

A Matrix Integral Solution to $[P, Q] = P$ and Matrix Laplace Transforms

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Abstract

In this paper we solve the following problems: (i) find two differential operators P and Q satisfying $[P, Q] = P$, where P flows according to the KP hierarchy $\partial P / \partial t_n = [(P^{n/p})_+, P]$, with $p := \text{ord } P \geq 2$; (ii) find a matrix integral representation for the associated τ -function. First we construct an infinite dimensional space $\mathcal{W} = \text{span}_{\mathbb{C}}\{\psi_0(z), \psi_1(z), \dots\}$ of functions of $z \in \mathbb{C}$ invariant under the action of two operators, multiplication by z^p and $A_c := z \partial / \partial z - z + c$. This requirement is satisfied, for arbitrary p , if ψ_0 is a certain function generalizing the classical Hankel function (for $p = 2$); our representation of the generalized Hankel function as a double Laplace transform of a simple function, which was unknown even for the $p = 2$ case, enables us to represent the τ -function associated with the KP time evolution of the space $\mathcal{W}$ as a “double matrix Laplace transform” in two different ways. One representation involves an integration over the space of matrices whose spectrum belongs to a wedge-shaped contour $\gamma := \gamma^+ + \gamma^- \subset \mathbb{C}$ defined by $\gamma^{\pm} = \mathbb{R}_+ e^{\pm \pi i / p}$. The new integrals above relate to the matrix Laplace transforms, in contrast with the matrix Fourier transforms, which generalize the Kontsevich integrals and solve the operator equation $[P, Q] = 1$.

Introduction

It is a long-standing puzzle in the theory of $2d$ -gravity to find an adequate description of gravitational coupling of (p, q) minimal models. One part of it is to find two differential operators P and Q of order p and q respectively, such that $[P, Q] = f(P)$ for some function f . In the simplest case of $q = 1$ and $f \equiv 1$, such description is provided by 1-matrix models, especially by the Kontsevich integral and their generalizations; see [1, 19, 25]. Going along the chain, $2d$ -gravity $\rightarrow$ equilateral triangles $\rightarrow$ discrete matrix models $\rightarrow$ Kontsevich models, this approach has lead to the discovery of integrable structures for non-perturbative partition functions, which take the form of τ -functions of the KP hierarchy (see [7, 25, 31] for review and references). While similar results are believed to be true in the general (p, q) -case, the Kontsevich integral counterparts are still unknown. Note that a minor modification of the generalized Kontsevich integral can be interpreted as a duality transformation between (p, q) and (q, p) -models [18].

So far the most promising approach for finding integrable structures in the general (p, q) -case seems to be the one initiated by Kac–Schwarz in the case $q = 1$ and $f = 1$. So, the general problem comes in two stages: (1) Find a point in Sato’s Grassmannian invariant under two symmetry operators,

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satisfying some commutation relation; the existence of such a plane leads to a system of differential equations specifying the wave function Ψ and thus to an algebra of constraints for the τ -function. (2) Find a matrix integral representation for this τ -function. Note a matrix representation, beyond the case $q = 1$ and $f = 1$, if it exists at all, was unknown.

The purpose of this paper is to find a τ -function and a matrix integral representation for the equation $[P, Q] = P$ for $q = 1$ and arbitrary p . Remarkably, the matrix integral representation can still be found, but it is far less straightforward and considerably more involved, than the ordinary Kontsevich integral.

The message is the following: whereas the case $[P, Q] = 1$ is described by general *matrix Fourier transforms*, a solution to $[P, Q] = P$ is related to *double Laplace transforms*. While it is not known whether this solution has immediate physical relevance, it may help to shed some light on the (p, q) -case and on the matrix representations of the corresponding τ -functions. In particular, what are the proper multimatrix generalizations of the Kontsevich integrals?

Note this problem has come up in the physical literature, in various different contexts: unitary matrix models have been written down, leading to equations $[P, Q] = P$ for differential operators P and Q in the double scaling limit; see the studies of Dalley, Johnson, Periwai, Minahan, Morris, Shevitz, and Watterstam [4, 5, 27, 28, 22, 23]). In the mathematical context (inverse scattering and monodromy preserving transformations), see Ablowitz, Flaschka, Fokas and Newell [11, 9, 10]). The solution provided in our paper is new and does *not* require any scaling limit.

Consider the problem of finding a differential operator P of order p and another differential operator Q satisfying

$$[P, Q] = f(P), \quad \text{with } 0 \neq f(z) \in \mathbb{C}[z]. \quad (1)$$

When P is (formally) deformed with respect to the KP flows, i.e., $\partial P / \partial t_n = [(P^{n/p})_+, P]$, one can introduce the corresponding deformation of Q which preserves Eq. (1). Hence (1) can be considered as a condition on a solution of the p -reduced KP hierarchy.

The basic ingredients of this construction are¹

- $\psi_0 \in 1 + z^{-1}\mathbb{C}[[z^{-1}]]$,
- $A: \mathbb{C}((z^{-1})) \rightarrow \mathbb{C}((z^{-1}))$ which increases the order of an element of $\mathbb{C}((z^{-1}))$ in z exactly by one,

so that $\mathcal{W} := \text{span}_{\mathbb{C}}\{\psi_0, A\psi_0, A^2\psi_0, \dots\}$ belongs to the big stratum of the Sato Grassmannian and satisfies $A\mathcal{W} \subset \mathcal{W}$, such that

- ψ_0 satisfies the differential equation $v(z)\psi_0 = F(A)\psi_0$ for some $v(z) \in \mathbb{C}((z^{-1}))$ and $F(Z) \in \mathbb{C}[Z]$, so that $v(z)\mathcal{W} \subset \mathcal{W}$ also holds.

Let Ψ be the KP wave function corresponding to $\mathcal{W}$. The above conditions lead to the existence of differential operators Q and P in x such that $Q\Psi = A\Psi$ and $P\Psi = v(z)\Psi$. If A coincides with $\partial/\partial v = (1/v')\partial/\partial z$ up to the conjugation by a function, then we have $[P, Q] = 1$. And if ψ_0 is defined by a Fourier transform and the action of A on it can be expressed in a suitable way, then the corresponding Hermitian *matrix* Fourier transform, properly normalized, is the corresponding τ -function. See Sect. 3 for details.

The matrix integral approach to (1) has so far needed $\text{ord } Q = 1$ at the initial point of the formal KP time flows, requiring $\deg_z f(z) \leq 1$. The degree 0 case can be reduced to $[P, Q] = 1$. In this paper, we provide a solution to the degree 1 case, or the next simplest instance of (1), which can clearly be reduced to

$$[P, Q] = P, \quad (2)$$

with differential operators P and Q . As in the case of $[P, Q] = 1$, we write the τ -function of its formal KP deformation explicitly in terms of a matrix integral.

¹ $\mathbb{C}[[x]] := \{\sum_{n=0}^{\infty} a_n x^n \mid a_n \in \mathbb{C}\}$ is the ring of formal power series in x , and $\mathbb{C}((x)) := \{\sum_{-\infty \ll n < \infty} a_n x^n \mid a_n \in \mathbb{C}\}$ is the ring of formal Laurent series in x .

Definition 1 Let $-1 < c < 0$, $p \in \mathbb{Z}$, $p \geq 2$. Let $\mathcal{W}$ be the linear span

$$\mathcal{W} = \text{span}_{\mathbb{C}} \{ \psi_0(z), \psi_1(z), \psi_2(z), \dots \},$$

of generalized Hankel functions,

$$\psi_k(z) = \frac{p^{c+1}}{\Gamma(-c)} \int_1^\infty \frac{z^{-c}(uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du, \quad k = 0, 1, 2, \dots, \quad (3)$$

also representable as double Laplace transforms

$$\psi_k(z) = \frac{p^{c+1}}{2\pi i} z^{(p-1)(c+1)} e^z \int_0^\infty dx x^c e^{-xz^p} \int_0^\infty dy f_k(y) e^{-xy^p} \quad (4)$$

of the functions

$$f_k(y) = (\zeta^{k+1} e^{-\zeta y} - \zeta^{-k-1} e^{-\zeta^{-1} y}) y^k, \quad k = 0, 1, 2, \dots, \quad \text{with } \zeta := e^{\pi i/p}. \quad (5)$$

Using the asymptotic expansion $\psi_k(z) = z^k (1 + O(1/z)) \in \mathbb{C}((z^{-1}))$ as $\Re z \rightarrow \infty$, $\mathcal{W}$ defines a point of the Sato Grassmannian Gr . Let Ψ and τ be the wave (formal Baker–Akhiezer) function and τ -function, respectively, associated with the KP time evolution $\mathcal{W}^t = e^{-\sum t_i z^i} \mathcal{W}$; see Sects. 1 and 2. Then we have

Theorem 1

$$\Psi(x, 0, z) = e^{xz} \psi_0((1-x)z), \quad (6)$$

and it satisfies

$$\left(L \left(x - 1, \frac{\partial}{\partial x} \right) - z^p \right) \Psi(x, 0, z) = 0 \quad \text{and} \quad \left(L \left(z, \frac{\partial}{\partial z} - 1 \right) - (x-1)^p \right) \Psi(x, 0, z) = 0, \quad (7)$$

where $L(z, \partial/\partial z)$ is the monic differential operator

$$L \left(z, \frac{\partial}{\partial z} \right) := \frac{1}{z^p} \left(\prod_{i=0}^{p-1} \left(z \frac{\partial}{\partial z} + c - i \right) - cp \prod_{i=0}^{p-2} \left(z \frac{\partial}{\partial z} + c - i \right) \right) = \left(\frac{\partial}{\partial z} \right)^p + \dots$$

Note that for $p = 2$, $L(z, \partial/\partial z) = (\partial/\partial z)^2 - (c^2 + c)/z^2$.

Theorem 2 Let $\mathcal{H}_N$ be the space of $N \times N$ Hermitian matrices, and $\mathcal{H}_N^+$ the subspace of $\mathcal{H}_N$ of positive definite Hermitian matrices. The corresponding τ -function evaluated at

$$t_n := -\frac{1}{n} \text{tr } Z^{-n}, \quad \text{for } n = 1, 2, \dots, \quad \text{and with an } N \times N \text{ diagonal } Z,$$

is given by the following (normalized) double matrix Laplace transform:

$$\tau(t) = S_1(t) \frac{\int_{\mathcal{H}_N^+} dX \det X^c e^{-\text{tr } Z^p X} \int_{\mathcal{H}_N^+} dY S_0(y) e^{-\text{tr } XY^p}}{\int_{\mathcal{H}_N} dX \exp \text{tr} \left(-\frac{(X+Z)^{p+1}}{p+1} \right)_2},$$

where $(\cdot)_2$ denotes the terms quadratic in X ,

$$S_0(y) := \frac{\Delta(y^p)}{\Delta(y)^2} \det(f_{k-1}(y_i))_{1 \leq i, k \leq N} \quad \text{and} \quad S_1(t) := \det(Z^{(p-1)(c+1/2)}) e^{\text{tr } Z},$$

where $y = (y_1, \dots, y_N)$ are the eigenvalues of Y , $y^p = (y_1^p, \dots, y_N^p)$, and $\Delta(y) := \prod_{i>j} (y_i - y_j) = \det(y_i^{j-1})_{i,j}$, and f_{k-1} are as in (5).

The function $\tau(t)$ also has the following matrix integral representation

$$\tau(t) = \frac{\int_{\mathcal{H}_N^\gamma} m(dW) \int_{\mathcal{H}_N^+} dX \det X^c (\Delta(w^p)/\Delta(w)) e^{\text{tr}(Z-W)} e^{\text{tr} X(W^p - Z^p)}}{\int_{\mathcal{H}_N} dX \exp \text{tr} \left(-\frac{(X+Z)^{p+1}}{p+1} \right)_2},$$

integrated over the space of matrices

$$\mathcal{H}_N^\gamma = \{W = UD_\gamma U^{-1} \mid U \in \mathbf{U}(N), D_\gamma := \text{diag}(w_1, \dots, w_N) \in (\gamma)^N\},$$

where γ denotes a wedge-shaped contour in $\mathbb{C}$, defined in Sect. 4 (see Fig. 1), in terms of a complex-valued measure

$$m(dW) = dU dw \prod_{1 \leq i < j \leq N} (w_i - w_j)^2.$$

Theorem 3 (i) The algebra of stabilizers of $\mathcal{W}$,

$$S_{\mathcal{W}} := \{\phi(z, \partial/\partial z) \in \mathbb{C}((z^{-1}))[\partial/\partial z] \text{ such that } \phi\mathcal{W} \subset \mathcal{W}\},$$

is generated by $A_c := z \frac{\partial}{\partial z} - z + c$, z^p and $\xi := z^{-p} F(A_c)$, where $F(u) = \prod_0^{p-1} (u-i) - cp \prod_0^{p-2} (u-i)$:

$$S_{\mathcal{W}} = \mathbb{C}[A_c, z^p, \xi] \subset \mathbb{C}((z^{-1}))[\partial/\partial z].$$

Moreover, $\mathcal{W} = \mathbb{C}[A_c]\psi_0$, and ψ_0 satisfies the differential equation

$$F(A_c)\psi_0 = (-z)^p \psi_0(z). \quad (8)$$

(ii) A family of solutions to the operator equation $[P, Q] = P$ is given by the differential operators P and Q , defined equivalently by

$$P\Psi = z^p \Psi, \quad Q\Psi = \frac{1}{p} A_c \Psi, \quad (9)$$

or by

$$P = S \left(\frac{d}{dx} \right)^p S^{-1} \quad \text{and} \quad Q = \frac{1}{p} (MP^{1/p} - P^{1/p} + c),$$

where $M = S \left(\sum_1^\infty k \bar{t}_k (d/dx)^{k-1} \right) S^{-1}$, $\bar{t}_k = t_k + \delta_{k,1} x$, with wave operator

$$S = \frac{\tau(\bar{t} - [(d/dx)^{-1}])}{\tau(\bar{t})}.$$

(iii) The function $\tau(t)$ satisfies, in terms of the W -generators in Eq. (20), the following constraints

$$\sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq i}} \alpha_{m,i} \binom{i}{j} \frac{(-1)^{i-j}}{j+1} W_{i+np-j}^{(j+1)} \tau(t) = a_{m,n,c} \tau(t), \quad m, n = 0, 1, 2, \dots, \quad (10)$$

for some constants $a_{m,n,c}$, where the constants $\alpha_{n,i}$ are defined by the formula² $(x \cdot d/dx)^n = \sum_{i=0}^n \alpha_{n,i} x^i (d/dx)^i$. In particular, setting $m = 1$, $\tau(t)$ satisfies Virasoro constraints of the form (with $W_{np}^{(2)} = \sum_{i+j=np} :J_i^{(1)} J_j^{(1)}:$)

$$\left(\frac{1}{2} W_{np}^{(2)} - \frac{\partial}{\partial t_{np+1}} - a_{1,n,c} \right) \tau = 0, \quad n = 0, 1, 2, \dots \quad (11)$$

Remark 1 The constants $a_{m,n,c}$ in (10) can all be calculated; in particular, the Virasoro constraint (11) for $n = 0$ becomes:

$$\left(\sum_1^\infty i t_i \frac{\partial}{\partial t_i} - \frac{\partial}{\partial t_1} - \frac{c(1+c)(p-1)}{2} \right) \tau = 0.$$

² More explicitly, $\alpha_{n,i} = \frac{1}{i!} \sum_{j=0}^i \binom{i}{j} (-1)^{i-j} j^n$. Note that it vanishes if $n > 0$ and $i = 0$.

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1 The KP Hierarchy

Throughout, x is a formal scalar variable near 0, and z is a formal scalar variable near ∞ . If $g(z) = cz^q(1 + O(z^{-1}))$, $c \neq 0$, then $\text{ord}_z g(z) := q$ is the *order* of $g(z)$.

Throughout, we denote $\partial/\partial x$ by D . The algebra of ordinary pseudodifferential operators in x is denoted by $\mathcal{D}$ (the word “in x ” may be dropped if there is no fear of confusion), with its splitting $\mathcal{D} = \mathcal{D}_+ + \mathcal{D}_-$ into the subalgebras of ordinary differential operators and of ordinary pseudodifferential operators of negative order:

$$\mathcal{D} = \left\{ \sum_{-\infty < i \leq n} a_i D^i \mid n \in \mathbb{Z} \text{ arbitrary, } a_i = a_i(x) \right\},$$

$$A = \sum a_i D^i \in \mathcal{D} \quad \Rightarrow \quad A_+ = \sum_{i \geq 0} a_i D^i \in \mathcal{D}_+ \quad \text{and} \quad A_- = A - A_+ \in \mathcal{D}_-.$$

The ring $\mathcal{D}$ acts on the space of functions of the form $\sum_{-\infty < i \leq \infty} a_i(x) z^i e^{xz}$ simply by extending the formulas $D^n e^{xz} = z^n e^{xz}$ and $A(Be^{xz}) = (A \circ B)e^{xz}$, $A, B \in \mathcal{D}$. When $A \in \mathcal{D}_+$, this definition of $A(Be^{xz})$ coincides with the usual action of A , as a differential operator, on Be^{xz} as a formal series in x with z -dependent coefficients.

A pseudodifferential operator in x may depend on the KP time variables $t = (t_1, t_2, \dots)$ introduced below, but not on z unless otherwise noted. We are not specific about the regularity of the coefficients of pseudodifferential operators. The operators S , L , M etc., associated to a point $\mathcal{W}$ of the big stratum Gr^0 of the Sato Grassmannian (see below) have regular (i.e., formal power series) coefficients; otherwise, the singularities of those operators can be controlled by the Schubert stratum to which $\mathcal{W} \in \text{Gr}$ belongs. In particular, there exist $n, m \geq 0$ such that $x^n S$ and $S^{-1} x^m$ at $t = 0$ have regular coefficients. See [29] for details.

As in [2], we set $\bar{t} = (x + t_1, t_2, t_3, \dots)$, and

$$\bar{\partial} = \left(\frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \frac{1}{3} \frac{\partial}{\partial t_3}, \dots \right).$$

The elementary Schur functions p_n are defined by $\exp(\sum_1^\infty t_n z^n) = \sum_0^\infty p_n(t) z^n$.

1.1 KP hierarchy

The operator $L = L(t) = D + \sum_{j=-\infty}^{-1} a_j(x, t) D^j \in \mathcal{D}$, with $t = (t_1, t_2, \dots)$, subjected to the KP equations

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n = 1, 2, \dots,$$

is known to have the following representation in terms of an operator $S \in 1 + \mathcal{D}_-$ called the wave operator, and the associated, formally infinite order pseudodifferential operator

$$W := S e^{\sum_{i=1}^{\infty} t_i D^i},$$

as follows:

$$\begin{aligned} L &= S D S^{-1} = W D W^{-1}, \\ \frac{\partial S}{\partial t_n} &= -(L^n)_- S, \quad \text{and} \quad \frac{\partial W}{\partial t_n} = (L^n)_+ W. \end{aligned} \tag{12}$$

The wave function

$$\Psi(t, z) := \Psi(x, t, z) := W e^{xz} = S e^{\sum_{i=1}^{\infty} \bar{t}_i z^i}, \tag{13}$$

where $\bar{t}_i = t_i + \delta_{i,1} x$, satisfies

$$L \Psi = z \Psi \quad \text{and} \quad \frac{\partial \Psi}{\partial t_n} = (L^n)_+ \Psi, \tag{14}$$

and has the following representation in terms of a scalar-valued function associated to S called the tau function τ :

$$\begin{aligned} \Psi(t, z) &= \frac{\tau(\bar{t} - [z^{-1}])}{\tau(\bar{t})} e^{\sum_{i=1}^{\infty} \bar{t}_i z^i} \\ &= \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial}) \tau(\bar{t})}{\tau(\bar{t})} z^{-n} e^{\sum_{i=1}^{\infty} \bar{t}_i z^i} \\ &= \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial}) \tau(\bar{t})}{\tau(\bar{t})} D^{-n} e^{\sum_{i=1}^{\infty} \bar{t}_i z^i}, \end{aligned}$$

implying in view of (13)

$$S = \frac{\tau(\bar{t} - [D^{-1}])}{\tau(\bar{t})} := \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial}) \tau(\bar{t})}{\tau(\bar{t})} D^{-n}. \tag{15}$$

Moreover, using (13), we have

$$\frac{\partial}{\partial z} \Psi = \frac{\partial}{\partial z} W e^{xz} = W \frac{\partial}{\partial z} e^{xz} = W x e^{xz} = W x W^{-1} \Psi,$$

thus leading to the operator

$$\begin{aligned} M &:= W x W^{-1} = S e^{\sum t_k D^k} x e^{-\sum t_k D^k} S^{-1} = S \left(x + \sum_1^{\infty} k t_k D^{k-1} \right) S^{-1} \\ &= S \left(\sum_1^{\infty} k \bar{t}_k D^{k-1} \right) S^{-1} \end{aligned} \tag{16}$$

satisfying

$$M \Psi = (\partial / \partial z) \Psi \quad \text{and} \quad [L, M] = W [D, x] W^{-1} = 1,$$

and for any formal series $f = f(x, \xi)$,

$$f(M, L) = W f(x, D) W^{-1}. \tag{17}$$

1.2 Symmetries

Consider the Lie algebra w_∞ of operators

$$w_\infty := \mathbb{C}[z, z^{-1}][d/dz] = \text{span}_{\mathbb{C}} \left\{ z^\alpha \left(\frac{\partial}{\partial z} \right)^\beta \mid \alpha, \beta \in \mathbb{Z}, \beta \geq 0 \right\},$$

and its completion $\overline{w}_\infty := \mathbb{C}((z^{-1}))[\partial/\partial z]$ in the z^{-1} -adic topology, for the customary commutation relation $[\cdot, \cdot]$. Acting on Ψ , we have

$$z^\alpha (\partial/\partial z)^\beta \Psi = M^\beta L^\alpha \Psi, \quad (18)$$

motivating the definition of the following vector fields, called symmetries, on Ψ :

$$\mathbb{Y}_{z^\alpha (\partial/\partial z)^\beta} \Psi := (M^\beta L^\alpha)_- \Psi.$$

We require that these flows act trivially on parameters x, t , and hence on $S^{-1}MS = \sum k \bar{t}_k D^{k-1}$, for instance.

Lemma 1 *There is an injective homomorphism of Lie algebras*

$$\begin{aligned} \overline{w}_\infty / \mathbb{C} &\longrightarrow \left\{ \begin{array}{l} \text{Lie algebra of vector fields} \\ \text{on the manifold of wave functions } \Psi \\ \text{commuting with the KP flows } \partial/\partial t_n \end{array} \right\} \\ z^\alpha \left(\frac{\partial}{\partial z} \right)^\beta &\longmapsto \mathbb{Y}_{z^\alpha (\partial/\partial z)^\beta} \Psi = (M^\beta L^\alpha)_- \Psi, \end{aligned}$$

i.e.,

$$\left[\mathbb{Y}_{z^\alpha (\partial/\partial z)^\beta}, \mathbb{Y}_{z^{\alpha'} (\partial/\partial z)^{\beta'}} \right] = \mathbb{Y}_{[z^\alpha (\partial/\partial z)^\beta, z^{\alpha'} (\partial/\partial z)^{\beta'}]}.$$

This definition differs from the one in [2] by the sign. Here this definition is chosen to make it consistent with the natural action of $\overline{w}_\infty$ on the Grassmannian discussed in the next section, rather than its negative. These vector fields induce vector fields on S and $L = SDS^{-1}$, as

$$\mathbb{Y}_{z^\alpha (\partial/\partial z)^\beta} (S) = (M^\beta L^\alpha)_- S$$

and

$$\mathbb{Y}_{z^\alpha (\partial/\partial z)^\beta} (L) = [(M^\beta L^\alpha)_-, L].$$

Proposition 1 ([2]) *We have*

$$-\frac{(M^n L^{n+\ell})_- \Psi}{\Psi} = (e^{-\eta} - 1) \frac{\frac{1}{n+1} W_\ell^{(n+1)}(\tau)}{\tau} \Big|_{t_1 \rightarrow t_1 + x}, \quad n, \ell \in \mathbb{Z}, \quad n \geq 0, \quad (19)$$

where the $W_\ell^{(n+1)}$, the generators of the W_∞ -algebra, are the coefficients in the expansion of the vertex operator

$$\begin{aligned} X(t, \lambda, \mu) &:= \exp \left(\sum_{i=1}^{\infty} (\mu^i - \lambda^i) t_i \right) \exp \left(\sum_{i=1}^{\infty} \frac{\lambda^{-i} - \mu^{-i}}{i} \frac{\partial}{\partial t_i} \right) \\ &= \sum_{k=0}^{\infty} \frac{(\mu - \lambda)^k}{k!} \sum_{\ell=-\infty}^{\infty} \lambda^{-\ell-k} W_\ell^{(k)}, \quad \text{with } W_\ell^{(0)} = \delta_{\ell,0}. \end{aligned} \quad (20)$$

2 Grassmannian

Let $H := \mathbb{C}((z^{-1}))$, $H_+ := \mathbb{C}[z]$, and $H_- := z^{-1}\mathbb{C}[[z^{-1}]]$, so that $H = H_+ \oplus H_-$. We denote by Gr the Grassmannian manifold of linear subspaces $\mathcal{W}$ of H of relative dimension 0 with respect to H_+ , i.e., the natural map

$$\pi_{\mathcal{W}}: \mathcal{W} \hookrightarrow H \xrightarrow{\pi} H/H_- \simeq H_+$$

being Fredholm of index 0. $\text{Gr}^0 := \{\mathcal{W} \in \text{Gr} \mid \pi_{\mathcal{W}} \text{ is isomorphism}\}$ is the big (open) Schubert stratum of Gr .

Given a wave function $\Psi = \Psi(x, t, z)$, let $\mathcal{W}$ be the point of Gr defined by³

$$\begin{aligned} \mathcal{W} &= \text{span}_{\mathbb{C}} \left\{ \frac{\partial^j}{\partial x^j} \Psi(0, 0, z) \mid j = 0, 1, 2, \dots \right\} \\ &= \text{span}_{\mathbb{C}} \left\{ \frac{\partial^{j_1 + \dots + j_N}}{\partial t_1^{j_1} \dots \partial t_N^{j_N}} \Psi(0, 0, z) \mid N \geq 0, j_1, \dots, j_N \geq 0 \right\}. \end{aligned}$$

The first line guarantees $\mathcal{W} \in \text{Gr}$, and the second line follows from the first by using the second equation in (14), i.e., the KP time evolutions of Ψ . Hence up to the t -adic completion we have

$$\mathcal{W} = \text{span}_{\mathbb{C}} \left\{ \left(\frac{\partial}{\partial x} \right)^j \Psi(0, t, z) \mid j = 0, 1, 2, \dots \right\},$$

so that, letting $\psi = e^{-\sum t_i z^i} \Psi$ and

$$\mathcal{W}^t := e^{-\sum t_i z^i} \mathcal{W} = \text{span}_{\mathbb{C}} \left\{ (\partial/\partial x)^j \psi(0, t, z) \mid j = 0, 1, 2, \dots \right\},$$

we have $\psi = (\pi_{\mathcal{W}^t})^{-1}(1)$, i.e., ψ is the preimage of 1 by the map $\pi_{\mathcal{W}^t}: \mathcal{W}^t \rightarrow H_+$.

The corresponding τ -function $\tau(t)$ is the determinant of the composite map

$$\mathcal{W} \xrightarrow{g} \mathcal{W}^t \xrightarrow{\pi_{\mathcal{W}^t}} H/H_- \simeq H_+, \quad (21)$$

where g denotes the multiplication by $e^{-\sum t_i z^i}$. Given $\mathcal{W}$, the determinant is well-defined up to a constant which is determined by the choice of a basis $\{\psi_k\}_{k=0}^{\infty}$, $\psi_k = z^k(1 + O(z^{-1}))$ for $k \gg 0$, of $\mathcal{W}$. We take $\{z^k\}_{k=0}^{\infty}$ as the basis of H_+ . More specifically, $\tau(t)$ is defined as the limit as $n \rightarrow \infty$ of the determinant of

$$\mathcal{W}_n \hookrightarrow \mathcal{W} \rightarrow H_+ \rightarrow H_+/z^n H_+, \quad (22)$$

where the middle arrow is the composite map in (21), $\mathcal{W}_n = \text{span}_{\mathbb{C}} \{\psi_k\}_{k=0}^{n-1}$, and the determinant is computed with respect to the bases $\{\psi_k\}_{k=0}^{n-1}$ of $\mathcal{W}_n$ and $\{z^k\}_{k=0}^{n-1}$ of $H_+/z^n H_+$. The limit exists in the t -adic topology of $\mathbb{C}[[t]]$, i.e., for any multi-index α , there exists a positive integer n_{α} such that, if $n \geq n_{\alpha}$, then the coefficient of t^{α} in the determinant of (22) is independent of n , and gives the coefficient of t^{α} in $\tau(t)$. This finiteness property is an immediate consequence of the fact that, expanding $\tau(t)$ in terms of Schur functions, the coefficients give the Plücker coordinates of $\mathcal{W}$. See [29] for details.

The $\overline{w}_{\infty}$ -action on Ψ becomes the natural action of $\overline{w}_{\infty}$ on Gr : As an ordinary differential operator in z , each $A \in \overline{w}_{\infty}$ acts on H , which defines a vector field on Gr .

2.1 Stabilizers

Given $\mathcal{W} \in \text{Gr}$, we shall call

$$S_{\mathcal{W}} := \{Q := Q(z, \partial/\partial z) \in \overline{w}_{\infty} \mid Q\mathcal{W} \subset \mathcal{W}\}$$

the *stabilizer* of $\mathcal{W}$. In this subsection we shall observe basic properties of the stabilizer which can be obtained without referring to matrix integrals.

³ If Ψ is singular at $(x, t) = 0$, we need to replace $(\partial/\partial x)^j \Psi(0, 0, z)$ in the first line by $(\partial/\partial x)^j (x^n \Psi(x, 0, z)) \big|_{x=0}$ for some $n > 0$, and make a similar replacement in the second line (see [29] for details). We chose to write the formulas for $\mathcal{W} \in \text{Gr}^0$ for simplicity.

Lemma 2 Let $\mathcal{W} \in \text{Gr}$ and $A := \sum_{-\infty < i \ll \infty, 0 \leq j \ll \infty} c_{ij} z^i (\partial/\partial z)^j \in \overline{\mathcal{W}}_\infty$. If

$$A\mathcal{W} \subset \mathcal{W}, \quad (23)$$

then

$$Q_A := \sum_{\substack{-\infty < i \ll \infty \\ 0 \leq j \ll \infty}} c_{ij} M^j L^i \in \mathcal{D}_+.$$

Conversely, if $Q \in \mathcal{D}_+$ is of this form, i.e., $Q = Q_A$ for some $A \in \overline{\mathcal{W}}_\infty$, then this A satisfies (23).

Proof: We have

$$A\Psi(t, z) = Q_A\Psi(t, z) \quad (24)$$

by definition. Since $A\mathcal{W} \subset \mathcal{W}$, and since the Taylor coefficients (or Laurent coefficients if $\mathcal{W} \notin \text{Gr}^0$) in x of Ψ generates $\mathcal{W}$, $A\Psi$ is a $\mathbb{C}[[x, t]]$ -linear combination of $\Psi, D\Psi, D^2\Psi, \dots$, i.e., $A\Psi = Q\Psi$ for some $Q \in \mathcal{D}_+$. Hence, since (24) determines Q_A uniquely, Q_A itself must be in $\mathcal{D}_+$. Conversely, suppose $Q_A \in \mathcal{D}_+$, and let $\Psi(x, 0, z) = \sum f_n(z)x^n$ be the Taylor (or Laurent) expansion of $\Psi(x, 0, z)$ at $x = 0$. Then each Taylor coefficient in x of $Q_A\Psi$ is a linear combination of $\{f_n(z)\}$, and hence it belongs to $\mathcal{W}$, so that by (24) $Af_n \in \mathcal{W}$ for every n (the action of A on f_n is well-defined since A is a differential operator in z). Since $\{f_n\}$ is a basis of $\mathcal{W}$, we have $A\mathcal{W} \subset \mathcal{W}$. $\blacksquare$

Corollary 1 Let $p \neq 0$ be an integer, and let $Q \in \mathcal{D}_+$ such that $\text{ad}(L^p)^N Q = 0$ for $N \gg 0$. Then $Q = Q_A$ for some $A \in \overline{\mathcal{W}}_\infty$ such that $A\mathcal{W} \subset \mathcal{W}$ holds. In particular, a solution to the string equation (1) always comes from a pair of $A \in \overline{\mathcal{W}}_\infty$ and $\mathcal{W} \in \text{Gr}$, such that $A\mathcal{W} \subset \mathcal{W}$ (and $z^p\mathcal{W} \subset \mathcal{W}$ due to the extra assumption $P = L^p \in \mathcal{D}_+$).

Proof: Writing $Q = \sum_{ij} c_{ij} M^j L^i$, let $A = \sum_{ij} c_{ij} z^i (\partial/\partial z)^j$. Since $\text{ad}(L^p)^N Q = 0$ we have $\text{ad}(z^p)^N A = 0$, which implies that A is a differential operator in z . Hence the “converse” part of Lemma 2 applies. $\blacksquare$

Lemma 3 Let $A, B \in \overline{\mathcal{W}}_\infty$, $\psi_0 = 1 + O(z^{-1}) \in 1 + H_-$ and $\mathcal{W} \in \text{Gr}$. Suppose A acts on the monomials z^k , $k \in \mathbb{Z}$, as

$$Az^k = z^{k+1}(c_k + O(z^{-1})),$$

and $c_k \neq 0$ if $k \geq 0$. Then the following conditions are equivalent:

- (i) $\psi_0 \in \mathcal{W}$, $A\mathcal{W} \subset \mathcal{W}$ and $B\mathcal{W} \subset \mathcal{W}$;
- (ii) $\mathcal{W} = \text{span}_{\mathbb{C}}\{\psi_0, A\psi_0, A^2\psi_0, \dots\}$, and ψ_0 satisfies the differential equations

$$BA^n\psi_0 = F_n(A)\psi_0, \quad n = 0, 1, \dots \quad (25)$$

for some $F_n(s) \in \mathbb{C}[s]$.

In particular, under these conditions $\mathcal{W}$ belongs to the big stratum Gr^0 of Gr . If, moreover, A and B satisfy a commutation relation of the form

$$[A, B] = a(A)B + b(A) \quad (26)$$

for some $a(s), b(s) \in \mathbb{C}[s]$, then in (25) it suffices to assume only the $n = 0$ case, i.e.,

$$B\psi_0 = F(A)\psi_0 \quad (27)$$

for some $F(s) \in \mathbb{C}[s]$.

Proof: Since $\psi_0 \in \mathcal{W}$, $A\mathcal{W} \subset \mathcal{W}$ implies $\mathcal{W}' := \text{span}_{\mathbb{C}}\{\psi_0, A\psi_0, A^2\psi_0, \dots\} \subset \mathcal{W}$. Since $\psi_0 = 1 + O(z^{-1})$ and A raises the order of a function in z by 1, the map $\mathcal{W}' \rightarrow H_+$ is bijective, and $\mathcal{W}' \in \text{Gr}^0$. In particular, both $\mathcal{W}$ and $\mathcal{W}'$ are of relative dimension 0, so that $\mathcal{W} = \mathcal{W}'$. Conversely, $\mathcal{W} = \mathcal{W}'$ clearly implies $\psi_0 \in \mathcal{W}$ and $A\mathcal{W} \subset \mathcal{W}$. Assume these equivalent conditions. Then $B\mathcal{W} \subset \mathcal{W}$ if and only if $B\mathcal{W}' \subset \mathcal{W}'$ if and only if the differential equations of the form (25) are satisfied. Finally, when A and B satisfy a commutation relation of the form (26), the n^{th} equation in (25) implies the $(n+1)^{\text{st}}$ one, so that (27) suffices. $\blacksquare$

The following propositions take a closer look at the $[P, Q] = 1$ case and $[P, Q] = P$ case, to show that essentially those elements in $\overline{\mathcal{W}}_{\infty}$ which give rise to P and Q in the sense of Lemma 2, and their polynomials, are the only elements of the stabilizer.

Proposition 2 *Let $p \in \mathbb{Z}$, $p > 0$. Let $A \in \overline{\mathcal{W}}_{\infty}$ be such that $[A, z^p] = 1$. If $\mathcal{W} \in \text{Gr}$ satisfies $z^p\mathcal{W} \subset \mathcal{W}$ and $A\mathcal{W} \subset \mathcal{W}$, then the stabilizer of $\mathcal{W}$ is generated by z^p and A , i.e., $S_{\mathcal{W}} = \mathbb{C}[A, z^p]$.*

Proof: Since $[A, z^p] = 1$, A is a first order differential operator in z , so that any $C \in S_{\mathcal{W}}$ can be written as $C = \sum_{-\infty < i \leq \infty, 0 \leq j \leq N} a_{ij} z^i A^j$ for some $N \geq 0$. It suffices to prove that $a_{ij} = 0$ if $i < 0$ or if $i \not\equiv 0 \pmod{p}$. Suppose A raises the order of a function in z by k : $\text{ord}_z Az^{\ell} = \ell + k$. Let I be the set of pairs (i, j) such that $i < 0$ or $i \not\equiv 0 \pmod{p}$, $a_{ij} \neq 0$, and $i + kj$ is maximum among all such a_{ij} 's. We have $|I| < \infty$, and we only need to prove $|I| = 0$. Suppose this is not true. Let $C_0 := \sum_{(i,j) \in I} a_{ij} z^i A^j$. Noting

$$[A, z^i A^j] = [A, (z^p)^{i/p}] A^j = (i/p) z^{i-p} A^j,$$

so that $\text{ad}(A)^n(z^i A^j) = 0$ for $n \gg 0$ if and only if $i \geq 0$ and $i \equiv 0 \pmod{p}$, we see that for $n \gg 0$ the leading terms of $\text{ad}(A)^n C$ are $\text{ad}(A)^n C_0$, which lowers the order of a function in z , and does not annihilate the function for a general n . This cannot happen since $\text{ad}(A)^n C\mathcal{W} \subset \mathcal{W}$, and since in $\mathcal{W}$ the order of functions in z are bounded from below. $\blacksquare$

Proposition 3 *Let $p \in \mathbb{Z}$, $p > 0$. Let $A = z\partial/\partial z - a(z)$, where $a(z) \in z + \mathbb{C}[[z^{-1}]]$, and $\psi_0 = 1 + O(z^{-1}) \in 1 + H_-$. Let $\mathcal{W} \in \text{Gr}$ be the point of the Grassmannian determined by the conditions $\psi_0 \in \mathcal{W}$ and $A\mathcal{W} \subset \mathcal{W}$. Suppose $\mathcal{W}$ also satisfies $z^p\mathcal{W} \subset \mathcal{W}$. Let $F(s) = c \prod_{i=1}^p (s - c_i) \in \mathbb{C}[s]$, where $c_i \in \mathbb{C}$, $c \in \mathbb{C}^*$, be the polynomial of degree p as in (27) with $B = z^p$, i.e., ψ_0 satisfies the equation*

$$F(A)\psi_0 = z^p\psi_0. \quad (28)$$

Then if F satisfies the following genericity condition:

(G) *For any $n \not\equiv 0 \pmod{p}$, we have $(F) + n := \sum (c_i + n) \not\equiv (F) \pmod{p}$, i.e., $\pi_p((F) + n) \neq \pi_p(F)$, where $(F) = \sum_{i=1}^p (c_i)$ is the divisor of F , and $\pi_p: \mathbb{C} \rightarrow \mathbb{C}/p\mathbb{Z}$ is the natural projection,*

then the stabilizer of $\mathcal{W}$ is generated by A , z^p and $\xi := z^{-p}F(A)$, i.e.,

$$S_{\mathcal{W}} = \mathbb{C}[A, z^p, \xi]. \quad (29)$$

Remark 2 *Condition (G) is equivalent to*

(G') *There does not exist $n \mid p$, $0 < n < p$, and $H(s) \in \mathbb{C}[s]$ of degree n such that $F(s) = \prod_{i=0}^{p/n} H(s - in)$;*

and if it is not satisfied, i.e., if $F(s) = \prod_{i=0}^{p/n} H(s - in)$ for some $n \mid p$ and H , then taking such (n, H) of the smallest n , we observe from our proof below that $\mathbb{C}[A, z^p, \xi] \subset S_{\mathcal{W}} \subset \mathbb{C}[A, z^n, \xi']$, where $\xi' = z^{-n}H(A)$.

Remark 3 The right-hand side of (29) equals $\sum_{i,j,k \geq 0} \mathbb{C} a^i b^j c^k$, where (a, b, c) is any permutation of (A, z^p, ξ) ; the order does not matter because

$$[A, z^p] = pz^p, \quad [A, \xi] = -pF(A) \quad \text{and} \quad [z^p, \xi] = F(A) - F(A - p). \quad (30)$$

Remark 4 Condition (G) is satisfied by the F in Theorem 3: Since

$$F(s) = \left(\prod_{i=0}^{p-2} (s - i) \right) (s - (p - 1 + cp))$$

and $-1 < c < 0$, there is no period less than p in the divisor of F modulo p .

Proof of Prop. 3. Using the commutation relations (30), the definition of $\mathcal{W}$, and Eq. (28), we observe easily that $S_{\mathcal{W}} \supset \mathbb{C}[A, z^p, \xi]$. We prove the converse inclusion in two steps. Only Step 2 needs Condition (G).

Step 1. We observe that $S_{\mathcal{W}}$ is spanned by the z -homogeneous elements in $S_{\mathcal{W}}$, i.e., the elements of $S_{\mathcal{W}}$ of the form $z^n f(A)$, where $n \in \mathbb{Z}$ and $f(s) \in \mathbb{C}[s]$.

Indeed, let $S' \subset S_{\mathcal{W}}$ be the subspace of $S_{\mathcal{W}}$ spanned by the z -homogeneous elements, and suppose that $S'' := S_{\mathcal{W}} \setminus S' \neq \emptyset$. Let N be a nonnegative integer such that

$$S''^{(N)} := \{C \in S'' \mid \text{ord}_{\partial/\partial z} C \leq N\}$$

is nonempty. Let $C \in S''^{(N)}$ be such that, writing

$$C = \sum z^n f_n(A), \quad (31)$$

$n_0(C) := \max\{n \mid f_n \neq 0\}$ is the smallest in $S''^{(N)}$. Such a C exists because

Claim 1 $\{n_0(C) \mid C \in S''^{(N)}\}$ is bounded below.

Proof: Indeed it is bounded from below by $-2N + 1$: since $C \in S''^{(N)}$ is an ordinary differential operator of order $\leq N$, and since $\psi_0, A\psi_0, \dots, A^{N-1}\psi_0$ are linearly independent, we have $CA^i\psi_0 \neq 0$ for some i , $0 \leq i < N$. Since $A^i\psi_0 = (-1)^i z^i (1 + O(z^{-1}))$, since $C\mathcal{W} \subset \mathcal{W}$, and since $\mathcal{W}$ is a span of $A^j\psi_0$ for $j \geq 0$, we observe that C does not decrease the order of $A^i\psi_0$ in z by more than $N - 1$. This implies, using the notation of (31), that $n + \deg f_n \geq -(N - 1)$ for some n . Hence $n_0(C) \geq n \geq -\deg f_n - (N - 1) \geq -(2N - 1)$. $\blacksquare$

Now let

$$C' := [A, C] - n_0(C)C = \sum (n - n_0(C)) z^n f_n(A). \quad (32)$$

Clearly $C' \in S_{\mathcal{W}}$. We have $\text{ord}_{\partial/\partial z} C' \leq \text{ord}_{\partial/\partial z} C \leq N$, and $n_0(C') \leq n_0(C) - 1$. Hence by the minimality of $n_0(C)$, we must have $C' \notin S''^{(N)}$, so that $C' \in S'$. Thus each term $(n - n_0(C)) z^n f_n(A)$ in (32) belongs to S' , and only finitely many f_n are non-zero. As a finite linear combination of such, we have $C'' := C - z^{n_0(C)} f_{n_0(C)}(A) \in S'$, so that $z^{n_0(C)} f_{n_0(C)}(A) = C - C''$ must also belong to $S_{\mathcal{W}}$, and hence to S' , since it is z -homogeneous. This implies $C = C'' + (C - C'') \in S'$, which is a contradiction.

Step 2. Let $f(s) \neq 0$ be any constant coefficient polynomial, and let n be an integer. We prove that

$$z^n f(A) \in S_{\mathcal{W}} \quad \text{implies} \quad p \mid n,$$

and that, when $n < 0$, $z^n f(A) \in S_{\mathcal{W}}$ must have the form $\xi^k h(A)$ for $k := -n/p > 0$ and some $h(s) \in \mathbb{C}[s]$.

Suppose $z^n f(A) \in S_{\mathcal{W}}$. We assume $n \neq 0$ without loss of generality. Since $z^n f(A)\psi_0 \in \mathcal{W}$, by Lemma 3 there exists another polynomial $g(s) \in \mathbb{C}[s]$, such that

$$z^n f(A)\psi_0 = g(A)\psi_0. \quad (33)$$

First assume $n > 0$. Let $\ell > 0$ be the least common multiple of p and n . Noting

$$z^{2p}\psi_0 = z^p F(A)\psi_0 = F(A-p)z^p\psi_0 = F(A-p)F(A)\psi_0$$

etc., we have

$$\left(\prod_{i=0}^{\ell/p-1} F(A-ip) \right) \psi_0 = z^\ell \psi_0 \quad (34)$$

from (28), and

$$\left(\prod_{j=0}^{\ell/n-1} G(A-jn) \right) \psi_0 = z^\ell \psi_0 \quad (35)$$

from (33), where $G(s) = g(s)/f(s-n)$ is a rational function in s , and $G(A-jn)$ in (35) is understood as an element of the field of fractions of $\mathbb{C}[A]$; this makes sense because, since $\{A^n \psi_0\}_{n=0,1,\dots}$ is linearly independent, the representation

$$\mathbb{C}[s] \ni f(s) \mapsto f(A)\psi_0 \in \mathcal{W}$$

is faithful.

Comparing the left-hand sides of (34) and (35), we thus have the equality

$$\prod_{i=0}^{\ell/p-1} F(s-ip) = \prod_{j=0}^{\ell/n-1} G(s-jn) \quad (36)$$

of rational functions in s . Since the left-hand side of it is a polynomial of s , so is the right-hand side. Let D be the divisor of this polynomial, and let π_ℓ be the natural map $\mathbb{C} \rightarrow \mathbb{C}/\ell\mathbb{Z}$. From the left- (resp. right-) hand side of (36) the image $\pi_\ell(D)$ of divisor D on the cylinder $\mathbb{C}/\ell\mathbb{Z}$ is invariant under the translation by p (resp. n). But the genericity condition (G) implies that if $\pi_\ell(D)$ is invariant under the translation by $k \in \mathbb{Z}$, then $p \mid k$. Hence $p \mid n$.

Note here that, since ℓ is the least common multiple of p and n , this implies $\ell = n$, so that the right-hand side of (36) is $G(s)$ itself. Hence

$$g(s)/f(s-n) = G(s) = \prod_{i=0}^{n/p-1} F(s-ip).$$

In particular, $g(s)/f(s-n)$ is a polynomial.

In the case where $n < 0$, after rewriting (33) as

$$z^{-n}g(A)\psi_0 = f(A)\psi_0,$$

we switch the roles of f and g , and n and $-n$, to proceed exactly the same way to prove $p \mid n$ and

$$f(s)/g(s+n) = \prod_{i=0}^{-n/p-1} F(s-ip).$$

Thus we have

$$\begin{aligned} z^n f(A) &= z^n \left(\prod_{i=0}^{-n/p-1} F(A-ip) \right) g(A+n) \\ &= (z^{-p} F(A))^{-n/p} g(A+n) \\ &= \xi^k g(A+n) =: \xi^k h(A), \end{aligned}$$

proving the last assertion of Step 2, and hence completing the proof of Prop. 3.

2.2 Symmetric functions and matrix integrals

In this subsection, we prove a number of lemmas regarding symmetric functions.

Lemma 4 *Let s and N be positive integers. Let $F(x^{(1)}, \dots, x^{(s)})$ be a function which is symmetric in each $x^{(r)} := (x_1^{(r)}, \dots, x_N^{(r)}) \in \mathbb{C}^N$, $r = 1, \dots, s$; let $f_1, \dots, f_s$ be functions of two variables, and let $B(x^{(s)})$ be a skew-symmetric function of $x^{(s)}$. If $C_1, \dots, C_s$ denote s fixed contours in $\mathbb{C}$, then the integral*

$$\begin{aligned} \Phi(x^{(0)}) &:= \int \cdots \int_{(C_1)^N \times \cdots \times (C_s)^N} \prod_{r=1}^s \prod_{i=1}^N dx_i^{(r)} \cdot \\ &\quad \cdot F(x^{(1)}, \dots, x^{(s)}) B(x^{(s)}) \prod_{r=1}^s \det \left(f_r(x_i^{(r-1)}, x_j^{(r)}) \right)_{1 \leq i, j \leq N}, \end{aligned}$$

where $x^{(0)} \in \mathbb{C}^N$ comes in as the first argument of f_1 , is skew-symmetric in $x^{(0)}$, and

$$\begin{aligned} \Phi(x^{(0)}) &= (N!)^s \int \cdots \int_{(C_1)^N \times \cdots \times (C_s)^N} \prod_{r,i} dx_i^{(r)} \cdot \\ &\quad \cdot F(x^{(1)}, \dots, x^{(s)}) B(x^{(s)}) \prod_{r=1}^s \prod_{i=1}^N f_r(x_i^{(r-1)}, x_i^{(r)}). \end{aligned}$$

Proof: For any (good) functions $A = A(x^{(1)}, \dots, x^{(s)})$ and $h = h(x^{(1)}, \dots, x^{(s)})$, let

$$\langle Ah \rangle := \int \cdots \int_{(C_1)^N \times \cdots \times (C_s)^N} \prod_{r,i} dx_i^{(r)} \cdot A(x^{(1)}, \dots, x^{(s)}) h(x^{(1)}, \dots, x^{(s)}).$$

For any $\sigma_r \in \mathfrak{S}_N$, let $x_{\sigma_r}^{(r)} := (x_{\sigma_r,1}^{(r)}, \dots, x_{\sigma_r,N}^{(r)})$, and $h^{(\sigma_1, \dots, \sigma_s)}(x^{(1)}, \dots, x^{(s)}) := h(x_{\sigma_1}^{(1)}, \dots, x_{\sigma_s}^{(s)})$. Clearly $\langle Ah \rangle = \langle A^{(\sigma_1, \dots, \sigma_s)} h^{(\sigma_1, \dots, \sigma_s)} \rangle$. If, moreover, A is symmetric in each of $x^{(1)}, \dots, x^{(s-1)}$, and skew-symmetric in $x^{(s)}$, i.e., $A^{(\sigma_1, \dots, \sigma_s)} = (-1)^{\varepsilon(\sigma_s)} A$, then we have

$$\langle Ah \rangle = \langle A^{(\sigma_1, \dots, \sigma_s)} h^{(\sigma_1, \dots, \sigma_s)} \rangle = (-1)^{\varepsilon(\sigma_s)} \langle Ah^{(\sigma_1, \dots, \sigma_s)} \rangle \quad \forall \sigma_r \in \mathfrak{S}_N.$$

Applying this to $h(x^{(1)}, \dots, x^{(s)}) := \prod_r \prod_i f_r(x_i^{(r-1)}, x_i^{(r)})$, and summing it up over $(\sigma_1, \dots, \sigma_s) \in (\mathfrak{S}_N)^s$, we obtain

$$\begin{aligned} &(N!)^s \left\langle A \prod_r \prod_i f_r(x_i^{(r-1)}, x_i^{(r)}) \right\rangle \\ &= \left\langle A \sum_{\sigma_1, \dots, \sigma_s} (-1)^{\varepsilon(\sigma_s)} \prod_r \prod_i f_r(x_{\sigma_{r-1}i}^{(r-1)}, x_{\sigma_r i}^{(r)}) \right\rangle, \quad \text{with } \sigma_0 = \text{id}, \\ &= \left\langle A \sum_{\sigma_1, \dots, \sigma_s} \prod_r (-1)^{\varepsilon(\sigma_r) - \varepsilon(\sigma_{r-1})} \prod_i f_r(x_{\sigma_{r-1}i}^{(r-1)}, x_{\sigma_r i}^{(r)}) \right\rangle \\ &= \left\langle A \prod_r \sum_{\sigma \in \mathfrak{S}_N} (-1)^{\varepsilon(\sigma)} \prod_i f_r(x_i^{(r-1)}, x_{\sigma i}^{(r)}) \right\rangle \\ &= \left\langle A \prod_r \det \left(f_r(x_i^{(r-1)}, x_j^{(r)}) \right)_{i,j} \right\rangle. \end{aligned}$$

Setting here $A = F(x^{(1)}, \dots, x^{(s)}) B(x^{(s)})$ proves the identity in Lemma 4. Finally, $\Phi(x^{(0)})$ is skew-symmetric in $x^{(0)}$ since $\det \left(f_1(x_i^{(0)}, x_j^{(1)}) \right)$ is. ■

Lemma 5 (See [19, Lemma 4.2], [17, Eq. (2.21)], [26, Theorem 8.18].) Let

$$\mathcal{W} = \text{span}_{\mathbb{C}}\{\psi_0(z), \psi_1(z), \psi_2(z), \dots\} \in \text{Gr}$$

with functions

$$\psi_k(z) = \sum_{-\infty < j \leq k} a_{j,k} z^j, \quad k = 0, 1, 2, \dots,$$

such that $a_{kk} = 1$ for $k \gg 0$, i.e., $\text{ord}_z \psi_k(z) \leq k$, and $\psi_k(z) = z^k(1 + O(z^{-1}))$ for $k \gg 0$. Let $N > 0$ be any integer such that this condition holds for $k \geq N$. Let $z_1, \dots, z_N$ be formal scalar variables near ∞ . Then the τ -function $\tau(t)$ at

$$t_n := -\frac{1}{n} \sum_{i=1}^N z_i^{-n}, \quad n = 1, 2, \dots, \quad (37)$$

is given by

$$\tau(t) = \frac{\det(\psi_{j-1}(z_i))_{1 \leq i, j \leq N}}{\det(z_i^{j-1})_{1 \leq i, j \leq N}}. \quad (38)$$

Proof: Our proof is based on Kontsevich's idea in [19]; see [17, Sect. 2.3] for a proof using free fermions. To keep the notation simple, let us denote by $(1-z)^{-1}$ and $(-z+1)^{-1}$ the geometric series $\sum_0^\infty z^n$ and $-\sum_{-\infty}^{-1} z^n$, respectively. Let $\delta(z) := (1-z)^{-1} - (-z+1)^{-1} = \sum_{-\infty}^\infty z^n$, which plays the role of delta function, in the sense that

$$\delta(z/y)f(z) = \delta(z/y)f(y), \quad (39)$$

as is obvious by taking $f(z) = z^m$ (see [6]). Let $\sigma := \prod_{i=1}^N (-z_i) = (-1)^N z_1 \dots z_N$. Let $\sigma_i := 1/\prod_{j(\neq i)} (1 - z_i/z_j)$, $i = 1, \dots, N$, understood as rational functions of z_j 's, so that we have the following identity of formal power series in z :

$$\prod_{i=1}^N \left(1 - \frac{z}{z_i}\right)^{-1} = \sum_{i=1}^N \sigma_i \left(1 - \frac{z}{z_i}\right)^{-1}.$$

From (37) we have

$$\begin{aligned} g := \exp\left(-\sum_{n=1}^\infty t_n z^n\right) &= \prod_{i=1}^N \left(1 - \frac{z}{z_i}\right)^{-1} = \sum_{i=1}^N \sigma_i \left(1 - \frac{z}{z_i}\right)^{-1} \\ &= \sum_{i=1}^N \sigma_i \delta(z/z_i) + \sum_{i=1}^N \sigma_i \left(-\frac{z}{z_i} + 1\right)^{-1} \\ &= \sum_{i=1}^N \sigma_i \delta(z/z_i) + \prod_{i=1}^N \left(-\frac{z}{z_i} + 1\right)^{-1}, \end{aligned}$$

so that by using (39), we have

$$\begin{aligned} g\psi_j(z) &= \sum_{i=1}^N \sigma_i \delta(z/z_i) \psi_j(z) + \left(\prod_{i=1}^N \left(-\frac{z}{z_i} + 1\right)^{-1}\right) \psi_j(z) \\ &= \sum_{i=1}^N \sigma_i \delta(z/z_i) \psi_j(z_i) + z^{-N} (\sigma + O(z^{-1})) \psi_j(z). \end{aligned}$$

Denoting by B the matrix of the composite map in (21) with respect to the bases $\{\psi_j\}_{j=0}^\infty$ and $\{z^k\}_{k=0}^\infty$, we have thus $B = B^0 + B^1$, where

$$B^0 = \begin{pmatrix} 1 & \cdots & 1 \\ z_1^{-1} & \cdots & z_N^{-1} \\ z_1^{-2} & \cdots & z_N^{-2} \\ \vdots & \cdots & \vdots \end{pmatrix} S_N \begin{pmatrix} \psi_0(z_1) & \psi_1(z_1) & \cdots \\ \vdots & \vdots & \vdots \\ \psi_0(z_N) & \psi_1(z_N) & \cdots \end{pmatrix},$$

$$B^1 = \left(\begin{array}{ccc|c} \cdots & 0 & 0 & \overbrace{0 \cdots 0}^N \sigma \quad * \\ \cdots & 0 & 0 & 0 \quad \sigma \\ \cdots & \vdots & \vdots & \vdots \quad \ddots \quad \ddots \\ \cdots & \vdots & \vdots & \vdots \quad \cdots \quad \ddots \end{array} \right) \left(\begin{array}{cc|c} \vdots & \vdots & \cdots \\ a_{-2,0} & a_{-2,1} & \cdots \\ a_{-1,0} & a_{-1,1} & \cdots \\ \hline a_{00} & a_{01} & \cdots \\ 0 & a_{11} & \cdots \\ 0 & 0 & \ddots \\ \vdots & \vdots & \ddots \end{array} \right),$$

S_N is the diagonal matrix $\text{diag}(\sigma_1, \dots, \sigma_N)$, and a_{kj} , $-\infty < k < \infty$, $0 \leq j < \infty$, are the Laurent coefficients of $\psi_j = \sum_k a_{kj} z^k$.

Let us apply some column operations on B . Adding an appropriate linear combination of first N columns to the $(N+i)^{\text{th}}$ column ($i > 0$), we can eliminate the column ${}^t(\psi_{N+i}(z_1), \dots, \psi_{N+i}(z_N))$, $i > 0$, from B^0 . Since N is large enough so that $a_{jj} = 1$ for $j \geq N$, B^1 has the form

$$\left(\begin{array}{c|ccc} & \sigma & & * \\ & & \sigma & \\ O_{\infty \times N} & & & \ddots \\ & 0 & & \end{array} \right),$$

so that the “*” part can be eliminated by further column operations on columns $N+1, N+2, \dots$, which do not alter the B^0 -part. Here $O_{m \times n}$ is the $m \times n$ zero matrix. The matrix B can thus be reduced to $B' = B^0 + B'^1$, where

$$B^0 = \left(\begin{array}{ccc} 1 & \cdots & 1 \\ z_1^{-1} & \cdots & z_N^{-1} \\ z_1^{-2} & \cdots & z_N^{-2} \\ \vdots & \cdots & \vdots \end{array} \right) S_N \left(\begin{array}{ccc|c} \psi_0(z_1) & \cdots & \psi_N(z_1) & \\ \vdots & \vdots & \vdots & \\ \psi_0(z_N) & \cdots & \psi_N(z_N) & \end{array} \middle| O_{N \times \infty} \right),$$

$$B'^1 = \left(O_{\infty \times N} \middle| \sigma I_{\infty} \right).$$

Let $n, n \geq N$, be an integer. Note that the column operations needed to bring B into B' only adds linear combinations of lower numbered columns to higher ones. Hence, denoting by B_n, B'_n, B_n^0 and $B_n'^1$ the matrices of the first n rows and columns in B, B', B^0 and B'^1 , respectively, we have $\det B_n = \det B'_n = \det(B_n^0 + B_n'^1)$, with

$$B_n^0 = \left(\begin{array}{ccc} 1 & \cdots & 1 \\ z_1^{-1} & \cdots & z_N^{-1} \\ \vdots & \cdots & \vdots \\ z_1^{-n+1} & \cdots & z_N^{-n+1} \end{array} \right) S_N \left(\begin{array}{ccc|c} \psi_0(z_1) & \cdots & \psi_N(z_1) & \\ \vdots & \vdots & \vdots & \\ \psi_0(z_N) & \cdots & \psi_N(z_N) & \end{array} \middle| O_{N \times (n-N)} \right),$$

and

$$B_n'^1 = \left(\begin{array}{c|c} O_{(n-N) \times N} & \sigma I_{n-N} \\ \hline O_{N \times N} & O_{N \times (n-N)} \end{array} \right).$$

Since the last $n-N$ columns of B_n^0 are 0, we have

$$B_n' = \left(\begin{array}{c|c} * & \sigma I_{n-N} \\ \hline Z & O_{N \times (n-N)} \end{array} \right),$$

where Z consists of the last N rows and the first N columns of B_n^0 :

$$Z = \left(\begin{array}{ccc} z_1^{-n+N} & \cdots & z_N^{-n+N} \\ \vdots & \cdots & \vdots \\ z_1^{-n+1} & \cdots & z_N^{-n+1} \end{array} \right) S_N \left(\begin{array}{ccc} \psi_0(z_1) & \cdots & \psi_N(z_1) \\ \vdots & \vdots & \vdots \\ \psi_0(z_N) & \cdots & \psi_N(z_N) \end{array} \right).$$

Hence we have, using $\sigma = (-1)^N z_1 \dots z_N$,

$$\begin{aligned} \det B_n = \det B'_n &= (-1)^{N(n-N)} \det Z \det(\sigma I_{n-N}) \\ &= (z_1 \dots z_N)^{n-N} \det Z \\ &= (z_1 \dots z_N)^{1-N} \det Z', \end{aligned}$$

where

$$Z' = \begin{pmatrix} z_1^{N-1} & \dots & z_N^{N-1} \\ \vdots & \dots & \vdots \\ z_1^1 & \dots & z_N^1 \\ 1 & \dots & 1 \end{pmatrix} S_N \begin{pmatrix} \psi_0(z_1) & \dots & \psi_N(z_1) \\ \vdots & \vdots & \vdots \\ \psi_0(z_N) & \dots & \psi_N(z_N) \end{pmatrix}.$$

Noticing

$$\det(z_j^{N-i})_{1 \leq i, j \leq N} = (-1)^{N(N-1)/2} \det(z_j^{i-1})_{1 \leq i, j \leq N},$$

and

$$\det S_N = \prod_1^N \sigma_i = \frac{\left(\prod_{j=1}^N z_j\right)^{N-1}}{\prod_{i,j \neq i} (z_j - z_i)} = \frac{(z_1 \dots z_N)^{N-1}}{(-1)^{N(N-1)/2} \det(z_j^{i-1})_{1 \leq i, j \leq N}^2},$$

we observe that $\det B_n$ coincides with the right-hand side of (38). Since $n \geq N$ is arbitrary, this completes the proof of Lemma 5. $\blacksquare$

Lemma 6 *Let $Z := \text{diag}(z_1, \dots, z_N)$. Let $\lambda := ((p-1)(N-1), (p-1)(N-2), \dots, p-1)$. For a polynomial $f(y, z)$, let us denote by $(f(y, z))_2$ the terms in $f(y, z)$ which are quadratic in y . Then we have⁴*

$$\begin{aligned} \frac{\Delta(z^p)}{\Delta(z)} &= F_\lambda \left(-\text{tr } Z, -\frac{1}{2} \text{tr } Z^2, -\frac{1}{3} \text{tr } Z^3, \dots \right) \\ &= c \prod_i z_i^{-\frac{p-1}{2}} \left(\int_{\mathcal{H}_N} dY \exp \text{tr} \left(-\frac{(Y+Z)^{p+1}}{p+1} \right)_2 \right)^{-1}, \end{aligned}$$

where c is a non-zero constant which depends only on N and p .

Proof: The Schur function associated with the partition λ is given by (see [21])

$$F_\lambda \left(-\sum_1^N y_i, -\frac{1}{2} \sum_1^N y_i^2, -\frac{1}{3} \sum_1^N y_i^3, \dots \right) := \frac{\Delta_{\lambda+\delta}(y)}{\Delta_\delta(y)},$$

where $\delta = (N-1 > N-2 > \dots > 1 > 0)$ and $\Delta_\mu(y) = \det(y_i^{\mu_j})_{1 \leq i, j \leq N}$. Therefore we have, with $\lambda + \delta = (p(N-1) > p(N-2) > \dots > p > 0)$,

$$\frac{\Delta(z^p)}{\Delta(z)} = \frac{\Delta_{\lambda+\delta}(z)}{\Delta_\delta(z)} = F_\lambda \left(-\sum_1^N z_i, -\frac{1}{2} \sum_1^N z_i^2, -\frac{1}{3} \sum_1^N z_i^3, \dots \right),$$

establishing the first equality of Lemma 6. In order to establish the second one, note

$$\begin{aligned} \text{tr} \left(\frac{(Y+Z)^{p+1}}{p+1} \right)_2 &= \frac{1}{2} \text{tr}(Y^2 Z^{p-1} + Y Z Y Z^{p-2} + \dots + Y Z^{p-1} Y) \\ &= \frac{1}{2} \sum_{i,j} Y_{ij} Y_{ji} (z_i^{p-1} + z_i^{p-2} z_j + \dots + z_j^{p-1}) \\ &= \frac{1}{2} \sum_{i,j} Y_{ij} Y_{ji} \left(\frac{z_i^p - z_j^p}{z_i - z_j} \right). \end{aligned}$$

⁴ F_λ is the Schur function for the partition λ .

Hence, performing a Gaussian integration, we find

$$\begin{aligned}
\int dY \exp \operatorname{tr} \left(-\frac{(Y+Z)^{p+1}}{p+1} \right)_2 &= \int dY \exp \left(-\frac{1}{2} \sum_{i,j} Y_{ij} Y_{ji} \frac{z_i^p - z_j^p}{z_i - z_j} \right) \\
&= (2\pi)^{N^2/2} \left(\prod_{1 \leq i,j \leq N} \frac{z_i - z_j}{z_i^p - z_j^p} \right)^{1/2} \\
&= \frac{(2\pi)^{N^2/2}}{p^{N/2}} \prod_{1 \leq i < j \leq N} \frac{z_i - z_j}{z_i^p - z_j^p} \prod_1^p z_i^{-\frac{p-1}{2}} \\
&= \frac{(2\pi)^{N^2/2}}{p^{N/2}} \frac{\Delta(z)}{\Delta(z^p)} \prod_1^N z_i^{-\frac{p-1}{2}},
\end{aligned}$$

establishing Lemma 6. ■

Remark 5 *In general we have*

$$\int_{\mathcal{H}} dY e^{-\operatorname{tr}(V(Y+Z))_2} = (2\pi)^{N^2/2} \frac{\Delta(z)}{\Delta(V'(z))} \frac{1}{\sqrt{\prod_1^N V''(z_i)}}.$$

The following lemma is due to Harish Chandra, Bessis–Itzykson–Zuber and Duistermaat–Heckman among others:

Lemma 7 *Given $N \times N$ -diagonal matrices X and Y , we have*

$$\int_{\mathbf{U}(N)} e^{\operatorname{tr} XUYU^\dagger} dU = (2\pi)^{\frac{N(N-1)}{2}} \frac{\det(e^{x_i y_j})_{1 \leq i,j \leq N}}{\Delta(X)\Delta(Y)}.$$

A proof can be found in [13].

3 Matrix Fourier Transforms

In this section we explain how generalized Kontsevich integrals (see [19, 1, 24]) are closely related to the theory of Fourier transforms. Indeed, if $V(x)$ grows sufficiently at infinity, any *Fourier transform*

$$a(y) = \int_{-\infty}^{\infty} e^{-V(x)+xy} dx \tag{40}$$

leads to a linear space of functions $\mathcal{W}$ invariant under two operators A and $V'(z)$ satisfying $[A, V'(z)] = 1$.

(i) The point is that $a(y)$ satisfies the differential equation

$$V' \left(\frac{\partial}{\partial y} \right) a(y) = ya(y), \tag{41}$$

as seen from

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty} \frac{\partial}{\partial x} e^{-V(x)+xy} dx = \int_{-\infty}^{\infty} (-V'(x) + y) e^{-V(x)+xy} dx \\
&= \left(-V' \left(\frac{\partial}{\partial y} \right) + y \right) a(y).
\end{aligned}$$

Thus setting $y = V'(z)$ in (41) and $A_0 := V''(z)^{-1} \partial / \partial z = \partial / \partial y|_{y=V'(z)}$, the function $a(V'(z))$ satisfies the differential equation

$$V'(A_0) a(V'(z)) = V'(z) a(V'(z)). \tag{42}$$

(ii) The method of stationary phase applied to integrals (40) and their derivatives leads to the following estimate, upon Taylor expanding $V(x)$ around $x = z$,

$$\begin{aligned}
& \left(\frac{\partial}{\partial y} \right)^n a(y) \Big|_{y=V'(z)} \\
&= \int_{-\infty}^{\infty} x^n e^{-V(x)+xV'(z)} dx \\
&= \int_{-\infty}^{\infty} x^n e^{-(V(z)+(x-z)V'(z)+(1/2)(x-z)^2V''(z)+O(x-z)^3)+xV'(z)} dx \\
&= e^{-V(z)+zV'(z)} \int_{-\infty}^{\infty} x^n e^{-(1/2)(x-z)^2V''(z)(1+(V'''/V'')O(x-z))} dx \\
&= e^{-V(z)+zV'(z)} \frac{1}{\sqrt{V''}} \left(\int_{-\infty}^{\infty} \left(\frac{y}{\sqrt{V''}} + z \right)^n e^{-y^2/2} dy + O(1/z) \right) \\
&= \rho(z)^{-1} z^n (1 + O(1/z)), \tag{43}
\end{aligned}$$

with

$$\rho(z) = \frac{1}{\sqrt{2\pi}} e^{V(z)-zV'(z)} \sqrt{V''(z)}.$$

Therefore defining

$$A := \rho(z) \frac{\partial}{\partial y} \Big|_{y=V'(z)} \circ \rho(z)^{-1}$$

and

$$\psi_n(z) := A^n \psi_0(z) := \rho(z) \frac{\partial^n}{\partial y^n} a(y) \Big|_{y=V'(z)}, \quad n = 0, 1, \dots,$$

the differential equation (42) implies

$$V'(A)\psi_0(z) = V'(z)\psi_0(z).$$

This, combined with (43), proves that the linear span

$$\mathcal{W} := \text{span}_{\mathbb{C}} \{ \psi_k(z) = z^k (1 + O(1/z)) \mid k = 0, 1, 2, \dots \}$$

is invariant under the operators A and $V'(z)$, i.e.,

$$A\mathcal{W} \subset \mathcal{W} \quad \text{and} \quad V'(z)\mathcal{W} \subset \mathcal{W}, \quad \text{with} \quad [A, V'(z)] = 1.$$

(iii) By Lemma 5, the τ -function corresponding to $\mathcal{W}$, at time t as in (37), is given by

$$\begin{aligned}
\tau(t) &= \frac{\det(A^{j-1}\psi_0(z_i))_{1 \leq i, j \leq N}}{\det(z_i^{j-1})_{1 \leq i, j \leq N}} \\
&= \frac{1}{\Delta(z)} \det \left(\rho(z_i) \left(\frac{\partial}{\partial y} \right)^{j-1} \int_{-\infty}^{\infty} e^{-V(x)+xy} dx \Big|_{y=V'(z_i)} \right)_{1 \leq i, j \leq N} \\
&= \frac{\prod_1^N \rho(z_i)}{\Delta(z)} \int_{\mathbb{R}^N} dx e^{-\sum_1^N V(x_i)} \Delta(x) \prod_1^N e^{x_\alpha V'(z_\alpha)} \\
&= \frac{\prod_1^N \rho(z_i)}{N! \Delta(z)} \int_{\mathbb{R}^N} dx e^{-\sum_1^N V(x_i)} \Delta(x) \det(e^{x_\alpha V'(z_\beta)})_{1 \leq \alpha, \beta \leq N}, \\
&\quad \text{using Lemma 4 with } s = 1 \text{ and the skew-symmetry of } \Delta(x), \\
&= \frac{\prod_1^N \rho(z_i)}{N! \Delta(z) / \Delta(V'(z))} \int_{\mathbb{R}^N} dx e^{-\sum_1^N V(x_i)} \Delta^2(x) \frac{\det(e^{x_\alpha V'(z_\beta)})_{1 \leq \alpha, \beta \leq N}}{\Delta(x) \Delta(V'(z))}
\end{aligned}$$

$$\begin{aligned}
&= c \frac{\prod_1^N \rho(z_i)}{\Delta(z)/\Delta(V'(z))} \int_{\mathbb{R}^N} dx e^{-\sum_1^N V(x_i)} \Delta^2(x) \int_{\mathbf{U}(N)} dU e^{\text{tr} UXU^{-1}V'(Z)}, \\
&\quad \text{using Lemma 7, with } X = \text{diag}(x), \\
&= c' e^{\text{tr}(V(Z) - ZV'(Z))} \frac{\int_{\mathcal{H}_N} dX e^{-\text{tr} V(X)} e^{\text{tr} XV'(Z)}}{\int_{\mathcal{H}_N} dX e^{-\text{tr}(V(X+Z))_2}}, \quad \text{using Lemma 6,} \\
&= c'' \frac{\int_{\mathcal{H}_N} dY e^{-\text{tr}(V(Y+Z))_{\geq 2}}}{\int_{\mathcal{H}_N} dY e^{-\text{tr}(V(Y+Z))_2}}, \quad \text{upon setting } X = Y + Z,
\end{aligned}$$

for some constants c , c' and c'' depending on N .

4 Generalized Hänkel Functions, Differential Equations and Laplace Transforms

This section deals with the properties of Hänkel functions and their generalizations.

Lemma 8 *The family of integrals*

$$\psi_k(z) = \frac{p^{c+1}}{\Gamma(-c)} \int_1^\infty \frac{z^{-c}(uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du, \quad \begin{aligned} -1 < c < 0, \\ k = 0, 1, \dots, p = 2, 3, \dots \end{aligned} \quad (44)$$

admits, for large $z > 0$, an asymptotic expansion in $\mathbb{C}((z^{-1}))$ of the form

$$\psi_k(z) = z^k (1 + O(1/z)), \quad (45)$$

with $\psi_0(z)$ satisfying the differential equation

$$e^z z^{-c} \left(\prod_{i=0}^{p-1} \left(z \frac{\partial}{\partial z} - i \right) - cp \prod_{i=0}^{p-2} \left(z \frac{\partial}{\partial z} - i \right) \right) z^c e^{-z} \psi_0(z) = (-z)^p \psi_0(z), \quad (8)$$

or equivalently

$$e^z z^{-c} \left(z^p \left(\frac{\partial}{\partial z} \right)^p - cp z^{p-1} \left(\frac{\partial}{\partial z} \right)^{p-1} \right) z^c e^{-z} \psi_0(z) = (-z)^p \psi_0(z). \quad (8')$$

Moreover $\psi_k(z)$ admits the following representation in terms of a double integral⁵

$$\begin{aligned}
\psi_k(z) &= \frac{p^{c+1}}{2\pi i} z^{(p-1)(c+1)} \int_{\gamma} dw \int_0^\infty dx e^{z-w} w^k x^c e^{x(w^p - z^p)} \\
&= \frac{p^{c+1}}{2\pi i} z^{(p-1)(c+1)} e^z \int_0^\infty dx x^c e^{-xz^p} \int_0^\infty dy f_k(y) e^{-xy^p}, \quad (46)
\end{aligned}$$

where, in the first integral, $\gamma := \gamma^+ + \gamma^- \subset \mathbb{C}$ denotes the contour consisting of two half-lines $\gamma^\pm = \mathbb{R}_+ \zeta^{\pm 1}$, $\zeta := e^{\pi i/p}$, through the origin making an angle $\pm \pi/p$ with the positive real axis, with the orientation given as to go from $\zeta^{-1} \cdot \infty$ to 0 to $\zeta \cdot \infty$ (see Fig. 1 (a)), and where in the second integral,

$$f_k(y) = (\zeta^{k+1} e^{-\zeta y} - \zeta^{-k-1} e^{-\zeta^{-1} y}) y^k = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} a_{j+k+1} y^{j+k},$$

where $a_n = \zeta^n - \zeta^{-n} = 2i \sin(n\pi/p)$.

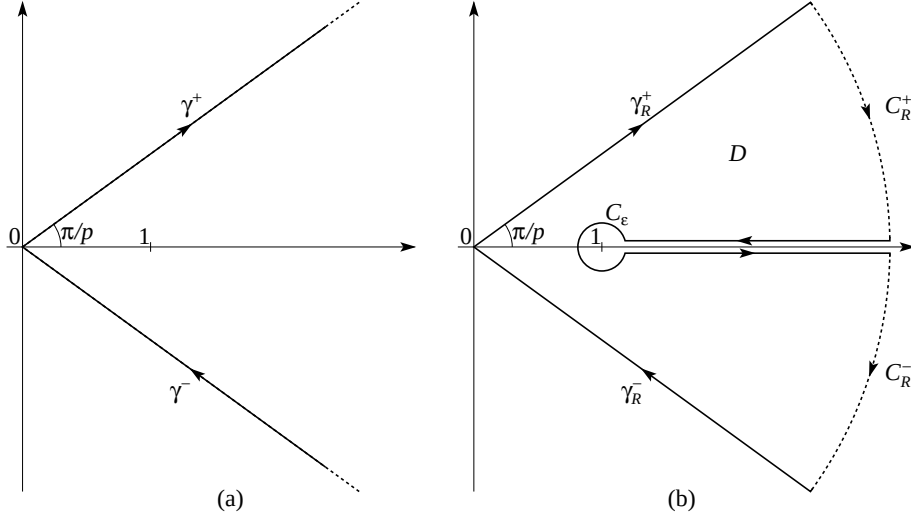


Figure 1: Contours of integration: (a) contour γ for (46); (b) closed contour for (48)

Proof: Setting $v = (u - 1)z$, and using

$$\Gamma(-c) = \int_0^\infty \frac{e^{-v}}{v^{c+1}} dv \quad \text{for } c < 0,$$

we first observe that for each $n \geq 0$,

$$\begin{aligned} \psi_k(z) &= \frac{p^{c+1} z^k}{\Gamma(-c)} \int_0^\infty \frac{(1 + v/z)^k e^{-v}}{v^{c+1} p^{c+1} \left(1 + \frac{1}{p} \left(\sum_{i=2}^p \binom{p}{i} (v/z)^{i-1}\right)\right)^{c+1}} dv \\ &= z^k \left(1 + \tilde{b}_{k,1} z^{-1} + \dots + \tilde{b}_{k,n} z^{-n} + O(1/z^{n+1})\right) \end{aligned}$$

as $z \rightarrow \infty$, where the $\tilde{b}_{k,i} := (\Gamma(-c + i)/\Gamma(-c)) b_{k,i} = \left(\prod_{j=0}^{i-1} (-c + j)\right) b_{k,i}$ are obtained from the coefficients $b_{k,i}$ of the expansion⁶

$$\frac{(1 + s)^k}{\left(1 + \frac{1}{p} \left(\sum_{i=2}^p \binom{p}{i} s^{i-1}\right)\right)^{c+1}} = 1 + \sum_{i=1}^\infty b_{k,i} s^i, \quad (47)$$

confirming the asymptotic expansion (45).

Moreover, setting

$$\varphi_0(z) = \int_1^\infty \frac{z^{-c} e^{-uz}}{(u^p - 1)^{c+1}} du,$$

we have for $c < 0$ and $\text{Re } z > 0$,

$$\begin{aligned} 0 &= -z^{p-1-c} \frac{e^{-uz}}{(u^p - 1)^c} \Big|_{u=1}^{u=\infty} \\ &= -z^{p-1} \int_1^\infty \frac{\partial}{\partial u} \left((u^p - 1) \frac{z^{-c} e^{-uz}}{(u^p - 1)^{c+1}} \right) du \end{aligned}$$

⁵ If $p = 2$, so that γ becomes the imaginary axis, these integrals should be interpreted by replacing ζ by $\zeta_\varepsilon = e^{(\pi i/2) - \varepsilon}$, and γ by $\mathbb{R}_+ \zeta_\varepsilon + \mathbb{R}_+ \zeta_\varepsilon^{-1}$, and then taking the limit as $\varepsilon \downarrow 0$.

⁶ Noting that the radius of convergence of this power series is $|\zeta - 1|$, one can get a precise growth estimate of the coefficients of $\psi_k(z)$ which implies that, in particular, as always with the string equation, $\mathcal{W}$ does not belong to the L^2 -Grassmannian of Segal–Wilson [30].

$$\begin{aligned}
&= (-1)^p \int_1^\infty ((-zu)^p - cp(-zu)^{p-1} - (-z)^p) \frac{z^{-c} e^{-uz}}{(u^p - 1)^{c+1}} du \\
&= (-1)^p z^{-c} \left(z^p \left(\frac{\partial}{\partial z} \right)^p - cp z^{p-1} \left(\frac{\partial}{\partial z} \right)^{p-1} - (-z)^p \right) z^c \varphi_0(z) \\
&= (-1)^p z^{-c} \left(\prod_{i=0}^{p-1} \left(z \frac{\partial}{\partial z} - i \right) - cp \prod_{i=0}^{p-2} \left(z \frac{\partial}{\partial z} - i \right) - (-z)^p \right) z^c \varphi_0(z),
\end{aligned}$$

using in the last line the operator identity

$$\prod_{i=0}^{p-1} \left(z \frac{\partial}{\partial z} - i \right) = z^p \left(\frac{\partial}{\partial z} \right)^p,$$

thus showing that $\psi_0(z)$ satisfies the differential equation (8) or (8').

Consider a bounded domain $D \subset \mathbb{C}$, whose boundary consists of the lines $\gamma_R^\pm$, making an angle $\pm\pi/p$ with the positive real axis, two circle segments $C_R^\pm$, about the origin, of large enough radius R and a small circle about 1 of radius ε connected to $C_R^\pm$, as in Fig. 1 (b). The function $e^{-uz}/(u^p - 1)^{c+1}$ is univalued in D and all its singularities lie outside D . By Cauchy's theorem we have

$$\left(\int_{\gamma_R^-} + \int_{\gamma_R^+} + \int_{C_R^+} + \int_R^{1+\varepsilon} + \int_{C_\varepsilon} + \int_{1+\varepsilon}^R + \int_{C_R^-} \right) \frac{(uz)^k e^{-(u-1)z}}{(u^p - 1)^c} du = 0. \quad (48)$$

Observe that, for $z > 0$ and $p > 2$, we have $z \cos \theta \geq z \cos(\pi/p) > 0$ for $0 \leq \theta \leq \pi/p$, implying

$$\int_{C_R^\pm} \frac{(uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du = O(R^{k-(c+1)p+1} e^{-Rz \cos(\pi/p)}) \rightarrow 0$$

as $R \uparrow \infty$. Since $c < 0$, we also have

$$\int_{C_\varepsilon} \frac{(uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du = O(\varepsilon^{-c}) \rightarrow 0$$

as $\varepsilon \downarrow 0$. So, taking limits as $\varepsilon \downarrow 0$ and $R \uparrow \infty$ leads to

$$\begin{aligned}
\int_\gamma \frac{(uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du &= - \left(\int_\infty^1 + \int_{1-i0}^{\infty-i0} \right) \frac{(uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du \\
&= (1 - e^{-2\pi i(c+1)}) \int_1^\infty \frac{(uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du \\
&= 2ie^{-\pi ic} \sin \pi c \int_1^\infty \frac{(uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du.
\end{aligned}$$

Note that, since $u^p - 1 < 0$ along γ , we have the following Γ -function representation

$$\frac{1}{(u^p - 1)^{c+1}} = -\frac{e^{-\pi ic}}{\Gamma(c+1)} \int_0^\infty dx x^c e^{x(u^p - 1)},$$

and thus

$$\begin{aligned}
\psi_k(z) &= \frac{p^{c+1}}{\Gamma(-c)} \int_1^\infty \frac{z^{-c} (uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du \\
&= \frac{p^{c+1} e^{\pi ic}}{2i \sin \pi c \Gamma(-c)} z^{-c} \int_\gamma \frac{(uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du \\
&= -\frac{p^{c+1} z^{-c}}{2i \sin \pi c \Gamma(-c) \Gamma(c+1)} \int_\gamma du (uz)^k e^{-(u-1)z} \int_0^\infty dx x^c e^{x(u^p - 1)} \\
&= \frac{p^{c+1}}{2\pi i} z^{-c} \int_\gamma du (uz)^k e^{-(u-1)z} \int_0^\infty dx x^c e^{x(u^p - 1)}, \\
&= \frac{p^{c+1}}{2\pi i} z^{(p-1)(c+1)} \int_\gamma dw \int_0^\infty dx e^{z-w} w^k x^c e^{xw^p} e^{-xz^p},
\end{aligned}$$

upon setting $w = uz$. Here we used the Γ -function duplication, $\Gamma(-c)\Gamma(c+1) = -\pi/\sin \pi c$, $-1 < c < 0$. Working out the integral over γ , interchanging the integrations and using $\zeta^{\pm p} = -1$, we find

$$\begin{aligned}\psi_k(z) &= \frac{p^{c+1}}{2\pi i} z^{(p-1)(c+1)} e^z \int_0^\infty dx x^c e^{-xz^p} \cdot \\ &\quad \cdot \left(\zeta^{-k-1} \int_\infty^0 dy e^{-\zeta^{-1}y} y^k e^{-xy^p} + \zeta^{k+1} \int_0^\infty dy e^{-\zeta y} y^k e^{-xy^p} \right) \\ &= \frac{p^{c+1}}{2\pi i} z^{(p-1)(c+1)} e^z \int_0^\infty dx x^c e^{-xz^p} \int_0^\infty dy f_k(y) e^{-xy^p}\end{aligned}$$

with

$$\begin{aligned}f_k(y) &= (\zeta^{k+1} e^{-\zeta y} - \zeta^{-k-1} e^{-\zeta^{-1}y}) y^k \\ &= \sum_{j=0}^\infty \frac{(-1)^j}{j!} (\zeta^{j+k+1} - \zeta^{-j-k-1}) y^{j+k},\end{aligned}$$

as announced in (46), thus ending the proof of Lemma 8. ■

Lemma 9 *The linear space spanned by the generalized Hankel functions,*

$$\mathcal{W} = \text{span}_{\mathbb{C}} \left\{ \psi_k(z) = \frac{p^{c+1}}{\Gamma(-c)} \int_1^\infty \frac{z^{-c}(uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du \mid k = 0, 1, 2, \dots \right\}$$

is invariant under

$$z^p \quad \text{and} \quad A_c := z^{-c} e^z z \frac{\partial}{\partial z} \circ e^{-z} z^c = z \frac{\partial}{\partial z} - z + c$$

(so that $[(1/p)A, z^p] = z^p$), with ψ_0 satisfying the differential equation (8).

Proof: The space $\mathcal{W}$ is invariant under A_c , because

$$\begin{aligned}A_c \psi_k(z) &= \frac{p^{c+1}}{\Gamma(-c)} z^{-c} e^z z \frac{\partial}{\partial z} z^c e^{-z} \int_1^\infty \frac{z^{-c}(uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du \\ &= \frac{p^{c+1}}{\Gamma(-c)} z^{-c} e^z z \frac{\partial}{\partial z} \int_1^\infty \frac{(uz)^k e^{-uz}}{(u^p - 1)^{c+1}} du \\ &= k \psi_k(z) - \psi_{k+1}.\end{aligned}$$

Moreover, the operator

$$\prod_{i=0}^{p-1} (A_c - i) - cp \prod_{i=0}^{p-2} (A_c - i)$$

has the form $\sum_0^p \alpha_j A_c^j$, with $\alpha_p = 1$. From Lemma 8, the solution to the differential equation

$$\left(\prod_{i=0}^{p-1} (A_c - i) - cp \prod_{i=0}^{p-2} (A_c - i) \right) \psi_0(z) = (-z)^p \psi_0(z)$$

is given by the function in (44) or (46) for $k = 0$. An asymptotic expansion of the form

$$\psi_0(z) = 1 + O(z^{-1})$$

follows from (45). ■

5 Proof of the Main Statements

5.1 Proof of Theorems 3 and 1 and Remark 1

In Lemma 9, we have constructed a space $\mathcal{W}$ and an operator $A = A_c$ such that

$$A\mathcal{W} \subset \mathcal{W} \quad \text{and} \quad z^p\mathcal{W} \subset \mathcal{W},$$

with the lowest order element $\psi_0 \in \mathcal{W}$ satisfying Eq. (8). Proposition 3 and Remark 4 imply that the stabilizer of $\mathcal{W}$ is $\mathbb{C}[A, z^p, z^{-p}F(A)]$, proving Theorem 3, Part (i).

Let Ψ and τ be the wave function and the τ -function, respectively, associated with the KP time evolution $\mathcal{W}^t = e^{-\sum t_i z^i} \mathcal{W}$ of $\mathcal{W}$. We now define the operators P and Q in the x -variable, via the operators A and z^p in the z -variable, by means of

$$z^p\Psi(t, z) = P\Psi(t, z) \quad \text{and} \quad (1/p)A\Psi(t, z) = Q\Psi(t, z).$$

According to Lemma 2, P and Q are differential operators. They satisfy $[P, Q] = P$ since $[(1/p)A, z^p] = z^p$. Note that P and Q can also be written:

$$P = L^p = SD^pS^{-1}$$

and

$$Q = \frac{1}{p}(ML - L + c) = \frac{1}{p}S \left(\sum_1^\infty k\bar{t}_k D^k - D + c \right) S^{-1},$$

where

$$S = \frac{\tau(t - [D^{-1}])}{\tau(t)}$$

in terms of the τ -function above, and L and M are as in (12) and (16), proving Theorem 3, Part (ii).

Since $(M - 1)L = pQ - c$ is a differential operator, we also have, using the notation α_{ij} as in the statement of Theorem 3,

$$\begin{aligned} ((M - 1)L)^m L^{np} &= \sum_{i=1}^m \alpha_{m,i} (M - 1)^i L^{i+np} \\ &= \sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq i}} \alpha_{m,i} \binom{i}{j} (-1)^{i-j} M^j L^{i+np} \end{aligned}$$

is a differential operator. Thus

$$\sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq i}} \alpha_{m,i} \binom{i}{j} (-1)^{i-j} (M^j L^{i+np})_- \Psi = 0,$$

implying (10), upon using (19), completing the proof of Theorem 3.

To prove Remark 1, we evaluate

$$\left(\sum i t_i \frac{\partial}{\partial t_i} - \frac{\partial}{\partial t_1} - a \right) \tau = 0$$

at $t = 0$ to find $-((\partial\tau/\partial t_1)/\tau)|_{t=0} = a$. Remember, on the one hand,

$$\Psi(0, 0, z) = \psi_0(z) = (1 + \tilde{b}_{01}z^{-1} + \dots),$$

and on the other hand

$$\begin{aligned} \Psi(0, 0, z) &= \frac{\tau(t_1 + x - z^{-1}, \dots)}{\tau(t_1 + x, \dots)} \Big|_{x=0, t=0} \\ &= \left(1 - \tau^{-1} \frac{\partial\tau}{\partial x} z^{-1} + \dots \right) \Big|_{t=0}. \end{aligned}$$

Therefore $a = \tilde{b}_{01} = (-c)b_{0,1} = c(1+c)(p-1)/2$ as stated in Remark 1, as implied by (47).

To prove Theorem 1, note that at $t = 0$,

$$\begin{aligned} Q|_{t=0} &= (1/p)S((x-1)(\partial/\partial x) + c)S^{-1} \\ &= (1/p)(x-1)(\partial/\partial x) + c + (\text{negative order terms}). \end{aligned}$$

Since Q must be a differential operator, the negative order terms vanish, and $Q|_{t=0} = (1/p)(x-1)(\partial/\partial x) + c$. Thus, from the second equation in (9), we have

$$0 = (Q|_{t=0} - A_c)\Psi(x, 0, z) = ((x-1)(\partial/\partial x) - z(\partial/\partial z - 1))\Psi(x, 0, z). \quad (49)$$

Since this is a first order equation and the line $x = 0$ is noncharacteristic, $\Psi(x, 0, z)$ is determined by (49) together with the initial condition $\Psi(0, 0, z) = \psi_0(z)$. It is easy to check that the right-hand side of (6) satisfies these conditions. Finally, (7) follows from (8): Writing (8) as

$$F\left(z\frac{\partial}{\partial z} + c\right)(e^{-z}\psi_0(z)) = (-z)^p e^{-z}\psi_0(z),$$

substituting $(1-x)z$ for z , using the scaling invariance of $z\partial/\partial z$, and dividing both sides by z^p , we get

$$\frac{1}{z^p}F\left(z\frac{\partial}{\partial z} + c\right)(e^{(x-1)z}\psi_0((1-x)z)) = (x-1)^p e^{(x-1)z}\psi_0((1-x)z). \quad (50)$$

Multiplying both sides of this formula by e^z , and using the identity $e^z(z\partial/\partial z + c) = (z\partial/\partial z - z + c) \circ e^z$, we get the second formula in (7). Next, switching the roles of z and $1-x$ in (50), we get

$$\frac{1}{(x-1)^p}F\left((x-1)\frac{\partial}{\partial x} + c\right)(e^{(x-1)z}\psi_0((1-x)z)) = z^p e^{(x-1)z}\psi_0((1-x)z).$$

Multiplying both sides of this formula by e^z , and using the fact that e^z commutes with $(x-1)\partial/\partial x + c$, we get the first formula in (7), completing the proof of Theorem 1.

5.2 Proof of Theorem 2

Setting $t_n = -\frac{1}{n} \sum_{i=1}^n z_i^{-n}$, $n = 1, 2, \dots$, and using Lemma 5, and Lemma 4 with $s = 2$, we have

$$\begin{aligned} \tau(t) &= \frac{\det(\psi_{k-1}(z_i))_{1 \leq k, i \leq N}}{\Delta(z)} \\ &= \frac{a^N}{\Delta(z)} \det \left(z_i^{(p-1)(c+1)} e^{z_i} \int_0^\infty dx \int_0^\infty dy x^c e^{-xz_i^p} f_{k-1}(y) e^{-xy^p} \right)_{k,i} \\ &= \frac{a^N S_2(t)}{\Delta(z)} \int_{\mathbb{R}_+^N} dx \int_{\mathbb{R}_+^N} dy \left(\prod_1^N x_i^c \right) \cdot \det(f_{k-1}(y_i))_{k,i} e^{-\sum_1^N x_i z_i^p} e^{-\sum_1^N x_i y_i^p} \\ &= \frac{a^N S_2(t)}{(N!)^2 \Delta(z)} \int_{\mathbb{R}_+^N} dx \int_{\mathbb{R}_+^N} dy \left(\prod_1^N x_i^c \right) \cdot \det(f_{k-1}(y_i))_{k,i} \det \left(e^{-x_i z_j^p} \right)_{i,j} \det \left(e^{-x_i y_j^p} \right)_{i,j} \\ &= \frac{a^N S_2(t) \Delta(z^p)}{(N!)^2 \Delta(z)} \int_{\mathbb{R}_+^N} dx \int_{\mathbb{R}_+^N} dy \left(\prod_1^N x_i^c \right) \Delta(x)^2 \Delta(y)^2 \cdot \frac{\det \left(e^{-x_i z_j^p} \right)_{i,j}}{\Delta(x) \Delta(z^p)} \frac{\det \left(e^{-x_i y_j^p} \right)_{i,j}}{\Delta(x) \Delta(y^p)}, \end{aligned}$$

where $a = p^{c+1}/2\pi i$,

$$S_2(t) = \prod_1^N \left(z_i^{(p-1)(c+1)} e^{z_i} \right),$$

and

$$S_0(y_1, y_2, \dots, y_N) = \frac{\Delta(y^p)}{\Delta(y)} \frac{\det(f_{k-1}(y_i))_{1 \leq i, k \leq N}}{\Delta(y)}.$$

So we have, for some constants C , C' and C'' depending on N , p and c ,

$$\begin{aligned} \tau(t) &= C \frac{S_2(t)\Delta(z^p)}{\Delta(z)} \int_{\mathbb{R}_+^N} dx \Delta(x)^2 \int_{\mathbb{R}_+^N} dy \Delta(y)^2 S_0(y) \cdot \\ &\quad \cdot \int_{\mathbf{U}(N)} dU_X e^{-\text{tr } Z^p U_X^{-1} x U_X} \int_{\mathbf{U}(N)} dV_Y e^{-\text{tr } x V_Y^{-1} y^p V_Y} \\ &\quad \text{using Lemma 7} \\ &= C \frac{S_2(t)\Delta(z^p)}{\Delta(z)} \int_{\mathbb{R}_+^N} dx \Delta(x)^2 \left(\prod_1^N x_i^c \right) \int_{\mathbb{R}_+^N} dy \Delta(y)^2 S_0(y) \cdot \\ &\quad \cdot \int_{\mathbf{U}(N)} dU_X e^{-\text{tr } Z^p U_X^{-1} x U_X} \int_{\mathbf{U}(N)} dU_Y e^{-\text{tr } U_X^{-1} x U_X U_Y^{-1} y^p U_Y} \\ &\quad \text{setting } U_Y = V_Y U_X \text{ for fixed } U_X \text{ in the last} \\ &\quad \text{integral and noting that } dU_X dU_Y = dU_X dV_Y \\ &= C \frac{S_2(t)\Delta(z^p)}{\Delta(z)} \int_{\mathbb{R}_+^N} dx \Delta(x)^2 \left(\prod_1^N x_i^c \right) \int_{\mathbf{U}(N)} dU_X e^{-\text{tr } Z^p U_X^{-1} x U_X} \cdot \\ &\quad \cdot \int_{\mathbb{R}_+^N} dy \Delta^2(y) S_0(y) \int_{\mathbf{U}(N)} dU_Y e^{-\text{tr } U_X^{-1} x U_X U_Y^{-1} y^p U_Y} \\ &= C' \frac{S_2(t)\Delta(z^p)}{\Delta(z)} \int_{\mathcal{H}_N^+} dX \det(X^c) e^{-\text{tr } Z^p X} \int_{\mathcal{H}_N^+} dY S_0(y) e^{-\text{tr } XY^p} \\ &= C'' S_1(t) \frac{\int_{\mathcal{H}_N^+} dX \det(X^c) e^{-\text{tr } Z^p X} \int_{\mathcal{H}_N^+} dY S_0(y) e^{-\text{tr } XY^p}}{\int_{\mathcal{H}_N} dX \exp \text{tr} \left(-\frac{((X+Z)^{p+1})_2}{p+1} \right)}, \end{aligned}$$

where we used Lemma 6 in the last equality, and the definition of $S_1(t)$ in Theorem 2. A similar calculation, outlined below, implies the second formula for τ , upon using the first representation of $\psi_k(z)$ in (46):

$$\begin{aligned} \tau(t) &= \frac{\det(A^{k-1}\Psi(0, z_i))_{1 \leq k, i \leq N}}{\Delta(z)}, \quad \text{with } t_n = -\frac{1}{n} \sum_{i=1}^{\infty} z_i^{-n}, \\ &= \frac{1}{\Delta(z)} \det \left(a e^{z_i} z_i^{(p-1)(c+1)} \int_{\gamma} dw \int_0^{\infty} dx e^{-w} w^{k-1} x^c e^{xw^p} e^{-xz_i^p} \right)_{k,i} \\ &= \frac{a^N}{\Delta(z)} e^{\sum z_i} \prod z_i^{(p-1)(c+1)} \cdot \\ &\quad \cdot \int_{\gamma} \dots \int_{\gamma} dw \int_0^{\infty} \dots \int_0^{\infty} dx e^{-\sum w_i} \prod x_i^c \Delta(w) \prod_{i=1}^N e^{-z_i^p x_i} \prod_{i=1}^N e^{x_i w_i^p} \\ &= \frac{a^N}{(N!)^2} e^{\sum z_i} \prod z_i^{(p-1)(c+1)} \frac{1}{\Delta(z)} \int_{\gamma^N} dw \int_{\mathbb{R}_+^N} dx e^{-\sum w_i} \prod x_i^c \Delta(w) \cdot \\ &\quad \cdot \det(e^{-z_i^p x_j})_{1 \leq i, j \leq N} \det(e^{x_i w_j^p})_{1 \leq i, j \leq N} \\ &= \dots \end{aligned}$$

$$= \frac{\int_{\mathcal{H}_N^\gamma} m(dW) \int_{\mathcal{H}_N^+} dX \det X^c (\Delta(w^p)/\Delta(w)) e^{\text{tr}(Z-W)} e^{\text{tr} X(W^p-Z^p)}}{\int_{\mathcal{H}_N} dX \exp \text{tr} \left(-\frac{(X+Z)^{p+1}}{p+1} \right)_2},$$

ending the proof of Theorem 2.

References

- [1] Adler, M., van Moerbeke, P.: A matrix integral solution to two-dimensional W_p -Gravity. Commun. Math. Phys. **147**, 25–56 (1992)
- [2] Adler, M., Shiota, T., van Moerbeke, P.: (a) From the w_∞ -algebra to its central extension: A τ -function approach. Phys. Lett. **A194**, 33–43 (1994);
(b) A Lax representation for the vertex operator and the central extension. Commun. Math. Phys. **171**, 547–588 (1995)
- [3] Crnković, C., Douglas, M., Moore, G.: Physical solutions for unitary matrix models. Nucl. Phys. **B360**, 507–523 (1991)
- [4] Dalley, S., Johnson, C.V., Morris, T.R., Wätterstam, A.: Unitary matrix models and 2D quantum gravity. Mod. Phys. Lett. **A 7**, (29) 2753–2762 (1992)
- [5] Dalley, S., Johnson, C.V., Morris, T.R.: Multicritical complex matrix models and non-perturbative two-dimensional quantum gravity. Nucl. Phys. **B368**, 625–654 (1992)
- [6] Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Transformation groups for soliton equations. In: Proceedings RIMS Symp. Non-linear integrable systems – classical theory and quantum theory (Kyoto 1981), Singapore: World Scientific, 1983, pp. 39–119
- [7] Dijkgraaf, R.: Intersection Theory, Integrable Hierarchies and Topological Field Theory. Lectures given at the Cargese Summer School on “New Symmetry Principles in Quantum Field Theory.” hep-th/9201003.
- [8] Fastré, J.: A Grassmannian version of the Darboux transformation. To appear in Bull. des Sciences Math.
- [9] Flaschka, H.: A commutator representation of Painlevé equations. J. Math. Phys. **21** (5), 1016–1018 (1980)
- [10] Flaschka, H., Newell, A.C.: Monodromy- and spectrum-preserving deformations I. Commun. Math. Phys. **76**, 65–116 (1980)
- [11] Fokas, A.S., Ablowitz, M.J.: On a unified approach to transformations and elementary solutions of Painlevé equations. J. Math. Phys. **23** (11), 2033–2042 (1982)
- [12] Gross, D.J., Newman, M.J.: Unitary and Hermitian matrices in an external field. Phys. Lett. **B266**, 291–297 (1991)
- [13] Harish Chandra: Differential operators on a semi-simple lie algebra. Am. J. Math. **79**, 87–120 (1957)
- [14] Hollowood, T., Miramontes, L., Pasquinucci, A., Nappri, C.: Hermitian versus anti-Hermitian one-matrix models and their hierarchies. Nucl. Phys. **B373**, 247–280 (1992)
- [15] Kac, V., Schwarz, A.: Geometric interpretation of partition function of 2D gravity. Phys. Lett. **B257**, 329–334 (1991);
Schwarz, A.: On some mathematical problems of 2d-gravity and W_h -gravity. Mod. Phys. Lett. **A 6**, 611–616 (1991)

- [16] Kazakov, V.A.: A simple solvable model of quantum field theory of open strings. *Phys. Lett.* **B237** (2), 212–215 (1990)
- [17] Kharchev, S., Marshakov, A., Mironov, A., Morozov, A., Zabrodin, A.: Towards unified theory of 2d gravity. *Nucl. Phys.* **B380**, 181–240 (1992)
- [18] Kharchev, S., Marshakov, A.: Topological versus non-topological theories. Preprint FIAN/TD-15/92. To appear in *Int. J. Mod. Phys. A*. See also Sect. 4.9 of the second paper in [24].
- [19] Kontsevich, M.: Intersection theory on the moduli space of curves and the matrix Airy function. *Commun. Math. Phys.* **147**, 1–23 (1992)
- [20] Kostov, I.K.: Exactly solvable field theory of $D = 0$ closed and open strings. *Phys. Lett.* **B238**, 181–186 (1990)
- [21] Macdonald, I.G.: Symmetric functions and Hall polynomials. Second Edition. Oxford: Clarendon Press; New York: Oxford University Press, 1995
- [22] Minahan, J.A.: Matrix models with boundary terms and the generalized Painlevé II equation. *Phys. Lett.* **B268**, 29–34 (1991)
- [23] Minahan, J.A.: Schwinger–Dyson equations for unitary matrix models with boundaries. *Phys. Lett.* **B265**, 382–388 (1991)
- [24] For a general discussion of string program and more details on matrix models in this context see:
 Morozov, A.: String theory: What is it? *Usp. Fiz. Nauk* **162** (8), 83–176 (1992) (*Soviet Physics Uspekhi* **35**, 671–714 (1992)), and
 Morozov, A.: Integrability and matrix models. *Usp. Fiz. Nauk*, **164** (1), 3–62 (1994) (*Physics Uspekhi*, **37**, 1–55 (1994), hep-th/9303139), and references therein.
- [25] Morozov, A.: Matrix Models as Integrable Systems. Presented at Banff Conference, Banff, Canada, August 15–23, 1994. ITEP-M2/95, hep-th/9502091
- [26] Mulase, M.: Algebraic theory of the KP equations. In: *Perspectives in Mathematical Physics*, Ed.: R. Penner and S.-T. Yau, International Press Company, 1994, pp. 157–223
- [27] Periwal, V., Shevitz, D.: Exactly solvable unitary matrix models: Multicritical potentials and correlations. *Nucl. Phys.* **B344**, 731–746 (1990)
- [28] Periwal, V., Shevitz, D.: Unitary-matrix models as exactly solvable string theories. *Phys. Rev. Lett.* **64**, (12) 1326–1329 (1990)
- [29] Sato, M.: Soliton equations and the universal Grassmann manifold (by Noumi in Japanese). *Math. Lect. Note Ser. No. 18*. Sophia University, Tokyo, 1984
- [30] Segal, G., Wilson, G.: Loop groups and equations of KdV type. *IHES Publ. Math.* **61**, 5–65 (1985)
- [31] van Moerbeke, P.: Integrable foundation of string theory. In: *Proceedings of the CIMPA school 1991*. Ed.: O. Babelon, P. Cartier, Y. Kosmann-Schwarzbach, Singapore: World Scientific, 1994, pp. 163–267
- [32] Wätterstam, A.: A solution to the string equation of unitary matrix models. *Phys. Lett.* **B263**, No 1, 51–58 (1991)