On the interaction between mutual insurance and the risk-taking behaviour of farm households

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Abstract

Empirical evidence on developing countries shows on the one hand that rich farm-households invest more in technology adoption and are higher risk takers than poor households. On the other hand, however, they are shown to be less vulnerable to income shocks than poor farmers. This paper provides a rationale for these observations. Risk averse, heterogeneously endowed, agents non-cooperatively decide on their level of subscription to mutual insurance and on the degree of individual crop risk they take. Rich households take more risks and fully subscribe to mutual insurance. Although mutual insurance allows all households to cope with idiosyncratic shocks, the risk-taking behavior of rich households increases the covariate component of poor households' income variance through mutual insurance, deterring the participation of the poor. These poor households in turn opt for safer but less productive crop plans.

JEL Codes: O12, O13, O17, O33

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1 Introduction

In developing countries, the ability of farm households to cope with risk is a key determinant of their daily livelihood. Poverty, food security and insurance issues are intrinsically linked to one another. This chapter elaborates on these issues, highlighting the relationship between self- and mutual insurance, and the way this relationship is affected by household wealth.

Self-insurance mechanisms are implemented before crop risks are realized. We can refer to them as ex-ante insurance. It covers income diversification, technology adoption and production choices or any individual means of mitigating risk ex ante. Kurosaki & Fafchamps (2002) provide empirical evidence that production choices belong to the set of risk-coping strategies in contexts of market

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imperfections. In line with our motivation, it has been empirically observed that household wealth does affect technology adoption, which tends to be low among the poorest households (Dercon & Christiaensen (2011)).

On the other hand, once crops, and potential shocks, are realized, consumption relies on the degree of ex-post insurance, which is mainly provided by households' involvement in mutual insurance mechanisms. Empirical evidence also suggests that poor households' consumption paths seem to be more affected by idiosyncratic shocks (Jalan & Ravallion (1999); Morduch (1995)). The general lack of access to ex post insurance in rural areas results from market failure and is well documented in the literature. To smooth consumption ex post, rural households can rely on informal insurance transfers. Due to moral hazard and imperfect commitment, full informal insurance has been rejected theoretically and empirically. On the one hand, from a theoretical perspective, the impact of limited commitment is widely acknowledged (Coate & Ravallion (1993); Ligon et al. (2002); Murgai et al. (2002)) while on the other hand, the hypothesis of full income pooling is often empirically rejected (Townsend (1994); Jalan & Ravallion (1999); Hoogeveen (2002); Hoogeveen (2002); Morduch (1995)). Households may also have invested in assets with buffer purposes in the past and use them as another means of smoothing consumption. It is however strongly dependent on household wealth.

One interpretation of the two above-mentioned stylized facts is that, considering the inability to smooth consumption ex post as given, poor farmers are reluctant to adopt high-return risky production strategies. However, we argue that a satisfactory explanation to this phenomenon can only be provided if one studies simultaneously both ex-ante and ex-post insurance decisions.

We present a model with agents heterogeneously endowed in terms of buffer stocks, who non-cooperatively decide on their risk-coping strategies, namely their level of subscription to the mutual insurance scheme and their production choices. Our theoretical contribution thus consists in considering both self- and mutual insurance as endogenous. The intuition behind our main result is the following: rich households tend to adopt risky production strategies that impact on the variance of the income pool. This gives an incentive to poorer farmers to partially withdraw from community level risk pooling. In addition, the model accounts for imperfect risk pooling without assuming neither information asymmetries nor imperfect commitment. Population heterogeneity is highlighted as a major determinant of incomplete community sharing.

Section 2 presents our assumptions regarding the risk environment and the way we model ex ante self-insurance strategies. In Section 3, the mutual insurance scheme is described. The Nash Equilibrium of the model is then computed in sections 4 and 5. Section 6 is aimed at providing a benchmark case in which population heterogeneity is removed. Section 7 concludes.

2 The model

In order to introduce the concepts gradually, we start by representing the farm households' objective in "autarky", that is, in the absence of informal insurance. The latter will be introduced in the next subsection.

2.1 The case of autarky

The economy is composed of a population of farm households, or agents, of measure one noted H. Agents are heterogenously endowed in wealth w, which follows a distribution $F(w)$. Incomes / production outputs are risky, and farm households can affect the shape of this risk by choosing their production plans / risk-taking strategies. The household's risk strategy $\sigma \in R_+$ captures the range of production choices that a farm household can take and highlights the following tradeoff: more risk-taking raises the expected income / value of the harvest, but also increases its variance.[1] Formally, the agent's income writes

$$Y = \mu(\sigma) + I + J, \tag{1}$$

where $\mu(\sigma)$ is an increasing, concave function of the risk strategy σ, and I and J are zero-mean random variables representing idiosyncratic and covariate shocks, respectively. Since the means of I and J are nil, the agent's expected income is simply

$$E(Y;\sigma) = \mu(\sigma). \tag{2}$$

Covariate shocks, such as drought or crop diseases, are assumed to be perfectly positively correlated across agents:

$$Corr(J_i, J_{i'}) = 1 \ \forall i, i' \in H. \tag{3}$$

Idiosyncratic shocks I, such as the birth, disease, or death of a household member, are independent across individuals as well as from J:

$$I_i \perp I_{i'} \ \forall i \neq i' \in H, \tag{4}$$

$$X_i \perp J_{i'} \ \forall i, i' \in H. \tag{5}$$

As explained above, the variance of these shocks is increasing in σ. More precisely, these variances are noted

$$VAR(I;\sigma) = (1-j)\,\sigma^2, \tag{6}$$

$$VAR(J;\sigma) = j\sigma^2, \tag{7}$$

[1] Considering a given activity, the farmer can adopt technologies with different implications in terms of risk and return, choosing between traditional and improved seeds, for instance. From a broader point of view, the household may specialize in a given production strategy and either gain from economies of scope or spread the allocation of its land and labour resources over a portfolio of activities in order to diversify income sources.

where $j \in [0;1]$ is a constant which represents the fraction of total risk which is joint, or covariate. Indeed, due to the independence between both types of shocks, the variance of income is

$$\begin{aligned} VAR(Y;\sigma) &= VAR(I+J;\sigma) = VAR(I;\sigma) + VAR(J;\sigma) \\ &= \sigma^2. \end{aligned}$$

The expected utility of farm households writes $Eu(c)$, where consumption c is based on wealth w and income Y:

$$c = w + Y.$$

Agents are risk averse, so that $u(c)$ is concave and has decreasing absolute risk aversion (DARA). Let $A(c) = -\frac{u''(c)}{u'(c)}$ define the coefficient of absolute risk aversion, with $A' < 0$ and $A'' > 0$ as is the case in a wide range of utility functions, such as the hyperbolic absolute risk aversion (HARA), which encompasses a large class of functions such as CARA, CRRA and quadratic utility functions. The expected utility derived from period-2 consumption can be rewritten in the following way:

$$Eu(c) = u(C),$$

where C is the certainty equivalent of the lottery c:

$$\begin{aligned} C &= E(c) - \Pi \\ &= w + \mu(\sigma) - \Pi, \end{aligned}$$

where Π is the risk premium:

$$\begin{aligned} \Pi &\approx A(E(c))\frac{VAR(c)}{2} \\ &\approx A(w+\mu(\sigma))\frac{\sigma^2}{2}. \end{aligned}$$

Having introduced the main concepts of autarky, we can now solve the household's problem. The timing is the following: the household chooses its production plan / risk-taking strategy σ, then shocks and income levels are realized and consumption takes place.

Proposition 1 *In the absence of mutual insurance, richer households engage in higher risk-taking if and only if*

$$\frac{\partial^2 \Pi}{\partial \sigma \partial w} < 0.$$

This condition is equivalent to the condition that the elasticity of the marginal risk aversion A' to risk-taking σ is smaller than the elasticity of $VAR(Y)$ to risk-taking σ:

$$\left|\epsilon_{A',\mu}\right| \epsilon_{\mu,\sigma} < \epsilon_{VAR(Y),\sigma} = 2.$$

The formal proof of this proposition is provided in the Appendix. Intuitively, this condition states that the marginal growth of the risk premium induced by risk-taking is lower for the rich than for the poor. This is the case if Briefly, we need to determine the sign of $\frac{d\sigma_i^*}{dw}$, whicj by the implicit function theorem writes:

$$\frac{d\sigma_i^*}{dw} = -\frac{\frac{\partial^2 C}{\partial\sigma\partial w}}{\frac{\partial^2 C}{\partial\sigma^2}},$$

where $\frac{\partial^2 C}{\partial\sigma^2} < 0$ by the second order condition. Therefore, the necessary and sufficient condition for richer households to take higher risks is that the cross derivative of expected utility with respect to risk-taking and wealth is positive. Formally,

$$\begin{aligned} \frac{\partial^2 C}{\partial\sigma\partial w} &> 0 \Leftrightarrow \frac{\partial^2 \Pi}{\partial\sigma\partial w} < 0 \\ &\Leftrightarrow \frac{\mu A''}{-A'}\frac{\sigma\mu'}{\mu} \equiv \left|\epsilon_{A',\mu}\right| \epsilon_{\mu,\sigma} < \epsilon_{VAR(Y),\sigma}, \end{aligned}$$

where $\epsilon_{VAR(Y),\sigma} \equiv \frac{\sigma(2\sigma)}{\sigma^2} = 2$. The condition stated in Proposition 1 is relatively mild, as shows the following corollary applied to CRRA preferences.

Corollary 2 *With CRRA preferences, richer households engage in higher risk-taking if the elasticity of expected income $\mu(\sigma)$ is smaller than one:*

$$\epsilon_{\mu,\sigma} \equiv \frac{\sigma\mu'}{\mu} \leq 1.$$

With CRRA preferences, $\left|\epsilon_{A',\mu}\right| = 2\frac{\mu}{w+\mu} < 2$, so that a sufficient condition is $\epsilon_{\mu,\sigma} \leq 1$. It is reasonable to assume that doubling the risk-taking σ does not result in more than double the expect return.

Let us now introduce mutual insurance.

2.2 The Mutual Insurance Scheme

In addition to risk-taking and precautionary savings, households also have the possibility to mitigate shocks by sharing risks within the community.[2] The timing of the game becomes the following. Households choose their levels of precautionary savings, risk-taking strategy and involvement in mutual insurance. Then, shocks are realized and transfers inside the community are made.

We model risk sharing as the contribution to a virtual community income pool. Then some proportions of this pool are reallocated to households, where these proportions depend on the shares of pre-transfer income that households commited to provide to the pool in the first period. This mutual insurance mechanism has the property of being actuarially fair in the sense that on average, households are neither net contributors nor net receivers in terms of transfers. In other

[2] In order to focus this discussion on the relation between self and mutual insurance, the enforceability of mutual insurance agreements is assumed satisfied.

words, average transfers are zero, for each household and independently of their subscription level. This design allows us to focus on pure risk-sharing by excluding redistribution mechanisms.

Each household, identified by its initial wealth w, commits to provide a share α_w of its income Y_w, so that the contribution to the pool of household $i \in H$ writes

$$\begin{aligned} T_{Oi} &= \alpha_i Y_i && (8) \\ &= \alpha_i \left(\mu\left(\sigma_i\right) + I_i + J_i\right). && (9) \end{aligned}$$

On the other hand, the household receives a transfer from the income pool, noted

$$T_{Ii} = r_i P, \tag{10}$$

where P is the value of the pool, that is, the sum of all agent's contributions:

$$P = \int \alpha_w Y_w dF(w).$$

For the insurance mechanism to be actuarially fair, this share r_i must be such that the average contribution to the pool of each agent equals the average transfer he receives: for all $i \in H$,

$$\begin{aligned} E\left(T_{Oi}\right) &= E\left(T_{Ii}\right) \iff \\ r_i &= \frac{\alpha_i E\left(Y_i\right)}{E\left(P\right)} \iff && (11) \\ r_i &= \frac{\alpha_i}{\overline{\alpha}} \frac{\mu\left(\sigma_i\right)}{\widetilde{\mu}}, && (12) \end{aligned}$$

where $\overline{\alpha}$ is the community's average contribution to the pool and $\widetilde{\mu}$ is the community's weighted average of expected incomes, with the relative contributions $\frac{\alpha_i}{\overline{\alpha}}$ as weights:

$$\begin{aligned} \overline{\alpha} &\equiv \int \alpha_w dF(w) && (13) \\ \widetilde{\mu} &\equiv \int \frac{\alpha_w}{\overline{\alpha}} \mu\left(\sigma_w\right) dF(w). && (14) \end{aligned}$$

The share of the income pool to which a household is entitled depends on two factors. On the one hand, it is an increasing function of the household's relative contribution to the income pool, $\frac{\alpha_i}{\overline{\alpha}}$. The higher the subscription to mutual insurance as compared to the others' subscriptions, the higher the transfer received. On the other hand, s_i is proportional to the ratio between the household's expected income, $\mu\left(\sigma_i\right)$ and the community's weighted average $\widetilde{\mu}$. In other words, the more agressive the risk-taking strategy, the higher the share of the pool an agent receives.

The insurance scheme defines the post-transfer income

$$\begin{aligned} X_i &= Y_i - T_{Oi} + T_{Ii} && (15) \\ &= \left(1 - \alpha_i\right) Y_i + s_i P. && (16) \end{aligned}$$

The actual consumption level is the sum of wealth / precautionary savings and this post-transfer income X_i: $c = w + X_i$. In order to derive the agent's utility under the mutual insurance scheme, let us therefore analyze the mean and variance of the post-transfer income X_i.

Lemma 3 *The mean and variance of the post-transfer income X_i write*

$$\begin{aligned} E(X_i) &= \mu(\sigma_i) \\ VAR(X_i) &\equiv \Sigma_i = VAR(X_{Ii}) + VAR(X_{Ji}), \end{aligned}$$

where

$$\begin{aligned} VAR(X_{Ii}) &= (1-j)(1-\alpha_i)^2 \sigma_i^2, \\ VAR(X_{Ji}) &= j\left[(1-\alpha_i)\sigma_i + \alpha_i \widetilde{\sigma}\frac{\mu(\sigma_i)}{\widetilde{\mu}}\right]^2 \\ &= j\sigma_i^2\left[(1-\alpha_i) + \alpha_i\left[\frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}}\right]\right]^2, \end{aligned}$$

where $\widetilde{\mu}$ is defined in (14) and

$$\widetilde{\sigma} = \int \frac{\alpha_w}{\overline{\alpha}}\sigma_w dF(w). \tag{17}$$

The (relatively complex) proof of this lemma is provided in the Appendix. On the one hand, all agents benefit from mutual insurance via a reduction in the idiosyncratic part of the income variance:

$$\frac{\partial VAR(X_{Ii})}{\partial \alpha_i} = -2(1-j)(1-\alpha_i)\sigma_i^2 < 0.$$

On the other hand, interestingly, the effect of α_i on the covariate part of the variance depends on the agent's degree of risk taking. This covariate component of the post-transfer income variance is a function of household's own production choices but also on the risk taking behaviours of all the other participants to the insurance scheme. In this sense, any agent who subscribes to the risk pool exposes himself to the variance level induced by the others' production decisions. In the other direction, the more a household engages in risk sharing, the more it impacts on the risk composition of the income pool. This mechanism is a key to the model and gives rise to strategic interaction. Let $h(\sigma) \equiv \mu(\sigma)/\sigma$, with $h'(\sigma) = \frac{\mu'\sigma - \mu}{\sigma^2} < 0$ since $\frac{\mu'\sigma}{\mu} \equiv \epsilon_{\mu,\sigma} < 1$, and let $\widehat{\sigma} \equiv h^{-1}(\widetilde{\mu}/\widetilde{\sigma})$.

Lemma 4 *The covariate part of the variance of post-transfer income $VAR(X_{Ji})$ increases with α_i if and only if $\sigma_i < \widehat{\sigma}$.*

Proof. The sign of the derivative of $VAR(X_{Ji})$ with respect to α_i is the same as the sign of

$$\begin{aligned} \frac{\partial\left[(1-\alpha_i) + \alpha_i\left[\frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}}\right]\right]}{\partial \alpha_i} &= -1 + \frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}} \\ &> 0 \iff \frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}} > 1 \\ &> 0 \iff \sigma_i < \widehat{\sigma}. \end{aligned}$$

■

This lemma implies that individuals taking more risks than $\widehat{\sigma}$ benefit from mutual insurance via a reduction of both idiosyncratic and covariate variances. In contrast, individuals taking less risks than $\widehat{\sigma}$ face a tradeoff using α between reducing idiosyncratic variance and increasing covariate variance. This adverse effect on covariate variance is due to a negative externality generated by agents taking agressive risks, which pollute the pool via covariate shocks.

Let us now introduce the main proposition of the paper, which rationalizes the twofold stylized facts that rich farm-households are higher risk-takers whereas they are less vulnerable to income shocks thanks to a higher participation in mutual insurance.

Proposition 5 *Rich agents take more risk than poorer agents if and only if*

$$\frac{\partial^2 \Pi}{\partial \sigma \partial w} < 0 \iff \left|\epsilon_{A',\mu}\right| \epsilon_{\mu,\sigma} > \epsilon_{\Sigma,\sigma},$$

where $\epsilon_{\Sigma,\sigma} < 2$ if and only if $\epsilon_{\mu,\sigma} < 1$. The effect of wealth on participation to mutual insurance has the same sign as that on risk-taking if and only if $\epsilon_{\mu,\sigma} < 1$.
In this case, rich agents whose risk-taking σ_i^* is larger than $\widehat{\sigma}$ fully subscribe to mutual insurance ($\alpha_i^* = 1$), whereas poorer agents, whose risk-taking is smaller than $\widehat{\sigma}$, only partially subscribe to mutual insurance ($\alpha_i^* < 1$).

Corollary 6 *If $\epsilon_{\mu,\sigma} < 1$, if rich agents take more risks than poor ones in autarky, so will they under mutual insurance, to which they will also participate more than poor agents.*

Corollary 7 *With CRRA preferences, $\epsilon_{\mu,\sigma} < 1$ is a sufficient condition in both cases of autarky and mutual insurance for the rich to take more risks and to participate more to mutual insurance (when it exists).*

Proof. The proof of Proposition 5 is structured as follows. We first identify the equilibrium conditions and discuss how their are related to each other. Then, we apply the multivariate version of the implicit function theorem to this pair of conditions to identify the sign of the effect of household wealth on the two endogenous variables, namely σ and α.
Let us start by analyzing the first order condition with respect to σ_i:

$$\begin{aligned} \frac{\partial u(C)}{\partial \sigma} &= u'(C)\frac{\partial C}{\partial \sigma} = 0 \iff \\ \frac{\partial C}{\partial \sigma} &= \mu' - \frac{1}{2}\left(A\frac{\partial \Sigma_i}{\partial \sigma_i} + A'\mu'\Sigma_i\right) = 0. \end{aligned}$$

As in the autarkic situation, a marginal increase in individual risk-taking results in two effects: on the one hand, an increase of the income mean , which results in an increase in the expected consumption level and a decrease in the absolute risk aversion, and on the other hand, an increase in the volatility of consumption:

$$\begin{aligned}\frac{\partial \Sigma_i}{\partial \sigma_i} &= 2\sigma \left[(1-j) \overbrace{\left[(1-\alpha)^2\right]}^{+} + j \overbrace{\left[\left(1-\alpha+\alpha\frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}}\right) \left(1-\alpha+\alpha\frac{\widetilde{\sigma}}{\widetilde{\mu}}\frac{\partial\mu(\sigma)}{\partial\sigma}\right)\right]}^{+} \right] \\ &> 0 \ \forall \sigma, \alpha.\end{aligned}$$

The above equation shows that an increase in risk-taking unambiguously increases both the idiosyncratic and covariate parts of income variance. As will be shown below, the household balances both effects on expected income and risk premium depending on its wealth level w and its subscription to mutual insurance.

Let us now analyze the agent's reaction function in terms of participation to mutual insurance.

$$\frac{\partial u(C)}{\partial \alpha_i} = -u'(C)\frac{A}{2}\frac{\partial \Sigma_i}{\partial \alpha_i},$$

where the effect on the variance is decomposed into an idiosyncratic and a covariate part:

$$\frac{\partial \Sigma_i}{\partial \alpha_i} = -2\left[(1-j)(1-\alpha_i)\sigma_i^2 - j\left[-1+\frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}}\right]\left[(1-\alpha_i)+\alpha_i\frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}}\right]\right].$$

Therefore, agents taking more risk than $\widehat{\sigma}$, that is, agents for whom $\frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}} < 1$, are at a corner solution in terms of participation to mutual insurance, whereas agents taking lower risks than $\widehat{\sigma}$ choose an interior level of α_i^*:

$$\begin{aligned}\alpha_i^* &= 1 - \frac{j\frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}}\left(\frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}}-1\right)}{j\left(\frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}}-1\right)^2+\sigma_i^2(1-j)} \\ &< 1 \iff \frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}} > 1 \iff \sigma_i < \widehat{\sigma} \\ &= 1 \iff \sigma_i^* \geq \widehat{\sigma}.\end{aligned}$$

It remains to prove that w has a positive effect on both α_i^* and σ_i^* for both agents at the corner and at the interior solution on α_i^*. The effect of w for agents at a corner solution on α_i^* is simply obtained by the single-equation implicit function theorem applied to the first order condition on σ. This is due to the fact that since α_i^* is at a corner, a marginal variation in w has no impact on α and only σ will react to w. This analysis is the same as that discussed in the proof of Proposition 1 for the case of autarky, where $VAR(Y_i)$ just needs to be replaced by Σ_i, but both are positively affected by σ and unaffected by w. Let us finish this proof by analyzing the effect of w for agents who are at an interior α_i^*. The pair of equilibrium conditions for (σ_i^*, α_i^*) is that

$$\begin{pmatrix} \frac{\partial C}{\partial \sigma} \\ \frac{\partial C}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By the bivariate version of the implicit function theorem on this pair of equations, we have that

$$\begin{aligned}\begin{pmatrix}\frac{d\sigma_i^*}{dw}\\ \frac{d\alpha_i^*}{dw}\end{pmatrix} &= -\begin{pmatrix}\frac{\partial^2 C}{\partial\sigma^2} & \frac{\partial^2 C}{\partial\sigma\partial\alpha}\\ \frac{\partial^2 C}{\partial\alpha\partial\sigma} & \frac{\partial^2 C}{\partial\alpha^2}\end{pmatrix}^{-1}\begin{pmatrix}\frac{\partial^2 C}{\partial\sigma\partial w}\\ \frac{\partial^2 C}{\partial\alpha\partial w}\end{pmatrix} &(18)\\ &= \frac{A\Omega^{-1}}{4}\left(A'\frac{\partial\Sigma}{\partial\sigma}+A''\Sigma\mu'\right)\begin{pmatrix}-\frac{\partial^2\Sigma}{\partial\alpha^2}\\ \frac{\partial^2\Sigma}{\partial\alpha\partial\sigma}\end{pmatrix}, &(19)\end{aligned}$$

where Ω is the determinant of the Hessian of C, and is positive for C to be at a maximum in (σ^*, α^*).[3] We therefore need to determine the signs of three elements, namely $\frac{\partial^2\Sigma}{\partial\alpha^2}$, $\frac{\partial^2\Sigma}{\partial\alpha\partial\sigma}$ and $\left(A'\frac{\partial\Sigma}{\partial\sigma}+A''\Sigma\mu'\right)$. Since we are analyzing the interior solution in α, the second order condition in α must be satisfied. This is the case if and only if $\frac{\partial^2\Sigma}{\partial\alpha^2}>0$:

$$\begin{aligned}\frac{\partial^2 u(C)}{\partial\alpha_i^2} &= u''(C)\left(\frac{A}{2}\overbrace{\frac{\partial\Sigma_i}{\partial\alpha_i}}^{=0}\right)^2-\frac{A}{2}u'(C)\frac{\partial^2\Sigma_i}{\partial\alpha_i^2}\\ &= -\frac{A}{2}u'(C)\frac{\partial^2\Sigma_i}{\partial\alpha_i^2}\\ &< 0 \iff \frac{\partial^2\Sigma_i}{\partial\alpha_i^2}>0.^4\end{aligned}$$

Making use of the fact that at the interior solution, $\frac{\partial\Sigma}{\partial\alpha}=0$, one can show that the cross derivative is always negative:

$$\begin{aligned}\frac{\partial^2\Sigma}{\partial\alpha\partial\sigma} &= \frac{\partial^2\Sigma}{\partial\alpha\partial\sigma}-\frac{2}{\sigma}\overbrace{\frac{\partial\Sigma}{\partial\alpha}}^{=0}\\ &= -2j\frac{\mu(\sigma_i)}{\widetilde{\mu}/\widetilde{\sigma}}\left(1-\frac{\sigma\mu'}{\mu}\right)\left(1+2\alpha\left(\frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}}-1\right)\right)\\ &< 0 \iff \frac{\sigma\mu'}{\mu}=\epsilon_{\mu,\sigma}<1.\end{aligned}$$

We therefore know that the impact of w has the same sign on both σ_i^* and α_i^* if and only if $\epsilon_{\mu,\sigma}<1$. If this is the case, this sign is determined by $-\left(A'\frac{\partial\Sigma}{\partial\sigma}+A''\Sigma\mu'\right)$. Equivalently,

$$\begin{pmatrix}\frac{d\sigma_i^*}{dw}\\ \frac{d\alpha_i^*}{dw}\end{pmatrix}>\begin{pmatrix}0\\0\end{pmatrix}\iff\frac{\mu A''}{-A'}\frac{\mu'}{\sigma\mu}<\frac{\sigma\frac{\partial\Sigma}{\partial\sigma}}{\Sigma}\iff\left|\epsilon_{A',\mu}\right|\epsilon_{\mu,\sigma}>\epsilon_{\Sigma,\sigma},$$

where

$$\begin{aligned}\epsilon_{\Sigma,\sigma} &= 2-\frac{2j\alpha\frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}}\sigma^2\left((1-\alpha)+\alpha\frac{\mu(\sigma_i)/\sigma_i}{\widetilde{\mu}/\widetilde{\sigma}}\right)\left(1-\frac{\sigma\frac{\partial\mu}{\partial\sigma}}{\mu(\sigma)}\right)}{\Sigma}\\ &< 2\iff\epsilon_{\mu,\sigma}<1.\end{aligned}$$

■

[3] The proof of this result is provided in the appendix.

3 Conclusion

This paper has been motivated by a twofold empirical observation: on the one hand, poorer farm households are reluctant to adopt modern technologies characterized by high risk and return levels. On the other hand, poor farmers tend to suffer from a higher exposure to idiosyncratic shocks. Unequal endowments in terms of buffer stocks only partially account for the latter observation since it appears that, somewhat paradoxically, poor households are less involved in informal insurance schemes. Our model, in which households non-cooperatively decide on self- and mutual insurance strategies, replicates these stylized facts. To the best of our knowledge, this model is the first to deal with ex ante and ex post risk-coping strategies in a unified framework. The following results can be highlighted.

We have shown that a pattern of differentiated production choices emerges as the outcome of population heterogeneity. At equilibrium, rich agents, who take higher risks than the average, have an incentive to fully participate in mutual insurance. Indeed, by subscribing to the community income pool, they free ride on the self insurance costs incurred by others, thereby reducing the covariate fraction of their income variance. On the contrary, poor agents, who take relatively cautious production decisions are faced with a trade-off since participation in mutual insurance, while reducing the extent of idiosyncratic shocks, entails an increase in the covariate fraction of their income variance. As a consequence, they partially withdraw from risk sharing mechanisms on a voluntarily basis.

Our framework offers a new rationale for incomplete risk pooling. Indeed, while classical moral hazard and imperfect commitment are ruled out, the model highlights population heterogeneity as a potential source of externalities. Partial subscription to mutual insurance follows.

To conclude this paper, our theoretical prediction is pessimistic with respect to poverty traps issues. Indeed, provided poverty is directly linked to the ability to cope with risk ex post, the model predicts that the lower is this ability, the less the household is likely to be involved in risk pooling. As a consequence, ex post means of insurance are weak on the aggregate and poor household have to rely on ex ante self-insurance mechanisms. Hence, their production decisions are highly affected by risk-coping considerations that make them reluctant to adopt the more profitable technologies.

References

Coate, S. & Ravallion, M. (1993). Reciprocity without commitment, characterization and performance of informal insurance arrangements. *Journal of Development Economics,*, 40, 1–24.

Dercon, S. & Christiaensen, L. (2011). Consumption risk, technology adoption, and poverty traps : evidence from ethiopia. *Journal of Development Economics*, 94(1).

Hoogeveen, H. (2002). Evidence on informal insurance in rural zimabwe. *Journal of African Economies*, 11(2), 249–278.

Jalan, J. & Ravallion, M. (1999). Are the poor less well insured? evidence on vulnerability to income risk in rural china. *Journal of Development Economics*, 58, 61–81.

Kurosaki, T. & Fafchamps, M. (2002). Insurance market efficiency and crop choices in pakistan. *Journal of Development Economics*, 67(2), 419–453.

Ligon, E., Thomas, J., & Worrall, T. S. (2002). Informal insurance arrangements with limited commitment: Theory and evidence from village economies. *The Review of Economic Studies*, 69, 209–244.

Morduch, J. (1995). Income smoothing and consumption smoothing. *Journal of Economic Perspectives*, 9(3), 103–14.

Murgai, R., Winters, P., Sadoulet, E., & Janvry, A. d. (2002). Localized and incomplete mutual insurance. *Journal of Development Economics*, 67(2), 245–274.

Townsend, R. M. (1994). Risk and insurance in village india. *Econometrica*, 62, 539–591.

4 Appendix

4.1 Proof of Proposition 1

Maximizing $Eu(c)$ boils down to

$$\begin{aligned} \max_{\sigma} C &= E\left(c\right) - \Pi \\ &\approx w + \mu\left(\sigma\right) - A\left(w + \mu\left(\sigma\right)\right) \frac{\sigma^2}{2}. \end{aligned}$$

The first order condition writes[5]

$$\begin{aligned} \frac{\partial E(c)}{\partial \sigma} - \frac{\partial \Pi}{\partial \sigma} &= 0, \\ \mu' - \left(A'\mu' \frac{\sigma^2}{2} + A\sigma \right) &= 0. \end{aligned}$$

The second order condition states that

$$\frac{\partial^2 C}{\partial \sigma^2} = \mu'' - \frac{1}{2}\sigma^2 \left(\mu'' A' + \mu^{2\prime} A'' \right) - A - 2\sigma A'\mu' < 0.$$

Let us now assess the effect of household wealth on risk taking.To this end, we simply need to apply the implicit function theorem in the following way:

$$\begin{aligned} \frac{d\sigma}{dw} &= -\frac{\frac{\partial^2 C}{\partial \sigma \partial w}}{\frac{\partial^2 C}{\partial \sigma^2}} > 0 \Leftrightarrow \\ \frac{\partial^2 C}{\partial \sigma \partial w} &= -\frac{1}{2}\left(2\sigma A' + A''\sigma^2 \mu' \right) > 0 \\ &\Leftrightarrow \frac{\mu A''}{-A'} \frac{\sigma \mu'}{\mu} < 2 \\ &\Leftrightarrow \epsilon_{A',\mu}\epsilon_{\mu,\sigma} < \epsilon_{VAR(Y),\sigma}. \end{aligned}$$

With CRRA preferences, $\frac{\mu A''(s+\mu)}{-A'(s+\mu)} = 2\frac{\mu}{s+\mu}$, so that the condition holds for $\frac{\sigma\mu'}{\mu} = \epsilon_{\mu,\sigma} \leq 1$.

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[5] The second order condition states that

$$\begin{aligned} \left(A'\mu'' \right) \frac{\sigma^2}{2} + A'\mu'\sigma &> 0 \\ 2 &< -\frac{\sigma \mu''}{\mu'} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 C}{\partial \sigma^2} &= \mu'' - \left(\left((\mu')^2 A'' + A'\mu'' \right) \frac{\sigma^2}{2} + A'\mu'\sigma \right) - \left(A'\mu'\sigma + A \right) \\ &= \mu'' - \left((\mu')^2 A'' + A'\mu'' \right) \frac{\sigma^2}{2} - 2A'\mu'\sigma - A \\ &= \mu'' - \left(A'\mu'' \right) \frac{\sigma^2}{2} - A - \sigma\mu' \left(A''\mu' \frac{\sigma}{2} + 2A' \right) \end{aligned}$$

2 ways to ensure the negativity of $\frac{\partial^2 C}{\partial \sigma^2}$:

A single sufficient condition (stronger than Assumption 1):

$$-2A'\mu'\sigma - A < 0 \Leftrightarrow -\frac{\mu A'}{A}\frac{\mu'\sigma}{\mu} < \frac{1}{2} \Leftrightarrow |\epsilon_{A,\mu}|\, \epsilon_{\mu,\sigma} < 1/2$$

A pair of sufficient conditions:

$$\begin{aligned} A'\mu'\sigma + A &> 0 \Leftrightarrow -\frac{\mu A'}{A}\frac{\mu'\sigma}{\mu} < 1 \Leftrightarrow |\epsilon_{A,\mu}|\, \epsilon_{\mu,\sigma} < 1 \\ \left(A'\mu'' \right)\frac{\sigma^2}{2} + A'\mu'\sigma &> 0 \Leftrightarrow -\frac{\sigma\mu''}{\mu'} > 2 \Leftrightarrow \epsilon_{\mu',\sigma} > 2 \end{aligned}$$

The pair of conditions is intuitive, the condition on $\epsilon_{\mu',\sigma}$ imposes a sufficient degree of concavity of μ.

4.2 Proof of Lemma 1

Proving the first part of the lemma is straightforward. Indeed, since the mutual insurance mechanism is actuarially fair,

$$\begin{aligned} E(X_i) &= E(Y_i) + E(R_i - T_i) \\ &= E(Y_i) = \mu(\sigma_i). \end{aligned}$$

Proving the second part of the Lemma is more requiring. Making use of equation (16), we can write

$$VAR(X_i) = (1-\alpha_i)^2 VAR(Y_i) + s_i^2 VAR(P) + 2(1-\alpha_i) s_i COV(Y_i, P). \tag{20}$$

To facilitate its reading, we decompose this proof in three parts by computing each of the three terms on the right hand side of (20).

4.2.1 The First Term of (20)

By (**??**), the first term simply writes

$$(1-\alpha_i)^2 VAR(Y_i) = (1-\alpha_i)^2 \sigma^2. \tag{21}$$

4.2.2 The Second Term of (20)

With respect to the second term, by the definition of the variance and making use of (**??**), the fact that X and Z are assumed to be zero mean random variables implies

$$VAR(P) = E\left[(P - E(P))^2\right] = E\left[\left(\int \alpha_\rho (X_\rho + Z_\rho) dF(\rho)\right)^2\right].$$

Expanding the right hand side gives

$$\begin{aligned} VAR(P) &= VAR\left(\int \alpha_\rho X_\rho dF(\rho)\right) + VAR\left(\int \alpha_\rho Z_\rho dF(\rho)\right) \\ &\quad +2COV\left(\int \alpha_\rho X_\rho dF(\rho), \int \alpha_\rho Z_\rho dF(\rho)\right). \end{aligned} \tag{22}$$

In order to compute (22), the following preliminary result is needed:

Lemma 8 *Suppose a vector of n random variables. If these variables are perfectly correlated, then the variance of any linear combination of these variables equals the squared linear combination of their standard deviations. Formally,*

$$\begin{aligned} Corr(Xi, Xj) &= 1, \forall i, j \in (1, 2, ...n) \\ &\Downarrow \\ VAR\left(\sum_{i=1}^{n} \alpha_i X_i\right) &= \left(\sum_{i=1}^{n} \alpha_i stdev(X_i)\right)^2, \forall (\alpha_1, \alpha_2, ...\alpha_n) \in R^n. \end{aligned}$$

Proof. *First note that, as a consequence of perfect correlation,*

$$COV(\alpha_i Xi, \alpha_j Xj) = \sqrt{VAR(\alpha_i Xi)\, VAR(\alpha_j Xj)}$$

Then, making use of this result, we obtain what follows:

$$\begin{aligned} VAR\left(\sum_{i=1}^{n} \alpha_i X_i\right) &= VAR(\alpha_n X_n) + VAR\left(\sum_{i=1}^{n-1} \alpha_i X_i\right) \\ &\quad +2COV\left(\alpha_n X_n, \sum_{i=1}^{n-1} \alpha_i X_i\right) \\ &= VAR(\alpha_n X_n) + VAR\left(\sum_{i=1}^{n-1} \alpha_i X_i\right) \\ &\quad +2\sqrt{VAR(\alpha_n X_n)\, VAR\left(\sum_{i=1}^{n-1} \alpha_i X_i\right)} \\ &= \left(stdev(\alpha_n X_n) + stdev\left(\sum_{i=1}^{n-1} \alpha_i X_i\right)\right)^2, \end{aligned}$$

Incidentally,

$$\begin{aligned} &VAR\left(\sum_{i=1}^{n-1} \alpha_i X_i\right) = \left(stdev(\alpha_{n-1} X_{n-1}) + stdev\left(\sum_{i=1}^{n-2} \alpha_i X_i\right)\right)^2 \\ &\Leftrightarrow stdev\left(\sum_{i=1}^{n-1} \alpha_i X_i\right) = stdev(\alpha_{n-1} X_{n-1}) + stdev\left(\sum_{i=1}^{n-2} \alpha_i X_i\right). \end{aligned}$$

Applying recursively, we end up with

$$VAR\left(\sum_{i=1}^{n} \alpha_i X_i\right) = \left(\sum_{i=1}^{n} \alpha_i stdev(X_i)\right)^2.$$

*By (**??**), lemma 1 applies to the first term on the right hand side of (22):*

$$\begin{aligned} VAR\left(\int \alpha_\rho X_\rho dF(\rho)\right) &= \left(\int \alpha_\rho stdev(X_\rho)\, dF(\rho)\right)^2 \\ &= \left(\int \alpha_\rho \sqrt{j} \sigma_\rho dF(\rho)\right)^2 \\ &= j\left[E(\alpha_\rho \sigma_\rho)\right]^2. \end{aligned}$$

*where use has been made of (**??**). Second, since the Law of Large Numbers applies to the case of i.n.i.d. (independent but not identically distributed) random variables, the second term on the right hand side of (22) vanishes: On the one hand, note that we are indeed in a case of i.n.i.d.*

*variables given that the variables are independent to one another (**??**) and taking into account that the variance and hence the distribution of* $\alpha_\rho Z_\rho$ *are allowed to differ between individuals. Indeed,*

$$VAR\left(\alpha_\rho Z_\rho\right) = \alpha_\rho^2 \left(1-j\right)^2 \sigma_\rho^2,$$

and both α_ρ *and* Z_ρ *belong to the players' strategy space. They may therefore diverge from one individual to the other. On the other hand, the fact that players belong to a continuum allows to consider* $\int \alpha_\rho Z_\rho dF(\rho)$ *as the average of a sample whose size tends to infinity. This last element completes the proof.* ■

Finally, the third term on the right hand side of (22) is also 0 by (**??**). It follows that

$$s_i^2 VAR\left(P\right) = j s_i^2 \left[E_\rho\left(\alpha_\rho \sigma_\rho\right)\right]^2. \tag{23}$$

4.2.3 The Third Term of (20)

By definition,

$$COV\left(Y_i, P\right) = E\left(Y_i P\right) - E\left(Y_i\right) E\left(P\right),$$

where

$$E\left(Y_i P\right) = E\left(Y_i \int \alpha_\rho Y_\rho dF(\rho)\right).$$

Making use of (**??**) and distributing the expectation operator, we obtain

$$\begin{aligned} E\left(Y_i P\right) &= \int \alpha_\rho \left(\mu_i\right) \left(\mu_\rho + E\left(X_\rho\right) + E\left(Z_\rho\right)\right) dF\left(\rho\right) \\ &\quad + E\left(X_i\right) \int \alpha_\rho \left(\mu_\rho\right) dF\left(\rho\right) + \int \alpha_\rho E\left(X_i X_\rho\right) dF\left(\rho\right) \\ &\quad + \int \alpha_\rho E\left(X_i Z_\rho\right) dF\left(\rho\right) + E\left(Z_i\right) \int \alpha_\rho \left(\mu_\rho\right) dF\left(\rho\right) \\ &\quad + \int \alpha_\rho E\left(Z_i X_\rho\right) dF\left(\rho\right) + \int \alpha_\rho E\left(Z_i Z_\rho\right) dF\left(\rho\right). \end{aligned}$$

Given that X and Z have zero means, we are left with

$$\begin{aligned} E\left(Y_i P\right) &= \left(\mu_i\right) \int \alpha_\rho \left(\mu_\rho\right) dF\left(\rho\right) + \int \alpha_\rho E\left(X_i X_\rho\right) dF\left(\rho\right) \\ &\quad + \int \alpha_\rho E\left(X_i Z_\rho\right) dF\left(\rho\right) + \int \alpha_\rho E\left(Z_i X_\rho\right) dF\left(\rho\right) + \int \alpha_\rho E\left(Z_i Z_\rho\right) dF\left(\rho\right). \end{aligned} \tag{24}$$

By the same argument, the mean cross products in (24) are covariances. Since the variables are perfectly correlated,

$$\frac{E\left(X_i X_\rho\right)}{\sqrt{VAR\left(X_i\right) VAR\left(X_\rho\right)}} = 1 \Leftrightarrow E\left(X_i X_\rho\right) = j \sigma_i \sigma_\rho.$$

It follows from the independence assumption (**??**) that the third and fourth terms are zero. Finally, the last term on the right hand side of (24) is also zero given that, on the one hand, the idiosyncratic shocks are independent between individuals (**??**) and that, on the other hand, the weight of individual i tends to zero in the integral. In other words, individual i is atomistic in the continuum. As a consequence of the above arguments,

$$\begin{aligned} E\left(Y_i P\right) &= \left(\mu_i\right)\int \alpha_\rho \mu_\rho dF\left(\rho\right) + \int \alpha_\rho j \sigma_i \sigma_\rho dF\left(\rho\right) \\ &= \left(\mu_i\right)\int \alpha_\rho \mu_\rho dF\left(\rho\right) + j\sigma_i \int \alpha_\rho \sigma_\rho dF\left(\rho\right) \end{aligned}$$

In addition,

$$E\left(Y_i\right) E\left(P\right) = \mu_i \int \alpha_\rho \mu_\rho dF\left(\rho\right),$$

hence,

$$COV\left(Y_i, P\right) = j\sigma_i E_\rho\left(\alpha_\rho \sigma_\rho\right).$$

The third and final term of (20) is therefore

$$2\left(1-\alpha_i\right) s_i COV\left(Y_i, P\right) = 2j\left(1-\alpha_i\right) s_i \sigma_i E_\rho\left(\alpha_\rho \sigma_\rho\right). \tag{25}$$

As a final step, we are now able to sum the three terms (21), (23) and (25) in order to find the variance of R_i:

$$\begin{aligned} VAR\left(R_i\right) &= \left(1-\alpha_i\right)^2 \sigma_i^2 + j s_i^2 \left[E_\rho\left(\alpha_\rho \sigma_\rho\right)\right]^2 + 2j\left(1-\alpha_i\right) s_i \sigma_i E_\rho\left(\alpha_\rho \sigma_\rho\right) \\ &= \left(1-j\right)\left(1-\alpha_i\right)^2 \sigma_i^2 + j\left[\left(1-\alpha_i\right)\sigma_i + s_i\left(E_\rho\left(\alpha_\rho \sigma_\rho\right)\right)\right]^2. \end{aligned}$$

$$\left(1-j\right)\left(1-\alpha_i\right)^2 \sigma_i^2 + j\left(1-\alpha_i\right)^2 \sigma_i^2 + j s_i^2 \left[E_\rho\left(\alpha_\rho \sigma_\rho\right)\right]^2 + 2j\left(1-\alpha_i\right) s_i \sigma_i E_\rho\left(\alpha_\rho \sigma_\rho\right)$$

Plugging (11) in the latter expression, one can easily find equation (**??**).

$$j\left(1-\alpha_i\right)^2 \sigma_i^2 + j s_i^2 \left(E_\rho\left(\alpha_\rho \sigma_\rho\right)\right)^2 + 2j\left(1-\alpha_i\right) s_i \sigma_i E_\rho\left(\alpha_\rho \sigma_\rho\right) = j\left(\left(1-\alpha_i\right)\sigma_i + s_i E_\rho\left[\alpha_\rho \sigma_\rho\right]\right)^2$$

$$\begin{aligned} VAR(R_\rho; \sigma_\rho, \alpha_\rho) &= \left(1-j\right)\left(1-\alpha_i\right)^2 \sigma_i^2 + j\left(\left(1-\alpha_i\right)\sigma_i + s_i E_\rho\left[\alpha_\rho \sigma_\rho\right]\right)^2 \\ &= \left(1-j\right)\left(1-\alpha_i\right)^2 \sigma_i^2 + j\left(\left(1-\alpha_i\right)\sigma_i + \alpha_i \frac{E_\rho\left[\alpha_\rho \sigma_\rho\right]\mu\left(\sigma_i\right)}{E_\rho\left[\alpha_\rho \mu\left(\sigma_\rho\right)\right]}\right)^2 \\ &= \left(1-j\right)\left(1-\alpha_i\right)^2 \sigma_i^2 + j\left(\left(1-\alpha_i\right)\sigma_i + \alpha_i \sigma_i\left(\frac{\mu\left(\sigma_i\right)}{\sigma_i} / \frac{E_\rho\left[\alpha_\rho \mu\left(\sigma_\rho\right)\right]}{E_\rho\left[\alpha_\rho \sigma_\rho\right]}\right)\right)^2 \end{aligned}$$

4.3 Proof of Proposition 5

We show here that

$$SIGN\begin{pmatrix} \frac{d\sigma_i^*}{dw} \\ \frac{d\alpha_i^*}{dw} \end{pmatrix} = SIGN\begin{pmatrix} -\frac{1}{4}A\frac{\partial^2\Sigma}{\partial\alpha^2}\left(A'\frac{\partial\Sigma}{\partial\sigma} + A''\Sigma\mu'\right) \\ \frac{1}{4}A\frac{\partial^2\Sigma}{\partial\alpha\partial\sigma}\left(A'\frac{\partial\Sigma}{\partial\sigma} + A''\Sigma\mu'\right) \end{pmatrix}.$$

First, note that

$$\begin{aligned} \begin{pmatrix} \frac{\partial^2 C}{\partial\sigma^2} & \frac{\partial^2 C}{\partial\sigma\partial\alpha} \\ \frac{\partial^2 C}{\partial\alpha\partial\sigma} & \frac{\partial^2 C}{\partial\alpha^2} \end{pmatrix}^{-1} &= \Omega^{-1}\begin{pmatrix} -\frac{1}{2}A\frac{\partial^2\Sigma}{\partial\alpha\partial\alpha} & \frac{1}{2}A\frac{\partial^2\Sigma}{\partial\alpha\partial\sigma} + \frac{1}{2}A'\mu'\frac{\partial\Sigma}{\partial\alpha} \\ \frac{1}{2}A\frac{\partial^2\Sigma}{\partial\alpha\partial\sigma} + \frac{1}{2}A'\mu'\frac{\partial\Sigma}{\partial\alpha} & -\frac{1}{2}A''\Sigma\mu'^2 - A'\frac{\partial\Sigma}{\partial\sigma}\mu' - \frac{1}{2}A\frac{\partial^2\Sigma}{\partial\sigma\partial\sigma} + \mu''\left(1-\frac{1}{2}A'\Sigma\right) \end{pmatrix} \\ &= \Omega^{-1}\begin{pmatrix} -\frac{1}{2}A\frac{\partial^2\Sigma}{\partial\alpha\partial\alpha} & \frac{1}{2}A\frac{\partial^2\Sigma}{\partial\alpha\partial\sigma} \\ \frac{1}{2}A\frac{\partial^2\Sigma}{\partial\alpha\partial\sigma} & -\frac{1}{2}A''\Sigma\mu'^2 - A'\frac{\partial\Sigma}{\partial\sigma}\mu' - \frac{1}{2}A\frac{\partial^2\Sigma}{\partial\sigma\partial\sigma} + \mu''\left(1-\frac{1}{2}A'\Sigma\right) \end{pmatrix}, \end{aligned}$$

with

$$\Omega = \left(\frac{\partial^2 C}{\partial \sigma^2} \frac{\partial^2 C}{\partial \alpha^2} - \left(\frac{\partial^2 C}{\partial \alpha \partial \sigma} \right)^2 \right),$$

Also,

$$\begin{aligned} \begin{pmatrix} \frac{\partial^2 C}{\partial \sigma \partial w} \\ \frac{\partial^2 C}{\partial \alpha \partial w} \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2} A' \frac{\partial \Sigma}{\partial \sigma} - \frac{1}{2} A'' \Sigma \mu' \\ -\frac{1}{2} A' \frac{\partial \Sigma}{\partial \alpha} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} \left(A' \frac{\partial \Sigma}{\partial \sigma} + \frac{1}{2} A'' \Sigma \mu' \right) \\ 0 \end{pmatrix}, \end{aligned}$$

since $\frac{\partial \Sigma}{\partial \alpha} = 0$ at equilibrium. For the solution to be a maximum, Ω needs to be positive, with both $\frac{\partial^2 C}{\partial \sigma^2}$ and $\frac{\partial^2 C}{\partial \alpha^2}$ negative. Applying the formula in (18), we finish the proof of the result:

$$\begin{aligned} \begin{pmatrix} \frac{d\sigma_i^*}{dw} \\ \frac{d\alpha_i^*}{dw} \end{pmatrix} &= -\begin{pmatrix} \frac{\partial^2 C}{\partial \sigma^2} & \frac{\partial^2 C}{\partial \sigma \partial \alpha} \\ \frac{\partial^2 C}{\partial \alpha \partial \sigma} & \frac{\partial^2 C}{\partial \alpha^2} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial^2 C}{\partial \sigma \partial w} \\ \frac{\partial^2 C}{\partial \alpha \partial w} \end{pmatrix} \\ &= \frac{A\Omega^{-1}}{4} \left(A' \frac{\partial \Sigma}{\partial \sigma} + A'' \Sigma \mu' \right) \begin{pmatrix} -\frac{\partial^2 \Sigma}{\partial \alpha^2} \\ \frac{\partial^2 \Sigma}{\partial \alpha \partial \sigma} \end{pmatrix} \end{aligned}$$