Abstract

Discontinuous Galerkin Method for 1D shallow water models

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In contrast to the complex multidimensional dynamics of deltas, lakes and estuaries, rivers generally exhibit a simpler dynamics. It can often be dealt with satisfactorily by having recourse to 1D models. The cross section integrated Saint-Venant equations are widely used in river modeling and engineering applications. In order to ensure the mass conservation the conservative form of the equations is preferred, in this case the flux and source terms may be formulated in several ways. It is seen, however, that not all of them lead to stable and accurate numerical results. The choice of the convenient unknown and intermediate variables allows getting an optimal stability with less adjustments. In nature, rivers may be connected to several streams. Regardless of the complex dynamics of those junction points, in the case of a subcritical flow the simple 1D model is able to predict accurately the flow rate division in the different branches. It is to be dealt with carefully that under the influence of tides the flow direction can be reversed, which mean that a junction can be at the same time a bifurcation and confluence depending on the flow direction. The aforementioned modules are implemented in the framework of a discontinuous Galerkin finite-element model, namely the Second-generation Louvain-la-Neuve Ice-ocean Model (SLIM, www.slim-ocean.be). Validation is performed by running the model in idealised configurations. Then, the river-lakes-delta continuum of the Mahakam River (Borneo, Indonesia) is simulated in a realistic manner.

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Introduction

River's dynamics can often be dealt with satisfactorily by having recourse to 1D models[3]. The physical equations expressing mass and momentum conservation can be written in several equivalent ways. Regarding the difference in treatment of the source and flux terms, this equivalence is lost in discrete form. The momentum conservation equation can include several intermediate variables (the water depth, the free surface level, ...). The variable involved in flux and source terms should be carefully defined, aiming at a more naturally stable formulation. Another issue to deal with in real rivers arise from the field data treatment, the interpolation of the discrete measured geometry data should be dealt with carefully. The discontinuous Galerkin finite element (DG) method is evaluated for modelling 1D flows in rivers, with three different formulations in order to reveal their strengths and weaknesses, with a focus on data interpolation options.

Numerical method

$$\boxed{\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F(U)}}{\partial x} = \mathbf{S(U)}}$$

Discontinuous Galerkin method :

for every Φ_i and each Ω_e in Ω:

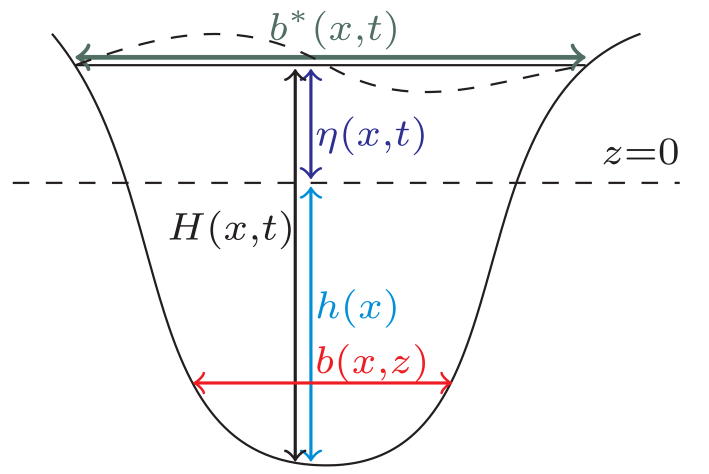
$$\int_{\Omega_e} U_t \Phi_i dx - \int_{\Omega_e} F.\nabla\Phi_i dx + [F\Phi_i]_{\partial\Omega_e} = \int_{\Omega_e} S\Phi_i dx$$

Linear approximation on local cells:

Optimal stability+ high order accuracy

Necessity of ensuring coherence between boundary & volumetric flux terms

Notations



Formulation I

Introducing the water level η, the Saint-Venant equations can be written: ($S = n^2 \frac{|Q|Qb^{*4/3}}{A^{7/3}}$)

$$\underbrace{\frac{\partial}{\partial t}\begin{bmatrix} A \\ Q \end{bmatrix}}_{\mathbf{U}} + \frac{\partial}{\partial x}\underbrace{\begin{bmatrix} Q \\ \frac{Q^2}{A} + gA\eta \end{bmatrix}}_{\mathbf{F(U)}} = \underbrace{\begin{bmatrix} 0 \\ g\eta\frac{\partial A}{\partial x} - gS \end{bmatrix}}_{\mathbf{S(U)}}$$

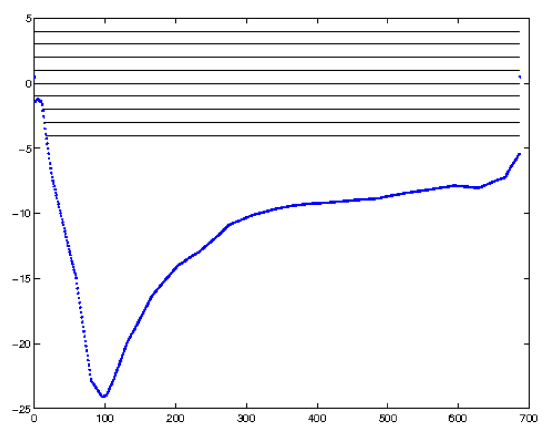
The free surface level η appears in three different terms $\int_{\Omega_e} F.\nabla\Phi_i dx$, $[F\Phi_i]_{\partial\Omega_e}$ and $\int_{\Omega_e} S\Phi_i dx$. Thus, a small shift in the choice of the reference level will lead to a different discrete form even when the water depth H is same, which can lead to instabilities.

What is the optimal reference level?

Formulation II

The free surface elevation can be replaced by the water depth ($H = \eta + h$) that is independent from the reference level :

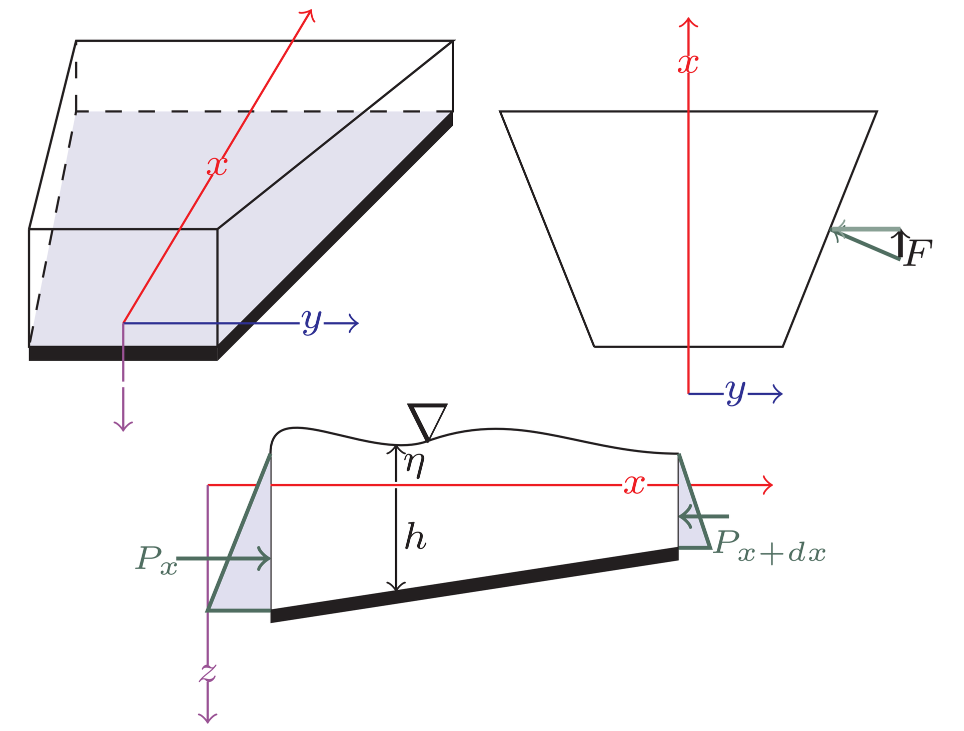
$$\frac{\partial}{\partial t}\begin{bmatrix} A \\ Q \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} Q \\ \frac{Q^2}{A} + gAH \end{bmatrix} = \begin{bmatrix} 0 \\ gH\frac{\partial A}{\partial x} + gA\frac{dh}{dx} - gS \end{bmatrix}$$



Cross section profiles such as that depicted above frequently occur, with $H_{mean} << H_{max}$. **Which bed level h and water depth H to use for a smooth source term?**

- defined to the deepest point:
  $\Longrightarrow$ high variation in source term
- Mean values:
  $\Longrightarrow$ what would be the optimal choice?

Final Formulation



Based on Newton's second law of motion:

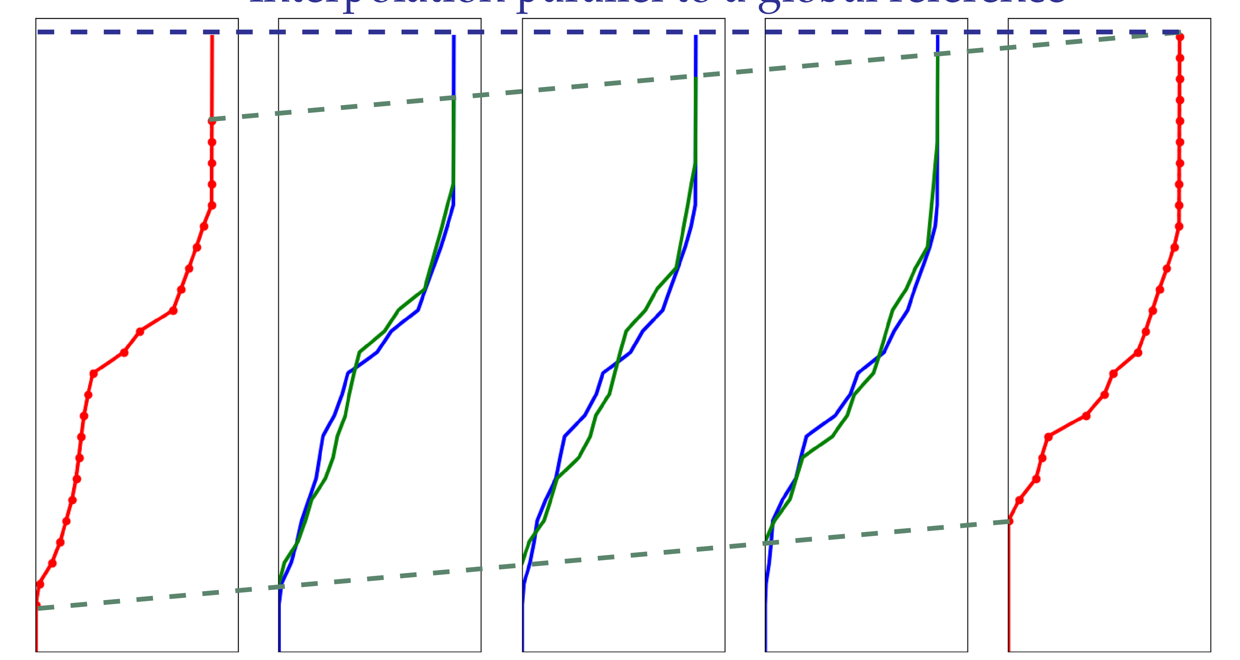
$$\frac{\partial}{\partial t}\begin{bmatrix} A \\ Q \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} Q \\ \frac{Q^2}{A} + \frac{P}{\rho} \end{bmatrix} = \begin{bmatrix} 0 \\ gA\frac{\partial h}{\partial x} - gS + \frac{F}{\rho} \end{bmatrix}$$

Pressure integrals are less sensitive neither to local geometry variation nor to the choice of the reference level:

$$P = \rho g \int_0^H (H - \xi) b(x,\xi) d\xi$$

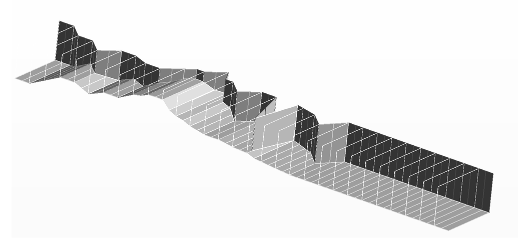
$$F = \rho g \int_0^H (H - \xi) \frac{\partial b(x,\xi)}{\partial x} d\xi$$

Width interpolation

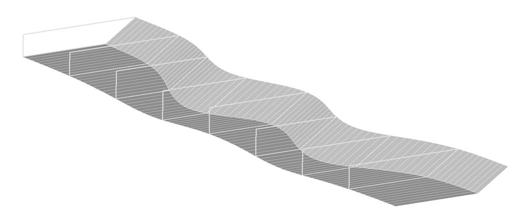


Validation test cases

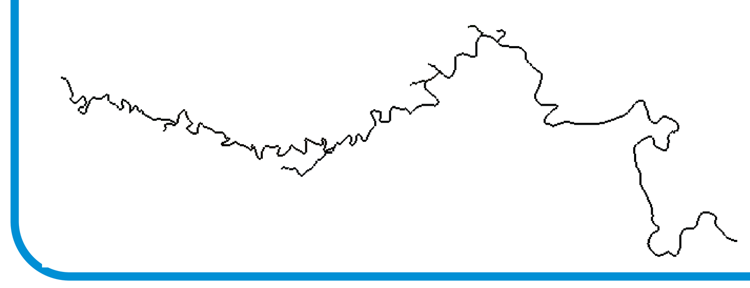
Lake-at-rest[1]



steady-state-flow[2]



Mahakam 1D



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Current Work

We are in the process of coupling the 1D and 2D modules of SLIM (www.slim-ocean.be), in order to study the Mahakam river-delta-sea continuum (Borneo Island, Indonesia).