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On Harsanyi Dividends and Asymmetric Values

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DISCUSSION PAPER

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Abstract

The concept of dividend of a coalition introduced by Harsanyi in 1959 within the framework of transferable utility games is a flexible and powerful concept that can be used to characterize different solution concepts, including random order values and weighted Shapley values. Many authors have contributed to that question. Here, we offer a synthesis of their work, with a particular attention to restrictions on dividend distributions, starting with the seminal contributions of Vasil'ev (1978), Hammer, Peled and Sorensen (1977) and Derks, Haller and Peters (2000), until the recent paper of van den Brink, van der Laan and Vasil'ev (2014).

Keywords: Harsanyi dividends, Weber set, weighted Shapley values, core

JEL Classification: C71

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1. Introduction

We begin with a chronological enumeration of the concepts that will be formally defined and related in the present paper.

The concept of *value* of a transferable utility game was introduced by Shapley in 1953. His initial idea was to answer the question of what a player may reasonably expect from playing a game. However, by requiring players' evaluations to be consistent in order to achieve efficiency (exact distribution of the social output) and symmetry (equal treatment of equals), the Shapley value is more of a normative tool.

The concept of *dividend* was introduced by Harsanyi in 1959. The idea is to associate to each coalition a dividend (positive or negative) that is distributed among its members to define an allocation of the social surplus. Harsanyi shows that the Shapley value coincides with the payoff that results from the equal division of dividends within each coalition. The set of allocations that results from all possible distributions of the dividends is an object that has been studied independently by Vasil'ev in papers published in Russian in 1975 and 1978, and by Hammer, Peled and Sorensen in a paper published in a Belgian operations research journal in 1977. While the latter used the name "selectope", here we shall retain the term *Harsanyi set*. Derks, Haller and Peters popularized the concept in a paper published in *International Journal of Game Theory* in 2000. At that time, they did not know about the contributions of Vasil'ev. These became known with the publication in 2002 by Vasil'ev and van der Laan of a paper containing all the results known by that time. Since then, a number of papers has been published in particular by Derks, van der Laan and Vasil'ev (2006, 2010), adding new results, including an axiomatization of the Harsanyi set.[1]

Weber has introduced in 1988 the notion of *probabilistic values* that allocates to each player his expected marginal contribution computed with respect to a probability distribution independent of the game's data. *Quasi-values* are then obtained by considering probability distributions ensuring efficiency, the Shapley value being the unique efficient and symmetric quasi-value.

In 1971, Shapley has characterized geometrically the core of a convex game using the concept of marginal contribution vector that associates allocations to players' orderings: for a given ordering, each player receives his marginal contribution, following the ordering. The core of a convex game is then the nonempty and convex polytope whose vertices are precisely the marginal contribution vectors. Weber (1988) has later defined the concept of *random order value* as the expected marginal contribution vector, given a probability distribution over players' orderings. Weber shows that random order values are quasi-value, the Shapley value

[1] Billot and Thisse (2005) have proposed an axiomatization of what turns out to be a much larger set of allocations by not imposing a positivity axiom.

being the random order value corresponding to the uniform probability distribution. Moreover, Weber shows that the core is a subset of the set of random order values, equivalently defined as the convex hull of the marginal contribution vectors. This set is known as the *Weber set* and, following Shapley's characterization of the core, the two sets coincide on the class of convex games.

In his 1953 paper, Shapley did consider the possibility for symmetric players to be treated differently. The *asymmetric* version of the value is obtained by introducing *exogenous* weights in order to cover asymmetries that are not included in the underlying game. The weighted Shapley value has been axiomatized later, in particular by Kalai and Samet (1987) without explicit reference to weights, by Hart and Mas-Colell (1989) using a generalized potential function and by Dehez (2011) in a cost sharing context, along the lines suggested by Shapley (1981). The value associated to positive weights turns out to result from a weighted division of dividends within each coalition and, as a consequence, weighted values are Harsanyi payoffs. Monderer, Samet and Shapley (1992) have shown that the set of all weighted values – the *Shapley set* – contains the core. To quote the authors, this inclusion "is somewhat surprising in light of the difference in concept behind these solutions". Weighted values being random order values, the Shapley set is contained in the Weber set and the three solution sets coincide when applied to convex games.

The Harsanyi set turns out to be the largest solution set. It includes the Weber set and, Derks, Haller and Peters (2000) have shown that it coincides with the core for positive games – games whose dividends are non-negative. Consequently, positive games being convex, the four solution sets – core, Shapley, Weber and Harsanyi sets – coincide when applied to positive games.

Solutions are usually characterized axiomatically. They can also be characterized starting with the Harsanyi set and then imposing restrictions on dividend distributions. A natural restriction is *monotonicity*. It consists in requiring that the share of a player in a coalition does not increase if the coalition is enlarged. Even if such restrictions reduce considerably the set of possible dividend distributions, it is not sufficient to generate a particular solution set. Billot and Thisse (2005) claim that the set of Harsanyi payoff vectors resulting from monotonic dividend distributions coincides with the core if the game is convex. We show that this is actually true only for three players! Assuming a monotonic dividend distribution, individual rationality obtains under in 3-player superadditive games and in 4-player convex games. Beyond four players, we show that there is no hope. Under a stronger monotonicity condition, Vasil'ev (1988) shows that Harsanyi payoffs vectors are random order values. If the distributions of dividends within coalitions are obtained from the distribution of dividends within the grand coalition by Bayesian updating, Derks et al. (2000) show that the resulting Harsanyi set coincides with the set of weighted Shapley values.

The paper is organized as follows. Section 2 introduces transferable utility games. Solution concepts are then defined and interrelated in Section 3, with a particular attention to weighted Shapley values and the case where some players are assigned a zero weight. In Section 4, we review the axiomatic characterizations of the Shapley, Weber and Harsanyi sets. We then look at the characterization of the Weber and Shapley sets by way of restrictions on dividend distributions. We finally consider the subsets of Harsanyi payoffs that result from restrictions linked to the structure of the game. The last section offers a few concluding remarks and an appendix gathers intermediary results.

2. Transferable utility games

2.1 Characteristic functions

Cooperative games cover situations in which a group of individuals cooperate on a common project with the objective of maximizing the resulting collective gain. It is assumed that *utility is transferable* through some commodity-money, allowing for transfers (side-payments) among players. A cooperative game with transferable utility is defined by a player set N and a *characteristic function* v that associates to each *coalition* $S \subset N$ a real number $v(S)$ that represents its (potential) worth, defined as the minimum gain that it can realize without the participation of the others. In particular, $v(i)$ is what player i can obtain alone and $v(N)$ is the maximum amount that the "grand coalition" is able to generate. By convention, $v(\emptyset) = 0$.

Notation: For any given set T, we denote by $\mathcal{C}(T) = \left\{ S \subset T \,\middle|\, S \neq \emptyset \right\}$ the collection of nonempty subsets of T and by $\mathcal{C}_i(T) = \left\{ S \subset T \,\middle|\, i \in S \right\}$ the collection of nonempty subsets of T containing i. Upper-case letters are always used to denote subsets and the corresponding lower-case letters are used to denote their sizes: $s = |S|, t = |T|, \ldots$ Coalitions $\{i,j,k,\ldots\}$ are sometime written as $ijk\ldots$ Set inclusion is denoted $\subset$ and *strict* inclusion by $\subsetneq$. For a given subset S, $S \setminus i$ denotes the subset obtained by *subtracting* i from S. In the same way, $S \cup i$ denotes the subset obtained by *adding* i to S. It will be convenient to write

$$x(S) = \sum_{i \in S} x_i, \; x_S = (x_i \mid i \in S) \text{ and } x = (x_S, x_{N \setminus S})$$

for any vector $x \in \mathbb{R}^n$ and nonempty subset $S \subset N$, with the convention $x(\emptyset) = 0$. In particular, $x^{-i} = x_{N \setminus i}$. Vectors are compared following the sequence $x \geq y$, $x > y$ and $x >> y$. In some instances, the summation sign Σ will be used, without reference to a set when there is no ambiguity. For a given finite set A, we denote by $\Delta(A)$ the set of all probability distributions on A.

Two games (N, v) and $(N, \tilde{v})$ on a common player set are said to be *strategically equivalent* if there exists $a > 0$ and $b \in \mathbb{R}^n$ such that

$$\tilde{v}(S) = a\, v(S) + b(S)$$

This defines an *equivalence relation*. In particular, a game (N,v) and its 0-normalization (N,v_0) defined by

$$v_0(S) = v(S) - \sum_{i \in S} v(i)$$

are strategically equivalent. A game can be restricted to a subset of its player set. Formally, given a game (N,v) and a subset $T \subset N$, the restriction to T is the game (T, v_T) defined by $v_T(S) = v(S)$ for all $S \subset T$.

2.2 Superadditivity and monotonicity

Throughout the paper, characteristic functions will be assumed to be *superadditive* i.e. for all S, T in N,

$$S \cap T = \varnothing \;\Rightarrow\; v(S) + v(T) \leq v(S \cup T)$$

This is a natural assumption that is satisfied in most economic and social situations: getting together is beneficial, or at least harmless. Superadditivity implies in particular the inequality $\sum v(i) \leq v(N)$. Interesting games are those for which a strict inequality holds. They are called *essential*. Obviously, a game that is strategically equivalent to a superadditive game is itself superadditive.

A characteristic functions is *monotone* if $S \subset T \;\Rightarrow\; v(S) \leq v(T)$. It implies that the larger surplus is generated by the grand coalition. There is no direct relation between superadditivity and monotonicity except for the following lemma.[2]

Lemma 1 Consider a game (N,v) such that $v(i) \geq 0$ for all $i \in N$. Superadditivity then implies that the characteristic functions v is monotone and positively valued.

As a consequence, the 0-normalization of a game is a monotone and positive-valued game.

2.3 Harsanyi dividends

We denote by $G(N)$ the set of all set functions on the finite set N. It is a *vector space* that can be identified with $\mathbb{R}^{2^n-1}$. Shapley (1953) shows that the collection of *unanimity games* (N,u_T) defined for all non-empty subsets $T \subset N$ by

$$\begin{aligned} u_T(S) &= 1 \text{ if } S \supset T \\ &= 0 \text{ otherwise} \end{aligned}$$

forms a basis of the vector space $G(N)$ i.e. there exists a unique $2^n - 1$ dimensional vector $\alpha(N,v) = (\alpha_T(N,v) \,|\, T \in \mathcal{C}(N))$ such that:

$$v(S) = \sum_{T \in \mathcal{C}(N)} \alpha_T(N,v)\, u_T(S) = \sum_{T \in \mathcal{C}(S)} \alpha_T(N,v) \tag{1}$$

[2] Proofs of lemmas are in Appendix.

with the convention $\alpha_\emptyset = 0$. Following Harsanyi (1959, 1963), the α_T's are interpreted as the dividends that accrue to coalition *T*. Indeed, according to (1), distributing the dividends is feasible within every coalition.[3]

The dividends can be defined recursively, starting with $\alpha_\emptyset = 0$, as follows:

$$\alpha_T = v(T) - \sum_{S \subsetneq T} \alpha_S \quad \text{for all } T \subset N \tag{2}$$

i.e.

$$\alpha_{\{i\}} = v(i)$$

$$\alpha_{\{ij\}} = v(ij) - v(i) - v(j)$$

$$\alpha_{\{ijk\}} = v(ijk) - v(ij) - v(ik) - v(jk) + v(i) + v(j) + v(k), \ldots$$

Alternatively, the collection of α_T's is the *unique* solution of the linear system (1):

$$\alpha_T(N,v) = \sum_{S \in \mathcal{C}(T)} (-1)^{t-s} v(S) \quad (T \subset N, T \neq \emptyset) \tag{3}$$

Remark 1 The dividends associated to the unanimity game (N, u_T) are given by:

$$\begin{aligned} \alpha_S(N,u_T) &= 1 \quad \text{if } S = T \\ &= 0 \quad \text{otherwise} \end{aligned}$$

A player *i* is *necessary* for player *j* in a game (N,v) if the marginal contributions of *j* are equal to zero in all coalition not containing *i*: $v(S) = v(S \setminus j)$ for all $S \not\ni i$. In particular, $v(j) = 0$. Using (2), van den Brink et al. (2014) prove the following Lemma.

Lemma 2 If *i* is necessary for *j* in (N,v), then $\alpha_T(N,v) = 0$ for all $T \subset N \setminus i$ such that $j \in T$.

2.4 Positive games

Dividends can be negative or positive. A game is (totally) *positive* if its dividends are all non-negative. Actually, given an arbitrary game (N,v), the dividends associated to its 0-normalization (N, v_0) are unchanged except for coalitions that are singletons:

$$\alpha_{\{i\}}(N, v_0) = 0 \qquad \text{for all } i \in N$$

$$\alpha_T(N, v_0) = \alpha_T(N,v) \qquad \text{for all } T \subset N, t \geq 2$$

The term *almost positive* is used to qualify a game whose dividends of multi-player coalitions are non-negative. Equivalently, a game is almost positive if its 0-normalization is positive. Obviously, 2-player games are almost positive under superadditivity.

Remark 2 Positive games are monotone. This can easily be seen from (1).

[3] To keep notation simple, the dependence of the α_T on the game will sometime be omited.

2.5 Marginal contributions

The *marginal contribution* of player i to coalition S is defined by $v(S)-v(S \setminus i)$. It is his value added to coalition S by player i and it is of course zero for all coalitions of which he is not a member. For coalition of which he is a member, superadditivity implies that they are bounded below by his individual worth:

$$v(S)-v(S\setminus i)\geq v(i) \text{ for all } S\subset N \text{ such that } i\in S \tag{4}$$

Two players i and j are *symmetric* in a game (N,v) if they contribute equally to all coalitions to which they belong: $v(S)-v(S\setminus i)=v(S)-v(S\setminus j)$ for all $S\subset N$ such that $i, j\in S$. A player i is *null* in a game (N,v) if he never contributes: $v(S)-v(S\setminus i)=0$ for all $S\subset N$.

Remark 3 A player is null *if and only if* all dividends associated to coalitions containing that player are all equal to zero. This is an immediate consequence of (2). See also Lemma 2.

Let Π_N denote the set of all players' orderings. The *marginal contributions vector* $\mu^{\pi}(N,v)$ associated to the players' ordering $\pi=(i_1,...,i_n)\in\Pi_N$ is the vector defined by:

$$\begin{aligned} &\mu^{\pi}_{i_1}(N,v)=v(i_1)-v(\varnothing)=v(i_1)\\ &\mu^{\pi}_{i_k}(N,v)=v(i_1,...,i_k)-v(i_1,...,i_{k-1}) \quad (k=2,...,n) \end{aligned} \tag{5}$$

i.e.

$$\mu^{\pi}_{i}(N,v)=v(\pi^i)-v(\pi^i\backslash i) \quad (i=1,...,n)$$

where π^i denotes the set of players preceding i in π, i included. There are $n!$ marginal contribution vectors, not necessarily all distinct. Looking at strategically equivalent games, if $\tilde{v}(S)=a\,v(S)+b(S)$ for some $a>0$ and $b\in\mathbb{R}^n$,

$$\mu^{\pi}_{i}(N,\tilde{v})=a\;\mu^{\pi}_{i}(N,v)+b_i \quad (i=1,\ldots,n) \tag{6}$$

2.6 Convex games

A game (N,v) is *convex* (or *supermodular*) if $v(S)+v(T)\leq v(S\cup T)+v(S\cap T)$ for all $S \text{ and } T\subset N$. Obviously, convexity implies superadditivity and a game strategically equivalent to a convex game is itself convex. It is easily verified that unanimity games are convex. As a consequence, positive games are convex as positive linear combinations of convex games, and almost positive games are convex as well by strategic equivalence.

Shapley (1971) shows that a game is convex *if and only if* players' marginal contributions do not decrease with coalition size:

$$i\in S\subset T \;\Rightarrow\; v(S)-v(S\setminus i)\leq v(T)-v(T\setminus i)$$

Hence convexity means *increasing returns to size* and marginal contributions are *maximal* at the grand coalition N.

For any given player set N, the set of superadditive games, the set of monotonic games and the set of convex games are *convex cones,* denoted $SG(N)$, $MG(N)$ and $CG(N)$ respectively. The set of positive games is a convex cone as well. It is denoted by $G^{+}(N)$. Following Remark 2, we have the following sequences of inclusions:

$$G^{+}(N) \subset CG(N) \subset SG(N) \text{ and } G^{+}(N) \subset MG(N)$$

An arbitrary game (N,v) can be decomposed in a difference between two positive (and convex) games. Indeed, (1) can be written as

$$v(S) = \sum_{T \in \mathcal{C}(N)} \alpha_T(N,v)\, u_T(S) = v^{+}(S) - v^{-}(S) \text{ for all } S \subset N \tag{7}$$

where

$$v^{+}(S) = \sum_{T:\alpha_T > 0} \alpha_T(N,v)\, u_T(S) \text{ and } v^{-}(S) = \sum_{T:\alpha_T < 0} -\alpha_T(N,v)\, u_T(S)$$

The α_T –coefficients associated to these two games are given by:

$$\alpha_T(N,v^{+}) = Max\left(0, \alpha_T(N,v)\right)$$

$$\alpha_T(N,v^{-}) = -Min\left(0, \alpha_T(N,v)\right)$$

Convex games form an interesting class of games because many solution concepts tend to agree when applied to convex games. Moreover, many interesting economic situations can be modeled as convex games, like production games with increasing returns, bankruptcy games (Aumann and Maschler, 1985) and airport games (Littlechild and Owen, 1973). Positive games also form a subset of convex and monotonic games with interesting properties and applications, like river games (Ambec and Sprumont, 2002), queuing games (Maniquet, 2003) and liability games (Dehez and Ferey, 2013).

3. Values and solutions

3.1 Basic requirements

Given a game (N,v), the problem is to allocate $v(N)$ between the n players. A *value* is a mapping that associates a payoff vector $\varphi(N,v) \in \mathbb{R}^n$ to a game (N,v). A *solution* is a mapping Φ that associates a *subset* $\Phi(N,v)$ of payoff vectors to a game (N,v). The basic properties that a solution should ideally possess are the following:

Nonemptiness: $\Phi(N,v) \neq \varnothing$

Efficiency: $x \in \Phi(N,v) \;\Rightarrow\; x(N) = v(N)$

Individual rationality: $x \in \Phi(N,v) \;\Rightarrow\; x_i \geq v(i)$ for all $i \in N$

Covariance: $x \in \Phi(N,v) \;\Rightarrow\; a\,x + b \in \Phi(N, a\,v + b)$ for all $a > 0$ and $b \in \mathbb{R}^n$

Convexity: $\Phi(N,v)$ is a convex set

Nonemptiness implies restrictions on the class of games on which the solution applies. Efficiency is in the spirit of Pareto: allocating less than $v(N)$ is not efficient and allocating more than $v(N)$ is not feasible. Individually rationality is a minimal requirement to be imposed on allocations: no player will ever accept to take part in a collective project if his remuneration falls short of what he could secure by himself. A solution is covariant if, once it has been applied to a game, it can be extended to all strategically equivalent games. Convexity is a natural requirement in a world of transferable utility.

Imputations are individually rational and efficient allocations. This defines the *imputation set*:

$$I(N,v) = \{x \in \mathbb{R}^n \mid x(N) = v(N),\ \ x(i) \geq v(i) \ \text{ for all } i \in N\}$$

The class of games (N,v) satisfying the inequality $\sum v(i) \leq v(N)$ is the largest class of games on which the imputation set is a well-defined solution. It includes superadditive games. If the game is essential, it is the regular simplex of dimension n–1. Otherwise, it reduces to the singleton $(v(1),...,v(n))$. The imputation set is the largest set satisfying the above requirements.

3.2 Stable allocations: the core

The *core* is the set of imputations that no coalition can improve upon:

$$C(N,v) = \left\{x \in \mathbb{R}^n \;\middle|\; x(N) = v(N),\ \ x(S) \geq v(S) \ \text{ for all } S \subset N \right\} \tag{8}$$

It extends rationality from individuals to coalitions: given an allocation, a coalition that receives less than what it could secure is in a position to object. In this sense, core allocations are "stable".[4]

The core is a convex subset of the imputation set and it may be empty. The largest class of games on which the core is a well-defined solution is the class of *balanced games*.[5] Superadditivity is neither necessary, nor sufficient, for a game to have a nonempty core. However, it is easily verified that a game (N,v) with a nonempty core satisfies the following weaker condition:

$$v(S) + v(N \backslash S) \leq v(N) \ \text{ for all } S \subset N$$

Remark 4 In the case of a 3-player game (N,v), balancedness is equivalent to the inequality $v(1,2) + v(1,3) + v(2,3) \leq 2v(1,2,3).$ It shows that intermediate size coalitions should not be too strong. Notice that almost positivity requires a strengthening of the inequality: $\alpha_{\{1,2,3\}} \geq 0$ if and only if $v(1,2) + v(1,3) + v(2,3) \leq v(1,2,3).$ This follows from Lemma 1.

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[4] The term "core" was introduced by Gillies (1953, 1959) in connection to the von Neumann-Morgenstern stable sets. It was later introduced as an independent solution concept by Shapley. The core has been axiomatized by Peleg (1986) using the reduced game property.

[5] See Bondareva (1963) and Shapley (1967). For a complete account, see Kannai (1992).

Remark 5 It can be easily verified that, if i and j are symmetric players, allocation obtained by exchanging x_i and x_j in a core allocation x are also core allocations. Hence, if nonempty, the core contains allocations that give to players i and j equal amounts. Furthermore, the core allocates zero to null players.

Shapley (1971) shows that convex games are balanced and that the core of a convex game is the convex polyhedron whose vertices are the marginal contribution vectors defined by (5). Ichiishi (1989) shows that this is actually a *necessary and sufficient* condition for convexity.

3.3 Probabilistic and quasi-values

The concept of value was introduced by Shapley in 1953 as a measure of what a player may expect from playing a game. The Shapley value belongs to the family of probabilistic values introduced by Weber (1988). Let q_i^S be the probability that player i leaves coalition S, i.e.

$$q_i^S \geq 0 \text{ for all } S \subset N, q_i^S = 0 \text{ if } i \notin S \text{ and } \sum_{S \subset N} q_i^S = 1$$

These may be subjective probabilities or objective probabilities resulting from some random mechanism. They depend on the coalitions and not on the game. Following Weber (1988), the *probabilistic value* associated to the probability distributions $(q_i^S \mid S \subset N, i \in N)$ defines for each player his expected marginal contribution:

$$\varphi_i(N,v) = \sum_{S \in \mathcal{C}(N)} q_i^S \left(v(S) - v(S \setminus i)\right) \quad (i = 1,\dots,n) \tag{9}$$

A probabilistic value does not necessarily define an efficient payoff vector because probability distributions are unrelated. *Quasi-values* instead are efficient probabilistic value. They are obtained from probability distributions satisfying the following conditions:

$$\sum_{i \in N} q_i^N = 1 \text{ and } \sum_{i \in S} q_i^S = \sum_{i \in N \setminus S} q_i^{S \cup i} \text{ for all } S \subsetneq N, N \neq \varnothing \tag{10}$$

Weber (1988) indeed proves that the probabilistic values defined in (9) satisfy efficiency under (10).[6] By requiring consistency of the probability distributions, quasi-values can be given a normative content.

The Shapley value is a particular quasi-value. Weber (1988) proves that it is the unique *symmetric* quasi-value. It is defined by the following probabilities:

$$q_i^S = \frac{(s-1)!(n-s)!}{n!}$$

[6] See also Derks (2005).

Hence,

$$SV_i(N,v) = \frac{1}{n!} \sum_{S \subset N} (s-1)!(n-s)!\big(v(S) - v(S \setminus i)\big) \tag{11}$$

They only depend on coalitions' sizes and they correspond to the following fair random mechanism:[7] first, a size between 1 and n is picked up at random and then each player receives his marginal contribution to a coalition picked up at random among the coalitions of the predetermined size, of which he is a member. All sizes have the same probability, namely 1/n, and the probability of picking up a coalition of size s containing a given player is given by $(s-1)!(n-s)!/(n-1)!$.

The Shapley value is a well-defined single-valued solution on the class of all games, hence including superadditive games. Following (4), superadditivity ensures that the marginal contribution vectors are imputations. Covariance follows from (6). Hence, the Shapley value defines an imputation that is not necessarily stable, independently of the core being empty or not. However, as an average of marginal contribution vectors, it defines a core allocation when applied to a convex game. In view of the geometric characterization of the core of a convex game, the Shapley value occupies a central position. It generally differs from the barycenter of the core introduced as a solution concept by Gonzáles-Díaz and Sánchez-Rodríguez (2007). It also differs from the simple average of core's vertices except for the particular case of convex games having distinct marginal contribution vectors.[8]

3.4 Random order values: the Weber set

The object obtained by considering *all* convex combinations of the marginal contribution vectors is called the *Weber set*. It is the convex hull of the marginal contribution vectors:[9]

$$W(N,v) = co\left\{ x \in \mathbb{R}^n \;\middle|\; x = \mu^\pi(N,v) \text{ for some } \pi \in \Pi_N \right\}$$

Marginal contribution vectors being imputations, it is a well-defined solution on the class of superadditive games.

An allocation in the Weber set may *equivalently* be regarded as the average marginal contribution computed with respect to some probability distribution on Π_N. For a given game (N,v), the *random order value* associated to the probability distribution $p \in \Delta(\Pi_N)$ is given by:

$$RV_i(N,v,p) = \sum_{\pi \in \Pi_N} p(\pi)\, \mu_i^\pi(N,v) \quad (i = 1,...,n) \tag{12}$$

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[7] A random allocation mechanism is "fair" if it treats ex ante all players equally.

[8] This characterizes what Shapley (1971) calls *strictly* convex games, games with increasing marginal contributions.

[9] The convex hull of a set A, denoted $co(A)$, is the smallest convex set containing A. See Rockafellar (1970).

Hence, the Weber set can alternatively be defined as the set of all random order values. A random order value is a quasi-value. Indeed, $RV_i(N,v,p)=\sum_{S\subset N} q_i^S \left(v(S)-v(S\setminus i)\right)$ where the probability distribution (q_i^S) defined by

$$q_i^S = \sum_{\pi:\pi^i=S} p(\pi)$$

satisfies (10). Actually, Weber (1988) proves that a solution is a quasi-value *if and only if* it is a random-order value. Hence, the Shapley value is also a particular random order value. It corresponds to random orders that are *uniformly* drawn i.e. $p(\pi)=1/n!$ for all π:

$$SV_i(N,v)=\frac{1}{n!}\sum_{\pi\in\Pi_N}\mu_i^\pi(N,v) \qquad (i=1,\ldots,n) \tag{13}$$

The Shapley value is then the average marginal contribution vector and it can then be seen as resulting from another fair random mechanism: first, players are ordered randomly and they then receive their marginal contribution, depending on their position in the order that has been picked up.

3.5 Dividend distributions: the Harsanyi set

A distribution of the Harsanyi dividends can be summarized by a matrix λ of dimension $n\times(2^n-1)$ whose columns are the non-negative vectors λ^T $(T\subset N, T\neq\varnothing)$ satisfying

$$\sum_{i\in N}\lambda_i^T = 1 \text{ and } \lambda_i^T = 0 \text{ for all } i\notin T$$

i.e. λ^T specifies how the dividend α_T is distributed within coalition *T*. In particular, $\lambda_i^{\{i\}}=1$ for all *i* and λ^N can be any vector in the unit simplex Δ_n. We denote by Λ_n the set of all distribution matrices in the case of *n* players.

For a given game (N,v), the *Harsanyi payoff vector* $h(N,v,\lambda)$ corresponding to a matrix $\lambda\in\Lambda_n$ is the value defined by the product $h(N,v,\lambda)=\lambda\cdot\alpha(N,v)$ i.e.

$$h_i(N,v,\lambda)=\sum_{T\in\mathcal{C}(N)}\lambda_i^T\alpha_T(N,v)=v(i)+\sum_{T\in\mathcal{C}(N)\setminus\{i\}}\lambda_i^T\alpha_T(N,v) \quad (i=1,\ldots,n) \tag{14}$$

It is an allocation. Indeed, we have:

$$\sum_{i\in N}h_i(N,v,\lambda)=\sum_{i\in N}\sum_{T\in\mathcal{C}(N)}\lambda_i^T\alpha_T(N,v)=\sum_{T\in\mathcal{C}(N)}\alpha_T(N,v)\sum_{i\in N}\lambda_i^T=v(N)$$

The set of all distributions of H-dividends defines the *Harsanyi set*: [10]

$$H(N,v)=\left\{x\in\mathbb{R}^n \,\middle|\, x=h(N,v,\lambda) \text{ for some } \lambda\in\Lambda_n\right\}$$

[10] The Harsanyi set was introduced as a solution concept by Vasile'v (1978, 1981) and by Hammers et al. (1977), independently. The later use the term *selectope*.

For a given subset T, α_T is allocated between the members of T and, depending on its sign, they receive a positive or a negative amount. Hence, the Harsanyi set can alternatively be written as:

$$H(N,v) = \sum_{T \in \mathcal{C}(N)} \{x \in \mathbb{R}^n \mid x(T) = \alpha_T,\, x(N \backslash T) = 0,\, sign(x_i) = sign(\alpha_T) \text{ for all } i \in N\} \tag{15}$$

It is obviously a nonempty and convex set. It is covariant. Indeed, if $\tilde{v}(S) = a\, v(S) + b(S)$ for some $a > 0$ and $b \in \mathbb{R}^n$, we have:

$$\begin{aligned}
\alpha_{\{i\}}(N,\tilde{v}) &= \tilde{v}(i) = a\, v(i) + b_i = a\, \alpha_{\{i\}}(N,v) + b_i \\
\alpha_{\{ij\}}(N,\tilde{v}) &= \tilde{v}(ij) - \tilde{v}(i) - \tilde{v}(j) \\
&= a\,(v(ij) + b_i + b_j) - (a\, v(i) + b_i) - (a\, v(j) + b_j) = a\, \alpha_{\{ij\}}(N,v), \ldots
\end{aligned}$$

i.e. the additive term affects only the coefficient associated to singletons. Therefore, we have:

$$\begin{aligned}
h_i(N,\tilde{v},\lambda) &= \sum_{T \in \mathcal{C}(N)} \lambda_i^T \alpha_T(N,\tilde{v}) \\
&= a\, \alpha_{\{i\}}(N,v) + b_i + a \sum_{T \in \mathcal{C}(N) \backslash \{i\}} \lambda_i^T \alpha_T(N,v) \\
&= a\, h_i(N,v,\lambda) + b_i \quad (i = 1,\ldots,n)
\end{aligned}$$

In general, H-payoff vectors are not necessarily imputations and it may even be that the Harsanyi set contains no imputation at all.[11] However, for almost positive games, H-payoff vectors are imputations. This is an immediate consequence of (14).

3.6 Weighted values: the Shapley set

The Shapley value relies on symmetry: equal amounts are allocated to "identical" players. Shapley (1953) derives (11) from the following formula

$$SV_i(N,v) = \sum_{T \in \mathcal{C}_i(N)} \frac{\alpha_T(N,v)}{t} \quad (i = 1,\ldots,n) \tag{16}$$

i.e. the Shapley value is the H-payoff vector associated to the *uniform distribution* of dividends within each coalition: $\lambda_i^T = 1/t$ for all $i \in T$ and $\lambda_i^T = 0$ for all $i \notin T$.

Dropping symmetry opens the possibility for symmetric players to be treated differently. The *asymmetric* version of the value is obtained by introducing exogenous weights in order to cover asymmetries that are not included in the underlying game. It was introduced by Shapley (1953). Weighted games are denoted by (N,v,w) where (N,v) is a transferable utility game and $w = (w_1,\ldots,w_n) \geq 0$ are individual weights.

[11] Derks, van der Laan and Vasil'ev (2010) give the exemple of a game that fails to be superadditive, whose Harsanyi set has no intersection with the imputation set.

We denote by $SV(N,v,w)$ the weighted Shapley value associated to the weighted game (N,v,w). It is the Harsanyi payoff vector associated to the distribution of dividends derived from the weights:

$$SV_i(N,v,w) = \sum_{T \in \mathcal{C}_i(N)} \frac{w_i}{w(T)} \alpha_T(N,v) \tag{17}$$

That definition is valid only if *at most one* of the w_i's is equal to zero. To show this, consider an arbitrary player, say player 1. When all weights are positive, (17) can be decomposed as follows:

$$SV_1(N,v,w) = \alpha_{\{1\}}(N,v) + \sum_{T \in \mathcal{C}_1(N)\setminus\{1\}} \frac{w_i}{w(T)} \alpha_T(N,v)$$

Letting $w_1 \to 0,$ we obtain a well-defined limit:

$$SV_1(N,v,w) = \alpha_{\{1\}}(N,v) + \lim_{w_1 \to 0} \sum_{T \in \mathcal{C}_1(N)\setminus\{1\}} \frac{w_i}{w(T)} \alpha_T(N,v)$$

When two or more weights are set to zero, there is indeterminacy: several values are associated to the same weight system. Applying (17) to the unanimity game $(N,u_N),$ we get:

$$SV_i(N,u_N,w) = \frac{w_i}{w(N)} \quad (i = 1,\ldots,n) \tag{18}$$

an expression that reduces to $1/n$ in the symmetric case where weights are equal.

The weighted Shapley value can alternatively be obtained as the random order value $RV(N,v,p_w)$ associated to a probability distribution p_w on players' orderings that depends on w. For an arbitrary ordering π, the marginal contributions of player i in the unanimity game (N,u_N) is defined by:

$$\begin{aligned} \mu_i^{\pi}(N,u_N) &= 1 \quad \text{if (and only if) } i \text{ comes } \textit{last} \text{ in } \pi \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Hence, (18) corresponds to the random order values associated to distributions such that $w_i/w(N)$ is the probability that player i comes *last* in an arbitrary ordering, i.e. $p \in \Delta(\Pi_N)$ should satisfy

$$RV_i(N,u_N,p) = \sum_{\pi^{-i} \in \Pi_{N\setminus i}} p(\pi^{-i}, i) = \frac{w_i}{w(N)} \quad \text{for all } i \in N$$

Let's assume that weights are positive and natural numbers, w_i being interpreted as the number of players of type i. We then compute the probability that a given ordering comes out through a sequence of $w(N)$ independent drawings, knowing that each time a player is drawn, he is removed and only the last player of a given type to be removed is placed in the ordering. To illustrate the process, let's take $n=3$ and $w=(1,2,3)$. There are 60 drawing sequences and for instance the sequence (3,2,3,3,1,2) leads to the ordering (3,1,2). In general, the number of possible sequences is given by $w(N)!\big/\prod w_i!$ and they all have the same probability of occurrence. Player j comes out last in a given ordering if and only if he is the last to be drawn. This happens with probability

$$\frac{\left(w(N)-1\right)!}{(w_j-1)!\prod_{i\neq j} w_i!}\frac{\prod w_i!}{w(N)!}=\frac{w_j}{w(N)}$$

The probability that player k comes next to last knowing that player j came last is given by:

$$\frac{\left(w(N\setminus j)-1\right)!}{(w_k-1)!\prod_{i\neq j,k} w_i!}\frac{\prod_{i\neq j} w_i!}{w(N\setminus j)!}=\frac{w_k}{w(N\setminus j)}$$

Using this argument repeatedly until the second position, the probability that the ordering $\pi=(i_1,\dots,i_n)$ comes out is given by:

$$p_w(\pi)=\frac{w_{i_2}}{w_{i_1}+w_{i_2}}\cdots\frac{w_{i_{n-1}}}{w_{i_1}+\dots+w_{i_{n-1}}}\frac{w_{i_n}}{w_{i_1}+\dots+w_{i_n}}=\prod_{k=2}^{n}\frac{w_{i_k}}{\sum_{j=1}^{k} w_{i_j}}$$

or

$$p_w(\pi)=\prod_{k=2}^{n}\frac{1}{1+\sum_{j=1}^{k-1}(w_{i_j}/w_{i_k})} \qquad (19)$$

This formula is then extended to the case where weights are real.[12]

The probability distributions p_w are homogeneous of degree zero. Weights may therefore be normalized and there is a one-to-one relationship between the set of *positively* weighted values and the (relative) *interior* of Δ_n, the unit simplex of $\mathbb{R}^n$: any positively weighted value is associated to a unique *normalized* weight system, and vice-versa.

If a player is assigned a zero weight, weighted values are obtained as limit of sequences of positively weighted values. If there is a *single* zero-weight player, say player i, the limit distribution is still uniquely defined: player i is *first* with probability 1 and receives his individual worth $v(i)$. When two players or more are assigned a zero weight, several values are associated to the same normalized weight system. Considering converging sequences of positive weights, the resulting value depends on the relative speeds of convergence.

[12] I am grateful to Gerard van der Laan for suggesting this procedure. I initially used the sequence of n drawings where, each time a player is drawn, all players of the same type are removed. It leads to a probability distribution where $w_i/\Sigma w_j$ is the probability that player i comes *first*. This is appropriate for cost games and duals of surplus sharing games (see Dehez, 2011). The distribution (19) is obtained by considering the reverse order.

Let us denote the set of zero-weight players by $Z_w = \{i \in N \mid w_i = 0\}$.[13] For a given set N of players and non-negative weights w, the set of *all* weighted values is obtained from the probability distributions in the set

$$F_N(w) = \left\{ p \in \Delta(\Pi_N) \mid p(\pi) = \lim_{w^k \to w} p_{w^k}(\pi) \text{ for some converging sequence } (w^k) > 0 \right\}$$

In view of (19), this is a well-defined set: for any positive sequence (w^k) converging to $w \geq 0$, the associated sequence of distributions p_{w^k} converges to a distribution $p_w \in \Delta(\Pi_N)$. In particular, for $w = 0$, any probability distribution on Π_N is possible: $F_N(0) = \Delta(\Pi_N)$.

We then define the *Shapley set* as the set of all weighted values:

$$WS(N,v) = \left\{ x \in \mathbb{R}^n \,\middle|\, x = RV(N,v,p) \text{ for some } p \in F_N(w) \text{ and } w \in \mathbb{R}^n_+ \right\} \tag{20}$$

Given a game (N,v), consider its restriction (Z, v_Z) on Z and the game $(N \backslash Z, \tilde{v})$ defined by $\tilde{v}(S) = v(Z \cup S) - v(Z)$. It is the game that concerns the subset of non-zero weight players, once $v(Z)$ has been distributed to the zero-weight players. The following proposition establishes that, in order to compute weighted values, nonzero-weight players and zero-weight players can be treated separately.

Proposition 1 The values of the weighted game (N,v,w) consists of the allocations $x = (x_Z, x_{N\backslash Z})$ where $x_Z \in W(Z, v_Z)$ and $x_{N\backslash Z} = SV(N\backslash Z, \tilde{v}, w_{N\backslash Z})$.

Proof Inspecting (19), we observe that the distributions in $F_N(w)$ assign a zero probability to orderings in which a nonzero-weight player is followed by a zero-weight player. Hence, only orderings of the form $\pi = (\pi', \pi'') \in (\Pi_Z \times \Pi_{N\backslash Z})$ do actually matter and the distributions $p_w \in F_N(w)$ are of the form

$$\begin{aligned} p_w(\pi) &= p_0(\pi')\, p_{w_{N\backslash Z}}(\pi'') && \text{for all } \pi = (\pi', \pi'') \in \Pi_N \times \Pi_{N\backslash Z} \\ &= 0 && \text{otherwise} \end{aligned}$$

for some probability distribution $p_0 \in \Delta(\Pi_Z)$. The corresponding allocation is given by:

$$x_i = \sum_{(\pi',\pi'') \in \Pi_Z \times \Pi_{N\backslash Z}} p_0(\pi')\; p_{w_{N\backslash Z}}(\pi'')\; \mu_i^{(\pi',\pi'')}(N,v)$$

Hence, for a player $i \in Z$, we have:

$$x_i = \sum_{\pi' \in \Pi_Z} p_0(\pi')\; \mu_i^{\pi'}(Z, v_Z) \sum_{\pi'' \in \Pi_{N\backslash Z}} p_{w_{N\backslash Z}}(\pi'') = \sum_{\pi' \in \Pi_Z} p_0(\pi')\; \mu_i^{\pi'}(Z, v_Z)$$

i.e. x_Z is a random order value of the game (Z, v_Z). Hence, $x_Z \in W(Z, v_Z)$.

[13] In what follows, we omit the dependance of Z on w.

Consider now the characteristic function $\tilde{v}$ defined on $N\backslash Z$ by $\tilde{v}(S) = v(Z \cup S) - v(Z)$. For an ordering $(\pi', \pi'') \in (\Pi_Z \times \Pi_{N\backslash Z})$ and a player $i \in N\backslash Z$, we have:

$$x_i = \sum_{\pi'' \in \Pi_{N\backslash Z}} p_{w_{N\backslash Z}}(\pi'')\, \mu_i^{\pi''}(N\backslash Z, v_{N\backslash Z}) \sum_{\pi' \in \Pi_Z} p_0(\pi') = \sum_{\pi'' \in \Pi_{N\backslash Z}} p_{w_{N\backslash Z}}(\pi'')\, \mu_i^{\pi''}(N\backslash Z, \tilde{v})$$

i.e. $x_{N\backslash Z}$ is the value associated to the weighted game $(N\backslash Z, \tilde{v}, w_{N\backslash Z})$. ♦

Remark 6 In practice, when more than one player are assigned a zero weight, it seems natural to treat them equally by considering converging sequences such that the ratios of their weights are equal to 1. The corresponding distribution is then given by $p_0(\pi) = 1/z!$ for all $\pi \in \Pi_Z$ and the resulting allocation is the symmetric Shapley value of the game (Z, v_Z).[14]

The Shapley set is clearly a non-empty subset of the Weber set and, as a solution, it is covariant. It is however not a convex set in general, as this was observed by Monderer, Samet and Shapley (1992), except for the 2-player case or when applied to convex games, as we shall see later.

Remark 7 Owen (1968) has been the first to notice that weighted values are not necessarily monotonic with respect to weights: an increase in the weight assigned to a player may indeed result in a decrease of his payoff. Weights being interpreted as measures of players' relative importance (Shapley talks about bargaining ability), this is an embarrassing fact. It is however no surprise in view of (17), knowing that dividends may be negative.[15] Monotonicity clearly holds for almost positive games. Monderer et al. (1992) have shown that it actually holds for (and only for) convex games. This can be explained intuitively by the link that exists between a characteristic of convex games and the probability distribution over orderings induced by the weights. Increasing the weight of a player means increasing his probability of arriving late and we know that marginal contributions are increasing with coalition size in convex games. Hence, increasing the weight of a player naturally increases his expected payoff.

3.7. Relation between solutions

We first notice that, except for the imputation set, the Weber set is the only solution that satisfies the five basic requirements under superadditivity. The question is to see how the core, the Weber set, the Shapley set and the Harsanyi set are interrelated? The following proposition is due to Derks et al. (2000).

Proposition 2 Marginal contribution vectors are Harsanyi payoff vectors.

---

[14] See Dehez and Tellone (2013) for an application of the weighted Shapley value with zero weight players.

[15] Owen (1968) suggests interpreting weights as reflecting slowness to reach a decision. An alternative definition of weighted value has been proposed by Haeringer (2006) in which an increase in the weight of a player leads to an increase in his share in positive dividends and a decrease of his share in negative dividends.

Proof Consider a game (N,v), an arbitrary ordering $\pi \in \Pi_N$ and the distribution matrix λ defined by

$$\begin{aligned} \lambda_i^T &= 1 \quad \text{if } i \in T \text{ and } T \subset \pi^i \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where π^i is the set of players preceding *i* in π and including *i*. Then, for any given coalition *T*, it gives a positive share *only* to the player in T that has the highest rank in π. As a consequence, $\sum_{i\in T}\lambda_i^T = 1$ for all $T \subset N$ and $\lambda \in \Lambda_n$. The corresponding H-payoff vector is then given by:

$$h_i(N,v,\lambda) = \sum_{T\in\mathcal{C}(N)} \lambda_i^T \alpha_T = \sum_{T\in\mathcal{C}_i(\pi^i)} \alpha_T = \sum_{T\subset\pi^i} \alpha_T - \sum_{T\subset\pi^i\setminus i} \alpha_T = v(\pi^i) - v(\pi^i \setminus i)$$

Hence, $h(N,v,\lambda)$ is the marginal contribution vector associated to the ordering π. ♦

As a consequence of Proposition 2, under superadditivity, the *Harsanyi imputation set* defined by

$$HI(N,v) = H(N,v) \cap I(N,v)$$

is a well-defined solution that obviously satisfies the five basic requirements.[16]

Proposition 3 The following sequence of inclusions holds for all superadditive games:

$$C(N,v) \subset WS(N,v) \subset W(N,v) \subset HI(N,v) \subset H(N,v)$$

Furthermore, when applied to convex games, the first three solutions coincide.

Proof Weber (1988) has shown that the core is a subset of the Weber set and that, when applied to convex games, the two solutions coincide. Actually this coincidence is a *necessary and sufficient* condition for convexity. We have already seen that weighted values are random order values.[17] Monderer Samet and Shapley (1992) have shown that the core is a subset of the set of weighted values. By Proposition 1, random order values are convex combinations of Harsanyi imputation vectors. The sequence of inclusions then follows from convexity of the Harsanyi set. ♦

Proposition 4 All solutions coincide on the set of almost positive games:

$$C(N,v) = WS(N,v) = W(N,v) = HI(N,v) = H(N,v)$$

This is a corollary of the following proposition due to Hammer et al. (1977) and Vasil'ev (1978).

[16] Notice that for 2-player games, the Harsanyi set coincides with the imputation set.
[17] In fact, the Shapley set is in general a dimensionally small subset of the Weber set.

Proposition 5 The core and the Harsanyi set coincide on the set of almost positive games.

Proof Let (N,v) be an almost positive (and thereby convex) game. Looking at its 0-normalization, we have:

$$v_0(S) = \sum_{T\in\mathcal{C}(N)} \alpha_T(N,v_0)\, u_T(S)$$

where the α_T are all non-negative. For any given $T \subset N,$ the core of the game $(N,\alpha_T u_T)$ is given by:

$$C(N,\alpha_T u_T) = \left\{x \in \mathbb{R}^n_+ \mid x(T) = \alpha_T \text{ and } x_i = 0 \text{ for all } i \notin T\right\}$$

Indeed $u_T(i) = 0$ for all $T \neq \{i\},$ $u_{\{i\}}(i) = 1$ and $u_{\{i\}}(j) = 1$ for all $j \neq i.$ The core is additive on the class of convex games. This follows from the following two lemmas:

Lemma 3 The core is a superadditive solution (Peleg, 1986).

Lemma 4 The Weber set is a subadditive solution (Dragan, Potters and Tijs, 1989).

Hence, we have:

$$C(N,v_0) = \sum_{T\in\mathcal{C}(N)} C(N,\alpha_T u_T) = \sum_{T\in\mathcal{C}(N)} \{x \in \mathbb{R}^n_+ \mid x(T) = \alpha_T \text{ and } x(N\backslash T) = 0\}$$

where the right hand side is the Harsanyi set of the game $(N,v_0).$ ♦

Actually, Hammer et al. (1977) prove that the core and the Harsanyi set coincide if *and only if* they apply to almost positive games. Knowing that core allocations (if any) are H-payoff vectors, another way to prove Proposition 5 consists in showing that the reverse inclusion $H(N,v) \subset C(N,v)$ holds for almost positive games. Indeed, using (14), we have:

$$\begin{aligned}\sum_{i\in S} h_i(N,v,\lambda) &= \sum_{i\in S}\sum_{T\in\mathcal{C}(N)} \lambda_i^T \alpha_T = \sum_{i\in S}\sum_{T\in\mathcal{C}(S)} \lambda_i^T \alpha_T + \sum_{i\in S}\sum_{\substack{T\subset\mathcal{C}(N)\\ T\not\subset S}} \lambda_i^T \alpha_T \\ &= \sum_{T\in\mathcal{C}(S)} \alpha_T \sum_{i\in T} \lambda_i^T + \sum_{i\in S}\sum_{\substack{T\subset\mathcal{C}(N)\\ T\not\subset S}} \lambda_i^T \alpha_T = v(S) + \sum_{i\in S}\sum_{\substack{T\subset\mathcal{C}(N)\\ T\not\subset S}} \lambda_i^T \alpha_T \geq v(S)\end{aligned}$$

At this stage, we can conclude that the core, the Shapley set, the Weber set and the Harsanyi imputation set all satisfy the five basic requirements when applied to convex games. In particular, the Shapley set is convex in this case.

Remark 8 Concerning the Shapley set, assuming convexity, there is a homeomorphism between the *relative interior* of the unit simplex and the relative interior of the core. This homeomorphism cannot be extended to non-negative weights and boundary core allocations, except when the game is strictly convex.

3.8 An illustration: liability games

Liability games have been introduced by Dehez and Ferey (2013). They cover situations where damage has been caused to a victim by several tortfeasors ("players") following the natural order 1,2,…,*n*. Each player is responsible for an additional damage, d_i for player *i*. The associated game (N,v) is then given by:

$$
\begin{aligned}
&v(S)=0 && \text{if } 1\notin S\\
&v(S)=d_1 && \text{if } 1\in S \text{ and } 2\notin S\\
&v(S)=d_1+d_2 && \text{if } 1,2\in S \text{ and } 3\notin S
\end{aligned}
$$

and so on… Defining $T_i=\{1,...,i\}$ as the set of subsequent players, starting with 1 and ending with *i*, the characteristic function can be written as:

$$v(S)=\sum_{i\in N} d_i\, u_{T_i}(S)$$

The problem is to divide the total damage $v(N)=d_1+...+d_n$ among the *n* players. The resulting allocation specifies the compensation that each player must pay to the victim. The Harsanyi dividends of a liability game are given by:

$$
\begin{aligned}
\alpha_T(N,v) &= d_i && \text{if } T=T_i\\
&=0 && \text{otherwise}
\end{aligned}
\tag{21}
$$

Hence, liability games are positive and thereby convex and all solutions coincide with the core.[18] In the 3-player case, the vector of dividends is given by $\alpha(N,v)=(d_1,0,0,d_2,0,0,d_3)$.

Using (8), it can be verified that the core of a liability game can be written as:

$$C(N,v)=\left\{x\in\mathbb{R}_+^n \,\middle|\, x(N)=d(N) \text{ and } \sum_{j=i}^{n}x_j\le\sum_{j=i}^{n}d_j \text{ for all } i\in N\right\}$$

By convexity, it is the convex polytope whose 2^{n-1} vertices are the marginal contribution vectors as defined by (5). In the 3-player case, there are four distinct marginal contribution vectors:

$$
\begin{aligned}
&\mu^{(1,2,3)}=(d_1,d_2,d_3)\\
&\mu^{(1,3,2)}=\mu^{(3,1,2)}=(d_1,d_2+d_3,0)\\
&\mu^{(2,1,3)}=(d_1+d_2,0,d_3)\\
&\mu^{(2,3,1)}=\mu^{(3,2,1)}=(d_1+d_2+d_3,0,0)
\end{aligned}
$$

[18] Liability games are dual of airport games, known to be positive games.

We observe that the allocation that imposes to the first player to pay the entire damage as well as the allocation $x = d$ that imposes to players to pay exactly their additional damages are core allocations. The core is illustrated in Figure 1.

In the context of Tort Law, weights can be used to reflect the degree of misconduct or negligence. Using (17) and (21), the weighted Shapley value associated to given positive weights $(w_1,...,w_n)$ defines the following apportionment rule:

$$SV_i(N,v,w) = \sum_{j=i}^{n} \frac{w_i}{w(T_j)} d_j \tag{22}$$

In the 3-player case, we obtain the following allocation:

$$x_1 = \frac{w_1}{w_1+w_2+w_3} d_3 + \frac{w_1}{w_1+w_2} d_2 + d_1$$

$$x_2 = \frac{w_2}{w_1+w_2+w_3} d_3 + \frac{w_2}{w_1+w_2} d_2$$

$$x_3 = \frac{w_3}{w_1+w_2+w_3} d_3$$

This triangular formula shows the one-to-one relationship that exists under convexity between the relative interior of the core and the relative interior of the unit simplex: to each core allocation, there exists one and only one weight vector in $w \in \operatorname{int}\Delta_n$ and vice versa. For boundary core allocations, limits have to be considered. For instance, the allocation $x = d$ corresponds to sequences $(w^k) \subset \operatorname{int}\Delta_n$ where w_1^k and w_2^k converge to zero and the ratio w_1^k / w_2^k converges to zero as well. Many boundary core allocations imply the exemption of some players. For instance, the allocation $(d_1+d_2+d_3, 0, 0)$ that exempts players 2 and 3 is associated to the weight vector (1,0,0).

Using (22), the symmetric Shapley value is given by:

$$SV_i(N,v) = \sum_{j=i}^{n} \frac{1}{j} d_j$$

In the 3-player case, we get the following allocation:

$$x_1 = \frac{1}{3} d_3 + \frac{1}{2} d_2 + d_1$$

$$x_2 = \frac{1}{3} d_3 + \frac{1}{2} d_2$$

$$x_3 = \frac{1}{3} d_3$$

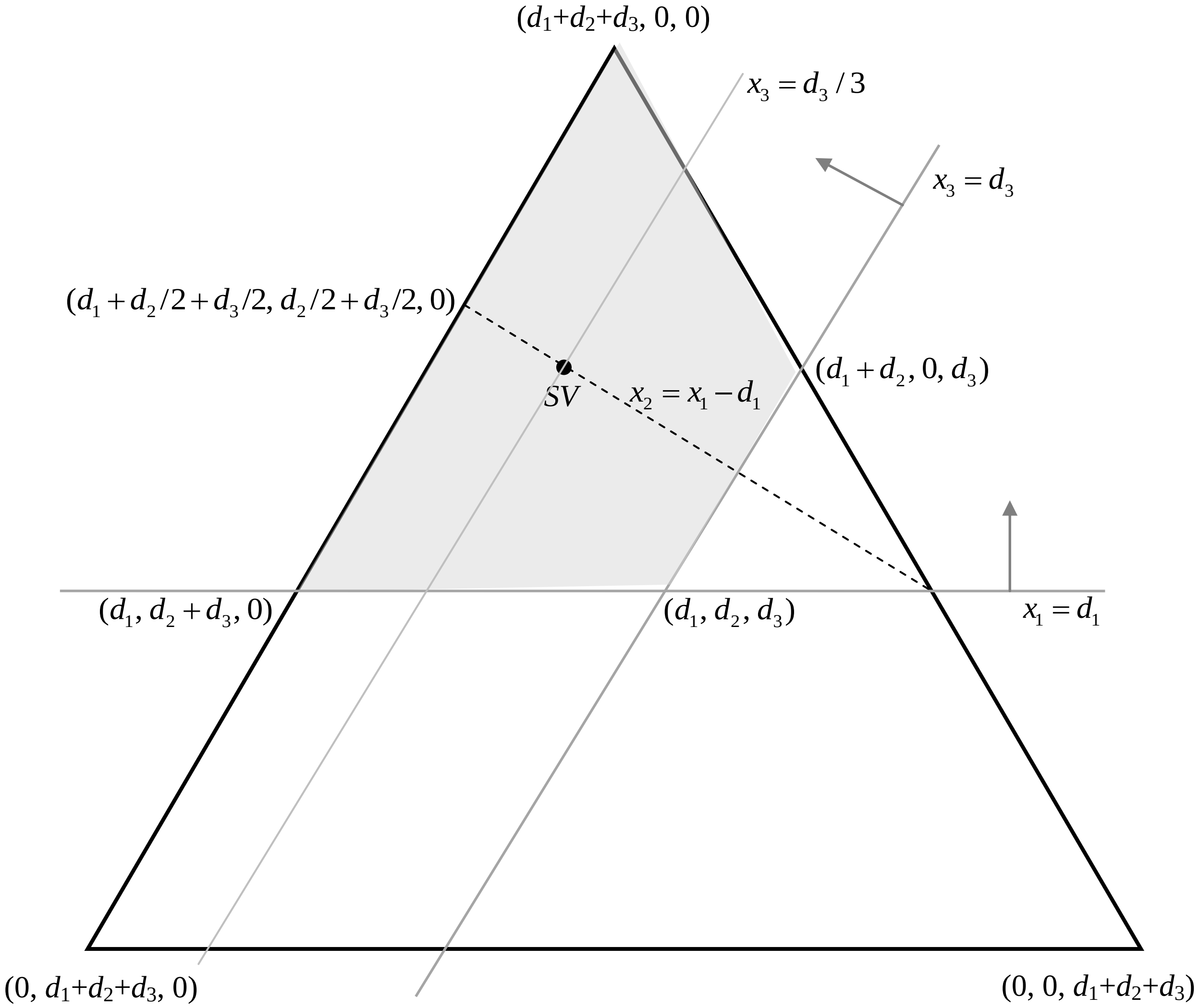

Figure 1: The core of a 3-player sequential liability game

Harsanyi payoff vectors have the same structure. They are given by

$$h_i(N,v,\lambda)=\sum_{j=i}^{n}\lambda_i^{T_j}d_j \tag{23}$$

where only the $\lambda_i^{T_j}$ matter. In the 3-player case, they are given by:

$$\begin{aligned}
x_1 &= \lambda_1^{123}d_3+\lambda_1^{12}d_2+d_1\\
x_2 &= \lambda_2^{123}d_3+\lambda_2^{12}d_2=\lambda_2^{123}d_3+(1-\lambda_1^{12})d_2\\
x_3 &= \lambda_3^{123}d_3=(1-\lambda_1^{123}-\lambda_2^{123})d_3
\end{aligned}$$

It is important to notice that, as a rule, the H-payoff vector and the weighted Shapley value are such that no one is liable for damage caused *downstream* in the liability sequence: what player i contributes depends only on $(d_i,...,d_n)$. This is a characteristic of the core. The two solutions however differ in the number of coefficients they require: weighted values require the specification of $n-1$ coefficients while Harsanyi payoff vectors require the specification of $n(n-1)/2$ coefficients. Hence, the later offer more degree of freedom and it may be more appropriate to use dividend allocations instead of weights to characterize judgments: a judgment would then require specifying how each additional damage d_i ($i = 1,\dots,n$) is to be allocated between the players 1 to i. However, for more than three players, there is no one-to-one correspondence between core allocations and dividend distributions.

4. Characterizing solutions

There are different ways to characterize solutions. Here we will consider two ways: by axioms and by restrictions on dividend distributions.

4.1 Characterizing solutions by axioms

Consider the following four properties:

Null player: i null in (N,v) $\Rightarrow$ $x_i=0$ for all $x\in\Phi(N,v)$

Additivity: $\Phi(N,v_1+v_2)=\Phi(N,v_1)+\Phi(N,v_2)$

Weak positivity: $v\in G^+(N)$ $\Rightarrow$ $\Phi(N,v)\subset\mathbb{R}_+^n$

Strong positivity: $v\in MG(N)$ $\Rightarrow$ $\Phi(N,v)\subset\mathbb{R}_+^n$

These are usual properties.[19] The following proposition due to Vasil'ev (1975, 2006) provides an axiomatization of the Harsanyi set.[20] We give here a simple proof.

Proposition 6 The Harsanyi set is the *unique* solution $\Phi:G(N)\to\mathbb{R}^n$ satisfying efficiency, null player, additivity and weak positivity.

---

[19] The term monotonicity is sometime used instead of positivity.

[20] Vasil'ev also requires homogeneity although only additivity is actually needed.

Proof It is easily verified that, for all games, the Harsanyi set defines an application that satisfies efficiency, null player, additivity and weak positivity. We have to verify unicity. Applied to the unanimity game $(N,\lambda u_T)$ where $T \subset N$ and $\lambda > 0,$ an application Φ satisfying efficiency, positivity and null player gives:

$$\Phi(N,\lambda u_T) = \{x \in \mathbb{R}^n_+ \mid x(N) = \lambda \text{ and } x_i = 0 \text{ for all } i \notin T\}$$

Indeed, the game $(N,\lambda u_T)$ is positive for $\lambda > 0$ and players outside T are null players. Consider a game (N,v) and its decomposition (7) as $v = v^+ - v^-$. Additivity implies:

$$\Phi(N,v^+) = \Phi(N,v^+ - v^-) + \Phi(N,v^-)$$

i.e. $\Phi(N,v) = \Phi(N,v^+) - \Phi(N,v^-)$ where (N,v^+) and (N,v^-) are positive games. Hence, there is a unique application Φ satisfying efficiency, weak positivity, null player and additivity. it is given by:

$$\begin{aligned}
\Phi(N,v) &= \sum_{T:\alpha_T>0} \Phi(N,\alpha_T u_T) - \sum_{T:\alpha_T<0} \Phi(N,-\alpha_T u_T) \\
&= \sum_{T:\alpha_T>0} \{x \in \mathbb{R}^n_+ \mid x(T) = \alpha_T \text{ and } x(N\backslash T) = 0\} \\
&\qquad - \sum_{T:\alpha_T<0} \{x \in \mathbb{R}^n_+ \mid x(T) = -\alpha_T \text{ and } x(N\backslash T) = 0\}
\end{aligned}$$

This is precisely the Harsanyi set of the game (N,v), as given by (15). ♦

Replacing weak positivity by strong positivity, results in the Weber set: the Weber set is the subset of H-payoff vectors that satisfy the strong positivity axiom.

To obtain the Shapley set, it is necessary to introduce an axiom that applies to values. Derks et al. (2000) use the following axiom:

Consistency: $i \in S \subset T \;\Rightarrow\; \varphi_i(N, \varphi(N,u_T)(S)\, u_S) = \varphi_i(N,u_T)$

It is weaker version of the *partnership* axiom introduced by Kalai and Samet (1987) to axiomatize the Shapley set. It is satisfied by the weighted value associated to positive weights. Indeed, we have:

$$i \in T \;\Rightarrow\; SV(N,u_T,w) = \frac{w_i}{w(T)}$$

and

$$i \in S \subset T \;\Rightarrow\; SV(N,\varphi(N,u_T)(S)\, u_T,w) = \frac{w_i}{w(S)}\frac{w(S)}{w(T)} = \frac{w_i}{w(T)}$$

The Shapley set is the subset of H-payoff vectors that satisfy the consistency axiom.

Remark 9 Combining Proposition 5 and the argument used in the proof of Proposition 6, we have the following identities:[21]

$$C(N,v^{+}) = H(N,v^{+}) \text{ and } C(N,v^{-}) = H(N,v^{-})$$

and

$$H(N,v) = C(N,v^{+}) - C(N,v^{-})$$

4.2 Characterizing solutions by restrictions on dividend distributions

Harsanyi payoff vectors are defined by distribution matrices in Λ_n without any further restrictions. A natural question concerns the identification of restrictions on distribution matrices such that the resulting set of H-payoff vectors corresponds to particular solutions.

Derks et al (2000) suggest to link distributions within a coalition to distributions within its sub-coalitions by requiring that a player's share in the dividend of a coalition cannot increase if the coalition gets larger:

$$i \in S \subset T \;\Rightarrow\; \lambda_i^S \geq \lambda_i^T \tag{24}$$

If a player leaves a coalition, that should not reduce the share of those remaining in the coalition. This monotonicity property imposes strong restrictions on distribution matrices. In particular, if the share of a player in a coalition is zero, his share must be equally zero for all larger coalitions. We denote by $H^m(N,v)$ the subset of Harsanyi payoffs vectors derived from distribution matrices satisfying (24). Derks et al. (2000) have shown that the distribution matrices associated to random order values are monotonic, i.e. $W(N,v) \subset H^m(N,v)$. There may however be H-payoff vectors derived from monotonic distribution matrices that are not random order value.

A stronger monotonicity requirement is needed. Vasil'ev (1988) and Derks et al. (2006) have proved the following proposition.

Proposition 7 The H-payoff vectors derived from distribution matrices λ satisfying the following inequalities

$$\sum_{S:S\supset T} (-1)^{s-t} \lambda_i^S \geq 0 \text{ for all } T \subset N \text{ and } i \in T \tag{25}$$

are random order values.

More precisely, if (q_i^S) and (λ_i^S) satisfy

$$q_i^S = \sum_{S:S\supset T} (-1)^{s-t} \lambda_i^S \text{ and } q_i^S \geq 0 \text{ for all } T \subset N \text{ and } i \in T$$

[21] See Lemma 3 in Derks, Haller and Peeters (2000).

the corresponding H-payoff vector and quasi-value coincide:

$$h_i(N,v,\lambda) = \sum_{S \subset N} q_i^S \left(v(S) - v(S \setminus i)\right) \quad (i = 1,\ldots,n)$$

Hence, $H^{sm}(N,v) = W(N,v)$ where $H^{sm}(N,v)$ denotes the set of Harsanyi payoff vectors derived from distribution matrices satisfying (25).

An even stronger restriction consists in assuming that the vector λ^N defines the "master" probability distribution from which all other distribution vectors λ^T for $T \subset N, t \geq 2,$ are derived by Bayesian updating:

$$i \subset T \subset N \;\Rightarrow\; \lambda_i^T = \frac{\lambda_i^N}{\lambda^N(T)} \text{ whenever } \lambda^N(T) > 0 \tag{26}$$

Derks et al. (2000) have proved the following proposition.

Proposition 8 The H-payoff vectors derived from distribution matrices λ satisfying (26) are weighted Shapley values.

Proof When normalized weights and shares in α_T are equal i.e. $w = \lambda^N$, we have:

$$h_i(N,v,\lambda) = \sum_{T \subset N} \lambda_i^T \alpha_T = \sum_{T \in \mathcal{C}_i(N)} \frac{\lambda_i^N}{\lambda^N(T)} \alpha_T = \sum_{T \in \mathcal{C}_i(N)} \frac{w_i}{w(T)} \alpha_T = SV_i(N,v,w) \qquad \blacklozenge$$

Hence, $H^b(N,v) = SV(N,v)$ where $H^b(N,v)$ denotes the set of H-payoff vectors derived from distribution matrices satisfying (26). It is easily verified that under (26), the distribution matrices satisfy the monotonicity requirements (24) and (25).

4.3 Implications of monotonic dividend distributions

Can we identify $H^m(N,v),$ the set of H-payoff vectors derived from distribution matrices satisfying the monotonicity conditions (24) ? We already know that $W(N,v) \subset H^m(N,v).$ Furthermore, $C(N,v) = H(N,v)$ if (and only if) (N,v) is almost positive, in which case monotonicity has no impact and core allocations can be written as H-payoff vectors derived from monotonic dividend distributions: $C(N,v) = H^m(N,v) = H(N,v).$

This is illustrated by liability games: core allocations correspond to H-payoffs vectors defined by monotonic dividend distributions and, vice versa, monotonic dividend distributions define core allocations. To see this, consider the 3-player case. The core is the convex hull its four vertices, $(d_1 + d_2 + d_3, 0, 0)$, $(d_1 + d_2, 0, d_3),$ (d_1, d_2, d_3) and $(d_1, d_2 + d_3, 0)$ i.e. core allocations are of the form

$$x = \left(d_1 + (\alpha_1 + \alpha_2)d_2 + \alpha_1 d_3, (\alpha_3 + \alpha_4)d_2 + \alpha_4 d_3, (\alpha_2 + \alpha_3)d_3\right) \text{ for some } \alpha \in \Delta_4$$

The corresponding dividend distributions are given by:

$$\lambda^{12} = (\alpha_1 + \alpha_2, \alpha_3 + \alpha_4)$$
$$\lambda^{123} = (\alpha_1, \alpha_4, \alpha_2 + \alpha_3)$$

They clearly satisfy the monotonicity condition (24).

Billot and Thisse (2005) claim that $H^m(N,v)$ coincides with the core if (and only if) the game is convex. This is actually only true for 3-player games! Consider a 3-player game and let *x* be the H-payoff vector corresponding to some monotonic dividend distribution. Individual rationality results from superadditivity:

$$\begin{aligned} x_1 - v(1) &= \lambda_1^{12}(v(12) - v(1) - v(2)) + \lambda_1^{13}(v(13) - v(1) - v(3)) \\ &\quad + \lambda_1^{123}(v(123) - v(12) - v(13) - v(23) + v(1) + v(2) + v(3)) \\ &\geq \lambda_1^{123}(v(123) - v(23) - v(1)) \geq 0 \end{aligned} \tag{27}$$

Considering 2-player coalitions, superadditivity implies the following inequality:

$$\begin{aligned} (x_1 + x_2) - v(12) &= \lambda_1^{13}(v(13) - v(1) - v(3)) + \lambda_2^{23}(v(23) - v(2) - v(3)) \\ &\quad + (\lambda_1^{123} + \lambda_2^{123})(v(123) - v(12) - v(13) - v(23) + v(1) + v(2) + v(3)) \\ &\geq \lambda_1^{123}(v(123) - v(12) - v(23) + v(2)) \\ &\quad + \lambda_2^{123}(v(123) - v(12) - v(13) + v(1)) \end{aligned}$$

where the last part is non-negative under convexity. The argument applies identically to all single players and 2-player coalitions, confirming that *x* is a core allocation.

For more than three players, convexity is not sufficient to ensure that H-payoff vectors are core allocations under monotonicity. Consider the 4-player convex game defined by $v(S) = s - 1.$ Its dividends are given by:

$$\alpha_{\{i\}} = 0$$
$$\alpha_{\{i,j\}} = 1$$
$$\alpha_{\{i,j,k\}} = -1$$
$$\alpha_{\{1,2,3,4\}} = 1$$

The H-payoffs associated to the monotonic matrix given by Table 1 are $(0.4, 0.7, 0.8, 1.1),$ an allocation that does not belong to the core: the coalition {1,2,3} indeed obtains only 1.9.

For 3-player games, (27) tells us that superadditivity is enough to ensure that monotonic dividend distributions define imputations. For 4-player games, adding convexity ensures individual rationality. To simplify and without loss of generality, let's assume $v(i)=0$ for all i. Considering an arbitrary distribution matrix λ, the H-payoff of player 1 is given by:

$$\begin{aligned} x_1 = {} & \lambda_1^{12}v(12)+\lambda_1^{13}v(13)+\lambda_1^{14}v(14) \\ & +\lambda_1^{123}(v(123)-v(12)-v(13)-v(23)) \\ & +\lambda_1^{124}(v(124)-v(12)-v(14)-v(24)) \\ & +\lambda_1^{134}(v(134)-v(13)-v(14)-v(34)) \\ & +\lambda_1^{1234}(v(1234)-v(123)-v(124)-v(134)-v(234) \\ & \qquad +v(12)+v(13)+v(14)+v(23)+v(24)+v(34)) \end{aligned}$$

By Lemma 1, $v(S)\geq 0$ for all S. Using convexity and applying (24), we obtain:

$$\begin{aligned} x_1 \geq {} & \lambda_1^{123}(v(123)-v(13)-v(23)) \\ & +\lambda_1^{124}(v(124)-v(12)-v(24)) \\ & +\lambda_2^{134}(v(134)-v(14)-v(34)) \\ & +\lambda_1^{1234}(v(1234)-v(123)-v(124)-v(134)-v(234) \\ & \qquad +v(12)+v(13)+v(14)+v(23)+v(24)+v(34)) \end{aligned}$$

Applying again (24) and making the appropriate simplifications, we finally get:

$$x_1 \geq \lambda_1^{1234}(v(1234)-v(234))\geq 0$$

Convexity is needed for the result to hold. Indeed, consider the following 4-player game and associated dividends.

$$\begin{array}{lcl} v(i)=0 & & \alpha_{\{i\}}=0 \\ v(i,j)=1 & & \alpha_{\{i,j\}}=1 \\ v(1,i,j)=1 & \rightarrow & \alpha_{\{i,j,k\}}=1-3=-2 \\ v(2,3,4)=2 & & \alpha_{\{2,3,4\}}=2-3=-1 \\ v(1,2,3,4)=2 & & \alpha_{\{1,2,3,4\}}=2-5+6=3 \end{array}$$

This game is superadditive but not convex. The distribution matrix given by Table 2, while satisfying the conditions of monotonicity, leads to an allocation that violates individual rationality. Player 1 is indeed allocated a negative amount: $x_1=2.1-4.2+1.65=-0.45<0$. Beyond four players, convexity does not ensure individual rationality under monotonicity.

Indeed, consider the following game:

$$\begin{array}{lcl} v(i)=0 & & \alpha_{\{i\}}=0 \\ v(i,j)=1 & & \alpha_{\{i,j\}}=1 \\ v(1,i,j)=2 & & \alpha_{\{1,i,j\}}=-1 \\ v(i,j,k)=3 & \rightarrow & \alpha_{\{i,j,k\}}=0 \\ v(1,i,j,k)=4 & & \alpha_{\{1,i,j,k\}}=1 \\ v(2,3,4,5)=5 & & \alpha_{\{2,3,4,5\}}=-1 \\ v(1,2,3,4,5)=6 & & \alpha_{\{1,2,3,4,5\}}=-1 \end{array}$$

Again here, player 1 is allocated a negative amount on the basis of the monotonic distribution matrix given by Table 3.[22]

4.4 Graph structures and restrictions on dividend distributions

Given a game, additional data can be used to place restrictions on dividend distribution. The structure of a game itself may also convey information on players leading to such restrictions. This is the case where players are ordered like for instance river games (Ambec and Sprumont, 2002) and liability games (Dehez and Ferey, 2014).

This question has been studied by van den Brink, van der Laan and Vasil'ev (2014) where the ordering of players is modeled by a *directed graph*. Given a finite set N, a directed graph $D \subset N \times N$ is a set of pairs (i, j) such that $(i,i) \notin D$. The set of directed graphs is denoted by $\mathscr{D}$. Nodes are players and $(i, j) \in D$ means that i precedes j. For a given game, the payoff of a player then depends not only upon the characteristic function but also on his position on the graph.[23]

More specifically, van den Brink et al. (2014) suggest that, for any given pair of players, the shares in the dividends of all coalitions containing them be larger or equal for the player that precedes the other:

$$i, j \in S \text{ and } (i, j) \in D \Rightarrow \lambda_i^S \geq \lambda_j^S \tag{28}$$

Given a game (N,v) and a graph $D \in \mathscr{D}$, this condition leads to a subset of the Harsanyi set that we denote by $HG(N,v,D)$. The authors consider only positive games in which case $HG(N,v,D)$ is a subset of the core by Proposition 5, called the *Harsanyi constrained core*. They prove the following two propositions, the second one applying only to positive games.

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[22] We only reproduce the shares of player 1. Shares can easily be allocated to the other players so as to satisfy monotonicity.

[23] Another instance of graph-driven restrictions on dividend distributions is given by van den Brink, van der Laan and Pruzhansky (2011) who consider games with communication graphs à la Myerson (1977). The idea is to link the dividend distribution to the "power" of the players as measured for instance by the size of their neighborhood.

	12	13	14	23	24	34	123	124	134	234	1234
1	0.5	0.4	0.3	0	0	0	0.4	0.3	0.3	0	0.2
2	0.5	0	0	0.5	0.3	0	0.3	0.3	0	0.2	0.2
3	0	0.6	0	0.5	0	0.3	0.3	0	0.3	0.2	0.2
4	0	0	0.7	0	0.7	0.7	0	0.4	0.4	0.6	0.4

Table 1

	12	13	14	23	24	34	123	124	134	234	1234
1	0.7	0.7	0.7	0	0	0	0.7	0.7	0.7	0	0.55
2	0.3	0	0	0.7	0.3	0	0.15	0.15	0	0.15	0.15
3	0	0.3	0	0.3	0	0.3	0.15	0	0.15	0.15	0.15
4	0	0	0.3	0	0.7	0.7	0	0.15	0.15	0.7	0.15

Table 2

12	13	14	15	123	124	125	134	135	145	1234	1235	1245	1345	12345
0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
0.9	0	0	0	0.45	0.45	0.45	0	0	0	0.33	0.33	0.33	0	0.25
0	0.9	0	0	0.45	0	0	0.45	0.45	0	0.33	0.33	0	0.33	0.25
0	0	0.9	0	0	0.45	0	0.45	0	0.45	0.33	0	0.33	0.33	0.25
0	0	0	0.9	0	0	0.45	0	0.45	0.45	0	0.33	0.33	0.33	0.25

Table 3

Proposition 9 For any given game (N,v) and graph $D \in \mathscr{D}$, the Shapley value $SV(N,v)$ is an element of $H(N,v,D)$. Furthermore, the Shapley value is the *unique* element of $H(N,v,D)$ if and only if D is the *complete* directed graph $\bar{D} = \{(i,j) \in N \times N \mid i \neq j\}$.

Proof The Shapley value is defined by $\lambda_i^S = 1/s$ for all $i \in S$ and $S \subset N$ and therefore (27) is verified for all graph D. If $D = \bar{D}$, $\lambda_i^S = \lambda_j^S$ for all $i, j \in S$ and $S \subset N$ and $\lambda_i^S = 1/s$ for all $i \in S$ and $S \subset N$. Hence, $HG(N,v,\bar{D}) = \{SV(N,v)\}$. Next, consider a directed graph D such that $(j,k) \notin D$ for some j and k, $j \neq k$, and the unanimity game $(N, u_{\{j,k\}})$. From Remark 1, the allocation $\bar{x} \in \mathbb{R}^n$ defined by $\bar{x}_k = 1$ and $\bar{x}_i = 0$ for all $i \neq k$ belongs to $HG(N, u_{\{j,k\}}, D)$. It differs from the Shapley value $SV(N, u_{\{j,k\}})$ which allocated 1/2 to j and k. ♦

Proposition 10 For any *positive* game $(N,v) \in G^+(N)$ and graph $D \in \mathscr{D}$,

$$H(N,v,D) = C(N,v) \text{ if and only if } D = \varnothing.$$

Proof When $D = \varnothing$, $H(N,v,D) = H(N,v) = C(N,v)$ as a result of Proposition 5. When instead $D \neq \varnothing$, there exist j and k such that $(j,k) \in D$ and the allocation $\bar{x}$ previously defined does not belong to $HG(N, u_{\{j,k\}}, D)$. However, it belongs to $C(N, u_{\{j,k\}})$. ♦

van den Brink et al. (2014) also characterize axiomatically the Harsanyi constrained core on the class $G^+(N)$ of positive games. A solution associates a subset $\Phi(N,v,D)$ to a game $(N,v) \in G^+(N)$ and directed graph D. They show that the Harsanyi constrained core is *maximal* among the solutions satisfying efficiency, null player property, additivity, weak positivity and the additional property of *structural monotonicity* defined by:

> For all game $(N,v) \in G^+(N)$ and directed graph $D \in \mathscr{D}$, all allocation $x \in \Phi(N,v,D)$ are such that $x_i \geq x_j$ if $(i,j) \in D$ and i is necessary to j in (N,v).

Liability games are positive and are associated to the directed graph

$$D = \{(i,j) \mid i = 1,\ldots,n-1 \text{ and } j = i+1\}$$

It reduces to $D = \{\{1,2\},\{2,3\}\}$ in the 3-player case. (28) implies that damage d_i is distributed among the players in $T_i = \{1,\ldots,i\}$ according to coefficients satisfying

$$\lambda_j^{T_i} \geq \lambda_{j+1}^{T_i} \text{ for } j = 1,\ldots,i-1$$

In the 3-player case, it means $\lambda_1^{12} \geq \lambda_2^{12} = 1 - \lambda_1^{12}$ and $\lambda_1^{123} \geq \lambda_2^{123} \geq \lambda_3^{123} = 1 - \lambda_1^{123} - \lambda_2^{123}$. These inequalities in turn imply $\lambda_1^{12} \geq 1/2$, $\lambda_1^{123} \geq 1/3$, $\lambda_2^{123} \leq 1/2$ and $\lambda_3^{123} \leq 1/3$ as shown in Figure 2. The resulting Harsanyi constrained core is shown in Figure 3.

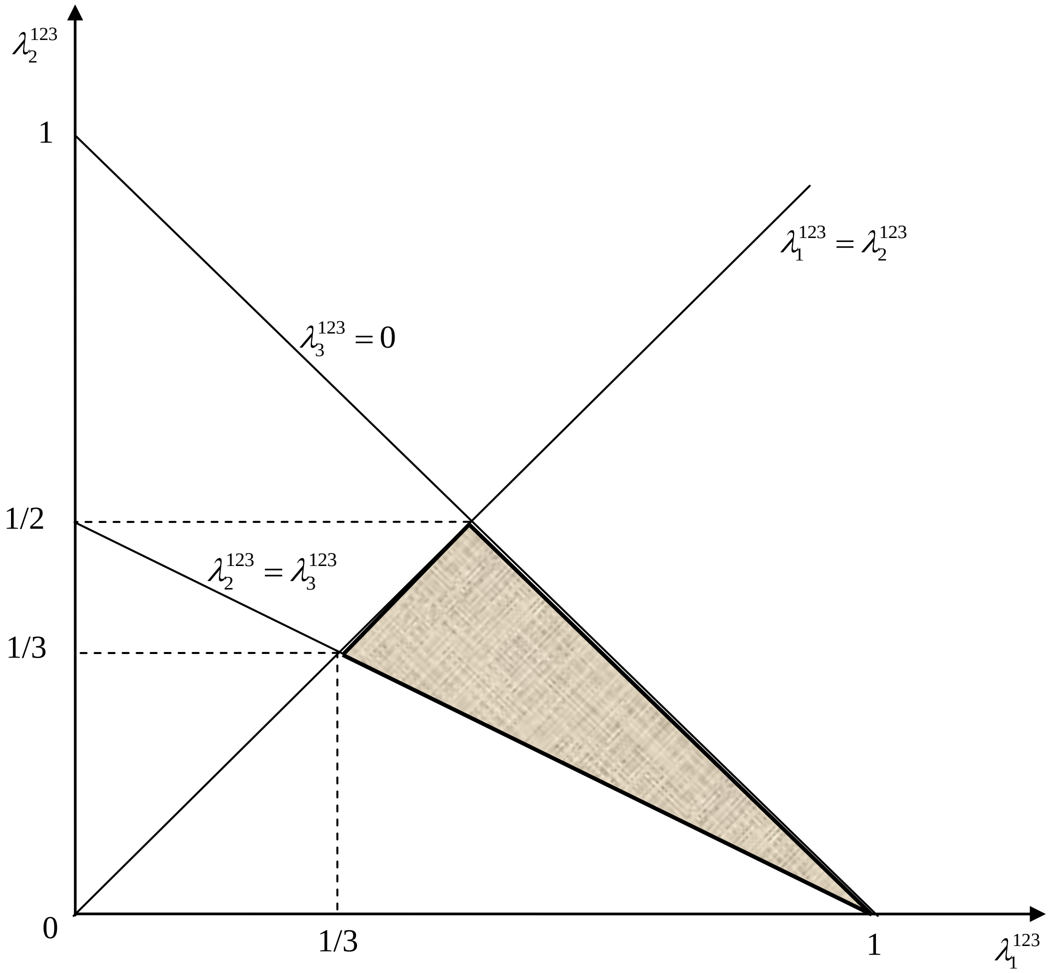

Figure 2: Admissible distributions of d_3

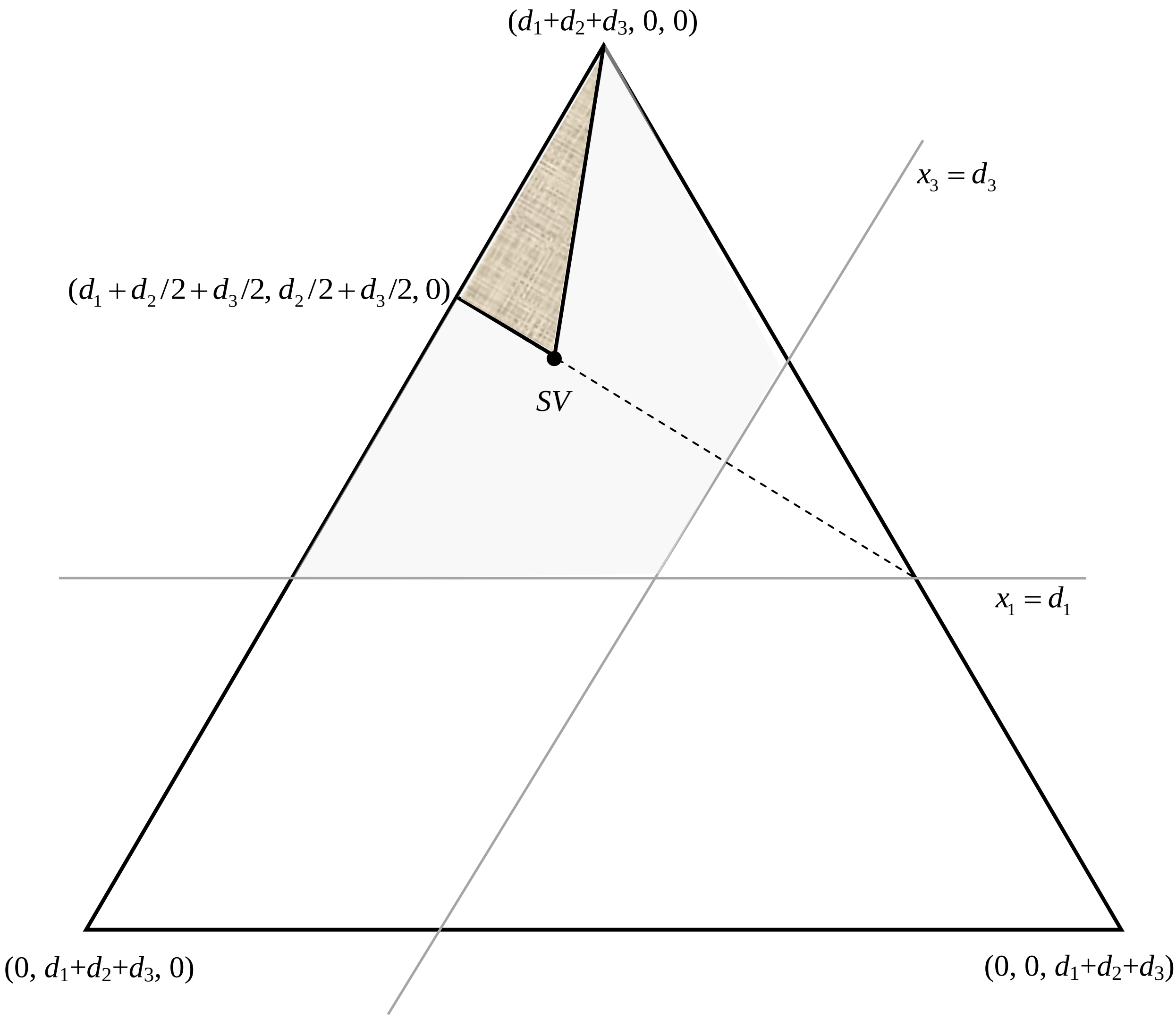

Figure 3: The Harsanyi constrained core of a 3-player sequential liability game

5. Concluding remarks

Other solution concepts could be considered, for instance the *nucleolus* introduced by Schmeidler (1969). When the core is nonempty, it is a core selection and we know that it is then a particular H-payoff vector. Can it be characterized in terms of dividend distributions? The answer is negative: liability games offer a counter-example. As shown in Dehez-Ferey (2013), the *nucleolus* of a 3-player liability game is given by

$$\begin{aligned}\varphi(N,v) &= \left(d_1+\frac{d_2}{2}+\frac{d_3}{4},\frac{d_2}{2}+\frac{d_3}{4},\frac{d_3}{2}\right) && \text{if } d_3 \le 2d_2 \\ &= \left(d_1+\frac{d_2+d_3}{3},\frac{d_2+d_3}{3},\frac{d_2+d_3}{3}\right) && \text{if } d_3 \ge 2d_2\end{aligned}$$

In the first case, it is average of core's vertices that is the H-payoff associated to $\lambda_1^{12} = 1/2$ and $\lambda_1^{123} = \lambda_2^{123} = 1/4.$ In the second case, it is the *equal loss* allocation to which it is not possible to associate an admissible distribution matrix. As an apportionment rule, the nucleolus violates the "downstream" condition: in the second case, what player 3 contributes depends upon damage caused by player 2.

Among the questions that remain open, there is the identification of restrictions on dividend distributions such that the resulting H-payoff vectors are imputations. At this stage, we have only learned that monotonicity is not sufficient even if convexity is assumed.

Appendix

Lemma 1 Consider a game (N,v) such that $v(i)\geq 0$ for all $i\in N$. Superadditivity then implies that the characteristic functions v is monotone and positively valued.

Proof Consider a coalition $S\subset N$ and a player $i\notin S$. By superadditivity, we have:

$$v(S\cup i)\geq v(S)+v(i)$$

Hence, if $v(i)\geq 0$ for all $i\in N$, adding a player to a coalition does not decrease its worth. This extends to the addition of any number of players. Positivity of v follows using the above inequality, starting with $S=\{j\},\ j=1,\ldots,n.$ ♦

Lemma 2 If i is necessary for j in (N,v), then $\alpha_T(N,v)=0$ for all $T\subset N\setminus i$ such that $j\in T$.

Proof Consider a coalition $T\subset N\setminus i$ not containing j. We already know that $\alpha_T=v(j)=0$ if $T=\{j\}$. Assume now that $\alpha_S=0$ for all $S\subset T$ such that $j\in S$. We proceed by induction using (2). Because i is necessary for j in (N,v), we have:

$$\alpha_T=v(T)-\sum_{S\subset T}\alpha_S=v(T)-\sum_{S\subset T\setminus j}\alpha_S=v(T)-v(T\setminus j)=0 \quad ♦$$

Lemma 3 The core is a superadditive solution (Peleg, 1986).

Proof Given a player set N, consider two set functions $v_1,v_2\in G(N)$ and an allocation $x\in C(N,v_1)+C(N,v_2)$. Then using (?), there exist $x^1\in C(N,v_1)$ and $x^2\in C(N,v_2)$ such that $x=x^1+x^2$ i.e. $x^1(S)\geq v_1(S)$ and $x^2(S)\geq v_2(S)$ for all $S\subset N$. Hence, we have:

$$x(S)=x^1(S)+x^2(S)\geq (v_1+v_2)(S) \text{ for all } S\subset N$$

i.e. $x\in C(N,v_1+v_2)$. ♦

Lemma 4 The Weber set is a subadditive solution (Dragan, Potters and Tijs, 1989).

Proof Given a player set N, consider two set functions $v_1,v_2\in G(N)$ and the corresponding marginal contribution vectors μ^1 and μ^2 as defined by (6). By definition of convex hull (Rockafellar, 1970), we have:

$$\begin{aligned} W(N,v_1)+W(N,v_2) &= co\{\mu^1(\pi)\mid \pi\in\Pi_N\}+co\{\mu^2(\pi)\mid \pi\in\Pi_N\} \\ &= co\left(\{\mu^1(\pi)\mid \pi\in\Pi_N\}+\{\mu^2(\pi)\mid \pi\in\Pi_N\}\right) \end{aligned}$$

The marginal contribution vectors associated to the game (N,v^1+v^2) are the sum of the marginal contribution vectors. Hence, $W(N,v^1+v^2)=co\{\mu^1(\pi)+\mu^2(\pi)\mid \pi\in\Pi_N\}$ where

$$\{\mu^1(\pi)+\mu^2(\pi)\mid \pi\in\Pi_N\}\subset\{\mu^1(\pi)\mid \pi\in\Pi_N\}+\{\mu^2(\pi)\mid \pi\in\Pi_N\}$$

Consequently, by the definition of the convex hull, we have:

$$co\{\mu^1(\pi)+\mu^2(\pi)\mid \pi\in\Pi_N\}\subset co\left(\{\mu^1(\pi)\mid \pi\in\Pi_N\}+\{\mu^2(\pi)\mid \pi\in\Pi_N\}\right) \quad ♦$$

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