On some properties of ocean ventilation timescales

Eric Deleersnijder[1], March 31, 2009

According to England (1995), the "World Ocean circulation at its largest scale can be thought of as a gradual renewal or ventilation of the deep ocean by water that was once at the sea surface". Thus, estimating the age as the time elapsed since leaving the ocean surface layers is likely to provide useful insight into the ventilation processes of the World Ocean. This is why the age is a popular diagnostic tool in this domain of interest.

The Constituent-oriented Age and Residence time Theory (CART[2]) provides a coherent set of partial differential equations from which various ages — and other timescales — can be evaluated in a Eulerian ocean model. This diagnostic tool has been instrumental in establishing a number of results about ventilation timescales. Some of them are summarised in the present working note.

The Constituent-oriented Age and Residence time Theory (CART)

Assume that a seawater sample of volume ΔV is taken at time t at location $\mathbf{x}$. Consider one of the water constituents in this water sample. Then, if the constant ρ denotes the density of the water[3], the mass of the constituent under consideration whose age lies in the interval $[\tau,\tau+\Delta\tau]$ tends to $\rho c(t,\mathbf{x},\tau)\Delta\tau\Delta V$ as $\Delta\tau\to 0$ and $\Delta V\to 0$; $c(t,\mathbf{x},\tau)$ is a distribution function that, inside the World Ocean, satisfies the boundary conditions

$$c(t,\mathbf{x},0) = 0 = c(t,\mathbf{x},\infty) . \tag{1}$$

Further assume that the constituent under study is subject to radioactive decay and no other physical, chemical or biological transformation. If γ^{-1} is the mean life of this tracer, then simple mass budget considerations in the four-dimensional $(\mathbf{x},\tau)$ space lead to the equation[4] governing the evolution of the distribution function (Delhez et al. 1999):

$$\frac{\partial c}{\partial t} = -\gamma c - \nabla\bullet(\mathbf{u}c-\mathbf{K}\bullet\nabla c) - \frac{\partial c}{\partial \tau} , \tag{2}$$

where $\mathbf{u}$ and $\mathbf{K}$ denote the velocity field and the diffusivity tensor, respectively. Obviously, by setting $\gamma=0$, the equation for a passive — or inert — constituent is obtained. In the present note, the ocean hydrodynamics will be assumed to be at a steady state, implying that the velocity $\mathbf{u}$ and the diffusivity tensor $\mathbf{K}$ are functions of the position only. The former is divergence-free, while the latter is symmetric and positive definite. Tensor $\mathbf{K}$ is associated

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[2] See www.climate.be/CART

[3] The Boussinesq approximation is used herein, as in all of the CART papers published so far. However, it is possible to adapt CART's equations to flows for which the Boussinesq approximation does not hold valid. Such a modification is rather straightforward.

[4] A entirely general equation for the distribution function has been established by Delhez et al. (1999) or Deleersnijder et al. (2001), i.e. an equation that can be applied to water constituents which are characterised by any type of production-destruction terms, including non-linear ones. Therefore, expression (2) is the simplified budget equation that is appropriate for a radioactive tracer.

only with the parameterisation of mixing processes, i.e. phenomena that cause the variance of the tracer concentration to decrease. Stirring is assumed to be included in the velocity **u**. In other words, the latter is the sum of the water velocity and the vector needed to represent stirring processes, if any (e.g. Griffies 2004).

In accordance with the so-called age-averaging hypothesis[5] (Deleersnijder et al. 2001), the mean age of all the particles[6] of the constituent under study that are present in the water sample taken at time t and location $\mathbf{x}$ is the ratio

$$a(t,\mathbf{x}) = \frac{\alpha(t,\mathbf{x})}{C(t,\mathbf{x})}, \tag{3}$$

where

$$C(t,\mathbf{x}) = \int_0^\infty c(t,\mathbf{x},\tau)\, d\tau \tag{4}$$

$$\alpha(t,\mathbf{x}) = \int_0^\infty \tau c(t,\mathbf{x},\tau)\, d\tau \tag{5}$$

denote the concentration and the age concentration, respectively.

The equation governing the concentration may be obtained by integrating (2) over τ,

$$\frac{\partial C}{\partial t} = -\gamma C - \nabla \bullet (\mathbf{u}C - \mathbf{K} \bullet \nabla C) . \tag{6}$$

Similarly, integrating over τ the product of (1) and τ yields after some manipulations the equation governing the evolution of the age concentration:

$$\frac{\partial \alpha}{\partial t} = C - \gamma\alpha - \nabla \bullet (\mathbf{u}\alpha - \mathbf{K} \bullet \nabla \alpha) . \tag{7}$$

To obtain these two equations, it was necessary to take into the account the boundary conditions (1).

Equations (6) and (7) contain similar — advective and diffusive — transport terms. Therefore, to solve numerically the age concentration equation, no new discretised differential

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[5] Consider, for instance, two particles that are identified by means of subscripts "A" and "B". Their mass and age are denoted m^X and a^X, respectively, with X=A,B. The mass of the system "A+B" obvioulsy is $m^{A+B} = m^A + m^B$. This is in agreement with basic physical principles that stipulate that mass is an additive quantity. No such principle exists for the age. Therefore, to obtain the mean age of the system "A+B", an arbitrary decision is to be made. The latter was formulated as the so-called age-averaging hypothesis (Deleersnijder et al., 2001), which is the only arbitrary element of CART. It stipulates that the mean age of a system made up of various particles is the mass-weighted average of the ages of the particles. Accordingly, the mean age of the system "A+B", a^{A+B}, satisfies the follow relation: $m^{A+B}a^{A+B} = m^A a^A + m^B a^B$. Clearly, the age is not an additive quantity, but the age content is — the age content of a particle being defined as the product of its mass and its age. Ages other than CART's, such as the carbon-14 age, implicitly or explicitly rely on an age-averaging hypothesis (Deleersnijder et al., 2001). However, it is believed that CART's age-averaging hypothesis is the simplest one could think of.

[6] CART's equations are formulated in the Eulerian framework. This is desirable since CART is a set of diagnostic tools intended for Eulerian models. Moreover, several theoretical results that have been established in the Eulerian formalism would have been difficult, or even impossible, to obtain in the Lagrangian one. However, for the sake of simplicity, many verbal explanations are given in Lagrangian terms, using words such as "particle", "trajectory", etc. This is based on the equivalence between the Eulerian and the Lagrangian approaches. See Spivakovskaya et al (2007) and references therein.

operator need be developed. This does not hold true for the equation governing the age. The latter is obtained by combining (3), (6)-(7):

$$\frac{\partial a}{\partial t} = 1 - \nabla \bullet (\mathbf{u}a - \mathbf{K} \bullet \nabla a) + 2C^{-1} \nabla C \bullet \mathbf{K} \bullet \nabla a \, . \tag{8}$$

The last term in the right-hand member of (8) is a differential operator which has no counterpart in the concentration and age concentration equations. This is why it is probably not advisable to solve numerically the age equation. Instead, it is likely to be more appropriate to solve equation (6) and (7), and eventually compute the age by means of expression (3).

The age equation (8) may be rewritten in advective form, i.e.

$$D_t a = 1 + \nabla \bullet (\mathbf{K} \bullet \nabla a) + 2C^{-1} \nabla C \bullet \mathbf{K} \bullet \nabla a \, , \tag{9}$$

where $D_t = \partial / \partial t + \mathbf{u} \bullet \nabla$ denotes the material derivative operator. If there is no mixing (**K**=0), then all the constituents in a water parcel will age at the same pace, so that all ages will be equal if they are subject to similar initial and boundary conditions.

Water ages

CART's equations may be applied to any seawater constituent or group of constituents, i.e. an aggregate (Delhez et al. 1999, Deleersnijder et al. 2001). In ventilation studies, the aggregate that is worth considering is the seawater itself. The concentration of the latter, $C_w(t,\mathbf{x})$, is equal to unity. Therefore, the equation obeyed by the age of seawater, $a_w(t,\mathbf{x})$, is

$$\frac{\partial a_w}{\partial t} = 1 - \nabla \bullet (\mathbf{u}a_w - \mathbf{K} \bullet \nabla a_w) \, . \tag{10}$$

This equation, which is equivalent to that used by Thiele and Sarmiento (1990) or England (1995), may be derived from (7) or (8) under the assumption that $C_w(t,\mathbf{x}) = 1$. At the initial instant, t=0, the age is zero. To estimate ventilation rate, the age must be prescribed to be zero on the ocean-atmosphere interface, which is denoted Γ^s. The rest of the domain boundary, Γ^i, is impermeable. So, given that $[\mathbf{u} \bullet \mathbf{n}]_{\mathbf{x} \in \Gamma^i} = 0$, the initial and boundary conditions to be enforced to solve the water age equation (9) are

$$a_w(0,\mathbf{x}) = 0 \, , \; \left[a_w(t,\mathbf{x})\right]_{\mathbf{x} \in \Gamma^s} = 0 \, , \; \left[(\mathbf{K} \bullet \nabla a_w) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^i} = 0 \, , \tag{11}$$

where **n** is the outward unit normal to the ocean boundary.

It is fruitful to split the water particles into two categories, i.e. the surface water and the water of the interior of the ocean, or interior water for short. The first type of water consists of all the water particles that have touched the ocean surface at least once, while the second refers to those water particles that have not yet touched the upper boundary of the domain of interest. The surface water age, $a_{sw}(t,\mathbf{x}) = \alpha_{sw}(t,\mathbf{x}) / C_{sw}(t,\mathbf{x})$ (Figures 1, 2), and that of the interior water, $a_{iw}(t,\mathbf{x}) = \alpha_{iw}(t,\mathbf{x}) / C_{iw}(t,\mathbf{x})$, are governed by the following relations:

$$\frac{\partial C_\eta}{\partial t} = -\nabla \bullet (\mathbf{u}C_\eta - \mathbf{K} \bullet \nabla C_\eta) \, , \quad \eta = sw, iw \, , \tag{12}$$

$$C_{sw}(0,\mathbf{x}) = 0 \, , \; \left[C_{sw}(t,\mathbf{x})\right]_{\mathbf{x} \in \Gamma^s} = 1 \, , \; \left[(\mathbf{K} \bullet \nabla C_{sw}) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^i} = 0 \, , \tag{13}$$

$$C_{iw}(0,\mathbf{x}) = 1 \, , \; \left[C_{iw}(t,\mathbf{x})\right]_{\mathbf{x} \in \Gamma^s} = 0 \, , \; \left[(\mathbf{K} \bullet \nabla C_{iw}) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^i} = 0 \, , \tag{14}$$

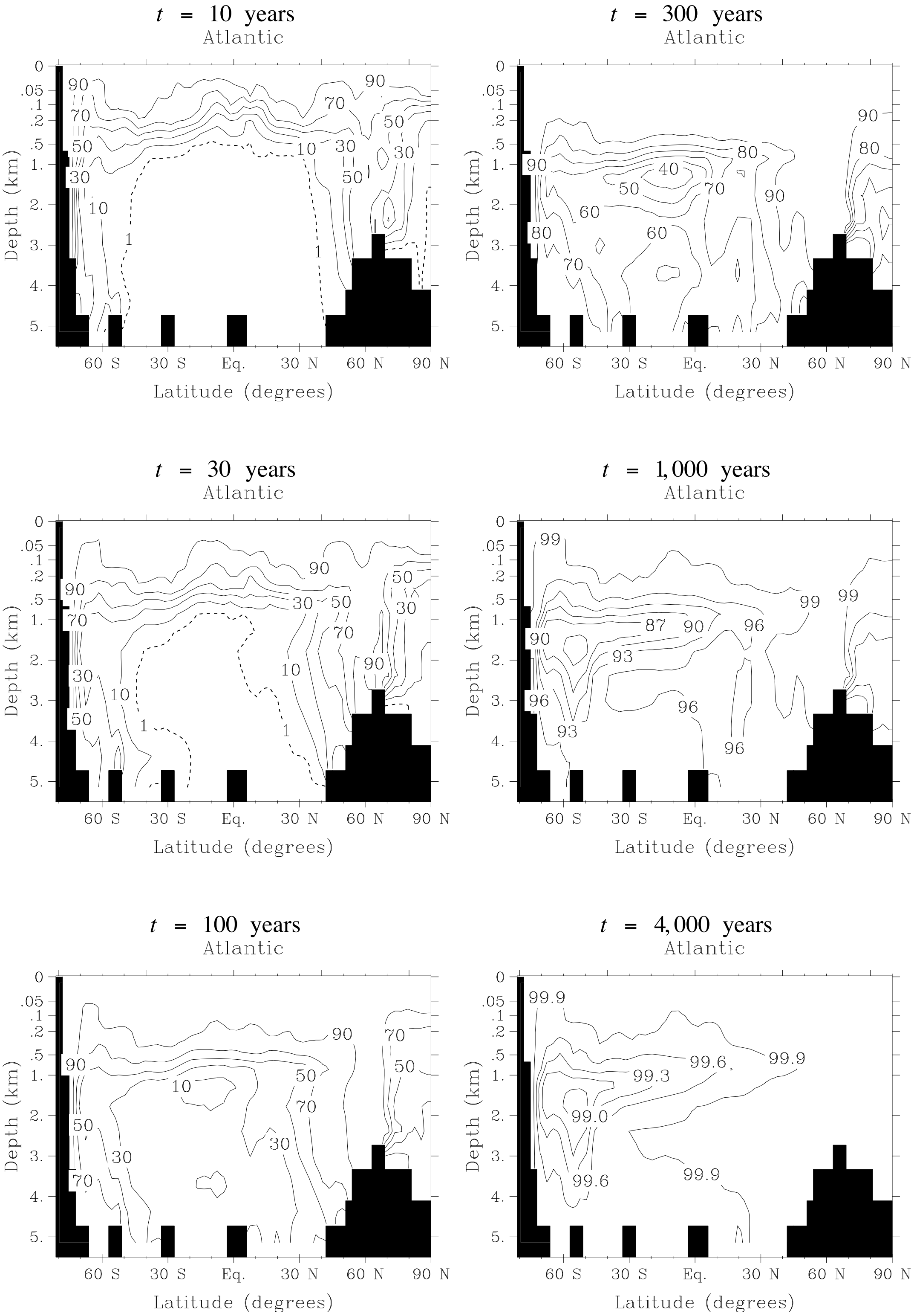

Figure 1. Zonal average, taken over the Atlantic basin, of the surface water concentration ($10^2 \times C_{sw}$), as simulated numerically by a low-resolution model at t=10, 30, 100, 300, 1000 and 4000 years. The dotted isoline corresponds to the value $10^2 \times 10^{-2} = 1$. This figure is adapted from Deleersnijder et al. (2001).

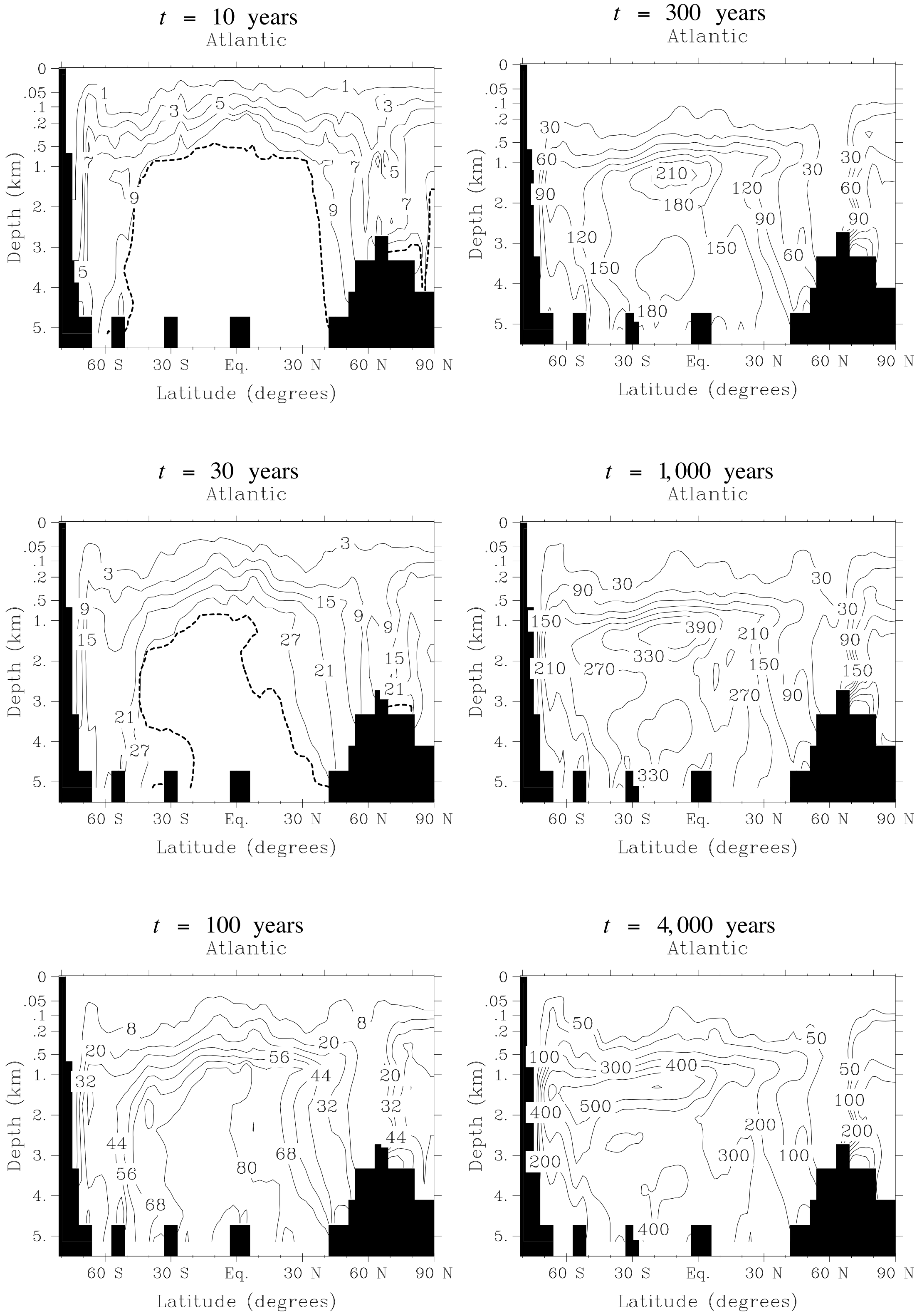

Figure 2. Zonal average, taken over the Atlantic basin, of the surface water age (in years), as simulated numerically by a low-resolution model at *t*=10, 30, 100, 300, 1000 and 4000 years. This figure is adapted from Deleersnijder et al. (2001).

and

$$\frac{\partial \alpha_\eta}{\partial t} = C_\eta - \nabla \bullet (\mathbf{u}\alpha_\eta - \mathbf{K} \bullet \nabla \alpha_\eta) \ , \ \eta = sw, iw \ , \tag{15}$$

$$\alpha_\eta(0,\mathbf{x}) = 0 \ , \ \left[\alpha_\eta(t,\mathbf{x})\right]_{\mathbf{x}\in\Gamma^s} = 0 \ , \ \left[(\mathbf{K} \bullet \nabla \alpha_\eta) \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \ , \ \eta = sw, iw \ , \tag{16}$$

The definition of the surface and interior waters imply that

$$C_w = C_{sw} + C_{iw} = 1 \ . \tag{17}$$

When applied to the water classes considered herein, the age-averaging hypothesis yields

$$C_w \, a_w = C_{sw} a_{sw} + C_{iw} a_{iw} \ . \tag{18}$$

Clearly, the age of the interior water must be equal to the elapsed time, i.e.

$$a_{iw}(t,\mathbf{x}) = t \ . \tag{19}$$

Then, combining (17)-(18) leads to (Deleersnijder et al. 2002):

$$\boxed{a_w(t,\mathbf{x}) = a_{sw}(t,\mathbf{x}) + [1 - C_{sw}(t,\mathbf{x})][t - a_{sw}(t,\mathbf{x})]} \tag{20}$$

This result can be obtained without having recourse to equations (11)-(17); physical intuition and common sense are sufficient. However, it is easily verified that (18)-(20) actually are consistent with (11)-(17), which is reassuring as to the well-foundedness of CART.

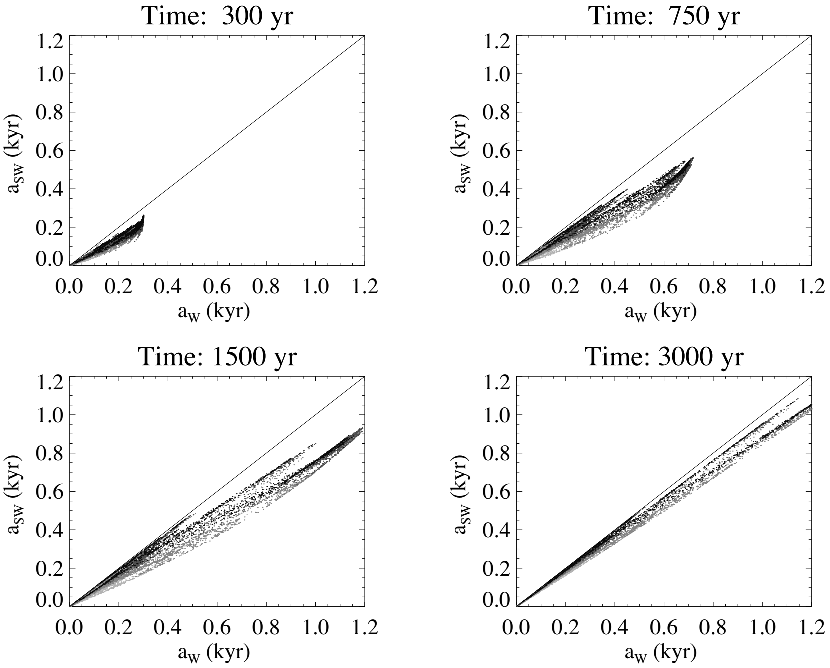

Figure 3. Scatterplot of the age of the surface water, a_{sw}, *versus* the age of the water, a_w, as simulated numerically at every grid point of a coarse-grid ocean model. The ages are expressed in 10^3 years, and are displayed at t=300, 750, 1500 and 3000 years. That the age of the water is larger than that of the surface water and that both ages tend to the same limit as time increases is clearly illustrated. This is Figure 1 of Deleersnijder et al. (2002).

According to Deleersnijder et al. (2001), any age that is zero at the initial instant (*t*=0) will remain smaller than or equal to the elapsed time, i.e. $a_{sw}(t,\mathbf{x}) \le t$. It may be seen that at any time and position $C_{sw}(t,\mathbf{x}) \le 1$. As a consequence, the three water ages considered in this Section satisfy the inequalities (Figure 3)

$$a_{sw}(t,\mathbf{x}) \;\le\; a_w(t,\mathbf{x}) \;\le\; a_{iw}(t,\mathbf{x}) \;. \tag{21}$$

In addition, as time progresses, the amount of water particles having touched at least once the ocean surface increases monotonically, implying that $C_{sw}(\infty,\mathbf{x}) = 1$. Thus, by virtue of (20), in the limit $t \to \infty$, the age of the water tends to that of the surface water, i.e.

$$a_w(\infty,\mathbf{x}) \;=\; a_{sw}(\infty,\mathbf{x}) \;. \tag{22}$$

On the other hand, it may also be seen that

$$a_w(t,\mathbf{x}) \;=\; \int_0^t [1 - C_{sw}(t',\mathbf{x})]\, dt' \;. \tag{23}$$

This result has yet to be published and given a physical interpretation.

Radioactive tracers

Using radioactive decay for dating purposes is common practice in archaeology. The best-known method probably is that having recourse to $^{14}\mathrm{C}$. A key condition for such an approach to be valid is that the system whose age is to be estimated has exchanged no matter with its environment since the instant the age is prescribed to be zero. This constraint is not satisfied by a seawater parcel, for mixing processes are not zero in the ocean. Nonetheless, this has not prevented many from estimating ages in the World Ocean, i.e. ventilation timescales, from field measurements of radioactive tracer[7] concentrations. For rapidly decaying tracers, the results of such studies may be regarded as somewhat questionable, since the biases caused by mixing are unlikely to be that negligible. This is briefly documented in this Section.

Consider two radioactive tracers that are identified by subscripts “1” and “2”. The rate of decay of the former is smaller than that of the latter, i.e.

$$\gamma_2 \;>\; \gamma_1 \;. \tag{24}$$

The tracers under study obey the following initial and boundary conditions:

$$C_j(0,\mathbf{x}) = 0 \;,\; \left[C_j(t,\mathbf{x})\right]_{\mathbf{x}\in\Gamma^s} = 1 \;,\; \left[(\mathbf{K}\bullet\nabla C_j)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \;,\; j = 1,2 \;. \tag{25}$$

At any time $t > 0$ and any point that does not belong to the ocean surface, the concentrations obey the inequality

$$C_1(t,\mathbf{x}) \;>\; C_2(t,\mathbf{x}) \;. \tag{26}$$

Next, seeking inspiration in the numerous studies that estimated ventilation timescales from radio-isotopes data, Delhez et al. (2003) suggested the following “radio-age”

$$r_{1,2}(t,\mathbf{x}) \;=\; \frac{1}{\gamma_2 - \gamma_1} \log \frac{C_1(t,\mathbf{x})}{C_2(t,\mathbf{x})} \;. \tag{27}$$

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[7] Herein, a constituent whose concentration is very small is termed a tracer. Therefore, most constituents of seawater may be regarded as tracers, except pure water. The concentration of the latter is close to unity. Needless to say, the sum of all constituent concentrations is equal to unity.

The latter satisfies the evolution equation

$$D_t r_{1,2} = 1 + \nabla\bullet(\mathbf{K}\bullet\nabla r_{1,2}) + \frac{\nabla C_1\bullet\mathbf{K}\bullet\nabla C_1}{(\gamma_2-\gamma_1)C_1^2} - \frac{\nabla C_2\bullet\mathbf{K}\bullet\nabla C_2}{(\gamma_2-\gamma_1)C_2^2}\,. \tag{28}$$

If mixing processes are negligible ($\mathbf{K}=0$), then any constituent — or water — age defined according to CART, $a(t,\mathbf{x})$, obeys a transport equation similar to that of the radio-age defined above:

$$D_t r_{1,2} = 1 = D_t a\ . \tag{29}$$

Thus, the ages will be equal if they are subject to equivalent initial and boundary conditions. These considerations were frequently invoked — explicitly or implicitly — as a justification for contenting oneself with radio-ages. Unfortunately, mixing is not negligible. This is why discrepancies between various age estimates may be rather significant (Figure 4).

Delhez et al. (2008) established the following inequalities

$$\boxed{a_2(t,\mathbf{x}) < r_{1,2}(t,\mathbf{x}) < a_1(t,\mathbf{x})} \tag{30}$$

The latter are valid at any time and point inside the ocean, and suggest that a radio-age may be considered, to a certain measure, as an approximation of the age of the two tracers needed to evaluate the radio-age. Obviously, the difference between the three ages involved in (30) decreases as $\gamma_2-\gamma_1$ decreases.

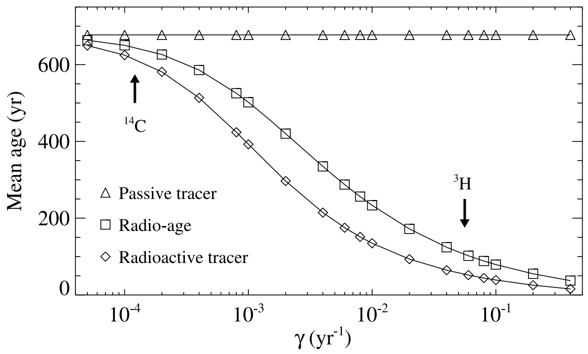

Figure 4. Average over the World Ocean of the age of a passive tracer, of a radioactive tracer for various decay rates γ, and the corresponding radio-age as obtained from (31). The means ages are calculated from the steady-state results of a coarse-grid ocean model. The two vertical arrows indicate the decay rate of radiocarbon and tritium, respectively. The passive tracer age is equivalent to that of the water or the surface water, as steady-state values are used. The latter age probably is the only true ventilation timescale. This is Figure 1 of Delhez et al. (2003).

One should now address the question as to whether a radio-age is a valid approximation to the surface water wage, which presumably is the most relevant ventilation timescale. Setting $\gamma_1=0$ implies that tracer associated with subscript "1" is equivalent to the surface water.

Next, the second tracer is the only one that is radioactive, suggesting that subscript "2" may be omitted or replaced by "r", whenever appropriate. Then, the radio-age now reads

$$r(t,\mathbf{x}) = \frac{1}{\gamma}\log\frac{C_{sw}(t,\mathbf{x})}{C_r(t,\mathbf{x})} . \tag{31}$$

The theoretical and numerical results of Delhez et al. (2003) suggest that the difference between the age of the surface water and the radio-age, $a_{sw} - r$, scales as γ as $\gamma \to 0$. In other words, a radio-age using a tracer that is decaying sufficiently slowly may provide a fairly good approximation of the surface water age. This does not hold true for a rapidly decaying tracer. For such a tracer, the corresponding radio-age tends to zero as $\gamma^{-1/2}$ as $\gamma \to \infty$.

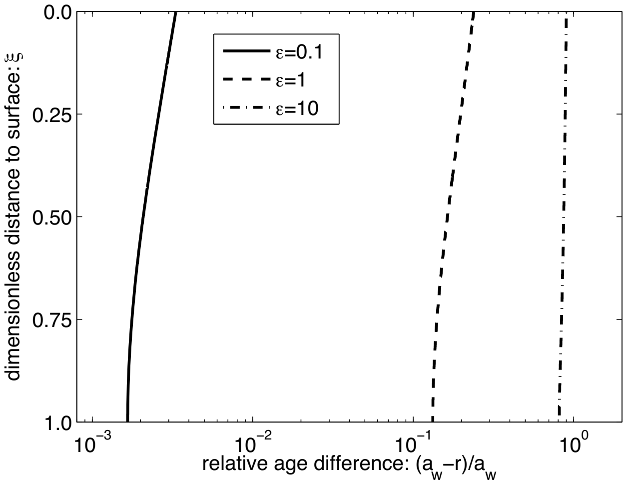

Figure 5. The relative difference between the age of the water and the radio-age at a steady state in a one-dimensional, purely diffusive ocean model. The age difference is computed according to relation (35). The faster the radioactive decay, the larger the dimensionless parameter ε. Thus, in the present idealised model, no matter the distance to the surface, only slowly-decaying tracers yield radio-ages that are close the surface water age.

It is sometimes argued that rapidly decaying tracers may lead to radio-ages yielding relevant ventilation timescales in the region where their concentration is significant, i.e. in the upper layers of the ocean. This is unlikely to be correct. All of the ages introduced herein tend to zero as the ocean surface is approached and so do the differences between ages. However, the relative differences between the ages, e.g. $(a_w - r)/a_w$, do not. This can be seen in OGCM results (not shown here), in the one-dimensional analytical solutions of Table 2 of Deleersnijder et al. (2001) and in the idealised problem below.

Consider a one-dimensional, purely diffusive ocean, whose surface and bottom are located at $\xi = 0$ and $\xi = 1$, where ξ is a dimensionless vertical coordinate that is increasing downwards. A steady state is obtained in the limit $t \to \infty$. The corresponding age of the water and radio-age are

$$a_w(\infty,\xi) = \frac{h^2}{2K}(2\xi - \xi^2) \tag{32}$$

and

$$r(\infty,\xi) = \frac{1}{\gamma}\log\frac{\cosh\varepsilon}{\cosh[\varepsilon(1-\xi)]} , \tag{33}$$

with

$$\varepsilon = \sqrt{\frac{\gamma h^2}{K}} , \tag{34}$$

where the positive constants h and K denote the ocean depth and the vertical eddy diffusivity, respectively. Then, as the ocean surface is approached ($\xi \to 0$), the relative age difference behaves as follows:

$$\frac{a_W(\infty,\xi) - r(\infty,\xi)}{a_W(\infty,\xi)} \sim \frac{\varepsilon - \tanh\varepsilon}{\varepsilon} - \frac{\sinh\varepsilon\cosh\varepsilon - \varepsilon}{2\varepsilon[\cosh\varepsilon]^2}\xi . \tag{35}$$

Thus, the surface value of the relative age difference, $(\varepsilon - \tanh\varepsilon)/\varepsilon$, is small only if the dimensionless parameter ε is also small, i.e. if the tracer decays slowly. For a rapidly decaying tracer, ε is of the order of unity or larger, leading to a value of the relative age difference of the order of unity. This idealised example reinforces the notion that rapidly-decaying tracers are unsuitable for estimating ventilation timescales in the vicinity of the ocean surface — or anywhere else.

The probability that a radio-element particle undergoes a nuclear reaction causing the emission of radiation is independent of its age. Therefore, that a radioactive tracer is younger than a slower-decaying tracer — as indicated by (30) — is not due to radioactive decay that would preferentially destroy older tracer particles, thus preventing rapidly-decaying from growing old. The correct explanation is subtler. The faster the radioactive decay, the smaller the tracer concentration at a certain depth. Thus, the gradient of the concentration tends to increase as the rate of decay increases, thereby increasing the impact of diffusive processes, enhancing the diffusive flux of young tracer particles originating from the ocean surface. In other words, it is diffusion that causes a quicker-decaying tracer to be younger.

The leaky funnel metaphor

At a steady state[8], the ventilation timescale, i.e. the age of the water or the surface water[9], presumably depends on a number of dimensionless parameters characterising the transport processes in the ocean. In the simplest approach, only one parameter may be taken into account. The latter should probably be the Peclet number,

$$Pe = \frac{UL}{K} , \tag{36}$$

where U, L and K denote a suitable velocity, length and diffusivity scale, respectively. It is assumed that U is associated with horizontal motions, implying that L and K also refer to essentially horizontal processes.

Let $\langle a_W \rangle$ represents the average over the World Ocean of the steady-state water age:

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[8] In this Section, only steady-state dependent variables are considered. Therefore, the time will not appear anymore as part of the independent variables.

[9] At a steady-state, the water and surface water are equivalent, this the concentration of the latter is equal to unity.

$$\left\langle a_W \right\rangle = \frac{1}{\Omega}\int_{\Omega} a_W(\mathbf{x})\, d\mathbf{x} \ . \tag{37}$$

Then, according to the Buckingham π theorem, only two dimensionless numbers may be constructed from $\left\langle a_W \right\rangle$, U, L and K, for instance $U\left\langle a_W \right\rangle/L$ and Pe. There should be a relationship between these numbers, which can be cast into the generic form

$$F\left(\frac{\left\langle a_W \right\rangle}{L/U}, Pe\right) = 0 \ . \tag{38}$$

Unfortunately, the Buckingham π theorem gives no indication as to the form of F. Physical arguments are now needed about the functioning of the ventilation at its largest scale.

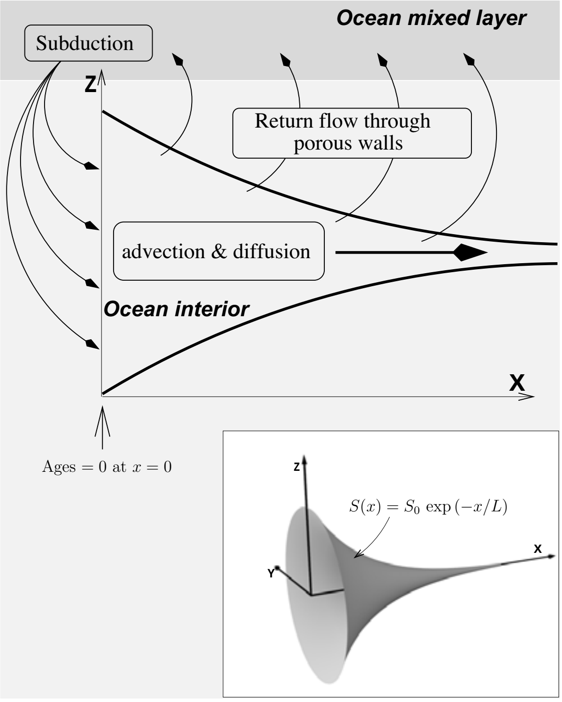

Figure 6. Schematic representation of the leaky funnel, which extends from $x = 0$ to $x = \infty$. The surface water enters the leaky funnel at $x = 0$, is advected at a constant speed U and is subjected to diffusive processes characterised by a constant eddy coefficient K. The water progressively escapes the funnel through its porous walls. As is depicted in the inset, the funnel section decreases exponentially, the constant L being the associated e-folding length scale. The latter may also be seen to be the mean length of the trajectory of water parcels in the funnel. This is Figure 1 of Mouchet and Deleersnijder (2008).

The classical Stommel description of the World Ocean circulation has it that “the horizontal circulation in the actual ocean may be thought to be a consequence of localized sinking and generalized upwelling” (Warren 1981). Along this line of thinking, Mouchet and Deleersnijder (2008) suggested an idealisation of the ventilation processes in which the interior of the ocean is represented as a semi-infinite pipe with porous walls (Figure 6). Let x and $S(x)$ denote the distance to the entrance and the section of the pipe, respectively. The water flux escaping the pipe through the porous walls is $-UdS/dx$. For this flux to actually represent in an idealised manner the recirculation toward the ocean surface, it is essential that dS/dx be negative. In other words, the section of the pipe must decrease as the distance to the entrance increases, which is why the expression “leaky funnel” was coined by Mouchet and Deleersnijder (2008) to refer to this pipe. To enable the evaluation of domain-averaged variables, it is necessary that the volume of the pipe be finite — though its length is infinite. Therefore, the section of the pipe must decrease sufficiently rapidly.

In the leaky funnel, the water age $\hat{a}_w(x)$[10] is the solution of

$$\frac{d}{dx}\left(SK\frac{d\hat{a}_w}{dx} - SU\hat{a}_w \right) + S = 0 . \qquad (39)$$

If the section of the funnel decreases exponentially with the e-folding length scale L, i.e.

$$S(x) = S_0\, e^{-x/L} , \qquad (40)$$

then the age equation (39) simplifies to

$$\frac{d}{dx}\left(K\frac{d\hat{a}_w}{dx} - U'\hat{a}_w \right) + 1 = 0 , \qquad (41)$$

with $U' = U(1+Pe^{-1})$. There is probably no closed-form solution to (39) for a general expression of the section of the funnel. However, (41) can be solved easily. Therefore, if it is for a crucial physical reason that the section of the pipe is assumed to be a decreasing function of x, it is only for obtaining an analytical solution that this function is assumed to an exponential. In addition, the volume of the pipe is finite and simply is S_0L. The value of section at the entrance, S_0, will be seen to be unimportant.

Under the assumption that the age is zero at the funnel entrance, i.e. $\tilde{a}_w(0) = 0$, the solution to (41) reads

$$\hat{a}_w(x) = \frac{x}{U'} = \frac{x}{U}\frac{Pe}{1+Pe} , \qquad (42)$$

and the domain-averaged age is

$$\left\langle \hat{a}_w \right\rangle = \frac{1}{S_0L}\int_0^\infty S(x)\,\hat{a}_w(x)\,dx = \frac{L}{U'} = \frac{L}{U}\frac{Pe}{1+Pe} . \qquad (43)$$

This immediately leads to the explicit expression of (38) that could not be derived from the Buckingham π theorem, i.e.

$$\boxed{\frac{\left\langle \hat{a}_w \right\rangle}{L/U} = \frac{Pe}{1+Pe}} \qquad (44)$$

For the leaky funnel is of any relevance to the World Ocean ventilation, the value of its three parameters must be deemed to be acceptable. In fact, they are not independent. Only two of them are. This is why Mouchet and Deleersnijder (2008) prescribed *a priori* U as the mean value of the horizontal velocity of the OGCM they used, i.e.

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[10] Circumflexes are used to distinguish variables associated with the leaky funnel from those related to the World Ocean.

$$U \approx 3\times10^{-3} \text{ ms}^{-1} . \tag{45}$$

Then, a series of sensitivity experiments was carried out by means of the numerical model in which the velocity or the eddy diffusivity were multiplied at any location by coefficients $\mu^{\mathbf{u}}$ and $\mu^{\mathbf{K}}$, respectively, with $\mu^{\mathbf{u}} \in [8^{-1},2]$ and $\mu^{\mathbf{K}} \in [1,6]$. The same multipliers were applied to the leaky-funnel parameters. Then, *U* and *K* were optimised in such a way that the behaviour of the domain-averaged age in the OGCM and the leaky funnel be as similar as possible. This led to

$$L \approx 9.6\times10^{7} \text{ m} , \quad K \approx 8.6\times10^{4} \text{ m}^2\text{s}^{-1} . \tag{46}$$

The length scale *L* is the mean length of the water parcel trajectories in the leaky funnel. It should also be indicative of the mean length of the water parcel trajectories in the World Ocean interior. In this respect the value obtained by Mouchet and Deleersnijder (2008) seems to be fairly decent. And so is the value of the diffusivity, which is in agreement with Okubo's ocean diffusion diagram (1971) — for diffusion at the largest scale in an ocean basin.

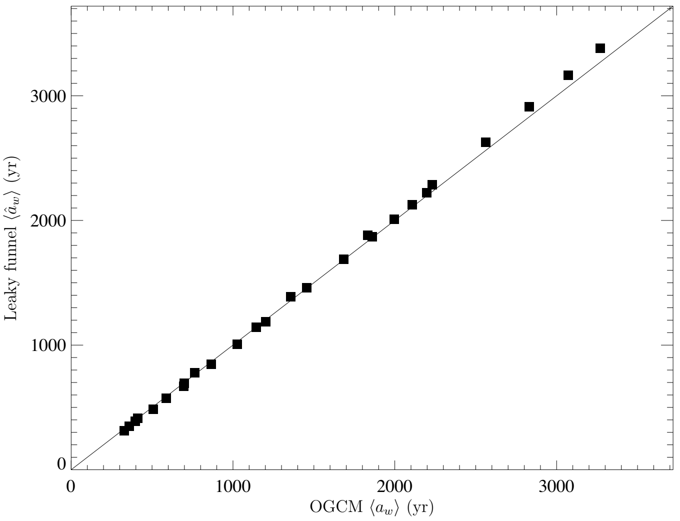

Figure 7. Scatter plot of the domain-averaged leaky funnel age *versus* that of a coarse-grid OGCM. The former is computed by means of (43), while formula (37) is resorted to in order to obtain the latter. The linear correlation coefficient associated with the two age series is 0.9998 and the corresponding rms difference is 40.87 years. These values are surprisingly good. This is Figure 3 of Mouchet and Deleersnijder (2008).

Though the leaky funnel physics is rather simple, it is capable of representing very well the behaviour of the domain-averaged water age of a coarse-grid OGCM (Figure 7). This holds also true for radio-ages and radioactive tracer ages; these results are not shown here but are displayed and discussed in Mouchet and Deleersnijder (2008).

To calibrate the values of *L* and *K*, Mouchet and Delersnijder (2008) had to conduct sensitivity runs whose physics may be questionable, for the velocity or diffusivity are modified in a way that is not consistent with the momentum equation — while the continuity equation is preserved. To circumvent this difficulty Deleersnijder et al. (2009) suggest another approach to the calibration of the leaky funnel parameters. The velocity scale is left unchanged, but the length and diffusivity scales are computed in such a way that the difference between the global water age distribution of the OGCM and that of the leaky funnel be as small as possible. This no longer demands a series of questionable sensitivity runs.

The global water distribution of the World Ocean, $\phi(\tau)$, is such that the fraction of the volume of the water particles whose age lies in the interval $[\tau, \tau + \Delta\tau]$ tends to $\phi(\tau)\Delta\tau$ as $\Delta\tau \to 0$. It is readily understood that $\phi(\tau)$ is

$$\phi(\tau) = \frac{1}{\Omega} \int_{\Omega} c_W(\mathbf{x},\tau)\, d\mathbf{x} \ , \tag{47}$$

where the water age distribution $c_W(\mathbf{x},\tau)$ is the solution of the following partial differential problem:

$$\frac{\partial c_W}{\partial \tau} = -\nabla \bullet (\mathbf{u} c_W - \mathbf{K} \bullet \nabla c_W) \ , \tag{48}$$

$$c_W(0,\mathbf{x}) = 0 \ , \ \left[c_W(t,\mathbf{x})\right]_{\mathbf{x}\in\Gamma^s} = \delta(\tau - 0) \ , \ \left[(\mathbf{K} \bullet \nabla c_W) \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \ , \tag{49}$$

where δ denotes the Dirac function. It is readily seen that the solution of this problems satisfies at any location the following constraint:

$$\int_0^{\infty} c_W(\mathbf{x},\tau)\, d\tau = 1 \ . \tag{50}$$

Therefore, as expected, the global water age distribution satisfies a similar constraint:

$$\int_0^{\infty} \phi(\tau)\, d\tau = 1 \ . \tag{51}$$

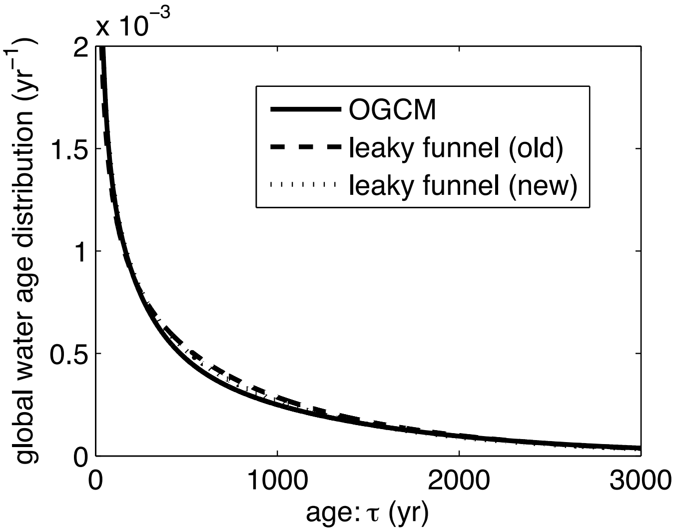

Figure 8. The global water age distribution of a coarse-grid OGCM and that of the leaky funnel with the parameters determined by Mouchet and Deleersnijder (2008) and Deleersnijder et al. (2009). The penultimate and last curves are referred to "old" and "new" above.

In the leaky funnel, the water age distribution and the global water age distribution are (Deleersnijder et al. 2009)

$$\hat{c}_w(x,\tau) \;=\; \frac{x \exp\left[-\dfrac{(x-U'\tau)^2}{4K\tau}\right]}{\sqrt{4\pi K\tau^3}} \tag{52}$$

and

$$\hat{\phi}(\tau) \;=\; \sqrt{\frac{K}{\pi L^2 \tau}}\exp\left(-\frac{U'^2\tau}{4K}\right) \;+\; \frac{1}{\theta}\left[1+\operatorname{erf}\left(\frac{1}{\theta}\sqrt{\frac{L^2\tau}{K}}\right)\right]\exp\left(-\frac{U\tau}{L}\right), \tag{53}$$

with $\dfrac{1}{\theta}=\dfrac{U'}{2L}\left(1-\dfrac{2}{Pe'}\right)$ and $Pe' = U'L/K$. These results are yet to be published.

Substituting into (53) the parameters *U*, *L* and *K* determined by Mouchet and Deleersnijder (2008) leads to a global water age distribution that is very close to that of a coarse grid OGCM. However, an even better agreement may be obtained by optimising the values of *L* and *K* so that a suitable measure of the difference $\hat{\phi}(\tau)-\phi(\tau)$ be as small as possible. Doing this yields values that are of the same order of magnitude as those of Mouchet and Deleersnijder (2008), i.e.

$$L \;\approx\; 10^8\ \mathrm{m}\ ,\quad K \;\approx\; 1.3\times10^5\ \mathrm{m^2\,s^{-1}}\ , \tag{54}$$

No matter the set of values of the leaky-funnel length and diffusivity scales selected, the difference $\hat{\phi}(\tau)-\phi(\tau)$ is surprisingly small (Figure 8).

Final remarks

It must be stressed that (6) and (7) are not the most general concentration and age concentration equations of CART. The latter are able to deal with any type of constituent, such as, for instance, the variables of ecological models, which are often associated with complex, non-linear production-destruction terms (Delhez et al. 2004).

A detailed physical interpretation of the leaky funnel global water age distribution will be given in Deleersnijder et al. (2009).

That the leaky funnel such a good and robust idealisation of the ventilation processes may be due to the fact that it has been tested only against the results of a coarse grid model. The discrepancies might be larger if a high-resolution model were used, as the latter probably is less diffusive.

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