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MODELS OF INTUITIONISTIC TT AND NF

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Abstract. Let us define the *intuitionistic part* of a classical theory T as the intuitionistic theory whose proper axioms are identical with the proper axioms of T. For example, Heyting arithmetic HA is the intuitionistic part of classical Peano arithmetic PA.

It's a well-known fact, proved by Heyting and Myhill, that ZF is identical with its intuitionistic part.

In this paper, we mainly prove that TT, Russell's Simple Theory of Types, and NF, Quine's "New Foundations," are not equal to their intuitionistic part. So, an intuitionistic version of TT or NF seems more naturally definable than an intuitionistic version of ZF.

In the first section, we present a simple technique to build Kripke models of the intuitionistic part of TT (with short examples showing bad properties of finite sets if they are defined in the usual classical way).

In the remaining sections, we show how models of intuitionistic NF_2 and NF can be obtained from well-chosen classical ones. In these models, the excluded middle will not be satisfied for some non-stratified sentences.

§1. Models of intuitionistic TT.

1.1. The axioms of TT. The language $\mathscr{L}_{\mathrm{TT}}$ of TT is a many-sorted language including variables $x^i, y^i, \ldots$ for each $i \in \mathbb{N}$. The atomic formulæ of $\mathscr{L}_{\mathrm{TT}}$ are of the form $x^i \in y^{i+1}$ or $x^i = y^i$, for each $i \in \mathbb{N}$.

We define *intuitionistic TT* as the intuitionistic theory whose proper axioms are exactly the proper axioms of classical TT (see [1] for more details about classical TT):

- extensionality axioms: for each $i \in \mathbb{N}$,
$$(\forall x^{i+1})(\forall y^{i+1})((\forall z^i)(z^i \in x^{i+1} \leftrightarrow z^i \in y^{i+1}) \rightarrow x^{i+1} = y^{i+1});$$
- comprehension axioms: $(\exists x^{i+1})(\forall z^i)(z^i \in x^{i+1} \leftrightarrow \varphi)$, for each formula φ in which x^{i+1} does not occur free and each $i \in \mathbb{N}$.

1.2. Models of intuitionistic TT. We shall now introduce a very simple technique to obtain Kripke models of intuitionistic TT, in which the reader might recognize ideas from [3] or [7].

Consider a model $\mathscr{M} = \langle M, \in^{\mathscr{M}} \rangle$ of classical ZF (a similar construction could be undertaken within models of other set theories, including classical TT, for example; see section 1.4 below). Within $\mathscr{M}$, we are going to define a Kripke structure $\mathscr{N} = \left\langle (\mathscr{N}_k)_{k \in K}, \langle K, \leqslant, \mathbf{0} \rangle \right\rangle$, where

- $\langle K, \leqslant, \mathbf{0} \rangle$ is a partial ordering such that $(\forall k \in K)(\mathbf{0} \leqslant k)$;

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- for each $k \in K$, $\mathscr{N}_k = \langle N_k^0, N_k^1, \ldots ; \in^{\mathscr{N}_k}, =^{\mathscr{N}_k} \rangle$ is a classical $\mathscr{L}_{\mathrm{TT}}$-structure, except that the equality relation $=^{\mathscr{N}_k}$ does not need to be standard;
- as usually in Kripke structures, $\mathscr{N}_k \subseteq \mathscr{N}_l$ whenever $k \leqslant l$ ($\mathscr{N}_k \subseteq \mathscr{N}_l$ means that the domain and relations of $\mathscr{N}_k$ are respectively *included* in the domain and relations of $\mathscr{N}_l$; it does not mean that $\mathscr{N}_k$ is a substructure of $\mathscr{N}_l$).

We want $\mathscr{N}$ to be a model of intuitionistic TT.

So let $\langle K, \leqslant, \mathbf{0} \rangle \in M$ such that $\mathscr{M} \models (\langle K, \leqslant, \mathbf{0} \rangle$ is a partial ordering). For each $i \in \mathbb{N}$, we are going to define, within $\mathscr{M}$, a class N^i of functions whose domain is a subset of K. Then, we shall define N_k^i as $\{x \in N^i \mid k \in \mathrm{dom}(x)\}$.

On the other hand, the elements of each N^i will contain all the information needed to characterize the $\in^{\mathscr{N}_k}$'s and $=^{\mathscr{N}_k}$'s. For example, if $x, y \in N_k^0$, we shall define $x =^{\mathscr{N}_k} y$ as $\mathscr{M} \models x(k) = y(k)$. On the other hand, if $x \in N_k^i$ and $y \in N_k^{i+1}$, we would like to define $x \in^{\mathscr{N}_k} y$ as $\mathscr{M} \models x \in y(k)$. But if we want $\mathscr{N}$ to satisfy the extensionality axiom, the definition of $\in^{\mathscr{N}_k}$ should be a little bit more sophisticated, as we shall see below.

Here are the details. Let N^0 be any non empty class of functions $x \in M$ such that the following properties are satisfied in $\mathscr{M}$:

- $x \neq \varnothing$ and $\mathrm{dom}(x) \subseteq K$;
- $(\forall k \in \mathrm{dom}(x))(\forall k' \in K)(k' \geqslant k \ \rightarrow\ k' \in \mathrm{dom}(x))$;
- $(\forall x' \in N^0)(\forall k \in K)(x(k) = x'(k) \ \rightarrow\ (\forall k' \geqslant k)(x(k') = x'(k')))$.

For each $k \in K$, we define $N_k^0 = \{x \in N^0 \mid k \in \mathrm{dom}(x)\}$.

Then the remaining N^i's are inductively defined from N^0 in the following way. Within $\mathscr{M}$, we define N^{i+1} as the class of *all* functions x such that

- $x \neq \varnothing$ and $\mathrm{dom}(x) \subseteq K$;
- if $k \in \mathrm{dom}(x)$, then $x(k) \subseteq N_k^i$;
- if $k \in \mathrm{dom}(x)$, then $(\forall k' \geqslant k)(k' \in \mathrm{dom}(x) \ \wedge\ x(k) \subseteq x(k'))$.

For each $k \in K$, we define, as above, $N_k^{i+1} = \{x \in N^{i+1} \mid k \in \mathrm{dom}(x)\}$.

Finally, $\mathscr{N} \Vdash_k x \in y$ (i.e., $x \in^{\mathscr{N}_k} y$) and $\mathscr{N} \Vdash_k x = y$ (i.e., $x =^{\mathscr{N}_k} y$) are defined as follows, by induction on the type of x and y:

- if $x, y \in N_k^0$, then $\mathscr{N} \Vdash_k x = y$ if and only if $\mathscr{M} \models x(k) = y(k)$;
- if $x \in N_k^i$ and $y \in N_k^{i+1}$, then $\mathscr{N} \Vdash_k x \in y$ if and only if $(\exists z \in N_k^i)(\mathscr{N} \Vdash_k x = z \ \wedge\ \mathscr{M} \models z \in y(k))$;
- if $x, y \in N_k^{i+1}$, then $\mathscr{N} \Vdash_k x = y$ if and only if $\mathscr{N} \Vdash_k \mathrm{Eq}[x, y]$, where $\mathrm{Eq}(x^{i+1}, y^{i+1}) \equiv (\forall z^i)(z^i \in x^{i+1} \leftrightarrow z^{i+1} \in y^{i+1})$, and $\Vdash_k$ is the usual satisfiability relation (forcing) for Kripke structures (for example, $\mathscr{N} \Vdash_k (\varphi \ \rightarrow\ \psi)$ is defined as $(\forall l \geqslant k)((\mathscr{N} \Vdash_l \varphi) \ \rightarrow\ (\mathscr{N} \Vdash_l \psi)))$.

It is easy to check that $\mathscr{N}$ is indeed a Kripke structure ($\mathscr{N}_k \subseteq \mathscr{N}_l$ when $k \leqslant l$), and that $\mathscr{N}$ satisfies the axioms of equality.

It is worth noting that, for a given $\mathscr{M}$, $\mathscr{N}$ is totally characterized by N^0.

It is also easy to prove the following *Definability Lemma*.

LEMMA 1. *Let $\varphi(x^i, y^j, \ldots)$ be a formula of $\mathscr{L}_{TT}$. Then there exists a formula $\Delta_\varphi(p_1, p_2, p_3, p_4, x, y, \ldots)$ in the language of ZF such that, for all $k \in K$, $a \in N_k^i$, $b \in N_k^j, \ldots,$*

$$\mathscr{N} \Vdash_k \varphi[a, b, \ldots] \leftrightarrow \mathscr{M} \models \Delta_\varphi[K, \leqslant, N^0, k, a, b, \ldots].$$

Now we can prove the main property of $\mathcal{N}$.

THEOREM 2. *$\mathcal{N}$ is a model of intuitionistic TT.*

PROOF. Let $k \in K$.

By definition of $\mathcal{N} \Vdash_k x = y$, it is clear that $\mathcal{N} \Vdash_k \varphi$, if φ is an extensionality axiom.

The comprehension is a little bit more difficult. We want to prove that, for any formula φ, and any values $a, b, \dots$ of its parameters, $\mathcal{N} \Vdash_k (\exists x^{i+1})(\forall z^i)(z^i \in x^{i+1} \leftrightarrow \varphi)[a, b, \dots]$.

$\mathcal{M}$ satisfies the comprehension schema of ZF. Thus, by the Definability Lemma, there exists some $x \in N_k^{i+1}$ such that, for all $l \geqslant k$,

$$\mathcal{M} \models x(l) = \{z \in N_l^i \mid \mathcal{N} \Vdash_l \varphi[z, a, b, \dots]\}.$$

For all $l \geqslant k$,

$$\begin{aligned}
\mathcal{N} \Vdash_l z \in x \quad &\Longleftrightarrow \quad (\exists z' \in N_l^i)(\mathcal{N} \Vdash_k z = z' \ \wedge\ \mathcal{M} \models z' \in x(l)) \\
&\Longleftrightarrow \quad (\exists z' \in N_l^i)(\mathcal{N} \Vdash_k z = z' \ \wedge\ \mathcal{N} \Vdash_l \varphi[z', a, b, \dots]) \\
&\Longleftrightarrow \quad (\mathcal{N} \Vdash_l \varphi[z, a, b, \dots]) \qquad \text{(equality axioms).}
\end{aligned}$$

Then, it is clear that $\mathcal{N} \Vdash_k (\exists x^{i+1})(\forall z^i)(z^i \in x \leftrightarrow \varphi(z^i))[a, b, \dots]$. $\dashv$

1.3. Application: a short study of finiteness. In this section, we plan to demonstrate how the technique described above can be used to show that the usual definition of finiteness in classical TT satisfies some "bad properties" in intuitionistic TT.

Finite sets in intuitionistic TT. As in classical TT, we define Fin^2 as $\bigcap\{E^2 \mid E^2$ is inductive$\}$, i.e., the smallest inductive set of type 2, where a set E^2 is *inductive* if and only if $\varnothing^1 \in E^2$ and $(\forall x^1 \in E^2)(\forall y^0)(x^1 \cup \{y^0\} \in E^2)$ (in an intuitionistic framework, it may be useful to state precisely that $\varnothing^1 = \{x^0 \mid \neg(x^0 = x^0)\}$, and $\{y^0\} = \{x^0 \mid x^0 = y^0\}$). As expected, the following induction principle can then be proved in intuitionistic TT:

$$\text{(1)} \quad \left[\varphi(\varnothing^1) \wedge \left(\forall x^1 \in \mathrm{Fin}^2\right)\left(\varphi(x^1) \rightarrow (\forall y^0)(\varphi(x^1 \cup \{y^0\}))\right)\right] \rightarrow (\forall x^1 \in \mathrm{Fin}^2)\varphi(x^1).$$

The first "bad" property of Fin is that

$$\text{(2)} \qquad (\forall x^1 \in \mathrm{Fin}^2)(\forall y^1)(y^1 \subseteq x^1 \rightarrow y^1 \in \mathrm{Fin}^2)$$

cannot be proved in intuitionistic TT. Before constructing a Kripke model of TT where (2) is not satisfied, it is worth noting that (2) is rather strong, as shown by Proposition 3.

From now on, we omit the type indices to improve readability of the formulæ. There should be no ambiguity; the type of Fin is always assumed to be 2.

PROPOSITION 3. *Let σ_1, σ_2, σ_3 and σ_4 be the following sentences:*

$$\begin{array}{ll} \sigma_1. & (\forall x \in Fin)(\forall y \subseteq x)(y \in Fin) \\ \sigma_2. & (\forall x)(\forall y \subseteq \{x\})(y \in Fin) \\ \sigma_3. & (\forall x)(\forall y \subseteq \{x\})[(\exists z)(z \in y) \vee \neg(\exists z)(z \in y)] \\ \sigma_4. & (\exists x)(\forall y \subseteq \{x\})[(\exists z)(z \in y) \vee \neg(\exists z)(z \in y)]. \end{array}$$

Then σ_1, σ_2, σ_3 and σ_4 are provably equivalent in intuitionistic TT. Furthermore, $\sigma_i \rightarrow (\varphi \vee \neg\varphi)$ can be proved in intuitionistic TT, for each $i \in \{1,2,3,4\}$ and each formula φ.

PROOF.
$\sigma_1 \rightarrow \sigma_2$: Trivial because $\{x\} \in$ Fin.
$\sigma_2 \rightarrow \sigma_3$: It is easy to prove that $(\forall x \in \text{Fin})[(\exists z)(z \in x) \vee \neg(\exists z)(z \in x)]$, by induction on $x \in$ Fin, using (1).
$\sigma_3 \rightarrow \sigma_1$: Prove σ_1 by induction on $x \in$ Fin, using σ_3 and $(\forall x, y \in \text{Fin})(x \cup y \in \text{Fin})$, which can be proved by induction on x.
$\sigma_4 \rightarrow (\varphi \vee \neg\varphi)$: This is a part of folklore (see for example [9]). Suppose that x and z do not occur free in φ, and define $E = \{z \mid z = x \wedge \varphi\}$. $E \subseteq \{x\}$. But $(\exists z)(z \in E)$ implies φ, while $\neg(\exists z)(z \in E)$ implies $\neg\varphi$. So $\sigma_4 \rightarrow (\varphi \vee \neg\varphi)$.
$\sigma_3 \leftrightarrow \sigma_4$: One direction is trivial, and the other one is a consequence of the previous point. ⊣

First example (constant domains, standard equality for type 0 objects). We are now going to build a Kripke model $\mathscr{N}$ of TT which does not satisfy σ_4. By Proposition 3, $\mathscr{N}$ will also fail to satisfy (2).

Consider any model $\mathscr{M}$ of ZF. Take any elements $\mathbf{0}, \alpha \in M$ and define $K = \{\mathbf{0}, \alpha\}$, with $\mathbf{0} \leqslant \alpha$. Finally, define $N^0 = \{x\}$, where x is any function whose domain is *equal* to K. K and N^0 are easily seen to exist in $\mathscr{M}$, as a consequence of the comprehension schema of ZF. From N^0, define $\mathscr{N}$ as in section 1.2.

Now consider the following two elements s and s' of N^1: s is a function such that $\mathscr{M} \models (s(\mathbf{0}) = s(\alpha) = \{x\})$, while $\mathscr{M} \models (s'(\mathbf{0}) = \varnothing \wedge s'(\alpha) = \{x\})$. Clearly, $\mathscr{N} \Vdash_{\mathbf{0}} (s = \{x\})$ and $\mathscr{N} \Vdash_{\mathbf{0}} (s' \subseteq s)$. But $\mathscr{N} \nVdash_{\mathbf{0}} (\exists z)(z \in s)$. Furthermore, $\mathscr{N} \Vdash_{\mathbf{0}} \neg\neg(s' = s)$ and so $\mathscr{N} \nVdash_{\mathbf{0}} \neg(\exists z)(z \in s')$. Thus $\mathscr{N}$ satisfies neither σ_4, nor (2).

An axiom of infinity. In classical TT, the axiom of infinity is defined to be $V^1 \notin \text{Fin}^2$, where V^1 is the set $\{x^0 \mid x^0 = x^0\}$, i.e., the universe of type 1. But in intuitionistic TT, $V \notin$ Fin does not seem appropriate as an axiom of infinity. For example, $V \notin$ Fin does *not* imply the existence of finite sets as large as you want. More precisely, in intuitionistic TT, $V \notin$ Fin does not imply

$$(3) \qquad (\forall x \in \text{Fin})(\exists y)(y \notin x).$$

This can be seen by means of the following example.

Second example (variable domains, standard equality for type 0 objects). Take any model $\mathscr{M}$ of ZF, and any $K = \{\mathbf{0}, \alpha\} \in M$, with $\mathbf{0} \leqslant \alpha$. In $\mathscr{M}$, define

$$N^0 = \{a\} \cup \{a_i \mid i \in \omega\},$$

where a is the function $\{(\mathbf{0},0),(\alpha,0)\}$, and each a_i is the function $\{(\alpha,i)\}$. From N^0, define $\mathscr{N}$ in $\mathscr{M}$ as described in section 1.2. So $N^0_{\mathbf{0}} = \{a\}$ and $N^0_\alpha = N^0$; in $N^0_{\mathbf{0}}$ and N^0_α, the equality is standard.

As a is the only element in N^0_α, $\mathscr{N} \nVdash_{\mathbf{0}} (\exists y)(y \notin \{a\})$. So $\mathscr{N} \nVdash_{\mathbf{0}} (\forall x \in \mathrm{Fin})(\exists y)(y \notin x)$.

On the other hand, $\mathscr{N}_\alpha$ may be considered as a classical structure, and, for any formula φ, $\mathscr{N} \Vdash_\alpha \varphi$ if and only if $\mathscr{N}_\alpha \models \varphi$. It is easy to check that $\mathscr{N}_\alpha \models V \notin \mathrm{Fin}$, and then to see that this implies $\mathscr{N} \Vdash_{\mathbf{0}} V \notin \mathrm{Fin}$.

So (3) is not a consequence of $V \notin \mathrm{Fin}$ in intuitionistic TT.

Third example (constant domains, non standard equality for type 0 objects). The model we have shown in the preceding section satisfies

$$(\forall x \in \mathrm{Fin})\neg\neg(\exists y)(y \notin x). \tag{4}$$

Nevertheless, intuitionistic TT does not prove that $V \notin \mathrm{Fin}$ implies (4).

A model of $V \notin \mathrm{Fin}$ where (4) is not satisfied is a little bit more difficult to obtain. To that aim, take $\mathscr{M}$ to be any model of ZF. Suppose $\mathscr{M} \models (K = \omega)$. In $\mathscr{M}$, let $\leqslant$ be the usual order on ω, and $\mathbf{0} = 0$. Then, in $\mathscr{M}$, define N^0 to be $\{a_n \mid n \in \omega\}$, where, for each $n \in \omega$,

$$a_n(k) = \begin{cases} \{n\} & \text{if } k < n, \\ \{0,\dots,n,\dots,k\} & \text{if } k \geqslant n. \end{cases}$$

So, for every $k \in K$, $N^0_k = N^0$. Now define $\mathscr{N}$ from N^0, as described in section 1.2.

If $k \geqslant n$, then $\mathscr{N} \Vdash_k a_n = a_k$. Thus, for all $a_n, a_k \in N^0_{\mathbf{0}}$, $\mathscr{N} \Vdash_{\mathbf{0}} \neg\neg(a_n = a_k)$. This implies

$$\mathscr{N} \Vdash_{\mathbf{0}} \Big(\forall x\Big)\Big((\exists z)(z \in x) \rightarrow (\forall y)\neg\neg(y \in x)\Big),$$

and then $\mathscr{N} \nVdash_{\mathbf{0}} (\forall x \in \mathrm{Fin})\neg\neg(\exists y)(y \notin x)$.

On the other hand, it is harder to check that $\mathscr{N} \Vdash_{\mathbf{0}} V \notin \mathrm{Fin}$. Here are the main steps of the proof. In $\mathscr{M}$, there exists a set $\mathscr{F}$ such that

$$\mathscr{M} \models \Big(\forall E\Big)\Big(E \in \mathscr{F} \leftrightarrow (E \in N^1) \wedge (\forall k \in K)(|E(k)| < \omega)\Big).$$

Define $F \in N^2$ to be the function such that, for all $k \in K$, $F(k) = \mathscr{F}$.

First prove that $\mathscr{N} \Vdash_{\mathbf{0}}$ (F is inductive). So $\mathscr{N} \Vdash_{\mathbf{0}}$ ($\mathrm{Fin} \subseteq F$). Then prove that $\mathscr{N} \Vdash_{\mathbf{0}} (V \notin F)$, by using, mainly, the fact that $\mathscr{M} \models (|N^0_k/(=^{\mathscr{N}_k})| = \omega)$. Thus $\mathscr{N} \Vdash_{\mathbf{0}} (V \notin \mathrm{Fin})$.

From the above examples, it is clear that the excluded middle is not a consequence of the axioms of intuitionistic TT. We leave open the question about the right way to define the set of finite sets in intuitionistic TT and to state the axiom of infinity. Our intention was simply to give the reader some examples of models of intuitionistic TT.

1.4. Remarks about the construction of models. (1) In the inductive definition of the N^i's, we defined N^{i+1} as "the class of *all* functions x such that ...". N^{i+1} could be smaller: it must simply be large enough to include the x required in the proof of Theorem 2. Remark also that K, $\leqslant$, and all the N^i's (including N^0) can be proper classes; nevertheless, they must be definable, so that the Definability Lemma can be proved.

(2) We defined our models $\mathscr{N}$ within models $\mathscr{M}$ of ZF. We could have chosen another set theory than ZF. The main requirement is that $\mathscr{M}$ must satisfy a comprehension schema strong enough to prove Theorem 2. In particular, $\mathscr{M}$ could be a model of classical TT. Nevertheless, for the construction to work in that case, one has to be careful with types. For example, if $x \in N^0$, then x is a function whose domain is a subset of K. So, in $\mathscr{M}$, the type of x and the type of N^0 are higher than the type of K. But if $x \in N^1$, then x should be a function whose domain is a subset of K, and whose range is a subset of N^0. This is a problem in TT because, with the usual definition of a function as a set of Kuratowski pairs, the domain of a function should have the same type as its range. This problem can be avoided by using another definition for pairs, or by replacing, in the definition of each N^k, K with $\mathrm{USC}^k(K)$, for some suitable $k \in \mathbb{N}$ (recall that $\mathrm{USC}(K) = \{\{x\} \mid x \in K\}$). Those type raising technicalities make the definitions more complex, of course.

§2. Elementary extensions of Kripke structures. Before we exhibit some Kripke models of intuitionistic NF_2 and NF, we need to present some definitions and results about elementary extensions of Kripke structures.

From now on, we suppose that, if $\left\langle (\mathscr{M}_k)_{k\in K}, \langle K, \leqslant, \mathbf{0}\rangle \right\rangle$ is a Kripke structure, then $\langle K, \leqslant, \mathbf{0}\rangle$ is an ω-*tree*, i.e.,

- $\leqslant$ is a partial order relation on K;
- for each $k \in K$, $\mathbf{0} \leqslant k$;
- for each $k \in K$, $\{l \in K \mid l \leqslant k\}$ is finite and totally ordered by $\leqslant$.

In other words, K is a tree, with root $\mathbf{0}$, whose height is less or equal to ω. A completeness theorem for predicate calculus can still be proved if this restriction on K is added.

Let Σ be a set of $\mathscr{L}$-formulæ, for some language $\mathscr{L}$. We define Σ to be *closed under subformulæ* if and only if any subformula of a formula $\varphi \in \Sigma$ also belongs to Σ.

Now consider $\mathscr{M} = \langle (\mathscr{M}_k)_{k\in K}, \langle K, \leqslant, \mathbf{0}\rangle\rangle$ and $\mathscr{N} = \langle (\mathscr{N}_k)_{k\in K'}, \langle K', \leqslant', \mathbf{0}'\rangle\rangle$. Then we say that a function $i : K \to K'$ *induces a Σ-elementary embedding of $\mathscr{M}$ into $\mathscr{N}$* if and only if:

- for every $k, l \in K$, $k \leqslant l$ implies $i(k) \leqslant' i(l)$;
- for every $k \in K$, $\mathscr{M}_k \subseteq \mathscr{N}_{i(k)}$ (in a more general definition, we could replace $\subseteq$ with an embedding);
- for each $k \in K$, each $\varphi \in \Sigma$, and each $a_1, \dots, a_n \in M_k$,

$$\mathscr{M} \Vdash_k \varphi[a_1, \dots, a_n] \iff \mathscr{N} \Vdash_{i(k)} \varphi[a_1, \dots, a_n].$$

Then we shall say that $\mathscr{N}$ is a Σ-*elementary extension* of $\mathscr{M}$ (noted $\mathscr{N} \prec_\Sigma \mathscr{M}$) if and only if there exists a function inducing a Σ-elementary embedding of $\mathscr{M}$ into $\mathscr{N}$.

Replacing truncations with elementary extensions. Let $\mathscr{M} = \langle (\mathscr{M}_k)_{k\in K}, \langle K, \leqslant, \mathbf{0}\rangle\rangle$ be a Kripke structure. If $k \in K$, the *truncation of $\mathscr{M}$ at k*, $\mathscr{M}^{\geqslant k}$, is the subtree of $\mathscr{M}$ which is above k :

$$\mathscr{M}^{\geqslant k} = \left\langle (\mathscr{M}_k)_{k\in K^{\geqslant k}}\,,\ \langle K^{\geqslant k}, \leqslant\restriction_{K^{\geqslant k}}, k\rangle \right\rangle,$$

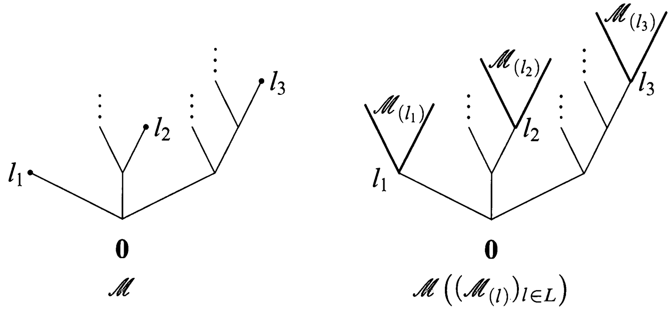

FIGURE 1. A picture of $\mathscr{M}\left((\mathscr{M}_{(l)})_{l\in L}\right)$.

where $K^{\geqslant k} = \{l \in K \mid l \geqslant k\}$.

We shall use the following result: if some $\mathscr{M}^{\geqslant k}$'s are replaced with Σ-elementary extensions of those $\mathscr{M}^{\geqslant k}$'s, then the resulting structure is also a Σ-elementary extension of the initial structure $\mathscr{M}$.

Let us state this result in a more precise way. Given $\mathscr{M}$, we define L to be the set of *leaves* of K :

$$L = \{l \in K \mid \neg(\exists k \in K)(k > l)\}.$$

Now consider a family of Kripke structures $(\mathscr{M}_{(l)})_{l\in L}$, where $\mathscr{M}_{(l)} = \left\langle (\mathscr{M}_{(l)k})_{k\in K_l}, \langle K_l, \leqslant_l, l\rangle\right\rangle$, for each $l \in L$ (remark that $\mathscr{M}_{(l)l}$ is the "root" structure of $\mathscr{M}_{(l)}$). We say that this family is *compatible* with $\mathscr{M}$ if and only if

- for every $l \in L$, $\mathscr{M}_l \subseteq \mathscr{M}_{(l)l}$;
- if $l, l' \in L$, then $l \neq l'$ implies $K_l \cap K_{l'} = \varnothing$.

$\mathscr{M}((\mathscr{M}_{(l)})_{l\in L})$ is the structure obtained by attaching the $\mathscr{M}_{(l)}$'s "on the top" of $\mathscr{M}$ (see Figure 1) :

$$\mathscr{M}\left((\mathscr{M}_{(l)})_{l\in L}\right) = \left\langle (\overline{\mathscr{M}}_k)_{k\in\overline{K}}, \langle \overline{K}, \preccurlyeq, \mathbf{0}\rangle\right\rangle,$$

where

- $\overline{K} = K \cup \bigcup_{l\in L} K_l$;
- $\preccurlyeq\; = \;\leqslant \cup \bigcup_{l\in L} \leqslant_l$;
- if $k \in K \setminus L$, then $\overline{\mathscr{M}}_k = \mathscr{M}_k$, and if $k \in K_l$, then $\overline{\mathscr{M}}_k = \mathscr{M}_{(l)k}$.

So, if $(\mathscr{M}_{(l)})_{l\in L}$ and $(\mathscr{M}'_{(l)})_{l\in L}$ are two families of Kripke structures, compatible with $\mathscr{M}$, then $\mathscr{M}((\mathscr{M}'_{(l)})_{l\in L})$ can be considered as $\mathscr{M}((\mathscr{M}_{(l)})_{l\in L})$ where some subtrees (the $\mathscr{M}_{(l)}$'s) have been replaced with other Kripke structures (the $\mathscr{M}'_{(l)}$'s). Now we can state the result we need.

THEOREM 4. *Let $\mathscr{M} = \langle(\mathscr{M}_k)_{k\in K}, \langle K, \leqslant, \mathbf{0}\rangle\rangle$ be a Kripke structure where $\langle K, \leqslant, \mathbf{0}\rangle$ is an ω-tree, and let L be the set of leaves of K. Let $(\mathscr{M}_{(l)})_{l\in L}$ and $(\mathscr{M}'_{(l)})_{l\in L}$ be two families of Kripke structures, compatible with $\mathscr{M}$. Let also Σ be a set of* *formulæ which is closed under subformulæ. If $\mathscr{M}_{(l)} \prec_\Sigma \mathscr{M}'_{(l)}$ for every $l \in L$, then $\mathscr{M}((\mathscr{M}_{(l)})_{l\in L}) \prec_\Sigma \mathscr{M}((\mathscr{M}'_{(l)})_{l\in L})$.*

PROOF. First, let us introduce some notations. For each $l \in L$, let

$$\mathscr{M}_{(l)} = \left\langle (\mathscr{M}_{(l)k})_{k\in K_l}, \langle K_l, \leqslant_l, l\rangle \right\rangle \text{ and } \mathscr{M}'_{(l)} = \left\langle (\mathscr{M}'_{(l)k})_{k\in K_l}, \langle K'_l, \leqslant'_l, l\rangle \right\rangle.$$

Also, let

$$\begin{aligned} \mathscr{M}\left((\mathscr{M}_{(l)})_{l\in L}\right) &= \left\langle (\overline{\mathscr{M}}_k)_{k\in\overline{K}}, \langle \overline{K}, \preccurlyeq, \mathbf{0}\rangle \right\rangle \text{ and} \\ \mathscr{M}\left((\mathscr{M}'_{(l)})_{l\in L}\right) &= \left\langle (\overline{\mathscr{M}}'_k)_{k\in\overline{K}'}, \langle \overline{K}', \preccurlyeq', \mathbf{0}\rangle \right\rangle. \end{aligned}$$

As $\mathscr{M}_{(l)} \prec_\Sigma \mathscr{M}'_{(l)}$, there exists a function $i_l : K_l \to K'_l$ inducing a Σ-elementary embedding of $\mathscr{M}_{(l)}$ into $\mathscr{M}'_{(l)}$. Then the following function i induces a Σ-elementary embedding of $\mathscr{M}((\mathscr{M}_{(l)})_{l\in L})$ into $\mathscr{M}((\mathscr{M}'_{(l)})_{l\in L})$:

$$i : \overline{K} \longrightarrow \overline{K'} : k \longmapsto \begin{cases} k & \text{if } k \in K \setminus L \\ i_l(k) & \text{if } k \in K_l, \text{ for some } l \in L. \end{cases}$$

We should now prove that, if $k \in \overline{K}$, $a_1, \dots, a_n \in \overline{M}_k$ and $\varphi \in \overline{\Sigma}$, then

$$\mathscr{M}((\mathscr{M}_{(l)})_{l\in L}) \Vdash_k \varphi[a_1, \dots, a_n] \iff \mathscr{M}((\mathscr{M}'_{(l)})_{l\in L}) \Vdash_{i(k)} \varphi[a_1, \dots, a_n].$$

This can be proved by induction on the length of φ. The proof is easy but tedious. So we shall simply give one of the (sub)cases, which should be convincing enough.

Suppose φ is $\neg\psi$ and $k \in K \setminus L$ (other cases for k are trivial). Then

$$\begin{aligned} &\mathscr{M}((\mathscr{M}_{(l)})_{l\in L}) \Vdash_k \neg\psi[a_1, \dots, a_n] \\ &\quad\iff (\forall k' \succcurlyeq k)(\mathscr{M}((\mathscr{M}_{(l)})_{l\in L}) \nVdash_{k'} \psi[a_1, \dots, a_n]) \\ &\quad\iff (\forall k' \in K \setminus L)(k' \succcurlyeq k \to \mathscr{M}((\mathscr{M}_{(l)})_{l\in L}) \nVdash_{k'} \psi[a_1, \dots, a_n]) \\ &\qquad\quad \wedge\ (\forall l \in L)(\forall k' \geqslant_l l)(\mathscr{M}((\mathscr{M}_{(l)})_{l\in L}) \nVdash_{k'} \psi[a_1, \dots, a_n]) \\ &\qquad\quad (\text{definition of } \overline{K} \text{ and } \preccurlyeq) \\ &\quad\iff (\forall k' \in K \setminus L)(k' \succcurlyeq k \to \mathscr{M}((\mathscr{M}_{(l)})_{l\in L}) \nVdash_{k'} \psi[a_1, \dots, a_n]) \\ &\qquad\quad \wedge\ (\forall l \in L)(\mathscr{M}_{(l)} \Vdash_l \neg\psi[a_1, \dots, a_n]) \\ &\quad\iff (\forall k' \in K \setminus L)(k' \succcurlyeq i(k) \to \mathscr{M}((\mathscr{M}'_{(l)})_{l\in L}) \nVdash_{k'} \psi[a_1, \dots, a_n]) \\ &\qquad\quad \wedge\ (\forall l \in L)(\mathscr{M}'_{(l)} \Vdash_l \neg\psi[a_1, \dots, a_n]) \\ &\qquad\quad (i(k) = k, \text{ induction hypothesis and } \mathscr{M}_{(l)} \prec_\Sigma \mathscr{M}'_{(l)}) \\ &\quad\iff \mathscr{M}((\mathscr{M}'_{(l)})_{l\in L}) \Vdash_{i(k)} \neg\psi[a_1, \dots, a_n] \\ &\qquad\quad (\text{as above}). \end{aligned}$$

Notice that, to be able to use the induction hypothesis, ψ must belong to Σ; that is why Σ is required to be closed under subformulæ. ⊣

§3. A model of intuitionistic NF_2. Intuitionistic NF is the intuitionistic theory whose language is the language of ZF, and whose proper axioms are the usual proper axioms of classical NF:

- extensionality axiom: $(\forall x)(\forall y)((\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y)$;
- comprehension axioms: $(\exists x)(\forall z)(z \in x \leftrightarrow \varphi)$, for each stratified formula φ in which x does not occur free; a formula φ is *stratified* if its variables can be given type indices so as to obtain a well-formed formula in the language of TT. (Note for readers familiar with details of NF: in this paper, it does not matter whether "stratified" means "strongly stratified" or "weakly stratified".)

In other words, the axioms of NF are exactly the axioms of TT where the type indices have been "erased."

NF_2 is a fragment of NF. Its comprehension schema is restricted to the 2-stratified comprehension axioms of NF. A formula is said to be *2-stratified* if it can be obtained by "erasing" type indices in a formula of the language of TT where at most two different types occur. So NF_2 is the "typeless version" of TT_2, i.e., of the fragment of TT restricted to types 0 and 1.

For more information about NF, NF_2 and TT_2, we refer the reader to [1] and [5].

If the underlying logic is classical logic, the celebrated technique of Specker allows us to obtain a model of NF_2 from a suitable model of TT_2. Roughly, this can be stated as follows.

Let $\mathscr{M} = \langle M^0, M^1, \in_{\mathscr{M}}, =\rangle$ be a *classical* model of TT_2 (so $\in_{\mathscr{M}} \subseteq M^0 \times M^1$ and $= \subseteq (M^0 \times M^0) \cup (M^1 \times M^1)$). $\mathscr{M}$ is said to be a *shifting model* if and only if there exists a one-one function f mapping M^0 onto M^1.

If $\mathscr{M}$ is shifting, then $\mathscr{M}$ can be transformed into a model $\mathscr{M}(f)$ of NF_2 by "collapsing types":

$$\mathscr{M}(f) = \langle M^0; \in_f, =\rangle$$

where $x \in_f y$ if and only if $\mathscr{M} \models x \in f(y)$.

$\mathscr{M}(f)$ is a model of NF_2 because $\mathscr{M}$ is a model of TT_2: in some sense, $\mathscr{M}$ and $\mathscr{M}(f)$ satisfy the "same" 2-stratified formulæ. More precisely, this remark can be formalized as follows (an analogous lemma can be proved in the intuitionistic case).

LEMMA 5. *Let $\varphi(x^0, y^0, \dots, t^1, u^1, \dots)$ be a formula in the language of TT_2. Let $\varphi^o(x, y, \dots, t, u, \dots)$ be the formula in the language of NF obtained by "erasing" the type indices in φ. Assume that $x, y, \dots, t, u, \dots$ are distinct variables. If $a, b, \dots, c, d, \dots \in M^0$, then*

$$\mathscr{M}(f) \models \varphi^o[a, b, \dots, c, d, \dots] \iff \mathscr{M} \models \varphi[a, b, \dots, f(c), f(d), \dots].$$

PROOF. The proof is nothing but an easy induction on the length of φ. ⊣

So to find a model of NF_2, it is enough to find a shifting model of TT_2. And it has been proved (see [2]) that such shifting models can be exactly identified with the atomic Boolean algebras which have the same cardinality as their set of atoms. For example, consider $B = \{I \subset \omega \mid I \text{ is finite or } I \text{ is cofinite}\}$. The set of atoms of B is $A = \{\{n\} \mid n \in \omega\}$. As both A and B are countable, $\mathscr{M} = \{A, B; \subseteq, =\}$ is a classical shifting model of TT_2.

We want to build a Kripke model of intuitionistic NF_2. Of course, we do not want a degenerated Kripke model which would appear to be equivalent to a classical model! The model we are going to exhibit satisfies the excluded middle for 2-stratified formulæ. But we shall also explicitly give a sentence σ, which is not 2-stratified and such that $\sigma \vee \neg\sigma$ is not satisfied in the model.

The idea is simple. Take two classical models $\mathscr{M}_0$ and $\mathscr{M}_1$, for the language of NF. If $\mathscr{M}_0 \subseteq \mathscr{M}_1$, this pair of models can be used to define a Kripke model:

$$\mathscr{M}_0 \nearrow \mathscr{M}_1 = \left\langle (\mathscr{M}_k)_{k \in \{\mathbf{0},1\}}, \langle \{\mathbf{0}, 1\}, \leqslant, \mathbf{0} \rangle \right\rangle$$

(where $\mathbf{0} \leqslant 1$).

Now take a classical model $\mathscr{M}_0$ of NF_2. $\mathscr{M}_0$ can be considered as the Kripke model

$$\left\langle (\mathscr{M}_k)_{k \in \{\mathbf{0}\}}, \langle \{\mathbf{0}\}, \leqslant, \mathbf{0} \rangle \right\rangle$$

(where $\leqslant$ is the trivial order relation on $\{\mathbf{0}\}$). For any formula φ in the language of NF,

$$\text{(5)} \qquad \mathscr{M}_0 \Vdash_{\mathbf{0}} \varphi[\vec{a}] \iff \mathscr{M}_0 \models \varphi[\vec{a}],$$

for any $\vec{a} \in M$. So this Kripke model satisfies the excluded middle for all formulæ.

Furthermore, it is easy to see that $\mathscr{M}_0$ satisfies exactly the same formulæ as $\mathscr{M}_0 \nearrow \mathscr{M}_0$:

$$\text{(6)} \qquad (\mathscr{M}_0 \nearrow \mathscr{M}_0) \Vdash_{\mathbf{0}} \varphi[\vec{a}] \iff \mathscr{M}_0 \models \varphi[\vec{a}],$$

for every formula φ and every $\vec{a} \in M_0$.

But here comes the trick: consider a classical structure $\mathscr{M}_1$ such that $\mathscr{M}_0 \prec_{\Sigma^2} \mathscr{M}_1$, where Σ^2 is the set of 2-stratified formulæ. When $\mathscr{M}_0$ and $\mathscr{M}_1$ are considered as Kripke structures, $\mathscr{M}_0 \prec_{\Sigma^2} \mathscr{M}_1$ remains true. Thus, by Theorem 4,

$$(\mathscr{M}_0 \nearrow \mathscr{M}_0) \prec_{\Sigma^2} (\mathscr{M}_0 \nearrow \mathscr{M}_1).$$

So $\mathscr{M}_0 \nearrow \mathscr{M}_1$ remains a Kripke model of NF_2. Furthermore, we are going to prove that $\mathscr{M}_0$ and $\mathscr{M}_1$ can be chosen in such a way that $(\mathscr{M}_0 \nearrow \mathscr{M}_0) \not\prec (\mathscr{M}_0 \nearrow \mathscr{M}_1)$. More precisely, we shall find a sentence $\sigma \vee \neg\sigma$ which is (of course) satisfied in the "classical" structure $\mathscr{M}_0 \nearrow \mathscr{M}_0$, but not in $\mathscr{M}_0 \nearrow \mathscr{M}_1$. This sentence is necessarily not 2-stratified.

$\mathscr{M}_0$ and $\mathscr{M}_1$ will be obtained from shifting models of TT_2.

LEMMA 6. *There exist two classical models $\mathscr{M}_0$ and $\mathscr{M}_1$ of NF_2 such that $\mathscr{M}_0 \prec_{\Sigma^2} \mathscr{M}_1$, and $\mathscr{M}_0 \not\models \sigma$ and $\mathscr{M}_1 \models \sigma$, where $\sigma \equiv (\exists x)(x = \{x\})$.*

PROOF. Consider a countable shifting model $\mathscr{M}$ of TT_2. Let f be the 1–1 function mapping M^0 onto M^1. First, we are going to transform f into f', so that, for all $x \in M^0, \mathscr{M} \models f'(x) \neq \{x\}$. To that aim, let

$$S = \left\{\{x, x'\} \subset M^0 \mid \mathscr{M} \models f(x) = \{x\} \text{ and } \mathscr{M} \models f(x') = V \setminus \{x\}\right\}.$$

Now define $f' : M^0 \to M^1$ as follows:

$$\begin{array}{ll} f'(x) = f(x) & \text{if there is no } x' \in M^0 \text{ such that } \{x, x'\} \in S \\ f'(x) = f(x') & \text{if } \{x, x'\} \in S. \end{array}$$

Then it is easy to check that f' is a 1–1 function mapping M^0 onto M^1. And for all $x \in M^0$, $\mathscr{M} \models f'(x) \neq \{x\}$.

Consider $\mathscr{M}(f')$. If $x \in M(f')$, we have:

$$\begin{array}{rcl} \mathscr{M}(f') \models x = \{x\} & \iff & \mathscr{M}(f') \models (\forall y)(y \in x \leftrightarrow y = x) \\ & \iff & \mathscr{M} \models (\forall y^0)(y^0 \in f'(x) \leftrightarrow y^0 = x) \\ & \iff & \mathscr{M} \models f'(x) = \{x\}. \end{array}$$

So $\mathscr{M}(f') \not\models (\exists x)(x = \{x\})$, and we can set the $\mathscr{M}_0$ we want equal to $\mathscr{M}(f')$.

We want now to build a structure $\mathscr{M}'$ such that the $\mathscr{M}_1$ we want can be defined as $\mathscr{M}'(f'')$, for some suitable f''.

Let us come back to $\mathscr{M}$. Let Δ be the elementary diagram of $\mathscr{M}$. Take $a_0, a_1, \ldots$ to be a countable list of new type 0 constant symbols. Define $T = \Delta \cup \{a_i \neq a_j \mid i \neq j\}$. T is easily seen to be consistent, by the compactness theorem. So T has a countable model $\mathscr{M}'$. As $\mathscr{M}' \models \Delta$, $\mathscr{M} \prec \mathscr{M}'$. Furthermore, $M'^0 \setminus M^0$ is countable, because it contains the interpretations of the a_i's. And $M'^1 \setminus M^1$ is also countable, because it contains countably many y's such that $\mathscr{M}' \models y = \{a_i\}$ for some $i \in \omega$. So there exists a 1–1 function $f'' : M'^0 \to M'^1$ such that f'' is onto and *extends* f' : such an f'' is simply $f' \cup g$, where g is any 1–1 function mapping $M'^0 \setminus M^0$ onto $M'^1 \setminus M^1$. We may suppose that $\mathscr{M}' \models f''(x) = \{x\}$, for some $x \in M'^0$ (the argument is a simpler variant of the argument used above to transform f into f').

Let $\mathscr{M}_1 = \mathscr{M}'(f'')$. Then $\mathscr{M}_1 \models (\exists x)(x = \{x\})$.

Furthermore, as f'' extends f', Lemma 5 can be used to deduce $\mathscr{M}_0 \prec_{\Sigma^2} \mathscr{M}_1$ from the fact that $\mathscr{M} \prec \mathscr{M}'$. ⊣

With $\mathscr{M}_0$ and $\mathscr{M}_1$ as defined in the above lemma, it is easy to build a "non-classical" Kripke model of intuitionistic NF_2: simply put $\mathscr{M}_1$ above $\mathscr{M}_0$!

PROPOSITION 7. *There is a Kripke model of intuitionistic NF_2 which satisfies the excluded middle for all 2-stratified sentences, but does not satisfy $\sigma \vee \neg\sigma$, where $\sigma \equiv (\exists x)(x = \{x\})$.*

PROOF. By Lemma 6, we can find two classical models $\mathscr{M}_0$ and $\mathscr{M}_1$ of NF_2 such that $\mathscr{M}_0 \prec_{\Sigma^2} \mathscr{M}_1$, $\mathscr{M}_1 \not\models \sigma$ and $\mathscr{M}_0 \models \sigma$.

We claim that $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \nVdash_0 \sigma$. Otherwise, there would exist some $a \in N^0$ such that $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \Vdash_0 a = \{a\}$. But $y = \{x\}$ is a 2-stratified formula and $(\mathscr{M}_0 \nearrow \mathscr{M}_0) \prec_{\Sigma^2} (\mathscr{M}_0 \nearrow \mathscr{M}_1)$. So $(\mathscr{M}_0 \nearrow \mathscr{M}_0) \Vdash_0 a = \{a\}$. And thus, by (6), $\mathscr{M}_0 \models a = \{a\}$, contradicting the hypothesis.

On the other hand, $\mathscr{M}_1 \models \sigma$. In other words, $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \Vdash_1 \sigma$. So $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \nVdash \neg\sigma$.

All this implies that $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \nVdash \sigma \vee \neg\sigma$.

Furthermore, as $(\mathscr{M}_0 \nearrow \mathscr{M}_0) \prec_{\Sigma^2} (\mathscr{M}_0 \nearrow \mathscr{M}_1)$, then by (6), $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \Vdash NF_2$, and $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \Vdash \tau \vee \neg\tau$, for all 2-stratified sentences τ. ⊣

§4. A model of intuitionistic NF. Suppose that (classical) NF is consistent. We would like to adapt the techniques of the previous section to build a non trivial Kripke model of intuitionistic NF. To state it more precisely, we would like to find two models $\mathscr{M}_0$ and $\mathscr{M}_1$ of NF such that $\mathscr{M}_0 \prec_{\Sigma^\infty} \mathscr{M}_1$ but $\mathscr{M}_0 \not\prec \mathscr{M}_1$, where Σ^∞ is the set of all stratified formulæ.

This is not so easy. Nevertheless, remark that the method of permutation models (see [12] and [5, § 3.1]) allows us to find, in a quite simple way, two models $\mathscr{M}$ and $\mathscr{M}'$ of classical NF which are not elementary equivalent, although they satisfy the same stratified sentences, i.e., $\mathscr{M} \equiv_{\Sigma^\infty} \mathscr{M}'$ and $\mathscr{M} \not\equiv \mathscr{M}'$. But when $\equiv$ is replaced with $\prec$, the problem is much more difficult.

In terms of types, this problem amounts to finding two shifting models $\mathscr{N}$ and $\mathscr{N}'$ of classical TT, such that $\mathscr{N} \prec \mathscr{N}'$ and also such that the shift function of $\mathscr{N}'$ *extends* the shift function of $\mathscr{N}$. Thomas Forster, with the help of André Pétry, first published a partial solution to this problem in [4]. But recently, Friederike Körner gave a full solution.

Using cofinal indiscernibles, she proved the following theorem (see [6]):

If (classical) NF is consistent, then there exist two classical models of NF, $\mathscr{M}_0$ and $\mathscr{M}_1$, such that $\mathscr{M}_0 \prec_{\Sigma^\infty} \mathscr{M}_1$, and $\mathscr{M}_0 \models \neg\sigma$ and $\mathscr{M}_1 \models \sigma$, where $\sigma \equiv (\exists n \in Nn)(\forall m \geqslant n)(m < Tm)$.

(Nn is the set of natural numbers and T the "type-raising" operation roughly defined by: $Tn = m$ if and only if for some x whose cardinality is n, m is the cardinality of $\mathrm{USC}(x)$; see [5, Chap. 2] for details.)

Now it is quite easy to build a non-trivial Kripke model of intuitionistic NF.

PROPOSITION 8. *If classical NF is consistent, then there exists a Kripke model of intuitionistic NF, which satisfies the excluded middle for all stratified sentences, but does not satisfy $\sigma \vee \neg\sigma$, where $\sigma \equiv (\exists n \in Nn)(\forall m \geqslant n)(m < Tm)$.*

PROOF. The proof is similar to the proof of Proposition 7, but a little bit more elaborate.

Consider the models $\mathscr{M}_0$ and $\mathscr{M}_1$ of Körner's theorem. $\mathscr{M}_0 \prec_{\Sigma^\infty} \mathscr{M}_1$ implies $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \prec_{\Sigma^\infty} (\mathscr{M}_0 \nearrow \mathscr{M}_1)$. But, as we remarked above, $\mathscr{M}_0 \models \varphi[\vec{a}]$ if and only if $(\mathscr{M}_0 \nearrow \mathscr{M}_0) \Vdash_0 \varphi[\vec{a}]$, for every formula φ (stratified or not), and every $\vec{a} \in M_0$. So, for every *stratified* formula φ and every $\vec{a} \in M_0$,

$$\mathscr{M}_0 \models \varphi[\vec{a}] \iff (\mathscr{M}_0 \nearrow \mathscr{M}_1) \Vdash_0 \varphi[\vec{a}]. \tag{7}$$

We claim that $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \nVdash_0 \sigma$. Indeed, if $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \Vdash_0 \sigma$, then, by definition of the $\Vdash$ relation, this would imply

$$\Big(\exists n \in M_0\Big)\Big[(\mathscr{M}_0 \nearrow \mathscr{M}_1) \Vdash_0 n \in \mathrm{Nn} \\ \wedge \Big(\forall m \in M_0\Big)\Big((\mathscr{M}_0 \nearrow \mathscr{M}_1) \Vdash_0 m \geqslant n \rightarrow (\mathscr{M}_0 \nearrow \mathscr{M}_1) \Vdash_0 m < Tm\Big)\Big].$$

The formulæ $x \in \mathrm{Nn}$, $x \geqslant y$ and $x < Ty$ are stratified formulæ, so by (7),

$$\Big(\exists n \in M_0\Big)\Big(\mathscr{M}_0 \models n \in \mathrm{Nn} \wedge (\forall m \in M_0)(\mathscr{M}_0 \models m \geqslant n \rightarrow \mathscr{M}_0 \models m < Tm)\Big).$$

In other words, $\mathscr{M}_0 \models \sigma$, which is absurd. So $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \nVdash_0 \sigma$.

On the other hand, if $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \Vdash_0 \neg\sigma$, then, in particular, $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \nVdash_1 \sigma$. But, as in (5), this is equivalent to $\mathscr{M}_1 \nvDash \sigma$, which is absurd. So $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \nVdash_0 \neg\sigma$.

From all this we conclude that $(\mathscr{M}_0 \nearrow \mathscr{M}_1) \nVdash_0 \sigma \vee \neg\sigma$. ⊣

We have just proved that if classical NF is consistent, then intuitionistic NF does not prove the excluded middle (at least for non-stratified formulæ). It is not known whether the consistency of intuitionistic NF implies the consistency of classical NF: for example, it seems that to find some double negation interpretation of classical NF into intuitionistic NF is much more difficult than for theories as PA/HA, ZF or TT: as the universe of NF is not well-founded, the constructions by induction used for ZF (see [11]) or TT (see [8] and [10]) cannot be reproduced.

Nevertheless, Thomas Forster pointed out to me the following easy remark.

COROLLARY 9. *If* intuitionistic *NF is consistent, then it does not prove the excluded middle (for non-stratified formulæ).*

PROOF. Suppose that intuitionistic NF is consistent, and suppose it proves the excluded middle (for all formulæ, including formulæ where some variables occur free). Then (intuitionistic NF + excluded middle) is consistent. In other words, this means that classical NF is consistent. So, by Proposition 8, intuitionistic NF does not prove the excluded middle. This is absurd and proves that intuitionistic NF does not proves the excluded middle, under the hypothesis that it is consistent. ⊣

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