Proceedings of the IASS Annual Symposium 2016
"Spatial Structures in the 21st Century"
26–30 September, 2016, Tokyo, Japan
K. Kawaguchi, M. Ohsaki, T. Takeuchi (eds.)

Graphical limit state analysis of hyperstatic structures

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Abstract

This paper suggests a methodology that applies *Graphic Statics* to the limit state analysis of bended structures within the framework of *Plastic Theory*. For rather simple statically indeterminate bended structures like beams or frames it shows that this approach could be an alternative method to classical algebraic methods – like applications of the *Principle of Virtual Works* – for evaluating the structural safety. By application of the *Lower Bound Theorem of Plasticity*, a 'safe' value for the collapse load factor can be graphically determined by means of specific constructions of the reciprocal *Form* and *Force Diagrams* that is expressed as a scale factor applicable to the latter in order to transform it into a collapse *Force Diagram*. The advantages of this methodology are outlined in case of analysing geometrically complex existing structures.

Keywords: graphic statics, plastic theory, limit analysis, hyperstatic structures, load factor, virtual works

1. Introduction

According to Timoshenko and Young [17], most of the methods for resolving problems of statics are based on two fundamental principles: the *Parallelogram of Forces* and the *Principle of Virtual Works*. The first one has been extensively developed in the nineteenth century *Graphic Statics* for statically determinate structures (Maxwell [11] [12]; Cremona [2]; Culmann [3]; Levy [9]; Ritter [13]) and for some specific indeterminate cases. The second principle is mostly used for solving statically indeterminate problems where equilibrium methods do not provide a sufficient amount of equations. Indeed, given an elastic linear behaviour of the material, these problems can be solved expressing the compatibility of displacements in additional equations. *Virtual Works* may be used to solve this aspect of structural analysis in a very elegant way. It is also the case when studying the behaviour of structures at collapse where it proved to be a very efficient tool (Thompson and Haywood [16]).

The main concern of this paper consists in presenting an alternative methodology for analysing statically indeterminate structures at collapse that is based on the fundamental principle of *Parallelogram of Forces* using its graphical expression. After reminding the fundamental hypotheses and theorems of limit analysis within the framework of *Plastic Theory*, the basic principles of *Graphic Statics* and specific notations used for graphical limit state analysis are introduced and illustrated with a basic statically determinate structure. Rather simple statically indeterminate bended structures – beams, frames – are then studied in order to show that 'safe' values for the collapse load factor λ_c can be defined graphically. Theses values expressed as scale factors for the *Force Diagram* are compared to those obtained by the classical algebraic methods of *Virtual Works* that give upper bound values for the collapse load factor. Advantages of this method are then discussed, in particular when applied for analysing complex existing buildings. Eventually some further developments are discussed.

2. Plastic Theory and limit state analysis

This section introduces synthetically the main theoretical concepts on which this graphical limit state analysis approach is based. More developments of the concepts linked with *Plastic Theory* and *Limit Analysis* may be found in literature, particularly in the works of Jacques Heyman referenced below.

2.1. Hypotheses and fundamental theorems of plasticity

Initially formulated in the 1930's for steel structures presenting a certain degree of statical indeterminacy (Heyman [6]), *Plastic Theory* provides a powerful theoretical framework for analysing the stability of existing structures as well as for design purposes. Providing the material shows a ductile behaviour and that no instability can occur in any structural member under working loads, three fundamental theorems can be used to study the behaviour of structures at collapse. A so-called *load factor* at collapse λ_c is defined (Heyman [7]) as the proportionality factor that can be applied to the magnitudes of all the component forces $\{F\}_a$ of the actual loading, in order to obtain the component forces $\{F\}_c$ of the loading leading to the collapse of the structure (Thompson and Haywood [17]). This proportional increase of the magnitudes is expressed in equation (1).

$$\lambda_c \cdot \{F\}_a = \{F\}_c \tag{1}$$

Using this concept of *load factor*, the well-known plastic theorems may be expressed as followed:

- *Kinematic theorem* (upper bound/unsafe theorem): any load factor $\lambda_{k,i}$ calculated on basis of a kinematically compatible mechanism is greater than or equal to the collapse load factor λ_c;
- *Static theorem* (lower bound/safe theorem): any load factor $\lambda_{s,j}$ calculated on basis of a statically compatible distribution of internal forces and applied loads that respect the yield conditions is lower than or equal to the collapse load factor λ_c;
- *Uniqueness theorem*: the collapse factor is unique and defined: $\lambda_c = min_{|i}\,(\lambda_{k,i}) = max_{|j}\,(\lambda_{s,j})$.

$$\lambda_{s,j} \leq \lambda_c \leq \lambda_{k,j} \tag{2}$$

Because of the plastic behaviour of the material at collapse, the superposition principle of structural actions may not be applied. This means that all the possible kinematically compatible mechanisms and their corresponding load factor $\lambda_{k,i}$ should be taken into account for obtaining the actual value of the load factor $\lambda_c = min_{/i}\,(\lambda_{k,i})$. Alternatively, all the possible load factors $\lambda_{s,j}$ obtained using a static approach that respects the yield conditions should be calculated in order to define $\lambda_c = max_{|j}\,(\lambda_{s,j})$. (Rjanitsyn [14])

2.2. Kinematic approach for Limit analysis using Virtual Works

Virtual Works are commonly used in literature (Heyman [7]; Thompson and Haywood [16]; Timoshenko and Young [17]) to study the behaviour of structures at collapse within the framework of *Plastic Theory* and in particular when applying the *kinematic theorem*. Despite the remarkable efficiency of this method for finding values for the load factors ($\lambda_{k,i}$) associated with collapse mechanisms, the great number of these possible mechanisms and their kinematical complexity disadvantage this method. Furthermore, the given solutions for these load factors are 'unsafe' in the sense that equation (2) states that they all are upper bound values for the actual collapse load factor λ_c. This means that taking into account a subset $\{i*\}$ of the total set of possible mechanisms $\{i\}$ may lead to overestimating the bearing capacity of the structure since λ_c may be lower than $min_{|i*}\,(\lambda_{k,i*})$.

Therefore, in this paper we try to put forward a static approach for structural limit analysis that uses *Graphic Statics* properties of the reciprocal diagrams to estimate 'safe' values for the collapse load factor. The basic principles of *Graphic Statics* and notations adopted are exposed in the next section.

3. Graphic Statics

3.1. Reciprocal figures

Based on the principle of *Parallelogram of Forces*, *Graphic Statics* determines the way a set of forces can respect the equilibrium conditions. The introduction of some properties relative to reciprocal figures – studied around the same time by J.C. Maxwell [10] [11] [12] and L. Cremona [2] – transformed graphic statics into a very effective tool: each segment in the so-called *Form Diagram* represents the action line of a force which magnitude is expressed in the reciprocal *Force diagram* by the length of the corresponding segment. Some fundamental reciprocal relationships of the two diagrams are adopted. The first one is that each segment in a diagram is related to one sole segment in the other diagram, parallel to the latter – Maxwell [10] showed that this property is not applicable in some specific cases. Furthermore, all the segments that are connected in a single point in one diagram need to form a closed polygon in its reciprocal. This second fundamental relationship ensures that the structure represented by the *Form Diagram* respects the rotational and translational equilibriums.

It must be underlined that, because of their reciprocity, each of the two diagrams can be considered as a *Form Diagram* as well as a *Force Diagram*. They constitute dual structures that provide graphical insight into each other, especially when assessing mechanisms and states of self-stress (Baker [1]).

3.2. Graphic Statics of bended structures using funicular polygons

Applying *Graphic Statics* to bended structures requires the use of funicular polygons because of the (semi-)parallelism between all the external forces applied on the structural members. According to Timoshenko and Young [17], a funicular polygon may be regarded as a system of hinged bars supported as to form an arch – or, inverted, a string – that is in equilibrium under the action of the applied forces. It is constructed in the same way as reciprocal figures for trusses: each intersection of three bars of the funicular polygon (*Form Diagram*) must correspond to a closed triangle in the *Force Diagram*. A crucial issue when dealing with bended structures, especially using graphic statics, is to express the value of the bending moment. Using the relationships between the two reciprocals diagrams, the bending moment can be expressed as a couple of forces which action lines are parallel to each other. The distance between these two action lines (μ) is measured on the *Form Diagram*, since the intensity of forces (H) is measured on the *Force Diagram*, perpendicular to the direction of μ, so that the intensity of the bending moment is given by the product of these two quantities – measured at certain scales, respectively n and k:

$$M = (\mu \cdot n) \cdot (H \cdot k) \tag{3}$$

This methodology is illustrated in Figure 1 with a simply supported beam submitted alternatively to a single force F_1 and to an embedding moment applied at point A. For both cases, two pairs of reciprocal diagrams are drawn according to the aforementioned rules. These ensure that the value of the bending moment (M ; M_A) resulting from the product of the perpendicular distances μ and H remains the same, independently of the arbitrary chosen position for point O, the so-called *pole* of the Force Diagram.

$$\begin{aligned} (nk) \cdot \mu \cdot H &= (nk) \cdot \mu' \cdot H' = M \\ (nk) \cdot \mu_A \cdot H_M &= (nk) \cdot \mu_A' \cdot H_M' = M_A \end{aligned} \tag{4}$$

3.3. Graphic Statics for statically indeterminate structures

The construction of the actual reciprocal *Force Diagram* of a statically indeterminate structure requires other hypotheses that the equilibrium considerations exposed above. Some authors have detailed a method for the analysis of statically indeterminate trusses (Ritter [13]) or beams (Levy [9]) that have not been implement efficiently until now. On the contrary, applying *Graphic Statics* within the frame of *Plastic Analysis* is far easier since there is no statical indeterminacy left. This approach will be presented in the next section with the study of some simple bended structures.

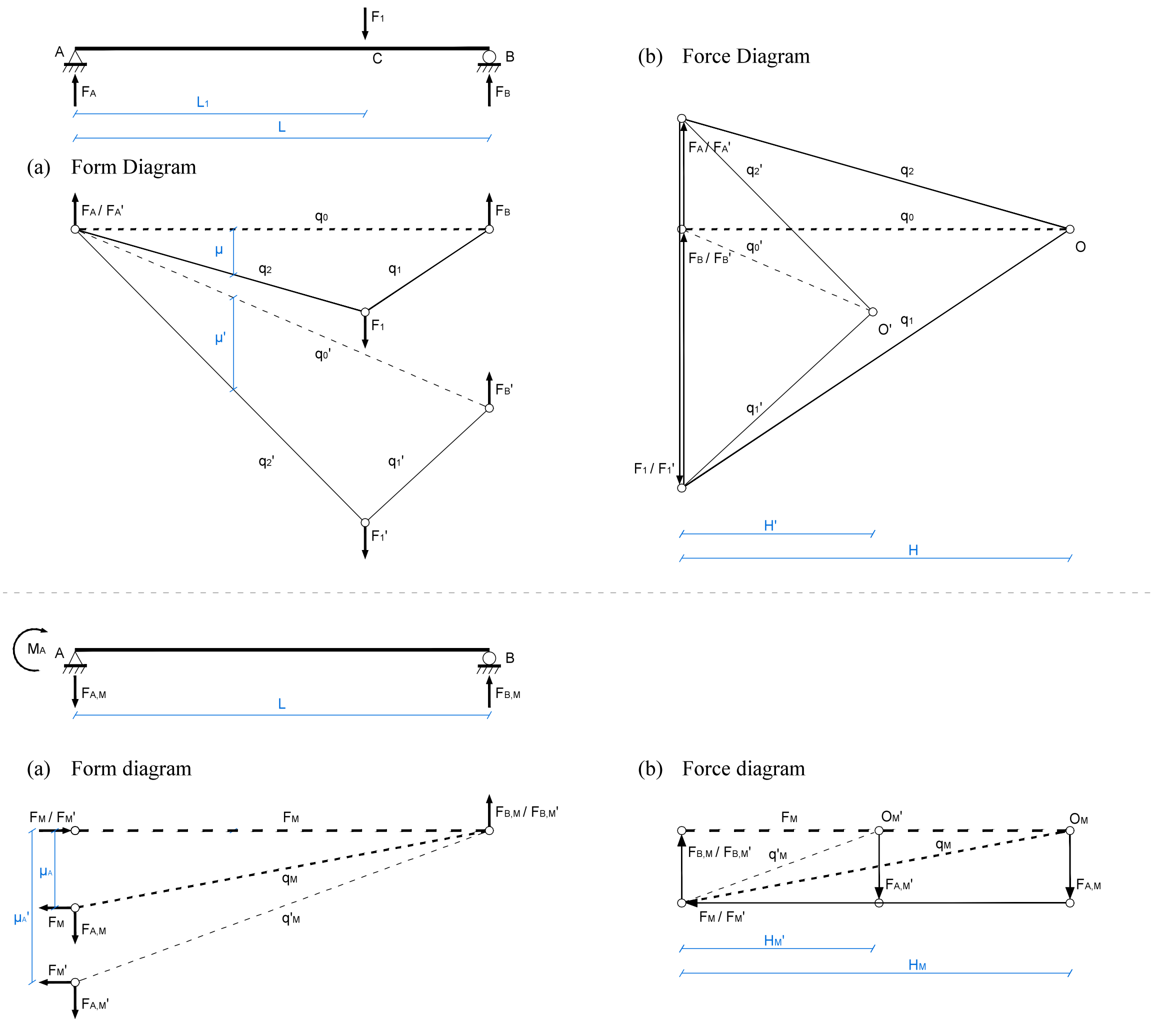

Figure 1: Construction of different reciprocal polygons when applying an external force F_1 or an embedding moment M_A on a simply supported beam.

4. Graphical limit state analysis of statically indeterminate bended structures

In this section some basic structures in bending are analysed using reciprocal polygons corresponding to their collapse limit states. The graphical construction rules ensure that the forces are in equilibrium (statically compatible). Providing that the yield conditions are respected, it means that the *safe theorem of plasticity* may be applied. For bended structures, yield conditions are expressed in terms of limiting the value of bending moments: $M \leq M_c$ where M_c is the value of the full plastic moment corresponding to the formation of a plastic hinge in one of the cross sections. Expressed in terms of *Graphic Statics*, this means that for a given value of H all the nodes of the funicular polygon in the *Form Diagram* must be enclosed in-between two straight lines parallel to the beam axis at a distance $\mu = M_c/H$ – supposing without loss of generalities that n=1 and k=1. Fixing $\mu = \mu_c$, a peculiar value for H can be defined so that $H_c = M_c/\mu_c$. At collapse, H_c is such that the funicular polygon reaches the limit value μ_c in the number of sections needed to transform it into a collapse mechanism. Each of the limit forces polygons is related to a specific pole O_j whose position $H_{s,j}$ can be measured. The *static theorem* ensures that the actual collapse configuration is the one corresponding to the lowest value of H so that:

$$\lambda_{s,c} = \max_{|j}\left(\lambda_{s,j}\right) = \max_{|j}\left(\frac{H_c}{H_{s,j}}\right) = H_c \cdot \frac{1}{\min_{|j}\left(H_{s,j}\right)} = \frac{H_c}{H_{s,c}} \tag{5}$$

In order to make the method more explicit, simple beams will first be analysed. These are submitted to different loadings, for which the value of the collapse factor $\lambda_{s,c}$ is determined graphically, but expressed in an algebraic form in order to compare its formulation to the one $\lambda_{k,c}$ obtained using the *Principle of Virtual Works* (*PVW*). Extension of this methodology to semi-rigid frames and pin-jointed frames is then discussed.

4.1. Graphical limit state analysis of beams

The following four study cases concern a simple beam of length L submitted to different loadings and abutment conditions. For each of them drawing the possible limit state reciprocal polygons leads to determining the related load factor $\lambda_{s,c.}$ For some of them, a comparative analysis is made with the load factor obtained by application of the *kinematic theorem of plasticity* using the *PVW*.

4.1.1. Simply supported beam – one force

Applying the *static theorem of plasticity* to a statically determinate simply supported beam gives a lower value for the load factor. Using the similar triangles relationships between the reciprocal diagrams of Figure 2(a)(b)(c), equation (6) gives the only possible value for $\lambda_{s,c}$.

$$\lambda_{s,c} = \frac{F_{1,c}}{F_{1,a}} = \frac{\mu_c}{\mu_1} \cdot \left(\frac{H_c}{H_c}\right) = \frac{M_c}{F_{B,a} \cdot (L - L_1)} = M_c \cdot \frac{L/L_1}{F_{1,a} \cdot (L - L_1)} \tag{6}$$

Since this structure is statically determinate, only one collapse equilibrium $\{q\}_c$ is possible and therefore the value of $H_{s,c}$ and $\lambda_{s,c} = H_{s,c}/H_a = H_c/H_a$ for a fixed value of μ_c.

Applying now the *Principle of Virtual Works* to the only possible mechanism, (7) gives the classical result for the value of the unique kinematic load factor $\lambda_{k,c}$ (Fig. 3d).

$$\lambda_{k,c} \cdot F_{1,a} \cdot \Delta_1 = M_c \cdot \left(\theta_A + \theta_B\right) = M_c \cdot \left(\frac{\Delta_1}{L_1} + \frac{\Delta_1}{L - L_1}\right) \Rightarrow \lambda_{k,c} = M_c \cdot \frac{L/L_1}{F_{1,a} \cdot (L - L_1)} \tag{7}$$

Since these two approaches give the same value for the load factor, equation (2) ensures that $\lambda_{s,c} = \lambda_{k,c} = \lambda_c$.

This simple example being statically determinate, the results obtained could be found using equilibrium relationships only, so that a plastic approach is not very relevant. This is not the case for the statically indeterminate structures analysed hereafter since equilibrium considerations only cannot give sufficient information to describe the inner forces acting inside the beam.

4.1.2. Propped-cantilever beam – one force

In order to construct the reciprocal polygons corresponding to this structure, the principle of superposition of two equilibrium systems is applied, corresponding to the ones analysed in 2.2. A force F_1 and a couple of forces F_M-F_M* separated by a distance μ_A (Fig. 1a) are applied alternatively to the beam. The forces F_M-F_M* have the same magnitude H_a as the other polar rays $\{q_l\}$ (Fig. 3b). It can be seen that the resulting *Force Diagram* presents a mechanism, corresponding to the state of self-stress in the *Form Diagram*. However no indeterminacy is kept at collapse since $\mu_A = \mu_1 = \mu_c$. The relations in similar triangles of Figure 3 are expressed in equation (8) and give the unique possible value for $\lambda_{s,c}$.

$$F_{1,a} = F_A + F_B = H_a \cdot \left(\frac{\mu_A + \mu_1}{L_1} + \frac{\mu_1}{L - L_1}\right) \Rightarrow \lambda_{s,c} = \frac{F_{1,c}}{F_{1,a}} = \frac{H_a}{F_{1,a}} \cdot \left(\frac{\mu_c + \mu_c}{L_1} + \frac{\mu_c}{L - L_1}\right) = \frac{M_c}{F_{1,a}} \cdot \left(\frac{2}{L_1} + \frac{1}{L - L_1}\right) \tag{8}$$

It has been proven that a model can be build up on proper constructions of reciprocal diagrams that defines in a geometric way a lower bound value for the load factor in case of statically indeterminate structures, since graphic statics rules ensure that the set of efforts in the beam is in equilibrium with the external loads. This value of $\lambda_{s,c}$ can be visualised on Figure 3(b) by the ratio between the horizontal length of the polar rays at collapse H_c and the actual length of the polar rays H_a.

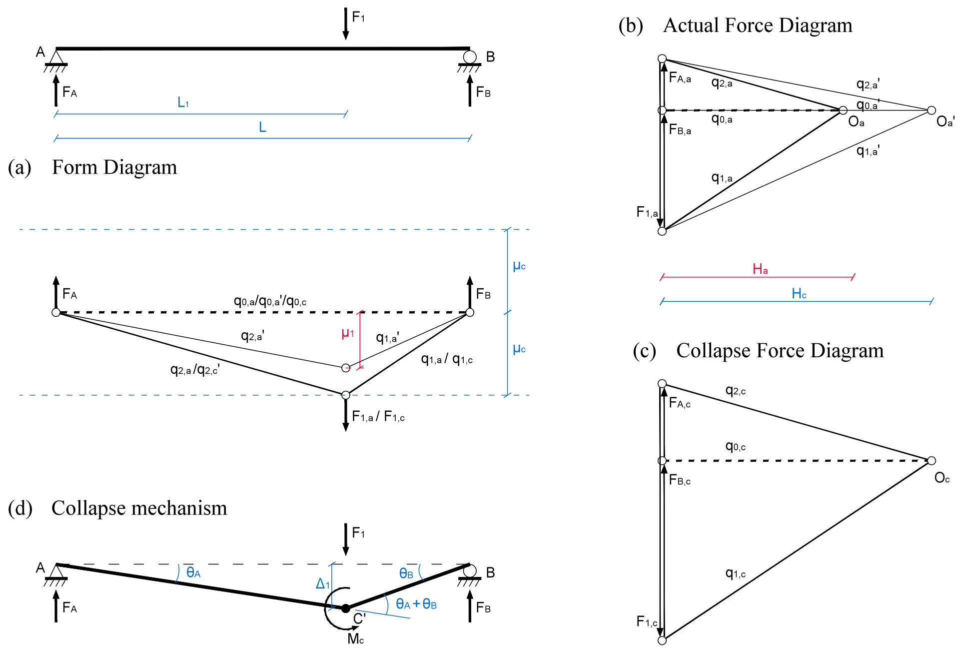

Figure 2: Simply supported beam loaded by one force F_1 – Limit state analysis

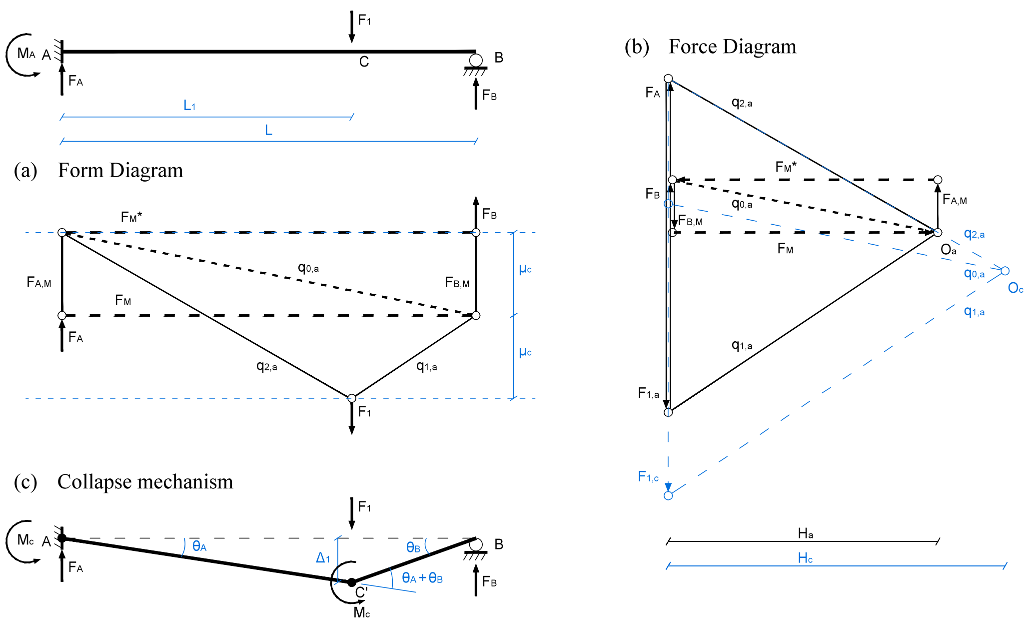

Figure 3: Propped cantilever beam loaded by one force F_1 – Graphical limit state analysis and visualisation of the value of the statically determined collapse load factor $\lambda_{s,c} = F_{1,c} / F_{1,a} = H_c / H_a$

Applying the *PVW* to the corresponding collapse mechanism (Fig. 3c), equation (9) gives once more the same value for the load factor as the one obtained in equation (8) by the static approach, so that $\lambda_{s,c} = \lambda_{k,c} = \lambda_c$.

$$\lambda_{k,c} \cdot F_{1,a} \cdot \Delta_1 = M_c \cdot \theta_A + M_c \cdot (\theta_A + \theta_B) = M_c \cdot \left(2\frac{\Delta_1}{L_1} + \frac{\Delta_1}{L - L_1}\right) \Rightarrow \lambda_{k,c} = \frac{M_c}{F_{1,a}} \cdot \left(\frac{2}{L_1} + \frac{1}{L - L_1}\right) \qquad (9)$$

Since there is only one possible mechanism, the solution may seem obvious. This is not the case for the following examples, for which limit analysis that fully uses the plastic theorems is thus needed.

4.1.3. Propped-cantilever beam – two forces

Adding a second force F_2 to the propped-cantilever beam from sub-section 4.1.2. implies that more than one pair of reciprocal diagrams may correspond to a collapse situation. The three possible funicular polygons are represented on Figure 4(a), corresponding to three different positions of the pole O in the *Force Diagram,* and a consequently different value for *H*. The formation of the two plastic hinges leading to collapse occurs when two nodes of the funicular polygon reach the borders of the yield surface (blue-dotted lines). The different possible polygons represent possible equilibriums of external and internal forces. Though, one of them $\{q\}_I$ violates the yield conditions and cannot be considered as a possible configuration for applying the *static theorem of plasticity*; this means that the corresponding static load factor $\lambda_{s,I}$ does not exist. Therefore, using the relations in similar triangles for both dual diagrams, equation (10) gives the values of the static load factors $\lambda_{s,II}$ and $\lambda_{s,III}$.

$$\lambda_{s,II}\frac{F_a + F_a}{H_c} = \frac{2\mu_c}{L} \Rightarrow \lambda_{s,II} = \frac{M_c}{FL};\lambda_{s,III}\frac{F_a + F_a}{H_c} = \frac{\mu_1 + \mu_2}{L} = \frac{5\mu_c/3 + 4\mu_c/3 - \mu_c/3}{L} \Rightarrow \lambda_{s,III} = \frac{4}{3}\frac{M_c}{FL} \qquad (10)$$

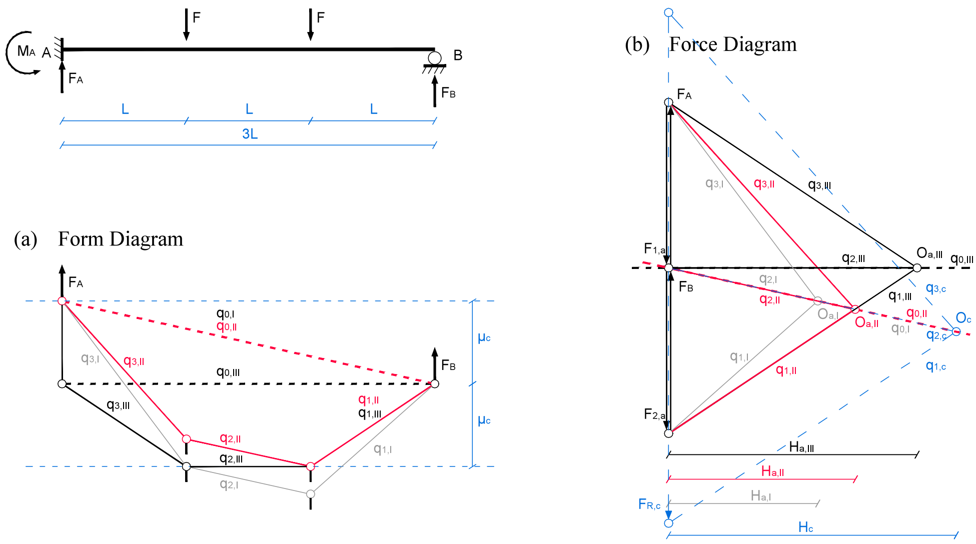

Figure 4: Propped-cantilever beam loaded by two forces *F* – Graphical limit state analysis: the red collapse polygons correspond to the maximum static load factor since $H_{a,II}$ shows a maximum length difference with H_c in the *Force Diagram* (b); the grey mechanism is excepted since it does not respect the yield conditions in (a).

Applying now the *lower bound theorem*, it follows that the actual collapse load factor λ_c is the highest value of the ones given by equation (10), so that $\lambda_c = max_{|j=I,II,III}\,(\lambda_{s,j}) = \lambda_{s,II}$. This corresponds also to the shortest *H* in the *Force Diagram* since for a given value of M_c and a fixed distance μ_c, $\lambda_c = \lambda_{s,II} = H_c/H_{a,II}$ is the scaling factor that may be applied to the *Force Diagram* to correspond to the collapse loading, as expressed by equation (11) with k_c the actual scale of the collapse *Force Diagram*.

$$M_c = (\mu_c \cdot n)\cdot(H\cdot k)\cdot\lambda_c = (\mu_c \cdot n)\cdot(H_c \cdot k_c) \qquad (11)$$

In order to assess the validity of this result, a kinematic analysis is made for which the three kinematic load factors corresponding to the only three possible mechanisms of Figure 5 have been calculated. Algebraic expressions of the values of these load factors are given in equation (12) and lead to the same conclusions as the static approach in determining the actual collapse mechanism (K_{II}) and the corresponding collapse load factor since $\lambda_c = \lambda_{k,II} = \lambda_{s,II}$.

$$\lambda_{k,I} = \frac{5M_c}{3FL};\lambda_{k,II} = \frac{4M_c}{3FL};\lambda_{k,III} = \frac{3M_c}{FL} \Rightarrow \lambda_c = \min_{|i=I,II,III}(\lambda_{k,i}) = \lambda_{k,II} < \lambda_{k,I} < \lambda_{k,III} \qquad (12)$$

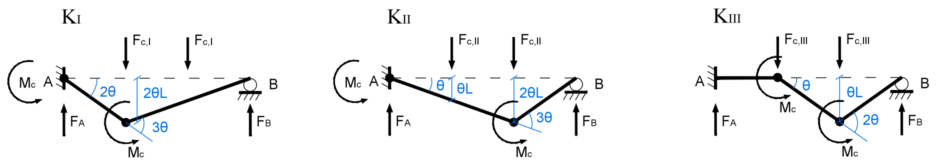

Figure 5: Propped-cantilever beam loaded by two forces F – Kinematic limit state analysis for the three possible collapse mechanisms $\{K_i\}$ and identification of the actual one K_{II}

4.1.4. Propped-cantilever beam – several forces

The same methodology can be applied to a more complex loading system, leading to similar conclusions, as shown in Figure 6. In this particular case only two collapse equilibrium configurations are possible while simultaneously respecting the yield conditions. The actual collapse mechanism corresponds to the *Form* and *Force polygons* $\{q\}_I$ because $H_{a,I} < H_{a,II}$.

For this particular case with five critic sections, a kinematic limit analysis would imply to study $C_5^2 = 5!/2! = 10$ collapse mechanisms resulting from the formations of two plastic hinges among five possible positions for them to occur. Another advantage of the static method is that it reduces the amount of configurations to study in comparison to the kinematic approach. Furthermore, the static approach is safe/conservative since values obtained for λ_s are lower or equal to the actual value λ_c, so that neglecting one of the possible collapse mechanisms does not have consequences on the structural safety. Since this is not the case for a kinematic approach, leaving out some mechanisms could lead to overestimating the load factor and thus the structural safety.

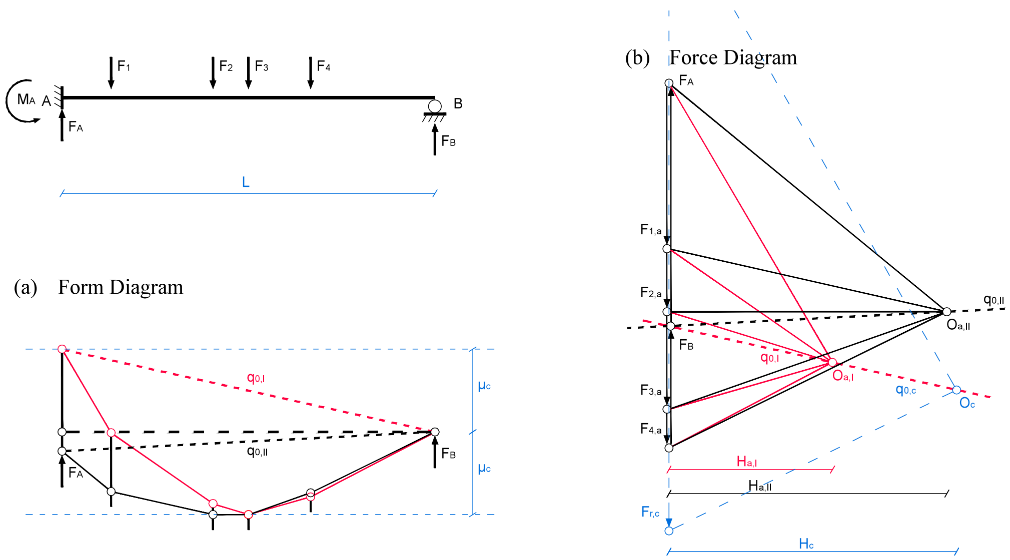

Figure 6: Propped-cantilever beam loaded by several forces – Graphical limit state analysis and identification of the collapse reciprocal polygons (red colour) corresponding to the maximum value of the static load factor

4.2. Graphical limit state analysis of frames

In this section, a methodology is proposed to extend the results obtained for single beams to bended frames. The basic idea is to define a pole and a funicular polygon for each of the structural members, which construction follows the rules proposed in 3.2. In particular, H has to be perpendicular to the axis line of the corresponding member. These polygons are not independent from each other in the sense that they must be constructed so that the global and local equilibrium conditions are respected: reaction forces must be in equilibrium with the external loads, and internal equilibrium of each section is ensured by equalizing left and right values of the bending moment. Furthermore, length H has to be the same for all the force polygons in order to define a unique scale for the *Force Diagram*. This methodology is illustrated in next sub-section by a deliberately simplified example. Extension to more complex structures is possible: studying frames with more structural members or presenting a complex geometry, using various values of μ_c for them, introducing additional external forces, etc. would be of great interest for assessing the efficiency of the abovementioned methodology.

4.2.1. Statically indeterminate half-frame – two single forces

The graphical limit state analysis presented in Figure 7 follows this methodology for a half-frame AB. It reveals that only one funicular construction $\{q\}_I$ (in colour on Figure 7) can statically correspond to a collapse mechanism, since the second one $\{q\}_{II}$ (grey) violates the yield conditions and the third one is statically incompatible and does not correspond to a regular construction of the funicular polygon. The kinematic analysis of the structure at collapse confirms this conclusion since the *Virtual Works* equations (13) give the lowest value among the set $\{\lambda_{k,j}\}$ that corresponds to the same mechanism.

$$\lambda_c = \min_{|i=I,II,III}(\lambda_{k,i}) = \lambda_{k,I} = \frac{3M_c}{FL} < \lambda_{k,II} = \frac{6M_c}{FL} < \lambda_{k,III} = \frac{8M_c}{3FL} \qquad (13)$$

(a) Form Diagram (b) Force Diagram

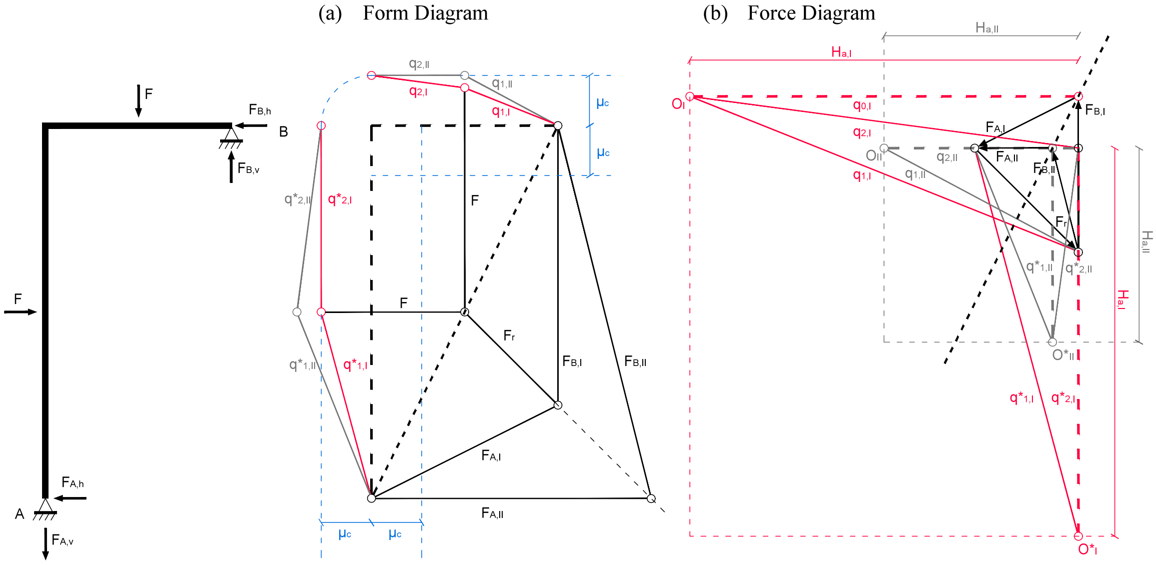

Figure 7: Half-frame loaded by two forces F – Graphical limit state analysis and collapse polygons (red)

4.2.2. Pin-jointed trusses

Pin-jointed trusses are specific cases of frames for which the direction of the bars of both polygons is given by the geometry of the structural members themselves. Therefore, collapse does not occur when plastic hinges are formed but when a sufficient number of bars plastically deform when the normal stress reaches the plastic value. This question has already been studied in ancient textbooks outside the framework of *Plastic Theory* (Levy [9]; Ritter [13]).

5. Conclusion and perspectives

This paper presents an application of *Graphic Statics* to the limit analysis of statically indeterminate bended structures, and proposes extensions to different types of structures. The research is placed within the theoretical framework of *Plasticity* taking advantage of its three fundamental theorems. The defined methodology uses the properties of reciprocal *Form* and *Force Diagrams* of *Graphic Statics* to evaluate the collapse load factor of rather simple bended structures. The way these reciprocal polygons are constructed and constrained secures the structural equilibrium and ensures that the yield conditions are fulfilled. These two conditions guarantee that the static theorem of plasticity may be applied to determine lower bound values for the collapse load factor. These can be compared to upper bound values for the collapse load factor obtained by applying a kinematic approach by means of the *Principle of Virtual Works*.

Further work will consider potentialities offered by parametric-oriented CAO software to construct and constrain the reciprocal diagrams (Fivet [4]; Jasienski [8]). This will enable to apply this static – and safe – approach to complex geometry structures. The methodology is also very appropriate for cases where reliable information about the actual mechanical properties of the structural material is not available, such as in old masonry buildings (Rondeaux [15]). Since Heyman [7] has shown that historical masonry structures can be studied within the framework of *Plastic Theory,* graphical limit state analysis would be an adequate tool for the analysis of this particular type of structures.

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