Two-step nilpotent Leibniz algebras

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Leibniz algebras

- Let $\mathbb{F}$ be a field with $char(\mathbb{F}) \neq 2$. A *left Leibniz algebra* over $\mathbb{F}$ is a vector space $\mathfrak{g}$ over $\mathbb{F}$ endowed of a bilinear map (called *commutator* or *bracket*) $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the *left Leibniz identity*

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]], \ \forall x,y,z \in \mathfrak{g}.$$

- We can define a *right Leibniz algebra*, using the right Leibniz identity

$$[[x,y],z] = [[x,z],y] + [x,[y,z]], \ \forall x,y,z \in \mathfrak{g}.$$

- A Leibniz algebra that is both left and right is called *symmetric Leibniz algebra*.
- We assume that $\dim_{\mathbb{F}} \mathfrak{g} < \infty$.

- Every Lie algebra is a Leibniz algebra and every Leibniz algebra with skew-symmetric commutator is a Lie algebra.
- There is an adjunction between the category $\mathbf{LieAlg}_{\mathbb{F}}$ and the category $\mathbf{LeibAlg}_{\mathbb{F}}$.
- The left adjoint of the inclusion functor

$$\mathrm{i} : \mathbf{LieAlg}_{\mathbb{F}} \to \mathbf{LeibAlg}_{\mathbb{F}}$$

is

$$\pi : \mathbf{LeibAlg}_{\mathbb{F}} \to \mathbf{LieAlg}_{\mathbb{F}}$$

defined by

$$\mathfrak{g} \to \frac{\mathfrak{g}}{\mathrm{Leib}(\mathfrak{g})}$$

- The *Leibniz kernel*

$$\mathrm{Leib}(\mathfrak{g}) = \mathrm{Span}_{\mathbb{F}} \{[x, x] \mid x \in \mathfrak{g}\}$$

  is is the smallest bilateral ideal of $\mathfrak{g}$ such that $\mathfrak{g}/\mathrm{Leib}(\mathfrak{g})$ is a Lie algebra.

- The *left* and the *right center* of $\mathfrak{g}$ are

$$\mathrm{Z}_l(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] = 0\}$$

$$\mathrm{Z}_r(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [\mathfrak{g}, x] = 0\}$$

- The *center of* $\mathfrak{g}$ is $\mathrm{Z}(\mathfrak{g}) = \mathrm{Z}_l(\mathfrak{g}) \cap \mathrm{Z}_r(\mathfrak{g})$.

- $\mathrm{Leib}(\mathfrak{g}) \subseteq \mathrm{Z}_l(\mathfrak{g})$.

Solvable and Nilpotent Leibniz algebras

Definition

Let $\mathfrak{g}$ be a left Leibniz algebra and let

$$\mathfrak{g}^0 = \mathfrak{g},\ \mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k],\ \forall k \geq 0,$$

be the *derived series of* $\mathfrak{g}$. $\mathfrak{g}$ is $n-$*step solvable* if $\mathfrak{g}^{n-1} \neq 0$ and $\mathfrak{g}^n = 0$.

Definition

Let $\mathfrak{g}$ be a left Leibniz algebra and let

$$\mathfrak{g}^{(0)} = \mathfrak{g},\ \mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}],\ \forall k \geq 0,$$

be the *lower central series of* $\mathfrak{g}$. $\mathfrak{g}$ is $n-$*step nilpotent* if $\mathfrak{g}^{(n-1)} \neq 0$ and $\mathfrak{g}^{(n)} = 0$.

Levi decomposition

Theorem

Let $\mathbb{F}$ *be a field with* $\mathrm{char}(\mathbb{F}) = 0$ *and let* $\mathfrak{g}$ *be a Leibniz algebra over* $\mathbb{F}$. *Then there exists a Leibniz subalgebra* $\mathfrak{s}$ *(which is a semisimple Lie algebra) of* $\mathfrak{g}$ *such that*

$$\mathfrak{g} = \mathfrak{s} \oplus_s \mathfrak{r}$$

where $\mathfrak{r} = \mathrm{rad}(\mathfrak{g})$ *is the radical of* $\mathfrak{g}$.

Two-step nilpotent Leibniz algebras

Proposition

If $\mathfrak{g}$ *is a left two-step nilpotent Leibniz algebra, then* $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathsf{Z}(\mathfrak{g})$ *and* $\mathfrak{g}$ *is symmetric.*

Proposition

If $\mathfrak{g}$ *is a left nilpotent Leibniz algebra with* $\dim_{\mathbb{F}} \mathfrak{g}^{(1)} = 1$*, then* $\mathfrak{g}^{(1)} \subseteq \mathsf{Z}(\mathfrak{g})$ *and* $\mathfrak{g}$ *is symmetric.*

- Let $\mathfrak{g}$ be a two-step nilpotent Leibniz algebra with $[\mathfrak{g}, \mathfrak{g}] = Span_{\mathbb{F}}\{z_1, \ldots, z_t\}$, then

$$[x, y] = \phi_1(x, y)z_1 + \cdots + \phi_t(x, y)z_t, \ \ \forall x, y \in \mathfrak{g}$$

  where $\phi_i : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ is a bilinear form, $\forall i = 1, \ldots, t$.

- Our purpose is to reduce such a given bilinear form to a suitable canonical representation.

- Let $\mathfrak{g}$ be a nilpotent Leibniz algebra with $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}z$ and

$$[x, y] = \phi(x, y)z, \;\; \forall x, y \in \mathfrak{g}$$

- If $\mathfrak{g}$ is not a Lie algebra, then $0 \neq \mathrm{Leib}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathrm{Z}(\mathfrak{g})$.
- $\phi = \phi^- + \phi^+$, where

$$\phi^- = \frac{\phi - \phi^t}{2}, \;\; \phi^+ = \frac{\phi + \phi^t}{2}.$$

- We use the simultaneous reduction to canonical form of a pair of bilinear forms.

Definition

A Kronecker module is a quadruple (V_1, V_2, f_1, f_2), where V_1, V_2 are vector spaces over $\mathbb{F}$ and $f_1, f_2 : V_1 \to V_2$ are linear maps.

- Given a pair of bilinear forms $\phi^-, \phi^+ : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ as above, we define the Kronecker module $\left(\mathfrak{g}, \mathfrak{g}^*, \overline{\phi^-}, \overline{\phi^+}\right)$, where

$$\overline{\phi^-}(x) : z \mapsto \phi^-(x, z), \quad \overline{\phi^+}(y) : z \mapsto \phi^+(y, z), \ \forall z \in \mathfrak{g}.$$

- $\mathfrak{g}$ is *decomposable* if $\mathfrak{g} = U \oplus U^\perp$.
- We can assume that $\mathfrak{g}$ is indecomposable.

The indecomposable modules $\left(\mathfrak{g}, \mathfrak{g}^*, \overline{\phi^-}, \overline{\phi^+}\right)$ turn out to be one of the following three pairs

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} 0 & J_1 \\ -J_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J_2 \\ J_2^t & 0 \end{pmatrix} \tag{3}$$

where $A \in \mathrm{M}_n(\mathbb{F})$ and J_1, J_2 are the matrices associated with the linear maps $F_1, F_2 : \mathbb{F}^n \to \mathbb{F}^{n+1}$ defined by

$$F_1(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0),$$

$$F_2(x_1, \ldots, x_n) = (0, x_1, \ldots, x_n).$$

Proposition

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Leibniz algebras of dimension n with one-dimensional commutator ideals $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathbb{F}z_1$ and $[\mathfrak{g}_2, \mathfrak{g}_2] = \mathbb{F}z_2$, and let ϕ_1 and ϕ_2 be the bilinear forms associated with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. One can fix bases of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that, if Φ_1 and Φ_2 are the matrices of ϕ_1 and ϕ_2 respectively, then $\mathfrak{g}_1$ is isomorphic to $\mathfrak{g}_2 \iff \Phi_1$ is congruent to Φ_2.

If $X \in \mathrm{GL}_n(\mathbb{F})$, then

$$\begin{pmatrix} X & 0 \\ 0 & \left(X^{-1}\right)^t \end{pmatrix}$$

induces a change of basis that transforms (1) and (2) respectively in

$$\left(\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & XAX^{-1} \\ \left(XAX^{-1}\right)^t & 0 \end{pmatrix} \right),$$

$$\left(\begin{pmatrix} 0 & XAX^{-1} \\ -\left(XAX^{-1}\right)^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \right),$$

so we can always reduce A in its rational canonical form.

- If $A \in \mathrm{GL}_n(\mathbb{F})$, then the second canonical pair is equivalent to the first one:

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix}^t = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$
$$\begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix}^t = \begin{pmatrix} 0 & A^{-1} \\ \left(A^{-1}\right)^t & 0 \end{pmatrix}.$$

- If $A \notin \mathrm{GL}_n(\mathbb{F})$, then $A \sim J_0$ and we have a unique Kronecker module of type (2) up to isomorphisms.

The indecomposable modules $\left(\mathfrak{g}, \mathfrak{g}^*, \overline{\phi^-}, \overline{\phi^+}\right)$ turn out to be one of the following three pairs

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & C_{f(x)^m} \\ C^t_{f(x)^m} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & J_0 \\ -J^t_0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & J_1 \\ -J_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J_2 \\ J^t_2 & 0 \end{pmatrix}$$

where $f(x) \in \mathbb{F}[x]$ is a monic irreducible polyonomial and J_1, J_2 are the matrices associated with

$$F_1, F_2 : \mathbb{F}^n \to \mathbb{F}^{n+1}$$

defined by

$$F_1(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0),$$

$$F_2(x_1, \ldots, x_n) = (0, x_1, \ldots, x_n).$$

Definition

Let $f(x) \in \mathbb{F}[x]$ be a monic irreducible polynomial. Let $n \in \mathbb{N}$ and let $A = C_{f(x)^n}$. The *Heisenberg* Leibniz algebra $\mathfrak{l}_{2n+1}^{A}$ is the $(2n+1)$-dimensional indecomposable Leibniz algebra with associated Kronecker module of type (1).

- If $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{F})$ and $\{e_1, \ldots, e_n, f_1, \ldots, f_n, h\}$ is a basis of $\mathfrak{l}_{2n+1}^{A}$, then the non-trivial commutators are

$$[e_i, f_j] = (\delta_{ij} + a_{ij})\, h, \ [f_j, e_i] = (-\delta_{ij} + a_{ij})\, h, \ \forall i, j = 1, \ldots, n$$

- If $(a_{ij}) = 0$, then we obtain the classical Heisenberg algebra $\mathfrak{h}_{2n+1}$.

Definition

Let $n \in \mathbb{N}$ and let $A = C_{x^n}$. The *Kronecker* Leibniz algebra $\mathfrak{k}_n$ is the $(2n+1)$-dimensional indecomposable Leibniz algebra with associated Kronecker module of type (2).

Definition

The *Dieudonné* Leibniz algebra $\mathfrak{d}_n$ is the $(2n+2)-$dimensional Leibniz algebra with associated Kronecker module of type (3).

The case $\mathbb{F} = \mathbb{C}$

Let $n \in \mathbb{N}$ and let $a \in \mathbb{C}$. Then

$$C_{(x-a)^n} \sim J_a = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 1 & a \end{pmatrix}$$

Thus $\mathfrak{l}_{2n+1}^{A} \cong \mathfrak{l}_{2n+1}^{J_a}$ and the non-trivial Leibniz brackets are

$$[e_i, f_i] = (1 + a)h,\ [f_i, e_i] = (-1 + a)h\ \forall i = 1, \dots, n$$

$$[e_i, f_{i-1}] = h,\ [f_{i-1}, e_i] = h,\ \forall i = 2, \dots, n$$

where $\{e_1, \dots, e_n, f_1, \dots, f_n, h\}$ is a basis of $\mathfrak{l}_{2n+1}^{J_a}$.

Proposition (*G. La Rosa, M. M.*, 2022)

Let $a \in \mathbb{C}$. The Heinseberg Leibniz algebras $\mathfrak{l}_{2n+1}^{J_a}$ and $\mathfrak{l}_{2n+1}^{J_{-a}}$ are isomorphic.

Proof.

- $\mathfrak{l}_{2n+1}^{J_a} \cong \mathfrak{l}_{2n+1}^{-J_a^t}$ via the linear map φ defined by

$$\varphi(e_i) = f_i', \;\; \varphi(f_i) = e_i', \;\; \varphi(h) = -h', \;\; \forall i = 1, \cdots, n$$

- $-J_a^t \sim J_{-a} \Rightarrow \mathfrak{l}_{2n+1}^{J_a} \cong \mathfrak{l}_{2n+1}^{J_{-a}}$.

□

Proposition (*G. La Rosa, M. M.*, 2022)

Let $a, a' \in \mathbb{C}$. Then $\mathfrak{l}_3^a \cong \mathfrak{l}_3^{a'} \iff a' = \pm a$.

Thank you for your attention!

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