Pitfalls in calculating the age in a one-dimensional, semi-infinite domain

Eric Deleersnijder, 4 July 2019

Abstract. One-dimensional advection-diffusion of a passive tracer is investigated in a semi-infinite domain. The age is used as a diagnosis measuring the time elapsed since leaving the upstream boundary of the domain, where the concentration is prescribed to be equal to unity. The time- and position-dependent age distribution function is obtained, from which the concentration, age concentration and mean age are derived in line with CART (www.climate.be/cart). The concentration tends to unity as time progresses. The mean age tends to a steady-state limit (the ratio of distance to velocity), except if advection is negligible. In the latter case, although the age distribution function is a fast-decreasing function of the age, the mean age grows indefinitely as time progresses. This is because, in the absence of advection, the particles are not flushed away from the upstream domain boundary, allowing them to visit an indefinite number of times any location of the domain. The zero-diffusion solution is also obtained, exhibiting no thought-provoking result as opposed to the purely diffusive case.

A variant of this problem is also tackled (Appendix B): at the upstream boundary, the Dirichlet boundary condition is replaced by a Robin boundary condition, prescribing that the tracer particles entering the domain (with a constant flux) have zero age. The mean age at the incoming boundary is no longer zero. A steady-state solution exists, unless advection is disregarded. The purely diffusive case admits no steady-state concentration, age concentration and mean age.

The abovementioned tracer-related variables exhibit radically different behaviours, depending on the boundary conditions and the presence or absence of advection.

Motivation

Seemingly elementary advection-diffusion problems admitting analytical solutions are often resorted to in order to help gain insight into transport processes in complex fluid flows (e.g. Hall and Plumb 1994, Waugh et al. 2002, Waugh et al. 2003, Lucas et al. 2009, Mouchet et al. 2012, Andutta et al. 2014). Unfortunately, some analytical solutions entail pitfalls, which may be due to the model being too simple (e.g. Nihoul 1994). A rather well known example and a couple of variants are tackled hereinafter (e.g. Hall and Plumb 1994, Hall and Haine 2002), consisting in calculating tracer-related variables in a one-dimensional, semi-infinite domain.

No novel results are derived, but the link with the Constituent-oriented Age and Residence time Theory (CART, www.climate.be/cart) is explicitly established.

Problem description and general solution

Let independent variables t, x and τ denote the time, the distance to the upstream boundary and the age, respectively. The one-dimensional, semi-infinite domain of interest is defined by inequalities $0 \le x < \infty$ (Figure 1). The age of a passive tracer particle is defined as the time elapsed since leaving $x = 0$. At the initial instant (t=0), there is no tracer in the domain. If positive constants U and K represent the velocity and the diffusivity, the equation obeyed by age distribution function[1] $c(t,x,\tau)$ is (e.g. Delhez et al. 1999)

$$\frac{\partial c}{\partial t} + U\frac{\partial c}{\partial x} = K\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial \tau} \quad . \tag{1}$$

This equation[2] is to be dealt with under the following initial and boundary conditions

$$c(0,x,\tau)=0\ ,\quad c(t,0,\tau)=\delta(\tau)\ ,\quad c(t,x,0)=0\ ,\quad c(t,\infty,\tau)<\infty\ , \tag{2}$$

where δ is the Dirac delta function. The solution reads (Figure 2) (see also Appendix A)

$$c(t,x,\tau) = \frac{x}{\sqrt{4\pi K\tau^3}}\exp\left[-\frac{(x-U\tau)^2}{4K\tau}\right]\Upsilon(t-\tau) \tag{3}$$

where Υ is the Heaviside step function. Thus, function $\Upsilon(t-\tau)$ is equal to unity (zero) according to whether $t>\tau$ ($t<\tau$).

In appendix B, the impact of an alternative incoming boundary condition (constant flux of tracer particles with zero age) is investigated, which is inspired by the study of Deleersnijder (2017) of timescales in shallow reservoirs. Though it might be more useful for practical purposes, this variant is less well documented in the literature than problem (1)-(2). This may be due to the fact that the latter problem is somewhat easier to deal with.

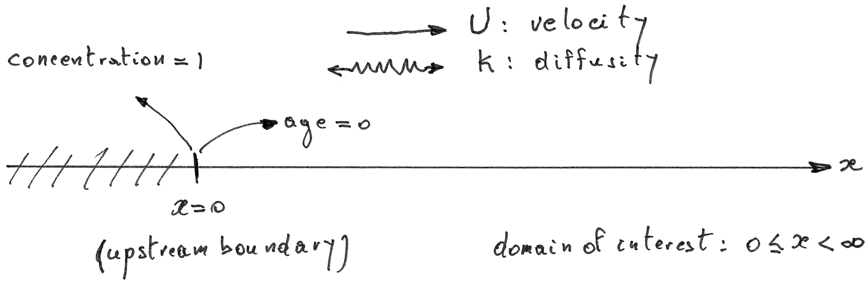

Figure 1. The one-dimensional, semi-infinite domain of interest, in which advection and diffusion of a passive tracer is to be investigated. At x=0, the tracer concentration is prescribed to be equal to unity (Dirichlet boundary condition). The age is a diagnosis measuring the time elapsed since leaving the upstream boundary (x=0).

The concentration of the tracer is

[1] Age distribution function $c(t,x,\tau)$ is defined as follows (e.g. Deleersnijder et al. 2001): at time t, in the elemental control "volume" $[x,x+\delta x]$ ($\delta x \to 0$), the fraction of the tracer particles whose age lies in the age interval $[\tau,\tau+\delta\tau]$ ($\tau\in[0,\infty[$) tends to $c(t,x,\tau)\delta\tau$ as $\delta\tau\to 0$. In other words, the age distribution function may be regarded as the histogram of the ages of the tracer particles present in the aforementioned control "volume". Clearly, the physical dimension of τ and that of $c(t,x,\tau)$ are time and time^{-1}, respectively.

[2] The equation governing the evolution of the age distribution function is obtained from mass budget considerations under the easily-acceptable assumption that tracer particles are advected and diffused irrespective of their age (e.g. Delhez et al. 1999, Deleersnijder et al. 2001). Thus, this equation leaves little room for arbitrariness.

$$C(t,x) \;=\; \int_0^\infty c(t,x,\tau)\,d\tau \;=\; \int_0^t \frac{x}{\sqrt{4\pi K\tau^3}}\exp\left[-\frac{(x-U\tau)^2}{4K\tau}\right]d\tau \quad , \tag{4}$$

whilst its age concentration reads

$$\alpha(t,x) \;=\; \int_0^\infty \tau\, c(t,x,\tau)\,d\tau \;=\; \int_0^t \frac{\tau\, x}{\sqrt{4\pi K\tau^3}}\exp\left[-\frac{(x-U\tau)^2}{4K\tau}\right]d\tau \quad . \tag{5}$$

The age concentration is readily seen to satisfy

$$\begin{aligned}\alpha(t,x) \;&=\; \int_0^t \frac{\tau\, x}{\sqrt{4\pi K\tau^3}}\exp\left[-\frac{(x-U\tau)^2}{4K\tau}\right]d\tau \\ &\leq\; t\int_0^t \frac{x}{\sqrt{4\pi K\tau^3}}\exp\left[-\frac{(x-U\tau)^2}{4K\tau}\right]d\tau \;=\; t\,C(t,x)\end{aligned} \tag{6}$$

Therefore, the mean age[3] of the tracer,

$$a(t,x) \;=\; \frac{\alpha(t,x)}{C(t,x)} \quad , \tag{7}$$

is smaller than or equal to elapsed time t, i.e. $a(t,x) \leq t$. This is in agreement with elementary physical intuition.

It is noteworthy that the mean age could have been obtained without explicitly calculating the age distribution function. The tracer concentration and age concentration are governed by the following partial differential problems (e.g. Delhez et al. 1999)

$$\begin{cases}\dfrac{\partial C}{\partial t} \;+\; U\dfrac{\partial C}{\partial x} \;=\; K\dfrac{\partial^2 C}{\partial x^2} \\ C(t,0)=1\ ,\quad C(0,x)=0\ ,\quad C(t,\infty)<\infty\end{cases} \tag{8}$$

and

[3] Consider, for instance, two particles that are identified by means of subscripts "A" and "B". Their mass and age are denoted m^X and a^X , respectively, with $X=A,B$. The mass of the system "A+B" obvioulsy is $m^{A+B} = m^A + m^B$. This is in agreement with basic physical principles that stipulate that mass is an additive quantity. No such principle exists for the age. Therefore, to obtain the mean age of the system "A+B", an arbitrary decision is to be made. The latter was formulated as the so-called age-averaging hypothesis (Deleersnijder et al., 2001), which is the only really arbitrary element of CART. It stipulates that the mean age of a system made up of various particles is the mass-weighted average of the ages of the particles. Accordingly, the mean age of the system "A+B", a^{A+B}, satisfies the follow relation: $m^{A+B}a^{A+B} = m^A a^A + m^B a^B$. Clearly, the age is not an additive quantity, but the age content is (the age content of a particle is defined as the product of its mass and its age). Ages other than CART's, such as the carbon-14 age, implicitly or explicitly rely on an age-averaging hypothesis (Deleersnijder et al., 2001). However, it is believed that CART's age-averaging hypothesis is the simplest one could think of.

$$
\begin{cases}
\dfrac{\partial \alpha}{\partial t} + U\dfrac{\partial \alpha}{\partial x} = K\dfrac{\partial^2 \alpha}{\partial x^2} + C \\
\alpha(t,0)=0\ ,\ \ \alpha(0,x)=0\ ,\ \ \lim\limits_{x\to\infty}\dfrac{\alpha(t,x)}{x}<\infty
\end{cases}
\tag{9}
$$

As expected, the solutions of (8)-(9) are (4)-(5), which is in line with CART's "machinery".

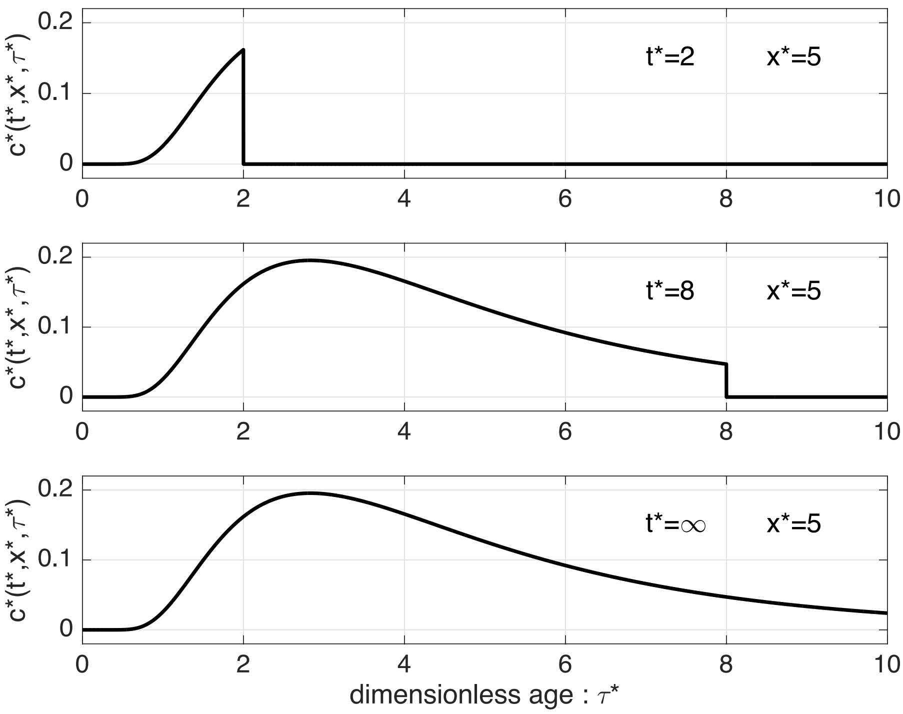

Figure 2. Illustration of age distribution function (3). Dimensionless variables are used. They are identified by asterisks and are defined as follows: $t^* = U^2 t / K$, $x^* = Ux / K$, $\tau^* = U^2\tau / K$ and $c^* = Kc / U^2$. The dimensionless age distribution function[4] is plotted at $x^* = 5$ as a function of the age at different instants.

Existence of a steady-state limit

In the limit $t \to \infty$, the age distribution function (Figure 2) (see also Appendix A), the concentration, the age concentration and the mean age are

$$
c_\infty(x,\tau) = \lim_{t\to\infty} c(t,x,\tau) = \frac{x}{\sqrt{4\pi K\tau^3}} \exp\left[-\frac{(x-U\tau)^2}{4K\tau}\right] \quad , \tag{10}
$$

$$
C_\infty(x) = \lim_{t\to\infty} C(t,x) = 1 \quad , \tag{11}
$$

[4] $c^*(t^*,x^*,\tau^*) = \dfrac{x^*}{\sqrt{4\pi(\tau^*)^3}} \exp\left[-\dfrac{(x^*-\tau^*)^2}{4\tau^*}\right] \Upsilon(t^*-\tau^*)$

$$\alpha_\infty(x) \;=\; \lim_{t\to\infty} \alpha(t,x) \;=\; \frac{x}{U} \quad , \tag{12}$$

$$a_\infty(x) \;=\; \lim_{t\to\infty} a(t,x) \;=\; \frac{x}{U} \quad . \tag{13}$$

The age distribution function satisfies

$$\lim_{\substack{\tau\to\infty \\ t>\tau}} \tau^n\, c_\infty(t,x,\tau) \;=\; 0 \;\;,\quad n = 0,1,2,... \tag{14}$$

Unsurprisingly, the steady-state concentration and age concentration can also be obtained from the steady-state versions of (8)-(9):

$$\begin{cases} U\dfrac{dC_\infty}{dx} \;=\; K\dfrac{d^2C_\infty}{dx^2} \\ C_\infty(0) = 1\,,\quad C(\infty) < \infty \end{cases} \tag{15}$$

and

$$\begin{cases} U\dfrac{d\alpha_\infty}{dx} \;=\; K\dfrac{d^2\alpha_\infty}{dx^2} \;+\; C_\infty \\ \alpha_\infty(0) = 0\,,\quad \displaystyle\lim_{x\to\infty} \frac{\alpha_\infty(t,x)}{x} < \infty \end{cases} \tag{16}$$

The zero diffusion limit

If diffusivity K is zero, then the age distribution function is

$$c(t,x,\tau) \;=\; \delta(\tau - x/U)\Upsilon(t-\tau) \;\;, \tag{17}$$

implying that the concentration and age concentration are

$$C(t,x) = \Upsilon(t - x/U)\,,\quad \alpha(t,x) = \frac{x}{U}\Upsilon(t - x/U) \quad . \tag{18}$$

Therefore, the mean age is x/U for $t > x/U$ and is undefined otherwise. These expressions can also be obtained by setting $K = 0$ in (8)-(9) and dealing with the resulting equations in the sense of distributions.

The zero advection limit

Setting $U = 0$ in (3) leads to the age concentration function for the purely diffusive case:

$$c(t,x,\tau) \;=\; \frac{x}{\sqrt{4\pi K\tau^3}}\exp\!\left(-\frac{x^2}{4K\tau}\right)\Upsilon(t-\tau) \quad . \tag{19}$$

This expression does not satisfy (14). It obeys a more restrictive version of it, i.e.

$$\lim_{\substack{\tau\to\infty \\ t>\tau}} \tau^n\, c(t,x,\tau) \;=\; \begin{cases} 0 \;\;, & n = 0,1 \\ \infty \;\;, & n = 2,3,... \end{cases} \tag{20}$$

Therefore, CART's concentration and age concentration equations still hold valid.

The concentration and age concentration are

$$C(t,x) \;=\; \int_0^\infty c(t,x,\tau)\,d\tau \;=\; \operatorname{erfc}\left(\frac{x}{\sqrt{4\pi Kt}}\right) \tag{21}$$

and

$$\alpha(t,x) \;=\; \int_0^\infty \tau\, c(t,x,\tau)\,d\tau \;=\; \sqrt{\frac{x^2 t}{\pi K}}\,\exp\left(-\frac{x^2}{4Kt}\right) - \frac{x^2}{2K}\operatorname{erfc}\left(\frac{x}{\sqrt{4\pi Kt}}\right) \;, \tag{22}$$

where erfc denotes the complementary error function[5]. These expressions can also be obtained by solving (8)-(9) under the assumption that $U = 0$.

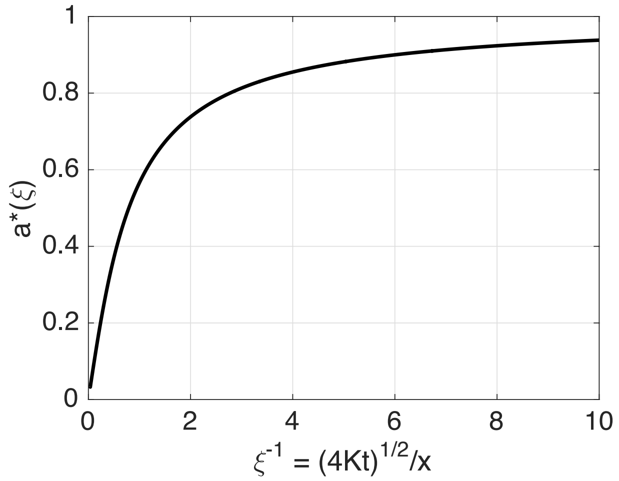

Figure 3. Representation of the age in the purely diffusive case (zero velocity)[6]. Dimensionless, self-similar solution (26) is displayed: normalised age a^* is plotted against $\xi^{-1} = \sqrt{4Kt}\ / x$. For a fixed value of space coordinate *x,* the age increases proportionally with $\sqrt{t}$ as $t \to \infty$, in agreement with (25). This is equivalent to asymptotic expression $a^* \sim 1-(\sqrt{\pi}-2/\sqrt{\pi})/\xi^{-1}$ for $\xi^{-1} \to \infty$.

The passive tracer concentration admits a steady-state limit:

$$C_\infty(x) \;=\; \lim_{t\to\infty} C(t,x) \;=\; 1 \quad . \tag{23}$$

The age concentration, however, does not admit a steady-state limit. As a result, the mean age (Figure 3),

$$a(t,x) \;=\; \frac{\alpha(t,x)}{C(t,x)} \;=\; \sqrt{\frac{x^2\, t}{\pi K}}\;\frac{\exp\left(-\dfrac{x^2}{4Kt}\right)}{\operatorname{erfc}\left(\dfrac{x}{\sqrt{4\pi Kt}}\right)} \;-\; \frac{x^2}{2K} \quad , \tag{24}$$

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[5] $\operatorname{erfc}(\varsigma) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_\varsigma^\infty e^{-\upsilon^2}\,d\upsilon$

[6] At first glance, this graph is not easy to understand. It must be underscored, however, that it contains all the information on self-similar solution (26)-(27).

grows indefinitely as time progresses,

$$a(t,x) \sim \sqrt{\frac{x^2 t}{\pi K}} \ , \quad \frac{x}{\sqrt{4Kt}} \to 0 \tag{25}$$

This behaviour is in sharp contrast with that of the age related to the advection-diffusion problem.

Clearly, it is advection that, by eventually flushing tracer particles in the downstream direction, causes the steady-state limit of the mean age to exist. Without advection, nothing prevents a tracer particle from coming back indefinitely to any location in the domain.

Interestingly, the mean age may be cast into a dimensionless, self-similar form,

$$a^*(\xi) = \sqrt{\frac{\pi K}{x^2 t}}\, a(t,x) = \frac{e^{-\xi^2}}{\mathrm{erfc}(\xi)} - \sqrt{\pi}\,\xi \quad , \tag{26}$$

with

$$\xi = \frac{x}{\sqrt{4Kt}} \quad . \tag{27}$$

Relation (26) is displayed in Figure 3.

Concluding remarks

In the purely diffusive case, that $\tau c(t,x,\tau)$ tends to zero in the limit $\tau \to \infty$ does not prevent the age from growing indefinitely as time progresses. It is diffusion and the absence of advection that prevent the steady-state limit from existing. Some may argue that this flies in the face of physical intuition.

Simple reasonings and seemingly obvious conclusions must be considered with caution. It is thus desirable that the mathematical properties of the solution of the problem at hand, however trivial it may seem to be, be derived thoroughly prior to embarking into possibly ill-conditioned analytical or numerical calculations and, subsequently, drawing potentially erroneous conclusions (e.g. Beckers et al. 2001, Hall and Haine 2004, Delhez et al. 2014)[7]. Doing so is unlikely to be easy, even for simple transport problems, but it is worth the effort as was shown above.

Appendix A

Steady-state age distribution function $c_\infty(x,\tau)$ is the solution of the following differential equation

[7] In their seminal article on diagnostic timescales, Bolin and Rodhe (1973) stated what should have been the obvious: “To avoid misunderstandings and even erroneous conclusions it is important to introduce precise definitions and to use them wit care”. Surprisingly (or not), this wise piece of advice was ignored by many. This led to a situation half-jokingly referred to as the Tower of Babel by Viero and Defina (2016), i.e. a wide range of poorly defined diagnostic timescales used rather carelessly, eventually causing misleading interpretations and conclusions to be produced.

$$U\frac{\partial c_\infty}{\partial x} = K\frac{\partial^2 c_\infty}{\partial x^2} - \frac{\partial c_\infty}{\partial \tau} \tag{A.1}$$

and boundary conditions

$$c_\infty(0,\tau)=\delta(\tau)\,,\quad c_\infty(x,0)=0\,,\quad c_\infty(\infty,\tau)<\infty \quad . \tag{A.2}$$

As indicated above, the solution to (A.1) reads

$$c_\infty(x,\tau) = \frac{x}{\sqrt{4\pi K\tau^3}}\exp\left[-\frac{(x-U\tau)^2}{4K\tau}\right] \quad . \tag{A.3}$$

After some calculations, one obtains

$$U\frac{\partial c_\infty}{\partial x} = \frac{-2Ux^2\tau+2U^2x\tau^2+4UK\tau^2}{4Kx\tau^2}c_\infty \tag{A.4}$$

and

$$K\frac{\partial^2 c_\infty}{\partial x^2} - \frac{\partial c_\infty}{\partial \tau} = \frac{-2Ux^2\tau+2U^2x\tau^2+4UK\tau^2}{4Kx\tau^2}c_\infty \tag{A.5}$$

The right-hand sides of (A.4) and (A.5) are equal to each other, implying that expression (A.3) satisfies equation (A.1). Then, it is easily seen that (A.3) obeys boundary conditions (A.2). As a consequence, it is proven that (A.3) is the solution to partial differential problem (A.1)-(A.2).

By comparing (3) and (10), it readily appears that time-dependent age distribution function (3) may be cast into the following form[8]

$$c(t,x,\tau) = c_\infty(x,\tau)\ \Upsilon(t-\tau) \tag{A.6}$$

Clearly, (A.6) satisfies initial and boundary conditions (2). Next, substituting (A.6) into (1) and transferring all the terms in the left-hand side yields

$$\underbrace{\left(U\frac{\partial c_\infty}{\partial x}-K\frac{\partial^2 c_\infty}{\partial x^2}+\frac{\partial c_\infty}{\partial \tau}\right)}_{=0\text{ , see (A.4)-(A.5)}}\Upsilon(t-\tau) + \underbrace{c_\infty\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)\Upsilon(t-\tau)}_{\substack{=0\text{ , in the sense}\\ \text{of distributions}}} = 0 \quad . \tag{A.7}$$

Thus, (A.6) satisfies (1), hence (A.6) or (3) is the solution to partial differential problem (1)-(2).

It has been demonstrated that, for the problem under consideration, the time-dependent age distribution function is easily derived from the solution of the steady-state one. This is because the velocity and diffusivity are time-independent, and the initial and boundary condition are favourable. Many, more complex fluid flow problems satisfy a similar property.

Appendix B

A variant of the problem tackled above is dealt with in the present Appendix. The Dirichlet boundary condition imposed at the upstream end of the domain is replaced by a Robin boundary condition: in line with Deleersnijder (2017), the incoming flux of tracer, $\boldsymbol{\Phi}$, is

[8] I have been unable to find out who derived this result for the first time. However, it is noteworthy that expressions bearing some similarities with (A.6) may be found in various treatises, e.g., Section (23) of Carslaw (1921) or Section (4.1) of Duffy (2001).

assumed to be constant and the age of the incoming particles is prescribed to be zero.

$$\begin{cases} \dfrac{\partial c}{\partial t} + U\dfrac{\partial c}{\partial x} = K\dfrac{\partial^2 c}{\partial x^2} - \dfrac{\partial c}{\partial \tau} \\ c(0,x,\tau)=0\,,\ \left[cU - K\dfrac{\partial c}{\partial x}\right]_{x=0} = \Phi\,\delta(\tau)\,,\ \ c(t,x,0)=0\,,\ \ c(t,\infty,\tau)<\infty \end{cases} \tag{B.1}$$

Setting $\Phi = U$ (so that the steady-state concentration is equal to unity), the age distribution function is found to be

$$c(t,x,\tau) = \left\{ \frac{U}{\sqrt{\pi K \tau}} \exp\left[-\frac{(x-U\tau)^2}{4K\tau}\right] - \frac{U^2 e^{Ux/K}}{2K} \operatorname{erfc}\left[\sqrt{\frac{U^2\tau}{4K}} + \sqrt{\frac{x^2}{4K\tau}}\right] \right\} \Upsilon(t-\tau) \tag{B.2}$$

The present age distribution function has a longer tail than that associated with a Dirichlet boundary condition at the incoming boundary of the domain (Figure 4).

In the presence of advection, it is unsurprising that a steady-state concentration equal to unity is reached as time progresses:

$$C_\infty(x) = \lim_{t\to\infty} \int_0^\infty c(t,x,\tau)\,d\tau$$

$$= \underbrace{\int_0^\infty \frac{U}{\sqrt{\pi K\tau}} \exp\left[-\frac{(x-U\tau)^2}{4K\tau}\right] d\tau}_{=2} - \underbrace{\int_0^\infty \frac{U^2 e^{Ux/K}}{2K} \operatorname{erfc}\left[\sqrt{\frac{U^2\tau}{4K}} + \sqrt{\frac{x^2}{4K\tau}}\right] d\tau}_{=1} = 1 \tag{B.3}$$

The age concentration tends to the following limit:

$$\alpha_\infty(x) = \lim_{t\to\infty} \int_0^\infty \tau\, c(t,x,\tau)\,d\tau$$

$$= \underbrace{\int_0^\infty \tau \frac{U}{\sqrt{\pi K\tau}} \exp\left[-\frac{(x-U\tau)^2}{4K\tau}\right] d\tau}_{=\frac{4K+2Ux}{U^2}} - \underbrace{\int_0^\infty \tau \frac{U^2 e^{Ux/K}}{2K} \operatorname{erfc}\left[\sqrt{\frac{U^2\tau}{4K}} + \sqrt{\frac{x^2}{4K\tau}}\right] d\tau}_{=\frac{3K+Ux}{U^2}} \tag{B.4a}$$

$$= \frac{K+Ux}{U^2}$$

which simplifies to

$$\alpha_\infty(x) = \frac{K+Ux}{U^2}\ . \tag{B.4b}$$

The related steady-state mean age reads

$$a_\infty(x) = \frac{\alpha_\infty(x)}{C_\infty(x)} = \frac{K}{U^2} + \frac{x}{U}\ . \tag{B.5}$$

The mean age is not zero at the upstream boundary. This is because, at this point, there are particles that are entering the domain (their age is zero) and also particles that have been in the domain for some time (their age is positive). This is why, at $x=0$, the age distribution function is not equal to Dirac delta function $\delta(\tau)$,

$$c(t,0,\tau) = \left\{ \frac{U}{\sqrt{\pi K \tau}} \exp\left[-\frac{U^2 \tau}{4K} \right] - \frac{U^2}{2K} \operatorname{erfc}\left(\sqrt{\frac{U^2 \tau}{4K}} \right) \right\} \Upsilon(t-\tau) \quad . \tag{B.6}$$

However, it is singular

$$c(t,0,\tau) \sim \frac{U}{\sqrt{\pi K \tau}} \quad , \quad \frac{U^2 \tau}{4K} \to 0 \quad . \tag{B.7}$$

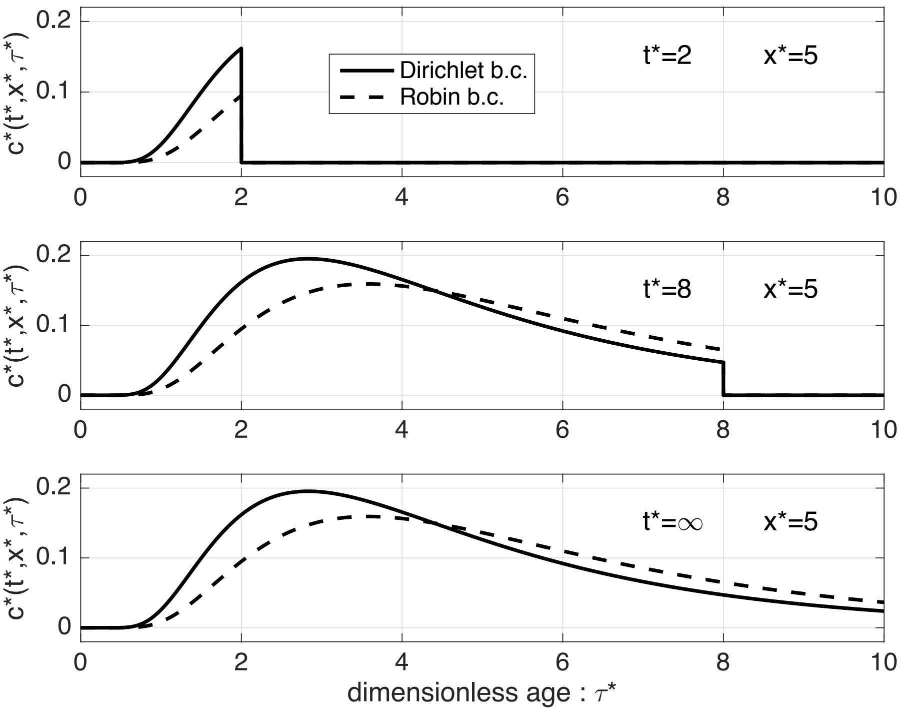

Figure 4. Illustration of age distribution functions (3) (solid line) and (B.2) (dashed line), which are obtained under Dirichlet and Robin boundary counditions at the incoming boundary of the domain ($x=0$). Dimensionless variables are used. They are identified by asterisks and are defined as follows: $t^* = U^2 t / K$, $x^* = Ux / K$, $\tau^* = U^2 \tau / K$ and $c^* = Kc / U^2$. The dimensionless age distribution functions[9] are plotted at $x^* = 5$ as functions of the age at different instants.

[9] $$\begin{cases} c^*(t^*,x^*,\tau^*) = \dfrac{x^*}{\sqrt{4\pi(\tau^*)^3}} \exp\left[-\dfrac{(x^*-\tau^*)^2}{4\tau^*} \right] \Upsilon(t^*-\tau^*) \quad \text{(Dirichlet b.c. at } x=0\text{)} \\ c^*(t^*,x^*,\tau^*) = \left\{ \dfrac{1}{\sqrt{\pi \tau^*}} \exp\left[-\dfrac{(x^*-\tau^*)^2}{4\tau^*} \right] - \dfrac{e^{x^*}}{2} \operatorname{erfc}\left[\sqrt{\dfrac{\tau^*}{4}} + \sqrt{\dfrac{(x^*)^2}{4\tau^*}} \right] \right\} \Upsilon(t^*-\tau^*) \quad \text{(Robin b.c. at } x=0\text{)} \end{cases}$$

If the velocity is zero ($U = 0$) (purely diffusive case), then the age distribution function reads

$$c(t,x,\tau) = \frac{\Phi}{\sqrt{\pi K \tau}} \exp\left(-\frac{x^2}{4K\tau}\right) \Upsilon(t-\tau) \tag{B.7}$$

and the associated tracer concentration is

$$C(t,x) = \Phi \sqrt{\frac{4t}{\pi K}} \exp\left(-\frac{x^2}{4Kt}\right) - \frac{2\Phi x}{K} \operatorname{erfc}\left(\sqrt{\frac{x^2}{4Kt}}\right) \quad . \tag{B.8}$$

Since incoming tracer flux Φ is constant, the tracer concentration grows unboundedly:

$$C(t,x) \sim \Phi \sqrt{\frac{4t}{\pi K}} - \frac{\Phi x}{K} \quad , \quad \frac{x}{\sqrt{4Kt}} \to 0 \quad . \tag{B.9}$$

On the other hand, the age concentration is

$$\alpha(t,x) = \Phi \sqrt{\frac{t}{9\pi K^3}} \, (2Kt - x^2) + \frac{\Phi x^3}{6K^2} \operatorname{erfc}\left(\sqrt{\frac{x^2}{4Kt}}\right) \quad . \tag{B.10}$$

Unsurprisingly, the mean age admits no steady-state limit:

$$a(t,x) \sim \frac{t}{3} + \sqrt{\frac{\pi x^2 t}{36K}} \quad , \quad \frac{x}{\sqrt{4Kt}} \to 0 \quad . \tag{B.11}$$

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