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Optimal liquidation under indirect price impact with propagator

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Abstract

We propose in this paper a new framework of optimal liquidation strategies for a trader seeking to liquidate his large inventory based on a jump-dependent price impact model with propagator. This new jump-dependent price impact model allows to best reproduce the empirical direct and indirect effects of market orders on the transaction price. More precisely, different choices of propagators are proposed and their implications in terms of temporary, permanent and transient impacts on the transaction price are discussed. For each choice of such kernels, we formulate the most relevant optimal liquidation problem faced by the trader, derive explicitly the related Hamilton-Jacobi-Bellman equation and solve it numerically. Moreover, we show how to extend our price impact model so to include the possibility for the trader to also use limit orders. We hence manage in this paper to propose an alternative more realistic and flexible description of the order book's dynamic and to make a bridge between high-frequency price models and optimal liquidation problems.

Keywords: Optimal liquidation, HJB equation, Price impact model, Market impact, High-frequency trading

1. Introduction

A classical problem in finance is how an agent can sell (or buy) a large amount of shares and yet minimize adverse price movements generated by her own trades. *Large* means here that the amount the agent is interested in selling is too big to execute in a single trade. The agent must then take into account the balance between price impact (trade quicker) and price risk (take longer to complete all trades). Indeed, if an agent trades quickly, then her orders will walk through the buy side of the limit order book (LOB) and she will obtain lower prices for her orders (temporary price impact). Even if she splits each order into smaller trades so that each one does not walk the book and executes them rapidly in the market, then other traders will notice an excess of sell orders (net order flow imbalance) and will adjust their quotes inducing again a negative price impact, which is in this case transient and/or permanent. On the other hand, if she trades slowly so as to avoid this price impact, then she will face the uncertainty of future prices, which could have dropped to a less desirable level by the time the agent executes the whole order. The aim of optimal liquidation is hence to propose an optimal strategy such as to balance these two factors based on a coherent price impact model reproducing the underlying asset price and the impact of the agent's transactions on this price. We refer to

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[24, 18, 26, 14] for a review of the most important results concerning optimal execution problems.

Typically, when modeling the price of an asset, three different time scales can be considered. At low frequencies, the dynamic of stock prices can often be reproduced by a diffusive process. At the other end, when dealing with very high frequencies, empirical features and statistical properties of the LOB dynamics have to be modeled (long-range dependence of trades, volatility clustering, high-frequency resilience of the price, etc.). At such very high frequencies, one then has to describe precisely the LOB dynamic. A variety of such models have been proposed in [1, 28, 19, 21] or recently in Bacry et al. [9, 10] with a tick-by-tick price model ruled by Hawkes processes, which describes jointly the order flow and the price dynamics and which reproduces numerous empirical properties of high-frequency markets. In between of low and high-frequency models, price impact models consider an intra-day intermediate time scale, anywhere from seconds to hours. They tend to discard most of the LOB events such as limit orders, cancellations, market orders, etc. and instead concentrate on describing the price impact of the agents' transactions. Their goal is to be more tractable than high-frequency models and to offer analytical results on practical issues such as the optimal execution problems studied in this paper. The usual setup is well described in Gatheral [22] where the execution price S^ν of a transaction is given for $t \in [0,T]$ with T the trading horizon by

$$S_t^\nu = S_0 + \sigma W_t + \int_0^t f(\nu_s) G(t-s) ds \tag{1.1}$$

where ν_s is the rate of trading of the liquidating agent at time s, $f(\nu)$ represents the instantaneous price impact of an agent trading at speed ν, G is called a *decay kernel* or *propagator* and W is a noise process (usually a Brownian motion). As explained in [3], $f(\nu)G(0)$ captures the temporary/instantaneous impact of trades on the price, $f(\nu)G(+\infty)$ the permanent impact and $f(\nu)[G(0+) - G(+\infty)]$ the transient impact. That is why the two extreme cases where G is a Dirac's delta $\delta(t-s)$ on $[0,T]$ and when $G \equiv 1$ are referred in the literature as temporary price impact and permanent price impact, respectively. They are core features in the well-known Almgren-Chriss model [13, 6, 7], up to a multiplicative constant. Moreover, the price impact model (1.1) also allows to generalize most of the standard processes considered in the optimal liquidation literature, see [15, 8, 35, 4, 31]. Since we are interested in this paper in optimal execution problems, we will then work with such price impact models observed at middle frequency. However, we aim at expanding these models by including more specific events and features of the LOB. One of our main goal is hence to make a bridge between high-frequency price models and the optimal execution framework by proposing a tractable model reproducing some empirical evidence of high-frequency microstructure markets. We refer to Alfonsi and Blanc [3] who were the first to propose such mixed-framework models incorporating both high-frequency and middle-frequency features.

As mentioned above and in [22, 14, 3], we have that price impact can be temporary, transient and/or permanent. Market orders (MOs) that walk the LOB have a price impact because they obtain average execution prices worse than the best quote and this price impact is temporary because it is assumed that: the LOB will replenish very quickly and, after restocking, there is no effect on the midprice of the asset. On the other hand, investors' trading activity has a transient and permanent effect on midprices because traders' order-flow (*i.e.* the number and the size of MOs) conveys information that is impounded on the midprice of the asset. In general, when there is positive net order-flow (more buy than sell volume) the midprice tends to drift up, and when there is negative net order-flow (more sell than buy volume) the midprice tends to drift down. Thus, when an investor is executing a large market order, this adds one-sided pressure to the market's order-flow and hence on midprices: up if the investor is buying or down if selling. In this paper, this downward permanent (and transient) pressure on the price will be modeled by an increase of the (mean-reverting) intensity of a non-homogeneous Poisson process with negative jump size, that conveys the information about MOs price impacts. This jump process will replace the term $f(\nu_s)ds$ in the standard setting (1.1) as we will see in Section 2 with equation (2.1). We will also investigate various choices of decay kernel G so as to combine different ways of modeling these temporary, transient and permanent price impacts.

More precisely, we observe empirically at high-frequency that at any one time, the number of shares displayed and available in the market at the quoted bid/ask is limited, see Figure 1.1 and 1.2. The large trader will then split its sell order in different trade sizes, some of which could be very small at a particular time compared to the depth of the LOB. If a market order at time t of the large trader does not walk the LOB, *i.e.* the size of this transaction is smaller than the depth at the best bid, there will not be any direct change in the stock's mid-price *(no temporary direct price impact from walking the LOB)*, even if this will lead to more pressure towards a price decrease *(indirect permanent and transient impact from the net order flow imbalance)*. This downward pressure in the price will be modeled in the sequel by an increase in the mean-reverting intensity of a jump process with negative jump size. Including a jump process for representing MO's price impact makes sense since the depth at the best bid is random/unpredictable at this time t: most of the time a small sell MO will not lead to a direct shift in price (no jumps) but for whatever reason, if the quantity available at the best bid is really low at this time, this small order of the large trader will however walk the LOB and lead to a shift in the mid price (jump). Similarly, if the LOB is particularly deep at that time t, even a large market sell order could not lead to a direct shift in the mid price (no temporary price impact) but it will however put a lot of downwards pressure in the price (indirect permanent/transient price impact). This will hence translate in our model by a strong increase in the intensity of the jump process. These features of the LOB are confirmed in [18, 17, 14] and by the following Figure 1.1 which represents the LOB for the AMZN share on June 21, 2012 at 10:36:47.016 and the related temporary price impact of a sell MO in function of volume. Similar figures and results are depicted in Fig. 6.1 of [18] and in [17] for SMH stock price.

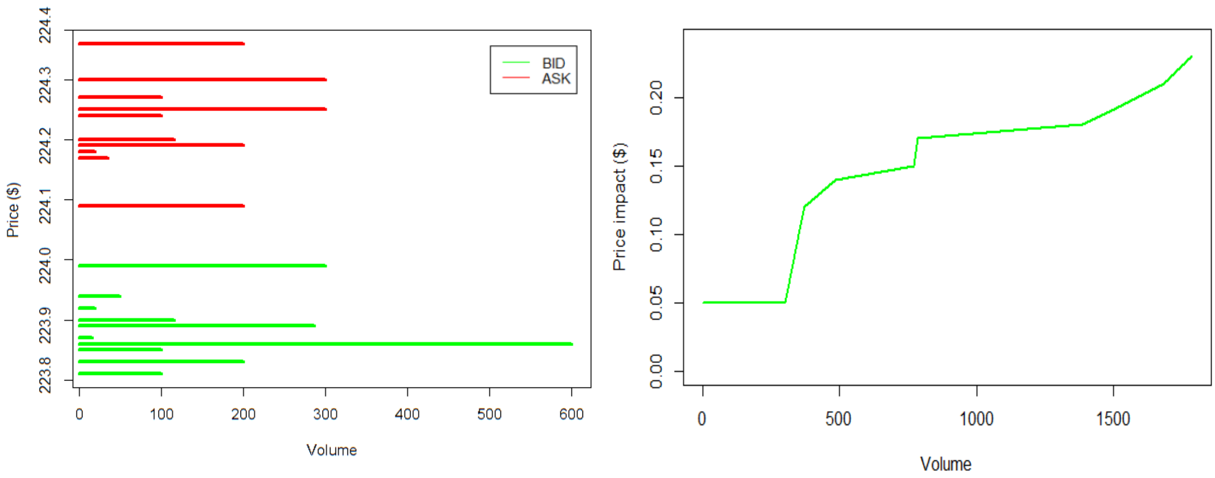
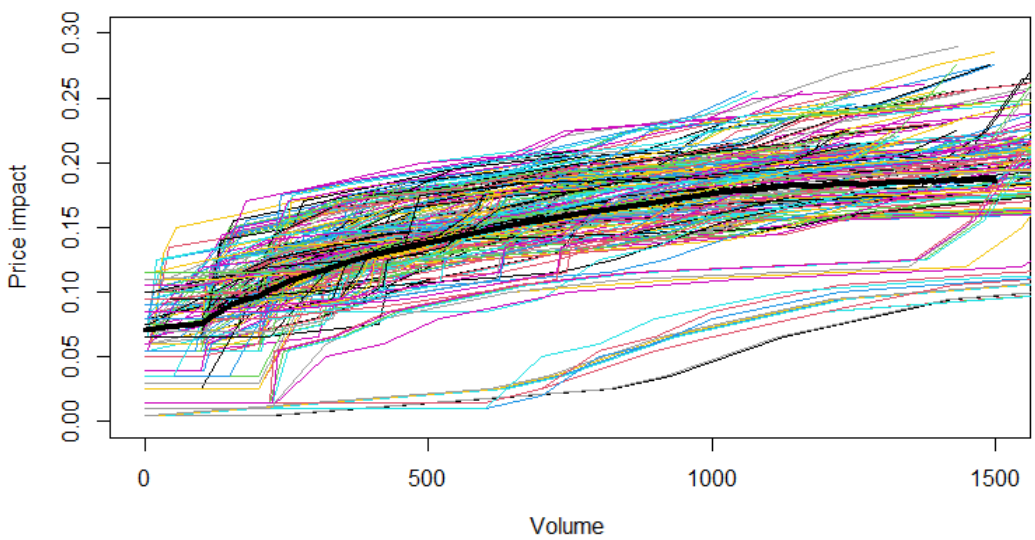


Figure 1.1: Limit order book of AMZN on the 21/06/2012 at 10:36:47.016 and related temporary price impact of a market sell order in function of the volume sold. In green, bid prices and in red, ask prices.

We clearly see a plateau in the temporary price impact for small volumes, meaning that such small MOs do not walk the LOB (*i.e.* sell volume < depth) and hence do not lead to any temporary price impact. Indeed for sell market orders with a volume below 300, there is no price impact (besides the bid-ask spread). For volumes above 300, we then observe a power-law relation between the (interpolated) price impact and the sell volume. This is confirmed by the following Figure 1.2 where multiple temporary price impact curves in function of the volume sold are depicted inside the same minute for the AMZN stock price.



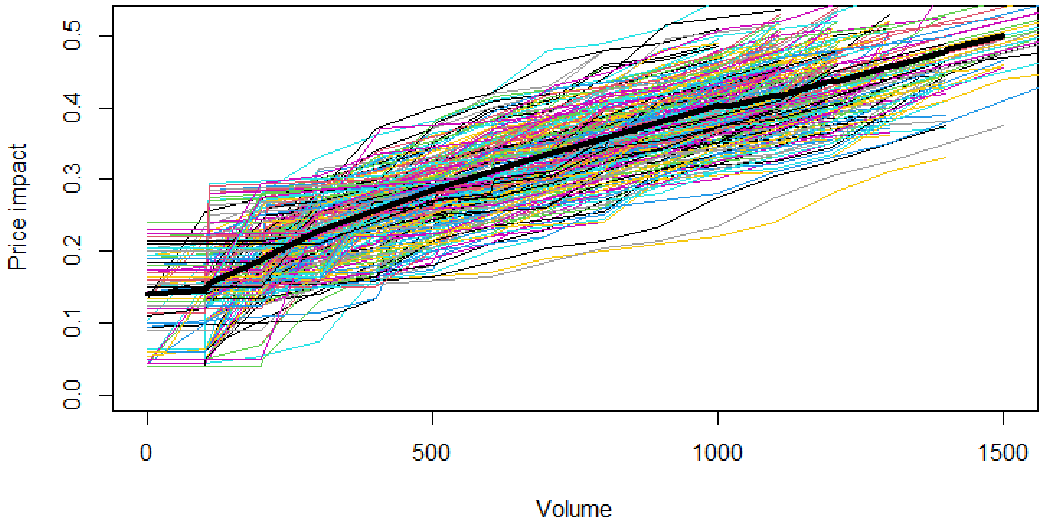


Figure 1.2: Temporary price impact in function of volume for AMZ on Jun 21, 2012 for every millisecond between 10:36 and 10:37 am (upper plot) and between 09:43 and 09:44 am (lower plot). In black wider line is depicted the mean temporary price impact over the corresponding minute.

In each price impact curve, we again see plateaus for small volumes, translating the fact that small enough sell MOs (with volume lower than the depth at the best bid) do not lead to any direct temporary price impact besides the bid-ask spread. More importantly, we see in both plots that even the average temporary impact curve over the minute (bold line) remains constant for sell volumes below 100. Moreover, once the volume sold is large enough for triggering a price impact, this temporary impact seems to be best reproduced by a power-law function with exponent below 1 and more precisely between 0.4 and 0.8. This is consistent with the literature and the so-called square-root law as investigated in [8, 40, 22, 15]. Such behaviors of the temporary impact curves justify the use of an indirect price impact model based on a jump process, as we will introduce in the following Section 2.1. This point process will indeed jump whenever the sell order is sufficiently large (here above 100) and the size of the negative jump will then scale as a power-law function of the volume sold. As mentioned above, the intensity of this jump process will also depend on the volume of the sell MO as each sell order puts a permanent downwards pressure on the price through the net order flow imbalance, even if there is no direct temporary impact (no jump). Existing price impact models neglect this absence of temporary negative impact for sufficiently low volumes. However, being able to trade with MOs without having any temporary price impact for small enough sell orders has a strong impact on the optimal strategy used by the liquidating trader.

Finally, we raise the point that optimal liquidation problems based on jump processes are already studied extensively in the literature, see *e.g.* [27, 11, 36], Chap 8.1-8.3 of [18] and references therein. However, the jump processes in these papers are systematically used to model the occurrence of limit orders sent by traders that provide liquidity. In our setting, the jump process allows instead to model the occurrence of the temporary/permanent/transient price impacts arising from MOs since all trades do not lead to a direct negative impact on the execution price of the transaction, see Figure 1.2. Moreover, we propose in Section 3 an extension of our price impact model so as to include strategies using both market and limit orders (based on a second jump process). This hence provides a more complete and realistic price impact model, allowing to best reproduce the underlying dynamic of the LOB.

The contributions of this paper are therefore multiple. We first introduce a new jump-dependent price impact model with propagator allowing to best reproduce the empirical direct and indirect effects of market orders on the transaction price, as described above and in Figure 1.2. We then discuss in Section 2 various ways of modeling in our setting the temporary, permanent and/or transient price impact of trades based on different choices of propagators G. For each choice of such kernels, we formulate the most relevant optimal liquidation problem faced by the agent, derive explicitly the related HJB equation and solve it numerically. This allows to obtain the optimal trading strategies in each scenario and to discuss the implications of each model (and its parameters) on the optimal trading rate of the liquidating agent. We finally extend in Section 3 the initial price impact model based solely on market orders such as to incorporate the possibility for the agent to use both market and limit orders. We hence provide with this paper a more realistic description of the underlying LOB dynamic that allows to include specific high-frequency features of the order book.

The remainder of this study is organized as follows. Section 2 introduces the indirect price impact model with propagator. Section 2.1, 2.2 and 2.3 consider various assumptions on the propagator G with, respectively, a temporary, a permanent and a transient form for G. Section 3 generalizes the model by including limit orders. Section 4 concludes.

2. The indirect price impact model with propagator

We introduce the following liquidation model in continuous time. We assume at first that only MOs are posted by the trader (limit orders are then discussed in Section 3). Moreover, no bid-ask spread is taken into account in what follows but the model can be extended directly so as to include it.

- T is the time horizon required for liquidating the shares, *i.e.* the trading horizon.
- $\nu = (\nu_t)_{0 \leq t \leq T}$ is the trading rate, the (positive) speed at which the agent is liquidating shares using MOs and which can be seen as the volume sold on each infinitesimal step of time.
- Q^ν is the agent's inventory of remaining shares to liquidate, affected by how fast she trades and given by $dQ^\nu_t = -\nu_t dt$ with $Q^\nu_0 = Q$. We do not assume here that the agent liquidates all the shares during the horizon T. Hence, it could be that $Q^\nu_T \geq 0$, *i.e.* the agent falls short of her target to liquidate all

the shares before T but she incurs in this case a terminal penalty (whose form will be detailed below). However, we do assume that the agent cannot liquidate more than the Q initial shares such that we cannot have $Q^\nu_T < 0$ (see below (2.10) with the stopping time τ).

- S^ν is the execution price affected by the optimal trading rate.
- X^ν is the agent's cash process resulting from her execution strategy and defined as $X^\nu_t := \int_0^t S^\nu_s \nu_s \, ds$.

We fix a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}\right)$ where T is the deterministic trading horizon. As an extension of the price impact model of Gatheral in [22], we further assume that the execution price S^ν_t at time t follows

$$S^\nu_t := S_t + \int_0^t G(t-s)\, dL^\nu_s\,, \tag{2.1}$$

where $S_t = S_0 + \sigma W_t$ is a continuous martingale representing the unaffected stock price and the decay kernel $G : (0, +\infty) \mapsto [0, +\infty]$ is such that G is a measurable function, square-integrable on $[0, T]$ and $G(0) := \lim_{t\to 0} G(t)$ exists. We will consider throughout the paper a $(\mathcal{F}_t)_{0\leq t\leq T}$-Brownian motion as the continuous martingale $W = (W_t)_{t\geq 0}$. We further define the jump process

$$L^\nu_t := -\int_0^t \kappa\, \nu^\gamma_s \, dN^\nu_s\,, \quad t \in [0, T]\,,$$

independent from W, where the size of the jumps depends on the size of the trading rate through the exponent γ and where N^ν is a Poisson process on $[0, T]$ with intensity defined by

$$\lambda^\nu_t := \lambda_0 + \int_0^t e^{-\alpha(t-s)}\, h(\nu_s) ds\,.$$

We assume in the sequel for simplicity a linear function $h(\nu) = \eta\nu$, $\eta \geq 0$, which gives the following mean-reverting intensity

$$\lambda^\nu_t = \lambda_0 + \eta \int_0^t e^{-\alpha(t-s)}\, \nu_s \, ds \qquad \text{or} \qquad d\lambda^\nu_t = (\eta\nu_t - \alpha\lambda^\nu_t)\, dt\,. \tag{2.2}$$

We must hence impose that ν_t remains positive for all $t \geq 0$ so that the intensity λ^ν_t remains non-negative. In practice, this means that purchasing shares is not allowed in our model at any time during the trading horizon. Moreover, this model induces that the execution price of a transaction depends both *indirectly* on the strategy ν chosen by the trader via the intensity λ^ν of N^ν as well as *directly* on ν through the size of the jumps $(-\kappa\nu^\gamma)$ of the point process L^ν and through the propagator G in (2.1). Indeed, high trading rates increase the probability of jumps and hence of a negative impact on the transaction price, which is then proportional to the size of that trade and scales more precisely as a power law function of ν with exponent $\gamma \leq 1$, as seen in Section 1 and Figure 1.2. Empirically, it makes sense that λ^ν_t depends on the past trading rates $(\nu_s)_{0\leq s\leq t}$ since a sell MO puts a permanent and transient downward pressure on the price, which translates in our model by a higher probability of a negative jump (see again Section 1). Hence, in this model, we always have a *permanent and transient* indirect impact of MOs on the price through the mean-reverting intensity λ^ν even if the decay kernel G ruling the direct price impact can be temporary (*i.e.* formally a Dirac's delta, cf. Section 2.1) or take other forms (Section 2.2 and 2.3). This allows to obtain a more flexible setting than many liquidation models proposed in the literature. Note that the mean-reversion property of the intensity (2.2) induces an (indirect) transient price impact in our model while the fact that the mean-reversion level of λ^ν depends on ν leads to a permanent impact of trades. Finally, as argued in [22] or [23], considering the martingale $S_t = S_0 + \sigma W_t$ (with W Brownian motion) for the unaffected price allows to implicitly incorporate the cumulative impact of others' trading (the trading crowd in the terminology of Huberman and Stanzl [29]) and is quite standard in the literature. Indeed, negative prices only occur with negligible probability at the considered time scales. Moreover, the drift term of S is usually omitted for such small time scales. Finally, the agent's cash process X^ν satisfies $dX_t^\nu = S_t^\nu \nu_t \, dt = \left(S_t + \int_0^t G(t-s) dL_s^\nu\right) \nu_t \, dt$ and therefore, the expected revenues (or *proceeds*) from the sale at time t is denoted

$$R_t^\nu = \mathbb{E}[X_t^\nu] = \mathbb{E}\left[\int_0^t S_s^\nu \nu_s \, ds\right] = \mathbb{E}\left[\int_0^t \left(S_s + \int_0^s G(s-u) dL_u^\nu\right) \nu_s \, ds\right] . \tag{2.3}$$

2.1. Temporary price impact kernel

We consider in this section only a temporary direct impact of trading on the execution price, meaning that a sell MO at a time t only has a direct effect on the price for this transaction but not on the price of subsequent transactions. Recall however that a permanent and transient impact of MOs is always included indirectly in our setting via the mean-reverting intensity λ^ν of the jump process in equation (2.2). In this temporary case, it is best to start by defining the dynamic of the wealth process X^ν. The best way to model this temporary direct impact of trades on the transaction price is to define the wealth process (at a stopping time τ, see more precisely (2.10) below) by

$$X_\tau^\nu := \int_0^\tau S_s \nu_s ds - \int_0^\tau \kappa \nu_s^{\gamma+1} dN_s^\nu = \int_0^\tau S_s \nu_s ds - \sum_{j=1}^{N_\tau^\nu} \kappa \nu_{t_j}^{\gamma+1} . \tag{2.4}$$

We then find using integration by parts with $Q_\tau^\nu = Q_t^\nu - \int_t^\tau \nu_s ds$ and with $\tau \geq t$,

$$\int_t^\tau S_s \nu_s ds = S_t Q_t^\nu - S_\tau Q_\tau^\nu + \int_t^\tau \sigma Q_s^\nu \, dW_s . \tag{2.5}$$

Since Q^ν is bounded, the third term is a martingale and since $dX_t^\nu = S_t^\nu \nu_t dt$ we find

$$\mathbb{E}[X_\tau^\nu \mid \mathcal{F}_t] = X_t^\nu + S_t Q_t^\nu + \mathbb{E}\left[-S_\tau Q_\tau^\nu - \kappa \int_t^\tau \nu_s^{\gamma+1} \lambda_s^\nu \, ds \,\middle|\, \mathcal{F}_t\right] . \tag{2.6}$$

As ν is non-negative and $\mathcal{F}$-predictable (see below the set (2.12) of admissible controls $\mathcal{A}$), we indeed have from [32] that $\mathbb{E}\left[\int_t^\tau \kappa \nu_s^{\gamma+1} dN_s^\nu \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^\tau \kappa \nu_s^{\gamma+1} \lambda_s^\nu \, ds \,\middle|\, \mathcal{F}_t\right]$. This expression (2.4) for the agent's wealth appears to be the only reasonable way for modeling this temporary direct impact on the price. Indeed, since trading occurs continuously during the trading horizon and since the direct impact of a trade only affects the transaction price at this time and not the subsequent ones, we have that the wealth is simply the proceeds of the sale at the unaffected price S minus the sum of these negative impulsive impacts $\kappa\nu_{t_j}^{\gamma}$ (each time they occur) times the quantity sold ν_{t_j} at that time. As explained in [2] and [22], considering only a temporary direct price impact amounts to use formally a Dirac's delta distribution $G(t-s)=\delta(t-s)$ on $[0,T]$ as propagator in the execution price S^{ν} (2.1). We can indeed write formally in this case the execution price as

$$\begin{aligned} S_t^{\nu} = S_t + \int_0^t G(t-s)\, dL_s^{\nu} &= S_t - \kappa \int_0^t G(t-s)\, \nu_s^{\gamma}\, dN_s^{\nu} \\ &= S_t - \kappa \sum_{j=1}^{N_T^{\nu}} \nu_{t_j}^{\gamma}\, \delta(t-t_j) \qquad (2.7) \\ &= S_t + dL_t^{\nu}/dt\,, \qquad (2.8) \end{aligned}$$

where t_j $(j=1,\ldots,N_T^{\nu})$ are the jump times of N^{ν} over $[0,T]$. This indeed leads to the expression (2.4) of the wealth process $X_{\tau}^{\nu} = \int_0^{\tau} S_s\nu_s ds + \int_0^{\tau} \frac{dL_s^{\nu}}{ds}\nu_s ds = \int_0^{\tau} S_s\nu_s ds - \int_0^{\tau} \kappa\nu_s^{\gamma+1} dN_s^{\nu}$. Such process S^{ν} for the execution price (2.7) is sometimes called a "Dirac process" and has already been investigated in [30] and [39] as arising from the time derivative of a Poisson process. This form of the execution price S^{ν} is not interesting per se and cannot be used to define our model (just as a Brownian motion alone is not particularly useful, similarly a Dirac process is only useful for what can be done with it). However, this price impact model (2.7) is justified by the fact that it is consistent with the definition of the wealth process (2.4), which is the only reasonable form for X^{ν} in the temporary case and which is the process of interest in the optimization program below (the state variables in the temporary case are indeed S and X^{ν}, but never S^{ν}).

Now suppose that the agent is willing to optimize the following performance criterion

$$H^{\nu}(t,x,S,q,N,\lambda) = \mathbb{E}_{t,x,S,q,N,\lambda}\left[X_{\tau}^{\nu} + Q_{\tau}^{\nu}\,(S_{\tau} - \pi\lambda_{\tau}^{\nu}Q_{\tau}^{\nu}) - \phi\int_t^{\tau} (Q_u^{\nu})^2\, du\right], \qquad (2.9)$$

where $\mathbb{E}_{t,x,S,q,N,\lambda}[.]$ denotes the expectation conditional to $X_{t^-}^{\nu}=x$, $S_t=S$, $Q_t^{\nu}=q$, $\lambda_t^{\nu}=\lambda$, $N_{t^-}^{\nu}=N$ and where τ is the stopping time defined by

$$\tau := T \wedge \inf\{t \geq 0 : Q_t^{\nu} = 0\}\,, \qquad (2.10)$$

which is the minimum of T and the first time that the inventory hits zero (because then no more trading is necessary), see *e.g.* Chapter 7.2, 7.4 and 8.4 of [18]. This stopping time τ is therefore defined as the last trading time of the agent (after which the wealth X does not vary anymore). The corresponding value function is then

$$H(t,x,S,q,N,\lambda) = \sup_{\nu\in\mathcal{A}} \mathbb{E}_{t,x,S,q,N,\lambda}\left[X_{\tau}^{\nu} + Q_{\tau}^{\nu}\,(S_{\tau} - \pi\lambda_{\tau}^{\nu}Q_{\tau}^{\nu}) - \phi\int_t^{\tau} (Q_u^{\nu})^2 du\right], \qquad (2.11)$$

where $\mathcal{A}$ is the admissible set of trading strategies in which ν is non-negative, $\mathcal{F}$-predictable and uniformly bounded from above

$$\mathcal{A} = \left\{(\nu_t)_{t\in[0,T]} \mid \nu \geq 0,\ \mathcal{F}\text{-predictable},\ \nu \in L^\infty(\Omega \times [0,T])\right\}. \tag{2.12}$$

We see that if the agent's strategy reaches the terminal date T with inventory left, then she must execute a final order to sell the required number of shares Q for a total revenue at T of $Q^\nu_T(S_T - \pi\lambda_T Q^\nu_T)$, where π is the terminal liquidation penalty parameter[1] that includes the cost of walking the book at T and any other additional penalties that the agent must incur for the execution of the trade at the terminal date. The terminal penalty $Q^\nu_T(S_T - \pi Q^\nu_T)$ without the term λ^ν_T is classical in the literature when the price impact is given by the model (1.1) and does not depend on a jump process (see Chap. 6 of [18] and references therein). However, it is necessary in our context to adapt this terminal penalty. Indeed, depending on the shape of the LOB at maturity T, it could be that liquidating the Q^ν_T remaining shares using MOs does not lead to a negative price impact and hence, the occurrence of this negative terminal penalty must again depend on the jump process N^ν with intensity λ^ν. Consistently with the literature where the temporary price impact in terms of the trading rate $\kappa\nu^\gamma$ is replaced by a penalty πQ^ν_T at terminal time T (see [17, 16, 18]), we hence introduce our penalized price S^π_T at T again as a sum of Dirac's delta according to equations (2.1)-(2.7) but with jump size $-\pi Q^\nu_T$ instead of $-\kappa\nu^\gamma$ in the following (formal) way

$$S^\pi_T := S_T + dL^\pi_T/dt = S_T - \pi Q^\nu_T \sum_{j=1}^{N^\nu_T} \delta(T - t_j), \tag{2.13}$$

where $L^\pi_T := -\pi \int_0^T Q^\nu_s\, dN^\nu_s$ is a new random variable for the terminal penalty with N^ν defined as above by (2.2). Note that even when the direct price impact is not temporary (see further sections with other kernels G), the terminal penalty remains a temporary impact since it only affects the price of the transaction at time T and not the other ones. With a penalized price of the form (2.13), we retrieve the terminal penalty

$$\mathbb{E}_{t,x,S,q,N,\lambda}\left[Q^\nu_T S^\pi_T\right] = \mathbb{E}_{t,x,S,q,N,\lambda}\left[Q^\nu_T\left(S_T - \pi\lambda^\nu_T Q^\nu_T\right)\right] = \mathbb{E}_{t,x,S,q,N,\lambda}\left[Q^\nu_\tau\left(S_\tau - \pi\lambda^\nu_\tau Q^\nu_\tau\right)\right]. \tag{2.14}$$

Again the expression (2.13) cannot be used to define our model but is induced and justified by the chosen form of the terminal penalty in (2.11). The last term in (2.11) is a running inventory penalty, which is not a financial cost to the agent's strategy (and hence does not depend on λ^ν) but which allows to incorporate the agent's urgency for executing the trade. It has been shown in [16] that this penalty is equivalent to a form of risk aversion from the agent. An alternative risk aversion penalty is also discussed in Appendix B. Finally, as seen above in Figure 1.2 and as shown in [40, 8, 22, 18], we have empirically that the *temporary* (or instantaneous) price impact depends on the square root of the trading rate ($\gamma = 1/2$) and more generally

[1] We consider here S_T and not S^ν_T since the direct impact is temporary, the terminal penalty being imputed here in place of this temporary impact.

on the power-law ν^γ with $\gamma \leq 1$. Our setup in this section is in fact similar in terms of temporary impact as the one of [8, 5, 22] where they also use $G(t-s) = \delta(t-s)$ but without jump process and based instead on equation (1.1) with $f(\nu) = -\kappa\nu^\gamma$ and $\gamma \in [0.4, 0.8]$.

2.1.1. Maximization of expected wealth with mean-reverting intensity

We first consider that the jump process N^ν is ruled by the mean-reverting intensity (2.2). We will then show in Section 2.1.2 that the optimization problem (2.11) can be simplified by choosing an alternative expression for λ^ν. Recall first that the set $\mathcal{A}$ of admissible trading rates ν consists in uniformly bounded, non-negative and $\mathcal{F}$-predictable strategies. From equation (2.4) and from (2.6), we can then rewrite the expected wealth (2.3) in the following way at the last trading time τ :

$$\mathbb{E}_{t,x,S,q,\lambda,N}[X_\tau^\nu] = x + qS + \mathbb{E}_{t,S,q,\lambda}\left[-S_\tau Q_\tau^\nu - \int_t^\tau \kappa\nu_s^{\gamma+1}\lambda_s^\nu\, ds\right],$$

with $X_{t^-} = x, S_t = S, Q_t^\nu = q, \lambda_t^\nu = \lambda, N_{t^-}^\nu = N$. We note that when considering formally a Dirac delta $G(t-s) = \delta(t-s)$ in equation (2.1) for the execution price S^ν and using a Fubini's theorem for distribution (Ch 4, Sec. 3, Thm. IV. of [38]), we find equivalently from equation (2.3) that

$$\begin{aligned}\mathbb{E}_{t,x,S,q,\lambda,N}[X_\tau^\nu] &= x + \mathbb{E}_{t,S,q,\lambda}\left[\int_t^\tau S_s\nu_s ds - \int_t^\tau \left(\int_0^s G(s-u)\kappa\nu_u^\gamma\lambda_u^\nu du\right)\nu_s\, ds\right]\\ &= x + qS + \mathbb{E}_{t,S,q,\lambda}\left[-S_\tau Q_\tau^\nu - \int_t^\tau \kappa\nu_s^{\gamma+1}\lambda_s^\nu\, ds\right].\end{aligned}$$

Since the terminal penalty is given by equation (2.14), we can rewrite the performance function (2.9) as

$$\begin{aligned}H^\nu(t,x,S,q,\lambda,N) &= \mathbb{E}_{t,x,S,q,\lambda,N}\left[X_\tau^\nu + Q_\tau^\nu\left(S_\tau - \pi\lambda_\tau^\nu Q_\tau^\nu\right) - \phi\int_t^\tau (Q_s^\nu)^2 ds\right]\\ &= x + qS + \mathbb{E}_{t,q,\lambda}\left[-\int_t^\tau \kappa\,\nu_s^{\gamma+1}\lambda_s^\nu\, ds - \pi\lambda_\tau^\nu\,(Q_\tau^\nu)^2 - \phi\int_t^\tau (Q_s^\nu)^2 ds\right].\end{aligned}$$

And we finally have for the value function (2.11) :

$$\begin{aligned}H(t,x,S,q,\lambda,N) &= x + qS + \sup_{\nu\in\mathcal{A}}\mathbb{E}_{t,q,\lambda}\left[-\int_t^\tau \kappa\,\nu_s^{\gamma+1}\lambda_s^\nu\, ds - \pi\lambda_\tau^\nu\,(Q_\tau^\nu)^2 - \phi\int_t^\tau (Q_s^\nu)^2 ds\right]\\ &= x + qS - \inf_{\nu\in\mathcal{A}}\mathbb{E}_{t,q,\lambda}\left[\int_t^\tau \kappa\,\nu_s^{\gamma+1}\lambda_s^\nu\, ds + \pi\lambda_\tau^\nu\,(Q_\tau^\nu)^2 + \phi\int_t^\tau (Q_s^\nu)^2 ds\right]\\ &= x + qS - \inf_{\nu\in\mathcal{A}} h^\nu(t,q,\lambda) = x + qS - h(t,q,\lambda).\end{aligned}$$

Note that since $Q_t^\nu = Q_0 - \int_0^t \nu_s\, ds$ and $\lambda_t^\nu = \lambda_0 + \int_0^t e^{-\alpha(t-s)}\nu(s)ds$, the objective function h^ν is continuous in ν and hence, it admits an optimal solution $\nu^* \in \mathbb{R}^+$. This solution ν^* is unique since h^ν is strictly convex in ν for $\gamma > 0$, see equation (2.15). Considering $h = \inf_\nu h^\nu$, standard results from dynamic programming imply that h is the unique viscosity solution of the following Hamilton-Jacobi-Bellman (HJB) equation (see[2] *e.g.* Fleming and Soner 2006, Chapter V) with $\gamma > 0$:

$$\partial_t h + \phi q^2 + \inf_{\nu} \left\{ (\eta\nu - \alpha\lambda)\partial_\lambda h - \nu \partial_q h + \kappa \nu^{\gamma+1}\lambda \right\} = 0\,, \tag{2.15}$$

subject to terminal and boundary conditions

$$h(T, q, \lambda) = \lambda \pi q^2 \quad \text{and} \quad h(t, 0, \lambda) = 0\,, \tag{2.16}$$

since we must have $H(T, x, S, q, \lambda) = x + qS - \lambda\pi q^2$ and $H(t, x, S, 0, \lambda) = x$. The boundary condition is implied by the non-negativity of ν^* and by the fact that when the remaining inventory hits $q = 0$, the agent stops acquiring shares as of that stopping time and there is no penalty (she thus simply walks away with x in cash). The first term of H is the accumulated cash of the strategy, the second is the marked-to-market book value of the remaining inventory and h is an extra penalty due to price impact stemming from optimally liquidating the rest of the shares. Note that when λ_t^ν tends to 0, the agent can liquidate all her remaining shares without risking to have an adverse impact on price, which is optimal, and we thus find the additional boundary condition $\lim_{\lambda\to 0} h(t, q, \lambda) = 0$ for the variable λ. However, specifying $\lambda_0 > 0$ prevents this situation from happening with the mean-reverting intensity (2.2), even if this boundary condition is useful numerically. The first order condition in (2.15) implies the following feedback form for the optimal trading rate

$$\nu^*(t, q, \lambda) = \left(\frac{\partial_q h(t, q, \lambda) - \eta\, \partial_\lambda h(t, q, \lambda)}{(\gamma + 1)\lambda\kappa} \right)^{1/\gamma}\,, \tag{2.17}$$

which is well defined since $\lambda > \lambda_0 > 0$. Hence, among all the uniformly bounded, non-negative and $\mathcal{F}$-predictable admissible solutions for the trading strategy, the optimal one ν^* is deterministic as it does not depend on S, W nor N^ν. Upon substitution back in the previous equation, we find

$$\partial_t h(t, q, \lambda) + \phi q^2 - \frac{(\partial_q h(t, q, \lambda) - \eta\, \partial_\lambda h(t, q, \lambda))^{\frac{\gamma+1}{\gamma}}}{(\lambda\kappa(\gamma + 1))^{1/\gamma}} \left(\frac{\gamma}{\gamma + 1} \right) - \alpha\lambda\, \partial_\lambda h(t, q, \lambda) = 0\,. \tag{2.18}$$

For the case $\gamma = 1$, this gives $\nu^*(t, q, \lambda) = (\partial_q h - \eta \partial_\lambda h) \,/\, (2\lambda\kappa)$, and

$$\partial_t h(t, q, \lambda) + \phi q^2 - \frac{1}{4\lambda\kappa} \left(\partial_q h(t, q, \lambda) - \eta\, \partial_\lambda h(t, q, \lambda)\right)^2 - \alpha\lambda\, \partial_\lambda h(t, q, \lambda) = 0\,, \tag{2.19}$$

and for the square-root law $\gamma = 1/2$, we find $\nu^*(t, q, \lambda) = 4 \left(\partial_q h - \eta \partial_\lambda h\right)^2 / \left(3\lambda\kappa\right)^2$ and

$$\partial_t h(t, q, \lambda) + \phi q^2 - \frac{4}{27(\lambda\kappa)^2} \left(\partial_q h(t, q, \lambda) - \eta\, \partial_\lambda h(t, q, \lambda)\right)^3 - \alpha\lambda\, \partial_\lambda h(t, q, \lambda) = 0\,. \tag{2.20}$$

Choosing the square-root law $\gamma = 1/2$ as empirically motivated in the literature for temporary price impact ensures that the optimal trading rate is non-negative. The PDE (2.18) cannot be solved analytically but is

[2]The state dynamics (dQ_s^ν, $d\lambda_s^\nu$) and reward function ($\kappa \nu_s^{\gamma+1} \lambda_s^\nu$) are indeed globally Lipschitz in the state variable while the running and terminal costs ($\phi(Q_s^\nu)^2$, $\pi\lambda_\tau^\nu (Q_\tau^\nu)^2$) are continuous in the state variables and are bounded below and above on $[0, T]$. Moreover, the state domain is bounded and the control set $\mathcal{A}$ is compact.

solved easily using a finite difference scheme. This scheme however requires that λ_0 is sufficiently large to ensure numerical convergence ($\lambda_0 > 0.1$). Since the optimal rate of trading is deterministic, the exact optimal strategy ν^* and Q^{ν^*} can be computed at the inception $t = 0$ and are depicted in the following figures 2.1, 2.2, 2.3 for various sets of parameters. If π and ϕ are sufficiently large (large terminal and running penalty), we see on the first figure that the agent has an incentive to liquidate her shares as soon as possible and we thus find in this case that $Q_T^{\nu^*} = 0$ (e.g $\pi = 10$, $\phi = 5$). When π and ϕ are small, the agent has instead an incentive to liquidate her shares much slower and to keep a remaining quantity at maturity. Moreover, we observe on Figure 2.2 that when the temporary penalty κ incurred at each trade increases, the optimal agent has an incentive to liquidate less shares so as to mitigate the negative impact on the transaction price and to face instead the terminal penalty, which becomes small relative to this large temporary impact κ.

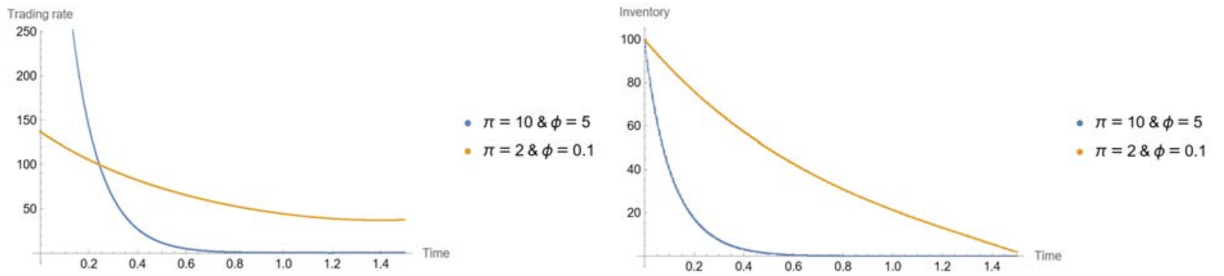


Figure 2.1: Optimal trading rate ν^* and inventory Q^{ν^*} from a temporary price impact with $\pi = 10$ and $\phi = 5$ compared with $\pi = 2$ and $\phi = 0.1$ with $\kappa = 0.1$, $\eta = 0.002$, $\alpha = 0.5$, $\lambda_0 = 0.5$, $T = 1.5$ and $Q = 100$.

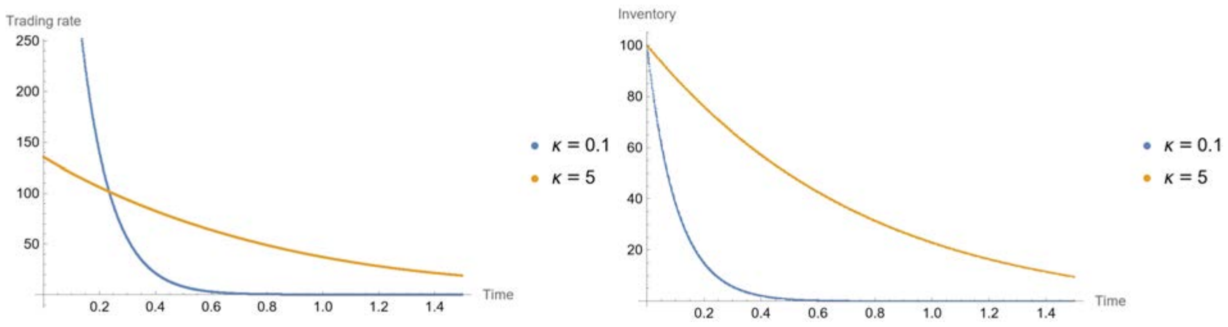


Figure 2.2: Optimal trading rate ν^* and inventory Q^{ν^*} from a temporary price impact with different values of κ with $\pi = 10$, $\phi = 5$, $\eta = 0.001$, $\alpha = 0.5$, $\lambda_0 = 0.5$, $T = 1.5$ and $Q = 100$.

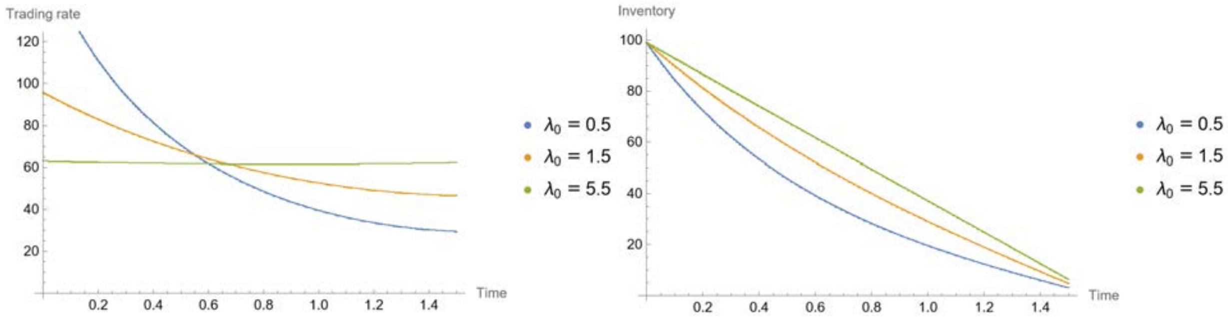


Figure 2.3: Optimal trading rate ν^* and inventory Q^{ν^*} from a temporary price impact with $\pi = 10$ and $\phi = 2$ for various levels of λ_0.

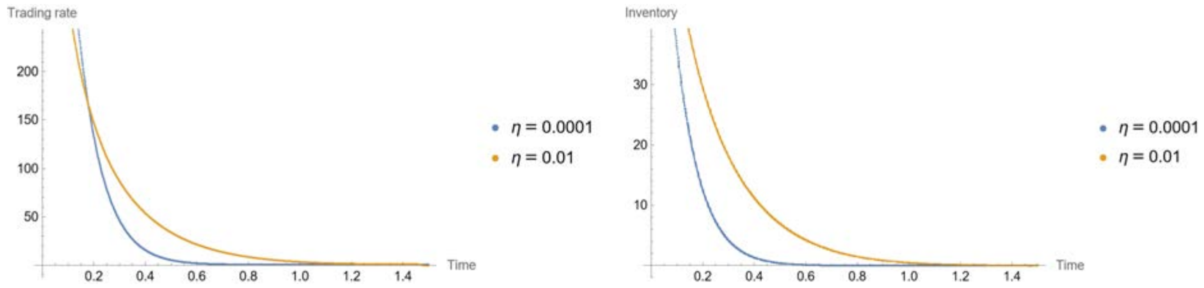


Figure 2.4: Optimal trading rate ν^* and inventory Q^{ν^*} from a temporary price impact with different values of η with $\pi = 10$, $\phi = 5$, $\kappa = 0.1$, $\alpha = 0.5$, $\lambda_0 = 0.5$, $T = 1.5$ and $Q = 100$.

Moreover, when adjusting the initial level λ_0 of the intensity (2.2), we observe significant variations in the trend of the trading rate even if it does not impact the value of the terminal inventory $Q_T^{\nu^*}$. Finally, when η is smaller and for large values of ϕ, we see that the remaining inventory decreases much faster at the beginning of the trading horizon. Indeed, since the (permanent) impact on the intensity λ is much smaller, the agent can trade a lot at the beginning while keeping the intensity rather low (which is in accordance with the large urgency to trade ϕ). The impact of η depends however on the value of ϕ: when $\phi = 0$, we observe the opposite behavior (see the geometric intensity case 2.1.2 below with no running penalty) since in this case there is no urgency to trade from the agent and she thus has no incentive to liquidate a lot at the beginning of the trading horizon even when η is low.

2.1.2. Maximization of expected wealth with geometric intensity

We now consider that the intensity λ^ν of the jump process N^ν follows instead the dynamic

$$d\lambda_t^\nu = \eta\lambda_t^\nu \nu_t dt \quad \text{such that} \quad \lambda_t^\nu = \lambda_0^\nu \exp\left(\int_0^t \eta\nu_s ds\right), \tag{2.21}$$

or in a more realistic setting (which again includes a transient and a permanent indirect impact)

$$d\lambda_t^\nu = \eta\lambda_t^\nu \nu_t dt + \theta\lambda_t^\nu d\widehat{W}_t \quad \text{such that} \quad \lambda_t^\nu = \lambda_0 \exp\left(\int_0^t (\eta\nu_s - \theta^2/2)ds + \theta\widehat{W}_t\right), \tag{2.22}$$

where $\widehat{W}$ is a $(\mathcal{F}_t)_{0\leq t\leq T}$-Brownian motion, independent from W. We need to impose $\eta > 0$ so that a higher volume sold ν leads to an increase of the occurrence probability of a negative price impact. We also assume no more running penalty for detaining stocks throughout the trading horizon ($\phi = 0$). The agent is then willing to optimize the following performance criterion

$$H(t,x,S,q,N,\lambda) = \sup_{\nu\in\mathcal{A}} \mathbb{E}_{t,x,S,q,N,\lambda}\left[X^\nu_\tau + Q^\nu_\tau\left(S_\tau - \pi\lambda^\nu_\tau Q^\nu_\tau\right)\right] \tag{2.23}$$

$$= x + qS - \inf_{\nu\in\mathcal{A}} \mathbb{E}_{t,q,\lambda}\left[\int_t^\tau \kappa\,\nu_s^{\gamma+1}\lambda^\nu_s\,ds + \pi\lambda^\nu_\tau\left(Q^\nu_\tau\right)^2\right]$$

$$= x + qS - \inf_{\nu\in\mathcal{A}} h^\nu(t,q,\lambda) = x + qS - h(t,q,\lambda)\,, \tag{2.24}$$

with again $h(T,q,\lambda) = \pi\lambda q^2$, $h(t,0,\lambda) = 0$ and $\lim_{\lambda\to 0} h(t,q,\lambda) = 0$ (since when the intensity is null, the agent can liquidate in an optimal way all her remaining shares at zero cost). Again, specifying $\lambda_0 > 0$ in (2.22) prevents this situation from happening even if the limit $\lambda \to 0$ provides a boundary condition with respect to the variable λ necessary for solving numerically the HJB equation. We then find that h^ν is again continuous and strictly convex in ν for $\gamma > 0$ and that the value function h is the unique viscosity solution of the HJB equation

$$\partial_t h(t,q,\lambda) + \frac{1}{2}\lambda^2\theta^2\partial_{\lambda\lambda}h(t,q,\lambda) + \inf_\nu\left\{\eta\lambda\nu\partial_\lambda h(t,q,\lambda) - \nu\partial_q h(t,q,\lambda) + \lambda\kappa\nu^{\gamma+1}\right\} = 0\,. \tag{2.25}$$

Taking $h(t,q,\lambda) = \lambda h(t,q)$, we have that equation (2.25) becomes

$$\partial_t h(t,q) + \inf_\nu\left\{\eta\nu h(t,q) - \nu\partial_q h(t,q) + \kappa\nu^{\gamma+1}\right\} = 0\,, \tag{2.26}$$

and hence the optimization problem becomes independent of the intensity λ. This leads to

$$\nu^*(t,q) = \left(\frac{\partial_q h(t,q) - \eta h(t,q)}{(\gamma+1)\kappa}\right)^{1/\gamma}\,, \tag{2.27}$$

and finally to

$$\partial_t h(t,q) - \frac{\left(\partial_q h(t,q) - \eta h(t,q)\right)^{\frac{\gamma+1}{\gamma}}}{\left((\gamma+1)\kappa\right)^{1/\gamma}}\left(\frac{\gamma}{\gamma+1}\right) = 0\,, \tag{2.28}$$

with $h(T,q) = \pi q^2$ and $h(t,0) = 0$. Hence, if we do not consider a running penalty anymore ($\phi = 0$) with the geometric intensity (2.22), the optimal rate ν^* does not depend on λ and is again purely deterministic (as it does not depend on W, $\widehat{W}$, N^ν). We then find the following optimal strategies for various sets of parameters using the square-root law $\gamma = 1/2$.

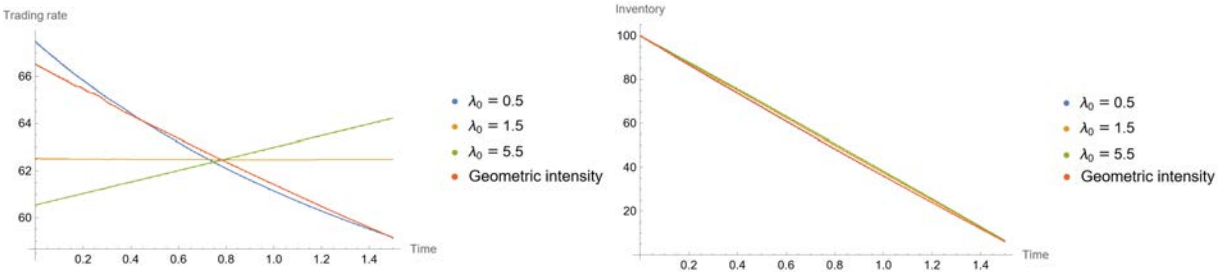


Figure 2.5: Optimal trading rate ν^* and inventory Q^{ν^*} from a temporary price impact with the mean-reverting intensity (2.2) for various λ_0 ($\phi = 0$), compared with the geometric intensity model (2.21) in red. Both models use $\pi = 8$, $\kappa = 0.8$, $\eta = 0.0025$, $\alpha = 0.05$, $Q = 100$ and $T = 1.5$.

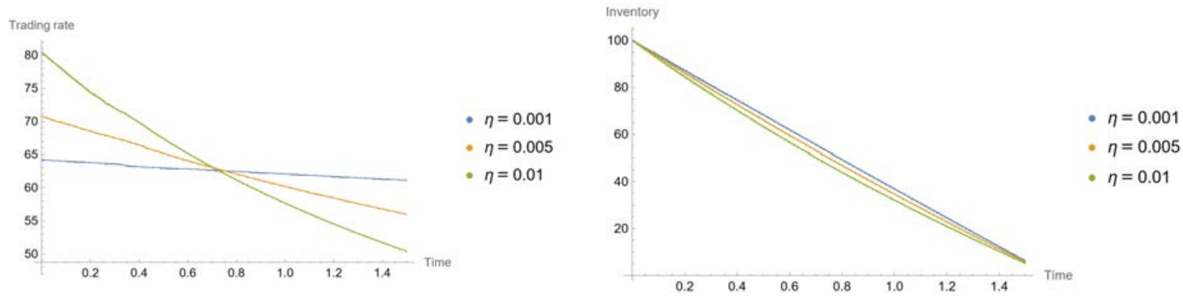


Figure 2.6: Optimal trading rate ν^* and inventory Q^{ν^*} from a temporary price impact with geometric intensity (2.21) for various values of η with $\pi = 8$, $\kappa = 0.8$, $Q = 100$ and $T = 1.5$.

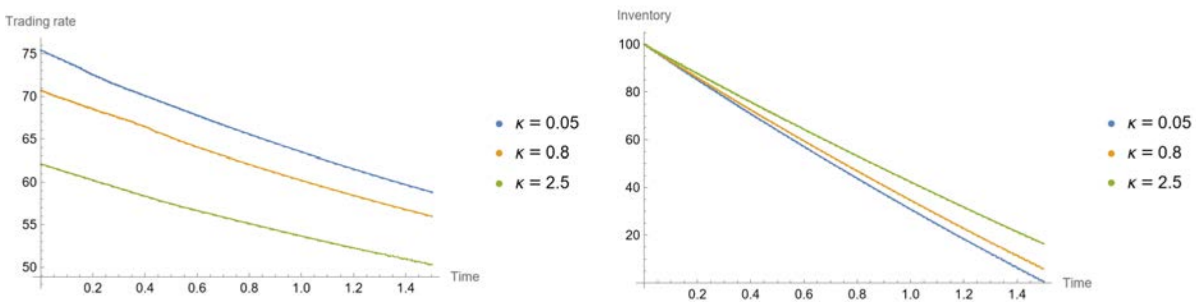


Figure 2.7: Optimal trading rate ν^* and inventory Q^{ν^*} from a temporary price impact with geometric intensity (2.21) for various values of κ with $\pi = 8$, $\eta = 0.005$, $Q = 100$ and $T = 1.5$.

We see in Figure 2.5 that the mean-reverting intensity model (2.2) of Section 2.1.1 with $\lambda_0 = 0.5$ and the geometric intensity model (2.22) from this section lead to similar results. However, the concavity is more pronounced when taking a mean-reverting intensity λ^ν in the optimization problem while the slope of the trading rate tends to remain constant for the geometric intensity. For various values of λ_0 in the mean-reverting intensity (2.2), we see that the slope of ν^* with respect to time can change drastically leading to different kind of strategies, while for the geometric intensity the trading rate is always decreasing with time. As already mentioned, since we consider no running penalty ($\phi = 0$), an increase of η leads to a faster decrease of the shares to liquidate. Finally, we have logically that a higher temporary impact κ leads to a lesser trading rate throughout the trading horizon since the agent prefers in this case to face the terminal penalty so as to avoid these higher negative impacts κ.

This temporary setting (2.4) appears as the most natural/consistent with the observations of Section 1 and Figure 1.2 since it allows to disentangle the temporary direct price impact that arises when MOs walk the LOB from the indirect permanent and transient impacts coming from the downward pressure in price due to changes in net order-flow imbalance. We however propose in Section 2.2 and Section 2.3 other propagators that will allow to combine differently these temporary, transient and permanent (direct and indirect) effects.

2.2. Permanent price impact kernel

We consider in this section that the direct price impact ruled by the kernel G is permanent in the sense that a sell order permanently affects the price of the asset and hence the price at which all subsequent transactions are executed. This corresponds to setting $G \equiv 1$ in (2.1) which gives

$$dX_t^\nu = S_t^\nu \nu_t dt = \left(S_t - \int_0^t \kappa \nu_s^\gamma \, dN_s^\nu \right) \nu_t dt := (S_t - Z_t)\nu_t dt \,, \tag{2.29}$$

where $dS_t^\nu = dS_t - dZ_t = \sigma dW_t - \kappa \nu_t^\gamma dN_t^\nu$. Z_t is thus the cumulative negative price impact from trades at time t. Notice that the execution price $S^\nu = S - Z$ here makes sense per se when $G \equiv 1$, compared with the temporary case (2.7). Moreover, this setting allows to include both a permanent indirect impact (as before via λ^ν) as well as a permanent direct impact on prices via this propagator $G \equiv 1$. We again consider an agent willing to maximize the expected wealth at the last trading time τ on $[0, T]$:

$$\widetilde{H}(t, x, S^\nu, q, N, \lambda) = \sup_{\nu \in \mathcal{A}} \mathbb{E}_{t,x,S^\nu,q,N,\lambda} \left[X_\tau^\nu + Q_\tau^\nu \left(S_\tau^\nu - \pi \lambda_\tau^\nu Q_\tau^\nu \right) - \phi \int_t^\tau (Q_u^\nu)^2 du \right] . \tag{2.30}$$

We now consider that S^ν is a state variable (instead of S in the optimization (2.11) above) since the direct price impact in this section is permanent and hence, the price at which the agent can sell the Q_T^ν remaining shares at the end of the trading horizon is S_T^ν (including the cumulative negative impacts) minus the terminal penalty. Finally, it is logical to assume that this terminal penalty is again of the form of equation (2.13) since the terminal penalty is a temporary impact (as it only affects the last trade at T) and hence does not depend in a direct way on the previous negative jumps (price impacts) of N^ν, even if it still depends indirectly on the past trades $(\nu_s)_{0 \le s \le T}$ through the intensity λ_T^ν. This leads to the final penalty $\mathbb{E}_{t,x,S^\nu,q,N,\lambda} \left[Q_T^\nu \left(S_T^\nu - \pi \lambda_T^\nu Q_T^\nu \right) \right] = \mathbb{E}_{t,x,S^\nu,q,N,\lambda} \left[Q_\tau^\nu \left(S_\tau^\nu - \pi \lambda_\tau^\nu Q_\tau^\nu \right) \right]$. From equation (2.29), we can rewrite the wealth process at τ by

$$X_\tau^\nu = X_t^\nu + \int_t^\tau (S_s - Z_s) \, \nu_s \, ds \, .$$

Using integration by parts with $dQ_t^\nu = -\nu_t dt$, $dS_t = \sigma dW_t$ and $dZ_t = \kappa \nu_t^\gamma dN_t^\nu$, we have

$$X_\tau^\nu = X_t^\nu - S_\tau Q_\tau^\nu + S_t Q_t^\nu + Z_\tau Q_\tau^\nu - Z_t Q_t^\nu + \int_t^\tau \sigma Q_s^\nu dW_s - \int_t^\tau \kappa Q_s^\nu \nu_s^\gamma dN_s^\nu \,, \tag{2.31}$$

which leads to the *proceeds* R^ν_τ of the strategy at τ (after which no more trading occurs),

$$
\begin{aligned}
R^\nu_\tau &:= X^\nu_\tau + Q^\nu_\tau \left(S^\nu_\tau - \pi\lambda^\nu_\tau Q^\nu_\tau\right) \\
&= X^\nu_t + S_t Q^\nu_t - Z_t Q^\nu_t + \int_t^\tau \sigma Q^\nu_s dW_s - \int_t^\tau \kappa Q^\nu_s \nu^\gamma_s dN^\nu_s - \pi\lambda^\nu_\tau (Q^\nu_\tau)^2 \,. \qquad (2.32)
\end{aligned}
$$

Again from the boundedness of Q^ν, the non-negativity of $Q^\nu\nu^\gamma$ and the $\mathcal{F}$-predictability of ν (see [32]), we finally find

$$
\mathbb{E}_{t,x,S^\nu,q,N,\lambda}\left[X^\nu_\tau + Q^\nu_\tau \left(S^\nu_\tau - \pi\lambda^\nu_\tau Q^\nu_\tau\right)\right] = x + qS^\nu + \mathbb{E}_{t,q,\lambda}\left[-\int_t^\tau \kappa Q^\nu_s \nu^\gamma_s \lambda^\nu_s \, ds - \pi\lambda^\nu_\tau (Q^\nu_\tau)^2\right].
$$

We can then rewrite the value function (2.30) for $t \in [0,\tau]$ in the following way

$$
\begin{aligned}
\widetilde{H}(t,x,S^\nu,q,N,\lambda) &= \sup_{\nu\in\mathcal{A}} \mathbb{E}_{t,x,S^\nu,q,N,\lambda}\left[X^\nu_\tau + Q^\nu_\tau \left(S^\nu_\tau - \pi\lambda^\nu_\tau Q^\nu_\tau\right) - \phi\int_t^\tau (Q^\nu_s)^2 ds\right] \\
&= x + qS^\nu - \inf_{\nu\in\mathcal{A}} \mathbb{E}_{t,q,\lambda}\left[\int_t^\tau \kappa Q^\nu_s \nu^\gamma_s \lambda^\nu_s \, ds + \pi\lambda^\nu_\tau (Q^\nu_\tau)^2 + \phi\int_t^\tau (Q^\nu_s)^2 ds\right] \\
&= x + qS^\nu - \inf_{\nu\in\mathcal{A}} \widetilde{h}^\nu(t,q,\lambda) = x + qS^\nu - \widetilde{h}(t,q,\lambda)\,, \qquad (2.33)
\end{aligned}
$$

where $\widetilde{h}(t,q,\lambda)$ is again the cumulative penalty from optimally liquidating the rest of the shares. We have that the value function $\widetilde{h}^\nu$ is continuous and strictly convex in ν provided that $\gamma > 1$. Hence, the square-root law $\gamma = 1/2$ which holds in the temporary case cannot apply anymore to the permanent case $G \equiv 1$.

2.2.1. *Permanent impact with mean-reverting intensity*

As in Section 2.1.1, we first assume that the intensity λ^ν of N^ν is mean-reverting and satisfies (2.2). We then obtain that $\widetilde{h}$ is the unique viscosity solution of the following HJB equation with $\gamma > 1$

$$
\partial_t \widetilde{h} + \phi q^2 + \inf_{\nu\in\mathcal{A}}\left\{(\eta\nu - \alpha\lambda)\partial_\lambda \widetilde{h} - \nu\partial_q \widetilde{h} + \kappa q \nu^\gamma \lambda\right\} = 0\,, \qquad (2.34)
$$

subject to terminal conditions

$$
\widetilde{h}(T,q,\lambda) = \pi\lambda q^2 \quad \text{since} \quad \widetilde{H}(T,x,S^\nu,q,N,\lambda) = x + qS^\nu - \pi\lambda q^2\,,
$$

and boundary conditions

$$
\widetilde{h}(t,0,\lambda) = 0 \quad \text{and} \quad \lim_{\lambda\to 0} \widetilde{h}(t,q,\lambda) = 0\,.
$$

Again the first boundary condition comes from the non-negativity of ν and the stopping time (2.10), leading to $\widetilde{H}(t,x,S^\nu,0,N,\lambda) = x$, and the second boundary condition is necessary for solving numerically (2.34) even if specifying $\lambda_0 > 0$ prevents us from attaining this boundary. First order condition in (2.34) leads to

$$
\nu^*(t,q,\lambda) = \left(\frac{\partial_q \widetilde{h}(t,q,\lambda) - \eta\partial_\lambda \widetilde{h}(t,q,\lambda)}{\kappa q \lambda \gamma}\right)^{\frac{1}{\gamma-1}}\,, \qquad (2.35)
$$

which is again deterministic (no dependence on N^ν nor W) and highly similar to (2.17) except that we now have the remaining inventory q in the denominator which accounts for the permanent direct impact on the transaction price. We could indeed denote $\gamma = \gamma' + 1$ where γ' is the exponent in (2.17) (since $\gamma' > 0$ in the previous section while now we have $\gamma > 1$). Again since $\lambda > \lambda_0 > 0$ and since by construction we have $\nu^*(t, 0, \lambda) = 0$ for $q = 0$, $\forall t \geq 0, \forall \lambda > \lambda_0$ due to the non-negativity of ν (no purchase of shares allowed) and the stopping time (2.10) (no more than Q shares sold), the above expression of ν^* is well-defined. Upon substitution back in (2.34), we find

$$\partial_t \widetilde{h}(t, q, \lambda) + \phi q^2 - \frac{\left(\partial_q \widetilde{h}(t, q, \lambda) - \eta\, \partial_\lambda \widetilde{h}(t, q, \lambda)\right)^{\frac{\gamma}{\gamma - 1}}}{(\kappa \gamma q \lambda)^{1/(\gamma - 1)}} \left(\frac{\gamma - 1}{\gamma}\right) - \alpha \lambda\, \partial_\lambda \widetilde{h}(t, q, \lambda) = 0\,.$$

Choosing $\gamma = 3/2$ ensures the non-negativity of the optimal control ν^* since $\nu^* = \frac{4}{9}\left(\frac{\partial_q \widetilde{h} - \eta \partial_\lambda \widetilde{h}}{\kappa q \lambda}\right)^2$ and

$$\partial_t \widetilde{h}(t, q, \lambda) + \phi q^2 - \frac{4}{27(\kappa q \lambda)^2}\left(\partial_q \widetilde{h}(t, q, \lambda) - \eta\, \partial_\lambda \widetilde{h}(t, q, \lambda)\right)^3 - \alpha \lambda\, \partial_\lambda \widetilde{h}(t, q, \lambda) = 0\,,$$

which is seemingly identical to the HJB equation in the temporary setting with square-root law (2.20) except again that the quantity q is now in the denominator $(\kappa q \lambda)^2$. Choosing $\gamma = 3/2$ is indeed consistent with the square root law $\gamma' = 1/2$ of the previous temporary/instantaneous price impact setting (Section 2.1) since we now take $G \equiv 1$ and hence consider instead the total/aggregated (direct) price impact of each MO. This choice leads to the following figures of optimal strategies for various parameters. Another reasonable assumption would be to take γ close to 1 and hence consider a linear (direct) permanent impact whenever L^ν jumps so as to be consistent with the linear permanent impact function as investigated in [8].

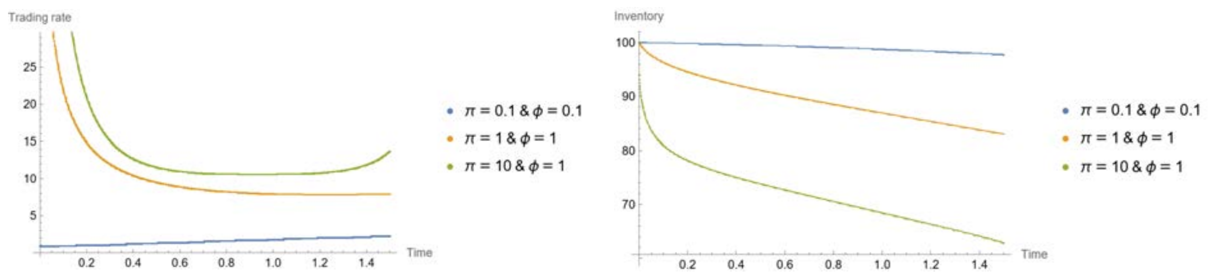


Figure 2.8: Optimal trading rate ν^* and inventory Q^{ν^*} from a permanent price impact with mean-reverting intensity and $\gamma = 3/2$ for various values of π and ϕ with $\kappa = 1$, $\eta = 0.1$, $\alpha = 0.5$, $\lambda_0 = 0.5$, $Q = 100$ and $T = 1.5$.

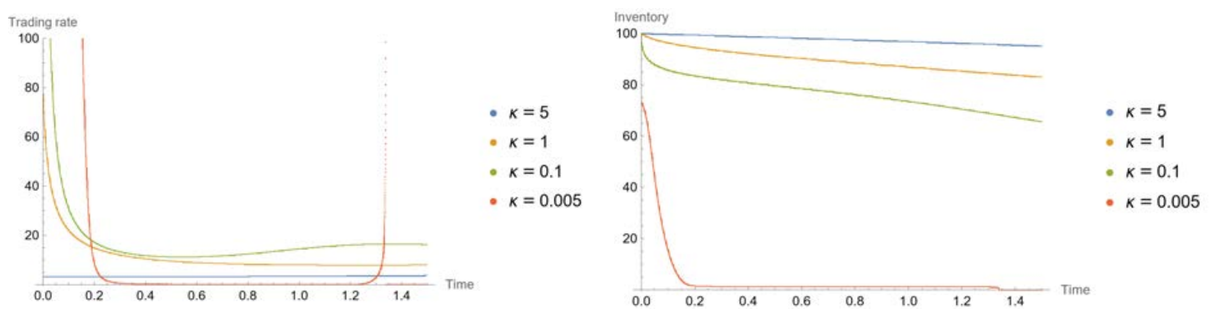


Figure 2.9: Optimal trading rate ν^* and inventory Q^{ν^*} from a permanent price impact with mean-reverting intensity and $\gamma = 3/2$ for various values of κ with $\pi = 1$, $\phi = 1$, $\eta = 0.1$, $\alpha = 0.5$, $\lambda_0 = 0.5$, $Q = 100$ and $T = 1.5$.

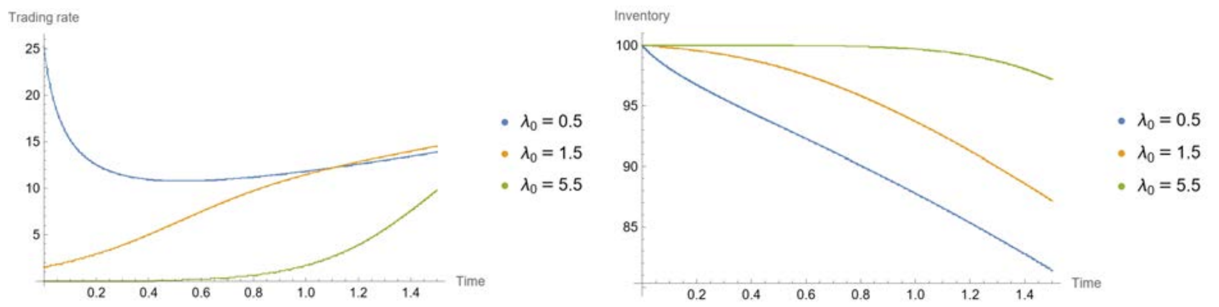


Figure 2.10: Optimal trading rate ν^* and inventory Q^{ν^*} from a permanent price impact with mean-reverting intensity and $\gamma = 3/2$ for various values of λ_0 with $\pi = 1$, $\phi = 1$, $\kappa = 1$, $\eta = 0.1$, $\alpha = 1$, $Q = 100$ and $T = 1.5$.

Figures 2.8 and 2.9 again lead to financially consistent results when varying the penalty terms ϕ, π and κ. Moreover, when choosing really low values of κ, or high values of π and ϕ such as to liquidate all the shares before T, the optimal strategy is to liquidate most of the shares right at the beginning of the horizon. However, changing the intensity-related parameters λ_0, α or η again strongly impacts the trend of the optimal strategy (see *e.g.* Figure 2.10). Therefore, no clear pattern of optimal strategy arises with a permanent direct impact and a mean-reverting intensity.

2.2.2. *Permanent impact with geometric intensity*

Secondly, we now assume for this permanent case $G \equiv 1$ no more running penalty ($\phi = 0$) and a geometric intensity linear in ν (as in Section 2.1.2) :

$$d\lambda_t^\nu = \eta\lambda_t^\nu \nu_t \, dt + \theta\lambda_t^\nu d\widehat{W}_t \quad \text{such that} \quad \lambda_t^\nu = \lambda_0 \exp\left(\int_0^t (\eta\nu_s - \frac{1}{2}\theta^2)\, ds + \theta\widehat{W}_t\right).$$

We find in this case that $\widetilde{h}$ is the unique viscosity solution of

$$\partial_t\widetilde{h} + \frac{1}{2}\lambda^2\theta^2\partial_{\lambda\lambda}\widetilde{h} + \inf_{\nu\in\mathcal{A}}\left\{\eta\nu\lambda\partial_\lambda\widetilde{h} - \nu\partial_q\widetilde{h} + \kappa q\nu^\gamma\lambda\right\} = 0\,, \tag{2.36}$$

with $\gamma > 1$. Taking $\widetilde{h}(t,q,\lambda) = \lambda\widetilde{h}(t,q)$ leads to

$$\partial_t\widetilde{h}(t,q) + \inf_{\nu\in\mathcal{A}}\left\{\eta\nu\widetilde{h}(t,q) - \nu\partial_q\widetilde{h}(t,q) + \kappa q\nu^\gamma\right\} = 0\,, \tag{2.37}$$

with $\widetilde{h}(T,q) = \pi q^2$ and $\widetilde{h}(t,0) = 0$. First order condition in (2.37) leads to

$$\nu^*(t,q) = \left(\frac{\partial_q\widetilde{h}(t,q) - \eta\widetilde{h}(t,q)}{\kappa q\gamma}\right)^{\frac{1}{\gamma-1}}\,, \tag{2.38}$$

so that the optimal trading rate ν^* is independent of λ (and again deterministic). The non-negativity of ν and the stopping time (2.10) lead by construction to $\nu^*(t,0) = 0$. Again, the obtained solution is highly similar to (2.27), except with the remaining inventory q in the denominator. Upon substitution back in the previous equation, we find

$$\partial_t\widetilde{h}(t,q) - \frac{\left(\partial_q\widetilde{h}(t,q) - \eta\widetilde{h}(t,q)\right)^{\frac{\gamma}{\gamma-1}}}{(\kappa\gamma q)^{1/(\gamma-1)}}\left(\frac{\gamma-1}{\gamma}\right) = 0\,. \tag{2.39}$$

This model is numerically stable and gives the following strategies for various sets of parameters starting at time $t = 0$. Choosing $\gamma = 3/2$ is the most sensible choice so that the total market impact is consistent with the square-root law used in the temporary case. It also ensures the non-negativity of ν^* in (2.38).

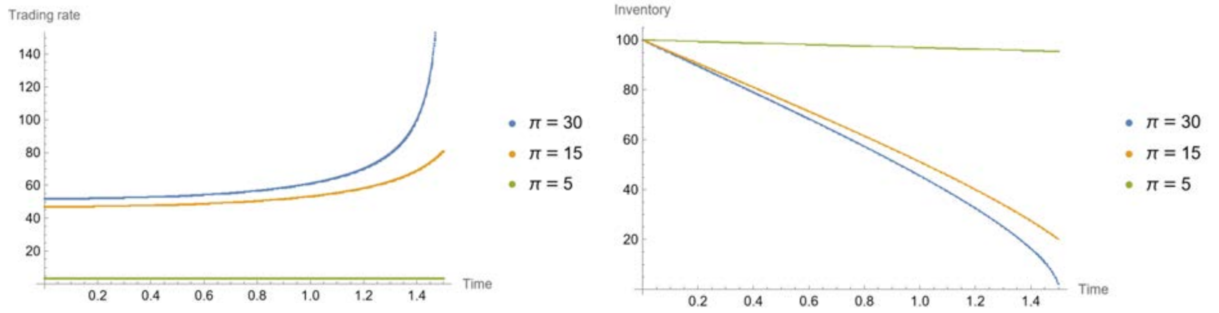


Figure 2.11: Optimal trading rate ν^* and inventory Q^{ν^*} from a permanent price impact with geometric intensity and $\gamma = 3/2$ for various values of π with $\kappa = 2$, $\eta = 0.01$, $Q = 100$ and $T = 1.5$.

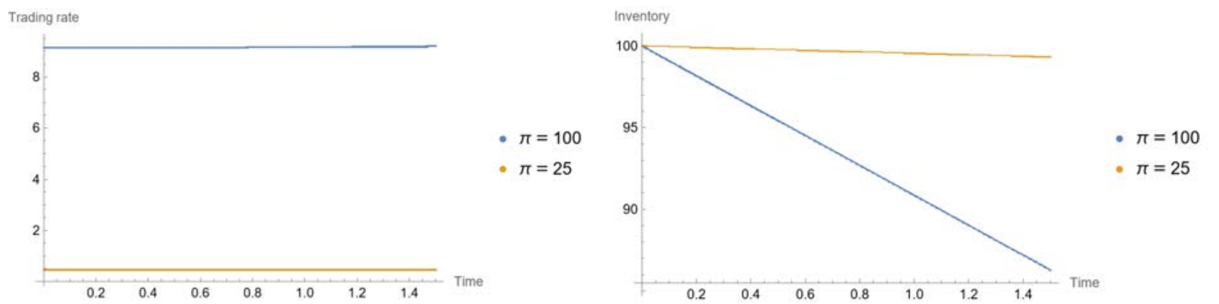


Figure 2.12: Optimal trading rate ν^* and inventory Q^{ν^*} from a permanent price impact model with geometric intensity and $\gamma = 3/2$ for various values of π with $\kappa = 25$, $\eta = 0.01$, $Q = 100$ and $T = 1.5$.

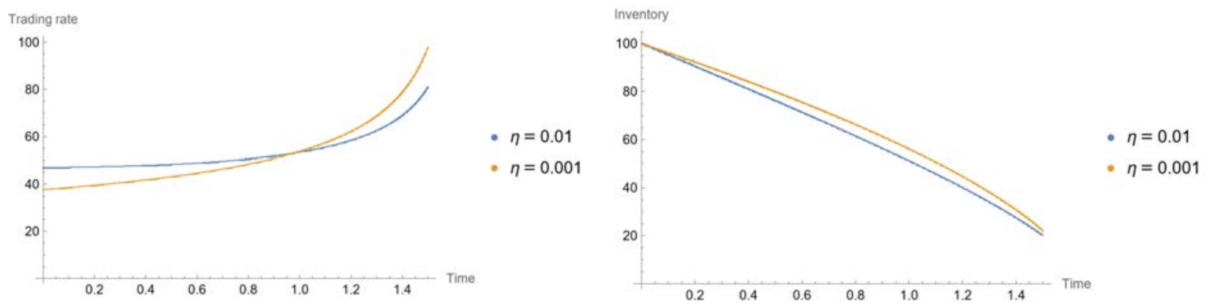


Figure 2.13: Optimal trading rate ν^* and inventory Q^{ν^*} from a permanent price impact model with geometric intensity and $\gamma = 3/2$ for various values of η with $\kappa = 2$, $\pi = 15$, $Q = 100$ and $T = 1.5$.

We see in the first Figure 2.11 that when the terminal penalty π is sufficiently large and the permanent impact κ sufficiently small, the agent must liquidate all her shares before the trading horizon T so as to avoid the final penalty and to only incur the low permanent price impact κ during the trading horizon. However, to mitigate this (cumulative) permanent price impact on the whole trading horizon, the trading rate is increasing with time, which is the opposite strategy than in the temporary setting 2.1.2. Moreover, when κ is sufficiently low (or π large) so as to liquidate all the shares before T, the optimal strategy consists here in liquidating most of these shares at the end of the trading horizon. We also observe that when π becomes small relative to κ (i.e. the ratio π/κ is sufficiently small), the agent prefers to incur the terminal penalty and to liquidate less shares during the trading horizon so as to mitigate the permanent impact. We can indeed see in Figure 2.11 (green line) a low constant trading rate coming from the very small terminal penalty. We also observe a similar constant trading rate in Figure 2.12 where the permanent impact κ is so large that the agent still prefers to incur the terminal penalty π even if it is quite high. Of course, when the penalty π increases, the trading rate tends to increase as well so as to end up with a lower terminal inventory $Q_T^{\nu^*}$. Finally, we again have that the trend of the optimal strategy strongly depends on the intensity-related parameter (here η).

A mean-variance optimization setting can also be considered instead of the expected wealth maximization (2.30) studied so far. Results for the permanent direct impact kernel $G \equiv 1$ are relegated in Appendix A.

2.3. Transient impact function

In between the temporary case $G \equiv \delta$ and the permanent one $G \equiv 1$, we now propose two different ways of describing a transient (direct) price impact of the agent's transactions.

2.3.1. First alternative

We consider at first a direct impact kernel of the form $G(t-s) = e^{-\beta(t-s)}$. In a similar way as Section 2.2.1, we can rewrite equation (2.29) by taking S and Z as state variables

$$dX_t^\nu = \left(S_t - \kappa \int_0^t e^{-\beta(t-s)} \nu_s^\gamma \, dN_s^\nu \right) \nu_t \, dt = (S_t - Z_t) \, \nu_t \, dt \,, \tag{2.40}$$

where $dZ_t = -\beta Z_t dt + \kappa \nu_t^\gamma dN_t^\nu$ and Z is again the cumulative negative impact from trades on the transaction price. As in previous sections, we aim at maximizing the expected wealth of the agent given the set (2.12) of admissible trading strategies $\mathcal{A}$:

$$H^*(t,x,S,q,\lambda,Z) = \sup_{\nu \in \mathcal{A}} \mathbb{E}_{t,x,S,q,\lambda,Z} \left[X_\tau^\nu + Q_\tau^\nu \left(S_\tau^\nu - \pi \lambda_\tau^\nu Q_\tau^\nu \right) - \phi \int_t^\tau (Q_s^\nu)^2 ds \right]. \tag{2.41}$$

$S^\nu = S - Z$ is again used to define the terminal penalty since at the end of the trading horizon, the price at which we can sell the Q_T^ν remaining shares is S_T^ν (minus the penalty $\pi \lambda_T^\nu Q_T^\nu$) due to the transient direct price impact from G. We find the following expression for the proceeds R_τ^ν using (2.40) and integration by parts

$$\begin{aligned} R_\tau^\nu &= X_\tau^\nu + Q_\tau^\nu \left(S_\tau^\nu - \pi \lambda_\tau^\nu Q_\tau^\nu \right) & (2.42) \\ &= X_t^\nu + S_t Q_t^\nu - Z_t Q_t^\nu + \int_t^\tau \sigma Q_s^\nu dW_s + \int_t^\tau Q_s^\nu \left(\beta Z_s ds - \kappa \nu_s^\gamma dN_s^\nu \right) - \pi \lambda_\tau^\nu (Q_\tau^\nu)^2 \,, & (2.43) \end{aligned}$$

since $\int_t^\tau Z_s\nu_s ds = -Z_\tau Q_\tau^\nu + Z_t Q_t^\nu + \int_t^\tau Q_s^\nu\left(-\beta Z_s ds + \kappa \nu_s^\gamma dN_s^\nu\right)$ and $\int_t^\tau S_s\nu_s ds = -S_\tau Q_\tau^\nu + S_t Q_t^\nu + \int_t^\tau Q_s^\nu \sigma dW_s$. We again have that the last term $\int_t^\tau Q_s^\nu\sigma dW_s$ is martingale since Q^ν is bounded and that $\mathbb{E}_{t,q,\lambda,Z}\left[\int_t^\tau \kappa Q_s^\nu \nu_s^\gamma dN_s^\nu\right] = \mathbb{E}_{t,q,\lambda,Z}\left[\int_t^\tau \kappa Q_s^\nu \nu_s^\gamma \lambda_s^\nu ds\right]$, see [32]. We can hence rewrite equation (2.30) as

$$
\begin{aligned}
H^*(t,x,S,q,\lambda,Z) &= x + S^\nu q + \sup_{\nu\in\mathcal{A}} \mathbb{E}_{t,q,\lambda,Z}\left[\int_t^\tau Q_s^\nu(\beta Z_s - \kappa\nu_s^\gamma\lambda_s^\nu)\, ds - \pi\lambda_\tau (Q_\tau^\nu)^2 - \phi\int_t^\tau (Q_s^\nu)^2 ds\right] \\
&= x + S^\nu q - \inf_{\nu\in\mathcal{A}} \mathbb{E}_{t,q,\lambda,Z}\left[\int_t^\tau Q_s^\nu(\kappa\nu_s^\gamma\lambda_s^\nu - \beta Z_s)\, ds + \pi\lambda_\tau (Q_\tau^\nu)^2 + \phi\int_t^\tau (Q_s^\nu)^2 ds\right] \\
&= x + S^\nu q - \inf_{\nu\in\mathcal{A}} h^{*,\nu}(t,q,\lambda,Z) = x + S^\nu q - h^*(t,q,\lambda,Z)\,.
\end{aligned}
$$

The value function $h^*(t,q,\lambda,Z)$ again satisfies the following HJB equation for $\gamma > 1$:

$$
\begin{aligned}
\partial_t h^* + \phi q^2 - \beta Z\partial_Z h^* - q\beta Z + \inf_{\nu\in\mathcal{A}} \Big\{ &(\eta\nu - \alpha\lambda)\partial_\lambda h^* - \nu\partial_q h^* + q\kappa\nu^\gamma\lambda \\
&+ \lambda\left[h^*(t,q,\lambda,Z+k\nu^\gamma) - h^*(t,q,\lambda,Z)\right]\Big\} = 0\,,
\end{aligned}
\tag{2.44}
$$

with $h^*(T,q,\lambda,Z) = \pi\lambda q^2$, $h^*(t,0,\lambda,Z) = 0$ and $\lim_{\lambda\to 0} h^*(t,q,\lambda,Z) = 0$. No explicit solution is available for the optimal trading rate ν^* since $\kappa\nu^\gamma$ is inside of $h^*(t,q,\lambda,\cdot)$ in (2.44). This makes the solving of (2.44) numerically very intensive and unstable when using classical numerical schemes such as finite differences or finite elements. Deep learning methods (see *e.g.* [12, 34, 33, 37]) offer ways of solving such 4-dimensional HJB equation (without explicit solution for ν^*) and will be investigated more thoroughly in a subsequent paper. Moreover, since the optimal strategy ν^* now depends on Z (and hence of N^ν), it is not deterministic anymore.

2.3.2. *Second alternative*

We now give some alternative (and more specific) assumptions on the transient price impact and on the terminal penalty allowing to obtain the optimal trading rate explicitly (and deterministically). We first assume that the dynamic of the price impact Z is now given by

$$
dZ_t = -\beta Z_{t^-} dt + \kappa Z_{t^-}\nu_t^\gamma dN_t^\nu\,, \tag{2.45}
$$

and using Itô's lemma for jump processes, we obtain with $Z_0 > 0$

$$
Z_t = Z_0 e^{-\beta t}\prod_{0\le s\le t}^{\Delta N_s^\nu \neq 0}(1+\kappa\nu_s^\gamma)\,,
$$

leading to an execution price of the form $S_t^\nu = S_t - Z_t + Z_0$ (so as to ensure $S_0^\nu = S_0$). However, we cannot write anymore the execution price in the form (2.1) based on a propagator G. We further assume that the intensity λ^ν of the jump process N^ν follows the mean-reverting dynamic $d\lambda_t^\nu = (\eta\nu_t - \alpha\lambda_t)dt$ as in (2.2). We finally assume that the terminal penalty when $Q_T^\nu > 0$ is a given multiple π' of the total negative impact Z_T at time T. The penalized price $S_T^{\pi'}$ at the end of the trading horizon is thus defined formally as in (2.13) by the random variable

$$S_T^{\pi'} := S_T^\nu + dL_T^{\pi'}/dt = S_T^\nu - \pi' Z_T\, Q_T^\nu \sum_{j=1}^{N_T^\nu} \delta(T - t_j)\,, \tag{2.46}$$

where $L_T^{\pi'} := -\pi' \int_0^T Q_s^\nu\, Z_{s^-}\, dN_s^\nu$ with N^ν defined again as in (2.2). We see in (2.46) that π is replaced by $\pi' Z_T$ and it makes sense to impose $\pi' > 0$ since we expect additional penalties that the agent must incur for the execution of the trade at the terminal date on top of the cumulative (transient) negative impact $Z_T - Z_0$. Hence, we find as terminal penalty

$$\begin{aligned}\mathbb{E}_{t,x,S,q,N,\lambda,Z}\left[Q_T^\nu S_T^{\pi'}\right] &= \mathbb{E}_{t,x,S,q,N,\lambda,Z}\left[Q_T^\nu\left(S_T^\nu - \pi' Z_T \lambda_T^\nu Q_T^\nu\right)\right] \\ &= \mathbb{E}_{t,x,S,q,N,\lambda,Z}\left[Q_\tau^\nu\left(S_\tau^\nu - \pi' Z_\tau \lambda_\tau^\nu Q_\tau^\nu\right)\right]. \end{aligned}\tag{2.47}$$

Following the exact same reasoning (2.42)-(2.43) on the dynamic (2.45) with $S^\nu = S - Z + Z_0$, we then obtain the following optimization problem assuming $\phi = 0$ (no urgency to trade) :

$$\begin{aligned}H^*(t,x,S,q,N,\lambda,Z) &= \sup_{\nu\in\mathcal{A}} \mathbb{E}_{t,x,S,q,N,\lambda,Z}\left[X_\tau^\nu + Q_\tau^\nu\left(S_\tau^\nu - \pi' Z_\tau \lambda_\tau^\nu Q_\tau^\nu\right)\right] \\ &= x + S^\nu q - \inf_{\nu\in\mathcal{A}} \mathbb{E}_{t,q,\lambda,Z}\left[\int_t^\tau Q_s^\nu(\kappa Z_s \nu_s^\gamma \lambda_s^\nu - \beta Z_s)\, ds + \pi' Z_\tau \lambda_\tau^\nu (Q_\tau^\nu)^2\right] \\ &= x + S^\nu q - \inf_{\nu\in\mathcal{A}} h^{*,\nu}(t,q,\lambda,Z) = x + S^\nu q - h^*(t,q,\lambda,Z)\,. \end{aligned}$$

We again have that the HJB equation associated with this optimization problem is given in a similar way as (2.44) by

$$\begin{aligned}\partial_t h^* - \beta Z \partial_Z h^* - q\beta Z - \alpha\lambda\partial_\lambda h^* + \inf_{\nu\in\mathcal{A}} \Big\{ &\eta\nu\partial_\lambda h^* - \nu\partial_q h^* + q\kappa Z \nu^\gamma \lambda \\ &+ \lambda\left[h^*(t,q,\lambda,Z + kZ\nu^\gamma) - h^*(t,q,\lambda,Z)\right]\Big\} = 0\,,\end{aligned}\tag{2.48}$$

but now with $h^*(T,q,\lambda,Z) = \pi\lambda Z q^2$, $h^*(t,0,\lambda,Z) = 0$, $h^*(t,q,\lambda,0) = 0$ (boundary condition in Z from (2.45)) and $\lim_{\lambda\to 0} h^*(t,q,\lambda,Z) = 0$. We then logically assume the form

$$h^*(t,q,\lambda,Z) = Z h_2^*(t,q,\lambda)\,,$$

with $h_2^*(T,q,\lambda) = \pi\lambda q^2$, $h_2^*(t,0,\lambda) = 0$ and $\lim_{\lambda\to 0} h_2^*(t,q,\lambda) = 0$. Indeed, recall that in the previous Sections 2.1 and 2.2 on temporary and permanent (direct) impacts, the value functions h and $\widetilde{h}$ could be seen as an extra penalty due to price impact stemming from optimally liquidating the rest of the shares. In this new transient setting where the terminal penalty is a multiple of the total negative impact, we therefore have that the value function h_2^* is still the extra penalty from optimally selling the remaining shares but expressed as a proportion of the total negative impact Z at that time. Hence, the value function h_2^* satisfies

$$\begin{aligned}&Z\partial_t h_2^*(t,q,\lambda) - \beta Z h_2^*(t,q,\lambda) - q\beta Z - \alpha\lambda Z \partial_\lambda h_2^*(t,q,\lambda) \\ &\quad + \inf_{\nu\in\mathcal{A}}\left\{\eta\nu Z\partial_\lambda h_2^*(t,q,\lambda) - \nu Z\partial_q h_2^*(t,q,\lambda) + q\kappa Z\nu^\gamma\lambda + \lambda\kappa Z\nu^\gamma h_2^*(t,q,\lambda)\right\} = 0\,,\end{aligned}$$

and therefore,

$$\partial_t h_2^*(t,q,\lambda) - \beta h_2^*(t,q,\lambda) - q\beta - \alpha\lambda\partial_\lambda h_2^*(t,q,\lambda) + \inf_{\nu\in\mathcal{A}} \left\{ \eta\nu\partial_\lambda h_2^*(t,q,\lambda) - \nu\partial_q h_2^*(t,q,\lambda) + \lambda\kappa\nu^\gamma \left(q + h_2^*(t,q,\lambda)\right)\right\} = 0\,,$$

which implies that $h_2^{\nu,*}$ is continuous and strictly convex in ν provided that $\gamma > 1$. First order condition leads to

$$\nu^*(t,q,\lambda) = \left(\frac{\partial_q h_2^*(t,\lambda,q) - \eta\partial_\lambda h_2^*(t,\lambda,q)}{\lambda\kappa\gamma\left(q + h_2^*(t,\lambda,q)\right)}\right)^{\frac{1}{\gamma-1}}\,. \tag{2.49}$$

We finally obtain the following explicit PDE with $\gamma > 1$:

$$\partial_t h_2^*(t,q,\lambda) - \beta(h_2^*(t,q,\lambda) + q) - \alpha\lambda\partial_\lambda h_2^*(t,q,\lambda) - \frac{\left(\partial_q h_2^*(t,q,\lambda) - \eta\,\partial_\lambda h_2^*(t,q,\lambda)\right)^{\frac{\gamma}{\gamma-1}}}{\left(\lambda\kappa\gamma(q + h_2^*(t,q,\lambda))\right)^{1/(\gamma-1)}}\left(\frac{\gamma-1}{\gamma}\right) = 0. \tag{2.50}$$

Again, we have that $\lambda > \lambda_0 > 0$ due to the intensity (2.2) above and that $\nu^*(t,0,\lambda) = 0$ by construction from (2.10), which ensures that ν^* is well defined. Choosing $\gamma = 3/2$ as in Section 2.2 on permanent impact is consistent with the square-root temporary impact and ensures the non-negativity of the optimal trading rate since we obtain $\nu^* = \frac{4}{9}\left(\frac{\partial_q h_2^* - \eta\partial_\lambda h_2^*}{\lambda\kappa(q+h_2^*)}\right)^2$ and

$$\partial_t h_2^* - \beta(h_2^* + q) - \alpha\lambda\partial_\lambda h_2^* - \frac{4\left(\partial_q h_2^* - \eta\,\partial_\lambda h_2^*\right)^3}{27\left(\lambda\kappa(q + h_2^*)\right)^2} = 0\,.$$

This leads to the following plots of optimal strategies in the transient price impact model (2.45).

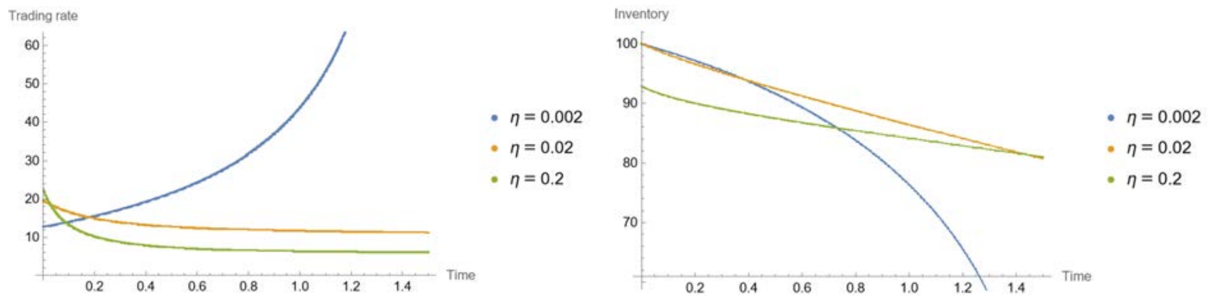


Figure 2.14: Optimal trading rate ν^* and inventory Q^{ν^*} from a transient geometric price impact (2.45) with mean-reverting intensity and $\gamma = 3/2$ for various values of η with $\kappa = 0.006$, $\alpha = 0.5$, $\beta = 1$, $\pi = 1000$, $\lambda_0 = 0.5$, $Q = 100$ and $T = 1.5$.

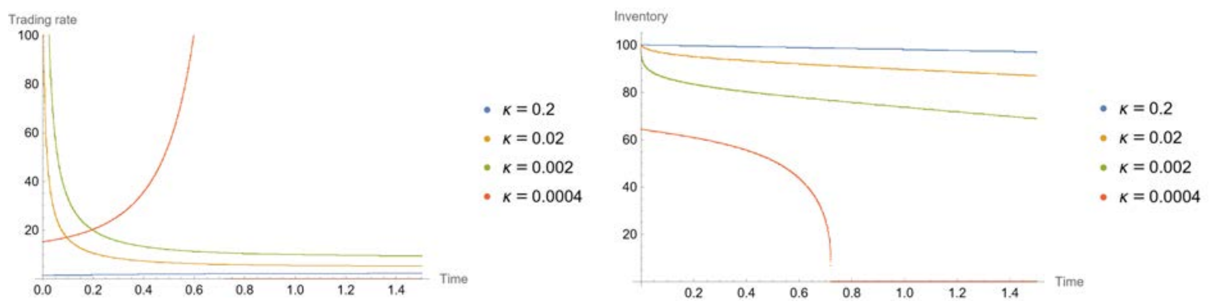
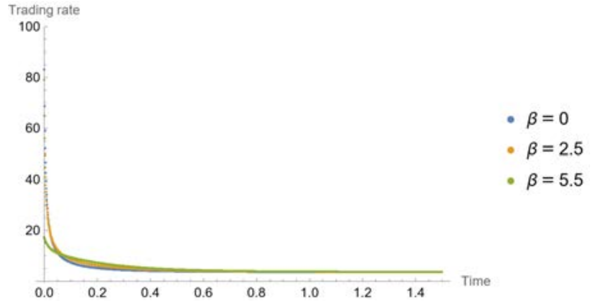


Figure 2.15: Optimal trading rate ν^* and inventory Q^{ν^*} from a transient geometric price impact (2.45) with mean-reverting intensity and $\gamma = 3/2$ for various values of κ with $\alpha = 0.5$, $\eta = 0.1$, $\beta = 1$, $\pi = 1000$, $\lambda_0 = 0.5$, $Q = 100$ and $T = 1.5$.



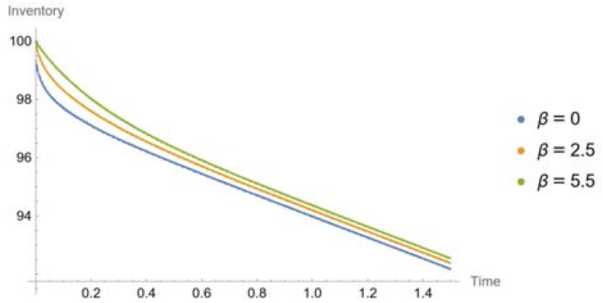


Figure 2.16: Optimal trading rate ν^* and inventory Q^{ν^*} from a transient geometric price impact (2.45) with mean-reverting intensity and $\gamma = 3/2$ for various values of β with $\alpha = 1$, $\eta = 0.5$, $\kappa = 0.05$, $\pi = 100$, $\lambda_0 = 0.5$, $Q = 100$ and $T = 1.5$.

Again, we see in Figure 2.14 that the intensity-related parameters strongly impact the trend of the optimal strategy. Moreover, when choosing a sufficiently small penalty κ such as to liquidate all the shares before T, we observe in Figure 2.15 that the optimal strategy consists in selling most of the shares right at the beginning of the trading horizon. Finally, the value of the drift β in (2.45) mostly influences the slope of the strategy at the beginning of the trading scheme.

3. Indirect optimal liquidation with limit and market orders

So far, we have considered that the liquidating agent could only post market (sell) orders. We now want to add the possibility for the seller to also post limit sell orders at a price $S_t + \delta_t$ that can be filled by MOs of buyers. The positive depth $\delta = (\delta_t)_{t \in [0,T]}$ at which the limit order is posted is hence also a control process and we again aim at maximizing the expected wealth of the selling agent. Based on [27, 26, 20] and Chapter 8.2 of [18], we expand the previous setting $dQ_t^\nu = -\nu_t dt$ of Section 2 by proposing instead

$$dQ_t^{\nu,\delta} = -\nu_t dt - \Delta dN_t^\delta \,, \tag{3.1}$$

where N^δ is a new (controlled) counting process corresponding to the number of buy MOs which lift the agent's offer, *i.e.* MOs which walk the sell side of the book to a price greater than or equal to $S_t + \delta_t$. Each filled limit order is of size $\Delta > 0$. We can consider that the number of arriving market buy orders (from other traders) follows a Poisson process with constant intensity λ^δ (δ being here an index). The probability that the agent's LO will be lifted when a buy MO arrives is then given by $P(\delta) = e^{-\rho\delta}$, $\rho > 0$. The model (3.1) hence induces that the agent is continuously posting MOs at a rate ν and that her limit orders at price $S_t + \delta_t$ are filled whenever N_t^δ jumps. For simplicity, we first assume as in Section 2.1 that G is purely temporary such that the execution price is given by $S_t^\nu = S_t - \kappa\nu_t^\gamma dN_t^\nu/dt$. The wealth process is then obtained accordingly by

$$dX_t^{\nu,\delta} = (S_t + \delta_{t-})\Delta dN_t^\delta + (S_t^\nu - \xi)\nu_t dt \,,$$

where ξ is the half bid-ask spread (which is set to zero in the sequel, w.l.o.g.). We thus find for the wealth process at the last trading time τ on $[0,T]$:

$$X_\tau^{\nu,\delta} := X_t^{\nu,\delta} + \int_t^\tau (S_s + \delta_{s-})\Delta dN_s^\delta + \int_t^\tau S_s \nu_s ds - \int_t^\tau \kappa \nu_s^{\gamma+1} dN_s^\nu \,. \tag{3.2}$$

The optimization problem becomes in this setting

$$\begin{aligned}
H(t,x,S,q,N^\nu,\lambda^\nu,N^\delta) &= \sup_{(\nu,\delta)\in\mathcal{A}''} H^{\nu,\delta}(t,x,S,q,N^\nu,\lambda^\nu,N^\delta) \\
&= \sup_{(\nu,\delta)\in\mathcal{A}''} \mathbb{E}_{t,x,S,q,N^\delta,\lambda^\nu,N^\nu}\left[X_\tau^{\nu,\delta} + Q_\tau^{\nu,\delta}\left(S_\tau - \pi\lambda_\tau^\nu Q_\tau^{\nu,\delta}\right) - \phi\int_t^\tau (Q_s^\nu)^2 ds\right],
\end{aligned}$$

where $\mathcal{A}''$ is defined as the set $\mathcal{A}$ given by (2.12), but enlarged with $\mathcal{F}$-predictable, bounded from below depths δ. The notation $\mathbb{E}_{t,x,S,q,N^\nu,\lambda^\nu,N^\delta}$ represents expectation conditional on $X_{t^-}^{\nu,\delta} = x$, $S_t = S$, $Q_{t^-}^{\nu,\delta} = q$, $N_{t^-}^\nu = N^\nu$, $\lambda_t^\nu = \lambda^\nu$ and $N_{t^-}^\delta = N^\delta$. If the agent has inventory remaining at the end of the trading horizon, she liquidates it using a MO for which she obtains worse prices than the midprice (terminal penalty). As in Section 2.1, we again need to include the intensity λ_T^ν in this terminal penalty and the terminal price at which she can sell the $Q_T^{\nu,\delta}$ shares is S_T (not S_T^ν) since the direct impact of MOs is here temporary. Using integration by parts based on (3.1), we first find

$$\int_t^\tau S_s\left(\nu_s ds + \Delta dN_s^\delta\right) = -S_\tau Q_\tau^{\nu,\delta} + S_t Q_t^{\nu,\delta} + \int_t^\tau \sigma Q_s^{\nu,\delta}\, dW_s \,.$$

Hence, we directly obtain that

$$\begin{aligned}
&\mathbb{E}_{t,x,S,q,N^\nu,\lambda^\nu,N^\delta}\left[X_\tau^{\nu,\delta} + Q_\tau^{\nu,\delta}\left(S_\tau - \pi\lambda_\tau^\nu Q_\tau^{\nu,\delta}\right) - \phi\int_t^\tau (Q_s^\nu)^2 ds\right] \\
&= \mathbb{E}_{t,x,S,q,N^\nu,\lambda^\nu,N^\delta}\Bigg[X_t^{\nu,\delta} + S_t Q_t^{\nu,\delta} + \int_t^\tau \sigma Q_s^{\nu,\delta}\, dW_s - \int_t^\tau \kappa\nu_s^{\gamma+1}\, dN_s^\nu + \int_t^\tau \delta_{s^-}\Delta\, dN_s^\delta \\
&\qquad\qquad - \pi\lambda_\tau^\nu\left(Q_\tau^{\nu,\delta}\right)^2 - \phi\int_t^\tau (Q_s^\nu)^2 ds\Bigg] \\
&= x + Sq - \mathbb{E}_{t,q,\lambda^\nu}\left[\int_t^\tau \kappa\nu_s^{\gamma+1}\lambda_s^\nu ds - \int_t^\tau \delta_s\Delta\lambda^\delta e^{-\rho\delta}\, ds + \pi\lambda_\tau^\nu\left(Q_\tau^{\nu,\delta}\right)^2 + \phi\int_t^\tau (Q_s^\nu)^2 ds\right] \\
&= x + Sq - h^{\nu,\delta}(t,q,\lambda^\nu)\,,
\end{aligned}$$

which leads to

$$H(t,x,S,q,N^\nu,\lambda^\nu,N^\delta) = x + qS - \inf_{(\nu,\delta)\in\mathcal{A}''} h^{\nu,\delta}(t,q,\lambda^\nu) = x + qS - h(t,q,\lambda^\nu)\,. \tag{3.3}$$

Following Section 2.1 and from [27, 18], it can be shown that $h(t,q,\lambda^\nu)$ is solution of the following HJB equation

$$\begin{aligned}
\partial_t h + \phi q^2 - \alpha\lambda^\nu\partial_{\lambda^\nu} h + &\inf_\nu\left\{\eta\nu\partial_{\lambda^\nu} h - \nu\partial_q h + \kappa\nu^{\gamma+1}\lambda^\nu\right\} \\
&+ \inf_\delta\left\{\lambda^\delta e^{-\rho\delta}\left[h(t,q-\Delta,\lambda^\nu) - h(t,q,\lambda^\nu)\right] - \delta\Delta\lambda^\delta e^{-\rho\delta}\right\} = 0\,,
\end{aligned}$$

with terminal and boundary conditions $h(T,q,\lambda^\nu) = \pi\lambda^\nu q^2,\ h(t,0,\lambda^\nu) = 0$ and $\lim_{\lambda^\nu \to 0} h(t,q,\lambda^\nu) = 0$. Indeed, $h^{\nu,\delta}$ is again continuous and strictly convex in ν and δ for $\gamma > 0$, $\delta < 2/\rho$. Finally[3], first order condition leads to the exact same expression for ν^* as equation (2.17) :

$$\nu^*(t,q,\lambda^\nu) = \left(\frac{\partial_q h(t,q,\lambda^\nu) - \eta\,\partial_{\lambda^\nu} h(t,q,\lambda^\nu)}{(\gamma+1)\lambda^\nu\kappa}\right)^{1/\gamma}, \tag{3.4}$$

and for the optimal depth δ^* to

$$\delta^*(t,q,\lambda^\nu) = \frac{1}{\rho} + \frac{h(t,q-\Delta,\lambda^\nu) - h(t,q,\lambda^\nu)}{\Delta}. \tag{3.5}$$

Substituting these optimal expressions back into the HJB equation (3.4), we finally find explicitly

$$\begin{aligned}\partial_t h(t,q,\lambda^\nu) + \phi q^2 - \alpha\lambda^\nu\,\partial_{\lambda^\nu} h(t,q,\lambda^\nu) - \frac{(\partial_q h(t,q,\lambda^\nu) - \eta\,\partial_{\lambda^\nu} h(t,q,\lambda^\nu))^{\frac{\gamma+1}{\gamma}}}{(\lambda^\nu\kappa(\gamma+1))^{1/\gamma}}\left(\frac{\gamma}{\gamma+1}\right)\\ - \frac{\lambda^\delta e^{-1}\Delta}{\rho}\exp\left(-\rho\left[\frac{h(t,q-\Delta,\lambda^\nu) - h(t,q,\lambda^\nu)}{\Delta}\right]\right) = 0\,,\end{aligned} \tag{3.6}$$

which can be solved numerically in a very efficient way using the method of lines (with a delayed PDE in q). Again, choosing the square root law $\gamma = 1/2$ appears to be the most consistent choice in the temporary case. As the remaining inventory q now depends on the counting process N^δ in (3.1), the optimal trading rate ν^* and the optimal depth in the LOB δ^* are not deterministic anymore. To conclude, we now have a complete price impact model where the liquidating agent can both use MOs and LOs. We then obtain the following optimal strategies ν^* and δ^* for various sets of parameters.

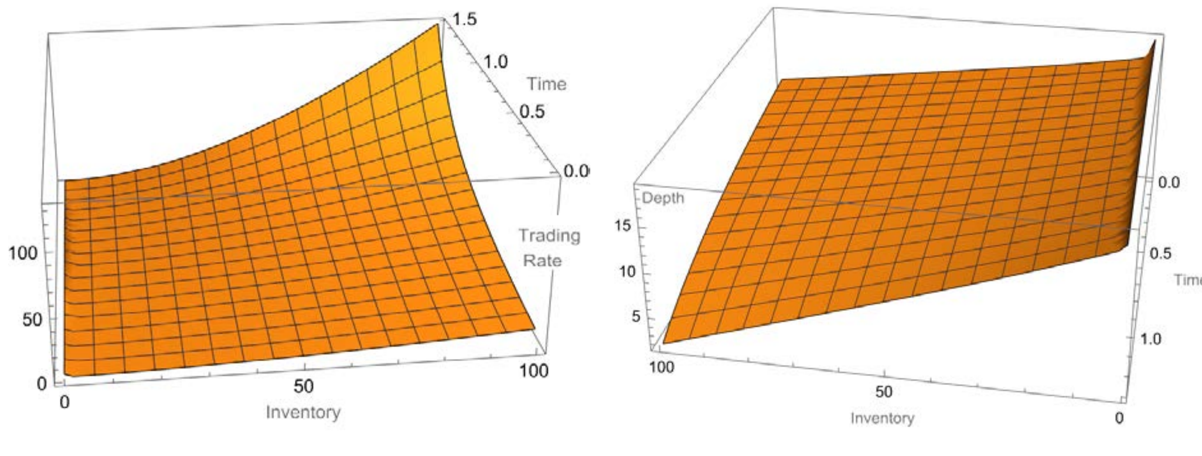


Figure 3.1: Optimal trading rate ν^* (left plot) and depth δ^* (right) in function of the remaining inventory q and time t at $\lambda = 2$ from a temporary price impact model allowing MOs and LOs with $\gamma = 1/2$, $\alpha = 0.5$, $\eta = 0.01$, $\kappa = 0.5$, $\pi = 0.05$, $\phi = 0$, $\lambda^\delta = 1$, $\rho = 0.08$, $\Delta = 1$, $\lambda_0 = 0.5$, $Q = 100$ and $T = 1.5$.

We see on the first plot that the trading rate ν^* is logically increasing with time (the closer the maturity, the more you need to trade so as to avoid the terminal penalty) and also increasing with the remaining inventory (the lower q, the less you need to trade so as to liquidate the remaining shares). We observe the

[3]Recall that ν and δ in λ^ν and λ^δ, respectively, are indices and not exponents.

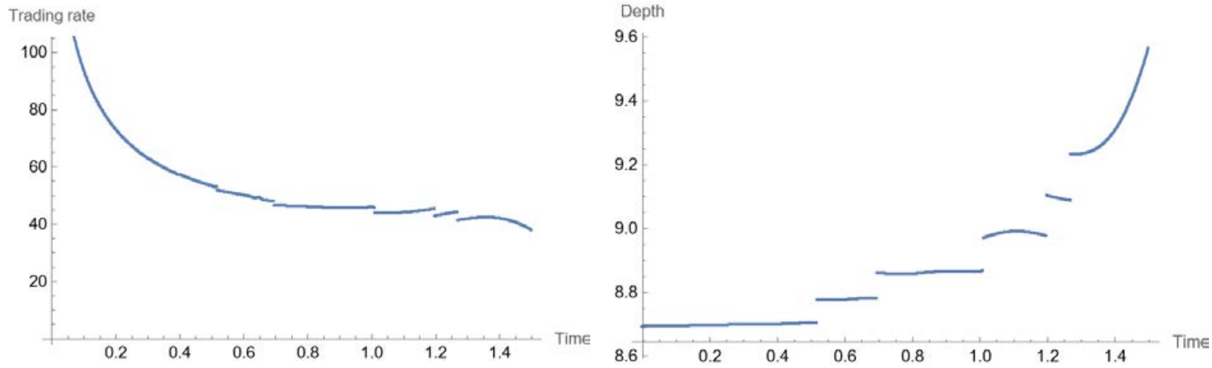


Figure 3.2: Optimal strategy of trading rate ν^* (left plot) and depth δ^* (right plot) from a temporary price impact model allowing MOs and LOs with $\gamma = 1/2$, $\alpha = 0.5$, $\eta = 0.01$, $\kappa = 0.5$, $\pi = 0.1$, $\phi = 0$, $\lambda^\delta = 1$, $\rho = 0.08$, $\Delta = 1$, $\lambda_0 = 0.5$, $Q = 100$ and $T = 1.5$.

opposite behavior for the optimal depth δ^* which is decreasing in time (due to the agent becoming more averse to holding inventories as the trading horizon approaches) and also decreasing in inventory. Indeed, if the agent's inventory is large, she is willing to accept a lower premium δ for providing liquidity so as to increase the probability that her order is filled. At the same time, this ensures that she may complete the liquidation of the Q shares before T and hence avoid paying a terminal penalty. However, if inventories are low, the agent prefers setting a large δ because with low inventory, the terminal penalty she incurs when crossing the spread with MOs will be moderate. On Figure 3.2, we see that the optimal trading rate and optimal depth both depend on the jumps of the Poisson process N^δ and are therefore not deterministic anymore.

This setting can be easily extended to a more general price impact model $S^\nu = S - Z$ by using instead a wealth process which depends on S^ν, *i.e.* :

$$dX_t^{\nu,\delta} = (S_t^\nu + \delta_{t-})\Delta dN_t^\delta + (S_t^\nu - \xi)\nu_t dt\,,$$

and by maximizing the following expected wealth at τ (with S^ν in the terminal penalty),

$$\begin{aligned}
&\mathbb{E}_{t,x,S,q,N^\nu,\lambda^\nu,N^\delta,Z}\left[X_\tau^{\nu,\delta} + Q_\tau^{\nu,\delta}\left(S_\tau^\nu - \pi\lambda_\tau^\nu Q_\tau^{\nu,\delta}\right) - \phi\int_t^\tau (Q_s^\nu)^2 ds\right] \\
&\quad = x + S^\nu q - \mathbb{E}_{t,q,\lambda^\nu,Z}\left[\int_t^\tau Q_s^{\nu,\delta}\, dZ_s - \int_t^\tau \delta_{s-}\Delta\lambda^\delta e^{-\rho\delta}\, ds + \pi\lambda_\tau^\nu\left(Q_\tau^{\nu,\delta}\right)^2 + \phi\int_t^\tau (Q_s^\nu)^2 ds\right] \\
&\quad = x + S^\nu q - \widehat{h}^{\nu,\delta}(t,q,\lambda^\nu,Z)\,.
\end{aligned}$$

In the permanent case 2.2 with $G \equiv 1$, we find for example

$$\begin{aligned}
&\widehat{h}^{\nu,\delta}(t,q,\lambda^\nu,Z) = \widehat{h}^{\nu,\delta}(t,q,\lambda^\nu) \\
&\quad = \mathbb{E}_{t,q,\lambda^\nu}\left[\int_t^\tau \kappa Q_s^{\nu,\delta}\nu_s^\gamma\lambda_s^\nu ds - \int_t^\tau \delta_{s-}\Delta\lambda^\delta e^{-\rho\delta}\, ds + \pi\lambda_\tau^\nu\left(Q_\tau^{\nu,\delta}\right)^2 + \phi\int_t^\tau (Q_s^\nu)^2 ds\right].
\end{aligned}$$

4. Conclusion

This paper introduces a new jump-dependent price impact model (2.1) with propagator allowing to best reproduce the empirical direct and indirect effects of market orders on the transaction price. In our setting, the jump process is used more precisely to model the occurrence of the (temporary/permanent/transient) price impact arising from MOs since all trades do not lead to a direct negative impact on the execution price of the transaction (see Figure 1.2), even if they systematically exert an (indirect) downward pressure on this price. This downward indirect effect on the price is here modeled by an increased mean-reverting intensity λ^ν of the jump process N^ν. Moreover, as the direct impact of market orders depends on the propagator G in (2.1), we discuss for each choice of G the most relevant optimal liquidation problem faced by the agent, derive the related HJB equation and solve it numerically. In the temporary case $G \equiv \delta$, we find that a square-root impact provides the most consistent results and obtain a deterministic decreasing optimal trading strategy that can be set up at time $t = 0$. Moreover, when choosing a geometric intensity (2.21), we show that the dependence of this optimal strategy to the intensity λ^ν disappears. Secondly, in the permanent setting (2.29) with $G \equiv 1$ and exponent $\gamma = 3/2$, we again find a deterministic optimal trading strategy whose trend strongly varies according to the model parameters and which does not depend on λ^ν when taking again the geometric intensity (2.21). Finally, in the transient case, alternative assumptions on the direct price impact and on the terminal penalty have to be assumed in order to obtain the optimal trading rate explicitly (and deterministically). These three scenarios based solely on MOs can then be expanded by allowing the agent to post limit orders in the LOB at a depth δ, which is also a control process to be optimized. Optimal strategies of trading rates ν^* and depths δ^* are again obtained explicitly and deterministically in this enhanced setting, leading to financially sound results. Our indirect price impact model with propagator therefore allows to offer sufficient tractability and flexibility such as to best reproduce the observed underlying dynamic of the LOB and to bridge the gap with high-frequency price models.

Subsequent papers should provide a thorough methodology to estimate the various model parameters based on real LOB data and to determine which kernel G (and which exponent γ) allows to best fit the data (provided that high-frequency data of sufficient quality can be obtained). Secondly, deep learning methods should be investigated and applied for solving high dimensional HJB equations without explicit solutions, as in (2.44) or (B.2). Finally, the no-dynamic-arbitrage properties of the introduced indirect price impact model should also be assessed for the various propagators G studied in this paper.

Appendix A. Mean-variance optimization with permanent price impact kernel

We still consider the permanent price impact setting (2.29) with a mean-reverting intensity (2.2) and we now restrict the set $\mathcal{A}'$ of admissible trading rates ν to deterministic, non-negative and uniformly bounded from above processes on $[0, T]$. This implies that λ^ν and Q^ν are also purely deterministic. In this mean-variance setting, the running penalty for detaining shares $\phi \int_t^\tau (Q_s^\nu)^2 ds$ is replaced instead by a penalty proportional to the variance of the terminal wealth at τ. Recall from equation (2.32) that the total proceeds R_τ at the last trading time τ for $G \equiv 1$ is defined and given in the following way

$$R_\tau^\nu := X_\tau^\nu + Q_\tau^\nu (S_\tau^\nu - \pi Q_\tau^\nu \lambda_\tau^\nu) = X_t^\nu + S_t Q_t^\nu - Z_t Q_t^\nu + \int_t^\tau \sigma Q_s^\nu dW_s - \int_t^\tau \kappa Q_s^\nu \nu_s^\gamma dN_s^\nu - \pi \lambda_\tau^\nu (Q_\tau^\nu)^2 ,$$

which directly leads with $S^\nu = S - Z$ and when ν is in $\mathcal{A}'$ to

$$\mathbb{E}_{t,x,S^\nu,q,\lambda}\left[R_\tau^\nu\right] = x + qS^\nu - \int_t^\tau \kappa\, Q_s^\nu\, \nu_s^\gamma \lambda_s^\nu\, ds - \pi \lambda_\tau^\nu (Q_\tau^\nu)^2 ,$$

and to

$$\mathbb{V}_{t,x,S^\nu,q,\lambda}\left[R_\tau^\nu\right] = \int_t^\tau \sigma^2 (Q_s^\nu)^2\, ds + \int_t^\tau \kappa^2 (Q_s^\nu)^2 \nu_s^{2\gamma} \lambda_s^\nu\, ds .$$

We now aim at optimizing the mean-variance criterion

$$\begin{aligned}
\widehat{H}(t, x, S^\nu, q, \lambda) &= \sup_{\nu \in \mathcal{A}'} \left\{ \mathbb{E}_{t,x,S^\nu,q,\lambda}\left[R_\tau^\nu\right] - \phi \mathbb{V}_{t,x,S^\nu,q,\lambda}\left[R_\tau^\nu\right] \right\} \\
&= \sup_{\nu \in \mathcal{A}'} \left\{ x + qS^\nu - \int_t^\tau \kappa\, Q_s^\nu\, \nu_s^\gamma \lambda_s^\nu\, ds - \pi \lambda_\tau^\nu (Q_\tau^\nu)^2 - \phi \int_t^\tau \sigma^2 (Q_s^\nu)^2\, ds - \phi \int_t^\tau \kappa^2 (Q_s^\nu)^2 \nu_s^{2\gamma} \lambda_s^\nu\, ds \right\} \\
&= x + qS^\nu - \inf_{\nu \in \mathcal{A}'} \left\{ \int_t^\tau \kappa\, Q_s^\nu\, \nu_s^\gamma \lambda_s^\nu\, ds + \pi \lambda_\tau^\nu (Q_\tau^\nu)^2 + \phi \int_t^\tau \sigma^2 (Q_s^\nu)^2\, ds + \phi \int_t^\tau \kappa^2 (Q_s^\nu)^2 \nu_s^{2\gamma} \lambda_s^\nu\, ds \right\} \\
&= x + qS^\nu - \inf_{\nu \in \mathcal{A}'} \widehat{h}^\nu(t, q, \lambda) = x + qS^\nu - \widehat{h}(t, q, \lambda) ,
\end{aligned} \tag{A.1}$$

with terminal and boundary conditions $\widehat{h}(T, q, \lambda) = \pi \lambda q^2$ and $\widehat{h}(t, 0, \lambda) = 0$ from the non-negativity of ν and from (2.10). We also have that $\widehat{h}(t, q, \lambda) \geq 0$ for all $t \geq 0$ since ν_t, Q_t^ν and λ_t^ν are non-negative and since $\kappa, \phi > 0$. When $\lambda_t^\nu \to 0$, the optimally-driven agent can liquidate all her remaining shares at this time t without incurring any penalty, and this hence leads to the boundary condition $\lim_{\lambda \to 0} \widehat{h}(t, q, \lambda) = 0$. Note that since the performance function $\widehat{h}^\nu$ is continuous and strictly convex in ν for $\gamma \geq 1$, it admits a unique optimum ν^*. Hence, from standard results in dynamic programming (see Chapter II of Fleming, Soner 2006), we find that $\widehat{h}$ is the unique viscosity solution of the following deterministic HJB equation

$$\partial_t \widehat{h}(t, q, \lambda) + \phi \sigma^2 q^2 + \inf_{\nu \in \mathcal{A}'} \left\{ (\eta \nu - \alpha \lambda) \partial_\lambda \widehat{h}(t, q, \lambda) - \nu \partial_q \widehat{h}(t, q, \lambda) + \kappa \nu^\gamma q \lambda + \phi \kappa^2 \nu^{2\gamma} q^2 \lambda \right\} = 0. \tag{A.2}$$

Choosing $\gamma = 3/2$ as in the previous Section 2.2 does not lead to an explicit solution for ν^* and the HJB equation above has to be solved numerically in an implicit way, see the review of [25]. The only value of γ allowing to have an explicit expression for ν^* is to choose $\gamma = 1$, which leads to a linear direct permanent impact. In this case, first order condition on ν gives

$$\nu^*(t, q, \lambda) = \frac{\partial_q \widehat{h}(t, q, \lambda) - \eta \partial_\lambda \widehat{h}(t, q, \lambda) - \kappa q \lambda}{2 \phi \kappa^2 q^2 \lambda} . \tag{A.3}$$

Again, $\nu^*(t, 0, \lambda) = 0$ for $t \geq 0$ and $\lambda > 0$ by construction since no purchase of shares is allowed and since trading is stopped at τ. Upon substitution in the HJB equation (A.2), we find

$$\partial_t \widehat{h}(t, q, \lambda) + \phi \sigma^2 q^2 - \alpha \lambda \partial_\lambda \widehat{h}(t, q, \lambda) - \frac{\left(\partial_q \widehat{h}(t, q, \lambda) - \eta \partial_\lambda \widehat{h}(t, q, \lambda) - \kappa q \lambda \right)^2}{4 \phi \kappa^2 q^2 \lambda} = 0 ,$$

with $\widehat{h}(T,q,\lambda) = \pi\lambda q^2$, $\widehat{h}(t,0,\lambda) = 0$ and $\lim_{\lambda\to 0}\widehat{h}(t,q,\lambda) = 0$. ϕ can again be viewed as a risk aversion parameter for detaining stocks and allows to incorporate the agent's urgency for executing the trade. Moreover, the form of the optimal trading rate (A.3) is similar as the one obtained for the permanent impact in the setting of expected wealth maximization (2.35).

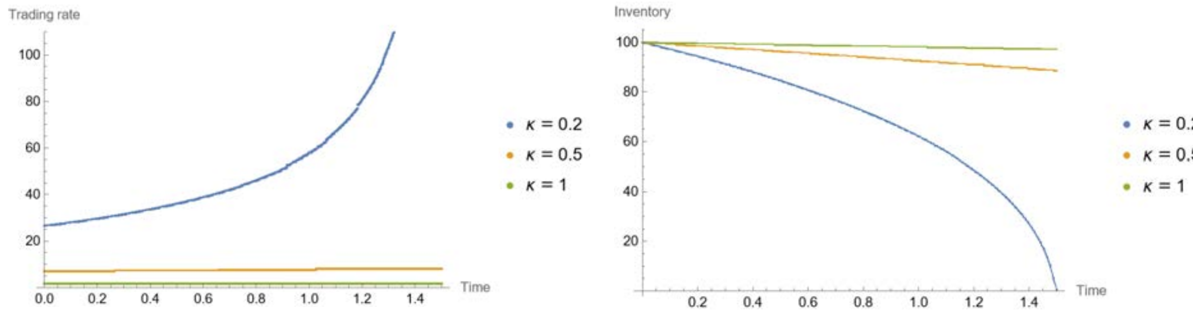


Figure A.1: Optimal trading rate ν^* and inventory Q^{ν^*} from a permanent price impact in the mean-variance setting (A.1) with mean-reverting intensity for various values of κ with $\gamma = 1$, $\alpha = 0$, $\eta = 0.0005$, $\phi = 2$, $\pi = 600$, $\lambda_0 = 0.06$, $\sigma = 0.7$, $Q = 100$ and $T = 1.5$.

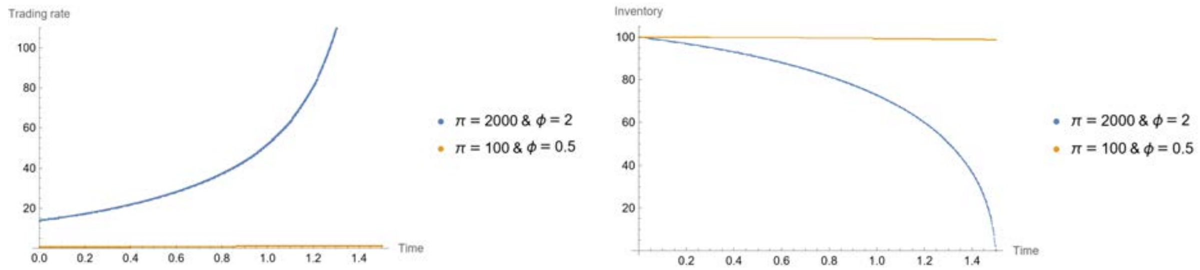


Figure A.2: Optimal trading rate ν^* and inventory Q^{ν^*} from a permanent price impact in the mean-variance setting (A.1) with mean-reverting intensity for various values of π and ϕ with $\gamma = 1$, $\alpha = 1$, $\eta = 0.0001$, $\kappa = 0.5$, $\lambda_0 = 0.06$, $\sigma = 0.7$, $Q = 100$ and $T = 1.5$.

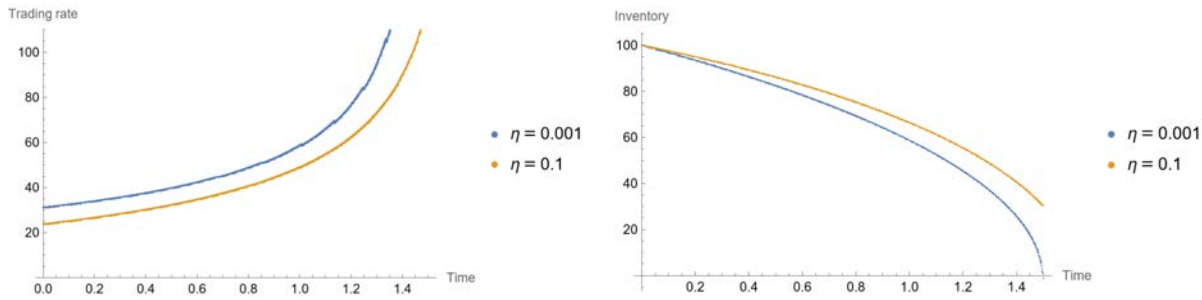


Figure A.3: Optimal trading rate ν^* and inventory Q^{ν^*} from a permanent price impact in the mean-variance setting (A.1) with mean-reverting intensity for various values of η with $\gamma = 1$, $\alpha = 0.1$, $\kappa = 0.9$, $\phi = 0.3$, $\pi = 1000$, $\sigma = 0.7$, $\lambda_0 = 10$, $Q = 100$ and $T = 1.5$.

On all the graphs, we have that the trading rate is increasing with time and is convex such as to mitigate the cumulative permanent price impact κ while diminishing the terminal penalty π. We see that the higher the permanent impact κ, the lesser the trading rate. The agent has indeed more incentive in this case to liquidate less shares and to face the terminal penalty instead. Similarly, for large values of the terminal penalty π and the risk aversion ϕ, it appears optimal for the agent to liquidate all the shares before maturity T. On the contrary, when the terminal penalty and the risk aversion are low, it is optimal for the agent to liquidate almost no shares during the trading horizon and to incur the full terminal penalty. Finally, when η is small, the impact of the trades on the intensity is small and the agent can hence trade a lot without making λ^ν too large, whereas she needs to reduce her trading rate for large values of η so as to not increase too much λ^ν.

Appendix B. Alternative risk-aversion penalty

As already explained in Section 2, the term $\int_t^\tau (Q_u^\nu)^2\, du$ in (2.11) can be seen as a form of risk-aversion (urgency to trade) of the agent. We now propose another form of agent's risk aversion based on an expected shortfall penalty allowing to limit the loss on the total revenue of the liquidating agent at the end of the trading τ. For simplicity, we consider the temporary setting of Section 2.1.1 but the following results can be directly extended to other propagators G :

$$H^\nu(t,x,S,q,\lambda) = \mathbb{E}_{t,x,S,q,N,\lambda}\left[X_\tau^\nu + Q_\tau^\nu\left(S_\tau - \pi\lambda_\tau^\nu Q_\tau^\nu\right) - \phi\left[K' - X_\tau^\nu\right]^+\right]. \tag{B.1}$$

We indeed penalize the agent only when the revenue X_τ^ν at the last trading time τ is below a threshold K'. This penalization term is Lipsichtz continuous in x and we obtain the following unique viscosity solution of the HJB equation with $h(t,x,S,q,\lambda) = \sup_{\nu\in\mathcal{A}} H^\nu(t,x,S,q,\lambda)$:

$$\begin{aligned}\partial_t h + \frac{1}{2}\sigma^2\partial_{SS}^2 h + \sup_\nu \Big\{ &(\eta\nu - \alpha\lambda)\partial_\lambda h - \nu\partial_q h \\ &+ \nu S\partial_x h + \lambda\left[h(t,x-\kappa\nu^{\gamma+1},S,q,\lambda) - h(t,x,S,q,\lambda)\right]\Big\} = 0\,,\end{aligned} \tag{B.2}$$

with $h(T,x,S,q,\lambda) = x - \phi[K'-x]^+ + qS - \pi\lambda q^2$. This 5-dimensional HJB equation is however hard to solve numerically since we only have an implicit solution for ν^*. We therefore consider instead the following optimization problem

$$H^\nu(t,x,S,q,N,\lambda) = \mathbb{E}_{t,x,S,q,N,\lambda}\left[X_\tau^\nu + Q_\tau^\nu\left(S_\tau - \pi\lambda_\tau^\nu Q_\tau^\nu\right) - \phi\int_t^\tau \nu_u\left[K - S_u\right]^+ du\right], \tag{B.3}$$

with again $h(t,x,S,q,\lambda) = \sup_{\nu\in\mathcal{A}} H^\nu(t,x,S,q,\lambda)$. Indeed, with this form of running penalty, we can show that if $(K' - X_\tau) < \int_t^\tau \nu_u(K - S_u)du$, then we have

$$(K' - X_\tau)^+ \leq \left(\int_t^\tau \nu_u(K - S_u)du\right)^+ \leq \int_t^\tau \nu_u(K - S_u)^+ du\,, \tag{B.4}$$

and thus that

$$\phi\int_t^\tau \nu(K - S_u)^+ du \geq \phi(K' - X_\tau)^+\,,$$

which ensures that the penalty in (B.3) is at least larger than the shortfall penalty (B.1) and hence, that the optimization (B.3) also allows to limit the loss on the revenue X_τ^ν. We therefore impose

$$K' - \int_t^\tau S_u \nu_u du + \int_t^\tau \kappa \nu_u^{\gamma+1} dN_u^\nu < \int_t^\tau \nu_u (K - S_u) du \tag{B.5}$$

$$\Leftrightarrow \int_t^\tau K \nu_u du - \int_t^\tau \kappa \nu_u^{\gamma+1} dN_u^\nu > K' \tag{B.6}$$

$$\Rightarrow K > \frac{K' + \kappa \nu_{max}^{\gamma+1} (N_\tau^\nu - N_t^\nu)}{Q_t^\nu - Q_\tau^\nu}, \tag{B.7}$$

since $\nu \in \mathcal{A}$ is bounded, $\gamma > 0$. As seen empirically in Section 1, every sell MO does not lead to a price impact and hence $(Q_t^\nu - Q_\tau^\nu) \gg (N_\tau^\nu - N_t^\nu)$. The condition (B.7) is systematically satisfied empirically when choosing $K > K'/Q_t^\nu$ as the agent aims to liquidate a large number of shares in a short interval of time T. In the permanent case 2.2, we can show that $K > S_0 + K'/Q_t^\nu$ also ensures a shortfall penalty on the expected revenue when liquidating a sufficiently large number of shares. We obtain the following HJB equation associated with the optimization problem (B.3) :

$$\partial_t h + \frac{1}{2}\sigma^2 \partial_{SS}^2 h + \sup_\nu \Big\{ -\phi\nu \,[K - S]^+ + (\eta\nu - \alpha\lambda)\partial_\lambda h - \nu \partial_q h + \nu S \partial_x h + \lambda \left[h(t, x - \kappa\nu^{\gamma+1}, S, q, \lambda) - h(t, x, S, q, \lambda) \right] \Big\} = 0\,.$$

Using $h(t, x, S, q, \lambda) = x + qS + \tilde{h}(t, S, q, \lambda)$, we find

$$\partial_t \widetilde{h} + \frac{1}{2}\sigma^2 \partial_{SS}^2 \widetilde{h} + \sup_\nu \left\{ -\phi\nu \,[K - S]^+ + (\eta\nu - \alpha\lambda)\partial_\lambda \widetilde{h} - \nu \partial_q \widetilde{h} - \lambda\kappa\nu^{\gamma+1} \right\} = 0\,,$$

and first order condition leads explicitly to $\nu^*(t, S, q, \lambda) = \left(\frac{-\phi[K-S]^+ + \eta\partial_\lambda \widetilde{h} - \partial_q \widetilde{h}}{(\gamma+1)\lambda\kappa} \right)^{1/\gamma}$, which finally gives

$$\partial_t \widetilde{h} + \frac{1}{2}\sigma^2 \partial_{SS}^2 \widetilde{h} + \frac{\left(-\phi[K - S]^+ + \eta \partial_\lambda \widetilde{h} - \partial_q \widetilde{h} \right)^{(1+\gamma)/\gamma}}{((\gamma+1)\lambda\kappa)^{1/\gamma}} \left(\frac{\gamma}{\gamma+1} \right) - \alpha\lambda\partial_\lambda \widetilde{h} = 0\,,$$

with $\widetilde{h}(T, S, q, \lambda) = -\pi\lambda q^2$. This explicit HJB equation is much more easy to solve than the implicit equation (B.2) based on the expected shortfall penalty (B.1). Moreover, the optimal trading rate ν^* is not deterministic anymore in this setting since it depends on S.

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