Necessary and Sufficient Conditions for Template-Dependent Ordering of Path-Complete Lyapunov Methods

Virginie Debauche
virginie.debauche@uclouvain.be
ICTEAM Institute, UCL
Louvain-la-Neuve, Belgium

Matteo Della Rossa
matteo.dellarossa@uclouvain.be
ICTEAM Institute, UCL
Louvain-la-Neuve, Belgium

Raphaël M. Jungers
raphael.jungers@uclouvain.be
ICTEAM Institute, UCL
Louvain-la-Neuve, Belgium

ABSTRACT

In the context of discrete-time switched systems, we study the comparison of stability certificates based on path-complete Lyapunov methods. A characterization of this general ordering has been provided recently, but we show here that this characterization is too strong when a particular *template* is considered, as it is the case in practice. In the present work we provide a characterization for templates that are closed under pointwise minimum/maximum, which covers several templates that are often used in practice. We use an approach based on abstract operations on graphs, called *lifts*, to highlight the dependence of the ordering with respect to the analytical properties of the template. We finally provide more preliminary results on another family of templates: those that are closed under addition, as for instance the set of quadratic functions.

CCS CONCEPTS

• **Theory of computation** → **Mathematical optimization**; • **Mathematics of computing** → *Graph theory*.

KEYWORDS

Switched systems, path-complete Lyapunov methods, stability analysis, lifts

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1 INTRODUCTION

The analysis of switched dynamical systems [14] have been intensely studied in past decades. Indeed, this framework provides a modeling structure for many physical phenomena. In this note, given a finite set of continuous vector fields $f_1, \dots, f_M \in C^0(\mathbb{R}^n, \mathbb{R}^n)$, we consider *discrete-time switched systems* defined by

$$x(k+1) = f_{\sigma(k)}(x(k)), \quad k \in \mathbb{N}, \tag{1}$$

where $\sigma : \mathbb{N} \to \{1, \dots, M\}$ is a time-dependent signal. As for general non-linear systems, Lyapunov theory provides a powerful tool for the stability analysis of (1), see for example the surveys [15, 21]. Indeed, it is well-known that (1) is stable for arbitrary switching signals σ, i.e. there exists a function α of class $\mathcal{K}_\infty$[1] such that for any switching signal $\sigma : \mathbb{N} \to \{1, \dots, M\}$ and any initial condition $x(0) \in \mathbb{R}^n$,

$$\forall k \in \mathbb{N}, \quad \|x(k)\| \ \le \ \alpha(\|x(0)\|),$$

if and only if there exists a common Lyapunov function (i.e. a single function which is a valid Lyapunov function for *each* sub-dynamics f_j), see [13]. In the case of linear sub-dynamics ($f_j(x) \equiv A_j x$ for some $A_j \in \mathbb{R}^{n\times n}$, $\forall\, j \in \{1, \dots, M\}$) this result can be strengthened: stability is equivalent to the existence of a Lyapunov function homogeneous of degree 1 and convex (i.e. a Lyapunov *norm*), see [5], [12]. Despite these appealing converse Lyapunov results, the nature of these "theoretical" Lyapunov functions/norms and the complexity in approximating them are often prohibitive [22], even in the linear case. For that reason, several alternative approaches have been proposed, most of them relying on the concept of *multiple* Lyapunov functions, see [6], [14], [11] and references therein.

For the purpose of understanding the relations between different multiple Lyapunov functions structures, *path-complete Lyapunov functions* have been proposed as a unifying and flexible approach, see [1], [18] and [10] for a partial extension in the continuous-time setting. In this framework, the inequalities relating the positive definite functions composing a given multiple-Lyapunov stability criterion are encoded in a directed and labeled *path-complete* graph $\mathcal{G}$, as we will clarify. Possibly increasing the number of nodes/edges of this underlying graph $\mathcal{G}$, it is possible to "asymptotically" reach necessary and sufficient criteria, even starting from a very structured set of candidate Lyapunov functions, as for example quadratics or polyhedral functions [1], [3], [8]. While this framework is now mature enough to provide effective criteria to approximate the decay rate (a.k.a. the *joint spectral radius* [12] in the linear subsystems case), the following open question naturally arises:

> *How to systematically compare different path-complete graph structures?*

More precisely, given a family of candidate Lyapunov functions $\mathcal{V} \subseteq C(\mathbb{R}^n, \mathbb{R})$ (namely a *template*) and any path-complete graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, we want to provide a condition ensuring whether or not the conditions arising from $\mathcal{G}_1$ are less/more conservative than the ones given by $\mathcal{G}_2$ if we restrict our search for a solution in $\mathcal{V}$. Such a result is a crucial challenge in the path-complete stability criteria framework, since it would provide, as side product, formal

[1] A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{K}_\infty$ if it is continuous, strictly increasing, $\alpha(0) = 0$ and unbounded.

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confidence intervals for the decay rate and, in practice, could guide the user, given a particular system setting, to choose in a proper and smart way the path-complete structure, see [17]. A partial result is provided in [19], without any hypothesis on the candidate Lyapunov functions template $\mathcal{V}$, relying on the notion of *simulation* of graphs. On the other hand, in [8], it is underlined how the comparison/ordering relations between graphs is strongly affected by the *analytical properties* of the chosen template $\mathcal{V}$. Although it may be counter-intuitive, there are indeed examples for which increasing the size of the graph does not improve the stability certificate if we consider a particular family of Lyapunov functions. In [8], we introduced formal transformations of graphs, called *template-dependent lifts*, in order to improve the performance of a path-complete criterion (by enlarging the underlying graph) following particular rules which rely on closure properties of $\mathcal{V}$.

In this work, we make use of the lift approach to provide *necessary and sufficient* conditions for the aforementioned comparison relation, focusing on templates closed under the binary operations of pointwise minimum and maximum. We show that, similarly to the above mentioned general case solved in [19], the characterization is given by a simulation relation. However in this setting, the characterization is more involved, as it considers a *lift* of the graph $\mathcal{G}_1$ (for proving the relation $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_2$ for any template $\mathcal{V}$ closed under pointwise minimum), not the graph $\mathcal{G}_1$ itself.

We then consider another family of templates, namely the templates closed under addition. For this latter family we show that the situation is more complicated, essentially because the binary operation on functions given by the sum is not idempotent, contrary to the minimum or the maximum. We then provide similar results, however with some limitations. Moreover we provide several examples in order to underline the subtleties and the effectiveness of our theoretical developments. We also illustrate how our work provides a general method and proof technique which can be used for more broader settings.

The rest of the manuscript is organized as follows: In Section 2 we recall the main definitions and the necessary tools arising from the path-complete Lyapunov literature. In Section 3 we present our main results, providing a complete characterization of the ordering relation between graphs, when considering templates closed under pointwise minimum/maximum. In Section 4 we provide partial extensions of our results on templates closed under addition. In Section 5 we summarize our developments and we illustrate possible directions for further research.

Notation: Given $M \in \mathbb{N}$, we denote $\langle M \rangle := \{1, \dots, M\}$. Given $n \in \mathbb{N}$, we define $\mathbb{R}^n_{>0} := \{x \in \mathbb{R}^n \mid \forall i \in \langle n \rangle, x_i > 0\}$, $\mathbb{R}^n_{\geq 0} := \{x \in \mathbb{R}^n \mid \forall i \in \langle n \rangle, x_i \geq 0\}$ and $C^0(\mathbb{R}^n, \mathbb{R}^n)$ denotes the set of continuous vector fields on $\mathbb{R}^n$, while we denote by $C^0_+(\mathbb{R}^n, \mathbb{R})$ the set of continuous, positive definite and radially unbounded functions. Given two functions $g_1, g_2 : \mathbb{R}^n \to \mathbb{R}$, the function $g := \min\{g_1, g_2\}$ denotes the *pointwise minimum function* (in short, the min) between g_1 and g_2 defined by $g(x) := \min\{g_1(x), g_2(x)\}$, for all $x \in \mathbb{R}^n$. The same notation is adopted for pointwise maximum.

2 PATH-COMPLETE LYAPUNOV METHODS AND TEMPLATE-DEPENDENT LIFTS

The path-complete Lyapunov framework generalizes previous Lyapunov techniques for discrete-time switched systems (1) where the Lyapunov inequalities are encoded by the edges of a directed and labeled graph. More precisely, throughout this manuscript, given $M \in \mathbb{N}$ we consider graphs of the form $\mathcal{G} = (S, E)$ where S is a finite set and $E \subseteq S \times S \times \langle M \rangle$ is the set of (labeled) edges. The property that captures the validity of the stability certificate induced by a graph is the *path-completeness*.

Definition 1 (Path-complete graph). *A graph $\mathcal{G} = (S, E)$ is* path-complete *if for any $k \geq 1$ and any sequence $\sigma = i_1 \dots i_k$, $i_j \in \langle M \rangle$, there is a path $(a_j, a_{j+1}, i_j)_{j=1,\dots,k}$ in the graph with $(a_j, a_{j+1}, i_j) \in E$.*

We also use the notation introduced in the following statement.

Definition 2 (Dual Graph). *Given a graph $\mathcal{G} = (S, E)$, its* dual graph *$\mathcal{G}^\top = (S^\top, E^\top)$ is defined by $S^\top = S$ and $(a, b, j) \in E \Leftrightarrow (b, a, j) \in E^\top$, i.e. reversing the direction of each edge.*

It is clear that $\mathcal{G}$ is path-complete if and only if $\mathcal{G}^\top$ is path-complete. In what follows we formally introduce the main concept studied in this manuscript, i.e. the notion of *path-complete Lyapunov functions*.

Definition 3 (Path-complete Lyapunov function). *Given a switched system $F := (f_i \mid i \in \langle M \rangle) \in C^0(\mathbb{R}^n, \mathbb{R}^n)^M$, a* path-complete Lyapunov function *(PCLF) of F is a pair $(\mathcal{G}, V)$ where $\mathcal{G} = (S, E)$ is a path-complete graph, and $V = (V_s \mid s \in S) \in C^0_+(\mathbb{R}^n, \mathbb{R})^{|S|}$ satisfy the following Lyapunov inequalities:*

$$\forall (a, b, i) \in E,\ \forall x \in \mathbb{R}^n : \ V_b(f_i(x)) \leq V_a(x). \tag{2}$$

If this is the case, we say that V is admissible for F and $\mathcal{G}$, *denoted by $V \in PCLF(\mathcal{G}, F)$.*

The existence of a PCLF for a switched system F has been proved to be a sufficient condition for stability in [1][Theorem 2.4] for the linear case and in [19][Theorem 2.5] for the general case. In practice, different path-complete graphs and different sets of candidate Lyapunov functions (i.e. *templates*) can be used to prove stability. Thus, the path-complete Lyapunov framework generates a broad range of stability certificates for switched systems. In what follows, we introduce ordering relations among path-complete graphs where a *template* is formally defined as a family of countably many sets of Lyapunov functions of fixed dimension, i.e.

$$\mathcal{V} := \bigcup_{n \in \mathbb{N}} \mathcal{V}_n,$$

where $\mathcal{V}_n \subseteq C^0_+(\mathbb{R}^n, \mathbb{R})$. As an example, the set of polyhedral functions (introduced and studied in [4]) is a template in the sense of this definition. A family of systems $\mathcal{F}$ is defined similarly, i.e. as a family of countably many sets of systems of fixed dimension

$$\mathcal{F} := \bigcup_{n \in \mathbb{N}} \mathcal{F}_n,$$

where $\mathcal{F}_n \subseteq C^0(\mathbb{R}^n, \mathbb{R}^n)$. For instance, the class of linear switched systems is a family of systems according to this definition. Following [8], we now define the ordering relations that are the main object of this work.

Definition 4 (Ordering relations between graphs). *Consider two path-complete graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$ on $\langle M \rangle$, a template $\mathcal{V} := \cup_{n\in\mathbb{N}}\mathcal{V}_n$ and a family $\mathcal{F} := \cup_{n\in\mathbb{N}}\mathcal{F}_n$.*

(a) *We say that*
$$\mathcal{G} \ \leq_{\mathcal{V},\mathcal{F}} \ \widetilde{\mathcal{G}} \tag{3}$$
if, for any $n \in \mathbb{N}$, for any $F \in {\mathcal{F}_n}^M$,
$$\left[\exists U \in {\mathcal{V}_n}^{|S|} \text{ s.t. } U \in PCLF(\mathcal{G}, F)\right] \Rightarrow$$
$$\left[\exists W \in {\mathcal{V}_n}^{|\widetilde{S}|} \text{ s.t. } W \in PCLF(\widetilde{\mathcal{G}}, F)\right].$$

(b) *We say that*
$$\mathcal{G} \ \leq_{\mathcal{V}} \ \widetilde{\mathcal{G}} \tag{4}$$
if the inequality (3) *is satisfied for $\mathcal{F} = \bigcup_{n\in\mathbb{N}} C^0(\mathbb{R}^n, \mathbb{R}^n)$.*

(c) *We say that*
$$\mathcal{G} \ \leq \ \widetilde{\mathcal{G}} \tag{5}$$
if for any template $\mathcal{V}$, the inequality (4) *is satisfied.*

Note that the relations (3), (4) and (5) are actually preorder relations. In particular, for any path-complete graphs $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$,
$$\left[(\mathcal{G} \leq \mathcal{G}') \wedge (\mathcal{G}' \leq \mathcal{G}'')\right] \Rightarrow (\mathcal{G} \leq \mathcal{G}''),$$
and the same remains true considering the relations $\leq_{\mathcal{V}}$ and $\leq_{\mathcal{V},\mathcal{F}}$, for any template $\mathcal{V}$ and any family of systems $\mathcal{F}$. Moreover, since relation (5) implies relations (3) and (4) for any template $\mathcal{V}$ and any family $\mathcal{F}$, the following implications hold:
$$\left[(\mathcal{G} \leq \mathcal{G}') \wedge (\mathcal{G}' \leq_{\mathcal{V}} \mathcal{G}'')\right] \Rightarrow (\mathcal{G} \leq_{\mathcal{V}} \mathcal{G}''),$$
and
$$\left[(\mathcal{G} \leq \mathcal{G}') \wedge (\mathcal{G}' \leq_{\mathcal{V},\mathcal{F}} \mathcal{G}'')\right] \Rightarrow (\mathcal{G} \leq_{\mathcal{V},\mathcal{F}} \mathcal{G}'').$$

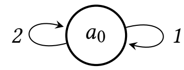
The general ordering relation (5) corresponds to the graph property of *simulation*, formally defined in what follows.

Definition 5 (Simulation). *A graph $\mathcal{G} = (S, E)$* simulates *a graph $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$ if there exists a function $R : \widetilde{S} \to S$ such that $\forall (a, b, i) \in \widetilde{E}$, $(R(a), R(b), i) \in E$.*

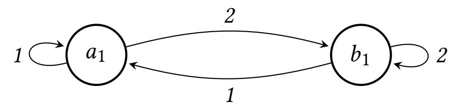
In terms of graphs, this combinatorial property consists in assigning a node of $\mathcal{G}$ to each node of $\widetilde{\mathcal{G}}$ while preserving the edges between the nodes. In terms of path-complete Lyapunov functions, it implies that if one considers any switched system $F \in C^0(\mathbb{R}^n, \mathbb{R}^n)^M$ for which there exists a solution $V \in PCLF(\mathcal{G}, F)$, then the set of Lyapunov functions
$$W := \left\{W_{\widetilde{s}} := V_{R(\widetilde{s})} \mid \widetilde{s} \in \widetilde{S}\right\}$$
is admissible for F and $\widetilde{\mathcal{G}}$. Thus, supposing that $\mathcal{G}$ simulates $\widetilde{\mathcal{G}}$, it follows immediately that $\mathcal{G} \leq \widetilde{\mathcal{G}}$, proving that the simulation is a sufficient condition for the general ordering (5). On the other hand, the reverse implication has been proved in [19, Theorem 3.5] recalled in the following statement.

Theorem 1 (Theorem 3.5 in [19]). *Consider two path-complete graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$. The following statements are equivalent:*

(1) *$\mathcal{G}$ simulates $\widetilde{\mathcal{G}}$.*
(2) *$\mathcal{G} \leq \widetilde{\mathcal{G}}$ in the sense of Definition 4(c).*



(a) The common Lyapunov function graph $\mathcal{G}_0 = (S_0, E_0)$.



(b) The graph $\mathcal{G}_1 = (S_1, E_1)$ in Examples 1 and 2.

Figure 1: The two graphs $\mathcal{G}_0$ and $\mathcal{G}_1$ in Example 1. Even if $\mathcal{G}_1 \not\leq \mathcal{G}_0$, $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_0$ for any template $\mathcal{V}$ closed under pointwise minimum.

When there is no simulation relation between two path-complete graphs $\mathcal{G}$ and $\widetilde{\mathcal{G}}$, it is possible that they might be comparable in terms of relations (3) and (4).

Example 1. *Consider the* common Lyapunov function graph *for 2 modes in Figure 1a denoted by $\mathcal{G}_0 = (\{s\}, \{(s, s, 1), (s, s, 2)\})$ and the graph $\mathcal{G}_1 = (S_1, E_1)$ in Figure 1b. One can easily prove that $\mathcal{G}_1$ does not simulate $\mathcal{G}_0$, but given a switched system F, if we assume that there exists $\{V_{a_1}, V_{b_1}\}$ admissible for F and $\mathcal{G}_1$, then the function $W_{a_0} := \min\{V_{a_1}, V_{b_1}\}$ is a common Lyapunov function admissible for F and $\mathcal{G}_0$. Then by Definition 4, $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_0$ for any template closed under pointwise minimum (in a sense that we will specify below in Definition 8). Therefore, one should not use $\mathcal{G}_1$ to prove the stability of a switched system with a template closed under pointwise minimum, since we have shown that $\mathcal{G}_1$ is equivalent, from the point of view of conservatism, to the common Lyapunov function graph, while requiring the feasibility of a large number of inequalities, for a larger number of functions variables.*

This example motivates us to investigate the ordering relations (3) and (4) for which some combinatorial tools called *lifts* have been introduced in [8].

Definition 6 (Lift). *Given $M \in \mathbb{N}$, we denote with $Graphs_M$ the set of directed and labeled graphs on $\langle M \rangle$. A function $L : Graphs_M \to Graphs_M$ is a* lift *if for any path-complete graph $\mathcal{G}$, $L(\mathcal{G})$ is path-complete.*

Following [9], we relate the *validity* of a lift L to the closure property of a template, and to the ordering relation in Definition 4 that it induces between a graph $\mathcal{G}$ and its lifted graph $L(\mathcal{G})$.

Definition 7 (Valid lifts). *We say that a lift $L : Graphs_M \to Graphs_M$ is:*

(a) valid with respect to a template $\mathcal{V}$ and a family $\mathcal{F}$ *if, for any path-complete graph $\mathcal{G}$,*
$$\mathcal{G} \ \leq_{\mathcal{V},\mathcal{F}} \ L(\mathcal{G}).$$

(b) valid with respect to a template $\mathcal{V}$ *if, for any path-complete graph $\mathcal{G}$,*
$$\mathcal{G} \ \leq_{\mathcal{V}} \ L(\mathcal{G}).$$

(c) valid *if, for any path-complete graph $\mathcal{G}$,*
$$\mathcal{G} \ \leq \ L(\mathcal{G}).$$

However it turns out that lifts can be used to provide further insight by characterizing the relations (3) and (4) for a specific class of templates and families of systems. In particular, we focus on templates that share a common *closure property*.

Definition 8 (Closure properties of a template). *Given a template $\mathcal{V} = \cup_{n\in\mathbb{N}}\mathcal{V}_n$ of candidate Lyapunov functions and a family of binary operations $\{\star_n : C^0_+(\mathbb{R}^n,\mathbb{R}) \times C^0_+(\mathbb{R}^n,\mathbb{R}) \to C^0_+(\mathbb{R}^n,\mathbb{R})\}_{n\in\mathbb{N}}$.*

(a) *For a fixed dimension $n \in \mathbb{N}$, we say that the set of functions $\mathcal{V}_n$ is* closed under the binary operation $\star_n$ *if for all $f_1, f_2 \in \mathcal{V}_n$, $f_1 \star_n f_2 \in \mathcal{V}_n$.*
(b) *We say that the* template $\mathcal{V}$ is closed under the family of binary operations $\{\star_n\}_{n\in\mathbb{N}}$ *if for all $n \in \mathbb{N}$, the set $\mathcal{V}_n$ is closed under $\star_n$.*

As an example, it can be seen that the template of polyhedral functions (i.e. norms whose sublevel sets are full-dimensional polyhedra) is closed under the binary operation given by the pointwise maximum of functions, which is intensively studied in the next section.

3 MIN AND MAX LIFTS

We know from [2] that the existence of a path-complete Lyapunov criterion implies the existence of a common Lyapunov function, defined as the composition of pointwise minima and maxima of some of the pieces of the path-complete Lyapunov function. In the framework of ordering and equivalence between graphs, this result means that for any template $\mathcal{V}$ closed under pointwise minimum and maximum, any path-complete graph is equivalent to the common Lyapunov function graph in the sense of the inequality (4). However, most of the templates used in the literature do not satisfy both closure properties. Therefore, we focus in this section on the *min lift* and its dual, the *max lift* that respectively exploit the pointwise minimum and maximum closure properties of a Lyapunov functions template.

Definition 9 (Min/Max lift). *Given a graph $\mathcal{G} = (S, E)$ on the alphabet $\langle M\rangle$. The* min *and* max *lifts, denoted by $\mathcal{G}_{\min} = (S_{\min}, E_{\min})$ and $\mathcal{G}_{\max} = (S_{\max}, E_{\max})$ respectively, are defined as follows:*

(1) *The sets of nodes $S_{\min}$ and $S_{\max}$ are both defined by*

$$S_{\min} = S_{\max} := \{S' \subseteq S \mid S' \neq \emptyset\}.$$

(2) *The sets of edges $E_{\min}$ and $E_{\max}$ are each defined as follows:*
   (a) *An edge $(A, B, i) \in E_{min}$ with $A, B \in S_{\min}$ and $i \in \langle M\rangle$ if and only if for all $a \in A$, there exists at least one $b \in B$ such that $(a, b, i) \in E$.*
   (b) *An edge $(A, B, i) \in E_{max}$ with $A, B \in S_{\max}$ and $i \in \langle M\rangle$ if and only if for all $b \in B$, there exists at least one $a \in A$ such that $(a, b, i) \in E$.*

In the following simple Lemma we note an important duality relation between the max and min lift.

Lemma 1. *Consider any path-complete graph $\mathcal{G} = (S, E)$ on $\langle M\rangle$. It holds that*

$$\left[(\mathcal{G}^\top)_{\max}\right]^\top = \mathcal{G}_{\min}, \tag{6a}$$

$$\left[(\mathcal{G}^\top)_{\min}\right]^\top = \mathcal{G}_{\max}. \tag{6b}$$

Proof. We use the notation $\mathcal{G}^\top = (S, E^\top)$, $\mathcal{G}_{\min} = (S_{\min}, E_{\min})$, $(\mathcal{G}^\top)_{\max} = (S^\top_{\max}, E^\top_{\max})$ and $[(\mathcal{G}^\top)_{\max}]^\top = ((S^\top_{\max})^\top, (E^\top_{\max})^\top)$. First of all, by Definitions 2 and 9 we have $S_{\min} = (S^\top_{\max})^\top (= \mathcal{P}(S) \setminus \{\emptyset\})$. Then, considering $S_1, S_2 \in S_{\min}$ and $i \in \langle M\rangle$, again by Definitions 2 and 9 we have

$$\begin{aligned}&(S_1, S_2, i) \in E_{\min} \;\Leftrightarrow\; \forall\, a \in S_1,\, \exists\, b \in S_2 \,:\, (a, b, i) \in E\\ &\Leftrightarrow\; \forall\, a \in S_1,\, \exists\, b \in S_2 \,:\, (b, a, i) \in E^\top \;\Leftrightarrow\; (S_2, S_1, i) \in E^\top_{\max}\\ &\Leftrightarrow\; (S_1, S_2, i) \in (E^\top_{\max})^\top,\end{aligned}$$

concluding the proof of (6a). Equation (6b) trivially follows with similar arguments. □

The validity of the min lift (in the sense of the relation (4) in Definition 4) has already been proved regarding the class of templates closed under pointwise minimum in [9]. Indeed, we have proved that for any switched system F, if there exists a solution $V := \{V_s \mid s \in S\}$ in such a template $\mathcal{V}$ admissible for F and a path-complete graph $\mathcal{G}$, then the graph $\mathcal{G}_{\min}$ also admits a solution in $\mathcal{V}$ for F, denoted by W. In practice, we showed that the path-complete Lyapunov function W can be defined as

$$W := \left\{W_P(x) := \min_{p\in P} V_p(x) \mid P \in S_{\min}, x \in \mathbb{R}^n\right\}. \tag{7}$$

The same result holds for the max lift and the corresponding pointwise maximum Lyapunov functions.

Theorem 2 (Theorem 3 in [9]). *Consider a family of binary operations $\{\star_n\}_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}$, $\star_n$ corresponds to the pointwise minimum (resp. pointwise maximum). The min (resp. max) lift is valid with respect to any template closed under $\{\star_n\}_{n\in\mathbb{N}}$.*

Inspired by the complete characterization of the general ordering relation (5) in Theorem 1, we tackle in this section the same problem for the less restrictive ordering relation (4) for a family of templates that all share the same closure property. In particular, we deal with the closure property of pointwise minimum in Theorem 3 and we prove that the validity of the relation (4) for any template closed under pointwise minimum is captured by the min lift.

Theorem 3. *Consider two path-complete graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$. The following statements are equivalent:*

(1) *$\mathcal{G}_{\min}$ simulates $\widetilde{\mathcal{G}}$.*
(2) *$\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$ for any template $\mathcal{V}$ closed under pointwise minimum.*

Our proof follows the same path of ideas as Theorem 3.5 in [19]. However, some modifications are needed to manage the closure property. Therefore, we split the proof of Theorem 3 in two parts. In Section 3.1 we prove a technical lemma that will be central in the main proof, which is presented in Section 3.2. In Section 3.3, we use duality to prove a similar theorem (see Theorem 4) for the max lift.

3.1 A technical lemma

We start by proving a technical result that states that, for any graph $\mathcal{G} = (S, E)$ on the alphabet $\langle M\rangle$, i.e. for any set of Lyapunov inequalities, it is possible to build a switched system F on M modes and a solution $V \in PCLF(\mathcal{G}, F)$ for which the associated pointwise minimum Lyapunov functions (7) satisfy the inequalities encoded by $\mathcal{G}_{\min}$ by construction but none of the non-existing edges of $\mathcal{G}_{\min}$ hold. This construction has been implemented in MATLAB and is available in [7].

LEMMA 2. *For any graph $\mathcal{G} = (S, E)$ on the alphabet $\langle M \rangle$, there exist an integer $n \geq 1$, a system $F := \{f_i \mid i \in \langle M \rangle\}$ in dimension n and $|S|$ candidate Lyapunov functions V_s, $s \in S$ for which*

$$\forall (p, q, i) \in E,\ \forall x \in \mathbb{R}^n:\ V_q(f_i(x)) \leq V_p(x), \quad (8)$$

$$\forall (P, Q, i) \in \overline{E_{\min}},\ \exists \overline{x} \in \mathbb{R}^n:\ \min_{q \in Q} V_q(f_i(\overline{x})) > \min_{p \in P} V_p(\overline{x}), \quad (9)$$

where $\overline{E_{\min}} = (S_{\min} \times S_{\min} \times \langle M \rangle) \setminus E_{\min}$ refers to the set of non-existing edges *of $\mathcal{G}_{\min}$.*

PROOF OF LEMMA 2. Given a graph $\mathcal{G} = (S, E)$ on the alphabet $\langle M \rangle$, we define a set of M block-diagonal $\{0, 1\}$-matrices $\{A_j \mid j \in \langle M \rangle\}$ of dimension $n = 2|\overline{E_{\min}}|$. Each block is associated to a non-existing edge $\tilde{e}$ of $\mathcal{G}_{\min}$, and is defined by

$$A_j[\tilde{e}] := \begin{cases} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \text{if } j = label(\tilde{e}), \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{otherwise,} \end{cases} \quad (10)$$

so that each matrix only acts on the blocks associated to the edges of the same label. We consider as template, a finite set of *weighted L_1 norms*, i.e. for any $s \in S, \forall x \in \mathbb{R}^n,\ V_s(x) := {v_s}^\top |x|$, where $v_s \in \mathbb{R}^n_{>0}$ and $|x|$ denotes the componentwise absolute value of $x \in \mathbb{R}^n$. In this context, since $A_j \in \mathbb{R}^{n \times n}_{\geq 0}$ for any $j \in \langle M \rangle$, satisfying a Lyapunov inequality $(s, d, i) \in E$ amounts to satisfying a set of $|\overline{E_{\min,i}}|$ scalar inequalities where $\overline{E_{\min,i}} := \{\tilde{e} \in \overline{E_{\min}} \mid label(\tilde{e}) = i\}$ since

$$\begin{aligned} & \forall x \in \mathbb{R}^n,\ V_d(A_i x) \ \leq\ V_s(x), \\ \Leftrightarrow \quad & {A_i}^\top v_d \ \leq_c\ v_s, \\ \Leftrightarrow \quad & \forall \tilde{e} \in \overline{E_{\min,i}},\ v_d[\tilde{e}]_2 \ \leq\ v_s[\tilde{e}]_1, \end{aligned} \quad (11)$$

where $\leq_c$ depicts a componentwise inequality, i.e. if $a, b \in \mathbb{R}^n$, $a \leq_c b \Leftrightarrow \forall i \in \langle n \rangle, a_i \leq b_i$. Given an edge $\tilde{e} = (P, Q, i) \in \overline{E_{\min}}$, we denote by $I(\tilde{e}) := \{p \in P \mid \forall q \in Q,\ (p, q, i) \notin E\}$. We define the blocks $v_s[\tilde{e}]$ for any $s \in S$ such that all the inequalities induced from E_i are satisfied but $\tilde{e}$ is violated. Therefore,

$$v_s[\tilde{e}]_1 := \begin{cases} 1 & \text{if } s \in I(\tilde{e}), \\ 3 & \text{otherwise.} \end{cases} \quad (12)$$

$$v_s[\tilde{e}]_2 := \begin{cases} 1 & \text{if } \exists l \in I(\tilde{e}) \text{ s.t. } (l, s, i) \in E, \\ 2 & \text{otherwise.} \end{cases} \quad (13)$$

We show now that this construction satisfies the expressions (8) and (9). We begin with the expression (8) that is equivalent to (11) holding for any $(s, d, i) \in E$ here. Given both edges $e = (s, d, i) \in E$ and $\tilde{e} = (P, Q, i) \in \overline{E_{\min}}$, the inequality in (11) would be violated only if $v_s[\tilde{e}]_1 = 1$ and $v_d[\tilde{e}]_2 = 2$. However, this cannot happen since it implies that $(s, d, i) \notin E$. In all the other configurations, the inequality (11) is satisfied.

We now focus on (9). Consider $\tilde{e}_1 = (P_1, Q_1, i) \in \overline{E_{\min}}$, by the same argument as in (11) and [8][Lemma 1], we have to prove that

$$\exists \tilde{e}_2 = (P_2, Q_2, i) \in \overline{E_{\min}}:\ \bigvee_{q \in Q_1} v_q[\tilde{e}_2]_2 \ >\ \bigvee_{p \in P_1} v_p[\tilde{e}_2]_1, \quad (14)$$

where $v_a \vee v_b$ denotes the componentwise minimum between the vectors v_a and v_b. We show that we can choose $\tilde{e}_2 = \tilde{e}_1$ to achieve this. Since $\tilde{e}_1 \in \overline{E_{\min}}$, $I(\tilde{e}_1)$ is not empty and the minimum of $v_p[\tilde{e}_1]_1$ over P_1 is 1. Moreover, for any $q \in Q_1$, $v_q[\tilde{e}_1]_2 = 2$ because for all $l \in I(\tilde{e}_1)$ and for all $q \in Q_1$, $(l, q, i) \notin E$ by definition of $I(\tilde{e}_1)$. Then, the minimum of $v_q[\tilde{e}_1]_2$ over Q_1 is 2. This concludes the proof of Lemma 2. □

3.2 Proof of the main theorem

Using Lemma 2, it is now possible to prove the main Theorem 3.

PROOF OF THEOREM 3. (1) ⟹ (2): By Theorem 2, the inequality $\mathcal{G} \leq_{\mathcal{V}} \mathcal{G}_{\min}$ holds for any template $\mathcal{V}$ closed under pointwise minimum. By assumption and recalling Theorem 1, we have $\mathcal{G}_{\min} \leq \widetilde{\mathcal{G}}$. By transitivity, we have that $\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$.
(2) ⟹ (1): Our proof follows the same path of ideas as Theorem 3.5 in [19]. Consider two graphs $\mathcal{G}$, $\widetilde{\mathcal{G}}$ and suppose that $\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$ for any template $\mathcal{V}$ closed under pointwise minimum. First, by applying Lemma 2 to the graph $\mathcal{G}$, we obtain a system $F := \{f_i \mid i \in \langle M \rangle\}$ and a set of candidate Lyapunov functions $\{V_s \mid s \in S\}$ such that the Lyapunov inequalities encoded by the edges of $\mathcal{G}$ are satisfied and none of the non-existing edges of $\mathcal{G}_{\min}$ are satisfied, i.e.

$$\forall (p_1, q_1, i) \in E,\ \forall x \in \mathbb{R}^n:\ V_{q_1}(f_i(x)) \leq V_{p_1}(x), \quad (15)$$

$$\forall (P_1, Q_1, i) \notin E_{\min},\ \exists x \in \mathbb{R}^n:\ \min_{q \in Q_1} V_q(f_i(x)) > \min_{p \in P_1} V_p(x). \quad (16)$$

Let us define the family $\mathcal{F} = \{F\}$ and the template

$$\mathcal{V} \ :=\ \{W_{P_1} := \min_{p \in P_1} V_p \mid P_1 \in S_{\min}\}$$

where $S_{\min} = \mathcal{P}(S) \setminus \emptyset$. Then, $\{V_s \mid s \in S\} \subset \mathcal{V}$ and the template $\mathcal{V}$ is closed under minimum such that $\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$. Obviously, there exists a solution admissible for $\mathcal{G}$ and F in $\mathcal{V}$. Then, by assumption, there exists a set of Lyapunov functions in the template $\mathcal{V}$ which are admissible for F and $\widetilde{\mathcal{G}}$, i.e. there exist $\{U_{\widetilde{s}} \mid \widetilde{s} \in \widetilde{S}\} \subseteq \mathcal{V}$ that satisfy the Lyapunov inequalities encoded by $\widetilde{\mathcal{G}}$. Since these functions belong to the template $\mathcal{V}$, we can associate a subset of S to each node of $\widetilde{\mathcal{G}}$, i.e. we can define a function $R : \widetilde{S} \to S_{\min}$ such that $U_{\widetilde{s}} = W_{R(\widetilde{s})}$. Finally, we just have to prove that this function R satisfies the definition of simulation, i.e.

$$\forall (p_2, q_2, i) \in \widetilde{E},\ (R(p_2), R(q_2), i) \in E_{\min}.$$

Assume by contradiction that there exists $(p_2, q_2, i) \in \widetilde{E}$ such that $(R(p_2), R(q_2), i) \notin E_{\min}$. Using Lemma 2, this means that there exists $\overline{x} \in \mathbb{R}^n$ such that

$$W_{R(q_2)}(\overline{x}) \ :=\ \min_{q \in R(q_2)} V_q(f_\sigma(\overline{x})) \ >\ \min_{p \in R(p_2)} V_p(\overline{x}) \ :=\ W_{R(p_2)}(\overline{x})$$

i.e. the set $\{U_{\widetilde{s}} \mid \widetilde{s} \in \widetilde{S}\}$ is not admissible. But it is by construction. Here is the contradiction. □

We illustrate Theorem 3 in the following example.

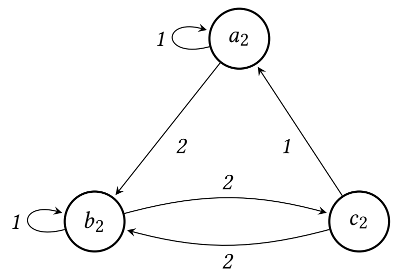


Figure 2: The graph $\mathcal{G}_2 = (S_2, E_2)$ in Example 2 such that $\mathcal{G}_1 \not\leq \mathcal{G}_2$ and $\mathcal{G}_2 \not\leq \mathcal{G}_1$ but $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_2$ and $\mathcal{G}_2 \leq_{\mathcal{V}} \mathcal{G}_1$ for any template $\mathcal{V}$ closed under pointwise minimum.

Example 2. *Consider the graph $\mathcal{G}_1 = (S_1, E_1)$ in Figure 1b and $\mathcal{G}_2 = (S_2, E_2)$ in Figure 2, and let us compare them thanks to Theorem 3. First, we prove that $\mathcal{G}_1$ does not simulate $\mathcal{G}_2$. Assume by contradiction that there exists a simulation function $R : S_2 \to S_1$. Because of the self-loops $(a_2, a_2, 1)$ and $(b_2, b_2, 1)$, we must have that $R(a_2) = R(b_2) := a_1$. The node c_2 admits an incoming edge with label 2, then $R(c_2) := b_1$ since the node a_1 only admits incoming edges with label* 1*. But, the edge $(a_2, b_2, 2) \in E_2$ and $(R(a_2), R(b_2), 2) = (a_1, a_1, 2) \notin E_1$, proving that R cannot be a simulation function. A similar approach can be applied to prove that $\mathcal{G}_2$ does not simulate $\mathcal{G}_1$. These two facts, recalling Theorem 1, are equivalent to $\mathcal{G}_1 \not\leq \mathcal{G}_2$ and $\mathcal{G}_2 \not\leq \mathcal{G}_1$.*

In contrast, we can prove with Theorem 3 that both graphs are equivalent in the sense of Definition (4) *for any template closed under pointwise minimum. Define the function $R_1 : S_1 \to S_{2\min}$ where $R_1(a_1) := \{a_2, b_2\}$ and $R_1(b_1) := \{b_2, c_2\}$. We can prove that R_1 is a simulation function. Let us take for instance the edge $(a_1, b_1, 2) \in E_1$. The edge $(R_1(a_1), R_1(b_1), 2) \in E_{2min}$ because $(a_2, b_2, 2)$ and $(b_2, c_2, 2) \in E_2$. By Theorem 3, it implies that $\mathcal{G}_2 \leq_{\mathcal{V}} \mathcal{G}_1$ for any template $\mathcal{V}$ closed under pointwise minimum. One can prove the reverse inequality with the function $R_2 : S_2 \to S_{1\min}$ where $R_2(a_2) = \{a_1\}$, $R_2(b_2) := \{a_1, b_1\}$ and $R_2(c_2) := \{b_1\}$.*

Consider the positive linear switched system $\mathcal{A} := \{A_1, A_2\}$ with

$$A_1 := \begin{bmatrix} 0.9 & 0.3 \\ 0.9 & 0.7 \end{bmatrix}, \; A_2 := \begin{bmatrix} 0.6 & 0.9 \\ 0.6 & 0.3 \end{bmatrix}, \tag{17}$$

and assume that we use the template of linear copositive norms (denoted by C) where the candidate Lyapunov functions are defined as a scalar product, i.e. $V(x) := v^\top x$, with $v \in \mathbb{R}^2_{>0}$. Since this template is closed under pointwise minimum, we expect that the approximations of the joint spectral radius (JSR) provided by $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\rho_{\mathcal{G}_1,C}(\mathcal{A})$ and $\rho_{\mathcal{G}_2,C}(\mathcal{A})$ respectively, are the same. Indeed, using the YALMIP toolbox [16] we get

$$\rho_{C,\mathcal{G}_1}(\mathcal{A}) \;=\; \rho_{C,\mathcal{G}_2}(\mathcal{A}) \;=\; 1.549.$$

If instead we consider the template of quadratic functions Q defined by

$$Q := \{f_Q : \mathbb{R}^n \to \mathbb{R} \mid Q \succ 0\},$$

with $f_Q(x) := \sqrt{x^\top Q x}$, which is not closed under pointwise minimum, it may happen that the results provided by the graphs are different. Indeed, for the system (17)*, we obtain*

$$\rho_{Q,\mathcal{G}_1}(\mathcal{A}) \;=\; 1.356 \;<\; 1.364 \;=\; \rho_{Q,\mathcal{G}_2}(\mathcal{A}). \tag{18}$$

This inequality proves that graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are not equivalent with respect to the family of quadratic Lyapunov functions.

3.3 Dual theorem for the max lift

We already noted the duality relation between the max and min lifts in Lemma 1. It turns out that we can obtain a result similar to Lemma 2 for the max lift.

Lemma 3. *For any graph $\mathcal{G} = (S, E)$ on the alphabet $\langle M \rangle$, there exist an integer $n \geq 1$, a system $F := \{f_i \mid i \in \langle M \rangle\}$ in dimension n and $|S|$ candidate Lyapunov functions U_s, $s \in S$ for which*

$$\forall (p, q, i) \in E, \; \forall x \in \mathbb{R}^n : \; U_q(f_i(x)) \leq U_p(x), \tag{19}$$

$$\forall (P, Q, i) \in \overline{E_{\max}}, \; \exists \, \overline{x} \in \mathbb{R}^n : \; \max_{q \in Q} U_q(f_i(\overline{x})) > \max_{p \in P} U_p(\overline{x}), \tag{20}$$

where $\overline{E_{\max}} = (S_{\max} \times S_{\max} \times \langle M \rangle) \setminus E_{\max}$ refers to the set of non-existing edges *of $\mathcal{G}_{\max}$.*

Proof. Our construction is derived from the one in the proof of Lemma 2 in Section 3.1. In particular, the (linear) system F given by matrices $\{\overline{A}_1, \ldots, \overline{A}_M\}$ and the functions U_s are obtained from the ones introduced in the proof of Lemma 2, simply defining

$$\overline{A}_j := A_j^\top, \quad j \in \langle M \rangle \quad \text{and} \quad U_s(x) := \max_{i \in \langle n \rangle} \left\{ \frac{|x_i|}{v_{si}} \right\}, \quad s \in S,$$

with matrices A_j defined in (10) and vectors v_s defined in (12) and (13). Indeed, the functions U_s are the conjugate functions ([20, page 104]) of the V_s in proof of Lemma 2, and by convex duality theory (see [20, Theorem 16.3]) we have that, given any convex functions $g_1, g_2 \in C^0(\mathbb{R}^n, \mathbb{R}^n)$ and $A \in \mathbb{R}^{n \times n}$, the following equivalence holds:

$$g_2(Ax) \leq g_1(x) \;\; \forall x \in \mathbb{R}^n \;\; \Leftrightarrow \;\; g_1^\star(A^\top x) \leq g_2^\star(x), \;\; \forall x \in \mathbb{R}^n,$$

where $g^\star$ denotes the convex conjugate of a convex and proper function g. □

We can now obtain a result similar to Theorem 3 for the max lift and the class of templates closed under pointwise maximum.

Theorem 4. *Consider two path-complete graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$. The following statements are equivalent:*

1. *$\mathcal{G}_{\max}$ simulates $\widetilde{\mathcal{G}}$.*
2. *$\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$ for any template $\mathcal{V}$ closed under pointwise maximum.*

Proof. We only sketch the proof, which follows the one of Theorem 3. We develop here, avoiding some details, the main ideas. Similarly to Theorem 3, the implication (1) ⇒ (2) is a simple consequence of Theorem 2. Regarding the reverse implication, we consider two path-complete graphs $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ such that $\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$ for any template $\mathcal{V}$ closed under pointwise maximum. Applying Lemma 3, we obtain a system F and a set of candidate Lyapunov functions $\{U_s \mid s \in S\}$ such that (19) and (20) are satisfied. Define the family $\mathcal{F} := \{F\}$ and the pointwise maximum closure template

$$\mathcal{V} \;:=\; \{Z_{P_1} := \max_{p \in P_1} U_p \mid P_1 \in S_{\max}\}.$$

By construction, there exists a solution in $\mathcal{V}$ admissible for F and $\mathcal{G}$. Then, by assumption, we can find a solution $\{Y_{\tilde{s}} \mid \tilde{s} \in \widetilde{S}\}$ in

$\mathcal{V}$ admissible for F and $\widetilde{\mathcal{G}}$ that implicitly defines a function $R : \widetilde{S} \to S_{\max}$ such that $Y_{\tilde{s}} := Z_{R(\tilde{s})}$. We conclude following the same reasoning by contradiction as in the proof of Theorem 3. □

4 T-SUM LIFT

In this section, we focus on the binary operation $\{\star_n\}_{n\in\mathbb{N}}$ (as in Definition 8) given by the sum/addition. Note that many usual candidate Lyapunov functions templates are closed under sum, as for instance the set of quadratic functions. To that end, we consider the *T-sum lift* and the *sum lift* that capture the Lyapunov inequalities between *sums* that are induced from an initial path-complete graph. In what follows, we define a *multi-set* as a set with possible repetitions and $Multi^T(S)$ denotes the set of multi-sets of cardinality $T \in \mathbb{N}$ with elements of S. For example, $\{a, a, b, b, b\}$ is a multi-set of $\{a, b\}$ of cardinality 5.

Definition 10 (T-sum/Sum lift). *Given $T \in \mathbb{N}$ and a graph $\mathcal{G} = (S, E)$ on the alphabet $\langle M\rangle$.*

(a) *The T-sum lift, denoted by $\mathcal{G}^{\oplus T} = (S^{\oplus T}, E^{\oplus T})$, is defined as follows :*
  (1) *The set of nodes $S^{\oplus T}$ is defined by*
$$S^{\oplus T} := Multi^T(S).$$
  (2) *For each $i \in \langle M\rangle$ and each multi-set of edges of E of the form $\{(a_1, b_1, i), \ldots, (a_T, b_T, i)\}$ such that $\{a_1, \ldots, a_T\}$ and $\{b_1, \ldots, b_T\} \in S^{\oplus T}$, the edge $(\{a_1, \ldots, a_T\}, \{b_1, \ldots, b_T\}, i) \in E^{\oplus T}$.*

(b) *The sum lift, denoted by $\mathcal{G}^{\oplus} = (S^{\oplus}, E^{\oplus})$, is defined as the infinite disjoint union of all the T-sum lifts, i.e.*
$$\mathcal{G}^{\oplus} := \bigcup_{T\in\mathbb{N}} \mathcal{G}^{\oplus T}.$$

Note that, given a path-complete graph $\mathcal{G} = (S, E)$, $T \in \mathbb{N}$, two multi-sets $A, B \in Multi^T(S)$ and $i \in \langle M\rangle$, the edge (A, B, i) belongs to the T-sum lift of $\mathcal{G}$ if and only if there exists a *perfect matching* in the bipartite graph (A, B, E_i), where E_i is the restriction of E to the edges of label i.

As expected, the validity of these lifts in the sense of (4) in Definition 4 holds for any template closed under sum. Indeed in [9], we proved that, for any $T \in \mathbb{N}$ and for any switched system F, if there exists a solution V in a template $\mathcal{V}$ closed under sum admissible for F and a path-complete graph $\mathcal{G}$, then the candidate path-complete Lyapunov function W defined as

$$W := \left\{ W_P := \sum_{p\in P} V_p \mid P \in S^{\oplus} \right\} \subseteq \mathcal{V}$$

is admissible for F and $\mathcal{G}^{\oplus}$. This proves that for any path-complete graph $\mathcal{G}$, $\mathcal{G} \leq_{\mathcal{V}} \mathcal{G}^{\oplus}$ for any template $\mathcal{V}$ closed under sum and therefore for any $T \in \mathbb{N}$, $\mathcal{G} \leq_{\mathcal{V}} \mathcal{G}^{\oplus T}$ for these templates as well.

Theorem 5 (Theorem 3 in [9]). *Consider $T \in \mathbb{N}$, the T-sum lift is valid with respect to any template closed under sum. Similarly, the sum-lift is valid with respect to any template closed under sum.*

Theorems 3 and 4 and their proofs suggest that we could use the sum lift in Definition 10 to characterize the ordering inequality (4) for the class of templates closed under sum.

Example 3. *In Example 2 we have shown that, considering the template of quadratic functions $\mathbf{Q}$, we have*

$$\rho_{\mathbf{Q},\mathcal{G}_1}(\mathcal{A}) < \rho_{\mathbf{Q},\mathcal{G}_2}(\mathcal{A})$$

for a positive switched system $\mathcal{A}$, recall (18). *This gap in the estimation of the JSR can be interpreted in a more general setting, by applying the sum lift.*

Indeed, one can prove that the 2-sum lift of $\mathcal{G}_2$ simulates $\mathcal{G}_1$ by defining a function $R : S_1 \to S_2^{\oplus 2}$ where $R(a_1) := \{a_2, b_2\}$ and $R(b_1) := \{b_2, c_2\}$. By Theorem 5 and transitivity, we conclude that for any template $\mathcal{V}$ closed under sum,

$$\mathcal{G}_2 \leq_{\mathcal{V}} \mathcal{G}_1.$$

This inequality implies that, given any linear switched system $\mathcal{A}$, the approximation of the JSR of $\mathcal{A}$ provided by $\mathcal{G}_2$ is at best equal to the one supplied by $\mathcal{G}_1$, provided that we consider a template of Lyapunov functions that is closed under sum. In particular, this inequality implies that for any linear switched system $\mathcal{A}$,

$$\rho_{\mathbf{Q},\mathcal{G}_2}(\mathcal{A}) \leq \rho_{\mathbf{Q},\mathcal{G}_1}(\mathcal{A}),$$

and for the system (17) *especially, as reflected in* (18). *Moreover, we can conclude from the result in* (18) *that*

$$\mathcal{G}_1 \not\leq_{\mathcal{V}} \mathcal{G}_2$$

for any template $\mathcal{V}$ closed under sum.

The ordering relation (4) for templates closed under sum has been tackled in [17]; In this work, the author raised an open question that, in the context of the T-sum lift, concerns the relevance of the use of multi-sets instead of subsets, as it is the case for the min and max lifts. More specifically, Question 6.1 in [17, pg. 184] conjectured the following open question.

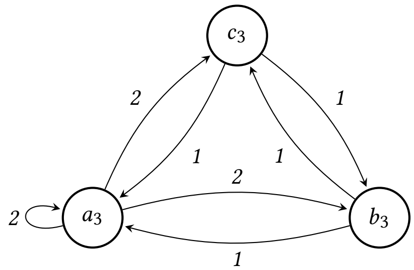
Question 1 ([17]). *Given two path-complete graphs $\mathcal{G}_1 = (S_1, E_1)$ and $\mathcal{G}_2 = (S_2, E_2)$ such that there exists a non-zero matrix $C \in \mathbb{R}_{\geq 0}^{|S_2|\times|S_1|}$ such that, $\forall n \in \mathbb{N}, \forall F \subseteq C^0(\mathbb{R}^n, \mathbb{R}^n)$,*

$$V \in PCLF(\mathcal{G}_1, F) \Rightarrow W := CV \in PCLF(\mathcal{G}_2, F),$$

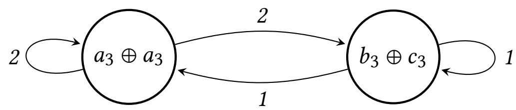
(which implies that $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_2$ for any template $\mathcal{V}$ closed under sum) then it is possible to find a matrix $\widetilde{C}$ with $\{0, 1\}$-elements satisfying the same implication.

In terms of lifts, Question 1 corresponds to restrict the set of nodes of the sum-lifted graph $\mathcal{G}^{\oplus}$ to the set of non-empty *subsets* of S in Definition 10; in other words it asks whether we can avoid to consider "repetitions" of nodes, thus avoiding to consider multi-sets. However, Example 4 shows that it is not enough to consider the power set of the initial set of nodes S in the sense that it is not possible to identify all the ordering relations between path-complete graphs.

Example 4 (Counter-example to Question 1). *Consider the graph $\mathcal{G}_3 = (S_3, E_3)$ on 2 modes in Figure 3a, and consider its 2-sum lift $\mathcal{G}_3^{\oplus 2} = (S_3^{\oplus 2}, E_3^{\oplus 2})$. If we restrict the nodes of this graph to the nodes associated to the non-empty subsets of S_3 of cardinality 2, i.e. $S_3^{\oplus 2} := \{\{a_3, b_3\}, \{a_3, c_3\}, \{b_3, c_3\}\}$, the lifted graph does not present any path-complete subgraphs. Therefore, one cannot establish any inequality between $\mathcal{G}_3$ and any path-complete graph $\widetilde{\mathcal{G}}$ in the sense of Definition 4 where the solution of $\widetilde{\mathcal{G}}$ would be expressed as the sum with $\{0, 1\}$-coefficients of the pieces of a solution of $\mathcal{G}_3$.*



(a) The graph $\mathcal{G}_3 = (S_3, E_3)$.



(b) The graph $\mathcal{G}_4 = (S_4, E_4)$, a strongly connected and path-complete component of ${\mathcal{G}_3}^{\oplus 2}$.

Figure 3: Counter-example to Question 1. As discussed in Example 4, $\mathcal{G}_4$ is a component of ${\mathcal{G}_3}^{\oplus 2}$ but not of its restriction to the subsets of the nodes of $\mathcal{G}_3$.

While considering ${S_3}^{\oplus 2}$ as in Definition 9, the 2-sum lifted graph contains the path-complete and strongly connected component $\mathcal{G}_4$ in Figure 3b. This means that $\mathcal{G}_3 \leq_{\mathcal{V}} \mathcal{G}_4$ for any template $\mathcal{V}$ closed under sum, but it would have been impossible to conclude this if we have restricted the definition of the sum lift to the non-empty subsets of S_3.

Note that in this example, the nodes associated to the multi-sets $\{a_3, b_3\}$, $\{a_3, c_3\}$, $\{b_3, b_3\}$ and $\{c_3, c_3\}$ are useless in the sense that $\mathcal{G}_4$ is the only strongly connected and path-complete component of ${\mathcal{G}_3}^{\oplus 2}$.

Similarly to the cases of min and max in Section 3, we provide, in the following statement, a similar result to Lemmas 2 and 3 for the T-sum lift. This construction has also been implemented in MATLAB and is available in [7].

LEMMA 4. *For any graph $\mathcal{G} = (S, E)$ on the alphabet $\langle M \rangle$ and any $T \in \mathbb{N}$, there exist an integer $n \geq 1$, a system $F := \{f_i \mid i \in \langle M \rangle\}$ in dimension n and $|S|$ candidate Lyapunov functions V_s, $s \in S$ for which*

$$\forall (p, q, i) \in E,\ \forall x \in \mathbb{R}^n : V_q(f_i(x)) \leq V_p(x), \quad (21)$$

$$\forall (P, Q, i) \in \overline{E^{\oplus T}},\ \exists \overline{x} \in \mathbb{R}^n : \sum_{q \in Q} V_q(f_i(\overline{x})) > \sum_{p \in P} V_p(\overline{x}), \quad (22)$$

where $\overline{E^{\oplus T}} = (S^{\oplus T} \times S^{\oplus T} \times \langle M \rangle) \setminus E^{\oplus T}$ refers to the set of non-existing edges of $\mathcal{G}^{\oplus T}$.

PROOF OF LEMMA 4. We follow the outline of the proof of Lemma 2. We consider M block-diagonal $\{0, 1\}$-matrices $\{A_i \mid i \in \langle M \rangle\}$ of dimension $n = 2\left|\overline{E^{\oplus T}}\right|$ for which each 2×2 block $A_j[\tilde{e}]$ associated to a non-existing edge $\tilde{e} \in \overline{E^{\oplus T}}$ is defined by (10). We use the template of weighted L_1 norms for which satisfying a Lyapunov inequality (s, d, i) amounts to satisfying (11). Moreover, proving that the Lyapunov inequality encoded by an edge $(P, Q, i) \in S^{\oplus T} \times S^{\oplus T} \times \langle M \rangle$ amounts to showing that a perfect matching exists in the bipartite graph (P, Q, E_i). Then, given an edge $\tilde{e} = (P, Q, i) \in \overline{E^{\oplus T}}$, Hall's marriage Theorem implies that there exists a non-empty multi-set denoted by $X(\tilde{e}) \subseteq Q$ such that $|X(\tilde{e})| > |N_{i,P}(X(\tilde{e}))|$, where $N_{i,P}(X)$ denotes the multi-set of the neighbourhood of X through edges of label i in P. We define the blocks $v_s[\tilde{e}]$ for any $s \in S$ such that all the inequalities induced from E_i are satisfied but the one induced by $\tilde{e}$ does not hold. Therefore,

$$v_s[\tilde{e}]_1 := \begin{cases} 2 & \text{if } \exists d \in X(\tilde{e}) \text{ s.t. } (s, d, i) \in E, \\ 1 & \text{otherwise.} \end{cases} \quad (23)$$

$$v_s[\tilde{e}]_2 := \begin{cases} 2 & \text{if } s \in X(\tilde{e}), \\ 1 & \text{otherwise.} \end{cases} \quad (24)$$

We show now that this construction satisfies the expressions (21) and (22). The expression (21) is equivalent to (11) holding for any $(s, d, i) \in E$ here. Given both edges $e = (s, d, i) \in E$ and $\tilde{e} = (P, Q, i) \in \overline{E^{\oplus T}}$, the inequality in (11) would be violated only if $v_s[\tilde{e}]_1 = 1$ and $v_d[\tilde{e}]_2 = 2$. However, this cannot happen since it implies that $(s, d, i) \notin E$. In all the other configurations, the inequality (11) is satisfied.

We now focus on (22). Consider $\tilde{e}_1 = (P_1, Q_1, i) \in \overline{E^{\oplus T}}$, by the same argument as in (11), we have to prove that

$$\exists \tilde{e}_2 = (P_2, Q_2, i) \in \overline{E^{\oplus T}} : \sum_{q \in Q_1} v_q[\tilde{e}_2]_2 > \sum_{p \in P_1} v_p[\tilde{e}_2]_1. \quad (25)$$

We show we can choose $\tilde{e}_2 = \tilde{e}_1$ to achieve this. Since $\tilde{e}_1 \in \overline{E^{\oplus T}}$, $X(\tilde{e}_1)$ is not empty and the sum of the second components of $v_q[\tilde{e}_1]$ over Q_1 is equal to $2|X(\tilde{e}_1)| + T - |X(\tilde{e}_1)|$. Moreover, the sum of the first components of $v_p[\tilde{e}_1]$ over P_1 is $2|N_{i,Q_1}(X(\tilde{e}_1))| + T - |N_{i,Q_1}(X(\tilde{e}_1))|$. Since $|X(\tilde{e}_1)| > |N_{i,Q_1}(X(\tilde{e}_1))|$ by assumption, it means that the sum of the second components of $v_q[\tilde{e}_1]$ over Q_1 is strictly larger than the sum of the first components of $v_p[\tilde{e}_1]$ over P_1. This concludes the proof of Lemma 4. □

Unfortunately, this lemma, while possibly being a crucial tool, is not enough to provide a characterization of the ordering relation (4) for the class of templates closed under sum via the sum lift. Indeed, if we want to follow ideas of the proof of Theorem 3, we need to consider the sum-closure template of a set of Lyapunov functions. However, this template will contain an infinite number of functions whereas both the min and max-closure templates admit a finite number of elements due to the absorbing character of these closure properties (i.e., $\min\{V, V\} \equiv \max\{V, V\} \equiv V$ for any $V \in \mathcal{C}^0_+(\mathbb{R}^n, \mathbb{R})$, while, in general, $V + V \not\equiv V$). This major difference is reflected by the fact the the sum lift is defined on an infinite number of nodes, while the min and max lifts have exactly $|\mathcal{P}(S)| - 1 = 2^{|S|} - 1$ nodes. In practice, this implies that we must know in advance for which appropriate $T \in \mathbb{N}$ we apply Lemma 3 such that the contradiction at the end of the proof holds. This represents the main obstacle for having a complete characterization of the ordering relation on path-complete graphs for templates closed under sum, and further discussions are provided in the subsequent Section 5.

To conclude this section, we provide a result that discusses the relation between the lifts that we have introduced. Indeed, an edge

(A, B, i) needs to satisfy more constraints to be added in the T-sum lift in Definition 10 than in the min and max lifts in Definition 9. The following proposition compares these lifts in the sense of Definition 4.

Proposition 1. *Given a graph $\mathcal{G} = (S, E)$. For any $T \in \mathbb{N}$, $\mathcal{G}_{\min}$ and $\mathcal{G}_{\max}$ simulate $\mathcal{G}^{\oplus T}$.*

Proof. Let us prove that $\mathcal{G}_{\min} = (S_{\min}, E_{\min})$ simulates $\mathcal{G}^{\oplus T} = (S^{\oplus T}, E^{\oplus T})$, and define a function $R : S^{\oplus T} \to S_{\min}$ such that any multi-set P of cardinality T in $S^{\oplus T}$ is mapped by R to the corresponding set by removing repetitions (e.g., $R(\{a, a, b, e, e, e\}) = \{a, b, e\}$). Let us prove now that the function R is a simulation relation. Consider an edge $(P, Q, i) \in E^{\oplus T}$. Then, for any $p \in P$, there exists $q \in Q$ such that $(p, q, i) \in E$. This implies that for any $p \in R(P), \exists q \in R(Q) : (p, q, i) \in E$, i.e. $(R(P), R(Q), i) \in E_{\min}$.

The proof that $\mathcal{G}_{\max}$ simulates $\mathcal{G}^{\oplus T}$ follows a similar argument. □

This result is not surprising. Indeed given a path-complete graph $\mathcal{G} = (S, E)$, the T-sum lift requires that a bijection in E_i exists between two multi-sets A an B to add the edge (A, B, i) in $E^{\oplus T}$. A contrario, we add an edge (P, Q, i) in the min lift (resp. the max lift) if there exists a function $f : P \to Q$ (resp. $f : Q \to P$) such that for all $p \in P, (p, f(p), i) \in E$ (resp. for all $q \in Q, (f(q), q, i) \in E$). Then, all the edges of the T-sum lift satisfy the condition of the min lift, while the reverse is not true.

The following result uses Proposition 1 and Theorems 3 and 4 to provide a sufficient condition for ordering.

Corollary 1. *Given two path-complete graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$. If there exists $T \in \mathbb{N}$ such that $\mathcal{G}^{\oplus T}$ simulates $\widetilde{\mathcal{G}}$, then $\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$ for any template $\mathcal{V}$ closed under pointwise minimum. The same result holds for the pointwise maximum.*

Proof. Consider $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ for which there exists $T \in N$ such that $\mathcal{G}^{\oplus T}$ simulates $\widetilde{\mathcal{G}}$. By Proposition 1, $\mathcal{G}_{\min}$ simulates $\mathcal{G}^{\oplus T}$. Then, by transitivity, $\mathcal{G}_{\min}$ simulates $\widetilde{\mathcal{G}}$ and Theorem 3 ends the proof.

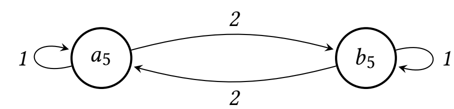
The proof for the pointwise maximum follows a similar argument. □

The following example illustrates Proposition 1.

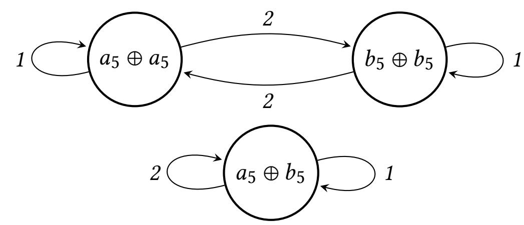
Example 5. *Consider the graph $\mathcal{G}_5 = (S_5, E_5)$ in Figure 4a, and its 2-sum lift $\mathcal{G}_5{}^{\oplus 2} = (S_5{}^{\oplus 2}, E_5{}^{\oplus 2})$ in Figure 4b. As outlined in Definition 10, $\mathcal{G}_5{}^{\oplus 2}$ admits a node for each multi-set of S_5 of cardinality 2, i.e. $\{a_5, a_5\}, \{a_5, b_5\}$ and $\{b_5, b_5\}$, and each multi-set of edges in E_5 leads to an edge in $\mathcal{G}_5{}^{\oplus 2}$. In particular, the multi-sets $\{(a_5, a_5, 1), (b_5, b_5, 1)\}$ and $\{(a_5, b_5, 2), (b_5, a_5, 2)\}$ provide the self-loops $(\{a_5, b_5\}, \{a_5, b_5\}, 1)$ and $(\{a_5, b_5\}, \{a_5, b_5\}, 2)$ in $\mathcal{G}_5{}^{\oplus 2}$. This implies that the common Lyapunov function graph, denoted by $\mathcal{G}_0 = (S_0, E_0)$, is a component of $\mathcal{G}_5{}^{\oplus 2}$.*

We claim that $\mathcal{G}_5{}^{\oplus 2}$, $\mathcal{G}_{5\min}$ and $\mathcal{G}_{5\max}$ are all equivalent to $\mathcal{G}_0$ in the sense of (5) in Definition 4.

First, the function $R : S_0 \to S_5{}^{\oplus 2}$ where $R(a_0) = \{a_5, b_5\}$ generates a simulation relation between $\mathcal{G}_5{}^{\oplus 2}$ and $\mathcal{G}_0$. By Theorem 1, this means that $\mathcal{G}_5{}^{\oplus 2} \leq \mathcal{G}_0$. The reverse inequality is trivially satisfied for the function that maps each node of $S_5{}^{\oplus 2}$ to the single node of $\mathcal{G}_0$. It follows that $\mathcal{G}_5{}^{\oplus 2}$ and $\mathcal{G}_0$ are equivalent in the sense of (5). By Proposition 1, we have that both $\mathcal{G}_{5\min}$ and $\mathcal{G}_{5\max}$ simulate $\mathcal{G}_5{}^{\oplus 2}$.



(a) The graph $\mathcal{G}_5 = (S_5, E_5)$.



(b) The 2-sum lift $\mathcal{G}_5{}^{\oplus 2} = (S_5{}^{\oplus 2}, E_5{}^{\oplus 2})$.

Figure 4: The graphs $\mathcal{G}_5$ and its 2-sum lift $\mathcal{G}_5{}^{\oplus 2}$ in Example 5.

Therefore, by transitivity, $\mathcal{G}_{5\min}$ and $\mathcal{G}_{5\max}$ are equivalent to $\mathcal{G}_0$ in the sense of (5).

In practice, it means that none of these three graphs should be used to prove the stability of a switched system since they will provide the same results as the common Lyapunov function, and they require more computations than $\mathcal{G}_0$.

Note that if we manage to characterize the ordering relation (4) for templates closed under sum by means of the sum lift, Proposition 1 would imply the following statement: given two path-complete graphs $\mathcal{G}$ and $\widetilde{\mathcal{G}}$, if $\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$ for any template $\mathcal{V}$ closed under sum, then $\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$ for any template $\mathcal{V}$ closed under pointwise minimum. The same statement would hold for the pointwise maximum.

5 CONCLUSION AND FURTHER DIRECTIONS

The path-complete Lyapunov framework generates a wide range of stability criteria for discrete-time switched systems by leveraging two degrees of freedom: the choice of the path-complete graph, and the nature of the candidate Lyapunov functions. In this paper, we have provided a characterization of the template-dependent ordering of path-complete graphs for the specific classes of templates closed under pointwise minimum and maximum (Theorems 3 and 4 respectively) by means of the combinatorial tool of simulation. In particular, given a switched system and a template, these results can help guiding the search of a better stability certificate by checking the existence of a simulation relation.

More broadly, the proof technique developed in this work seems to be generalizable (with appropriate minor modifications) and relevant for other settings, as the class of templates closed under addition introduced in Section 4. Moreover, the results provided in Section 3, even if they do not provide a general characterization for this kind of templates, suggest the following conjecture.

Conjecture 1. *Consider two path-complete graphs $\mathcal{G} = (S, E)$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{E})$. The following statements are equivalent:*

(1) *$\exists T \in \mathbb{N}$ such that $\mathcal{G}^{\oplus T}$ simulates $\widetilde{\mathcal{G}}$.*
(2) *$\mathcal{G} \leq_{\mathcal{V}} \widetilde{\mathcal{G}}$ for any template $\mathcal{V}$ closed under sum.*

Similarly to Theorems 3 and 4, the implication (1) ⇒ (2) holds by transitivity. The challenging part in Conjecture 1 lies in managing the infinite nature of this kind of templates, contrarily to the min and max closure properties that are absorbing. In the future, we plan to leverage the existing proofs to provide a general characterization for any closure property of template. This may also help us in the path-complete ordering characterization of a specific template, such as the set of quadratic functions, for instance.

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