AN INTERTEMPORAL DEMAND SYSTEM*

by

Louis PHLIPS and Daniel WEISERBS
Université Catholique de Louvain

Working Paper N° 7509 (Part I)

UNITE D'ANALYSE ECONOMIQUE (ANEC)
INSTITUT DES SCIENCES ECONOMIQUES
121 Parkstraat, B-3000 LOUVAIN (Belgium)

This paper reviews the progress made over recent years in the art of formulating and estimating dynamic demand systems, and reports some new results based on an intertemporal model. Section I is devoted to the economics, Section II moves on to the econometric front, while Section III describes the outcome of the battles with the data.

I. ECONOMICS

We first analyze the pioneering work by Houthakker and Taylor and then go on to develop it in an intertemporal framework and to present assumptions that make a generalization of the Slutsky conditions possible.

1.1. Myopic Systems

As far as we know, dynamic demand system originate in the work by Houthakker and Taylor (1970, Chapter 5), who proceed by maximizing an instantaneous "dynamic" utility function of the form

$$(1) \qquad u = u(q,s),$$

where q is an n-vector of purchases and s is the corresponding vector of state variables defined by the set of differential equations

$$(2) \qquad \dot{s} = q - Ds,$$

subject to the usual budget constraint

$$(3) \qquad p'q = y.$$

D is a diagonal matrix of n constant depreciation rates. For purposes of empirical implementation, Houthakker and Taylor specified (1) as the generalized quadratic utility function

$$(4) \qquad u(q,s) = q'a + s'b + \frac{1}{2}\, q'Aq + q'Bs + \frac{1}{2}\, s'Cs,$$

while Phlips (1972) used the generalized Stone-Geary function

$$u(q,s) = b'\log(q - a - Bs). \tag{5}$$

What are the strengths and weaknesses of this sort of models ?

The applied economist should be impressed by the possibility that is offered here to simultaneously compute not only a complete set of both short- and long-run price and income elasticities, but also a set of depreciation rates (δ) and partial adjustment coefficients (G) (for the state variables). In the dynamic linear expenditure system, an estimate of G is obtained from the estimates of D - B. In the Houthakker-Taylor model, we find[1]

$$G = D + A^{-1}B. \tag{6}$$

The initial enthusiasm is somewhat dampened, however, once it is realized that the utility function is additive both in the quantities and in the state variables.

The theoretician, on the other hand, might derive some intellectual satisfaction from the fact that the first-order conditions

$$\frac{\partial u}{\partial q_i} = \lambda p_i \qquad (i = 1, \ldots, n) \tag{7}$$

appear to be identical with the reduced forms one obtains by imposing a partial adjustment mechanism on the long-run equilibrium conditions, which in turn are formally identical with the static equilibrium conditions. A theoretical foundation for partial adjustments and the resulting distributed lags is thus available. It is not irrational for a consumer to distribute his response over time so as to approach his (static or long-run) equilibrium position gradually. On the contrary, by doing so he is maximizing utility.

The very same theoretician might immediately add, though, that models of this type stop short of attaining the theoretical standing he is used to. And recall that these models are presented as "models for durable goods", while what is being done is simply - to quote from Brown and Deaton (1972, p. 1225) - to "graft dynamic considerations into the static utility model rather than rooting them in an intertemporal maximisation process".

The objection is fundamental and entirely valid. The consumer who behaves according to equations (1) to (3) is looking backwards but not forwards. At each momemt of time, he maximizes an instantaneous utility function in which the influence of past decisions is imbedded. When he buys a durable, he takes his existing stock into account, but ignores the obvious fact that it is going to be there tomorrow, whether he still likes it or not, that it will need repairing or replacement, etc. The consumer who is developing habits acts in the same "myopic" way. That is why the adjective "myopic" has been used to describe the models discussed until now.

The use of the traditional budget constraint is open to the same criticism. The allocation of income between total expenditures and savings is still treated as exogenously given. Furthermore, the consumer is not allowed to borrow on future labour income. Nor has he access to second-hand markets, while a change in prices has no effect on his net worth.

In the next Section, we shall present a model that tries to incorporate some of the most important aspects of intertemporal utility maximization. However, before entering the intertemporal field, it may be worthwhile to say a few more words about other possible theoretical weaknesses of myopic models.

A theoretician may be disturbed by the presence of purchases rather than consumption services in the utility function. We are not too unhappy about this, given the interpretation in terms of partial adjustments to long-run equilibria. In the long-run, $\dot{s}$ being zero, quantities purchased are equal to depreciation which is equal to consumption. The utility function $u(q,s)$ is then a short-run rationalization of the partial adjustment, with the help of current purchases, to long-run desired consumption. But we wish to add that some refinements of the analysis, distinguishing between consumption and purchases in the short-run, can probably be incorporated in the Houthakker-Taylor approach, as is suggested in a recent paper by Taylor (1974).

A word should also be said about the comparative merits of the quadratic model (4) and the linear model (5). Summarizing the results of a comparative analysis, based on U.S. data for eleven commodities, carried out by Taylor and Weiserbs (1972), we may conclude that

a) although there is little difference between the statistical fit of the two models, forecasts made with the dynamic linear expenditure system are definitely superior to those made with the dynamic quadratic model ;
b) as satiation is not excluded in a quadratic formulation, one of the basic axioms of demand theory may be violated in the quadratic model (as is in fact suggested by a sharp decline of the estimated marginal utility of money at the end of the observation period).

One may wonder, therefore, why Houthakker and Taylor chose to work with the quadratic rather than with the Stone-Geary utility function, the more so as they started from linear state-adjustment equations, which can only be derived from the (generalized) Stone-Geary function.

1.2. An Intertemporal Model

The critique formulated above leads quite naturally to the construction of an intertemporal model.

We wish to emphasize, to begin with, that the Houthakker-Taylor model is set up in such a way that the techniques of optimal control are directly applicable. There is the distinction between state variables and control variables. In the terminology of control theory, the n vector of purchases $q = q(t)$ is a vector of decision variables or controls, while $s = s(t)$ remains an n vector of state variables. The latter describe the state of the system, whose law of motion is described by the familiar set of differential equations (2), which are called "transformation functions". We thus have n decision variables q_i and n state variables s_i related pairwise by n transformation functions. These describe a transitive system, as the entire past is described by the state s attained at time t.

The generalized utility function (1) remains the consumer's instantaneous time-invariant(twice differentiable and strictly concave) utility function. But now the consumer is supposed to maximize the functional

$$(8) \qquad U = \int_0^\infty e^{-\gamma t} u(q,s)\,dt$$

with respect to the decision variables. γ is a constant subjective rate of time preference reflecting impatience.

The choice of an infinite time horizon implies that the consumer is treated as a perpetual family of which each member (or head) is responsible for the management during his lifetime. This avoids the nasty problem of determining the contribution to utility of the final wealth (or bequests) that would inevitably be left over at the end of a finite time horizon.

Notice that the consumer is supposed to maximize (8) at each moment of time. Today, he determines an optimal program of commodity bundles q_t for the present period and for all subsequent periods. Tomorrow, he does his computations again, and so on. Whenever new information becomes available, the new optimal program is different from the optimal program determined yesterday. But even if all information about the present and the future remains unaltered, the program that appears as optimal tomorrow may be different from the program when viewed at an earlier date, e.g. because the discount function in the functional takes on a different value (for a particular time period) for the simple reason that time has evolved. Accordingly, observed purchases in any period is the initial vector of an optimal sequence, the sequence being in general different from period to period as the result of a continuous replanning. Observed behaviour is therefore seen not as following one single optimal path over time, but as a sequence of points (or segments) on successive and different optimal paths.

What, then, are the conditions required by consistency, i.e. under what conditions would, under perfect foresight, the sequence that appears as optimal today lie on the same path as the sequence that appeared as optimal in any previous period ? With a static instantaneous utility function, the intertemporal utility function would be strongly separable over time, and the constancy of γ in the discount function would guarantee consistency, as Strotz (1956) has shown.

However, with state variables in the utility function, a constant rate of time preference ceases to be the consistency condition. For one thing, the presence of s implies that u ceases to be separable from the past (backwards), given that s is the integral of all past purchases (the solution of (2)). As a result, the intertemporal function U ceases to be separable both backwards <u>and</u> forwards. To see this, redefine U as a sum over a number of discrete time periods, redefine x(t) as x_t and replace s(t) by the proxy x_{t-1}. The marginal rate of substitution between x_{it} and x_{it-1} is then seen to be dependent on x_{it+1}. On the other hand, changes in the state variables induce changes in tastes over time so that the Strotz approach cannot be relied upon, as was shown by Peleg and Yaari (1973). Fortunately, in the set-up of equation (8), there is "more" than a taste change : this change is in fact determined by past purchases. In other words, there is a "preference inheritance mechanism" such as the one described by Blackorby et al. (1973) which plays the same role as precommitment.

The subjective discount function in (8) reflects impatience. This impatience is necessarily present, even if the intertemporal utility function is not separable, in problems with infinite horizons, when the utility function in once continuously differentiable and the consumption space satisfies certain rather weak assumptions, as was shown by Burness (1973). The constancy of γ now becomes an assumption that is convenient for purposes of estimation, and can be relaxed (e.g. by considering different sub-periods with different γ's).

The next question is whether the static budget constraint makes sense in this intertemporal framework. We will show that it does, but only to the extent that the optimal path of total consumption expenditures is given, so that we have to handle first the problem of the lifetime allocation of savings and total consumption.

Let $w[=w(t)]$ represent non-human wealth, human wealth being defined as the discounted value of future labor income. With $\ell[=\ell(t)]$ representing current labor income, $r[=r(t)]$ the rate of interest on non-human wealth and $y[=y(t)]$ total consumption expenditures, we can write the wealth equation defined in Yaari (1964) and Lluch (1974) as

$$\dot{w} = rw + \ell - y.$$

If we were to maximize a functional similar to (8) with respect to total consumption, then the state variable associated with the latter would improperly summarize the physical stock of durables and the psychological stock of habits (characterizing the consumption of non-durables). Moreover, the stock of durables enters the non-human wealth but does not produce any interest. To take account of these remarks, we distinguish total expenditures of durables, $y_d(=p_d q_d)$, and of non-durables, $y_c(=p_c q_c)$, so that $y = y_c + y_d$. The price indices, p_d and p_c, are scalars as well as q_d and q_c. Next, we define two state variables s_d and s_c, associated respectively with q_d and q_c, as the solutions of the differential equations

(9a) $$\dot{s}_d = q_d - \delta_d s_d$$

and

(9b) $$\dot{s}_c = q_c - \delta_c s_c,$$

where δ_d is the rate of depreciation of the stock of durables and δ_c the rate at which habits to consume non-durables wear off. Accordingly, we redefine the wealth equation as

$$\dot{w} = r(w - p_d s_d) + \ell - y + p_d \dot{s}_d,$$

or

(10) $$\dot{w} = r(w - p_d s_d) + \ell - y_c - p_d \delta_d s_d.$$

It is worth noticing that, using Laplace transforms, (10) implies

$$w(o) + \int_o^\infty e^{-rt}\ell(t)\,dt = \int_o^\infty e^{-rt} y_c(t)\,dt + \int_o^\infty e^{-rt}(\delta_d + r)p_d(t)s_d(t)\,dt.$$

We are ready now to solve the problem of finding (P1) : the optimal path $y^* = y_d^* + y_c^*$ that maximizes[2]

(11) $$J_1 = \int_o^\infty e^{-rt}[u_d(q_d, s_d) + u_c(q_c, s_c)]\,dt$$

subject to (9), (10) and initial conditions on the state variables (s_d, s_c, w).

Notice that (P1) could have been formulated in terms of individual commodities. The use of the aggregates q_c and q_d is a simplification based on the assumption of separability between durables and non-durables. This raises the following question. Could the maximum conditions for (P1) in terms of individual commodities provide a theoretically plausible demand system, i.e. a system which satifies the restrictions of the classical theory of consumer demand ? The problem is that these restrictions are concerned with the instantaneous reactions of q_i to an instantaneous change in price or in total expenditure. In (P1) the reasoning is in terms of sequences. Although derivatives and elasticities could be computed, it seems difficult to interpret these in terms of the so-called Slutsky conditions (except possibly if the latter were redefined to include reactions to changes in sequences).

It is of some interest, therefore, to set up an analytical framework in which the old Slutsky conditions would still make sense, but without loosing the insights derived from an intertemporal approach. That is why we define a second problem (P2), in which optimal total expenditure y^* is supposed to be given as the solution of (P1), and is allocated among the n individual commodities. Differentiation of the maximum conditions in (P2) will lead to generalized Slutsky conditions that can be interpreted in the usual way and make the approach operational for the applied econometrician, as will be shown in Section 1.3. (while the implications for the structure of the error terms will be spelled out in Section 2.3).

After solving (P1), we thus have to find
(P2) : the optimal path of n goods q_i^* that maximizes (8) subject to (2), initial conditions on the state variables and the budget constraint :

$$p'q^* = y^*. \tag{12}$$

Notice that, if the utility functional (8) had been separable over time, we could have considered (P1) as the first step in a two-stage maximization procedure of an intertemporal utility tree, and then maximize the instantaneous utility function (1). This would have given a justification for the myopic approach, except for the fact that the latter supposes that <u>observed</u> total expenditure is optimal from an intertemporal point of view, as noted in Phlips (1975). Unfortunately, (8) is <u>not</u> separable over time. We have therefore to maximize the intertemporal function again in (P2).

It should be emphasized that the specific nature of the acquisition of durables has been incroporated in (P1). In the allocation problem (P2), there is nothing special with the purchases of durables as compared with the purchases of non-durables.

For simplicity, we ignore possible other constraints, such as non negativity of $y_d(t)$, $y_c(t)$, and $q_i(t).w(t)$, i.e. non-human wealth, is not constrained to be positive for all t, as was stressed by Arrow and Kurz (1970).

If an optimal program exists[3] and if all functions satisfy sufficiently strong smoothness conditions, the Maximum Principle gives the following necessary conditions[4].

Let $y_d^*[=y_d^*(t)]$ and $y_c^*[=y_c^*(t)]$ be an optimal choice for (P1). Then there exist three auxiliary variables $\phi_d[=\phi_d(t)]$, $\phi_c[=\phi_c(t)]$, $\mu[=\mu(t)]$, such that, for each t $(t \geqslant o)$:

(13) y_d^* $(=p_d q_d^*)$ and y_c^* $(=p_c q_c^*)$ maximize the current value of the Hamiltonian

$$H_1\ (q_d, q_c, s_d, s_c, \phi_d, \phi_c, \mu) = u_d + u_c + \phi_d \dot{s}_d + \phi_c \dot{s}_c + \mu \dot{w}$$

and

(14a) $$\dot{\phi}_d = \gamma \phi_d - \frac{\partial H_1}{\partial s_d}$$

(14b) $$\dot{\phi}_c = \gamma \phi_c - \frac{\partial H_1}{\partial s_c}$$

(15) $$\dot{\mu} = \gamma \mu - \frac{\partial H_1}{\partial w}$$

evaluated at $s_d = s_d(t)$, $s_c = s_c(t)$, $q_d = q_d^*(t)$, $q_c = q_c^*(t)$, $\phi_d = \phi_d(t)$, $\phi_c = \phi_c(t)$, $\mu = \mu(t)$ and $w = w(t)$.

Now, if max H_1 is concave in s_d and s_c for given ϕ_d, ϕ_c, μ and t, any policy satisfying (P1) and the transversality conditions

$$\lim_{t\to\infty} e^{-\gamma t}\phi_d s_d = \lim_{t\to\infty} e^{-\gamma t}\phi_d = o$$

$$\lim_{t\to\infty} e^{-\gamma t}\phi_c s_c = \lim_{t\to\infty} e^{-\gamma t}\phi_c = o$$

$$\lim_{t\to\infty} e^{-\gamma t}\mu w = \lim_{t\to\infty} e^{-\gamma t}\mu = o$$

is optimal.

According to conditions (13), (14) and (15), y_d and y_c are optimal for all t when

(16a) $$\frac{\partial u_d}{\partial q_d} = - \phi_d$$

(16b) $$\frac{\partial u_c}{\partial q_c} = \mu p_c - \phi_c \ ;$$

with

(17a) $$\dot{\phi}_d = (\gamma + \delta_d)\phi_d - \frac{\partial u_d}{\partial s_d} + (r + \delta_d)\mu p_d,$$

(17b) $\dot{\phi}_c = (\gamma + \delta_c)\phi_c - \frac{\partial u_c}{\partial s_c}$,

(18) $\dot{\mu} = (\gamma - r)\mu$.

Similarly, let ψ [$=\psi(t)$] be a vector of n auxiliary variables and λ [$=\lambda(t)$], the Lagrangian multiplier associated with (12), then q^* [$=q^*(t)$] will be the optimal allocation for (P2) - i.e. maximize the current value of the constrained Hamiltonian $H_2(q,s,\psi) + \lambda(y^* - p'q)$ - if the following conditions [5]

(19) $\frac{\partial u}{\partial q} = \lambda p - \psi$

(20) $\dot{\psi} = (\gamma - D)\psi - \frac{\partial u}{\partial s}$

(12) $y^* - p'q = o$

are satisfied together with (2) and the "transversality conditions"

$$\lim_{t\to\infty} \psi_i s_i e^{-\gamma t} = \lim_{t\to\infty} \psi_i e^{-\gamma t} = o \qquad (i=1,n)$$

In economic terms, a very nice interpretation can be given to these conditions.

The consumer is told to reduce his intertemporal problem to the maximization, in each period, of the Hamiltonian H_1 and H_2, the latter subject to (12). Clearly, the Hamiltonian is simply the current flow of utility from all sources, both enjoyed immediately (u) and anticipated to be enjoyed [$\phi_d\dot{s}_d + \phi_c\dot{s}_c$ in (P1), $\psi'\dot{s}$ in (P2)]. The auxiliary variables are thus defined as the marginal contribution of the corresponding state variable to the utility functional, given that the consumer intends to follow the optimal path. In other words, ϕ_d for instance is the implicit value attached to a marginal unit of s_d and μ is the implicit value attached to a marginal unit of wealth. ($\phi_d\dot{s}_d$) is therefore the rate of increase of utility due to the current rate of increase of state variable s_d. In a similar way, $(\mu\dot{w})$ is the rate of increase of utility due to the current rate of increase of non-human wealth .

Conditions (17) and (20) show that the change in the implicit price of a stock must be equal to minus its net contribution to utility.

However the optimal amount to invest in durables appears to be of a special nature. The analogy between equation (16a) and the optimal investment of the firm as in Gould (1968) is striking : the marginal utility of a unit of investment should be equal to its implicit marginal cost. In (17a), the change in the implicit price of a marginal unit of s_d is augmented, in comparison with (17b), by the alternative cost of investing in durables $(r\mu p_d)$ and the loss in terms of wealth resulting from the depreciation $(\delta_d \mu p_d)$.

Condition (15) also states that on the optimal path, the relative change of the implicit value of savings must be equal to the difference between the rate of time preference (γ) and the rate of interest. In long-run equilibrium, when $\dot{\mu}=o$, $\gamma=r$.

1.3. Generalized Slutsky Conditions

The solution of (P1) yields optimal total consumption expenditures, y^* $(=y_d^* + y_c^*)$, and optimal savings v^*. Indeed, if current labor income is corrected for transfers and taxes, one has, at each time period,

(21) $$v^* = rw + \ell - y^*,$$

and from (10)

(22) $$\dot{w}^* = v^* + p_d(\dot{s}_d - rs_d).$$

In words, the optimal rate of increase of non-human wealth is equal to optimal saving plus the net increase, in value, of the stock of durables minus the interest paid (or imputed) on the latter. Derivatives of y^* with respect to v^*, r, w and ℓ are easily deduced :

(23a) $$\frac{\partial y^*}{\partial v^*} = -1 ;$$

(23b) $$\frac{\partial y^*}{\partial r} = w - p_d s_d ;$$

(23c) $$\frac{\partial y^*}{\partial w} = r ;$$

(23d) $$\frac{\partial y^*}{\partial \ell} = 1.$$

We now turn to the allocation problem of y^* (P2). On solving (19) and (12), we get the system of n+1 equations

(24) $$q = q(y^*, p, s, \psi)$$
$$\lambda = \lambda(y^*, p, s, \psi).$$

The total differential of the vector of marginal utilities u_q is

$$du_q = U_{qq} d_q + U_{qs} ds$$

where U_{qq} is the matrix of second derivatives and $U_{qs} = [\partial^2 u/\partial q_i \partial s_j]$.

Total differentiation of (19) and (12) gives after rearrangement

(25) $$\begin{bmatrix} U_{qq} & p \\ p' & o \end{bmatrix} \begin{bmatrix} dq \\ -d\lambda \end{bmatrix} = \begin{bmatrix} o & \lambda I & -U_{qs} & -I \\ 1 & -q' & 0' & 0' \end{bmatrix} \begin{bmatrix} dy^* \\ dp \\ ds \\ d\psi \end{bmatrix}$$

where I is an identity matrix and 0 an n-vector with all elements equal to zero.

Total differentiation of (24) leads to

$$\begin{bmatrix} dq \\ \\ -dx \end{bmatrix} = \begin{bmatrix} q_{y^*} & Q_p & Q_s & Q_\psi \\ \\ -\lambda_{y^*} & -\lambda_p' & -\lambda_s' & -\lambda_\psi' \end{bmatrix} \begin{bmatrix} dy^* \\ dp \\ ds \\ d\psi \end{bmatrix}. \tag{26}$$

Substitution from (26) into (25) and solving for the derivatives gives the standard formulas of demand theory (and the resulting general restrictions). With respect to s we have the well known

$$\lambda_s = U_{qs} q_{y^*} \tag{27a}$$

$$Q_s = \frac{-1}{\lambda} K U_{qs}, \tag{27b}$$

where K is the matrix of the substitution effects. New are

$$\lambda_\psi = q_{y^*} \tag{28a}$$

$$Q_\psi = -\frac{1}{\lambda} K\ . \tag{28b}$$

Derivatives with respect to v^*, r, w and ℓ can be found using (23).

II. ECONOMETRICS

Before discussing the problems of estimation that arise with respect to the intertemporal model, it is of some interest to review the procedures used in a myopic framework as the experience acquired there may be directly relevant.

2.1. Myopic Models

The peculiarities of the estimation procedures stem from the fact that the state variables are unobservable and have therefore to be eliminated. In both the quadratic and the linear model, this elimination leads to demand equations that are highly non-linear and cannot be estimated directly, when due attention is given to problems of contemporaneous and serial correlation.

To overcome this difficulty, Houthakker and Taylor (1970, Chapter 5) used only the first n first-order conditions of equation (7), reformulated in discrete time and after the elimination of s, as estimating equations. These equations take the following very simple form :

$$q_{it} = K_{io} + K_{i1}q_{it-1} + K_{i2}\pi_t + K_{i3}\pi_{t-1}. \tag{29a}$$

π_t stands for $\lambda_t p_{it}$ in the quadratic model and for $(\lambda_t p_{it})^{-1}$ in the linear expenditure system. A recursive analytical expression for λ_t is derived by solving the first-order conditions (for λ) and is used in an iterative Gauss-Seidel procedure that is carried out until the estimated λ's are such that the budget constraint is satisfied. For example, the expression for λ_t in the quadratic model of Houthakker and Taylor is

$$\lambda_t = \frac{y - \sum_i K_{io} - \sum_i K_{i1}p_{it}q_{it-1} - \lambda_{t-1}\sum_i K_{i3}p_{it}p_{it-1}}{\sum_i K_{i2}p_{it}^2}\ . \tag{29b}$$

Since λ_t is defined up to a linear transformation, its initial value (λ_o) is arbitrary. All authors [Rossier (1972) excepted, see infra] set $\lambda_o = 1$, which amounts to iterating on λ_t/λ_o.

This procedure is attractive, if only because of its computational simplicity. However, the initial attempts[6] present deficiences on two accounts.

On the one hand, given that convergence is extremely slow once one gets close to satisfying the budget constraint[7], iteration is stopped when the estimated components are within .05 (or .1) percent of adding up to the observed total of expenditures. However the statistical properties of the estimated λ's depend upon those of the estimated regression coefficients, so that only the final estimates (when adding-up is exactly satisfied) of λ will take all restrictions into account.

On the other hand, although some attention was given by Houthakker and Taylor (1970) and Phlips (1972) to the single-equation autocorrelation problem, the "contemporaneous" correlation between equations was entirely neglected. The use of Zellner's seemingly unrelated regressions technique at each step of the Gauss-Seidel iteration on λ, as in Weiserbs (1972 or 1974) and Phlips (1974), was an improvement in this direction, but neglected in turn equation-by-equation serial correlation. This neglect is the more serious as the elimination of the state variables, leading to autoregressive estimating equations, makes the assumption of purely random errors rather hard to swallow.

An alternative approach, giving full attention to contemporaneous correlation, was developed by Rossier (1972). Although the first-order conditions are still used as estimating equations, the Gauss-Seidel iteration is replaced by a maximum-likelihood estimation of λ_o and of the regression coefficients, using the restricted Aitken estimator suggested by Court (1967) and Byron (1970). Here, the time sequence of λ_t is computed using an analytical solution for λ_t such as (29b), which appears indeed as a recursive relation between λ_t and λ_{t-1} and a function of all regression coefficients.

The trouble with this procedure is that, as mentioned above, λ_t is definedup to a linear transformation, so that its initial value is arbitrary. It does not make sense, therefore, to iterate on λ_o. To avoid this, Marquet and Phlips (1974) suggest to iterate on λ_1/λ_o. However, there are other difficulties, such as the practical necessity of using an algorithm that requires analytical derivatives (with respect to λ_1/λ_o and the other coefficients of the model), which in the present state of the art seems to preclude the possibility of taking simultaneously serial and contemporaneous correlation into account.

The Gauss-Seidel iteration on λ gives empirical results that are intuitively much more plausible than those derived by Rossier or Marquet and Phlips. Furthermore, in the Gauss-Seidel iteration the estimated system seems to become the more unstable the more one gets closer to satisfying the adding-up condition. This may be explained by the fact that imposing the budget constraint exactly, as it is done by the restricted Aitken estimator, leads into a logical inconsistency (once one tries to rationalize a myopic model from an intertemporal point of view) because non-separability over time is not compatible with an exact budget constraint defined in terms of observed total expenditure. It seems, therefore, that the Gauss-Seidel iteration on λ (which amounts to imposing the budget constraint approximately) has much to recommend it, on the condition that the estimators of the coefficients of the system [e.g. the K's in (29) or the underlying structural coefficients] have known desirable properties. Hendry's (1971) FIML procedure for systems with autocorrelated errors provides such estimators.

2.2. An Intertemporal Quadratic Model

Suppose that the average consumer is in fact behaving according to the rules developed in Section 1.2 and uses the conditions derived from the Maximum Principle as estimating equations. This amounts to supposing that the consumption data observed for period t are the result of planning decisions made in t such that these conditions be satisfied except for an error term.

To derive suitable estimating equations, we need to specify the instantaneous utility functions, to work out an approximation for finite time, and to eliminate the unobservable variables (s and Ψ). The dynamic Stone-Geary utility function (5) and the dynamic quadratic function (4) are natural candidates, with a preference for the former for reasons given above. Unfortunately, the non-linear character of the first-order derivatives of the Stone-Geary utility function makes the simultaneous elimination of Ψ and s very difficult, if not impossible[8]. Hence, for the time being, we shall have to concentrate our efforts on the quadratic model.

a) Determination of total expenditures (P1)

The following conditions describe the optimal path at each point in time :

(30a) $$a_d + A_d q_d + B_d s_d = -\phi_d$$

(30b) $$a_c + A_c q_c + B_c s_c = \mu p_c - \phi_c$$

(31a) $$\dot{\phi}_d = (\gamma + \delta_d)\phi_d - (b_d + B_d q_d + C_d s_d) + (r + \delta_d)\mu p_d$$

(31b) $$\dot{\phi}_c = (\gamma + \delta_c)\phi_c - (b_c + B_c q_c + C_c s_c),$$

together with (2), (10), (18) and the transversality conditions. Here, all parameters are scalars.

To derive our estimating equations in discrete time, we require a series of definitions and approximations.

Let

$$q_{dt} = \int_t^{t+1} q_d(t)dt, \tag{32}$$

and similarly for $q_{ct}, s_{dt}, s_{ct}, p_{dt}, p_{ct}, \phi_{dt}, \phi_{ct}, \mu_t, w_t, \ell_t$ and r_t. Let

$$s^o_{t+1} - s^o_t = \int_t^{t+1} \dot{s}(t)dt, \tag{33}$$

where the superscript ^{o} on s denotes the value of the variable at the beginning of the period, and similarly for $\dot{\phi}$, $\dot{\mu}$ and $\dot{w}$.

Suppose now that

$$s_t = \frac{1}{2}(s^o_{t+1} + s^o_t) \tag{34}$$

and similarly for ϕ_t, μ_t and w_t. This approximation will be the more accurate the closer the behavior of the variable is to linearity within the period [Houthakker and Taylor (1970) p.14].

With these assumptions, (30), (31) and (2) are rewritten in discrete time. A number of substitutions allow to eliminate ϕ and s and to end up with the estimating equations

$$y^*_{jt} = (p_j q^*_j)_t = p_{jt}[m_{jo} + m_{j1}q_{jt-1} + m_{j2}q_{jt-2} + m_{j3}\Pi_{jt} + m_{j4}\Pi_{jt-1} + m_{j5}\Pi_{jt-2}] \tag{35}$$

where j = c or d, Π_{jt} represents $\mu_t p_t$ when j = c and $(r_t + \delta)\mu_t p_t$ when j = d, and the m_j are functions of the structural parameters. The time shape of μ_t is determinated by (18), which can be rewritten in discrete time, using (32), (33) and (34) as

$$\mu_t = (2 - \gamma + r_t)^{-1}(2 + \gamma - r_{t-1})\mu_{t-1}. \tag{36}$$

The estimation method described below yields maximum likelihood estimates of γ and y^*.

b) The allocation model (P2)

Using definitions and approximations similar to (32), (33) and (34), we derive the following necessary conditions in discrete form for the allocation model :

$$(37) \qquad a + Aq_t + Bs_t = \lambda_t p_t - \psi_t$$

$$(38) \qquad 2(s_t - s_{t-1}) = q_t + q_{t-1} - D(s_t + s_{t-1})$$

$$(39) \qquad (2 - \gamma - D)\psi_t = (2 + \gamma + D)\psi_{t-1} - 2b - B(q_t + q_{t-1}) - c(s_t + s_{t-1})$$

together with the budget constraint and the "transversality conditions". Here small letters denote vectors, while capital letters denote diagonal matrices.

The elimination of the unobservable variables (ψ and s) yields the estimating system

$$\begin{aligned}(40) \qquad p_t q_t = {} & H^{-1}[2D\alpha p_t + [2A(D^2 + D\gamma + 4) + \beta] p_t q_{t-1} + [A(D - 2) \\ & (D + \gamma + 2) + \beta] p_t q_{t-2} + (D + 2)(2 - \gamma - D)\lambda_t p_t^2 + \\ & (D^2 + D\gamma + 4)\lambda_{t-1} p_t p_{t-1} + (2 - D)(D + \gamma + 2)\lambda_{t-2} p_t p_{t-2}] \\ & + \varepsilon_t\end{aligned}$$

where $H = A(2 + D)(2 - D - \gamma) - \beta$

$\alpha = 2[b + a(D + \gamma)]$

$\beta = B(\gamma + 2D) + C.$

To obtain estimates for α, β, A and D, it may be easier, from a programming point of view, to rearrange (40) and to estimate the system

$$(41) \qquad Q_t = K_1 X_{1t} + K_2 X_{2t} + K_3 X_{3t} + K_4 X_{4t} + \varepsilon_t$$

where

$$Q = p_t(q_t + 2q_{t-1} + q_{t-2})$$

$$X_1 = p_t$$

$$X_2 = p_t[(\gamma - 4)q_{t-1} + \gamma q_{t-2}]$$

$$X_3 = p_t(\lambda_t p_t + 2\lambda_{t-1}p_{t-1} + \lambda_{t-2}p_{t-2})$$

$$X_4 = p_t[(\gamma - 4)\lambda_{t-1}p_{t-1} + \gamma\lambda_{t-2}p_{t-2}]$$

and

$$K_1 = 2H^{-1}D\alpha$$

$$K_2 = -4H^{-1}A$$

$$K_3 = H^{-1}(4 - 2\gamma - D\gamma - D^2)$$

$$K_4 = 4H^{-1}$$

Then

$$\alpha = 2D^{-1}K_4^{-1}K_1$$

$$\beta = -4K_4^{-2}(K_2K_3 + K_4)$$

$$A = K_4^{-1}K_2$$

$$D = -\frac{1}{2}\gamma + \frac{1}{2}[(\gamma - 4)^2 - 16K_4^{-1}K_3]^{1/2}.$$

All structural coefficients are functions of the rate of time preference.

2.3. The error term

The consumer is certainly incapable of carrying out problem (P1), i.e. allocating his total expenditures over his lifetime, without error. Some years he spends more than planned, other years he saves more than planned. The car he purchases may turn out to be more expensive than planned ; a vacation trip typically costs more than expected... And nevertheless the family has to keep up its normal standard of living on other items. As a result, there will be an "intertemporal" error

(42) $$z_t = y_t - y_t^*.$$

Its estimate is obtained as $\hat{z}_t = \hat{u}_{dt} + \hat{u}_{ct}$, its components being estimates of the errors terms in (35), i.e. of

$$u_{dt} = y_{dt} - y^*_{dt}$$
$$u_{ct} = y_{ct} - y^*_{ct}.$$

Both u_{dt} and u_{ct} can be serially correlated. Furthermore, there may be some contemporaneous correlation between the two, although they are not related by a budget constraint, e.g. because of the business cycle affecting the rate of interest and labor income.

From the point of view of the allocation problem (P2), z_t is a variable which takes given values at each t and must be treated as non-stochastic. It has to be allocated among all commodities at each time period. We therefore decompose the error term of each demand equation, ε_{it}, as

$$(43) \qquad \varepsilon_{it} = \alpha_i z_t + \eta_{it}, \qquad (i = 1, \ldots, n)$$

and treat $\hat{z}_t$ as an observed variable. This makes it possible to reintroduce the budget constraint in terms of observed total expenditures (y_t) and to define the constraints on the system as

$$(44) \qquad \sum_i p_{it}\hat{q}_{it} = y_t$$

and

$$(45) \qquad \sum_i \alpha_i = 1,$$

which implies

$$(46) \qquad \sum_i \eta_{it} = 0$$

so that the contemporaneous variance-covariance matrix of the η's is non-diagonal and singular.

Although the term $\alpha_i z_t$ may take care of some autocorrelation in ϵ_{it}, we want to allow for the possibility of other sources of autocorrelation affecting η_{it}. We therefore specify the n vector η_t as

$$(47) \qquad \eta_t = \sum_{k=1}^{r} R_k \eta_{t-k} + \xi_t$$

where the R_k are matrices of autoregressive parameters and ξ_t is $NID(0,\Sigma)$.

FOOTNOTES

*This is a revised and enlarged version of a paper on "Dynamic Demand Systems" presented by the first author at the SSRC Conference on Consumer Behavior, The Royal Society, London, 26th October 1973, and in the Universities of Pennsylvania and Toronto, the Bureau of Labor Statistics and the Federal Reserve Board. Thanks are due to A.P. Barten, D. Hendry, J.D. Sagan, L.D. Taylor and J. Thisse for helpful discussions. We are also indebted to Mrs. M. Vuylsteke-Wauters, J.P. Lemaître, P. Rousseaux and D. Van Grunderbeeck for computational assistance.

[1]See Weiserbs (1972) or Phlips (1974).

[2]Lluch (1973a) has derived the aggregate consumption implied in this sort of intertemporal maximization, for the case of an instantaneous Stone-Geary function. In a more recent paper (1974), he has done the same for the dynamic functions (4) and (5), focusing attention on <u>subjective</u> planning time.

[3]A theorem of existence and uniqueness for this class of problems can be found in Thisse and Weiserbs (1973).

[4]See Arrow and Kurz (1970, Section II.6).

[5]In Phlips (1974, Chapter 10), only this second step intertemporal maximization was carried out, with y instead of y^* in the budget constraint.

[6]Houthakker and Taylor (1970, Chapter 5) and Phlips (1972).

[7]Tests carried out by F. Carlevaro (at the Centre d'Econométrie, Geneva) on the static linear expenditure system show that several thousands of iterations are needed to satisfy the budget constraint exactly. In practice, the iteration is stopped when the budget constraint is satisfied for all t to some predetermined degree.

[8]Efforts to construct an intertemporal linear expenditure system that can be estimated are reported in Phlips and Weiserbs (1972).

REFERENCES

ARROW, K. and M. KURZ (1970), *Public Investment, The Rate of Return, and Optimal Fiscal Policy,* Johns Hopkins, Baltimore.

BARTEN, A.P. (1969), Maximum Likelihood Estimation of a Complete System of Demand Equations, *European Economic Review,* 1, Fall 1969, 7-73.

BLACKORBY, C. and D. NISSEN, D. PRIMONT and R.R. RUSSELL (1973), Consistent Intertemporal Decision Marking, *Review of Economic Studies,* 40 (1973), 239-248.

BURNESS, H.S. (1973), Impatience and the Preference for Advancement in the Timing of Satisfactions, *Journal of Economic Theory,* 6 (1973), 495-507.

BROWN, J.A.C. and A. DEATON (1972), Surveys in Applied Economics : Models of Consumer Behaviour, *Economic Journal,* 82 (1972), 1145-1236.

BYRON, R.P. (1970), The Restricted Aitken Estimation of Sets of Demand Relations, *Econometrica,* 38 (1970), 816-830.

COURT, R.H. (1967), Utility Maximization and the Demand for New Zealand Meats, *Econometrica,* 35 (1967), 424-446.

GOULD, J. (1968), Adjustement Costs in the Theory of Investment of the Firm, *Review of Economic Studies,* (1968), 47-55.

HEIEN, D.M. (1972), Demographic Effects and the Multiperiod Consumption Function, *Journal of Political Economy,* 80 (1972), 125-138.

HENDRY, D.F. (1971), Maximum Likelihood Estimation of Systems of Simultaneous Regression Equations with Errors Generated by a Vector Autoregressive Process, *International Economic Review*, 12 (1971), 257-272.

HOUTHAKKER, H.S. and L.D. TAYLOR (1970), *Consumer Demand in the United States 1929-1970*, 2nd ed., Harvard University Press, Cambridge, Mass.

LLUCH, C. (1973a), The Extended Linear Expenditure System, *European Economic Review*, 4 (1973), 21-32.

LLUCH, C., (1974), Expenditure, Savings and Habit Formation, *International Economic Review*, 15 (1974), 786-797.

MARQUET, G. and L. PHLIPS (1974), Estimation par maximum de vraisemblance du système linéaire des dépenses dynamisé, *Recherches Economiques de Louvain*, 40 (1974), 45-53.

MODIGLIANI, F. (1966), The Life Cycle Hypothesis of Saving, the Demand for Wealth and the Supply of Capital, *Social Research*, 33 (1966), 160-217.

PELEG, B. and M.E. YAARI (1973), On the Existence of a Consistent Course of Action when Tastes are Changing, *Review of Economic Studies*, 40 (1973), 391-401.

PHLIPS, L. (1972), A Dynamic Version of the Linear Expenditure Model, *Review of Economics and Statistics*, 64 (1972), 450-458.

PHLIPS, L. and D. WEISERBS (1972), *An Intertemporal Version of the Linear Expenditure System*, Working Paper n° 7217, Institut des Sciences Economiques, Louvain, mimeo.

PHLIPS, L. (1974), *Applied Consumption Analysis*, Advanced Textbooks in Economics, North-Holland Publishing Cy, Amsterdam.

PHLIPS, L. (1975), Comment, *Annals of Economic and Social Measurement*, 4 (1975), 67-70.

ROSSIER, E. (1972), *Contribution aux explications dynamiques de la consommation semi-agrégée*, Doctoral Dissertation, Geneva.

STROTZ, R.H. (1966), Myopia and Inconsistency in Dynamic Utility Maximization, *Review of Economic Studies*, 23 (1956), 165-180.

TAYLOR, L.D. and D. WEISERBS (1972), On the Estimation of Dynamic Demand Functions, *Review of Economics and Statistics*, 54 (1972), 459-465.

TAYLOR, L.D. (1974), On the Dynamics of Dynamic Demand Models, *Recherches Economiques de Louvain*, 40 (1974), 21-31.

THISSE, J. and D. WEISERBS (1973), *Existence and Uniqueness of an Optimal Control with an Infinite Horizon*, Working Paper n° 7303, Institut des Sciences Economiques, Louvain, mimeo.

WEISERBS, D. (1972), *Contributions à une analyse dynamique des choix du consommateur*, Doctoral Dissertation, Anec, Louvain.

WEISERBS, D. (1974), More About Dynamic Demand Systems, *Recherches Economiques de Louvain*, 40 (1974), 33-43.

YAARI, M.E. (1964), On the consumer's lifetime allocation process, *International Economic Review*, 5 (1964), 304-317.