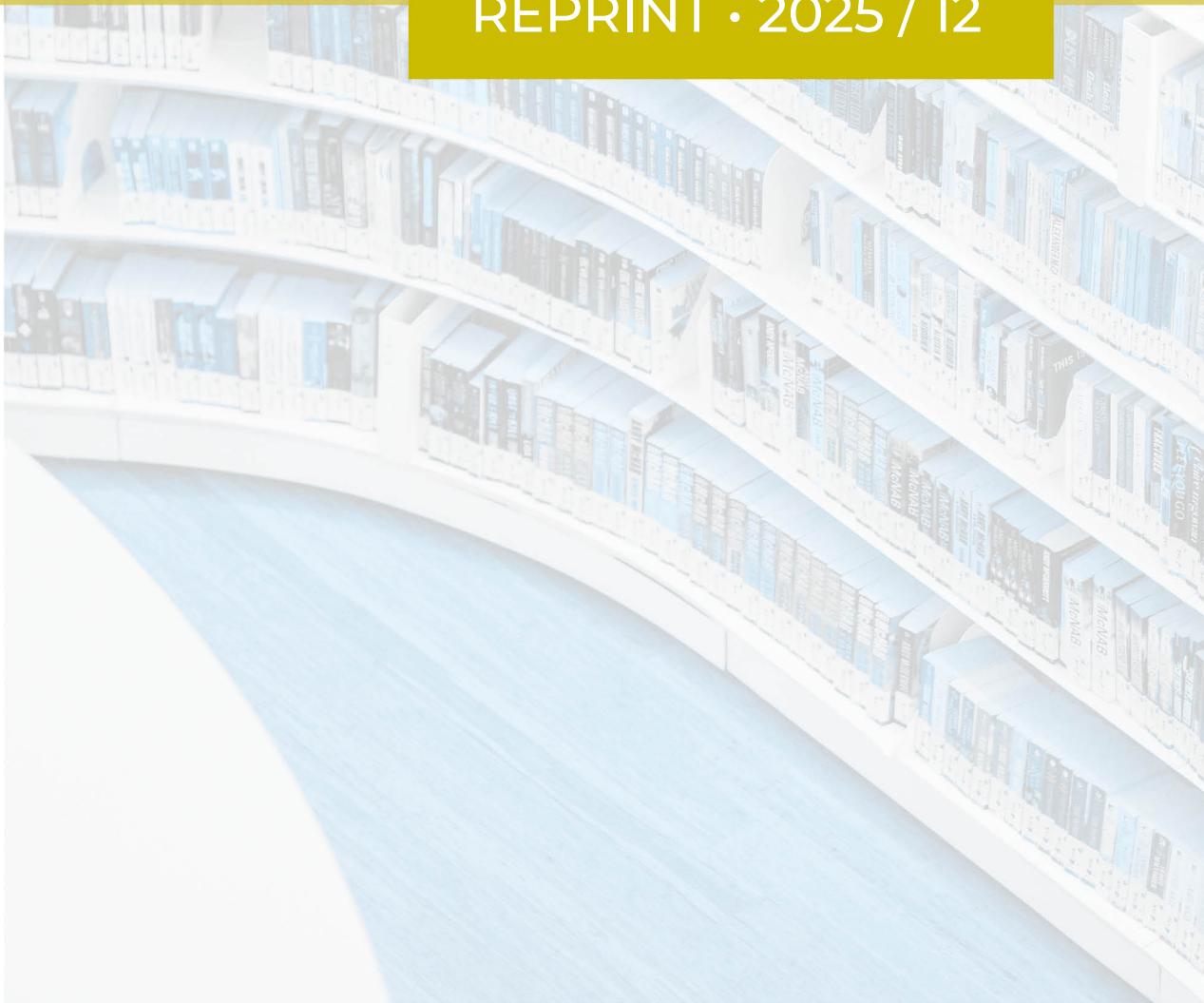


ON THE DISTRIBUTION OF THE INTEGRAL OF A FUNCTION WITH RESPECT TO A BROWNIAN BRIDGE

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On the distribution of the integral of a function with respect to a Brownian bridge

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Abstract

We derive a couple of results associated with the distribution of the integral J of a square-integrable function with respect to a Brownian bridge. These results are useful to design sampling algorithms and provide a tight upper bound for the distribution of the running supremum of J .

1 Introduction

An Itô integral $I := (I_t)_{t \geq 0}$ of a deterministic integrand is a stochastic integral of a function σ with respect to a Brownian motion W :

$$I_t := \int_0^t \sigma(u) dW_u . \quad (1)$$

It is well known that this process is a continuous local martingale with zero expectation. Moreover, under suitable integrability conditions of σ , it is a genuine martingale whose variance is given by Itô's isometry: $\mathbb{E}[I_t] = 0$ and $\mathbb{V}[I_t] = \int_0^t \sigma^2(u) du$; see, e.g., [8]. This type of integrals shows up in various applications related to stochastic calculus. For instance, this is the case when dealing with a geometric Brownian motion featuring a time-dependent deterministic volatility function, with the Ornstein-Uhlenbeck process or with Φ -martingales, to name but a few. More information about these processes can be found in [10], [4] and [7], respectively.

Often, one is led to study the behavior of the stochastic integral I when conditioning with respect to the value of W at some given time $T > 0$. This is the case, for example, when exploiting conditional independence combined with the tower law as well as for variance reduction. In quantitative finance, for instance, this is particularly useful to price exotic options (see, e.g., [10, p.114]). This leads us to study the behavior of I subject to $W_T = c$ or, equivalently, of a stochastic integral J of a deterministic function σ with respect to a Brownian bridge $B := (B_t)_{t \geq 0}$:

$$J_t := \int_0^t \sigma(u) dB_u \quad \text{where} \quad (B_0, B_T) = (0, c) \quad \text{and} \quad c \in \mathbb{R}, \quad T > 0 . \quad (2)$$

Indeed, by definition, the dynamics of B coincide with that of a Brownian motion W subject to the *terminal condition* $W_T = c$. Consequently, it holds that

$$J_t \sim I_t |_{W_T = c} . \quad (3)$$

Recall that B is a continuous Gaussian process with mean and covariance functions given, for any $s, t \leq T$, by (see, e.g., [8, Section 5.6.B] or [10, Section 4.7.2] for more details about the Brownian bridge)

$$\mathbb{E}[B_t] = c \frac{t}{T} \quad \text{and} \quad \text{Cov}[B_s, B_t] = s \wedge t - \frac{st}{T} . \quad (4)$$

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These expressions can be easily extended to the case where s, t could be beyond T : Assuming a grid $\mathcal{T}^{(n)} = (t_0, t_1, \dots, t_n)$ of strictly increasing times, sampling the process B at times in $\mathcal{T}^{(n)}$ yields the vector $\mathbf{B}_0^{(n)} = (B_{t_0}, B_{t_1}, \dots, B_{t_n})$, which is multivariate normal with parameters given by the functions

$$\mu^B(t) := \mathbb{E}[B_t] = c \frac{t \wedge T}{T} \quad \text{and} \quad \Sigma^B(s, t) := \text{Cov}[B_s, B_t] = s \wedge t - \frac{(s \wedge T)(t \wedge T)}{T}. \quad (5)$$

This extension amounts to consider that, beyond T , the bridge behaves as a Brownian motion. Although several construction schemes are possible, one stochastic representation for B is to set

$$B_t = \tilde{W}_t - \frac{\tilde{W}_T - c}{T}(t \wedge T) \quad (6)$$

where $\tilde{W}$ is a Brownian motion. This is the expression in [10, Def. 4.7.4], adjusted to accommodate the case $t > T$.

In what follows, we assume that σ is square-integrable up to any of the considered times and, in particular, up to T . Recall that a function $\sigma : \mathbb{R}^+ \rightarrow \mathbb{R}$ is said *square-integrable up to t* if

$$\|\sigma\|_t < \infty \quad \text{where} \quad \|\sigma\|_t^2 := \int_0^t \sigma^2(u) du.$$

We compute a couple of marginal and joint distributions related to J and $\mathbf{J}_0^{(n)} = (J_{t_0}, J_{t_1}, \dots, J_{t_n})$, and illustrate the results associated with the supremum using numerical simulations.

2 Marginal and conditional distributions of J

We define the following functions for further references :

$$\Sigma(t) := \int_0^t \sigma(u) du, \quad \Sigma(s, t) := \Sigma(t) - \Sigma(s), \quad \mu_T(s, t) := \frac{\Sigma(s, t)}{T}, \quad \mu_T(t) := \mu_T(0, t)$$

and

$$\gamma_T^2(s, t, u) := \|\sigma\|_t^2 - \|\sigma\|_s^2 - T\mu_T^2(s, u), \quad \gamma_T^2(t, u) := \gamma_T^2(0, t, u).$$

The next lemma gives the expression of the distribution of J_t .

Lemma 1. *Let J be given by (2) where σ is square-integrable up to $t \vee T$. Then, for all $t, T \geq 0$,*

$$J_t \sim \mathcal{N}(c\mu_T(t \wedge T), \gamma_T^2(t, t \wedge T)). \quad (7)$$

Proof:

The proof relies on the representation of the Brownian bridge B as a function of the Brownian motion $\tilde{W}$ given in (6). Note that B is not adapted to the natural filtration of $\tilde{W}$, but this is not a problem. An alternative proof that does not rely on this representation is given in the On-line appendix.

$$\begin{aligned} J_t &= \int_0^t \sigma(u) dB_u \\ &= \int_0^t \sigma(u) d\tilde{W}_u - \frac{\tilde{W}_T - c}{T} \int_0^t \sigma(u) d(u \wedge T) \\ &= \int_0^t \sigma(u) d\tilde{W}_u - \frac{\int_0^T d\tilde{W}_u - c}{T} \int_0^{t \wedge T} \sigma(u) du \\ &= \frac{c}{T} \Sigma(t \wedge T) + \int_0^{t \wedge T} \sigma(u) d\tilde{W}_u + \int_{t \wedge T}^t \sigma(u) d\tilde{W}_u - \frac{\Sigma(t \wedge T)}{T} \left(\int_0^{t \wedge T} d\tilde{W}_u + \tilde{W}_T - \tilde{W}_{t \wedge T} \right) \\ &= c\mu_T(t \wedge T) + \int_0^{t \wedge T} \left(\sigma(u) - \frac{\Sigma(t \wedge T)}{T} \right) d\tilde{W}_u + \int_{t \wedge T}^t \sigma(u) d\tilde{W}_u - (\tilde{W}_T - \tilde{W}_{t \wedge T}) \frac{\Sigma(t \wedge T)}{T}. \end{aligned} \quad (8)$$

Note that the sum of the last three terms in (8) is a centered Gaussian variable whose variance is the sum of the individual variances because they relate to non-overlapping increments of $\tilde{W}$. In particular, at most one of the last two terms is non-zero. Using Itô's isometry, the variance of J_t is thus

$$\begin{aligned}\mathbb{V}[J_t] &= \int_0^{t \wedge T} \left(\sigma(u) - \frac{\Sigma(t \wedge T)}{T} \right)^2 du + \int_{t \wedge T}^t \sigma^2(u) du + (T - (t \wedge T)) \frac{\Sigma^2(t \wedge T)}{T^2} \\ &= \|\sigma\|_{t \wedge T}^2 - 2 \frac{\Sigma^2(t \wedge T)}{T} + (t \wedge T) \frac{\Sigma^2(t \wedge T)}{T^2} + \|\sigma\|_t^2 - \|\sigma\|_{t \wedge T}^2 + (T - (t \wedge T)) \frac{\Sigma^2(t \wedge T)}{T^2} \\ &= \|\sigma\|_t^2 - T \mu_T^2(t \wedge T) .\end{aligned}$$

Adopting our notation, one obtains $J_t \sim \mathcal{N}(c\mu_T(t \wedge T), \gamma_T^2(t, t \wedge T))$, which is (7). $\square$

The next lemma gives the distribution of $I_s^t := I_t - I_s$ when conditioning with respect to the Brownian motion W at time T and possibly also at time $s \leq t \wedge T$.

Lemma 2. Define $J_s^t := J_t - J_s$, where σ is square-integrable up to $t \vee T$ and $0 \leq s \leq t, T$. Then,

$$J_s^t|_{B_s} \sim \mathcal{N}((c - B_s)\mu_{T-s}(s, t \wedge T), \gamma_{T-s}^2(s, t, t \wedge T)) . \quad (9)$$

Moreover,

$$J_s^t \sim \mathcal{N}(c\mu_T(s, t \wedge T), \gamma_T^2(s, t, t \wedge T)) . \quad (10)$$

Proof:

We define $\tilde{B}$ as the bridge satisfying $\tilde{B}_0 = a$ and $\tilde{B}_{T-s} = b$, which can be represented as (adapted from [10, p.175]):

$$\tilde{B}_u = a + \tilde{W}_u - \frac{\tilde{W}_{T-s} - (b - a)}{T - s}(u \wedge (T - s)) ,$$

where $\tilde{W}$ is a standard Brownian motion. An alternative proof that does not rely on this representation is given in the On-line appendix. We also introduce $\tilde{\sigma}(u) := \sigma(u + s)$ leading to $\tilde{\Sigma}(u) := \int_0^u \tilde{\sigma}(v) dv = \int_s^{u+s} \sigma(v) dv = \Sigma(s, u + s)$ and $\|\tilde{\sigma}\|_u^2 = \|\sigma\|_{u+s}^2 - \|\sigma\|_s^2$. Set $a = B_s$, $b = c$, $\tilde{I}_t := \int_0^t \tilde{\sigma}(u) d\tilde{W}_u$, and $t^\wedge := t \wedge T$ to shorten the expressions. Noting that $(t - s) \wedge (T - s) = t^\wedge - s$, one gets

$$\begin{aligned}J_s^t|_{B_s} &= \int_s^t \sigma(u) dW_u|_{(W_s, W_T)=(B_s, c)} \sim \int_0^{t-s} \tilde{\sigma}(u) d\tilde{W}_u|_{(\tilde{W}_0, \tilde{W}_{T-s})=(B_s, c)} = \int_0^{t-s} \tilde{\sigma}(u) d\tilde{B}_u \\ &= \tilde{I}_{t-s} + \frac{(b - a) - \tilde{W}_{T-s}}{T - s} \int_0^{t^\wedge - s} \tilde{\sigma}(u) du \\ &= \tilde{I}_{t^\wedge - s} + \tilde{I}_{t^\wedge - s}^{t-s} + \frac{(b - a) - \tilde{W}_{T-s}}{T - s} \tilde{\Sigma}(t^\wedge - s) \\ &= \frac{b - a}{T - s} \Sigma(s, t^\wedge) + \int_0^{t^\wedge - s} \left(\tilde{\sigma}(u) - \frac{\Sigma(s, t^\wedge)}{T - s} \right) d\tilde{W}_u + \tilde{I}_{t^\wedge - s}^{t-s} - (\tilde{W}_{T-s} - \tilde{W}_{t^\wedge - s}) \frac{\Sigma(s, t^\wedge)}{T - s} .\end{aligned} \quad (11)$$

The sum of the last three terms in (11) is a centered Gaussian variable whose variance is the sum of the individual variances because they relate to non-overlapping increments of W and that at most one of the last two terms is non-zero. Using Itô's isometry, the variance of $J_s^t|_{B_s}$ is thus

$$\begin{aligned}\mathbb{V}[J_s^t|_{B_s}] &= \int_0^{t^\wedge - s} \left(\tilde{\sigma}(u) - \frac{\Sigma(s, t^\wedge)}{T - s} \right)^2 du + \int_{t^\wedge - s}^{t-s} \tilde{\sigma}^2(u) du + (T - s - (t^\wedge - s)) \frac{\Sigma^2(s, t^\wedge)}{(T - s)^2} \\ &= \|\tilde{\sigma}\|_{t-s}^2 - 2 \frac{\Sigma^2(s, t^\wedge)}{T - s} + (t^\wedge - s) \frac{\Sigma^2(s, t^\wedge)}{(T - s)^2} + (T - s - (t^\wedge - s)) \frac{\Sigma^2(s, t^\wedge)}{(T - s)^2} \\ &= \|\sigma\|_t^2 - \|\sigma\|_s^2 - (T - s) \mu_{T-s}^2(s, t \wedge T) .\end{aligned}$$

Replacing (a, b) by (B_s, c) yields $J_s^t|_{B_s} \sim \mathcal{N}((c - B_s)\mu_{T-s}(s, t \wedge T), \gamma_{T-s}^2(s, t, t \wedge T))$, which is (9).

The second claim follows from the tower law, and noting that $\mathbb{E}[\Phi(\mu + \sigma Z)] = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)$ where $Z \sim \mathcal{N}(0,1)$ is a standard normal variable and Φ its cumulative distribution function. Indeed, the cumulative distribution of J_s^t becomes

$$\begin{aligned}
F_{J_s^t}(x) = \mathbb{E}[F_{J_s^t|B_s}(x)] &= \mathbb{E}\left[\Phi\left(\frac{x - (c - B_s)\mu_{T-s}(s, t \wedge T)}{\gamma_{T-s}(s, t, t \wedge T)}\right)\right] \\
&= \mathbb{E}\left[\Phi\left(\frac{x - c\mu_{T-s}(s, t \wedge T) + \mu_{T-s}(s, t \wedge T)\left(\frac{cs}{T} + \sqrt{s - \frac{s^2}{T}}Z\right)}{\gamma_{T-s}(s, t, t \wedge T)}\right)\right] \\
&= \mathbb{E}\left[\Phi\left(\frac{x - c\mu_{T-s}(s, t \wedge T)\frac{T-s}{T} - \mu_{T-s}(s, t \wedge T)\sqrt{s\frac{T-s}{T}}Z}{\gamma_{T-s}(s, t, t \wedge T)}\right)\right] \\
&= \Phi\left(\frac{x - c\mu_T(s, t \wedge T)}{\sqrt{\gamma_{T-s}^2(s, t, t \wedge T) + \mu_{T-s}^2(s, t \wedge T)s\frac{T-s}{T}}}\right) \\
&= \Phi\left(\frac{x - c\mu_T(s, t \wedge T)}{\sqrt{\|\sigma\|_t^2 - \|\sigma\|_s^2 - T\mu_T^2(s, t \wedge T)}}\right) \\
&= \Phi\left(\frac{x - c\mu_T(s, t \wedge T)}{\gamma_T(s, t, t \wedge T)}\right).
\end{aligned}$$

□

3 Joint distribution of J

As expected from the multivariate normal property of increments of B , the same applies to J when σ satisfies the squared-integrability condition.

Lemma 3. *Let σ be squared-integrable up to $s \vee t \vee T$. The stochastic integral J is a Gaussian process with mean and covariance functions*

$$\mu^J(t) := \mathbb{E}[J_t] = c\mu_T(t \wedge T) \quad \text{and} \quad \Sigma^J(s, t) := \text{Cov}[J_s, J_t] = \|\sigma\|_{s \wedge t}^2 - \frac{1}{T}\Sigma(s \wedge T)\Sigma(t \wedge T). \quad (12)$$

Proof. We prove that (J_s, J_t) is jointly normal with known μ and Σ . To this end, we start by considering that $s \leq t \leq T$, compute the marginal mean and variance, and then show that (J_s, J_s^t) is jointly normal. From Lemma 1, the elements of μ and the diagonal elements of Σ are necessarily given by

$$\mu_1 = \mathbb{E}[J_s] = c\mu_T(s), \quad \mu_2 = \mathbb{E}[J_t] = c\mu_T(t), \quad \Sigma_{1,1} = \mathbb{V}[J_s] = \gamma_T^2(s, s) \quad \text{and} \quad \Sigma_{2,2} = \mathbb{V}[J_t] = \gamma_T^2(t, t).$$

Let us now compute $\text{Cov}[J_s, J_s^t]$ when $s \leq t$. In this case, J_s is independent from J_s^t given B_s , and it holds from the tower law that

$$\mathbb{E}[J_s J_s^t] = \mathbb{E}[\mathbb{E}[J_s J_s^t | B_s]] = \mathbb{E}[\mathbb{E}[J_s | B_s] \mathbb{E}[J_s^t | B_s]].$$

On the other hand, $J_s|B_s$ has the same distribution as J_T when replacing T by s and c by B_s by (7) in Lemma 1, and the distribution of $J_s|B_s$ is given by (9) in Lemma 2. Hence,

$$\mathbb{E}[J_s | B_s] = \mu_s(s)B_s = \frac{\Sigma(s)}{s}B_s, \quad \mathbb{E}[J_s^t | B_s] = \mu_{T-s}(s, t)(c - B_s) = \frac{\Sigma(s, t)}{T - s}(c - B_s) \quad (13)$$

such that

$$\begin{aligned}
\mathbb{E}[J_s J_s^t] &= \mathbb{E}[\mathbb{E}[J_s|B_s] \mathbb{E}[J_s^t|B_s]] = \frac{\Sigma(s)\Sigma(s,t)}{s(T-s)}(c\mathbb{E}[B_s] - \mathbb{V}[B_s] - \mathbb{E}^2[B_s]) \\
&= \frac{\Sigma(s)\Sigma(s,t)}{s(T-s)}(c^2 \frac{s}{T} - s + \frac{s^2}{T} - \frac{c^2 s^2}{T^2}) \\
&= \frac{\Sigma(s)\Sigma(s,t)}{s(T-s)}(c^2 \frac{s}{T}(1 - \frac{s}{T}) - s(1 - \frac{s}{T})) \\
&= \frac{c^2 - T}{T^2} \Sigma(s)\Sigma(s,t) .
\end{aligned}$$

Using the mean of J_s^t available from (10) in Lemma 2, one obtains

$$\text{Cov}[J_s, J_s^t] = \frac{c^2 - T}{T^2} \Sigma(s)\Sigma(s,t) - c^2 \mu_T(s) \mu_T(s,t) = -\frac{1}{T} \Sigma(s)\Sigma(s,t) \quad \text{when } s \leq t . \quad (14)$$

We conclude that the covariance between J_s, J_t when $s \leq t$ is given by

$$\Sigma_{1,2} = \text{Cov}[J_s, J_t] = \mathbb{V}[J_s] + \text{Cov}[J_s, J_s^t] = \gamma_T^2(s, s) - \frac{1}{T} \Sigma(s)\Sigma(s, t) = \|\sigma\|_s^2 - \frac{1}{T} \Sigma(s)\Sigma(t) .$$

Although intuitive, we now show that (J_s, J_t) is bivariate normal. To this end, it is enough to prove that so is (J_s, J_s^t) . We have

$$\mathbb{P}(J_s \leq x, J_s^t \leq y) = \mathbb{E}[\mathbb{P}(J_s \leq x, J_s^t \leq y | W_s)] = \mathbb{E}[\mathbb{P}(J_s \leq x | W_s) \mathbb{P}(J_s^t \leq y | W_s)]$$

where, from Lemma 1 and Lemma 2, one respectively gets

$$\mathbb{P}(J_s \leq x | W_s) = \Phi\left(\frac{x - W_s \mu_s(s)}{\gamma_s(s, s)}\right) \quad \text{and} \quad \mathbb{P}(J_s^t \leq y | W_s) = \Phi\left(\frac{y - (c - W_s) \mu_{T-s}(s, t)}{\gamma_{T-s}(s, t, t)}\right) .$$

Setting $Z = W_s / \sqrt{s} \sim \mathcal{N}(0, 1)$, the joint probability becomes

$$\mathbb{P}(J_s \leq x, J_s^t \leq y) = \mathbb{E}\left[\Phi\left(\frac{x - (c \frac{s}{T} + \sqrt{s - \frac{s^2}{T}} Z) \mu_s(s)}{\gamma_s(s, s)}\right) \Phi\left(\frac{y - (c - c \frac{s}{T} - \sqrt{s - \frac{s^2}{T}} Z) \mu_{T-s}(s, t)}{\gamma_{T-s}(s, t, t)}\right)\right]$$

which can be written as

$$\mathbb{E}[\Phi(\mu + \sigma Z) \Phi(\tilde{\mu} + \tilde{\sigma} Z)] = \Phi_2\left(\frac{\mu}{\sqrt{1 + \sigma^2}}, \frac{\tilde{\mu}}{\sqrt{1 + \tilde{\sigma}^2}}, \frac{\sigma \tilde{\sigma}}{\sqrt{1 + \sigma^2} \sqrt{1 + \tilde{\sigma}^2}}\right) .$$

The values $\mu, \tilde{\mu}, \sigma, \tilde{\sigma}$ can be found by identification, and coincide with the moments of J_s and J_s^t :

$$\mu = \mathbb{E}[J_s] = c \mu_T(s) , \quad \tilde{\mu} = \mathbb{E}[J_s^t] = c \mu_T(s, t) , \quad \sigma^2 = \mathbb{V}[J_s] = \gamma_T^2(s, s) \quad \text{and} \quad \tilde{\sigma}^2 = \mathbb{V}[J_s^t] = \gamma_T^2(s, t, t) .$$

Having shown that (J_s, J_s^t) is bivariate normal, we conclude that so is $(J_s, J_t) = (J_s, J_s + J_s^t)$. As a check, one can verify that the mean vector μ satisfies

$$\mu_1 = \mu = \mu^J(s) , \quad \mu_2 = \mathbb{E}[J_s + J_s^t] = \mu + \tilde{\mu} = c \mu_T(t) = \mu^J(t)$$

and the covariance matrix Σ indeed corresponds to

$$\Sigma_{1,1} = \sigma^2 = \mathbb{V}[J_s] = \Sigma^J(s, s) , \quad \Sigma_{2,2} = \mathbb{V}[J_s + J_s^t] = \sigma^2 + \tilde{\sigma}^2 + 2 \text{Cov}[J_s, J_s^t] = \gamma_T^2(t, t) = \mathbb{V}[J_t] = \Sigma^J(t, t) ,$$

and

$$\Sigma_{1,2} = \text{Cov}[J_s, J_t] = \mathbb{V}[J_s] + \text{Cov}[J_s, J_s^t] = \gamma_T^2(s, s) - \frac{1}{T} \Sigma(s)\Sigma(s, t) = \|\sigma\|_s^2 - \frac{1}{T} \Sigma(s)\Sigma(t) = \Sigma^J(s, t) .$$

The case $s > t$ amounts to exchange the roles of s, t in $\Sigma_{1,2}$.

It remains to tackle the case where s, t can be larger than T . To this end, it suffices to adjust the expression of the moments. For the means (resp. variances), it is enough to cap the first argument of μ_T (resp. the second argument of γ_T^2) at T . For the covariance,

$$\begin{aligned}\mathbb{Cov}[J_s, J_T + J_T^t] &= \mathbb{Cov}[J_s, J_T] = \|\sigma\|_s^2 - \frac{1}{T}\Sigma(s)\Sigma(T) \quad \text{if } s \leq T \leq t \\ \mathbb{Cov}[J_t, J_T + J_T^s] &= \mathbb{Cov}[J_t, J_T] = \|\sigma\|_t^2 - \frac{1}{T}\Sigma(t)\Sigma(T) \quad \text{if } t \leq T \leq s \\ \mathbb{Cov}[J_s, J_s + J_s^t] &= \mathbb{V}[J_s] = \gamma_T^2(s, T) = \|\sigma\|_s^2 - \frac{1}{T}\Sigma(T)\Sigma(T) \quad \text{if } T \leq s \leq t \\ \mathbb{Cov}[J_t, J_t + J_t^s] &= \mathbb{V}[J_t] = \gamma_T^2(t, T) = \|\sigma\|_t^2 - \frac{1}{T}\Sigma(T)\Sigma(T) \quad \text{if } T \leq t \leq s\end{aligned}$$

where we have used that the increments of J after T are independent from J up to any time before T . All these expressions coincide with the definition of the covariance function $\Sigma^J(s, t)$. $\square$

Remark 1. The special case where $J = B$ is recovered by setting $\sigma \equiv 1$, in which case $\Sigma(s, t) = t - s$ and $\|\sigma\|_t^2 = \Sigma(t) = t$.

Lemma 4. Let $\underline{s} \leq s, \tilde{s} \leq \bar{s} \leq T$. The distribution of $(J_{\underline{s}}^s, J_{\bar{s}}^{\bar{s}})$ is jointly normal. The means and variances are given in (9) and the covariance is

$$\mathbb{Cov}\left[J_{\underline{s}}^s, J_{\bar{s}}^{\bar{s}}\right] = \|\sigma\|_s^2 - \|\sigma\|_{s \wedge \bar{s}}^2 - \frac{1}{T}(\Sigma(s \wedge T)(\Sigma(\bar{s} \wedge T) - \Sigma(\tilde{s} \wedge T)) + \Sigma(\underline{s} \wedge T)(\Sigma(\tilde{s} \wedge T) - \Sigma(\bar{s} \wedge T))) .$$

Proof:

We restrict ourselves to compute the covariance. Decomposing the increments of J ,

$$\mathbb{Cov}[J_s - J_{\underline{s}}, J_{\bar{s}} - J_{\tilde{s}}] = \mathbb{Cov}[J_s, J_{\bar{s}}] + \mathbb{Cov}[J_{\underline{s}}, J_{\tilde{s}}] - \mathbb{Cov}[J_s, J_{\tilde{s}}] - \mathbb{Cov}[J_{\underline{s}}, J_{\bar{s}}]$$

where we know from (14) that

$$\mathbb{Cov}[J_s, J_t] = \|\sigma\|_{s \wedge t}^2 - \frac{1}{T}\Sigma(s \wedge T)\Sigma(t \wedge T) .$$

Hence,

$$\begin{aligned}\mathbb{Cov}[J_s, J_{\bar{s}}] &= \|\sigma\|_s^2 - \frac{1}{T}\Sigma(s \wedge T)\Sigma(\bar{s} \wedge T) , \quad \mathbb{Cov}[J_{\underline{s}}, J_{\bar{s}}] = \|\sigma\|_{\underline{s}}^2 - \frac{1}{T}\Sigma(\underline{s} \wedge T)\Sigma(\bar{s} \wedge T) \\ \mathbb{Cov}[J_s, J_{\tilde{s}}] &= \|\sigma\|_{s \wedge \tilde{s}}^2 - \frac{1}{T}\Sigma(s \wedge T)\Sigma(\tilde{s} \wedge T) , \quad \mathbb{Cov}[J_{\underline{s}}, J_{\tilde{s}}] = \|\sigma\|_{\underline{s}}^2 - \frac{1}{T}\Sigma(\underline{s} \wedge T)\Sigma(\tilde{s} \wedge T) .\end{aligned}$$

Subtracting the sum of the first two covariances with the sum of the last two yields

$$\|\sigma\|_s^2 - \|\sigma\|_{s \wedge \bar{s}}^2 - \frac{1}{T}(\Sigma(s \wedge T)(\Sigma(\bar{s} \wedge T) - \Sigma(\tilde{s} \wedge T)) + \Sigma(\underline{s} \wedge T)(\Sigma(\tilde{s} \wedge T) - \Sigma(\bar{s} \wedge T))) .$$

$\square$

4 Exact sampling algorithms

Sampling the vectors $\mathbf{B}_0^{(n)}$ and $\mathbf{J}_0^{(n)}$ of B and J on a discrete time grid $\mathcal{T}^{(n)}$ can be achieved by exploiting the corresponding distributions. The algorithm consists in first sampling B on $\mathcal{T}^{(n)}$ from its distribution, and then sampling J using the conditional distributions $J_{t_k}^{t_{k+1}}|_{B_{t_k}, B_{t_{k+1}}}$. We assume that either $t < t_0$, $T \in \mathcal{T}^{(n)}$ or $T > t_n$ to ease the exposition. For any $s \leq t$ such that $t \leq T$ or $T \leq s$,

$$J_s^t|_{(B_s, B_t)} \sim I_s^t|_{(W_s, W_t)=(B_s, B_t)} ,$$

where the distribution of the random variable in the right-hand side can be recovered from (20) with $t = T$ and $c = B_t$. This leads to

$$J_s^t|_{(B_s, B_t)} \sim \mathcal{N}((B_t - B_s)\mu_{t-s}(s, t), \gamma_{t-s}^2(s, t, t)) \quad \forall 0 \leq s, t \quad \text{s.t.} \quad T \notin (s, t) .$$

Defining $\delta_k = t_{k+1} - t_k > 0$ and $\delta X_k = X_{t_{k+1}} - X_{t_k}$, the increments of J on the interval (t_k, t_{k+1}) are independent and distributed as

$$\delta J_k |_{B_{t_k}, B_{t_{k+1}}} \sim \mathcal{N}(\mu_k \delta B_k, \gamma_k^2), \quad k \in \{0, 1, 2, \dots, n-1\}$$

where

$$\mu_k := \mu_{\delta_k}(t_k, t_{k+1}) = \frac{\Sigma_k}{\delta_k}, \quad \gamma_k^2 := \gamma_{\delta_k}^2(t_k, t_{k+1}, t_{k+1}) = \|\sigma^2\|_{t_{k+1}} - \|\sigma^2\|_{t_k} - \frac{\Sigma_k^2}{\delta_k} \quad \text{and} \quad \Sigma_k := \Sigma(t_k, t_{k+1}).$$

5 Distribution of the supremum

We denote by $M_t^X := \sup_{s \in [0, t]} X_s$ the running supremum of a stochastic process X . We are interested in the distribution of M_t^J , where J is the integral of σ with respect to B . Although the analytical expression can be found for the distribution of the maximum of $\mathbf{J}_0^{(n)}$, the extension to M_t^J seems difficult. Yet, we can provide an upper bound which converges to the true value as $n \rightarrow \infty$, as we now explain.

5.1 Existing results for the Brownian motion

The distribution of the running maximum of a Brownian motion is known analytically; see, e.g., [9, p.630]:

$$F_{M_t^W}(x) := \mathbb{P}(M_t^W \leq x) = \mathbb{1}_{\{x \geq 0\}} \left[2\Phi\left(\frac{x}{\sqrt{t}}\right) - 1 \right].$$

Interestingly, every continuous local martingale M can be written as a time-changed Brownian motion where the clock is $t \rightarrow \langle M \rangle_t$, the quadratic variation of M [5]. Applying this to $M = I$, we conclude that I has the same dynamics as W time-changed with the clock $\Theta : t \mapsto \Theta(t) = \|\sigma\|_t^2$. Consequently,

$$F_{M_t^I}(x) = F_{M_{\|\sigma\|_t^2}^W}(x).$$

Unfortunately, this result cannot be recycled easily for J because of the condition $W_T = c$. While the stochastic integral I has the same dynamics as W time-changed with the clock Θ , this is no longer true for the conditional distribution of I_t or $W_{\Theta(t)}$ given W_T .

5.2 Existing results for the Brownian bridge

The joint density of (M_t^W, W_t) can be found in [8, Section 4.3.C]. This coincides with the density of M_t^B when $T = t$ and $c = W_t$. Therefore, the distribution of M_T^B is:

$$F_{M_T^B}(x) = \mathbb{1}_{\{x > c^+\}} \left[1 - \frac{\phi\left(\frac{2x-c}{\sqrt{T}}\right)}{\phi\left(\frac{c}{\sqrt{T}}\right)} \right].$$

The distribution of M_t^B is also known analytically when $t \neq T$. Noting $\Delta = T - t > 0$, we have (see [3, Theorem 2.1] and [3, Theorem 2.2])

$$F_{M_t^B}(x) = \mathbb{1}_{\{x > 0\}} \left[\Phi\left(\frac{xT - ct}{\sqrt{tT\Delta}}\right) - e^{-\frac{2x(x-c)}{T}} \Phi\left(\frac{2xt - xT - ct}{\sqrt{tT\Delta}}\right) \right], \quad t < T, \quad (15)$$

and

$$F_{M_t^B}(x) = \mathbb{1}_{\{x > c^+\}} \left[1 - e^{-\frac{2x(x-c)}{T}} - 2 \left(1 - e^{-\frac{2x(x-c)}{T}} \right) \Phi\left(\frac{c-x}{\sqrt{|\Delta|}}\right) \right], \quad t > T. \quad (16)$$

5.3 Distribution of the maximum J_t on a set of discrete times

Recall that if $\mathbf{X}^{(n)} = (X_1, X_2, \dots, X_n)$ is a vector of n normal variables with mean $\boldsymbol{\mu}^{(n)}$ and covariance matrix $\boldsymbol{\Sigma}^{(n)}$, its joint distribution is

$$F_n(\mathbf{x}; \boldsymbol{\mu}^{(n)}, \boldsymbol{\Sigma}^{(n)}) := \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) = \Phi_n(\mathbf{x}^*; \mathbf{R})$$

where $\Phi_n(\cdot; \mathbf{R})$ is the joint distribution of n standard normal variables with correlation matrix $\mathbf{R}$ and, for $1 \leq i, j \leq n$,

$$x_i^* := \frac{x_i - \mu_i^{(n)}}{\sigma_i}, \quad \sigma_i^2 = \Sigma_{i,i}^{(n)} \quad \text{and} \quad R_{i,j} = \frac{\Sigma_{i,j}^{(n)}}{\sigma_i \sigma_j}.$$

Applying this to $\mathbf{x} = x\mathbf{e}_n$ where $\mathbf{e}_n$ is an n -dimensional vector of ones yields the probability of the running maximum $M^{\mathbf{X}^{(n)}}$ of $\mathbf{X}^{(n)}$:

$$F_{M^{\mathbf{X}^{(n)}}}(x) = F_n(x\mathbf{e}_n; \boldsymbol{\mu}^{(n)}, \boldsymbol{\Sigma}^{(n)}) . \quad (17)$$

Corollary 1. *Let J be the integral of a deterministic function σ with respect to a Brownian bridge B satisfying $B_T = c$, $\mathbf{J}^{(n)} = (J_{t_1}, \dots, J_{t_n})$ and $\mathbf{J}_0^{(n)} = (J_{t_0}, \mathbf{J}^{(n)})$ where $t_0 = 0$. Then,*

$$F_{M^{\mathbf{J}_0^{(n)}}}(x) = \mathbb{1}_{\{x \geq 0\}} F_{M^{\mathbf{J}^{(n)}}}(x) = \mathbb{1}_{\{x \geq 0\}} F_n(x\mathbf{e}_n; \boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where for any $i, j \in \{1, 2, \dots, n\}$, $\mu_i = c\mu_T(t_i \wedge T)$ and $\Sigma_{i,j} = \|\sigma\|_{t_i \wedge t_j}^2 - \frac{1}{T}\Sigma(t_i \wedge T)\Sigma(t_j \wedge T)$.

The following corollary gives the corresponding result for the Brownian bridge.

Corollary 2. *Let B be a Brownian bridge satisfying $B_T = c$. Then,*

$$F_{M^{\mathbf{B}_0^{(n)}}}(x) = \mathbb{1}_{\{x \geq 0\}} F_{M^{\mathbf{B}^{(n)}}}(x) = \mathbb{1}_{\{x \geq 0\}} F_n(x\mathbf{e}_n; \boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where $\mu_i = \frac{c}{T}(t_i \wedge T)$ and $\Sigma_{i,j} = t_i \wedge t_j - \frac{1}{T}(t_i \wedge T)(t_j \wedge T)$ for all $1 \leq i, j \leq n$.

Observe that when there exists $i = i^*$ such that $t_i = T$, $\mu_{i^*}^{(n)} = c$ and $\Sigma_{i^*,j}^{(n)} = \Sigma_{j,i^*}^{(n)} = 0$, so that

$$F_{M^{\mathbf{B}_0^{(n)}}}(x) = \mathbb{1}_{\{x \geq c\}} F_{n-1}(x\mathbf{e}_{n-1}; \boldsymbol{\mu}^{i^*}, \boldsymbol{\Sigma}^{i^*})$$

where $\boldsymbol{\mu}^{i^*}$ is the $(n-1)$ -dimensional vector obtained by dropping row i^* of $\boldsymbol{\mu}$ in Corollary 2 and, similarly, $\boldsymbol{\Sigma}^{i^*}$ is the $(n-1)$ -by- $(n-1)$ covariance matrix obtained by dropping the i^* row and column of $\boldsymbol{\Sigma}$.

5.4 Upper bound to the distribution of the running maximum M_T^J

It is easy to see that the supremum M_t^X of a continuous-time stochastic process X on $[0, t]$ is bounded below by $\max_{t \in \mathcal{T}^{(n)}} X_t$ for any grid $\mathcal{T}^{(n)}$ made of increasing times $0 \leq t_0 < t_1 < \dots < t_n \leq t$ and $n \in \mathbb{N}$:

$$M_t^X = \sup_{s \in [0, t]} X_s \geq \max_{t \in \mathcal{T}^{(n)}} X_t = \max \mathbf{X}_0^{(n)} .$$

Assuming that $t_0 = 0$, $X_0 = x_0$ is known,

$$F_{M_t^X}(x) \leq \mathbb{1}_{\{x \geq x_0\}} F_{M^{\mathbf{X}^{(n)}}}(x)$$

with equality in the limit where $n \rightarrow \infty$ such that $\sup |t_i - t_{i-1}| \rightarrow 0$. This provides an upper bound for the distribution of the running maximum of a continuous-time stochastic process that can be infinitely tight. One can thus apply this result to J and B , setting $t_0 = B_0 = J_0 = 0$.

6 Example

We illustrate the above results for the supremum of the Brownian bridge B , of the integral J of the square-integrable function $\sigma(u) = e^{\kappa u}$ and of the Ornstein-Uhlenbeck process X driven by J . For the ease of exposition, we consider the running maxima up to $t < T$.

One can sample the Brownian bridge B on any grid $\mathcal{T}$ using the exact distribution (2). The analytical distribution of the maximum of B on $\mathcal{T}$ is given in Corollary 2, which is an upper bound for M_t^B . Note that the distribution of the running supremum of B is available from (15).

Similarly, one can sample the integral J on any grid $\mathcal{T}$ using the exact distribution provided in Lemma 3. Defining $g(x) = (e^x - 1)/x$, we get for the integrand $\sigma(u) = e^{\kappa u}$ that $\Sigma(t) = tg(\kappa t)$ and $\|\sigma\|_t^2 = tg(2\kappa t)$, so that the means and covariances are given by

$$\mu_i = c \frac{t_i g(\kappa t_i)}{T}, \quad \sigma_i^2 = t_i g(2\kappa t_i) - T \left(\frac{\mu_i}{c} \right)^2 \quad \text{and} \quad R_{i,j \neq i} = \frac{t_i \wedge t_j g(2\kappa t_i \wedge t_j) - t_i t_j g(\kappa t_i) g(\kappa t_j) / T}{\sigma_i \sigma_j}.$$

Injecting these expressions in Corollary 1 yields the distribution of the maximum of J on the grid $\mathcal{T}$, which is an upper bound for M_t^J .

Finally, the dynamics of the Ornstein-Uhlenbeck driven by the Brownian bridge B are given by

$$dX_t = \kappa(\theta - X_t)dt + \eta dB_t, \quad X_0 = x_0,$$

whose solution, at any time $t \in \mathcal{T}$, can be written as

$$X_t = x_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \eta e^{-\kappa t} J_t, \quad \text{where} \quad J_t = \int_0^t e^{\kappa u} dB_u.$$

Observe that X_t is a linear combination of J_t , hence, the joint distribution of X on a grid $\mathcal{T}$ can be found from that of J . The mean and variance of X_{t_i} are given by $\mu_i = x_0 e^{-\kappa t_i} + \theta(1 - e^{-\kappa t_i}) + \eta e^{-\kappa t_i} c \frac{t_i g(\kappa t_i)}{T}$ and $\sigma_i^2 = \eta^2 e^{-2\kappa t_i} \left(t_i g(2\kappa t_i) - T \left(\frac{\mu_i}{c} \right)^2 \right)$, while the correlations are the same as those of the J_{t_i} 's, i.e., are given by the matrix $\mathbf{R}$ above. The distribution of the maximum of X on $\mathcal{T}$ is found by injecting these expressions in (17).

The left panels of Figure 1 display the distribution of M_4^B , the running supremum of a Brownian bridge B on $[0, t]$ with $t = 4$, $T = 5$ and $c = 3$. The dotted blue curve corresponds to the theoretical expression, given in (15). The solid black curve shows the empirical distribution of $\max \mathbf{B}_0^{(n)} = \max(0, M^{\mathbf{B}^{(n)}})$ based on $M = 5,000$ trajectories on the grid $\mathcal{T}^{(n)} = \{0 = t_0, t_1, \dots, t_n\}$ where $n = 5$ (top), $n = 50$ (middle), $n = 100$ (bottom), $t_i = i\delta$ and $\delta = t/n$ generated by Monte Carlo simulations. The values of $\mathbf{B}^{(n)}$ on a same trajectory are sampled from the multivariate normal distribution with moments given in (4). The dashed red curve shows the analytical expression of the distribution of $\max \mathbf{B}_0^{(n)}$ provided in Corollary 2, and corresponds to the analytical expression that the black curve aims to estimate numerically. The solid gray curve is the same as the solid black curve but on a finer grid $\mathcal{T}^{(m)}$ with $m = 250$. It provides the Monte Carlo estimation of the distribution of the maximum of $\mathbf{B}^{(m)}$. For $m > n$, $F_{M^{\mathbf{B}^{(m)}}}$ is expected to be a tighter – but more costly – upper bound to $F_{M_t^B}$ compared to $F_{M^{\mathbf{B}^{(n)}}}$ and, for m sufficiently large, the former can be seen as a proxy for M_t^B . One can check that, indeed, the distribution of $M^{\mathbf{B}^{(m)}}$ (solid gray) is closer from M_t^B (dotted blue) compared to $M^{\mathbf{B}^{(n)}}$ (solid black). The center and right panels show similar results for M_4^J and M_4^X . We set the process parameters to $(x_0, \kappa, \theta, \eta) = (0, 0.2, 1, 0.2)$. As for B , the black curve shows the empirical distribution of $\max \mathbf{J}_0^{(n)}$ (resp. $\max \mathbf{X}_0^{(n)}$). Because we analyze both B and J in this exercise, we rely on the exact sampling scheme detailed in Section 4.¹ In this particular case,

$$\Sigma_k = (e^{\kappa t_{k+1}} - e^{\kappa t_k}) / \kappa.$$

The dashed red curve corresponds to the distribution of $M^{\mathbf{J}^{(n)}}$ and $M^{\mathbf{X}^{(n)}}$. Recall that the theoretical expressions of the distributions of the running supremums M_t^J and M_t^X are not available (justifying why the dotted blue curve is not shown) but, as explained above for B , they can be approached by $F_{M^{\mathbf{J}^{(m)}}}$

¹To double check the results, we provide in the On-line appendix Figure 2, which is the same as Figure 1 but when J is simulated using the Euler-Maruyama scheme $J_{t_i} - J_{t_{i-1}} = e^{\kappa t_{i-1}}(B_{t_i} - B_{t_{i-1}})$. The results are essentially the same, although the Euler scheme is biased for finite n .

and $F_{M^{\mathbf{x}(m)}}$ for large m (solid gray curves correspond to $m = 250$). We include in our comparison the method introduced in [1] and discussed in [2, pp.124-125] (solid green). The latter enhances the plain Monte Carlo approach discussed above by relying on four steps: (i) simulate the process (B , J or X) on the grid $\mathcal{T}^{(n)}$ using an Euler scheme, (ii) compute the analytical distribution of the maximum of the Gaussian approximation of that process on each interval conditional upon the initial and terminal values of the process at each interval found in the first step, (iii) sample maxima on each interval from the distributions found in the second step and, finally, (iv) take the maximum of those. The Gaussian approximation amounts to freeze the drift and volatility coefficients of the process at the beginning of each interval of $\mathcal{T}^{(n)}$, and is therefore perfect for B (whose dynamics coincides with the assumption) for all n . For $n = 5$, the results are quite good for J but less for X . This can be explained by the fact that the Gaussian approximation of X considers a variance for the increments of $\eta^2\delta$ which, for $n = 5$, is quite larger than the actual variance $\eta^2(1 - e^{-2\kappa\delta})/(2\kappa)$ (0.032 vs. 0.027 given the parameters at hand). This leads to an overestimation of large movements of the process within each interval. Note that there is no reason for the resulting CDF to dominate the exact one anymore, even asymptotically when $m \rightarrow \infty$.² Of course, all the approaches converge to the true distribution as δ decreases (n increases).

The upper bounds are evaluated using the function `pvnorm` available from the  package `mvtnorm` [6]. The fact that the red and the black curves coincide confirms that the upper bound actually corresponds to the true distribution of B and J sampled on the grid $\mathcal{T}^{(n)}$. The fact that the gray curves coincide with the blue curves on the left panels shows that the distribution of the supremum of B can be reasonably well estimated in Monte Carlo with $m = 250$, and hence can serve as a benchmark for J and X . The results confirm the validity of the upper bounds and the convergence toward the distribution of the running supremum. For the considered cases, the analytical approximations found using $n = 50$ look reasonable. The analytical estimation remains faster than Monte Carlo even in dimension $n = 100$.

7 Conclusion

We provide a couple of distributions related to the integral J of a deterministic, square-integrable function σ . We derive the analytical expression of the distribution of $M^{\mathbf{J}}$, the maximum of $\mathbf{J}$, defined as the process J sampled on a discrete grid $\mathcal{T}$. The distribution of $M^{\mathbf{J}}$ can serve as an upper bound to the distribution of M_t^J , the running supremum of J up to t . Moreover, the distribution of $M^{\mathbf{J}}$ converges to that of M_t^J when $\min \mathcal{T} = 0$, $\max \mathcal{T} = t$ and the largest time-step in $\mathcal{T}$ tends to zero. These results are useful in various applications, for example, to fasten convergence via importance sampling.

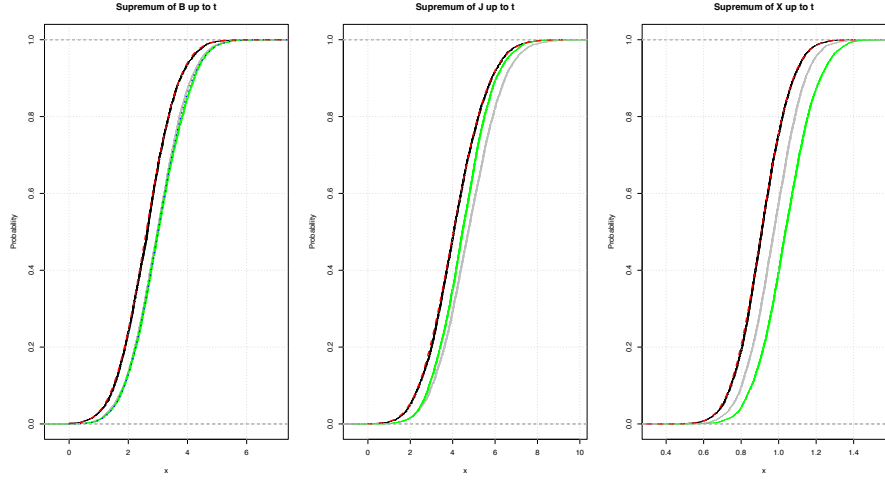
Acknowledgment

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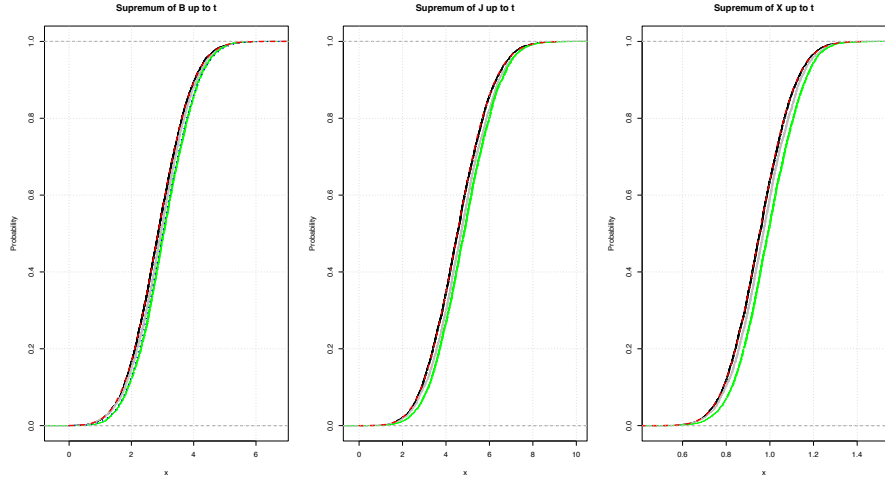
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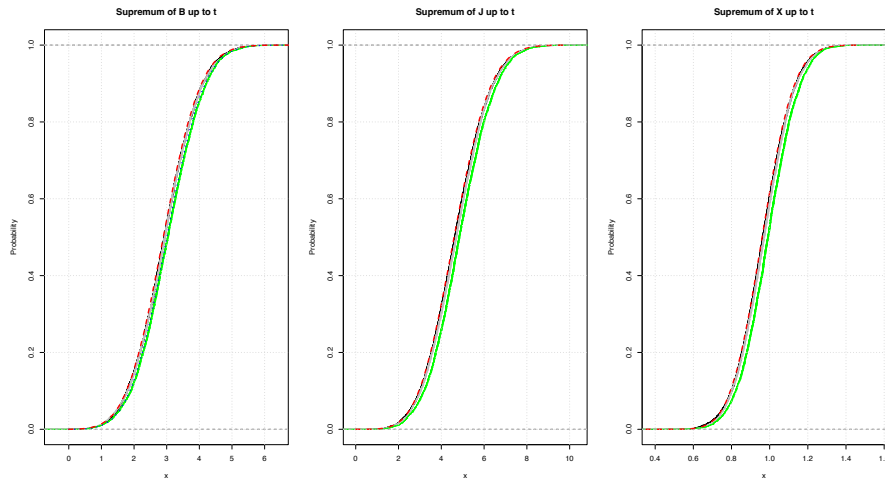
²The results are much better when, for example, the values of the parameters coincide with the interest rates application considered by the authors in their book (e.g., $(x_0, \kappa, \theta, \eta) = (0.02, 0.1, 0.02, 0.02)$); see Fig. 3 in the On-line appendix.



(a) $n = 5$



(b) $n = 50$



(c) $n = 100$

Figure 1: Running supremum M_t^B (left), M_t^J (center), and M_t^X (right). The black curve depicts the empirical distribution generated by Monte Carlo simulations where B is sampled from the multivariate normal distribution according to (4), and J is simulated using Section 4 with a time step of $\delta = t/n$ and $n = 5$ (top) $n = 50$ (middle) or $n = 100$ (bottom). The red curve shows the upper bound of the distribution of the supremum found using corollaries 1 and 2. The blue curve corresponds to the theoretical expression of M_t^B , given in (15).

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On-line appendix

7.1 Alternative proof to Lemma 1

Let us first show that

$$J_t \sim \mathcal{N}(c\mu_T(t), \gamma_T^2(t, t)) \quad \text{when } 0 \leq t \leq T. \quad (18)$$

We start from the definition of the Itô integral I in (1) as the limit of an Itô sum:

$$I_t^{(n)} \xrightarrow{n \rightarrow \infty} I_t \quad \text{with} \quad I_t^{(n)} := \sum_{i=1}^n \sigma(t_{i-1}) (W_{t_i} - W_{t_{i-1}}) \quad \text{and} \quad t_i := i\delta_n, \quad \delta_n = t/n, \quad i \in \{1, 2, \dots, n\}.$$

Now, define $Z_i := (W_{t_i} - W_{t_{i-1}})/\sqrt{\delta_n} \sim \mathcal{N}(0, 1)$ and $\tilde{Z} := (W_T - W_t)/\sqrt{T-t} \sim \mathcal{N}(0, 1)$, where

$$\sum_{i=1}^n \sqrt{\delta_n} Z_i + \sqrt{T-t} \tilde{Z} = (W_{t_n} - W_0) + (W_T - W_t) = W_T.$$

It is known from [11] that if $(Z_1, \dots, Z_{n+1})$ are $n+1$ i.i.d. standard Normal variables, then $(w_1 Z_1, \dots, w_{n+1} Z_{n+1})$ subject to $\mathbf{w}\mathbf{Z}' = c$ is distributed as $(X_1, \dots, X_n, c - \sum_{i=1}^n X_i)$ where $\mathbf{X}^{(n)} = (X_1, \dots, X_n)$ is normally distributed with known mean vector $\boldsymbol{\mu} \in \mathbb{R}^n$ and covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$:

$$\mathbf{X}^{(n)}|_{W_T} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad \text{where} \quad \mu_i = c \frac{w_i^2}{\|\mathbf{w}\|^2} \quad \text{and} \quad \Sigma_{i,j} = \frac{w_i^2}{\|\mathbf{w}\|^2} (\delta_{ij} \|\mathbf{w}\|^2 - w_j^2).$$

Exploiting this result with $(Z_1, \dots, Z_n, \tilde{Z})$, $\mathbf{w} := (\sqrt{\delta_n} \mathbf{e}_n, \sqrt{T-t})$ where $\mathbf{e}_n$ is an n -dimensional vector of ones and $c = W_T$, we have $w_{i,i \leq n}^2 = \delta_n$, $w_{n+1}^2 = T-t$ and $\|\mathbf{w}\|^2 = n\delta_n + (T-t) = T$, so that

$$\mu_i = \frac{W_T}{T} \delta_n \quad \text{and} \quad \Sigma_{i,j} = \frac{\delta_n}{T} \left(\mathbb{1}_{\{i=j\}} T - \frac{\delta_n}{T} \right) = \mathbb{1}_{\{i=j\}} \delta_n - \frac{\delta_n^2}{T}.$$

Hence,

$$\begin{aligned} F_{I_t^{(n)}|_{W_T}}(x) &= \mathbb{P} \left(\sum_{i=1}^n \sigma(t_{i-1}) X_i \leq x \mid W_T \right) \\ &= \Phi \left(\frac{x - \sum_{i=1}^n \sigma(t_{i-1}) \mu_i}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n \sigma(t_{i-1}) \sigma(t_{j-1}) \Sigma_{i,j}}} \right) \\ &= \Phi \left(\frac{x - \frac{W_T}{T} \sum_{i=1}^n \sigma(t_{i-1}) \delta_n}{\sqrt{\sum_{i=1}^n \sigma^2(t_{i-1}) \delta_n - \frac{1}{T} \sum_{i=1}^n \sum_{j=1}^n \sigma(t_{i-1}) \sigma(t_{j-1}) \delta_n^2}} \right). \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, the sums converge to Riemann integrals, and one finally gets

$$F_{I_t|_{W_T}}(x) = \Phi \left(\frac{x - \frac{W_T}{T} \Sigma(t)}{\sqrt{\|\sigma\|_t^2 - \frac{\Sigma^2(t)}{T}}} \right) = \Phi \left(\frac{x - W_T \mu_T(t)}{\sqrt{\|\sigma\|_t^2 - T \mu_T^2(t)}} \right) = \Phi \left(\frac{x - W_T \mu_T(t)}{\gamma_T(t, t)} \right).$$

From (3), replacing W_T by c in the above expression yields the distribution of J_t and proves (18).

Let us now show that

$$J_t \sim \mathcal{N}(c\mu_T(T), \gamma_T^2(T, T)) \quad \text{when } t \geq T. \quad (19)$$

We have from (18) that $J_T \sim \mathcal{N}(c\mu_T(T), \gamma_T^2(T, T))$. Moreover, $J_t - J_T$ is independent from B_T , hence $J_t - J_T \sim I_t - I_T|_{W_T=c} \sim I_t - I_T \sim \mathcal{N}(0, \int_T^t \sigma^2(u) du)$. Therefore, with $Z, Z^\perp$ two i.i.d. standard

normal variables independent from W_T ,

$$\begin{aligned}
F_{I_t|W_T}(x) &= \mathbb{P}(I_T|_{W_T} + (I_t - I_T)|_{W_T} \leq x) \\
&= \mathbb{P}\left(W_T\mu_T(T) + \gamma_T(T, T)Z + \sqrt{\int_t^T \sigma^2(u)du}Z^\perp \leq x \middle| W_T\right) \\
&= \mathbb{E}\left[\Phi\left(\frac{x - W_T\mu_T(T) - \sqrt{\|\sigma\|_t^2 - \|\sigma\|_T^2}Z^\perp}{\gamma_T(T, T)}\right) \middle| W_T\right] \\
&= \Phi\left(\frac{x - \frac{W_T}{T}\Sigma(T)}{\sqrt{\|\sigma\|_t^2 - T\mu_T^2(T)}}\right) \\
&= \Phi\left(\frac{x - W_T\mu_T(T)}{\gamma_T(t, T)}\right).
\end{aligned}$$

where we have used that $\mathbb{E}[\Phi(\mu + \sigma Z)] = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)$. Replacing W_T by c proves (19).

7.2 Alternative proof to Lemma 2

We focus on the first claim. Observe that $J_s^t|_{B_s} \sim I_s^t|_{(W_s, W_T)=(B_s, c)}$ and let us first show that

$$I_s^t|_{W_s, W_T} \sim \mathcal{N}((c - B_s)\mu_{T-s}(s, t), \gamma_{T-s}^2(s, t)) \quad \text{when } t \leq T. \quad (20)$$

We have

$$I_s^t|_{W_s, W_T} = \int_0^{t-s} \sigma(u+s)dW_{u+s}|_{W_s, W_T} \sim \int_0^{t-s} \tilde{\sigma}(u)d\tilde{W}_u|_{\tilde{W}_{T-s}=W_T-W_s}$$

where $\tilde{\sigma}(u) = \sigma(u+s)$ and $\tilde{W}_u = W_{u+s}$. The last integral above is similar to J_t where $t \leftarrow t-s$, $T \leftarrow T-s$ and $\sigma \leftarrow \tilde{\sigma}$. Hence, it is normally distributed with mean $\tilde{\mu}_{T-s}(t-s) := \tilde{\Sigma}(t-s)/(T-s)$ and variance $\tilde{\gamma}_{T-s}^2(t-s) := \|\tilde{\sigma}\|_{t-s}^2 - \frac{\tilde{\Sigma}^2(t-s)}{T-s}$. On the other hand,

$$\tilde{\Sigma}(t-s) = \int_0^{t-s} \tilde{\sigma}(u)du = \int_0^{t-s} \sigma(u+s)du = \int_s^t \sigma(u)du = \Sigma(s, t)$$

and

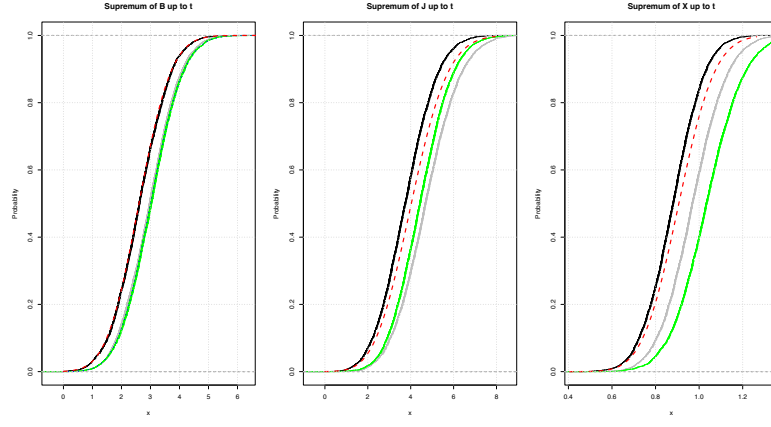
$$\tilde{\gamma}_{T-s}^2(t-s) = \int_0^{t-s} \tilde{\sigma}^2(u)du - \frac{\tilde{\Sigma}^2(t-s)}{T-s} = \int_s^t \sigma^2(u)du - \frac{\Sigma^2(s, t)}{T-s}.$$

This shows that $\tilde{\mu}_{T-s}(t-s) = \frac{\Sigma(s, t)}{T-s} = \mu_{T-s}(s, t)$ and $\tilde{\gamma}_{T-s}^2(t-s) = \gamma_{T-s}^2(s, t, t)$. Finally, for $t > T$,

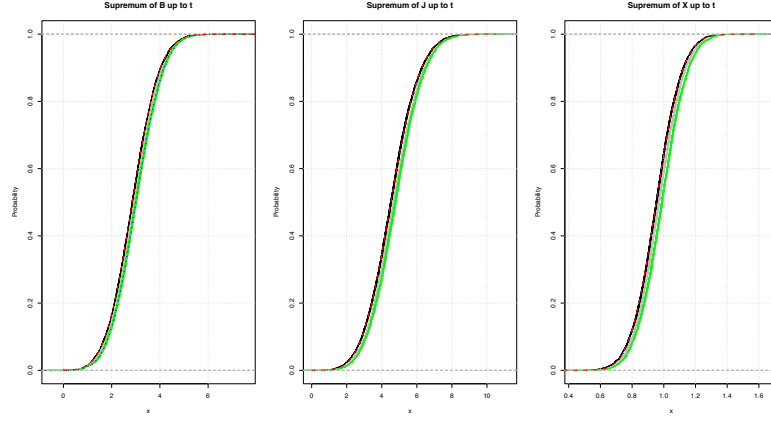
$$I_s^t|_{W_s, W_T} \sim I_s^t|_{W_T-W_s} \sim I_s^T|_{W_T-W_s} + I_T^t$$

where the last two variables are independent normal variables with means $\mu_{T-s}(s, T)(W_T - W_s)$ and 0, and variances $\gamma_{T-s}^2(s, T, T) = \|\sigma\|_T^2 - \|\sigma\|_s^2 - (T-s)\mu_{T-s}^2(s, T)$ and $\int_T^t \sigma^2(u)du$, respectively. The sum of the variances coincides with $\|\sigma\|_t^2 - \|\sigma\|_s^2 - (T-s)\mu_{T-s}^2(s, T) = \gamma_{T-s}^2(s, t, T)$.

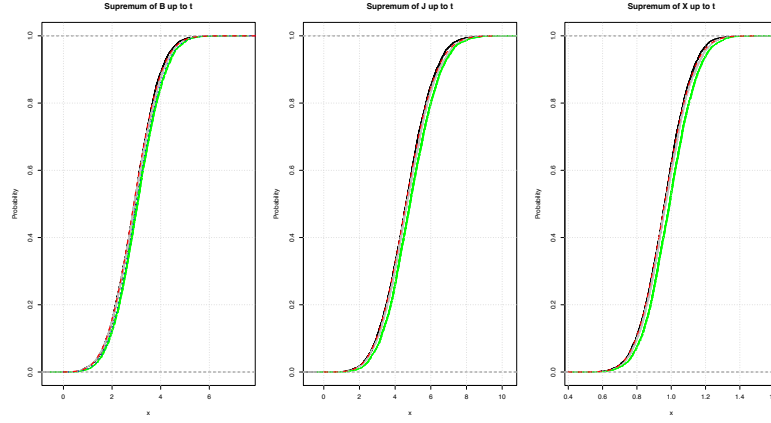
Complementary figures



(a) $n = 5$



(b) $n = 50$



(c) $n = 100$

Figure 2: Same as Figure 1 but J is simulated using the Euler scheme: $J_{t_i} - J_{t_{i-1}} = e^{\kappa t_{i-1}}(B_{t_i} - B_{t_{i-1}})$.

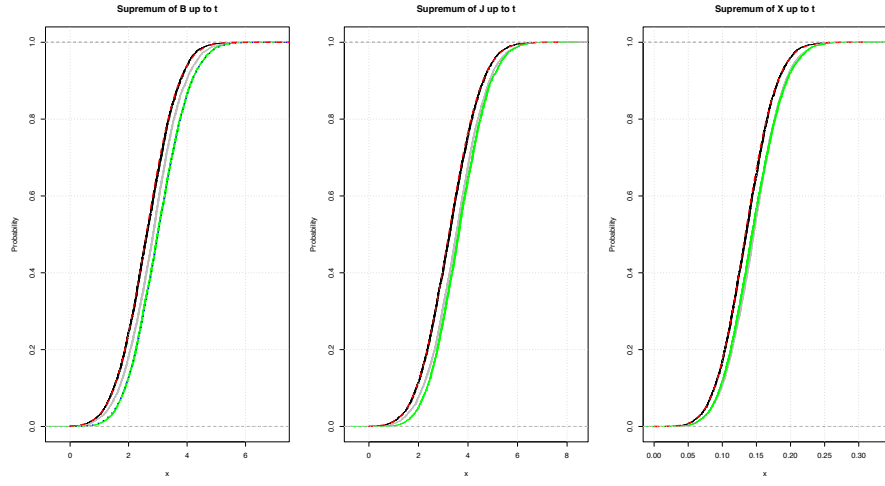


Figure 3: Same as top panels of Figure 1 ($n = 5$) but for the set of parameters $(x_0, \kappa, \theta, \eta) = (0.02, 0.1, 0.02, 0.02)$.