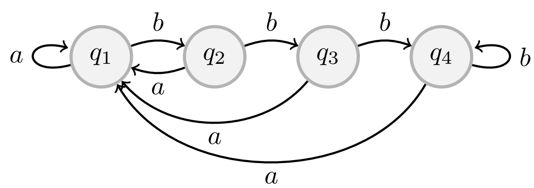
*Corollary 1:* Given a set of matrices $\mathcal{A}$, a template $\mathcal{T}$, and two path-complete graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ induced by non redundant coverings $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. If there exists a support set $\Pi$ of $\mathcal{G}_2$ with respect to $\mathcal{A}$ and $\mathcal{T}$ such that $\mathcal{C}_{2,\Pi}$ is finer than $\mathcal{C}_1$, then $\rho_{\mathcal{G}_2,\mathcal{T}}(\mathcal{A}) \leq \rho_{\mathcal{G}_1,\mathcal{T}}(\mathcal{A})$.

## VI. A Numerical Example

In this section, we highlight the importance of analyzing coverings of path-complete graphs and identifying suitable support sets for path-complete graphs by numerically comparing two graphs which do not have a simulation rela-




(a) Graph Presentation $\mathcal{G}_3$




(b) Graph Presentation $\mathcal{G}_4$

Fig. 2: The graph $\mathcal{G}_3$ statistically has a better performance than $\mathcal{G}_4$. For a random sample of $10^4$ pairs of $2 \times 2$ matrices $\mathcal{A} = \{A_1, A_2\}$, we observed that $\mathcal{G}_3$ outperforms $\mathcal{G}_4$ in 768 samples, while $\mathcal{G}_4$ outperforms $\mathcal{G}_3$ in only 627 samples. The rest of 8605 samples ended up in draw.

tion (as defined in Definition 11) but one graph is more likely to outperform the other. We demonstrate a numerical comparison between the graphs $\mathcal{G}_3 := (S^{(3)}, E^{(3)})$ and $\mathcal{G}_4 := (S^{(4)}, E^{(4)})$ which are given in Figure 2. The graph $\mathcal{G}_3$ is commonly referred to as the "two-step" memory De Bruijn graph (We refer the readers to the paper [18] for a complete formulation of generalized De Bruijn graphs) and its corresponding non-redundant covering is as follows:

$$\begin{aligned}\mathcal{C}_3 :&= \{C_{s_1}, C_{s_2}, C_{s_3}, C_{s_4}\} \\ &= \{[ab]_{-2,-1}, [bb]_{-2,-1}, [ba]_{-2,-1}, [aa]_{-2,-1}\}.\end{aligned}$$

The non-redundant covering for $\mathcal{G}_4$ is as follows:

$$\begin{aligned}\mathcal{C}_4 :&= \{C_{q_1}, C_{q_2}, C_{q_3}, C_{q_4}\} \\ &= \{[a]_{-1}, [ab]_{-2,-1}, [abb]_{-3,-1}, [bbb]_{-3,-1}\}.\end{aligned}$$

Note that neither $\mathcal{C}_3$ is finer than $\mathcal{C}_4$, nor $\mathcal{C}_4$ is finer than $\mathcal{C}_3$, and therefore due to Theorem 1, there is no simulation between the two graphs. Hence, the graphs are incomparable in terms of performance. However, one can see that $\mathcal{C}_3$ is closer to being finer than $\mathcal{C}_4$. The only element of $\mathcal{C}_3$ which is not contained in any element of $\mathcal{C}_4$ is $[bb]_{-2,-1}$. However, if we remove the cylinder sets $[abb]_{-3,-1}$ or $[bbb]_{-3,-1}$ from $[bb]_{-2,-1}$ (which amounts to removing a cylinder of measure $1/8$ with respect to the standard Cantor measure [14]), then this restricted covering of $\mathcal{C}_3$ will be finer than $\mathcal{C}_4$. On the contrary, to make a restricted covering of $\mathcal{C}_4$ finer than $\mathcal{C}_3$, one must remove the cylinder set $[ba]_{-2,-1}$ or $[aa]_{-2,-1}$ from the first element of $\mathcal{C}_4$ which are of measure $1/4$ each. Thus, we expect $\mathcal{G}_3$ to give a statistically better performance than $\mathcal{G}_4$.

For the numerical experiment, we sample $10^4$ random pairs of $2 \times 2$ matrices $\mathcal{A} = \{A_a, A_b\}$, where $\{a, b\}$ are the two modes of the linear discrete-time switched system (1). Each entry of the matrix is uniformly sampled in the interval $[-10, 10]$. We compute the performance index (as defined in Definition 7) for $\mathcal{G}_3$ and $\mathcal{G}_4$ in each sample using Mosek solver [15] from the Yalmip toolbox [11] in MATLAB. The code for generating simulation results is available in [23]. We have considered the template of quadratic functions $\mathcal{Q}$ defined as follows:

$$\mathcal{Q} := \{f_{\mathcal{Q}} : \mathbb{R}^n \to \mathbb{R} | \mathcal{Q} \succ 0\},$$

where $f_{\mathcal{Q}}(x) := \sqrt{x^\top \mathcal{Q} x}$. We observe in simulation results that out of 10000 samples, $\mathcal{G}_3$ performs better (i.e. has lower performance index) than $\mathcal{G}_4$ in 768 samples, whereas $\mathcal{G}_4$ performs better than $\mathcal{G}_3$ in 627 samples. On the remaining 8605 samples, both the graphs have equal performance. One can see why $\mathcal{G}_3$ is better prone to good performance than $\mathcal{G}_4$ based on their respective coverings. For instance, consider a set of paths $\Pi'$, contains all the edges from $\mathcal{G}_3$ except $(s_1, s_2, b)$. If $\Pi'$ happens to be a support set of $\mathcal{G}_3$, then the covering $\mathcal{C}_3$ restricted to $\Pi'$ is given by:

$$\mathcal{C}_{3,\Pi'} = \{[aa]_{-2,-1}, [ba]_{-2,-1}, [ab]_{-2,-1}, \{\cdots bbb \cdots\}\},$$

which is finer than $\mathcal{G}_4$. The corresponding partial simulation $F_{\Pi'} : S^{(3)} \to S^{(4)}$ is given by: $F(s_3) = q_1, F(s_4) = q_1, F(s_1) = q_2$ and $F(s_2) = q_4$. On the contrary, any support set of $\mathcal{G}_4$, such that $\mathcal{G}_3$ partially simulates $\mathcal{G}_4$ will have to exclude at least two edges from the edge set of $\mathcal{G}_4$. Further analysis is required on support sets for obtaining a numerical estimate of the relative performance between two graphs, but the above discussion provides a quantitative explanation. Moreover, to find a suitable support set, we need to consider the template of the candidate Lyapunov functions which remains to be addressed in future work.

## VII. Conclusions and Future Work

In this paper, we presented a framework for obtaining a simulation relation between path-complete Lyapunov functions (PCLF) via their induced coverings. We further utilized the concept of support sets to develop a weaker notion of simulation between PCLFs for the purpose of comparing their performances. We also demonstrated a numerical comparison between two PCLFs to show the utility of support sets and coverings. Future work includes analysis of graphs with redundant coverings, and obtaining a quantitative estimate of the relative performance of two graphs, i.e., computing a priori the comparison statistics for an arbitrary experiment. This contains several challenges, like predicting the frequency of support sets for a given graph, by taking into account the measure of sampled set of matrices.

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