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International Game Theory Review
Vol. 26, No. 1 (2024) 2350016 (13 pages)

DOI: 10.1142/S0219198923500160

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Received 23 June 2023
Revised 25 September 2023
Accepted 18 October 2023
Published 6 January 2024

A cooperative product game is a transferable utility game defined by associating the product of given players' coefficient to coalitions. This idea has been introduced by Rosales, D. ([2014] Cooperative product games, unpublished mimeo, www.academia.edu/40176429) where he investigates the core. Here, we analyze a corrected definition of a product game and cover the Shapley value to which we provide an axiomatic foundation.

Keywords: Product games; core; Shapley value.

Subject Classification: C71

1. Introduction

Consider a set of players and assume that, by joining a coalition, a player multiplies the coalition's payoff by some coefficient. Associating each coalition with the product of the coefficients of its members then defines a class of transferable utility games called "product cooperative games", following a terminology introduced by Rosales [2014].

Assuming that players' coefficients are larger than one, product games are convex and have therefore a nonempty core. They belong to the class of positive games and, in the absence of null player, they are strictly convex. The Shapley value is easily obtained from the Harsanyi dividends. It allocates to each player an amount that is *proportional* to his weight.

A product game can be viewed as a production game whose technology exhibits increasing returns to scale. The Shapley value then determines a fair remuneration for the participants. Health and climate are potential fields of application of product games where players are assimilated to factors which, when combined, favor the development of certain pathologies[a] or an aggravation of the climatic situation. The Shapley value then offers a measure of the actual share of each of these factors.

The paper is organized as follows. The definition and properties of product games are covered in Sec. 2. The core and the Shapley value are the objects of Secs. 3 and 4. Section 5 is devoted to an axiomatization of the Shapley value on the class of product games based on the proportionality property. Three extensions are considered in Sec. 6. The possibility that some players are necessary while not contributing can be accommodated as well as uncertainty by assuming that weights are random variables with known distributions. The scenario of a contraction is obtained when the coefficients are smaller than one. It gives rise to concave games. In these extensions, the proportionality of the Shapley value still holds. Finally, concluding remarks are given in Sec. 7.

2. Product Games: Definition and Properties

A transferable utility game is a pair (N, v) where $N = \{1, \ldots, n\}$ is the set of players and v is a function defined on the subsets of N, where $v(S)$ is the "worth" of coalition S that measures the minimum coalition S can hope to obtain if it decides to form. By convention, $v(\emptyset) = 0$.

Notation: Finite sets are denoted by upper-case letters and lower-case letters are used to denote their cardinals: $t = |T|, s = |S|, \ldots$ For a given vector x, $x(S)$ denotes the sum of its coordinates over the set S. By convention, a sum over an empty set is equal to 0 and the product over an empty set is equal to 1. Braces are sometimes omitted for coalitions, for instance, $v(i, j)$ replaces $v(\{i, j\})$.

The central question covered by the theory of transferable utility games concerns the allocation of the worth of the grand coalition among the players. For a given game, (N, v), an allocation $x \in \mathrm{R}^n$ is efficient if $x(N) = v(N)$. Requiring in addition that $x_i \geq v(i)$ for all i is a minimum requirement. It defines the *imputation set* $I(N, v)$, the set of individually rational and efficient allocations.

We consider a situation where a coefficient is attached to each player, a_i for player i. The coefficients are assumed to be real numbers not smaller than 1: $a_i \geq 1$ for all $i \in N$.

A product game is defined by the following (normalized) characteristic function[b]:

$$v_a(S) = \prod_{i \in S} a_i - 1 \quad \text{for all } S \subset N. \tag{1}$$

We denote by $A_n = [1, \infty[^n$ the set of admissible coefficients.

[a]See for instance Martinez and Sánchez-Soriano [2021].
[b]By convention, a product over an empty set is assumed to be equal 1.

Remark 1. Rosales [2014] actually defines a product game differently:

$$v_a(S) = \prod_{i\in S} a_i \quad \text{for all } S \subset N, \quad S \neq \emptyset$$
$$= 0 \quad \text{for } S = \emptyset.$$

This would be consistent if he had assumed that $v_a(\emptyset) = 1$, in which case the two games are strategically equivalent.

Example 1. The three-player game associated to the coefficient vector $a = (2,4,6)$ is defined by $v_a = (1,3,5\,|\,7,11,23\,|\,47)$.[c]

For any given two subsets S and T, we define $\Delta_a(S,T) = v_a(S\cup T) + v_a(S\cap T) - v_a(S) - v_a(T)$.

Proposition 1. *Product games are monotonic and superadditive.*

Proof. Monotonicity is immediate: $S \subset T$ implies $v_a(S) \leq v_a(T)$. Superadditivity as well. Knowing that $a_i \geq 1$ for all $i \in N, S\cap T = \emptyset$ implies:

$$\Delta_a(S,T) = \prod_{i\in S} a_i \prod_{i\in T} a_i - \prod_{i\in S} a_i - \prod_{i\in T} a_i + 1 = \left(\prod_{i\in S} a_i - 1\right)\left(\prod_{i\in T} a_i - 1\right) \geq 0. \quad \square$$

Proposition 2. *Product games are convex.*

Proof. For any two subsets S and T, we have

$$\begin{aligned}
\Delta_a(S,T) &= \prod_{i\in S\cup T} a_i + \prod_{i\in S\cap T} a_i - \prod_{i\in S} a_i - \prod_{i\in T} a_i \\
&= \prod_{i\in S\setminus T} a_i \prod_{i\in S\cap T} a_i \prod_{i\in T\setminus S} a_i + \prod_{i\in S\cap T} a_i \\
&\quad - \prod_{i\in S\setminus T} a_i \prod_{i\in S\cap T} a_i - \prod_{i\in T\setminus S} a_i \prod_{i\in S\cap T} a_i \\
&= \left(1 + \prod_{i\in S\setminus T} a_i \prod_{i\in T\setminus S} a_i - \prod_{i\in S\setminus T} a_i - \prod_{i\in T\setminus S} a_i\right) \prod_{i\in S\cap T} a_i \\
&= (v(S\setminus T \cup T\setminus S) - v(S\setminus T) - v(T\setminus S)) \prod_{i\in S\cap T} a_i \geq 0,
\end{aligned}$$

by applying superadditivity to the disjoint sets $T\setminus S$, and $S\setminus T$. $\square$

Remark 2. Convexity implies superadditivity. The proof of Proposition 2 shows that, for product games, convexity and superadditivity are actually *equivalent* properties.

[c]Definition of a three-player game follows the coalition order (1, 2, 3 | 12, 13, 23 | 123).

Remark 3. Referring to Remark 2, Rosales proves convexity within his formulation, under the assumption that $a_i \geq 2$ for all i.

Convexity can be defined in terms of marginal contributions. The *marginal contribution* of player i to coalition S is defined by $MC_i(S) = v(S) - v(S \backslash i)$. A game is convex if marginal contributions are nondecreasing with respect to set inclusion[d]: $MC_i(S) \leq MC_i(T)$ whenever $i \in S \subset T$. For a product game, if $i \in S$, we have

$$MC_i(S) = (a_i - 1) \prod_{j \in S \backslash i} a_j.$$

Hence, the term *weight* to designate $a_i - 1$: a player contributes once his weight is positive. Knowing that $a_i \geq 1$ for all i, marginal contributions are nondecreasing:

$$v_a(T) - v_a(T \backslash i) = (a_i - 1) \prod_{j \in T \backslash i} a_j \geq (a_i - 1) \prod_{j \in S \backslash i} a_j = v_a(S) - v_a(S \backslash i). \quad (2)$$

It confirms the convexity of product games.

Two players i and j are *substitutable* if their marginal contributions coincide: $i, j \in S$ implies $MC_i(S) = MC_j(S)$ for all $S \subset N$. A player is *null* if his marginal contributions are equal to zero: $MC_i(S) = 0$ for all $S \subset N$. In a product game, i and j are *substitutable* players if $a_i = a_j$ (identical weights) and i is a null player if $a_i = 1$ (zero weight).

A game is *strictly convex* if marginal contributions are strictly increasing with coalition size: $i \in S \subsetneq T \;\Rightarrow\; MC_i(S) < MC_i(T)$. According to (2), product games are strictly convex if (and only if) no player is null: $a_i > 1$ for all $i \in N$.

The set $G(N)$ of all characteristic functions on a set N of players is equivalent to the vector space $\mathrm{R}^{2^n - 1}$. In proving the uniqueness of his value, Shapley [1953] shows that the collection of *unanimity games* $(N, u_T)_{T \subset N, T \neq \varnothing}$ defined by

$$\begin{aligned} u_T(S) &= 1 \quad \text{if } T \subset S \\ &= 0 \quad \text{if } T \not\subset S, \end{aligned}$$

forms a basis of $G(N)$: for any given function $v \in G(N)$, there exists a unique $(2^n - 1)$-dimensional vector $\alpha = (\alpha_T)_{T \subset N}$ such that the characteristic function can be written as

$$v(S) = \sum_{T \subset N} \alpha_T \, u_T(S) = \sum_{T \subset S} \alpha_T \quad \text{for all } S \subset N, \quad (3)$$

assuming $\alpha_\varnothing = 0$. The α_T are the *Harsanyi dividends* [Harsanyi, 1959]. They can be defined recursively, starting with $\alpha_\varnothing = 0$ as follows:

$$\alpha_T = v(T) - \sum_{S \subsetneq T} \alpha_S \quad \Rightarrow \quad \alpha_T = \sum_{S \subset T} (-1)^{t-s} v(S). \quad (4)$$

[d]See Shapley [1971] and Ichiishi [1981].

The dividends of a product game are given by

$$\begin{aligned}\alpha_{\{i\}} &= a_i - 1,\\ \alpha_{\{i,j\}} &= a_i a_j - a_i - a_j + 1 = (a_i - 1)(a_j - 1),\\ \alpha_{\{i,j,k\}} &= a_i a_j a_k - a_i a_j - a_i a_k - a_j a_k + a_i + a_j + a_k - 1\\ &= (a_i - 1)(a_j - 1)(a_k - 1), \ldots .\end{aligned}$$

By extension, the dividend of a coalition is equal to the product of the weights of its members:

$$\alpha_T = \prod_{i \in T} (a_i - 1). \tag{5}$$

Dividends being nonnegative, we have the following proposition.

Proposition 3. *Product games are positive games.*

Positive games form an interesting class of games on which solution concepts tend to converge: the core coincides with the set of weighted Shapley values as well as the set of all distributions of Harsanyi dividends [for details, see Dehez, 2017].

Example 1. (*continued*) The Harsanyi dividends associated to the coefficient vector $a = (2, 4, 6)$ are given by $(1, 3, 5 \,|\, 3, 5, 15 \,|\, 15)$.

3. The Core of a Product Game

Imputations are individually rational and efficient allocations. Superadditivity ensures that the imputation set of a product game is nonempty. Product games are *essential* if $a_i > 1$ for at least one i. The difference

$$\prod_{i \in N} a_i - \sum_{i \in N} a_i + n - 1$$

is indeed increasing in a_i and equal to zero if $a_i = 1$ for all i. Hence, $\sum_{i \in N} v_a(i) < v_a(N)$.

Extending rationality from individuals to coalitions leads to the concept of *core*: it is the set of efficient allocations that gives to every coalition S at least its worth $v(S)$.[e] The core of a product game is then defined by

$$C(N, a) = \left\{ x \in \mathrm{R}^n \;\middle|\; \sum_{i \in N} x_i = \prod_{i \in N} a_i - 1,\ \sum_{i \in S} x_i \geq \prod_{i \in S} a_i - 1 \quad \text{for all } S \subset N \right\}.$$

If nonempty, the core is a convex polyhedron. Convex games have a nonempty (and typically large) core and its vertices are the *marginal contributions vectors*

[e]The term "core" was introduced by Gillies [1953] in connection with the notion of stable set. It was later introduced as an independent solution concept by Shapley [1955]. See Zhao [2018] for a clarification of the respective contributions of Gillies and Shapley as to the core.

associated to players' orderings. The vector $\mu^\pi(N, v)$ associated to the ordering $\pi = (i_1, i_2, \ldots, i_n)$ is defined by

$$\mu^\pi_{i_1}(N, v) = v(i_1),$$

$$\mu^\pi_{i_k}(N, v) = v(i_1, \ldots, i_k) - v(i_1, \ldots, i_{k-1}), \quad k = 2, \ldots, n-1,$$

$$\mu^\pi_{i_n}(N, v) = v(N) - v(i_1, \ldots, i_{n-1}).$$

There are $n!$ marginal contribution vectors, not necessarily distinct, except in the case of a strictly convex game. By convexity, the core coincides with the *Weber set* defined as the convex hull of the marginal contribution vectors [Weber, 1988].

For three-player product games, they are given by the following table:

π	1	2	3
123	$a_1 - 1 = 1$	$a_1(a_2 - 1) = 6$	$a_1 a_2(a_3 - 1) = 40$
132	$a_1 - 1 = 1$	$a_1 a_3(a_2 - 1) = 36$	$a_1(a_3 - 1) = 10$
213	$a_2(a_1 - 1) = 4$	$a_2 - 1 = 3$	$a_1 a_2(a_3 - 1) = 40$
231	$a_2 a_3(a_1 - 1) = 24$	$a_2 - 1 = 3$	$a_2(a_3 - 1) = 20$
312	$a_3(a_1 - 1) = 6$	$a_1 a_3(a_2 - 1) = 36$	$a_3 - 1 = 5$
321	$a_2 a_3(a_1 - 1) = 24$	$a_3(a_2 - 1) = 18$	$a_3 - 1 = 5$

Example 1. (*continued*) The game associated to the vector $a = (2, 4, 6)$ is strictly convex and, referring to the above table, its core vertices are given by (1, 6, 40), (1, 36, 10), (4, 3, 40), (24, 3, 20), (6, 36, 5) and (24, 18, 5).

4. The Shapley Value of a Product Game

Shapley [1953] proves that there is a unique efficient and additive allocation rule that allocates zero to null players and identical amounts to substitutable players. There are several formulations of the Shapley value. Here, we retain the formulation based on Harsanyi dividends.

Following Harsanyi, [1959], α_T is the *dividend* accruing to coalition T once all sub-coalitions have received their dividends. By (3), the sum of all dividends is equal to $v(N)$. An allocation can then be obtained by distributing the dividends of each coalition among its members.

Harsanyi shows that the Shapley value gives to each player the sum of the *per capita* dividend of the coalitions of which he is a member:

$$SV_i(N, v) = \sum_{T \subset N:\ i \in T} \frac{1}{t} \alpha_T(N, v)(i = 1, \ldots, n). \tag{6}$$

Introducing (5) into (6), we obtain the Shapley value of a product game:

$$SV_i(N, v_a) = \sum_{T \subset N:\, i \in T} \frac{1}{t} \prod_{j \in T} (a_j - 1) = (a_i - 1) \sum_{T \subset N:\, i \in T} \frac{1}{t} \prod_{j \in T \setminus i} (a_j - 1). \tag{7}$$

In the three-player case, the Shapley value is given by

$$\begin{aligned} SV_1(N, v_a) &= \frac{a_1 - 1}{6}(2 + a_2 + a_3 + 2a_2a_3), \\ SV_2(N, v_a) &= \frac{a_2 - 1}{6}(2 + a_1 + a_3 + 2a_1a_3), \\ SV_3(N, v_a) &= \frac{a_3 - 1}{6}(2 + a_1 + a_2 + 2a_1a_2). \end{aligned} \tag{8}$$

The average of the $n!$ marginal contribution vectors (whether distinct or not) is another way to define the Shapley value. Hence, convexity of product games ensures that their Shapley values define core allocations. Moreover, in the absence of null players, strict convexity applies and the Shapley value then coincides with the *simple* average of the marginal contribution vectors. If the order in which players act is given or constrained, the natural allocation is the average of the marginal contribution vectors corresponding to these orders. Clearly, marginal contributions being increasing with coalition size, the best is to act last while the worst is to act first.

Looking at (7), we observe that $SV_i(N, a)$ is *proportional* to his weight:

$$SV_i(N, v_a) = f(a_{-i})(a_i - 1) \;\;\Rightarrow\;\; \frac{SV_i(N, v_a)}{a_i - 1} = f(a_{-i}),$$

where $f : \mathrm{R}^{n-1} \to \mathrm{R}$ is a *player-independent* and *symmetric* function given by[f]

$$f(z) = 1 + \sum_{T \subset \{1,\ldots,n-1\})} \frac{1}{t+1} \prod_{j \in T} (z_j - 1). \tag{9}$$

Example 1. (*continued*) The Shapley value associated to the vector of coefficients $a = (2, 4, 6)$ is given by (10, 17, 20). If player 1 is the last to act while the other two players act in any order, the allocation is the average of the vectors corresponding to the orders 231 and 321. It is given by (24, 10.5, 12.5).

We know that the Shapley value satisfies symmetry: it allocates identical amount to players with identical weights. The converse is actually true. Indeed, we have

$$SV_i(N, p) - SV_j(N, p) = (a_i - a_j) \left[1 + \sum_{T \subset N \colon i, j \notin T} \frac{1}{t+1} \prod_{h \in T} (a_h - 1) \right],$$

where the term between brackets is positive. Hence, we have the following proposition.

Proposition 4. *The Shapley value of a product game satisfies the following property*:

$$SV_i(N, v_a) = SV_j(N, v_a) \quad \textit{if and only if} \quad a_i = a_j.$$

[f] A function is symmetric if permuting its arguments leaves its value unchanged.

Remark 4. We observe that the differences between the amounts allocated by the Shapley value are themselves proportional to the differences in weights.

5. Axiomatization of the Shapley Value on Product Games

We cannot rely on additivity simply because the sum of two product games is not a product game. It turns out that proportionality, together with efficiency and symmetry, characterize a unique allocation rule *on the class of product games.* It then must be the Shapley value since it satisfies these three properties. Given a player set N and a vector $a \in A_n$, we only retain the rules $\varphi : (N, v_a) \to \mathrm{R}^n$ that satisfy the following three properties.

A.1 *Efficiency*: $\sum_{i \in N} \varphi_i(N, v_a) = \prod_{i \in N} a_i - 1$.

A.2 *Symmetry*: $a_i = a_j \;\Rightarrow\; \varphi_i(N, v_a) = \varphi_j(N, v_a)$.

A.3 *Proportionality*: for all $i \in N, \varphi_i(N, v_a) = \lambda_i(a_i - 1)$ where λ_i is independent of a_i.

Proposition 5. *The Shapley value of a product game is the unique allocation rule that satisfies A.*1*–A.*3*.*

Proof. Consider a game (N, v_a) and an allocation rule φ satisfying A.1–A.3. According to A.3, there exist n functions $f_1, \ldots, f_n$ such that $\varphi_i(N, v_a) = f_i(a_{-i})(a_i - 1)$ for all i. We first show that, under A.2, the proportionality functions are independent of players' identities. We then show that under A.1, the common proportionality function f is uniquely defined and symmetric. As a result, φ must be the Shapley value and f is given by (9).

Consider the players $i \in \{1, \ldots, n-1\}$ and $j = i+1$. For any given vector $a \in A_n$ such that $a_i = a_j$, we have $a_{-i} = a_{-j} = b$ and $f_i(b) = f_j(b)$ by A.2. Independence then follows:

$$f_i(z_1, \ldots, z_{n-1}) = f(z_1, \ldots, z_{n-1}) \text{ for all } (z_1, \ldots, z_{n-1}) \in A_{n-1} \text{ and all } i \in N.$$

We now proceed iteratively to show that the function f is uniquely defined, using A.1 and A.3. We start by evaluating $f(z, \ldots, z)$ for some $z > 1$:

$$n\, f(z, \ldots, z)\,(z - 1) = z^n - 1$$
$$\Rightarrow f(z, \ldots, z) = \frac{z^n - 1}{n\,(z - 1)} = \frac{1}{n}(z^{n-1} + z^{n-2} + \cdots + z + 1). \tag{10}$$

For $n = 2, f(z) = (z + 1)/2$. For $n \geq 3$, we proceed to the evaluation of $f(z_1, z, \ldots, z)$ for some z_1 and $z > 1$:

$$f(z, \ldots, z)(z_1 - 1) + (n - 1) f(z_1, z, \ldots, z)(z - 1) = z_1 z^{n-1} - 1$$

or

$$f(z_1, z, \ldots, z) = \frac{z_1\, z^{n-1} - 1 - f(z, \ldots, z)\,(z_1 - 1)}{(n - 1)\,(z - 1)}. \tag{11}$$

We evaluate $f(z, z_2, \ldots, z)$ similarly. We then proceed to the evaluation of $f(z_1, z_2, z, \ldots, z)$ for z_1, z_2 and $z > 1$:

$$\begin{aligned} &f(z, z_2, \ldots, z)(z_1 - 1) + f(z_1, z, \ldots, z)(z_2 - 1) \\ &\quad + (n-2) f(z_1, z_2, z, \ldots, z)(z - 1) = z_1 z_2 \, z^{n-2} - 1 \end{aligned}$$

or

$$\begin{aligned} &f(z_1, z_2, z, \ldots, z) \\ &\quad = \frac{z_1 z_2 z^{n-2} - 1 - f(z, z_2, \ldots, z)(z_1 - 1) - f(z_1, z, \ldots, z)(z_2 - 1)}{(n-2)(z-1)}. \end{aligned}$$

And so on, until the evaluation of the complete function $f(z_1, z_2, z_3, \ldots, z_{n-1})$ on A_{n-1}. □

Note that the resulting function is symmetric: all along the proof, the relative positions of z_1, $z_2, \ldots$ are arbitrary.
Here is an illustration for different values of n. For $n = 2$, we know that $f(z) = (z+1)/2$. Hence,

$$SV_i(N, v_a) = f(a_j)(a_i - 1) = \frac{a_j + 1}{2}(a_i - 1), \quad i, j \in \{1, 2\}, \; i \neq j.$$

For $n = 3$, using (10) and (11), we have successively $f(z, z) = (z^2 + z + 1)/3$ and

$$\begin{aligned} f(z_1, z_2) &= \frac{z_1 z_2^2 - 1 - (z_2^2 + z_2 + 1)(z_1 - 1)/3}{2(z_2 - 1)} = \frac{1}{6}(2 + z_1 + z_2 + 2 z_1 \, z_2) \\ &= 1 + \frac{1}{2}(z_1 - 1) + \frac{1}{2}(z_2 - 1) + \frac{1}{3}(z_1 - 1)(z_2 - 1). \end{aligned}$$

We then retrieve (8): $SV_i(N, v_a) = f(a_j, a_k)(a_i - 1), i, j, k \in \{1, 2, 3\}, i \neq j, i \neq k, j \neq k$.

6. Extensions

6.1. *Case where some players do not directly contribute*

There may be situations where the presence of some players is necessary while they do not contribute to the multiplicative process. We can think of the production economy imagined by Shapley and Shubik [1967] where a player owns the technology but does not participate in the production.

Let $M \subsetneq N$ denote the subset of the necessary players whose coefficients equal one: $a_i = 1$ for all $i \in M$. The associated game is then described by the following characteristic function:

$$v_a(S) = \prod_{i \in S} a_i - 1 \quad \text{if } M \subset S$$
$$= 0 \quad \text{if } M \not\subset S.$$

Harsanyi dividends are similar except for coalitions that do not contain the necessary players:

$$\alpha_T = \prod_{j \in T \setminus M} (a_j - 1) \quad \text{if } M \subset T \quad \text{and} \quad t \geq m+1$$
$$= 0 \quad \text{otherwise.}$$

Using (6), the Shapley value is given by

$$SV_i(N, v_a) = \sum_{\substack{T \subset N: \\ T \supset M \\ t \geq m+1}} \frac{1}{t} \prod_{j \in T \setminus M} (a_j - 1) \quad \text{for all}\, i \in M,$$

$$SV_i(N, v_a) = \sum_{\substack{T \subset N: \\ T \supset M \\ i \in T}} \frac{1}{t} \prod_{j \in T \setminus M} (a_j - 1)$$

$$= (a_i - 1) \sum_{\substack{T \subset N: \\ T \supset M \\ i \in T}} \frac{1}{t} \prod_{j \in T \setminus (M \cup i)} (a_j - 1) \;\; \text{for } i \in N \setminus M.$$

We observe that proportionality holds for the productive players.

Example 2. Consider a five-player situation where the first two players are necessary but do not contribute while the last three are productive, with coefficients equal to 3, 4 and 5, respectively. The associated product game is defined by

$$v_a(1,2,3) = 2, \quad v_a(1,2,3,4) = 11,$$
$$v_a(1,2,4) = 3, \quad v_a(1,2,3,5) = 14,$$
$$v_a(1,2,5) = 4, \quad v_a(1,2,4,5) = 19, \quad v_a(1,2,3,4,5) = 59,$$

and $v_a(S) = 0$ for all remaining coalitions, those that do not contain the first two players. The Shapley value is given by (14.3, 14.3, 8.97, 10.3, 11.13). About half of the total payoff goes to the necessary players. In the symmetric case where productive players' weights are all equal, say three, the total product equals 26 and the Shapley value allocates 13.2 to the necessary players. The remaining amount is equally divided between the active players.

6.2. *Introducing uncertainty in weights*

Consider a situation where players' coefficients are random variables with known distributions. More precisely, we assign to each player a random variable with a finite mean and the interval $[1, +\infty[$ as support. We denote by X_i the random variable assigned to player i.

In the case where the random variables $X_1, \ldots, X_n$ are independent, the corresponding game is a product game whose coefficients are expected coefficients:

$$w(S) = \mathrm{E}\left[\prod_{i\in S} X_i\right] - 1.$$

Dropping the assumption of independence, we could instead rely on the *geometric* expectation. For an arbitrary random variable Y, it is defined by

$$\mathrm{E_G}[Y] = e^{\mathrm{E}[\log(Y)]}.$$

It is an alternative measure of the central tendencies of random variables defined on $]0, +\infty[$ that is used in different contexts.[g] The geometric expectation of a discrete random variable Y defined by $\{(r_1, q_1), \ldots, (r_m, q_m)\}$, where $r_h > 0$ for all h and $\sum q_h = 1$, is simply given by

$$\mathrm{E_G}[Y] = \prod_{h=1}^{m} r_h^{q_h},$$

hence the qualifier "geometric". Among the properties of the geometric expectation, three of them are worth mentioning:

$$\mathrm{E_G}[\alpha Y] = \alpha \mathrm{E_G}[Y] \quad \text{for all}\, \alpha > 0,$$

$$\mathrm{E_G}[Y] \leq \mathrm{E}[Y],$$

$$\mathrm{E_G}\left[\prod Y_i\right] = \prod \mathrm{E_G}[Y_i].$$

The first one is immediate, the second one is a direct consequence of Jensen inequality while the third one applies whether the random variables Y_i are independent or not. Hence, in the absence of independence, a product game can be defined by

$$w(S) = \mathrm{E_G}\left[\prod_{i\in S} X_i\right] - 1 = \prod_{i\in S} \mathrm{E_G}\left[X_i\right] - 1,$$

where the coefficients are the geometric expectations $\mathrm{E_G}[X_i]$.

Example 3. Consider a three-player situation where the coefficients of player i equals 2 with probability p_i and 4 with probability $1-p_i$, where $p = (1/2, 1/3, 2/3)$. The arithmetic and geometric means are respectively, given by (3, 10/3, 8/3) and $(2\sqrt{2},\ 2\sqrt[3]{4},\ 2\sqrt[3]{2})$. Using geometric means, the associated game is defined by the $(1.8, 2.2, 1.5 \,|\, 8, 6.1, 7 \,|\, 22)$ and its Shapley value is given by (7.22, 7.83, 6.58).

[g]There is a large literature on the geometric mean and its applications, in particular in finance and insurance. See for instance Jasiulewicz and Kordecki [2016].

6.3. *Case of a contraction*

Throughout the paper, weights have been assumed to be nonnegative. If instead we assume that $0 < a_i < 1$ for all $i \in N$, the game definition must be modified:

$$v_a(S) = 1 - \prod_{i \in S} a_i \quad \text{for all}\, S \subset N.$$

It defines a concave (and thereby subadditive) *cost sharing* game. Marginal contributions are indeed decreasing with coalition size:

$$v_a(S) - v_a(S\backslash i) = (1 - a_i) \prod_{j \in S\backslash i} a_j.$$

Dividends associated to coalitions with an *even* number of members are negative:

$$\alpha_T = (-1)^{t-1} \prod_{i \in T} (1 - a_i).$$

However, being a core allocation, the Shapley value is nonnegative. Indeed, in general, core allocations y associated to a cost sharing game (N, v) satisfy $y_i \geq v(N) - v(N\backslash i)$ for all $i \in N$ and we have $v_a(N) - v_a(N\backslash i) \geq 0$.

Example 4. The three-player game associated to the coefficient vector $a = (0.3, 0.6, 0.8)$ is defined by $v = (0.7, 0.4, 0.2 \,|\, 0.82, 0.76, 0.52 \,|\, 0.856)$. Its Shapley value is given by (0.509, 0.239, 0.109).

7. Concluding Remarks

A similar multiplicative structure appears in the probability games introduced by Hou *et al.* [2018] and later investigated by Dehez [2023]. The characteristic function associates to each coalition the probability that it succeeds in reaching a given target. In this framework, the Shapley value appears to be proportional as well.

About proportionality, it is remarkable that this simple property, together with efficiency and symmetry, characterizes the Shapley value on the class of product games: it is the property that differentiates the Shapley value from other efficient and symmetric allocation rules. Additivity does the same in the class of superadditive games. The coefficients identify the players: knowledge of the vector a is all that is needed to compute an allocation and, by Proposition 4, $SV_i(N, a) = SV_j(N, a)$ if and only if $a_i = a_j$. In addition, proportionality can be *equivalently* written as

$$a, b \in A_n \text{ such that } a_j = b_j \quad \text{for all}\, j \neq i \;\Rightarrow\; \frac{\varphi_i(N, a)}{\varphi_i(N, b)} = \frac{a_i - 1}{b_i - 1}.$$

Acknowledgments

The author is grateful to Françoise Forges, Reinhard John, Alan Miller, Hervé Moulin, Pier Mario Pacini and two referees for their useful comments and suggestions on earlier drafts. Computations have been made using the website *TUGlab*

created by Miguel Angel Mirás Calvo and Estela Sánchez Rodríguez of the University of Vigo and the Mathematica package *TuGames* created by Holger Meinhardt of the Karlsruhe Institute of Technology.

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References

Dehez, P. [2017] On Harsanyi dividends and asymmetric values, *Int. Game Theory Rev.* **19**, 1750012 [reprinted in Petrosian, L. A. and Yeung, D. W. K. (eds.) *Game Theoretic Analysis* (World Scientific Publishing, Singapore), pp. 523–558].

Dehez, P. [2023] Sharing a collective probability of success, *Math. Soc. Sci.* **123**, 122–127.

Gillies, D. B. [1953] *Some Theorems on n-Person Games*, Ph.D. Thesis, Princeton University.

Harsanyi, J.C. [1959] A bargaining model for the cooperative n-person game. Tucker, A.W., Luce, D.R. (eds.), *Contributions to the theory of games* (Vol. IV), Annals of Mathematics Study 40, Princeton University Press, pp. 325–355.

Hou, D., Xu, G., Sun, P. and Driessen, T. [2018] The Shapley value for the probability game, *Oper. Res. Lett.* **46**, 457–461.

Ichiishi, T. [1981] Super-modularity. Applications to convex games and to the greedy algorithm, *Journal of Economic Theory* **25**, 283–286.

Jasiulewicz, H. and Kordecki, W. [2016] Multiplicative parameters and estimators; applications in economics and finance, *Ann. Oper. Res.* **238**, 299–313.

Martinez, R. and Sánchez-Soriano, J. [2021] Mathematical indices for the influence of risk factors on the lethality of a disease, *Math. Biol.* **83**(6–7), 74.

Rosales, D. [2014] Cooperative product games, unpublished mimeo, www.academia.edu/40176429.

Shapley, L. S. [1953] A value for n-person games, in *Contributions to the Theory of Games II*, Annals of Mathematics Studies, Vol. 24, eds. Kuhn, H. & Tucker, A. W. (Princeton University Press, Princeton), pp. 307–317. Reproduced in Roth, A. E. (ed.) [1988] *The Shapley Value: Essays in Honor of Lloyd S. Shapley* (Cambridge University Press, Cambridge), pp. 31–40.

Shapley, L. S. [1955] Markets as cooperative games, RAND Discussion Paper P-629.

Shapley, L. S. [1971] Cores of convex games, *Int. J. Game Theory* **1**, 11–26.

Shapley L. S. and Shubik, M. [1967] Ownership and the production function, *Q. J. Econ.* **81**, 88–111.

Weber, R. J. [1988] Probabilistic values for games, in *The Shapley Value: Essays in Honor of Lloyd S. Shapley*, ed. Roth, A. E. (Cambridge University Press, Cambridge), pp. 101–119.

Zhao, J. [2018] Three little-known and yet still significant contributions of Lloyd Shapley, *Games Econ. Behav.* **108**, 592–599.