Real Analysis Exchange
Summer Symposium 2008, pp. 1–5

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THE MULTIDIMENSIONAL LUZIN THEOREM

Abstract

In 1912, N. Luzin showed that each finite measurable function $f : \mathbb{R} \to \mathbb{R}$ is equal a.e. to the derivative of a continuous, a.e. differentiable function $F : \mathbb{R} \to \mathbb{R}$. We explain how this can be improved in several dimensions.

This report exposes the results obtained in a joint work [3] with W.F. Pfeffer.

Before introducing this note, let us precise some notations.

Preliminaries and notations

Euclidean space $\mathbb{R}^n$ is endowed by the usual norm $|x| = \sqrt{x \cdot x}$. A set $I \subseteq \mathbb{R}^n$ is called an *interval* in case there exists nonvoid intervals $I_1, I_2, \ldots, I_n$ so that $I = I_1 \times I_2 \times \cdots \times I_n$. For an interval $I \subseteq \mathbb{R}^n$, we give $C(I)$ and $C_0(I)$ the usual meanings.

The *oscillation* (around 0) of $F : \mathbb{R}^n \to \mathbb{R}$ on $A \subseteq \mathbb{R}^n$ is the real number

$$\mathrm{osc}_0(F, A) := \sup\{|F(x)| : x \in A\}.$$

Mathematical Reviews subject classification: Primary: 26A24; Secondary: 28A15
Key words: Luzin theorem, Almost everywhere derivative
*The author is an *Aspirant* of the *Fonds de la Recherche Scientifique — FNRS*, Belgium.

Given sets $A \subseteq B \subseteq \mathbb{R}^n$ and a map $F : B \to \mathbb{R}$, we let $F \upharpoonright A$ denote the *restriction* of F to A. For $F : A \subseteq \mathbb{R}^n \to \mathbb{R}$ we denote by $\bar{F}$ its extension by 0 to the whole space $\mathbb{R}^n$.

Assuming that the sequence (F_k) of functions $F_k : A_k \subseteq \mathbb{R} \to \mathbb{R}$ is such that $F_k \upharpoonright (A_k \cap A_l) = F_l \upharpoonright (A_k \cap A_l)$ for each pair k, l, we let $\bigvee_k F_k$ denote the unique map $F : \bigcup_k A_k \to \mathbb{R}$ satisfying $F \upharpoonright A_k = F_k$ for each k.

1 Introduction

In 1912, N. Luzin [2] showed that given a finite measurable function $f : \mathbb{R} \to \mathbb{R}$, there exists an a.e. differentiable $F \in C(\mathbb{R})$ satisfying $F'(x) = f(x)$ for a.e. $x \in \mathbb{R}$.

Remark 1. Observe that given $\varepsilon > 0$, the continuous function F provided by Luzin's theorem can be chosen in such a way that $\operatorname{osc}_0(F, \mathbb{R}) \leqslant \varepsilon$. To show this, begin by writing $\mathbb{R}$ as a countable union of nonoverlapping compact intervals $I_0, I_1, I_2, \ldots$ such that $\operatorname{osc}_0(F, I_k) \leqslant \frac{\varepsilon}{2}$ for each k. Choose also for each integer k (see [5, Chapter III, §13.4]) a nondecreasing $H_k \in C(I_k)$ satisfying $H_k \upharpoonright \partial I_k = F \upharpoonright \partial I_k$ and $H_k'(x) = 0$ for a.e. $x \in I_k$ and observe that $\operatorname{osc}_0(F - H_k, I_k) \leqslant \varepsilon$. Let $H := \bigvee_k H_k$, notice that H is continuous and observe that $F - H \in C(\mathbb{R})$ is such that $(F - H)'(x) = F'(x)$ for a.e. $x \in \mathbb{R}$ while $\operatorname{osc}_0(F - H, \mathbb{R}) \leqslant \varepsilon$.

We will show in this note how this has been improved in several dimensions: given a finite measurable vectorfield $v : \mathbb{R}^n \to \mathbb{R}^n$ and any $\varepsilon > 0$, there exists an a.e. differentiable $F \in C(\mathbb{R}^n)$ satisfying $\nabla F(x) = v(x)$ for a.e. $x \in \mathbb{R}^n$ and $\operatorname{osc}_0(F, \mathbb{R}^n) \leqslant \varepsilon$.

2 Using W.F. Pfeffer's "Devil's platforms"

In [4], W.F. Pfeffer provides a skilful generalization of devil's staircases in dimension $n \geqslant 1$.

Theorem 2. *Given a compact interval $I \subseteq \mathbb{R}^n$ and a continuous $F_0 : \partial I \to \mathbb{R}$, there exists $F \in C(I)$ satifying the following conditions:*

1. *$F \upharpoonright \partial I = F_0$,*

2. *F is a.e. differentiable on I,*

3. *$\nabla F(x) = 0$ for a.e. $x \in I$.*

Using those devil's platforms, we can — proceding as in Remark 1 — reduce arbitrarily the oscillation of a continuous function without changing its gradient outside a negligible set.

Corollary 3. *Assume that $I \subseteq \mathbb{R}^n$ is an interval and fix an a.e. differentiable $F \in C(I)$. For any $\varepsilon > 0$ there exists an a.e. differentiable $\tilde{F} \in C_0(I)$ satisfying $\nabla F(x) = \nabla \tilde{F}(x)$ for a.e. $x \in I$ and $\operatorname{osc}_0(\tilde{F}, I) \leqslant \varepsilon$.*

Proof. Begin by writing I as a countable union of nonoverlapping compact intervals $I_0, I_1, I_2, \ldots$ such that $\operatorname{osc}_0(F, I_k) \leqslant \frac{\varepsilon}{2}$ for each k. For every k, choose $H_k \in C(I_k)$ associated to I_k and $F \restriction \partial I_k$ according to Theorem 2. Define $H \in C(I)$ by $H := \bigvee_k H_k$ and observe that $\tilde{F} := F - H$ is a suitable choice. ■

3 A Luzin theorem for gradients

The following result is an easy corollary of a very particular case of G. Alberti's "Luzin theorem for gradients" (see [1]).

Lemma 4. *Assume that $v : \mathbb{R}^n \to \mathbb{R}^n$ and $\varepsilon > 0$ are given. There exists a closed set $C \subseteq \mathbb{R}^n$ together with $H \in C^1(\mathbb{R}^n)$ satisfying the following conditions:*

1. *$|\mathbb{R}^n \setminus C| \leqslant \varepsilon$,*

2. *$\nabla H(x) = v(x)$ for each $x \in C$.*

Using the previous lemma and following Luzin's original argument, we prove our multidimensional Luzin theorem.

Theorem 5. *Fix a measurable $v : \mathbb{R}^n \to \mathbb{R}^n$ and $\varepsilon > 0$. There exists an a.e. differentiable $F \in C(\mathbb{R}^n)$ satisfying $\operatorname{osc}_0(F, \mathbb{R}^n) \leqslant \varepsilon$ and $\nabla F(x) = v(x)$ for a.e. $x \in \mathbb{R}^n$.*

SKETCH OF THE PROOF. Choose a closed set $C_0 \subseteq \mathbb{R}^n$ and $H_0 \in C^1(\mathbb{R}^n)$ associated to v and $\frac{\varepsilon}{2}$ according to Lemma 4. Apply Corollary 3 if necessary to find an a.e. differentiable $F_0 \in C(\mathbb{R}^n)$ satisfying $\operatorname{osc}_0(F_0, \mathbb{R}^n) \leqslant \frac{\varepsilon}{2}$ and $\nabla F_0(x) = \nabla H_0(x)$ for a.e. $x \in \mathbb{R}^n$. In particular, we have $\nabla F_0(x) = v(x)$ for a.e. $x \in C_0$.

According to Lemma 4, choose a closed set $C_1' \subseteq \mathbb{R}^n$ and a function $H_1' \in C^1(\mathbb{R}^n)$ such that $|\mathbb{R}^n \setminus C_1'| \leqslant \frac{\varepsilon}{2^2}$ and $\nabla H_1'(x) = v(x) - F_0(x)$ for each $x \in C_1'$. Write $\mathbb{R}^n \setminus C_0$ as a countable union of nonoverlapping compact intervals $I_0, I_1, I_2, \dots$ and for $k \in \mathbb{N}$, let $\delta_k := \operatorname{dist}(I_k, C_0) > 0$. Using Corollary 3, choose for each k an a.e. differentiable $H_{1,k} \in C_0(I_k)$ satisfying $\operatorname{osc}_0(H_{1,k}, I_k) \leqslant \frac{\varepsilon\delta_k}{2^{k+3}}$ and $\nabla H_{1,k}(x) = \nabla H_1'(x)$ for a.e. $x \in I_k$. Observe that the map $H_1 := \sum_{k=0}^{\infty} \bar{H}_{1,k}$ is continuous on $\mathbb{R}^n$, satisfies $\operatorname{osc}_0(H_1, \mathbb{R}^n) \leqslant \frac{\varepsilon}{2^2}$ and is a.e. differentiable outside C_0 with $\nabla H_1(x) = v(x) - \nabla F_0(x)$ for a.e. $x \in C_1' \setminus C_0$.

For $x \in C_0$ and an integer k, observe that $\bar{H}_{1,k}(x+h) = 0$ in case $x+h \notin I_k$. On the other hand if $x + h \in I_k$ we get $\delta_k \leqslant |h|$ so that

$$|H_{1,k}(x)| \leqslant \operatorname{osc}_0(H_{1,k}, I_k) \leqslant \frac{\varepsilon\delta_k}{2^{k+3}} \leqslant \frac{\varepsilon}{2^2} \cdot \frac{1}{2^{k+1}} \cdot |h|.$$

Define for $h \in \mathbb{R}^n$ a number $k(h) := \inf\{k \in \mathbb{N} : x + h \in I_k\}$. Computing for $h \in \mathbb{R}^n$

$$\begin{aligned} |H_1(x+h) - H_1(x)| = |H_1(x+h)| &\leqslant \sum_{k=k(h)}^{\infty} |\bar{H}_{1,k}(x+h)| \\ &\leqslant \frac{\varepsilon}{2^2} \left(\sum_{k=k(h)}^{\infty} \frac{1}{2^{k+1}} \right) |h| \qquad (1) \end{aligned}$$

and using the easy equality $\lim_{h\to 0} k(h) = +\infty$, we see that H_1 is differentiable at x and that $\nabla H_1(x) = 0$.

Consequently the map $F_1 \in C(\mathbb{R}^n)$ defined by $F_1 = F_0 + H_1$ is a.e. differentiable on $\mathbb{R}^n$, satisfies

$$\nabla F_1(x) = \nabla F_0(x) + \nabla H_1(x) = \nabla F_0(x) + [v(x) - \nabla F_0(x)] = v(x)$$

for a.e. $x \in C_1' \setminus C_0$ and

$$\nabla F_1(x) = \nabla F_0(x) + \nabla H_1(x) = v(x) + 0 = v(x)$$

for a.e. $x \in C_0$. Letting $C_1 := C_0 \cup C_1'$ we get $|\mathbb{R}^n \setminus C_1| \leqslant \frac{\varepsilon}{2^2}$ and $\nabla F_1(x) = v(x)$ for a.e. $x \in C_1$.

Continue this procedure, and construct a sequence $F_0, F_1, F_2, \dots$ of continuous functions together with an increasing sequence $C_0 \subseteq C_1 \subseteq C_2 \dots$ of closed sets satisfying, among other conditions, $|\mathbb{R}^n \setminus C_i| \leqslant \frac{\varepsilon}{2^{i+1}}$, $\nabla F_i(x) = v(x)$ for a.e. $x \in C_i$ and $\operatorname{osc}_0(F_{i+1} - F_i, \mathbb{R}^n) \leqslant \frac{\varepsilon}{2^{i+1}}$ for each integer i.

Simple computations then reveals that $F := \lim_i F_i$ is a well defined continuous function on $\mathbb{R}^n$ satisfying the desired conditions. ■

Remark 6. As we mentioned it before, Lemma 4 is only a particular case of G. Alberti's Luzin theorem for gradients [1]. Working first in an open set $U \subseteq \mathbb{R}^n$ having finite measure instead of $\mathbb{R}^n$, Alberti's result provides, given a Borel function $v : U \to \mathbb{R}^n$, a function $H \in C^1(U)$ and a closed set $C \subseteq U$ for which $|U \setminus C| \leqslant \varepsilon |U|$, $\nabla H(x) = v(x)$ for every $x \in C$ and

$$\operatorname{osc}_0(\nabla H, U) \leqslant \frac{C(n)}{\varepsilon} \operatorname{osc}_0(v, U).$$

As suggested by a referee, using the previous estimation can simplify considerably the proof of Theorem 5 by shortcutting the reference to Corollary 3.

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