On Path-Complete Lyapunov Functions : comparison between a graph and its expansion

Virginie DEBAUCHE
Université catholique de Louvain, Belgium
virginie.debauche@uclouvain.be

Raphaël JUNGERS
Université catholique de Louvain, Belgium
raphael.jungers@uclouvain.be

Abstract

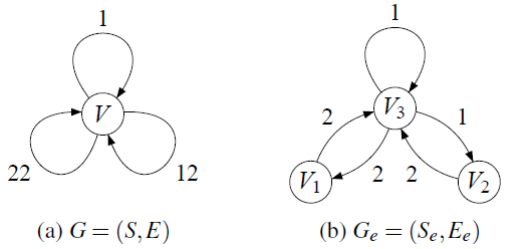
We study the stability of switching dynamical systems with the following dynamics:

$$x(t+1) = f_{\sigma(t)}\, x(t),$$

where $\sigma(t) \in \{1,\dots,M\} := \langle M \rangle$ is the mode of the system, with an integer $M > 0$. We use the *multiple path-complete Lyapunov function* approach introduced in [1] that provides sufficient conditions for the asymptotic stability of switching systems. This method is characterized by both algebraic and combinatorial components, namely a set $\mathscr{V}$ of candidate Lyapunov functions called a *template*, and a labeled directed graph $G = (S,E)$ with $E \subseteq S \times S \times \langle M \rangle$ that captures all the possible switching sequences, respectively. Figure 1a provides an example of a graph which captures all the switching sequences with two modes. According to the definition [2, Definition 2.4], a *Path-Complete Lyapunov function* is a pair $(G = (S,E), \{V_s \mid s \in S\})$ where the candidate Lyapunov functions $\{V_s \mid s \in S\}$ satisfy the following Lyapunov inequalities:

$$\forall (s,d,\sigma) \in E, \forall x \in \mathbb{R}^n : V_d(f_\sigma(x)) \leq V_s(x).$$

It turns out that different methods (i.e., different graphs) may result in more or less conservative conditions [2, 3] and it is still an open question to compare this conservativeness, for two given different graphs. Under the assumption that the template is closed under linear combination, a Linear Programming-based method has been developed in [3] to order two different graphs. This method only provides a sufficient condition to compare two graphs. The example [2, Example 3.9] also suggests checking compositional relations where the new candidate Lyapunov functions are defined as the composition between the old ones and the dynamics. All these order relations are defined for graphs whose edges admit label of length one. However, it has been known since [1, Definition 2.1] that any labeled directed graph $G(S,E)$ can be reduced to a graph with labels of length one thanks to its *expanded graph* denoted by $G_e(S_e,E_e)$. This graph is the outcome of a procedure whose purpose is to split each edge with a label of length larger than one in k new edges linking $k-1$ new nodes. An example of an expanded graph is provided in Figure 1. We generalize the technique [2, Example 3.9] by showing that any graph is equivalent to its expansion for any template closed under composition.



(a) $G = (S,E)$ (b) $G_e = (S_e,E_e)$

Figure 1: Example of a labeled directed graph that captures all the switching sequences of two modes, and its expansion.

Theorem 1 *Let $G(S,E)$ be a labeled directed graph, and $G_e(S_e,N_e)$ its expansion. Then, $G_e \leq G$ and $G \leq G_e$ for any template closed under composition.*

For the first inequality, one only has to keep the candidate Lyapunov function for each common node $s \in S \cap S_e$. For the second inequality, one just needs to define

$$V_{s_q} = V_j \circ f_{i_1} \circ \cdots \circ f_{i_{q-1}}$$

for each node s_q added from the expansion of the edge (i,j) with a label $i_1 i_2 \dots i_k$ of the initial graph. To the best of our knowledge, this result has not appeared in the literature before, as for instance [1] only mention the second inequality. We believe that this simple result is key for developing a decision algorithm allowing to compare two given path-complete Lyapunov function.

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References

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