Chapter 4: Boundary conditions

The previous chapter exposed how to calculate the fluxes through the interfaces of the finite-volume computational cells. In most of the cases, the state of the variables in the two neighbouring cells has to be known. The interfaces located at the upstream and downstream ends of the calculation domain have only one neighbouring cell, then the fluxes through these boundary interfaces require a specific treatment. An original boundary conditions treatment is proposed. To achieve the set of equations near the model boundaries, the common approach relies on characteristics (Abbott, 1979) and their related compatibility equations, which are combined with the boundary conditions. The direction of the characteristics determines the number of needed boundary conditions: at each characteristic originating outside the solution domain in the plane (x, t) a boundary condition must be substituted.

The eigenstructures of each model developed in chapter 2 are particularly useful to solve the equations at the boundaries. First, the boundary conditions treatment is applied to the simple case of the well-known Saint-Venant equations in pure hydrodynamics in order to exemplify the method. Then this approach is extended to the Saint-Venant – Exner equations and the two-layer model under various assumptions. The boundary conditions treatment applied to the two-velocity two-concentration model is also reported in Savary and Zech (2007). Finally, a discussion takes place to compare the different models on the basis of examples.

4.1. Eigenvalue analysis in pure hydrodynamic case

To exemplify the boundary condition strategy in a simple case, it is applied to the pure hydrodynamic case described by the Saint-Venant equations

(2.118). The two eigenvalues, slopes of the characteristics, whose sign depends on the Froude number Fr = u/c are

$$\lambda_1 = u + c \qquad \lambda_2 = u - c \tag{4.1}$$

Along the characteristics the Saint-Venant equations reduce to the compatibility relations

$$\left(\frac{dq}{dt}\right) + (-u + c)\left(\frac{dh}{dt}\right) = 0 \text{ along characteristic } \lambda_1 \tag{4.2a}$$

$$\left(\frac{dq}{dt}\right) + (-u - c)\left(\frac{dh}{dt}\right) = 0 \text{ along characteristic } \lambda_2 \tag{2.129b}$$

The compatibility relations link the value $h_{i+1/2}^{t+dt}$ and $q_{i+1/2}^{t+dt}$ at the interface i+1/2 at time t+dt with the value h_i^t, q_i^t, h_{i+1}^t and q_{i+1}^t coming from the neighbouring cells i and i+1 (in subcritical case) at time t (Fig.4.1).

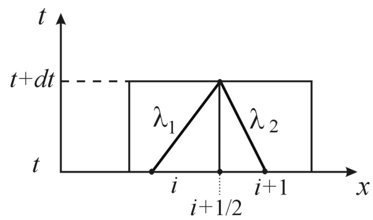

Figure 4.1. Characteristics in pure hydrodynamics

In case of an interface located at the extremity of the domain, some of the characteristics may generate outside the domain. A boundary condition has to be imposed to replace each of the characteristics generating outside the domain. The compatibility relations are combined with the boundary conditions, to find the value of the variables at the domain boundaries.

4.1.1. Downstream boundary conditions

The way to find the variables $h_{n+1/2}^{t+dt}$ and $q_{n+1/2}^{t+dt}$ at the last interface n+1/2 at time t + dt depends on the flow characteristics in the last cell n at time t.

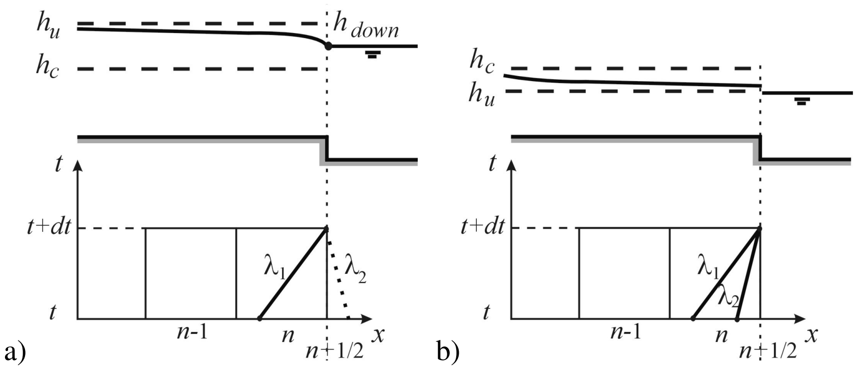

Figure 4.2. Boundary conditions in pure hydrodynamics: downstream end

In case of supercritical flow (Fr > 1), the characteristics linked to both eigenvalues are positive (inside the domain), they are relations between the variables at the interface and the variables in the last cell

$$\left\lfloor q_{n+1/2}^{t+dt} - q_n^t \right\rfloor_{\lambda_1} + \left(-u_n^t + c_n^t\right)\left\lfloor h_{n+1/2}^{t+dt} - h_n^t \right\rfloor_{\lambda_1} = 0 \tag{4.3a}$$

$$\left\lfloor q_{n+1/2}^{t+dt} - q_n^t \right\rfloor_{\lambda_2} + \left(-u_n^t - c_n^t\right)\left\lfloor h_{n+1/2}^{t+dt} - h_n^t \right\rfloor_{\lambda_2} = 0 \tag{4.3b}$$

For subcritical flow (Fr < 1), the characteristic λ_2 is negative and generates outside the domain, the corresponding compatibility relation (4.3b) has to be replaced by a boundary condition, typically the downstream water depth h_{down} (Fig.4.2).

$$h_{n+1/2}^{t+dt} = \left(h_{down}\right)_{n+1/2}^{t+dt} \tag{4.4}$$

In both subcritical and supercritical cases, a system composed of two linear relations allows to find the two unknowns $h_{n+1/2}^{t+dt}$ and $q_{n+1/2}^{t+dt}$.

4.1.2. Upstream boundary conditions

The parameters of the flow in the upstream cell 1 are concerned (Fig.4.3).

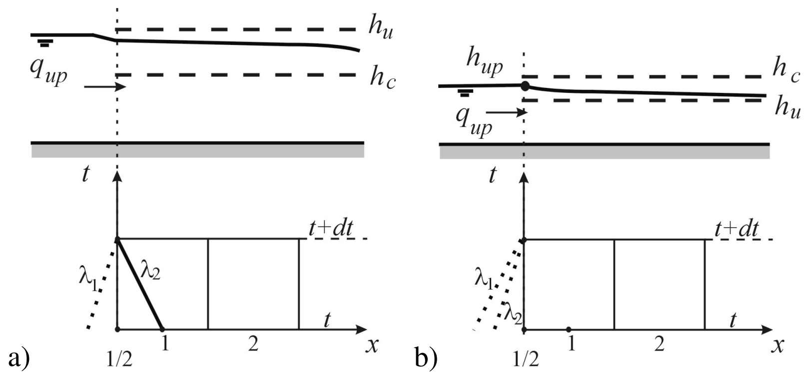

Figure 4.3. Boundary conditions in pure hydrodynamics: upstream end

In case of supercritical flow (Fr > 1), the characteristics link to both eigenvalues are positive. As they are outside the domain, the corresponding compatibility relations have to be replaced by boundary conditions, typically the upstream liquid discharge $q_{w,up}$ and the water depth $h_{w,up}$.

$$q_{1/2}^{t+dt} = \left(q_{up}\right)_{1/2}^{t+dt} \tag{4.5}$$

$$h_{1/2}^{t+dt} = \left(h_{up}\right)_{1/2}^{t+dt} \tag{4.6}$$

In case of subcritical flow (Fr < 1), the compatibility relation corresponding to the negative characteristic λ_2 coming from the first cell replaces the boundary condition (4.6).

$$\left\lfloor q_{1/2}^{t+dt} - q_1^t \right\rfloor_{\lambda_2} + \left(-u_1^t - c_1^t\right)\left\lfloor h_{1/2}^{t+dt} - h_1^t \right\rfloor_{\lambda_2} = 0 \tag{4.7}$$

4.1.3. Critical flow

The example of a control section at the downstream interface is illustrated (Fig.4.4).

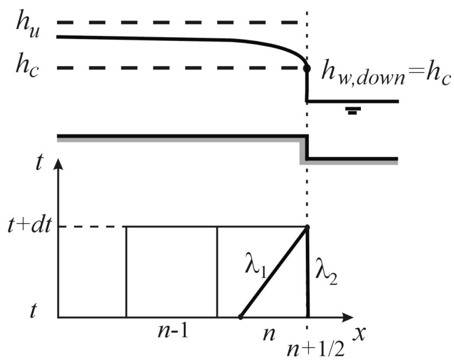

Figure 4.4. Boundary conditions in pure hydrodynamics: control section

The water depth imposed downstream h_{down} (4.4) should be the critical depth h_c depending on the discharge

$$h_{n+1/2}^{t+dt} = \left(h_{down}\right)_{n+1/2}^{t+dt} = \left(h_c\right)_{n+1/2}^{t+dt} \tag{4.8}$$

The specific link between the water depth and the discharge at a control section results from the expression Fr = 1. In this case, $\lambda_2 = 0$ and the related characteristic is vertical, which can be written under a linearised form

$$\lambda_{2\,n+1/2}^{\;t+dt} = \lambda_{2\,n}^{\;t} + \left(\frac{\partial \lambda_2}{\partial h}\right)_n \left[h_{n+1/2}^{t+dt} - h_n^t\right] + \left(\frac{\partial \lambda_2}{\partial q}\right)_n \left[q_{n+1/2}^{t+dt} - q_n^t\right] = 0 \tag{4.9}$$

Imposing this relation, linking the variables in a linear way, gives the same results and is easier to solve than imposing the critical depth (4.8), which is not a linear relation between the variables. This relation is completed with the compatibility relation (4.3a).

Identically, to impose a critical section at the upstream end, the condition on the water level (4.6) becomes (4.8) or should be replaced by (4.9).

4.1.4. Summary

In all cases, the solution at the extreme interfaces is calculated with a two-equation system. The equations composing the system depend on the Froude number at this interface (Table 4.1).

	Fr > 1 $\lambda_1, \lambda_2 > 0$	**Fr < 1** $\lambda_1 < 0$ **and** $\lambda_2 > 0$
Downstream boundary	• 2 positive characteristics and associated compatibility equations (total transmission of the flow)	• 1 downstream condition (generally the water depth *h* or a control section) • 1 positive characteristic
Upstream boundary	• 2 upstream conditions (generally the discharge *q* and the depth *h* or a control section)	• 1 upstream conditions (generally the discharge *q*) • 1 negative characteristic

Table 4.1. Boundary conditions: Saint-Venant

4.2. Saint-Venant – Exner

The method exemplified in the simple hydrodynamic case can be extended to the Saint-Venant – Exner equations (2.21-2.22). The number of boundary conditions depends on the signs of the eigenvalues at the domain boundaries. As previously presented in Figure 2.20 in section 2.2.2, independently of the Froude number, two eigenvalues (λ_1 and λ_2) are always positive and one (λ_3) is always negative. Then there are always two boundary conditions upstream and one downstream. The boundary conditions are combined with the compatibility relations giving information from inside the domain to find the variables h, u and z_b.

At the extreme upstream and downstream interfaces. The general expression of the compatibility relation linked to λ_i is

$$l_{i1}\left(\frac{dh}{dt}\right)_{\lambda_i}+l_{i2}\left(\frac{du}{dt}\right)_{\lambda_i}+l_{i3}\left(\frac{dz_b}{dt}\right)_{\lambda_i}=0 \tag{4.10}$$

where $l_{i,1\rightarrow3}$ are the components of the left eigenvector i.

4.2.1. Downstream boundary conditions

To find the variables $h_{n+1/2}$, $u_{n+1/2}$ and $z_{b,n+1/2}$ at the downstream end, the two compatibility relations associated to the positive characteristics λ_1 and λ_2 are written

$$l_{11}\left(h_{n+1/2}-h_n\right)_{\lambda_1}+l_{12}\left(u_{n+1/2}-u_n\right)_{\lambda_1}+l_{13}\left(z_{b,n+1/2}-z_{b,n}\right)_{\lambda_1}=0$$

$$l_{21}\left(h_{n+1/2}-h_n\right)_{\lambda_2}+l_{22}\left(u_{n+1/2}-u_n\right)_{\lambda_2}+l_{23}\left(z_{b,n+1/2}-z_{b,n}\right)_{\lambda_2}=0$$

(4.11a-b)

The third equation of the system is a boundary condition replacing the compatibility relation linked to the negative characteristic λ_3. The choice of the boundary condition to impose depends on the flow characteristics and the physical constraints (Fig.4.5).

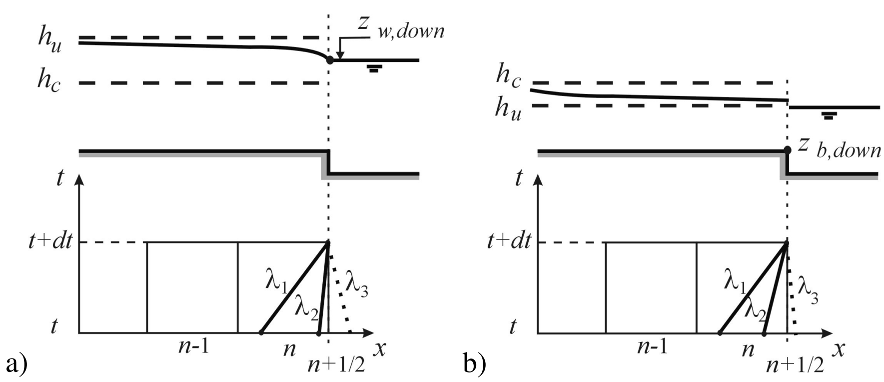

Figure 4.5. Boundary conditions for Saint-Venant – Exner: downstream end

The condition can be an expression combining several of the three variables. In case of subcritical flow, the left eigenvector analysis (chapter 2, section 2.2.2) reveals that the characteristic λ_3 propagates information about all the variables either hydrodynamic or sedimentologic. By analogy with the pure hydrodynamic case, the condition is usually imposed on hydrodynamic variables. The condition generally consists in imposing the water level z_w, for instance in the case of a junction with a lake (with level $z_{w,down}$) (Fig.4.5a).

$$\left(z_b\right)_{n+1/2}^{t+dt} + \left(h\right)_{n+1/2}^{t+dt} = \left(z_{w,down}\right)_{n+1/2}^{t+dt} \tag{4.12}$$

Another possible condition for subcritical flow could be a rating curve linking the depth h and the discharge q (using the main variable u), in case of a downstream pipe opening for instance. One has to remember that the variables h and u are related to the mixed flow (water and sediments). In some cases, approximation by reduction to pure hydrodynamic variables should be made with care.

For supercritical flow, according to the left eigenvector analysis (chapter 2, section 2.2.2), the characteristic λ_3 mainly concerns sediments (except in the critical zone). The corresponding compatibility relation should be replaced by a boundary condition on the sedimentologic variable z_b (Fig. 4.5b).

$$\left(z_b\right)_{n+1/2}^{t+dt} = \left(z_{b,down}\right)_{n+1/2}^{t+dt} \tag{4.13}$$

The problem of this sediment boundary condition is the fact that the bottom level z_b, is often difficult to impose in practice. In practical cases, the solid discharge q_s is often to be imposed, but this variable is not a main unknown of the set of equations (2.21-2.22). The solid discharge q_s is predicted as closure equation and thus has to be imposed externally. It implies the propagation of such an information will be driven by the arrangement of the computational cells rather than by the physics. At the end of this chapter, an example will illustrate the problem.

4.2.2. Upstream boundary conditions

The compatibility relations linked to the positive eigenvalue λ_1 and λ_2 are replaced by boundary conditions.

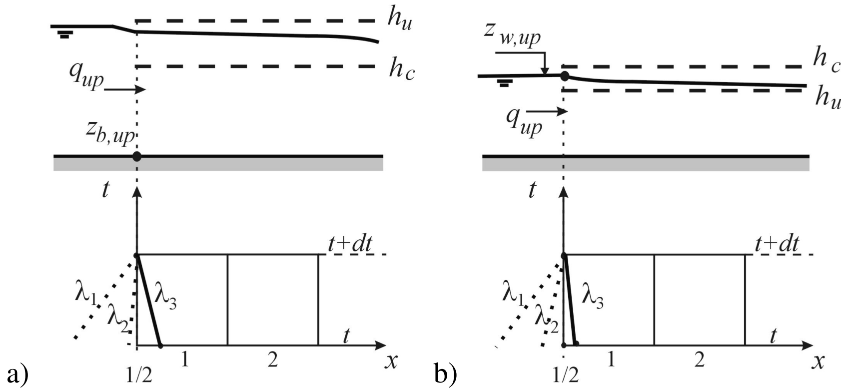

Figure 4.6. Boundary conditions for Saint-Venant – Exner: upstream end

In all cases, like in hydrodynamics, the compatibility relation linked to λ_1 is replaced by a boundary condition on the incoming total discharge q (sediment and water).

$$(q)_{1/2}^{t+dt} = \left(q_{w,up}\right)_{1/2}^{t+dt} + \left(q_{s,up}\right)_{1/2}^{t+dt} = \left(q_{up}\right)_{1/2}^{t+dt} \tag{4.14}$$

In case of subcritical flow, according to the left eigenvector analysis, the characteristic λ_2 mainly concerns sediments (except in the critical zone). The corresponding compatibility relation should be replaced by a boundary condition on the sedimentologic variable z_b (Fig.4.6).

$$\left(z_b\right)_{1/2}^{t+dt} = \left(z_{b,up}\right)_{1/2}^{t+dt} \tag{4.15}$$

which, as exposed before, not always corresponds to the physical behaviour.

In case of supercritical flow, the eigenvectors analysis does not reveal predominant variables for characteristic λ_2.

By analogy with the hydrodynamic case, the related compatibility relation is replaced by an hydrodynamic condition, generally on the water level.

$$(z_b)_{1/2}^{t+dt} + (h)_{1/2}^{t+dt} = (z_{w,up})_{1/2}^{t+dt} \tag{4.16}$$

The compatibility relation (4.10) associated to the negative characteristic λ_3 completes the system.

4.2.3. Critical flow

The presence of sediment changes the behaviour of the flow at the control section. As exposed in the eigenstructure analysis, none of the eigenvalues is equal to zero when $Fr = 1$. According to the approximated analytical expression of the eigenvalues (2.139), when $Fr = 1$, the eigenvalues λ_2 and λ_3 are identical in absolute value and opposite in sign. A downstream control section has thus to respect

$$(\lambda_2 + \lambda_3)_{n+1/2}^{t+dt} = 0 \tag{4.17}$$

which writes

$$\begin{aligned}(\lambda_2 + \lambda_3)_{n+1/2}^{t+dt} = (\lambda_2 + \lambda_3)_n^t &+ \left(\frac{\partial(\lambda_2 + \lambda_3)}{\partial h}\right)_n \left[(h)_{n+1/2}^{t+dt} - (h)_n^t\right] \\ &+ \left(\frac{\partial(\lambda_2 + \lambda_3)}{\partial q}\right)_n \left[(q)_{n+1/2}^{t+dt} - (q)_n^t\right] \\ &+ \left(\frac{\partial(\lambda_2 + \lambda_3)}{\partial z_b}\right)_n \left[(z_b)_{n+1/2}^{t+dt} - (z_b)_n^t\right] = 0\end{aligned} \tag{4.18}$$

where the derivative of the eigenvalues are calculated numerically or analytically on the basis of the approximated analytical expressions (2.139). This linear relation linking the three unknowns at the downstream interface can replace the condition on the water level (4.12) in the downstream boundary condition treatment.

Identically, to impose a control section at the upstream boundary, the expression (4.18) written for the first interface replaces the condition on the water level (4.16).

4.2.4. Summary

In all cases, the solution at the extreme interfaces is calculated with a three-equation system. There is always one boundary condition downstream and two upstream. The nature of the condition to impose depends on the flow regime at the interface (Table 4.2).

$\lambda_1, \lambda_2 > 0$ **$\lambda_3 < 0$**	**Fr > 1**	**Fr < 1**
Downstream boundary	• 2 compatibility relations • 1 boundary condition on sediments (bed level z_b)	• 2 compatibility relations • 1 boundary condition (for instance, water level z_w or a control section)
Upstream boundary	• 2 upstream conditions (generally the discharge q and the water level z_w or a control section) • 1 compatibility relation	• 2 upstream conditions 1 on sediments (bed level z_b) and the other generally on the discharge q. • 1 compatibility relation

Table 4.2. Boundary conditions: Saint-Venant – Exner

4.3. Two-layer model with assumption on velocity

4.3.1. Two-layer one-velocity model

The method can be applied to the two-layer equations in the case of identical velocity in both sediment transport and water layers.

Capart (2000) calculated the analytical eigenvalues of the system

The eigenvalues (2.145) of the system have been calculated analytically (Capart, 2000)

$$\lambda_1 = u_m \quad \lambda_2 = u_m + \sqrt{g\,\frac{h_m^2 + r\,h_s^2}{h_m + r\,h_s}} \quad \lambda_3 = u_m - \sqrt{g\,\frac{h_m^2 + r\,h_s^2}{h_m + r\,h_s}} \quad \lambda_4 = 0 \tag{4.19}$$

Once more the number of boundary conditions depends on the sign of the eigenvalues at the extreme interfaces. The eigenvalues λ_1 and λ_2 are always positive while λ_4 is equal to zero. The sign, positive or negative, of the remaining eigenvalue λ_3 depends on the flow regime. To find the variables h_m, h_s, z_b and u_m at the extreme interfaces, the boundary conditions are combined with compatibility relations of general expression

$$l_{i1}\left(\frac{dh_m}{dt}\right)_{\lambda_i} + l_{i2}\left(\frac{dh_s}{dt}\right)_{\lambda_i} + l_{i3}\left(\frac{dz_b}{dt}\right)_{\lambda_i} + l_{i4}\left(\frac{dq_m}{dt}\right)_{\lambda_i} = 0 \tag{4.20}$$

where $l_{i,1\rightarrow 4}$ are the components of the left eigenvector i linked to the eigenvalue λ_i detailed in the matrix (2.149).

The introduction of a sediment transport layer in the flow modifies its behaviour. The hydrodynamic Froude number is no longer sufficient to qualify the two-layer flow regime. A Froude number Fr_m linked to the two-layer flow can be defined (2.146), it is equal to unity when λ_3 is equal to zero.

4.3.1.1. Downstream boundary conditions

The number of boundary conditions to impose at the downstream end depends on the flow regime (Fig.4.7).

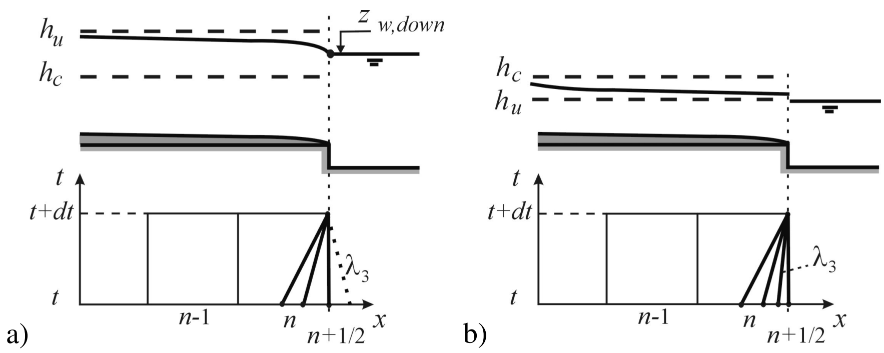

Figure 4.7. Boundary conditions for two-layer one-velocity model: downstream end

In the supercritical case ($\mathrm{Fr}_m > 1$) (Fig. 4.7b), the three compatibility relations corresponding to the positive eigenvalues λ_i ($i = 1$, 2 and 3), generated inside the domain, write

$$\begin{aligned} & l_{i1}\left[(h_m)_{n+1/2}^{t+dt} - (h_m)_n^t\right]_{\lambda_i} + l_{i2}\left[(h_s)_{n+1/2}^{t+dt} - (h_s)_n^t\right]_{\lambda_i} \\ & \quad + l_{i3}\left[(z_b)_{n+1/2}^{t+dt} - (z_b)_n^t\right]_{\lambda_i} + l_{i4}\left[(u_m)_{n+1/2}^{t+dt} - (u_m)_n^t\right]_{\lambda_i} = 0 \end{aligned} \tag{4.21}$$

One of the characteristics, related to λ_4, is always vertical. The corresponding compatibility relation (4.20) only concerns the bed level z_b, all the other component of the left eigenvector being equal to zero.

$$\left[(z_b)_{n+1/2}^{t+dt} - (z_b)_n^t\right]_{\lambda_4} = 0 \tag{4.22}$$

This compatibility relation express that z_b is not varying with time at the interface during the hyperbolic treatment. Of course, if there is erosion or deposition, the bed level changes, but this variation takes place during the source terms treatment. The initial value of the bed level at the interface $z_{b,n+1/2}^t$ has to be actualised at each time step in order to take into account the erosion or deposition process, which occurred in the source treatment of the previous time step. Generally, this evolution is deduced from the evolution in the last cell.

Resolving the system (4.21-4.22) leads to a total transmission, i.e. the variables at the downstream interface have the same values than the variables in the last cell.

In case of subcritical flow ($Fr_m < 1$) (Fig. 4.7b), λ_3 becomes negative and the corresponding compatibility relation has to be replaced by a boundary condition. By analogy with the pure hydrodynamic case and the Saint-Venant – Exner equations, the water level $z_{w,down}$ is generally imposed.

$$\left(z_b\right)_{n+1/2}^{t+dt} + \left(h_m\right)_{n+1/2}^{t+dt} = \left(z_{w,down}\right)_{n+1/2}^{t+dt} \tag{4.23}$$

4.3.1.2. *Upstream boundary conditions*

The number of boundary conditions to impose upstream (Fig.4.8) depends on the two-layer Froude number Fr_m.

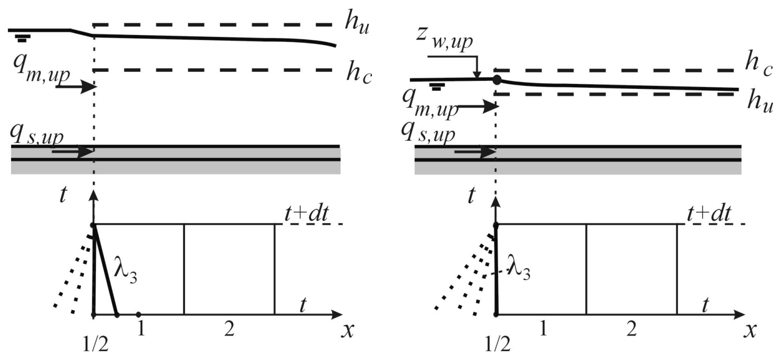

Figure 4.8. Boundary conditions for two-layer one-velocity model: upstream end

The characteristics corresponding to the positive eigenvalues λ_1 and λ_2 do not exist and have to be replaced by two upstream boundary conditions: for example the total discharge q_m and the solid discharge q_s. As they are not some of the main variables of the problem, they have to be expressed as functions of the main variables.

$$q_m = h_m u_m \qquad q_s = C_s h_s u_m \tag{4.24}$$

In order to obtain a linear relation depending on the main variables at the upstream interface. The corresponding boundary conditions may be linearised as

$$(q_m)_{1/2}^{t+dt} = (q_{m,\text{up}})_{1/2}^{t+dt} = (q_m)_1^t + \left(\frac{\partial q_m}{\partial h_m}\right)_1 \left[(h_m)_{1/2}^{t+dt} - (h_m)_1^t\right] + \left(\frac{\partial q_m}{\partial u_m}\right)_1 \left[(u_m)_{1/2}^{t+dt} - (u_m)_1^t\right] \tag{4.25}$$

$$(q_s)_{1/2}^{t+dt} = (q_{s,\text{up}})_{1/2}^{t+dt} = (q_s)_1^t + \left(\frac{\partial q_s}{\partial h_s}\right)_1 \left[(h_s)_{1/2}^{t+dt} - (h_s)_1^t\right] + \left(\frac{\partial q_s}{\partial u_m}\right)_1 \left[(u_m)_{1/2}^{t+dt} - (u_m)_1^t\right] \tag{4.26}$$

where the partial derivatives with variables not included in the expressions (4.24) have been removed, being equal to zero.

Like for the downstream boundary condition, the compatibility relation (4.22) linked to the vertical characteristic has to be taken into account.

In the case of a two-layer subcritical flow ($\text{Fr}_m < 1$), the eigenvalue λ_3 is negative and the corresponding compatibility relation (4.20) completes the system. Else ($\text{Fr}_m > 1$), a third boundary condition has to be added, the water level is usually imposed.

$$(z_b)_{1/2}^{t+dt} + (h_m)_{1/2}^{t+dt} = (z_{w,\text{up}})_{1/2}^{t+dt} \tag{4.27}$$

4.3.1.3. Critical flow

In the case of a one-velocity two-layer flow, the control section corresponds to a relation between the total depth h_m, the sediment transport depth h_s and the common velocity u_m corresponding to $\mathrm{Fr}_m = 1$.

$$u_m = \sqrt{g \frac{h_m^2 + r\, h_s^2}{h_m + r\, h_s}} \tag{4.28}$$

To impose this relation between the different variables at the downstream interface is similar, but more complicated than to impose $\lambda_3 = 0$ at this interface.

$$\begin{aligned} {\lambda_3}_{n+1/2}^{t+dt} = {\lambda_3}_n^t &+ \left(\frac{\partial \lambda_3}{\partial h_m}\right)_n \left[(h_m)_{n+1/2}^{t+dt} - (h_m)_n^t\right] \\ &+ \left(\frac{\partial \lambda_3}{\partial h_s}\right)_n \left[(h_s)_{n+1/2}^{t+dt} - (h_s)_n^t\right] + \left(\frac{\partial \lambda_3}{\partial z_b}\right)_n \left[(z_b)_{n+1/2}^{t+dt} - (z_b)_n^t\right] \\ &+ \left(\frac{\partial \lambda_3}{\partial u_m}\right)_n \left[(u_m)_{n+1/2}^{t+dt} - (u_m)_n^t\right] \end{aligned} \tag{4.29}$$

This condition has to replace the condition (4.23) on the water level. If a control section has to be imposed upstream, $\lambda_3 = 0$ has to be imposed at the upstream interface and replace (4.27).

4.3.1.4. Summary

The four-equation system to solve in order to find the variables at the extreme interfaces depends on the flow regime determined by the mixture Froude number Fr_m.

	$\mathbf{Fr}_m > 1$ **$\lambda_4 = 0$ and $\lambda_1, \lambda_2, \lambda_3 > 0$**	**$\mathbf{Fr}_m < 1$** **$\lambda_3 < 0$ $\lambda_4 = 0$ and $\lambda_1, \lambda_2 > 0$**
Downstream boundary	• 1 condition on the bed level z_b corresponding to the vertical characteristic • 3 positive characteristics and associated compatibility equations (total transmission of the flow)	• 1 condition on the bed level z_b corresponding to the vertical characteristic. • 1 downstream condition (for instance : velocity u_m, thickness h_m or h_s, water level z_w, or sediment discharge q_s) • 2 positive characteristics
Upstream boundary	• 1 condition on the bed level z_b corresponding to the vertical characteristic • 3 upstream conditions (for instance : mixture discharge q_m, solid discharge q_s, water level z_w, sediment level z_s)	• 1 condition on the bed level z_b corresponding to the vertical characteristic • 2 upstream conditions (mixture discharge q_m, solid discharge q_s, water level z_w) • 1 negative characteristic

Table 4.3. Boundary conditions: two-layer one velocity

4.3.2. Constant velocity ratio model

As it is the case for the one velocity model, the system (2.100-2.102) has four eigenvalues, but their analytical expression are too complicated to be developed and a numerical approach is required. Two of them are always positive, one is always equal to zero. The sign of the remaining one may change, depending of a mixed Froude number Fr_{sw} taking the sediment transport into account. The two-layer Froude number Fr_{sw}, used to determine the flow regime, is not identical to the one used previously for the one-velocity model. The compatibility relations completing the boundary conditions at the extreme interfaces have the general formulation

$$l_{i1}\left(\frac{dh_w}{dt}\right)_{\lambda_i}+l_{i2}\left(\frac{dh_s}{dt}\right)_{\lambda_i}+l_{i3}\left(\frac{dz_b}{dt}\right)_{\lambda_i}+l_{i4}\left(\frac{du_w}{dt}\right)_{\lambda_i}=0 \quad (4.30)$$

where $l_{i,1\to4}$ are the components of the left eigenvector i linked to the eigenvalue λ_i, they are numerically estimated.

The boundary condition treatment is completely identical to the one proposed for the one-velocity model, paying attention that the main variables are different. Moreover, the control section has a different expression and is, this time, defined by a relation issued from (2.154).

$$u_w^2\left(\rho_w h_w+\alpha^2\rho_s' h_s\right)=gh_w\left(\rho_w\left(h_w+h_s\right)\right)+gh_s\left(\rho_w h_w+\rho_s' h_s\right) \quad (4.31)$$

As in the one-velocity model (4.29), it corresponds to $\lambda_3 = 0$.

The boundary conditions, depending on the two-layer Froude number Fr_{sw}, are summarised in Table 4.4.

	$\mathbf{Fr}_{sw} > 1$ $\lambda_4 = 0$ **and** $\lambda_1, \lambda_2, \lambda_3 > 0$	$\mathbf{Fr}_{sw} < 1$ $\lambda_3 < 0$ $\lambda_4 = 0$ **and** $\lambda_1, \lambda_2 > 0$
Downstream boundary	• 1 condition on the bed level z_b corresponding to the vertical characteristic • 3 positive characteristics and associated compatibility equations (total transmission)	• 1 condition on the bed level z_b corresponding to the vertical characteristic. • 1 downstream condition (for instance: thickness h_w or h_s, water level z_w, or sediment discharge q_s) • 2 positive characteristics
Upstream boundary	• 1 condition on the bed level z_b corresponding to the vertical characteristic • 3 upstream conditions (for instance : water discharge q_w, solid discharge $q_s = \alpha C_s h_s u_w$, water level z_w, sediment level z_s)	• 1 condition on the bed level z_b corresponding to the vertical characteristic • 2 upstream conditions (generally: water discharge q_w and the solid discharge $q_s = \alpha C_s h_s u_w$) • 1 negative characteristic

Table 4.4. Boundary conditions: two-layer constant velocity ratio

4.4. Two-layer two-velocity model

When the velocity in the sediment transport and in the water layers are allowed to be distinct, the eigenstructure cannot be analytically determined. To handle the boundary conditions, the methods used are similar to those developed for the one-velocity model, using numerical evaluations where analytical derivatives do not exist. For further analysis, the eigenvalues are now ranked from the highest λ_1 to the lowest λ_5. The boundary conditions depend on the sign of the eigenvalues, which in turn depends on the parameter of water and sediment flow. They are completed by the compatibility relations of general expression

$$l_{i1}\left(\frac{dh_w}{dt}\right)_{\lambda_i}+l_{i2}\left(\frac{dh_s}{dt}\right)_{\lambda_i}+l_{i3}\left(\frac{dz_b}{dt}\right)_{\lambda_i}+l_{i4}\left(\frac{dq_w}{dt}\right)_{\lambda_i}+l_{i5}\left(\frac{dq_s}{dt}\right)_{\lambda_i}=0 \tag{4.32}$$

where $l_{i,1\to5}$ are the components of the left eigenvector i linked to the eigenvalue λ_i. Those components are also estimated numerically.

At the boundaries, only the characteristics inside the domain subsist (the negative ones upstream and the positive ones downstream). The other ones vanish and have to be replaced by a boundary condition. Such boundary conditions are combined with the subsisting compatibility relations to evaluate the variables h_w , h_s, q_w, q_s and z_b at the boundaries.

The number of these boundary conditions is thus directly linked to the signs of the eigenvalues. Once more, the water Froude number is not sufficient to determine the sign of the eigenvalues. As presented in the eigenstructure analysis (Table 2.1), the non-dimensional number Z_{sw} (2.163) has also to be considered.

4.4.1. Downstream boundary conditions

Some typical situations of downstream boundary conditions are illustrated in Figure 4.9, according to water and two-layer Froude number values.

The positive characteristics, inside the domain, provide compatibility equations similar to (4.32).

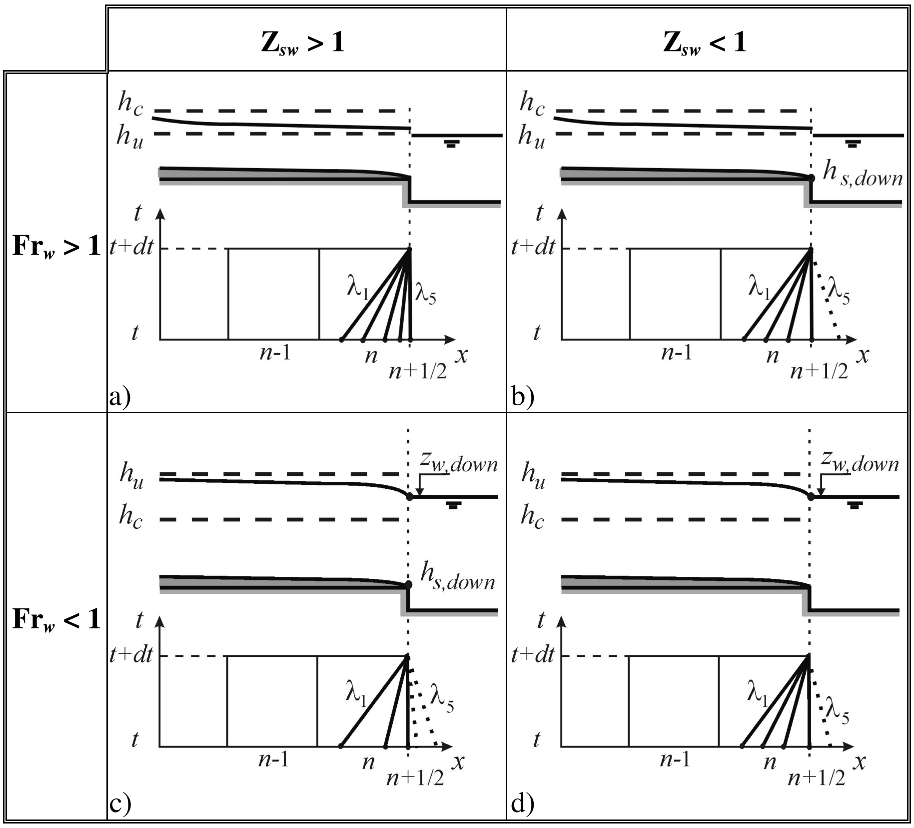

Figure 4.9. Boundary conditions for two-layer two-velocity model: downstream end

These compatibility relations result in relations between the value at the boundary (time $t + dt$, node n + ½) and the initial value in the downstream cell (time t, node n):

$$\begin{aligned} & l_{i1}\left[(h_w)_{n+1/2}^{t+dt} - (h_w)_n^t\right]_{\lambda_i} + l_{i2}\left[(h_s)_{n+1/2}^{t+dt} - (h_s)_n^t\right]_{\lambda_i} \\ & \quad + l_{i3}\left[(z_b)_{n+1/2}^{t+dt} - (z_b)_n^t\right]_{\lambda_i} + l_{i4}\left[(q_w)_{n+1/2}^{t+dt} - (q_w)_n^t\right]_{\lambda_i} \\ & \quad + l_{i5}\left[(q_s)_{n+1/2}^{t+dt} - (q_s)_n^t\right]_{\lambda_i} = 0 \end{aligned} \tag{4.33}$$

One of the characteristics, related to λ_j, is always vertical. The corresponding compatibility relation only concerns the bed level z_b and is expressed the same way than for the one-velocity model (4.22).

The other characteristics would be outside the domain and thus vanish. They have to be replaced by the appropriate boundary conditions. As observed, the number of downstream conditions to be imposed varies from 0 (Fig.4.9a) to 2 (Fig.4.9c). For instance, similarly to pure hydrodynamics, in subcritical flow cases the downstream water level $z_{w\ down}$ is imposed, which can be expressed in the following form:

$$(z_b)_{n+1/2}^{t+dt} + (h_s)_{n+1/2}^{t+dt} + (h_w)_{n+1/2}^{t+dt} = (z_{w,down})_{n+1/2}^{t+dt} \qquad (4.34)$$

In the case illustrated by Figures 4.9b and 4.9c, another type of boundary condition is to be imposed, The nature of this boundary condition depends on the problem to solve, but it will generally concern sediments. For example, the outgoing solid discharge q_s or the sediment transport layer depth h_s can be imposed. Sometimes, these conditions are not easy to impose in relation with physics, in this case, the equilibrium value of q_s or h_s , corresponding to uniform flow conditions, can be imposed. The equilibrium values obtained on the basis of uniform flow will be described in chapter 6.

4.4.2. Upstream boundary conditions

Some typical situations of upstream boundary conditions are illustrated in Fig., according to the Froude numbers values.

As observed, the number of upstream conditions to be imposed varies from 2 (Fig.4.10c) to 4 (Fig.4.10a). Two of them are obvious, those concerning the liquid and the solid discharge:

$$(q_w)_{1/2}^{t+dt} = (q_{w,\mathrm{up}})_{1/2}^{t+dt} \qquad (4.35)$$

$$(q_s)_{1/2}^{t+dt} = (q_{s,\mathrm{up}})_{1/2}^{t+dt} \qquad (4.36)$$

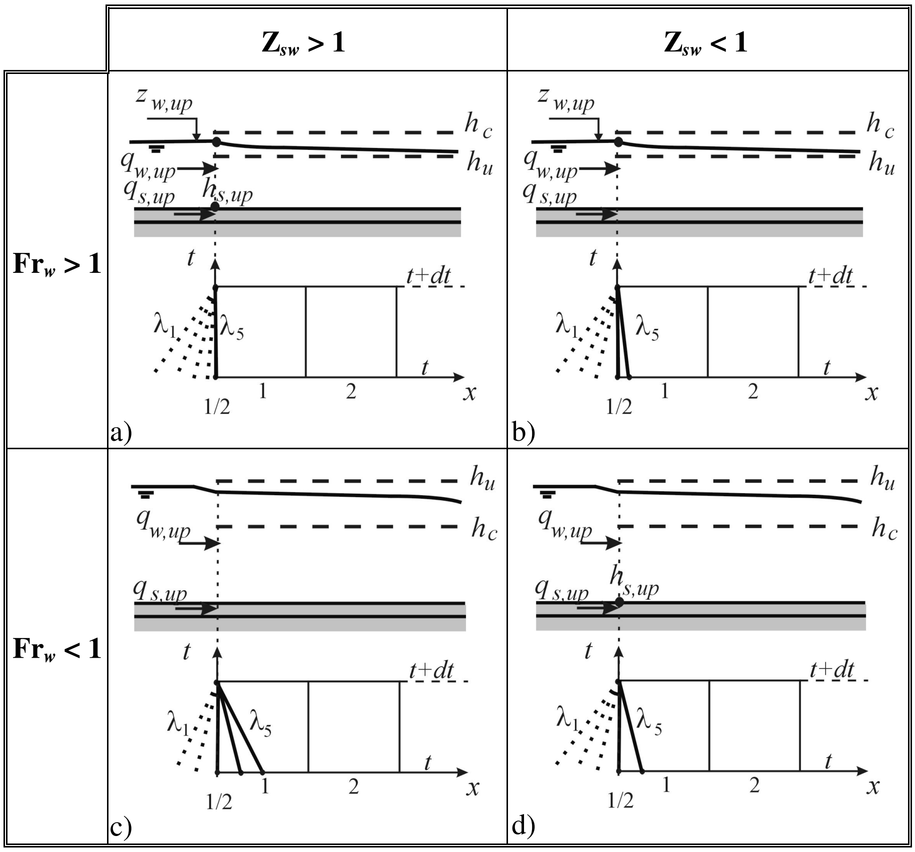

Figure 4.10. Boundary conditions for two-layer two-velocity model: upstream end

In case of supercritical flow (Figs.4.10a and 4.10b), like in pure hydrodynamics, an upstream level may be imposed:

$$\left(z_b\right)_{1/2}^{t+dt} + \left(h_s\right)_{1/2}^{t+dt} + \left(h_w\right)_{1/2}^{t+dt} = \left(z_{w,up}\right)_{1/2}^{t+dt} \tag{4.37}$$

In cases a and d of Fig. 4.10, an additional boundary condition is required. This could be for instance an imposed value of the sediment-layer depth h_s. Once more, if this condition does not correspond to physical constraints, the equilibrium value of h_s can be chosen.

The compatibility relations (4.32) corresponding to the eventual negative characteristics would complete the system. The compatibility relation corresponding to the vertical characteristic (4.22) closes the system.

4.4.3. Critical flow

If there is a change in regime at a boundary, for instance in case of a downstream weir, the common boundary condition imposing a control section is to be adapted to the two-layer case. Indeed, the critical depth defined in pure hydrodynamics is influenced by the presence of a sediments layer. Actually, the critical stage must be defined as corresponding to an additional eigenvalue equal to zero. The relation between h_w, h_s, u_w and u_s, derived from (2.162) is

$$gh_s\, gh_w \left(1 - \frac{\rho_w}{\rho_s{}'}\right) + u_s^2\, u_w^2 = gh_w\, u_s^2 + gh_s\, u_w^2 \tag{4.38}$$

It imposes that the product of the four non-zero eigenvalues is zero, leading to a second vertical characteristic. As a linear relation is easier to solve, such a relation may be replaced by imposing the eigenvalue equal to zero at the interface in a linearised form:

$$\begin{aligned}\lambda_{n+1/2}^{t+dt} = \lambda_n^t &+ \left(\frac{\partial\lambda}{\partial h_w}\right)_n \left[(h_w)_{n+1/2}^{t+dt} - (h_w)_n^t\right] \\ &+ \left(\frac{\partial\lambda}{\partial h_s}\right)_n \left[(h_s)_{n+1/2}^{t+dt} - (h_s)_n^t\right] + \left(\frac{\partial\lambda}{\partial z_b}\right)_n \left[(z_b)_{n+1/2}^{t+dt} - (z_b)_n^t\right] \\ &+ \left(\frac{\partial\lambda}{\partial q_w}\right)_n \left[(q_w)_{n+1/2}^{t+dt} - (q_w)_n^t\right] + \left(\frac{\partial\lambda}{\partial q_s}\right)_n \left[(q_s)_{n+1/2}^{t+dt} - (q_s)_n^t\right] = 0\end{aligned} \tag{4.39}$$

This condition can be imposed identically at the upstream boundary.

4.4.4. Summary

The above results are summarised in Tables 4.6 and 4.6, which shows the various possibilities depending on the value of the water Froude number Fr_w and of the two-layer Froude number Z_{sw}.

Supercritical flow	**$Fr_w > 1$ and $Z_{sw} > 1$** **$\lambda_1 = 0$ $\lambda_2, \lambda_3, \lambda_4, \lambda_5 > 0$**	**$Fr_w > 1$ and $Z_{sw} < 1$** **$\lambda_1 < 0$ $\lambda_2 = 0$ $\lambda_3, \lambda_4, \lambda_5 > 0$**
Downstream boundary	• 1 condition on the bed level z_b corresponding to the vertical characteristic. • 4 positive characteristics and associated compatibility equations (total transmission)	• 1 condition on the bed level z_b corresponding to the vertical characteristic. • 1 downstream condition (for instance : thickness h_s, or the sediment discharge q_s) • 3 positive characteristics
Upstream boundary	• 1 condition on the bed level z_b corresponding to the vertical characteristic • 4 upstream conditions (for instance : water discharge q_w, solid discharge q_s, water level z_w, sediment level z_s)	• 1 condition on the bed level z_b corresponding to the vertical characteristic • 3 upstream conditions (for instance : water discharge q_w, solid discharge q_s, water level z_w) • 1 negative characteristic

Table 4.5. Boundary conditions (supercritical flow): two-layer two velocity

Subcritical flow	$Fr_w < 1$ and $Z_{sw} < 1$ $\lambda_1 < 0$ $\lambda_2 = 0$ and $\lambda_3, \lambda_4, \lambda_5 > 0$	$Fr_w < 1$ and $Z_{sw} > 1$ $\lambda_1, \lambda_2 < 0$ $\lambda_3 = 0$ and $\lambda_4, \lambda_5 > 0$
Downstream boundary	• 1 condition on the bed level z_b corresponding to the vertical characteristic. • 1 downstream condition (for instance: the water level z_w or a control section) • 3 positive characteristics	• 1 condition on the bed level z_b corresponding to the vertical characteristic. • 2 downstream conditions (for instance : the water level z_w and the solid discharge q_s) • 2 positive characteristics
Upstream boundary	• 1 condition on the bed level z_b corresponding to the vertical characteristic • 3 upstream conditions (for instance: water discharge q_w, solid discharge q_s and sediment level z_s) • 1 negative characteristic	• 1 condition on the bed level z_b corresponding to the vertical characteristic • 2 upstream conditions (for instance : water discharge q_w, solid discharge q_s) • 2 negative characteristics

Table 4.6. Boundary conditions (subcritical flow): two-layer two velocity

4.4.5. Effect of the sediment concentration in the transport layer

The main effect of a lower sediment concentration in the sediment transport layer is the increase of the thickness of this layer. The induced variations of the sediment Froude number Fr_s and the non-dimensional number Z_{sw} implies variations of the number of boundary conditions to impose for similar flow characteristics. These changes are illustrated on an example of uniform flow (q_w = 10 m³/s/m), under equilibrium solid discharge conditions, computed for various bed-slopes and then Froude numbers. Two values of the concentration in sediment in the transport layer C_s are tested. The first one is equal to the concentration in the bed $C_s = C_b = 0.6$ and the second one is equal to the usual concentration in a bed-load layer (Sumer et al., 1996) $C_s = 0.22$. For both situations, the sediment Froude number Fr_s and the non-dimensional number Z_{sw} as a function of the water Froude number Fr_w are respectively plotted in Figures 4.11 and 4.12.

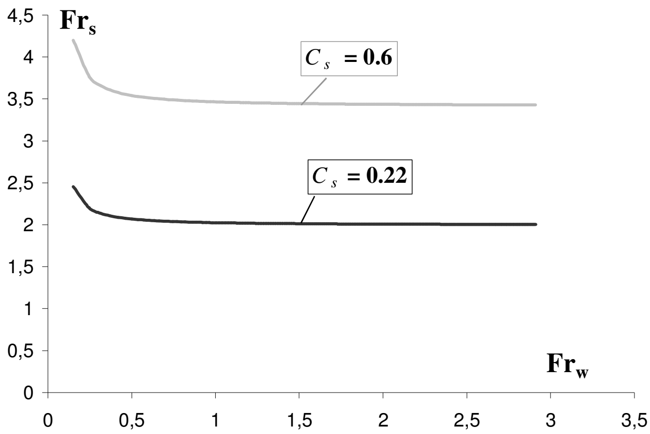

Figure 4.11. Variation of Fr_s with Fr_w and C_s under uniform flow conditions

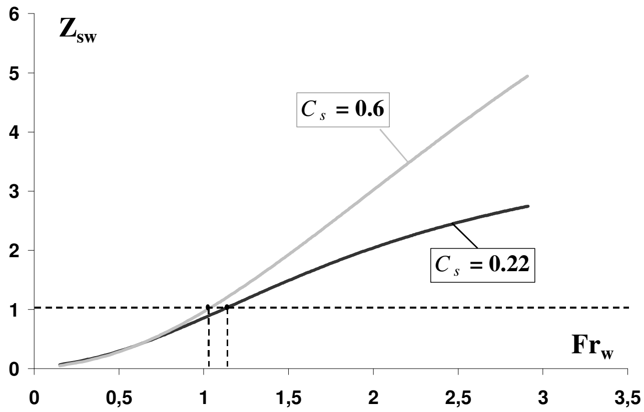

Figure 4.12. Variation of Z_{sw} with Fr_w and C_s under uniform flow conditions

The sediment Froude number decreases with the sediment concentration in the transport layer due to the rise of the transport layer thickness (Fig. 4.11). It induces a change in the non-dimensional number Z_{sw} (Fig.4.12), three parts have to be distinguished :

- $Fr_w < 1.02$: For both concentrations, $Z_{sw} < 1$, corresponding to three positive and one negative eigenvalues. It induces, not depending on the concentration, three upstream (i.e.: q_w, q_s, h_s) and one downstream (i.e.: z_w) boundary conditions have to be imposed.
- $1.02 < Fr_w < 1.15$: For $C_s = 0.22$, $Z_{sw} < 1$, there are still three positive and one negative eigenvalues corresponding to three upstream (i.e.: q_w, q_s, h_s) and one downstream (i.e.: z_w) boundary conditions. On the other hand, for $C_s = 0.6$, $Z_{sw} > 1$, all the eigenvalues are positive and all the boundary conditions have to be imposed upstream (i.e.: q_w, q_s, z_w, h_s).
- $Fr_w > 1.15$: For both concentrations, $Z_{sw} > 1$, corresponding to four positive eigenvalues. It induces, in all cases, four upstream boundary conditions (i.e.: q_w, q_s, z_w, h_s) have to be imposed.

When the concentration is lower ; the incidence of the sediment transport on the flow is more important. The control section due to the mixed flow ($Z_{sw} = 1$) corresponds to water Froude number more different from $Fr_w = 1$ when the concentration is low. For $1.02 < Fr_w < 1.15$, the one-concentration model ($C_s = 0.6$) calculates all information is coming from upstream, while the two-concentration one ($C_s = 0.22$) allows a downstream boundary condition.

4.5. Flux calculation

The aim of the boundary condition treatment is to calculate the fluxes **F*** through the extreme interfaces of the calculation domain allowing to calculate the main variables in the first and last cells with the finite-volume method (chapter 3). Once the main variables **U*** are calculated, as exposed above, at the extreme interfaces, the fluxes may be deduced by the linearised formulation

$$\mathbf{F}^* = \mathbf{F} + \sum_{i=1}^{n} \frac{\partial \mathbf{F}}{\partial \mathbf{U}_i^*}\left(\mathbf{U}_i^* - \mathbf{U}_i\right) \tag{4.40}$$

where $\mathbf{U}_{i\to n}$ are the n components of the main unknowns vector $\mathbf{U}$.

4.6. Comparison and discussion

The two-layer model is confronted to the Saint-Venant – Exner model on common examples in order to illustrate the main distinctions of both boundary condition approaches.

4.6.1. Example 1: lake level decrease

The example concerns a subcritical uniform flow joining a reservoir downstream, the bed-slope and the solid discharge correspond to the equilibrium state. The water level in the reservoir is supposed to decrease instantaneously, implying an M_2 water profile propagates in the channel (Fig.4.13).

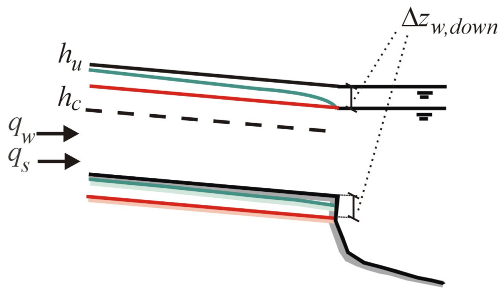

Figure 4.13. Lake level decrease: — initial profiles, — intermediate profiles, — final profiles

This example is numerically tested with each model on the following flow characteristics : the unit liquid discharge $q_w = 0.08$ m³/s/m, the initial bed-slope $S_0 = 0.005$. Initially the water level in the downstream reservoir corresponds to the uniform water depth $h_w = 0.106$ m , it is instantly decreased of 0.017 m. According to Jansen et al. (1979), the equilibrium state is reached when the bed-slope and the water depth are similar to the initial ones. Then, there is an erosion equivalent to the water level decrease in the reservoir.

In the case of the Saint-Venant – Exner model, two boundary conditions have to be imposed upstream, one concerns the total discharge q and the other the bed level z_b, and one downstream on the water level z_w (Fig.4.14).

$$q_{1/2} = q_{up} \qquad z_{b,1/2} = z_{b,up} \qquad z_{w,n+1/2} = z_{w,down} \tag{4.41}$$

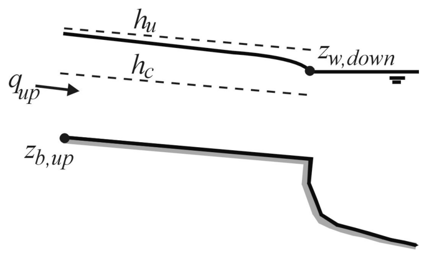

Figure 4.14. Lake level decrease: Saint-Venant – Exner boundary conditions

Two problems appear with the upstream boundary condition concerning the sediments. First, the bed level z_b should not be constant at the first interface as an erosion occurs all along the channel. The bed profile is artificially attached to its initial value at the upstream end. Moreover, there is no possibility to impose the solid discharge, which is not one of the main variable, at the upstream boundary. This problem is partially solved by imposing the solid discharge in the first cell instead of calculating it with the equilibrium transport formula.

In the case of the two-velocity model, two configurations are possible at each extremity depending on the value of the non-dimensional number Z_{sw}. At the upstream end, identically to the constant velocity ratio model, two conditions are always imposed on the water discharge q_w and the solid discharge q_s. As in the present case $Z_{sw} < 1$, one more condition is added on the transport layer thickness h_s, which has to correspond to the equilibrium transport thickness $h_{s,eq}$ obtained for the imposed solid discharge. At the downstream end, as $Z_{sw} < 1$, only one condition is imposed on the water level z_w (Fig.4.15).

$$q_{w,1/2} = q_{w,up} \qquad q_{s,1/2} = q_{s,up}$$

$$h_{s,1/2} = h_{s,eq,up} \qquad z_{w,n+1/2} = z_{w,down} \tag{4.42}$$

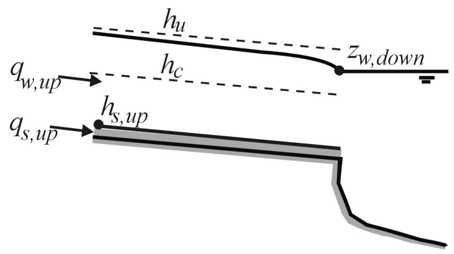

Figure 4.15. Lake level decrease: two-layer boundary conditions

In the simplified case of the constant velocity ratio, the boundary conditions are identical excepted the upstream condition on h_s, which is removed. The advantage of the two-layer model, is the fact that the bed level z_b is not directly affected by the fluxes treatment, and then by the boundary conditions treatment (as expressed by the vertical characteristic). Indeed, the bed level z_b is modified only during the second part of the algorithm dealing with the source terms. Another benefit is to be allowed to impose the solid discharge at the upstream end. The equilibrium solution obtained at the end of the computation is similar to the one proposed by Jansen et al. (1979).

4.6.2. Example 2: bed aggradation

This example is based on an experiment realised by Mourad Bellal (Capart et al.,1998) (Fig.4.16). As initial condition, the slope is 0.02, the flume is overfed with sediments, the solid discharge $q_s = 0.000147$ m³/s/m and the water discharge $q_w = 0.016$ m³/s/m. The bed slope increases until the equilibrium slope 0.03 is reached. The bed level at the downstream end corresponds to the rigid bottom of the flume, it is a fixed point during all the experiment. The flow remains supercritical during all the experiment.

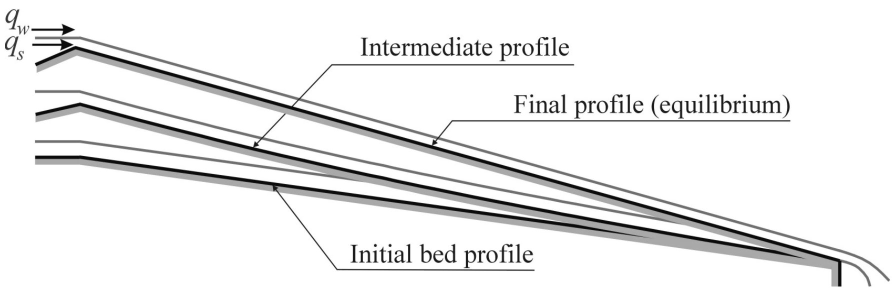

Figure 4.16. Bed aggradation, Capart et al. (1998)

In the case of the classical Saint-Venant – Exner model (Fig.4.17), the total discharge q and a control section (4.18) are imposed at the upstream boundary. The bed level z_b is imposed downstream and correspond, this time, to a physical constrain.

$$q_{1/2} = q_{up} \qquad \lambda_{2,1/2} + \lambda_{3,1/2} = 0 \qquad z_{b,n+1/2} = z_{b,down} \qquad (\mathbf{4}.43)$$

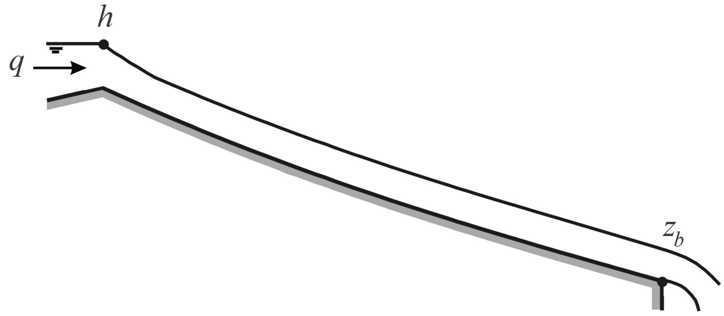

Figure 4.17. Bed aggradation: Saint-Venant – Exner boundary conditions

Once more, the solid discharge is imposed in the first computational cell instead of the upstream boundary.

In the case of the two-layer model (Fig.4.18), as $Z_{sw} > 1$ anywhere, four boundary conditions have to be imposed upstream: the liquid and solid discharges q_w and q_s, a control section (4.39) and the transport layer depth under uniform flow conditions $h_{s,eq}$. There is a total transmission downstream.

$$q_{w,1/2} = q_{w,up} \qquad q_{s,1/2} = q_{s,up} \qquad \lambda_{4,1/2} = 0 \qquad h_{s,1/2} = h_{s,eq,up} \quad (4.44)$$

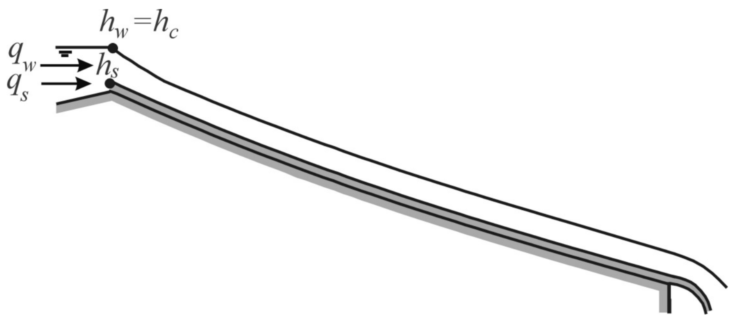

Figure 4.18. Bed aggradation: two-layer boundary conditions

As observed previously, if the constant velocity ratio model is used, the condition on h_s has to be removed.

4.6.3. Synthesis

The main characteristics of each model concerning the boundary conditions treatment are synthesised in Table 4.7.

The two-layer model offers more flexibility in the number and the kind of boundary conditions to impose than the Saint-Venant – Exner model. The two-layer two-velocity model allows to impose one additional boundary condition in comparison with the two-layer constant velocity ratio model. As exposed above, this additional boundary condition is often difficult to impose in accordance with the physical flow conditions. Another advantage of the two-layer model is the possibility to impose a condition on the solid discharge instead of the bed level.

For each model it was possible to determine a kind of Froude number taking sediments into account. The control section is obtained when this Froude number is equal to 1. In the case of the two-layer model, it occurs when one of the eigenvalue is equal to zero, contrarily to the Saint-Venant – Exner, for which it occurs when two of the eigenvalues are identical.

In the Saint-Venant – Exner scheme, in all cases, a part of the information comes from downstream, it is not always the case in the two-layer model. Nevertheless, the range of flow conditions for which a part of the information comes from downstream is enlarged in comparison with the hydrodynamic case, for some supercritical cases, the information may go upstream.

	Saint-Venant – Exner	Two-layer assumption on velocity	Two-layer two-velocity
Maximum upstream boundary conditions	2	3	4
Minimum upstream boundary conditions		2	2
Maximum downstream boundary conditions	1	1	2
Minimum downstream boundary conditions		0	0
Boundary condition on the solid discharge q_s	No	Yes	Yes
Boundary condition on the bed level z_b	Yes	Vertical characteristic	Vertical characteristic
Control section	Fr = 1	$\mathrm{Fr}_m = 1$ (2.146) $\mathrm{Fr}_{sw} = 1$ (2.155)	$Z_{sw} = 1$ (2.163)
Hyperbolicity	Yes	Yes	Depends on flow conditions

Table 4.7. Boundary conditions, comparison between models