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SDEs with uniform distributions: Peacocks, Conic martingales and mean reverting uniform diffusions

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Abstract

We introduce a way to design Stochastic Differential Equations of diffusion type admitting a unique strong solution distributed as a uniform law with general conic time-boundaries. We show that these processes are new diffusion martingales, hence peacocks, and recover two previously known special cases with square-root and linear time-boundaries. We study local time and activity of such processes. We further introduce general mean-reverting diffusion processes having a uniform law at all times evolving between constant boundaries. This may be used to model random probabilities, random recovery rates or random correlations. We verify via an Euler scheme simulation that they have the desired uniform behavior.

Keywords: Uniformly distributed Stochastic Differential Equation, Conic Martingales, Peacock Process, Uniformly distributed Diffusion, Uniform Martingale Diffusions, Mean Reverting Uniform Diffusion, Mean Reverting Uniform SDE, Maximum Entropy Stochastic Recovery Rates, Maximum Entropy Stochastic Correlation, Uniform SDE Simulation.

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1 Introduction

We introduce a new family of regular diffusion martingales

$$dX_t = \left(\mathbb{I}_{\{X_t \in (-b(t), b(t))\}} \frac{\dot{b}(t)}{b(t)} (b(t)^2 - X_t^2) \right)^{1/2} dW_t, \quad X_0 = 0,$$

for deterministic grounded support increasing boundaries b and W a Brownian motion. We show that this SDE admits a unique strong solution which is a peacock process with uniform distribution on the conically expanding boundary governed by b. This extends previously known results where $b(t)$ is either $\sqrt{t}$ (p. 523-525 in [9]) or t (p. 253 in [10]).

The above diffusion coefficient was initially guessed by informally inverting the forward Kolmogorov or Fokker-Planck equation, when forcing the density of the solution X to be uniform at all times with support $[-b, b]$, as initially sketched in the preprint [3]. This inversion technique was used in the past to construct diffusion processes with densities in exponential families [2, 4] and has been used more generally in a variety of contexts in mathematical finance. For example, [7] finds the diffusion coefficient ("local volatility") that is consistent with a probability law extrapolated from a surface of option prices. The paper [6] deals with designing a diffusion process consistent with a mixture of distributions for volatility smile modeling, whereas [5] inverts the Kolmogorov equation to show how two stochastic processes with indistinguishable laws in a time grid under the historical measure can lead to arbitrarily different option prices.

The paper is structured as follows. In Section 2 we formulate the problem. A solution is attempted in Section 3 along the lines of the above mentioned inversion. We then study the solution rigorously in Section 4 and prove that the related Stochastic Differential Equation (SDE) admits a unique strong solution. We further prove that the solution has indeed a uniform distribution with the desired conic boundary, and being it a martingale, the peacock property follows. In Section 5 we re-scale the conic diffusion martingale and study the related mean-reverting uniform diffusions, where now the uniform law is not conic but constant. Two special cases of interest are standard uniforms and uniforms in $[-1, 1]$, which can be used to model for example maximum entropy recovery rates or random probabilities and random correlations, respectively. In Section 6 we revisit the two previously known cases and hint at new choices for the boundaries. In the linear case we also study the process local time at the boundaries, to find it is zero, and the activity, finding that the pathwise activity of the mean reverting diffusion vanishes asymptotically. The behavior of the process is illustrated based on numerical simulations.

2 Conic diffusion martingales with uniform distribution

We set out to construct a martingale diffusion process X (zero drift) with marginal at time t having a uniform distribution in an interval $[a(t), b(t)]$. The martingale condition implies that $\mathbb{E}[X_t] = \mathbb{E}[X_0]$ for all $t \geq 0$, whereas the uniform distribution requirement implies that $\mathbb{E}[X_t] = [a(t) + b(t)]/2$ for all $t \geq 0$. Thus we have $a(t) + b(t) = a(0) + b(0)$ for all $t \geq 0$. We will assume $a(0) = b(0) = 0$, taking the initial condition X_0 to be deterministic and with value zero (Dirac delta law in 0). Hence $b(t) = -a(t)$ for all $t \geq 0$.

With such preliminaries in mind, we state the following

Problem 1 (Designing conic martingale diffusions with given uniform law)**.** *Consider the diffusion process*

$$dX_t = \sigma(X_t, t)dW_t, \quad X_0 = 0. \tag{1}$$

Find a diffusion coefficient $\sigma(x, t)$ such that

1. *The SDE* (1) *has a unique strong solution;*

2. *The solution of* (1) *at time $t > 0$ is uniformly distributed in $[-b(t), b(t)]$ for a non-negative strictly increasing continuous function $t \mapsto b(t)$ with $b(0) = 0$.*

In other terms, our aim is to build a diffusion martingale X as in (1) such that the process X has a density $p(x, t)$ at time $t > 0$ at the point x given by the uniform density

$$p(x, t) = \rho(x, t) := \mathbb{I}_{\{x \in [-b(t), b(t)]\}} / (2\ b(t)). \tag{2}$$

In Problem 1, b is restricted to be strictly increasing in time. The reason is that the tight upper (resp. lower) bound $b(t)$ (resp. $a(t)$) of any bounded martingale must be a non-decreasing (resp. non-increasing) function ([13]). Hence, X is a *conic martingale*; it is a martingale that exhibits a conic behavior. We will need strict monotonicity in the following derivation, so we assumed b to be strictly increasing in Problem 1.

3 Deriving the candidate SDE for a uniformly distributed martingale

Let us now guess a candidate solution σ for Problem 1. To do this, we write the forward Kolmogorov (or Fokker Planck) equation for the density p of (1), impose ρ to be a solution and derive the resulting σ. The derivation is informal but it is given full mathematical rigor by showing later that the resulting SDE (1) has a unique strong solution and confirming further, via moments analysis, that the density is indeed uniform.

The forward Kolmogorov eq. for (1) with ρ plugged in as a solution reads

$$\frac{\partial \rho(x, t)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma(x, t)^2 \rho(x, t)), \quad \rho(x, 0) = \delta_0(x). \tag{3}$$

Now we integrate twice both sides of (3) with respect to x and assume we can switch integration with respect to x and differentiation with respect to t (one can check *a posteriori* that the solution we find has a continuous partial derivative with respect to t so that Leibniz's rule can be used). We obtain

$$\frac{\partial}{\partial t}\left(\int_{-\infty}^{x}\left(\int_{-\infty}^{y}\rho(z,t)dz\right)dy\right)=\frac{1}{2}\sigma(x,t)^2\rho(x,t), \tag{4}$$

assuming the relevant first and second derivatives with respect to x on the right hand side vanish fast enough at minus infinity. Compute, substituting from (2),

$$\varphi(x,t):=\int_{-\infty}^{x}\left(\int_{-\infty}^{y}\rho(z,t)dz\right)dy=\begin{cases}0, & \text{if } x<-b(t)\\ \frac{(x+b(t))^2}{4b(t)}, & \text{if } x\in[-b(t),b(t)]\\ x, & \text{if } x>b(t)\end{cases}$$

and note that φ is continuous in x. We can write this more succinctly as

$$\varphi(x,t)=\frac{(x+b(t))^2}{4b(t)}\,\mathbb{1}_{\{x\in[-b(t),b(t)]\}}+x\,\mathbb{1}_{\{x>b(t)\}}\,. \tag{5}$$

Thus, rewriting (4) as

$$\frac{\partial\varphi(x,t)}{\partial t}=\frac{1}{2}\sigma(x,t)^2\rho(x,t), \tag{6}$$

and substituting (5) we are done. To do this, we need to differentiate φ with respect to time. The calculations are all standard but one has to pay attention when differentiating terms in (5) such as

$$\mathbb{1}_{\{x\in[-b(t),b(t)]\}}=\mathbb{1}_{\{x\geq -b(t)\}}-\mathbb{1}_{\{x>b(t)\}}$$

which can be differentiated in the sense of distributions. Let us show how to differentiate

$$\frac{d}{dt}\,\mathbb{1}_{\{x>b(t)\}}=\frac{d}{dt}\,\mathbb{1}_{\{t<b^{-1}(x)\}}=-\delta_{b^{-1}(x)}(t)$$

where the index in δ denotes the point where the Dirac delta distribution is centered. One can check that all terms involving δ's either offset each other or are multiplied by a function that vanishes at the point of evaluation.

In the end one gets, omitting time arguments and denoting differentiation with respect to time with a dot:

$$\frac{\partial\varphi(x,t)}{\partial t}=-\frac{-\dot{b}(2b)(x+b)+(2\dot{b})(x+b)^2}{2(2b)^2}\,\mathbb{1}_{\{x\in[-b,b]\}}\,.$$

We notice that $\dot{b}$ appears only in ratios $\dot{b}/b$, so that this quantity may be extended to time $t=0$ by continuity if needed.

The above quantity is the left hand side of (6). We can substitute ρ on the right hand side and we have that

$$-\frac{-\dot{b}(2b)(x+b)+(2\dot{b})(x+b)^2}{2(2b)^2}\,\mathbb{1}_{\{x\in(-b,b)\}}=\frac{1}{2}\frac{\sigma(x,t)^2}{2b}\,\mathbb{1}_{\{x\in[-b,b]\}}\,.$$

After some algebra, one obtains

$$\sigma^2(x,t) = 1\!\!\mathrm{I}_{\{x\in[-b(t),b(t)]\}} \frac{\dot b(t)}{b(t)}(b(t)^2 - x^2).$$

From the above development, we expect the diffusion coefficient $\sigma(x,t)$ defined as

$$\sigma(x,t) := 1\!\!\mathrm{I}_{\{x\in[-b(t),b(t)]\}} \sqrt{\frac{\dot b(t)}{b(t)}(b(t)^2 - x^2)}. \tag{7}$$

to be a valid candidate for the solution X of (1) to be a martingale with uniform $[-b,b]$ law. In order to rigorously show that, we prove in the next section that SDE (1) with diffusion coefficient (7) admits a unique strong solution and that this solution has indeed a uniform law at all times.

4 Analysis of the candidate SDE: solutions and distributions

Theorem 1 (Existence and Uniqueness of Solution for candidate SDE solving Problem 1)**.** *Let b a strictly increasing function defined on $[0,T]$ and of class C^1 in $(0,T]$, with $b(0)=0$ and T a positive real number. Assume the derivative of $b(t)^2$, namely $2b(t)\dot b(t)$, to be bounded in $(0,T]$ and to have a finite limit when $t\downarrow 0$. The stochastic differential equation*

$$dX_t = \left(1\!\!\mathrm{I}_{\{X_t\in(-b(t),b(t))\}} \frac{\dot b(t)}{b(t)}(b(t)^2 - X_t^2)\right)^{1/2} dW_t, \quad X_0 = 0, \tag{8}$$

whose diffusion coefficient is extended to $t=0$ by continuity in case the diffusion coefficient is not defined at $t=0$, admits a unique strong solution and its solution X is distributed at every point in time as a uniform distribution concentrated in $(-b(t),b(t))$. We thus have a conic diffusion martingale with the cone expansion controlled by the time function $b(t)$.

Proof. By continuity of diffusion paths, the solution X to the SDE (8), if it exists, belongs to $[-b(t),b(t)]$ almost surely. Indeed, the diffusion coefficient $\sigma(t,x)$ vanishes at the boundaries $\{-b(t),b(t)\}$. Because $b(t)$ is increasing, the process cannot exit the cone $[-b(t),b(t)]$. This will be further confirmed by our local time calculation in Theorem 5 for the specific case $b(t)=kt$.

It remains to prove that the solution X to (8) exists and is unique. To that end, it is enough to show that $\sigma(x,t)$ satisfies the linear growth bound and is Holder-1/2 for all $t\in[0,T]$ [11].

Clearly, $\sigma(x,t)$ in (7) satisfies the linear growth bound since it is uniformly bounded on $[0,T]$. To see this, notice that

$$\sigma^2(x,t) = 1\!\!\mathrm{I}_{\{-b(t)\le x\le b(t)\}}(\dot b(t)/b(t))(b^2(t)-x^2) \le \dot b(t)b(t) = \dot{b^2}(t)/2$$

and that $\dot{b}(t)b(t)$ is bounded on $(0,T]$ by assumption. Since $\sigma(x,t)$ is continuous and bounded on $(0,T]$, it admits a continuous extension at zero which is unique and bounded on $[0,T]$, showing that the extended $\sigma(x,t)$ is uniformly bounded on $[0,T]$.

We now proceed with the Holder continuity of σ. Recall that a function $f(x)$ is Holder-1/2 if there exists $K \in \mathbb{R}$ such that

$$|f(x) - f(y)| \leq K\sqrt{|x-y|}$$

for all x, y in $dom(f)$. Of course, $f(x) = \sqrt{|x|}$ is Holder-1/2 on $\mathbb{R}$ since $|\sqrt{|x|} - \sqrt{|y|}| \leq \sqrt{|x-y|}$ for all x, y. We now check that $\sigma(t,x)$ is Holder-1/2 for all t (we adopt the same strategy as in [14]). To that end, it is enough to check the Holder-1/2 continuity of

$$\eta(t,x) := \mathbb{I}_{\{-b(t)\leq x\leq b(t)\}}\sqrt{b^2(t)-x^2}\ , \qquad t \leq T$$

Define $I(t) := [-b(t), b(t)]$. We check the possible cases.

1. If $x, y \notin I(t)$, the diffusion coefficient vanishes and one gets $|\eta(t,x) - \eta(t,y)| = 0$
2. If $x, y \in I(t)$, using the Holder-1/2 continuity of $\sqrt{|x|}$:

$$\begin{aligned}
|\eta(t,x) - \eta(t,y)| &= |\sqrt{b^2(t)-x^2} - \sqrt{b^2(t)-y^2}| \\
&\leq \sqrt{|(b^2(t)-x^2) - (b^2(t)-y^2)|} \\
&= \sqrt{|y^2-x^2|} \leq \sqrt{|y+x|}\sqrt{|y-x|} \leq \sqrt{2b(T)}\sqrt{|x-y|}
\end{aligned}$$

3. If $x \in I(t),\ y > b(t)$:

$$|\eta(t,x)-\eta(t,y)| = |\eta(t,x)| = \sqrt{b^2(t)-x^2} = \sqrt{b(t)+x}\sqrt{b(t)-x} \leq \sqrt{2b(T)}\sqrt{|x-y|}$$

4. If $x \in I(t),\ y < -b(t)$ (so that $-y > b(t)$) :

$$\begin{aligned}
|\eta(t,x) - \eta(t,y)| &= |\eta(t,x)| \leq \sqrt{b(t)-x}\sqrt{b(t)+x} \leq \sqrt{2b(T)}\sqrt{x+b(t)} \\
&\leq \sqrt{2b(T)}\sqrt{x-y}
\end{aligned}$$

5. The case $x \notin I(t),\ y \in I(t)$ is similar to steps 3 and 4.

Hence, the solution X to (8) exists and is unique. Because it is bounded and evolves between $-b(t)$ and $b(t)$, it is a conic $[-b(t), b(t)]$-martingale.

□

We have proven that the SDE (8) has a unique strong solution. The SDE itself has been obtained by inverting the Kolmogorov equation for a uniform marginal density at time t in $[-b(t), b(t)]$, so we expect the density of the solution to be that uniform distribution. However, we haven't proven that the forward Kolmogorov equation for the density of (8) has a unique solution. To prove that our SDE (8) has the desired uniform distribution, we resort to a characterization of the uniform distribution by its moments, showing a theorem where we state that the moments of the solution of (8) are the same as the moments of the desired uniform law, and showing that this characterizes the uniform law. We start with the following

Definition 1. A probability measure μ is *determined by its moments* when it is the unique probability measure having this set of moments.

Lemma 1. *The continuous uniform distribution on* $[a, b]$*,* $-\infty < a < b < \infty$*, is determined by its moments.*

Proof. Let us note $\alpha_k(p) := \int_{-\infty}^{\infty} x^k p(x)dx$ the k-th moment associated to a probability density function p. From Theorem 30.1 of [1], it is known that if all the moments $\alpha_1(p), \alpha_2(p), \ldots$ are finite and are such that the series

$$S_r(p) := \sum_{k=1}^{\infty} \frac{\alpha_k(p) r^k}{k!}$$

admits a positive radius of convergence, then p is *determined by its moments.*

One concludes from this theorem that if a random variable X satisfies $\mathbb{E}(X^k) = \alpha_k(p)$ for all $k \in \mathbb{N}$, then $X \sim p$ provided that (i) $|\alpha_k(p)| < \infty$ for $k \in \mathbb{N}$ and (ii) there exists $r > 0$ such that the series $S_r(p)$ converges.

In particular, if the uniform density in $[a, b]$, $\rho(x) := \frac{1}{b-a} \mathbb{1}_{\{a \le x \le b\}}$, satisfies (i) and (ii), then any random variable X satisfying $\mathbb{E}(X^k) = \alpha_k(\rho)$ for all $k \in \{1, 2, \ldots\}$ is uniformly distributed on $[a, b]$.

Let us show that (i) and (ii) are satisfied for the uniform density in $[a, b]$. Condition (i) is clearly met since the moments of the uniform distribution are finite. In particular, defining $c := |a| \vee |b|$ one has $|\alpha_k(\rho)| \le c^k < \infty$. On the other hand, for $r > 0$,

$$0 \le \frac{|\alpha_k(\rho)| r^k}{k!} \le \frac{(cr)^k}{k!} \ .$$

Since the series

$$S'_r := \sum_{k=1}^{\infty} \frac{(cr)^k}{k!}$$

converges to $e^{cr} - 1$, the series $S_r(\rho)$ converges, too. This shows that both conditions (i) and (ii) are met for $p = \rho$, and completes the proof. □

Theorem 2. *The solution* X *to the SDE (8) is a uniform martingale on* $[-b(t), b(t)]$ *in the sense that for all* $t > 0$*,* X_t *is uniformly distributed on* $[-b(t), b(t)]$*.*

Proof. From the above results, it is enough to show that all moments of the random variable X_t $(t > 0)$ associated to eq. (8) coincide with those of the density $\mathbb{1}_{\{-b(t) \le x \le b(t)\}} \frac{1}{2b(t)}$.

Let X be a random variable uniformly distributed on $[a, b]$. Then,

$$\mathbb{E}(X^n) = \frac{1}{n+1} \sum_{i=0}^{n} (a)^i (b)^{n-i} .$$

In the special case where $a = -b$, this expression reduces to

$$\mathbb{E}(X^n) = \begin{cases} \frac{b^n}{n+1} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Let us now compute the moments of

$$X_t = \int_0^t \sigma(X_s, s) dW_s \ , \qquad \sigma(t, x) = \mathbb{1}_{\{-b(t) \le x \le b(t)\}} \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{b^2(t) - x^2}$$

solving eq. (8). By Itô's lemma:

$$X_t^n = n \int_0^t X_s^{n-1} dX_s + \frac{1}{2} n(n-1) \int_0^t X_s^{n-2} \sigma^2(X_s, s) ds$$

and we can compute the expression for the n-th moment, $n \ge 2$ using a recursion. Using the property that Itô 's integrals have zero expectation and exchanging integration and expectation operators, which is possible since $X_s^{n-2}\sigma^2(X_s, s)$ is bounded for all s and $n \ge 2$, we obtain

$$\begin{aligned} \mathbb{E}(X_t^n) &= n\mathbb{E}\left(\int_0^t X_s^{n-1} dX_s\right) + \frac{1}{2} n(n-1)\mathbb{E}\left(\int_0^t X_s^{n-2} \sigma^2(X_s, s) ds\right) \\ &= \frac{n(n-1)}{2} \left(\int_0^t b(s)\dot{b}(s)\mathbb{E}(X_s^{n-2}) ds - \int_0^t \frac{\dot{b}(s)}{b(s)} \mathbb{E}(X_s^n) ds\right) \end{aligned} \tag{9}$$

Notice that we have postulated in the last equality that the indicator $\mathbb{1}_{\{-b(s) \le X_s \le b(s)\}}$ in $\sigma(t, x)$ is always 1. This is a natural assumption: it says that X cannot stay on a boundary with a strict positive probability for a given period of time. This happens because in case X reaches $\pm b(t)$ at some time t, the process is locally frozen ($\sigma(t, x) = 0$) but the boundary $b(t)$ keeps on growing (see also the local time calculation in Theorem 5 for the specific case $b(t) = kt$).

Obviously, $\mathbb{E}(X_t) = X_0 = 0$ since X is a martingale and one concludes from eq. (9) that the n-th moment of X_t is zero when n odd. For n even, eq. (9) can be written as

$$f(t, n) = \frac{n(n-1)}{2} \left(\int_0^t b(s)\dot{b}(s) f(s, n-2) ds - \int_0^t \frac{\dot{b}(s)}{b(s)} f(s, n) ds\right)$$

with $f(t,n) := \mathbb{E}(X_t^n)$. This can be written as a recursive differential equation

$$\frac{\partial f(t,n)}{\partial t} = \dot{b}(t)\frac{n(n-1)}{2}\left(b(t)f(t,n-2) - \frac{1}{b(t)}f(t,n)\right)$$

with the constraint that $f(t,0) = \mathbb{E}(X_t^0) = 1$. The solution to this equation is $f(t,n) = b^n(t)/(n+1)$. One concludes that X_t is uniform on $[-b(t), b(t)]$ since all the odd moments are zero and all the even moments are given by

$$\mathbb{E}(X_t^n) = \frac{b^n(t)}{n+1}$$

and agree with those of a random variable uniformly distributed on $[-b(t), b(t)]$. □

Theorem 3 (Peacocks)**.** *Assumptions of Theorem 1 in force. The unique strong solution to the SDE* (8) *with general* $b(t)$ *is a peacock process.*

Proof. Recall that $X = (X_t)_{t\geq 0}$ is a Peacock process on $[0,T]$ if $\mathbb{E}[\Psi(X_t)] \geq \mathbb{E}[\Psi(X_s)]$ for all C^2 convex functions Ψ and all $0 \leq s \leq t \leq T$. The result follows immediately from the fact that our X is a diffusion martingale. Indeed, it is then enough to apply Ito's formula to conclude. □

The special case $b(t) = kt$ gives us a conic martingale with uniform distribution where the boundaries grow symmetrically and linearly in time. This example was considered originally in [3] and is also in [10] (see for instance ex. 6.5 p.253 with $\varphi(x) = x$ and $f(z) = 1/2\, \mathbb{I}_{\{-1-\leq z\leq 1\}}$). Another interesting case is $b(t) = k\sqrt{t}$. We will discuss such cases in the following sections and relate them to specific peacock processes literature [10].

5 Mean reverting uniform diffusions with constant boundaries

Consider the SDE (8) and define

$$Z_t = X_t/b(t), \quad X_t = b(t)Z_t.$$

Since X_t is uniform in $(-b(t), b(t))$, Z_t will be uniform in $[-1,1]$ for $t > 0$. We extend the distribution to a uniform $[-1,1]$ also at time 0. Thus, with this deterministic re-scaling, we have a process Z with fixed uniform distribution and fixed boundaries. We can differentiate Z to obtain its SDE. One has, via Leibnitz rule

$$dZ_t = -\frac{\dot{b}(t)}{b(t)}Z_t dt + \left(\frac{\dot{b}(t)}{b(t)}(1-Z_t^2)\right)^{1/2} \mathbb{I}_{\{Z_t\in[-1,1]\}}\, dW_t, \quad Z_0 = \zeta \sim U([-1,1]).$$

Here we assume the initial condition ζ to be independent of the driving Brownian motion.

If instead we aim at obtaining a standard uniform, in $[0,1]$, we adopt a slightly different transformation:

$$Y_t = (X_t/b(t)+1)/2 = X_t/(2b(t)) + 1/2$$

from which

$$X_t = 2b(Y_t - 1/2).$$

By Leibnitz rule we have the following

Theorem 4. *Consider the SDEs*

$$dY_t = \frac{\dot{b}}{b}(1/2 - Y_t)dt + \frac{1}{2b}\left(\mathbb{1}_{\{Y_t\in(0,1)\}}\, b(t)\dot{b}(t)(1-4(Y_t-1/2)^2)\right)^{1/2} dW_t, \quad Y_0 = \xi \sim U([0,1])$$

and

$$dZ_t = -\frac{\dot{b}(t)}{b(t)}Z_t dt + \left(\frac{\dot{b}(t)}{b(t)}(1-Z_t^2)\right)^{1/2} \mathbb{1}_{\{Z_t\in(-1,1)\}}\, dW_t, \quad Z_0 = \zeta \sim U([-1,1]) \tag{10}$$

with ξ and ζ independent of W. The solutions of these SDEs mean-revert to $1/2$ and 0 respectively with speed $\dot{b}/b$ and are distributed at any point in time as a standard uniform random variable and as a uniform $[-1,1]$ random variable respectively.

Y can be used for example to model the dynamics of recovery rates or probabilities in the case of no information (maximum entropy), whereas Z can be used as a model for stochastic correlation.

6 Specific choices of the boundary $b(t)$ and links with peacocks

In this section we present a number of different qualitative choices for $b(t)$.

6.1 The square-root case $b(t) = \sqrt{t}$

The case $b(t) = \sqrt{t}$ for (8), which leads to

$$dX_t = \frac{1}{\sqrt{2}}\sqrt{1-\frac{X_t^2}{t}}\,\mathbb{1}_{\{X_t\in[-\sqrt{t},\sqrt{t}]\}}\, dW_t, \quad X_0 = 0$$

corresponds exactly to the solution presented in [10], where the authors establish in formula (6.60) that the SDE

$$dX_t = \frac{1}{\sqrt{2}}\sqrt{1-\frac{X_t^2}{t}}dW_t$$

has a solution which is a continuous martingale such that X_t has the same law as $\sqrt{t}\,U$, U being a r.v. uniformly distributed on $[-1,+1]$. To do so, they study, for any $s \in \mathbb{R}$ (not necessarily positive) the SDE, for $s \le t$

$$Y_t = Y_s + \int_s^t \frac{1}{\sqrt{2}}\sqrt{1-Y_u^2}dW_u - \frac{1}{2}\int_s^t Y_u du$$

(which has a unique strong solution) and then show that $X_t = \sqrt{t}\,Y_{\log t}$, $X_0 = 0$ is a solution. We arrived to the same equation via the more general setting in $b(t)$ by inverting the Kolmogorov forward equation. Clearly, this $b(t)$ fits the assumption of Theorem 1 since the derivative of $b^2(t) = t$ is 1 and is thus bounded on $[0,\infty]$. Twenty sample paths from this process are shown on the left panel of Figure 2.

6.2 The linear case $b(t) = kt$: numerical examples, boundary behavior and activity

Again, $b(t) = t$ fits the assumption of Theorem 1 since the derivative of $b^2(t)$ is bounded on $[0,T]$ for any $T \in \mathbb{R}^+$. Our previous SDEs for X (8) and Z (10) specialize to

$$dX_t = 1\!\!\mathrm{I}_{\{X_t \in [-kt,kt]\}} \frac{1}{\sqrt{t}} \sqrt{(kt)^2 - X_t^2}\; dW_t, \quad X_0 = 0, \quad X_t \sim U([-kt,kt]) \;\text{ for all }\; t > 0 \tag{11}$$

and

$$dZ_t = -\frac{1}{t}\, Z_t\, dt + 1\!\!\mathrm{I}_{\{Z_t \in [-1,1]\}} \frac{1}{\sqrt{t}} \sqrt{1 - Z_t^2}\; dW_t, \;\; Z_0 \sim Z_t \sim U(-1,1) \;\text{ for all }\; t > 0. \tag{12}$$

As a numerical example we implement the Euler scheme for X. We know from [8] that under our assumptions the Euler scheme converges in probability. We thus implement a Euler scheme for the SDE for X and then plot a histogram of the density. This is shown in Figure 1. Moreover, we show in the right panel of Figure 2 a few sample paths of the process X.

We now study the behavior of the process X solving (11) near the conic boundary. To do so, we may consider the re-scaled process Z. The boundary behavior of X at $-kt$ and kt will be the same as the behavior of Z at -1 and 1. In particular, X will not spend time at the boundaries $-kt$ and kt if and only if Z will not spend time at -1 and 1.

We can use local time as an investigative tool for the behavior at the boundary.

Theorem 5 (Local time for the solution of the SDE (12) for $Z_t = X_t/kt$ with X_t in (11))**.** *The local time for Z at points -1 and 1 is zero.*

Proof. Using the fact that $d\langle Z\rangle_s = \frac{1-Z_s^2}{s} ds$, one has, for $t > \epsilon > 0$,

$$\begin{aligned} t \;&\ge\; \int_0^t 1\!\!\mathrm{I}_{\{Z_s > -1\}}\, ds = \int_0^t 1\!\!\mathrm{I}_{\{Z_s > -1\}} \frac{s}{1-Z_s^2} d\langle Z\rangle_s \ge \int_\epsilon^t 1\!\!\mathrm{I}_{\{Z_s > -1\}} \frac{s}{1-Z_s^2} d\langle Z\rangle_s \\ &\ge\; \epsilon \int_\epsilon^t 1\!\!\mathrm{I}_{\{Z_s > -1\}} \frac{1}{1-Z_s^2} d\langle Z\rangle_s \end{aligned}$$

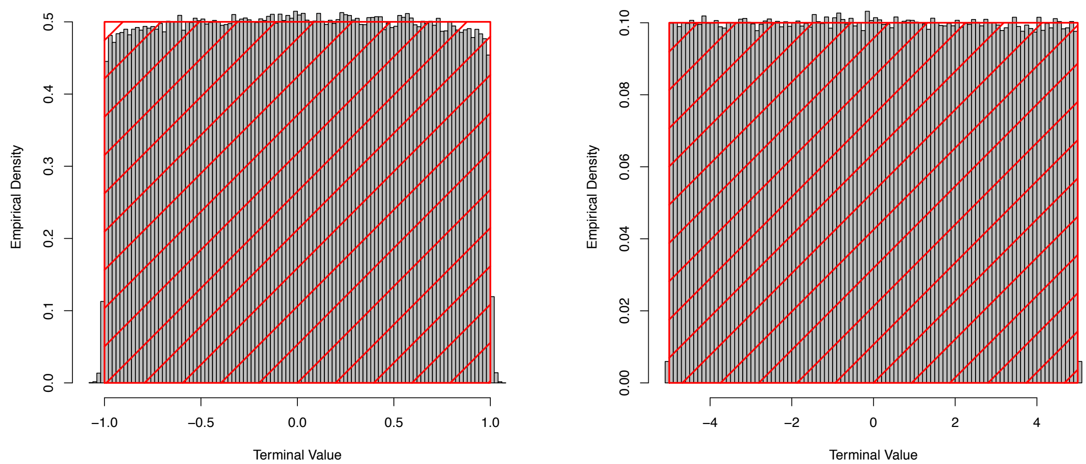

Figure 1: Histograms with 100 bins for the density of the SDE (11) at time t with $k = 1$ for 1 Million scenarios via Euler scheme: left $t = 1y$ and right $5y$. Time step is 0.01 years.

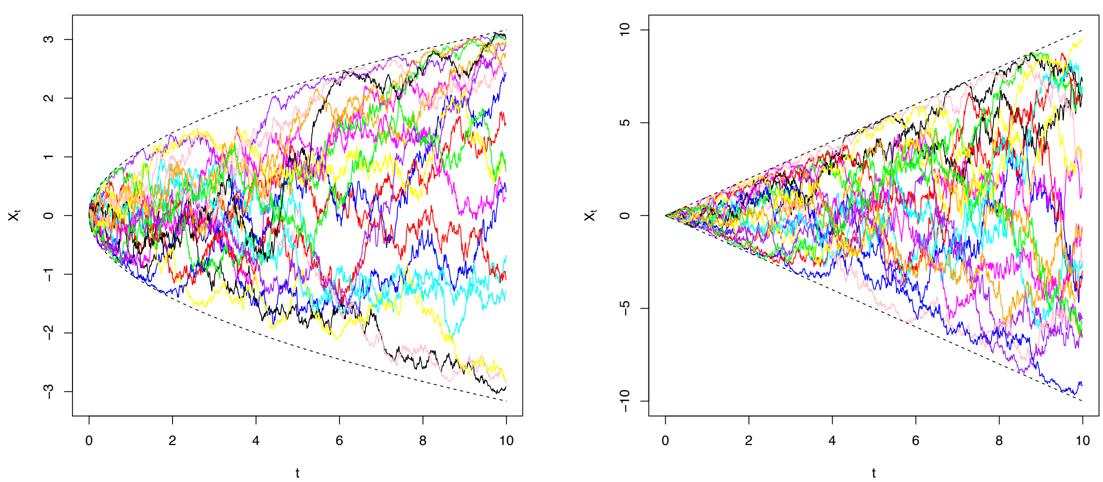

Figure 2: 20 paths of the SDE (11) at time $1y$ with $b(t) = t$ (left) and $b(t) = \sqrt{t}$ (right). Time step is 0.01 years. Euler Scheme.

From the occupation time formula [12, Cor. 1.6, Chapter VI]

$$\int_0^t \mathbb{I}_{\{Z_s>-1\}} \frac{1}{1-Z_s^2} d\langle Z\rangle_s = \int_{-1}^{\infty} L_t^z \frac{1}{1-z^2} dz$$

it follows that

$$t \geq \int_{-1}^{\infty} (L_t^z - L_\epsilon^z) \frac{1}{1-z^2} dz$$

where L^z is the local time of Z at level z. The quantity $(L_t^z - L_\epsilon^z)$ is non negative and the integral on the right hand side converges, hence $(L_t^{(-1)} - L_\epsilon^{(-1)}) = 0$. The local time being continuous, $L_\epsilon^{(-1)} \to 0$ when ϵ goes to 0, and it follows that $L_t^{(-1)} = 0$. By spatial symmetry, we have the same result for the point $+1$. □

More qualitatively, we observe that Z in (12) mean-reverts to 0 with speed $1/t$. The speed will be very large for small time but will become almost zero when time is large. The diffusion coefficient, similarly, is divided by $\sqrt{t}$, so it will tend to vanish for large t. This is confirmed by the following activity calculation. We may conclude that the process will not be absorbed in the boundary and will tend to "slow down" in time, while maintaining a uniform distribution.

We show that the pathwise activity of the uniform $(-1,1)$ process Z is vanishing for large t in the sense that the deviation of $Z_{t+\delta}(\omega)$ from $Z_t(\omega)$ collapses to zero for all $\delta > 0$, all $\omega \in \Omega$ as $t \to \infty$.

Lemma 2.

$$\forall \delta > 0\ ,\ \ \mathbb{V}\text{ar}(Z_{t+\delta} - Z_t) \to 0 \text{ as } t \to \infty\ .$$

Proof. Notice that for all $t > 0$, $\mathbb{E}(Z_t) = 0$ so that $\mathbb{V}\text{ar}(Z_t) = \mathbb{E}(Z_t^2) = \sigma^2$ where $\sigma^2 = \sqrt{2}/12$ is the variance of a zero-mean uniform random variable distributed on $[-1,1]$. Then,

$$\mathbb{V}\text{ar}(Z_{t+\delta} - Z_t) = \mathbb{V}\text{ar}(Z_{t+\delta}^2) + \mathbb{V}\text{ar}(Z_t^2) - 2\mathbb{C}\text{ov}(Z_t, Z_{t+\delta}) = 2\left(\sigma^2 - \mathbb{E}(Z_t Z_{t+\delta})\right)\ .$$

Since Z is bounded, one can rely on Fubini's theorem for all $t > 0$ and exchange time-integration and expectation:

$$\begin{aligned}
\mathbb{E}(Z_t Z_{t+\delta}) &= \mathbb{E}\left(Z_t\left(Z_t - \int_t^{t+\delta} \frac{Z_s}{s} ds + \int_t^{t+\delta} \sigma(s, Z_s) dW_s\right)\right) \\
&= \mathbb{E}(Z_t^2) - \mathbb{E}\left(Z_t \int_t^{t+\delta} \frac{Z_s}{s} ds\right) + \mathbb{E}\left(\int_t^{t+\delta} Z_t \sigma(s, Z_s) dW_s\right)) \\
&= \sigma^2 - \int_t^{t+\delta} \frac{\mathbb{E}\left(Z_t Z_s\right)}{s} ds\ .
\end{aligned}$$

Observing that $\mathbb{E}(Z_s Z_t) = \sqrt{\mathbb{V}\text{ar}(Z_t)\mathbb{V}\text{ar}(Z_s)}\rho(s,t) = \sigma^2 \rho(s,t)$ where $\rho(s,t)$ is the correlation between Z_t and Z_s, one gets

$$\mathbb{V}\mathrm{ar}(Z_{t+\delta} - Z_t) = 2\sigma^2 \int_t^{t+\delta} \frac{\rho(s,t)}{s} ds \leq 2\sigma^2 \int_t^{t+\delta} \frac{1}{s} ds = 2\sigma^2 \delta \frac{\delta + 2t}{t(t+\delta)^2} .$$

The RHS of the above expression tends to zero as $t \to \infty$ showing that the variance of increments of Z over a time interval of fixed length δ collapses to zero as $t \to \infty$.

□

The activity result can be generalized to the following

Lemma 3. *Let $X_t = x_0 + \int_0^t \theta_s dW_s$ and suppose $X = (X_t)_{t\geq 0}$ is a bounded non-vanishing martingale in the sense that for all $t \geq 0$, $a \leq X_t \leq b$ and $\mathbb{P}(\theta_t = 0) < 1$. Then, the path activity of X is collapsing to zero as time passes.*

Proof. Since martingales have independent increments, the variance of increments is the increment of the variances:

$$\mathbb{V}\mathrm{ar}(X_{t+\delta} - X_t) = \mathbb{V}\mathrm{ar}(X_{t+\delta}) + \mathbb{V}\mathrm{ar}(X_t) - 2\mathbb{C}\mathrm{ov}(X_t, X_{t+\delta}) = \mathbb{V}\mathrm{ar}(X_{t+\delta}) - \mathbb{V}\mathrm{ar}(X_t) .$$

Because the diffusion coefficient θ_s does not vanish on $(t, t+\delta)$, $\mathbb{V}\mathrm{ar}(X_{t+\delta} - X_t) = \int_t^{t+\delta} \mathbb{E}(\theta_s^2) ds > 0$ showing that the variance of X_t is monotonically increasing with respect to t. But the variance of a bounded process is bounded. In particular, it is easy to see that $\mathbb{V}\mathrm{ar}(X_t) \leq (x_0 - a)(b - x_0)$ since $\mathbb{E}(X_t) = x_0$ and the variance of any random variable Y with expectation μ_Y and taking values in $[a, b]$ is bounded from above by the variance of $a + (b-a)B$ where B is a Bernoulli random variable with parameter $\pi = (\mu_Y - a)/(b-a)$. Hence, $\mathbb{V}\mathrm{ar}(X_t)$ and $\mathbb{V}\mathrm{ar}(X_{t+\delta})$ are increasing to the same limit, proving that for all $\epsilon > 0$ there exists $t^\star$ such that $\mathbb{V}\mathrm{ar}(X_{t+\delta} - X_t) < \epsilon$ for all $t > t^\star$. □

Remark 1 (Other boundaries)**.** One could choose time-boundaries that are concave and converge asymptotically to a constant value B, e.g. $b(t) = BT/(t+\beta)$ or $b(t) = B(1 - e^{-\beta t})$ where $B > 0, \beta > 0$. It is also possible to use convex boundaries, like e.g. $b(t) = k(e^{\beta} - 1)$, $k > 0$.

7 Conclusions and further research

We introduced a way to design Stochastic Differential Equations of diffusion type admitting a unique strong solution distributed as a uniform law with conic time-boundaries. This leads to a new family of Peacock diffusion martingales. While the result with general boundary is new, the case with square-root boundaries had been dealt with previously. We further introduced general mean-reverting diffusion processes having a constant uniform law at all times. This may be used to model random probabilities, random recovery rates or random correlations.

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