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LOT-SIZING ON A TREE

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Abstract

For the problem of lot-sizing on a tree with constant capacities, or stochastic lot-sizing with a scenario tree, we present various reformulations based on mixing sets. We also show how earlier results for uncapacitated problems involving (Q, S_Q) inequalities can be simplified and extended. Finally some limited computational results are presented.

Keywords: Lot-Sizing, Scenario Tree, Mixing Sets.

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1 Introduction

In two recent papers Guan et al. have examined the problem of stochastic uncapacitated lot-sizing based on a tree of scenarios. In the first paper [3], using a mixed integer programming formulation of the problem, the authors developed a family of valid inequalities, called (Q, S_Q) inequalities, and carried out some computational experiments to demonstrate their utility. In the second [4] they showed that the (Q, S_Q) inequalities suffice to describe the convex hull of the set of solutions when there is just one period with uncertain outcomes. The goal in this note is to clarify and extend these results to capacitated problems using an underlying model of lot-sizing on a tree, and to show the role of mixing sets in providing strong relaxations for the problem.

Specifically in Section 2 we introduce the problem of lot-sizing on a tree. We then demonstrate two ways to obtain mixing set relaxations. In Section 3 we present a simple description of the (Q, S_Q) inequalities, and then demonstrate that these inequalities are all mixing inequalities. In Section 4 we view the special one period problem treated by Guan et al. as a discrete version of the one period newsboy problem in which demand, shortage and salvage cost functions vary by outcome, and the shortage cost can include fixed costs. Here we use recent results of Conforti et al. [1] to show that the results can be extended to cover production or shortage with constant capacities or batch sizes. We can also treat a slightly more general cost function using results of [2]. Finally in Section 5 we present limited computational results. Even though the default cutting planes (such as flow cover and path inequalities [9]) of the MIP solvers increase the lower bounds considerably on the instances tried, for a fixed running time the mixing set reformulations lead to significantly improved lower and upper bounds compared to those of the original formulation.

2 Lot-Sizing on a Tree and Mixing Sets

Given a rooted directed out-tree $T = (V, A)$, let $D(v)$ be the set of direct successors of v, $S(v)$ the set of all successors of v, and $P(j, k)$ with $k \in S(j)$ the set of nodes on the path from j to k. Node $r = 0 \in V$ is the root. $L = \{v \in V : S(v) = \emptyset\}$ are the leaves. We add a dummy node -1 and an arc $(-1, 0)$. Let $p(v)$ be the unique predecessor of v for all $v \in V$.

The lot-sizing problem on a tree *LS-C-TREE* is defined as the following mixed integer program,

$$\begin{aligned} &\min \textstyle\sum_{v\in V} (P_v x_v + Q_v y_v) + \sum_{v\in V\cup\{-1\}} H_v s_v \\ &s_{p(v)} + x_v = d_v + s_v \quad \text{for all } v \in V \\ &x_v \leq C_v y_v \quad \text{for all } v \in V \\ &s \in \mathbb{R}_+^{|V|+1}, x \in \mathbb{R}_+^{|V|}, y \in \{0,1\}^{|V|}, \end{aligned} \tag{1}$$

with production costs P_v, fixed costs Q_v, demands d_v and capacity C_v for all $v \in V$, and storage costs H_v for all $v \in V \cup \{-1\}$. Note the special form of the balance constraints (1), in which the flow s_v out of node $v \in V \setminus L$ is the inflow to each direct successor node $w \in D(v)$, see Figure 1.

Note that an alternative formulation is obtained if we replace the balance constraints (1) by the equations (we write s instead of s_{-1})

$$s + \sum_{u\in P(0,v)} x_u = d_{0v} + s_v \quad \text{for all } v \in V$$

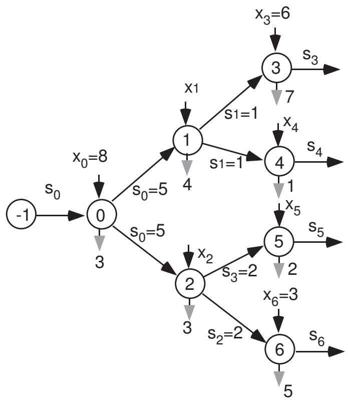

Figure 1: Lot-sizing on a tree.

or by the inequalities

$$s + \sum_{u \in P(0,v)} x_u \geq d_{0v} \quad \text{for all } v \in V$$

where $d_{uv} \equiv \sum_{w \in P(u,v)} d_w$.

To treat the stochastic lot-sizing problem, where the tree structure corresponds to the multistage structure of production decisions and demand observations, the problem of minimizing total expected cost can be modeled by weighting the terms in the objective function by appropriate probabilities.

Mixing Sets

Definition 1 *The set*

$$x + z_t \geq b_t \text{ for } t = 1, \ldots, n$$
$$x \in \mathbb{R}_+, z \in \mathbb{Z}^n_+$$

is called a mixing set*, denoted* X^{MIX}.

The convex hull of X^{MIX} has been studied in several papers. A valid inequality description is given in [5] (see Theorem 1 below), an $O(n \log n)$ separation algorithm is given in [8], and an extended formulation with $O(n)$ constraints and variables is given in [6].

The following theorem gives a linear description of the convex hull of a mixing set. For $t = 1, \ldots, n$ we define $f_t = b_t - \lfloor b_t \rfloor$.

Theorem 1 *[5] Let $T \subseteq \{1, \ldots, n\}$ and suppose that $i_1, \ldots, i_{|T|}$ is an ordering of T such that $0 = f_{i_0} \leq f_{i_1} \leq \cdots \leq f_{i_{|T|}}$. Then the* mixing inequalities

$$x \geq \sum_{t=1}^{|T|} (f_{i_t} - f_{i_{t-1}})(\lfloor b_{i_t} \rfloor + 1 - z_{i_t})$$

and

$$x \geq \sum_{t=1}^{|T|} (f_{i_t} - f_{i_{t-1}})(\lfloor b_{i_t} \rfloor + 1 - z_{i_t}) + (1 - f_{i_{|T|}})(\lfloor b_{i_1} \rfloor - z_{i_1})$$

are valid for X^{MIX}. Furthermore such inequalities (together with $z_t \geq 0$) give the convex hull of X^{MIX}.

Corollary 2 *Consider the set*

$$x + Mz_t \geq b_t \text{ for } t = 1, \ldots, n$$
$$x \in \mathbb{R}_+, z \in \mathbb{Z}_+^n$$

where $0 \leq b_1 \leq \cdots \leq b_n$ and M is a large number. Let $T \subseteq \{1, \ldots, n\}$ and assume that $i_1, \ldots, i_{|T|}$ is an increasing ordering of the elements of T. Then the mixing inequality

$$x \geq \sum_{t=1}^{|T|} (b_{i_t} - b_{i_{t-1}})(1 - z_{i_t})$$

is valid for the convex hull of the above set. Furthermore such inequalities give the convex hull of the above set.

2.1 Mixing Set Relaxations with Constant Capacities

Here we suppose that the capacities are constant at each node: $C_v = C$ for all $v \in V$.

Given two distinct nodes v, w, we let $\mu(v, w)$ be the common root, i.e the root of the smallest subtree containing both v and w. Also given a path $P(u, v)$, we let $\bar{P}(u, v) = P(u, v) \setminus \{u\}$.

Proposition 3 *For all $v \in V$, the mixing set $X_1^{MIX}(v)$*

$$s_v + Cz_{vw} \geq d_{0w} - d_{0v} \quad \text{for all } w \in W(v)$$
$$s_v \in \mathbb{R}_+^1, z_{vw} \in \mathbb{Z}_+ \quad \text{for all } w \in W(v)$$

where $W(v) = \{w \in V : d_{0w} - d_{0v} > 0\}$ and $z_{vw} = \sum_{u \in \bar{P}(\mu(v,w),w)} y_u$, is a relaxation of the set $X^{LS-TREE}$ defined by (2)-(4).

Proof: The inequality is obtained by combining the sum of the balance constraints (2) along the path $\bar{P}(\mu(v, w), v)$

$$s_{\mu(v,w)} + \sum_{u \in \bar{P}(\mu(v,w),v)} x_u = \sum_{u \in \bar{P}(\mu(v,w),v)} d_u + s_v$$

and the surrogate of the sum of the balance constraints along the path $\bar{P}(\mu(v, w), w)$

$$s_{\mu(v,w)} + \sum_{u \in \bar{P}(\mu(v,w),w)} Cy_u \geq \sum_{u \in \bar{P}(\mu(v,w),w)} d_u$$

along with the non-negativity of x_u, and the fact that $\sum_{u \in \bar{P}(\mu(v,w),w)} d_u - \sum_{u \in \bar{P}(\mu(v,w),v)} d_u = d_{0w} - d_{0v}$. □

Starting from the initial balance constraints, we can also include any subset of the x_v variables in the continuous variable in order to build other mixing relaxations.

Proposition 4 *For all $U \subseteq V$, the mixing set $X_2^{MIX}(U)$:*

$$s_U + C\zeta_{v,U} \geq d_{1v} \quad \text{for all } v \in V$$
$$s_U \in \mathbb{R}_+, \zeta_{v,U} \in \mathbb{Z}_+ \quad \text{for all } v \in V,$$

where $\zeta_{v,U} = \sum_{u \in P(0,v) \setminus U} y_u$ and $s_U = s + \sum_{u \in U} x_u$, is a relaxation of the set $X^{LS-TREE}$ defined by (2)-(4).

3 (Q, S_Q) Inequalities

Let $Q \subseteq V$ be such that no two nodes of Q lie on the same path from the root. This defines a unique rooted subtree $T_Q = (V_Q, A_Q)$ having the nodes of Q as leaves and 0 as the root node. We suppose that the $N+1$ nodes of this tree are (re-)numbered from 0 to N, where 0 corresponds to the root and the leaves following a prefix ordering are numbered $N-K+1, \ldots, N$, where $|Q| = K$. Note that given a planar representation of the tree, a prefix or infix ordering on the nodes always leads to the same ordering of the leaves. In addition we must satisfy the condition that $d_{0,N-K+1} < \cdots < d_{0,N}$.

For nodes $v \in V_Q$ of the tree T_Q, we use the notation:

- $m(v) = \max\{q \in Q \cap V_Q(v)\}$, where $V_Q(v)$ is the subtree of T_Q rooted at v.
- $\rho(v) = \max\{q \in Q : q < \min[q \in Q \cap V_Q(v)]\}$

If $\{q \in Q : q < \min[q \in Q \cap V_Q(v)]\} = \emptyset$ then $\rho(v)$ is undefined, but we set $d_{0,\rho(v)} = -\infty$. Note that this happens if and only if $v \in P(0, N-K+1)$. See Figure 2, where we assume that the order of the leaves is increasing starting from the top of the tree.

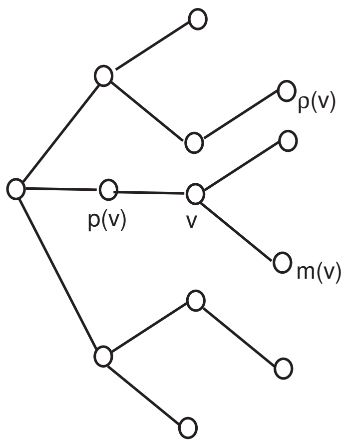

Figure 2: Tree Indicators

In the subsequent part of this section we will use the following property, which is easy to check: if $u \in P(0,v)$ then $\rho(u) \leq \rho(v)$. In particular, $d_{0,\rho(u)} \geq d_{0,\rho(p(u))}$ for every $u \neq 0$.

We can now present the (Q, S_Q) inequality.

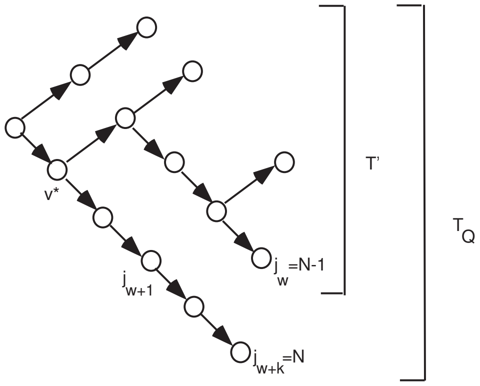

Figure 3: Induction Step

Proposition 5 *[3] For any subset $S_Q \subseteq V_Q$, the (Q, S_Q) inequality*

$$s + \sum_{u \in S_Q} x_u + \sum_{u \in V_Q \setminus S_Q} (d_{0,m(u)} - \max[d_{0,p(u)}, d_{0,\rho(u)}]) y_u \geq d_{0,m(0)} \tag{2}$$

is valid for the uncapacitated problem with $C_v = M$ large for all $v \in V_Q$.

We now show that this inequality is a mixing inequality.

We define $W = \{v \in V_Q : d_{0,v} > d_{0,\rho(v)}\}$. Clearly $Q \cup P(0, n - K + 1) \subseteq W$. We define an ordering $W = \{j_1, \dots, j_w\}$ of the nodes of W by nondecreasing values of d_{0,j_t}. We now explain how this order can be made unique.

Suppose there exist two indices i, k such that $j_i, j_k \in W$ and $d_{0,j_i} = d_{0,j_k}$. We show that then either $j_i \in P(0, j_k)$ or $j_k \in P(0, j_i)$. If neither of the two nodes is a successor of the other then $m(j_i) \neq m(j_k)$. We assume w.l.o.g. $m(j_i) < m(j_k)$. Then $m(j_i) < \min\{v \in Q \cap V_Q(j_k)\}$, which implies $\rho(j_k) \geq m(j_i)$. Then $d_{0,\rho(j_k)} \geq d_{0,m(j_i)} \geq d_{0,j_i} = d_{0,j_k}$ and thus $j_k \notin W$, a contradiction.

Therefore, j_i and j_k lie on the same path from the root, say $j_k \in P(0, j_i)$. Then we assume that the ordering satisfies $k < i$.

Lemma 6 *i. If $v \in W$ with $v = j_t$, then $v \in P(0, j_i)$ if and only if $t \leq i \leq k$ where $j_k = m(v)$.*
ii. If $v \notin W$, then $v \in P(0, j_i)$ if and only if $t \leq i \leq k$ where $j_{t-1} = \rho(v)$ and $j_k = m(v)$.

Proof: First we remark that if $v \notin W$ then $v \notin P(0, N - K + 1)$ and thus $\rho(v)$ is defined. Therefore the definition of t in the second part of the statement makes sense.

The proof uses induction on the number $|Q|$ of leaves. The case of $|Q| = 1$ is immediate. Consider now the tree T_Q. Let $v^* = \mu(N-1, N)$. The tree T' obtained from T_Q by removing the path $\bar{P}(v^*, N)$ has $|Q| - 1$ leaves.

We assume the inductive hypothesis for T', and show that it still holds for T_Q.

For all nodes of T' except those on the path $P(0, v^*)$, the values of ρ and m do not change. For nodes $v \in P(0, v^*)$, $m(v) = N - 1$ in T' and $m(v) = N$ in T_Q, but $\rho(v)$ is unchanged. For the new nodes in $\bar{P}(v^*, N)$, $m(v) = N$ and $\rho(v) = N - 1$. In addition the $k \geq 1$ nodes in $\bar{P}(v^*, N)$ for which $d_{0,v} > d_{0,N-1}$ are added to W giving $W = \{j_1, \dots, j_w, j_{w+1}, \dots, j_{w+k}\}$, see Figure 3.

Now it is easily checked that the conditions hold for all nodes on the path $P(0, N)$. $\square$

The above proof also shows the following.

Corollary 7 *For $i = 2, \ldots, |W|$, either $j_{i-1} = p(j_i)$ or $j_{i-1} = \rho(j_i)$.*

Proposition 8 *For any subset $S_Q \subseteq V_Q$, the (Q, S_Q) inequality is (dominated by) a mixing inequality.*

Proof: Define $M = \max\{C_v : v \in V\}$, $r = \max\{i : P(0, j_i) \subseteq S_Q\}$ and W as above. (If $\{i : P(0, j_i) \subseteq S_Q\} = \emptyset$, set $r = 0$.)

Proposition 4 implies that the following inequalities are valid:

$$s + \sum_{u \in S_Q} x_u + M \sum_{u \in P(0,j_i) \setminus S_Q} y_u \geq d_{0,j_i} \quad \text{for } i = r+1, \ldots, |W|. \tag{3}$$

Then, if we set $\bar{s} = s + \sum_{u \in S_Q} x_u - d_{0,j_r}$ and $z_i = \sum_{u \in P(0,j_i) \setminus S_Q} y_u$, we obtain that the following mixing set relaxation is valid:

$$\begin{aligned} \bar{s} + M z_i &\geq d_{0,j_i} - d_{0,j_r} \quad && \text{for } i = r+1, \ldots, |W| \\ \bar{s} \geq 0,\ z_i &\in \mathbb{Z} && \text{for } i = r+1, \ldots, |W| \end{aligned} \tag{4}$$

where $\bar{s} \geq 0$ follows from the definition of r.

The mixing inequality using all inequalities (4) is (see Corollary 2)

$$\bar{s} \geq \sum_{i=r+1}^{|W|} (d_{0,j_i} - d_{0,j_{i-1}})(1 - z_i),$$

where we set $d_{0,j_0} = 0$.

In the original variables, the inequality is

$$s + \sum_{u \in S_Q} x_u - d_{0,j_r} \geq \sum_{i=r+1}^{|W|} (d_{0,j_i} - d_{0,j_{i-1}})(1 - \sum_{u \in P(0,j_i) \setminus S_Q} y_u),$$

which is equivalent to

$$s + \sum_{u \in S_Q} x_u + \sum_{u \notin S_Q} [\sum_{i: i > r, u \in P(0,j_i)} (d_{0,j_i} - d_{0,j_{i-1}})] y_u \geq d_{0,m(0)}.$$

We show that this inequality dominates (2). Specifically, we show that for each $u \in V_Q \setminus S_Q$

$$\sum_{i: i > r, u \in P(0,j_i)} (d_{0,j_i} - d_{0,j_{i-1}}) \leq d_{0,m(u)} - \max\{d_{0,p(u)}, d_{0,\rho(u)}\}. \tag{5}$$

Fix $u \in V_Q \setminus S_Q$. If $\{i : i > r,\ u \in P(0, j_i)\} = \emptyset$ then condition (5) is trivially satisfied. So we assume $\{i : i > r,\ u \in P(0, j_i)\} \neq \emptyset$. Define $k(u) = \min\{i : i > r,\ u \in P(0, j_i)\}$. Then the left-hand-side of (5) is equal to $d_{0,m(u)} - d_{0,j_{k(u)-1}}$. So we have to prove that

$$d_{0,j_{k(u)-1}} \geq \max\{d_{0,p(u)}, d_{0,\rho(u)}\}. \tag{6}$$

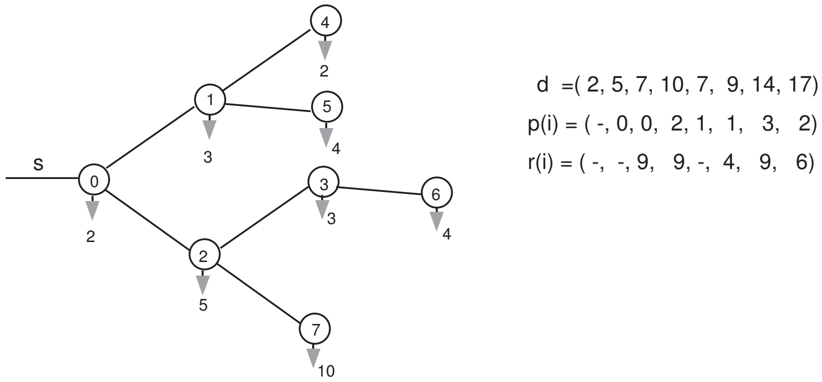

Figure 4: (Q, S_Q) tree

Suppose $k(u) = r+1$. Then Corollary 7 implies $d_{0,j_r} = \max\{d_{0,p(j_r)}, d_{0,\rho(j_r)}\}$. Then, since $u \in P(0, j_r)$, we have $d_{0,j_r} \geq d_{0,p(j_r)} \geq d_{0,p(u)}$ and $d_{0,j_r} \geq d_{0,\rho(j_r)} \geq d_{0,\rho(u)}$. Thus equation (6) is satisfied.

So we assume $k(u) > r+1$. Suppose first $u = j_{k(u)}$ and $d_{0,p(u)} > d_{0,\rho(u)}$. Then $d_{0,p(u)} > d_{0,\rho(p(u))}$, thus $p(u) \in W$. Corollary 7 implies $j_{k(u)-1} = p(u)$, thus (6) holds.

Suppose now $u = j_{k(u)}$ and $d_{0,p(u)} \leq d_{0,\rho(u)}$. Then, since $\rho(u) \in W$, Corollary 7 implies $j_{k(u)-1} = \rho(u)$, thus (6) holds.

Suppose $u \in W$ but $u \neq j_{k(u)}$. This implies $u = j_\ell$, with $\ell < k(u)$. Then $d_{0,j_{k(u)-1}} \geq d_{0,u} \geq \max\{d_{0,p(u)}, d_{0,\rho(u)}\}$.

Finally suppose $u \notin W$. Lemma 6 then implies $j_{k(u)-1} = \rho(u)$. Furthermore, since $u \notin W$, $d_{0,\rho(u)} \geq d_{0,u} \geq d_{0,p(u)}$. Then (6) is satisfied.

So inequality (6) holds for every $u \in V_Q \setminus S_Q$, thus concluding the proof. □

In the proof of Proposition 8 we have constructed a mixing inequality which dominates a given (Q, S_Q) inequality. In Section 4 we will give an example in which a mixing inequality is not implied by the (Q, S_Q) inequalities, thus showing that the mixing set relaxations give rise to inequalities which are tighter than the (Q, S_Q) inequalities.

Example 1 *We consider an instance of lot-sizing on a tree shown in Figure 4. The conditions required for a* (Q, S_Q) *inequality hold as* $d_{04} = 7 < d_{05} = 9 < d_{06} = 14 < d_{07} = 17$. *Calculating* $p = (-, 0, 0, 2, 1, 1, 3, 2)$, $m = (7, 5, 7, 6, 4, 5, 6, 7)$ *and* $\rho = (\emptyset, \emptyset, 5, 5, \emptyset, 4, 5, 6)$, *the* $(Q, \emptyset)$ *inequality is*

$s + (17-0)y_0 + (9 - \max[2, -\infty])y_1 + (17 - \max[2, 9])y_2 + (14 - \max[7, 9])y_3 + (7 - \max[5, -\infty])y_4 + (9 - \max[5, 7])y_5 + (14 - \max[10, 9])y_6 + (17 - \max[7, 14])y_7 \geq 17$, *or*

$$s \quad +17y_0 \quad +7y_1 \quad +8y_2 \quad +5y_3 \quad +2y_4 \quad +2y_5 \quad +4y_6 \quad +3y_7 \quad \geq \quad 17.$$

Now we generate the inequality as a mixing inequality. Note that $3 \in W$ *as* $d_{0,3} = 10 > d_{0,\rho(3)} = d_{0,5} = 9$, *and* $0, 1 \in W$ *as* $\rho(0), \rho(1)$ *are undefined. Thus* $W = Q \cup \{0, 1, 3\}$. *The ordering of* W *is* $\{0, 1, 4, 5, 3, 6, 7\}$ *with* $d_{0u} = (2, 5, 7, 9, 10, 14, 17)$.

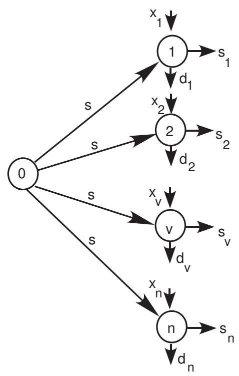

Figure 5: One Period Newsboy Problem

The mixing inequality is

$$
\begin{aligned}
s \geq \; & 2(1-y_0) \\
& +(5-2)(1-y_0-y_1) \\
& +(7-5)(1-y_0-y_1-y_4) \\
& +(9-7)(1-y_0-y_1-y_5) \\
& +(10-9)(1-y_0-y_2-y_3) \\
& +(14-10)(1-y_0-y_2-y_3-y_6) \\
& +(17-14)(1-y_0-y_2-y_7).
\end{aligned}
$$

which is the same inequality as above.

4 The One Period or Discrete Newsboy Problem

Here we examine a variant of the classical newsboy problem, see for instance Ch. 10 in Silver, Petersen and Pyke [7]. An initial amount s can be produced at time 0 at a unit cost of h without any fixed cost. Then with probability p_v the vth outcome in period 1 is observed. This consists of the demand d_v, as well as the new production costs involving a unit production cost c_v, a fixed cost q_v per batch of size C and a unit disposal cost of h_v for $v = 1, \ldots, n$. The corresponding lot-sizing tree is shown in Figure 5.

A mixed integer programming formulation for this discrete newsboy problem is now

$$
\begin{gathered}
\min hs + \sum_{v=1}^{n} p_v[c_v x_v + q_v y_v + h_v s_v] \\
s + x_v = d_v + s_v \text{ for } v = 1, \ldots, n \\
x_v \leq C y_v \text{ for } v = 1, \ldots, n \\
s \in \mathbb{R}_+^{n+1}, x \in \mathbb{R}_+^n, y \in \mathbb{Z}_+^n
\end{gathered}
$$

After elimination of the variables s_v for $v = 1, \ldots, n$, we obtain the feasible region, denoted

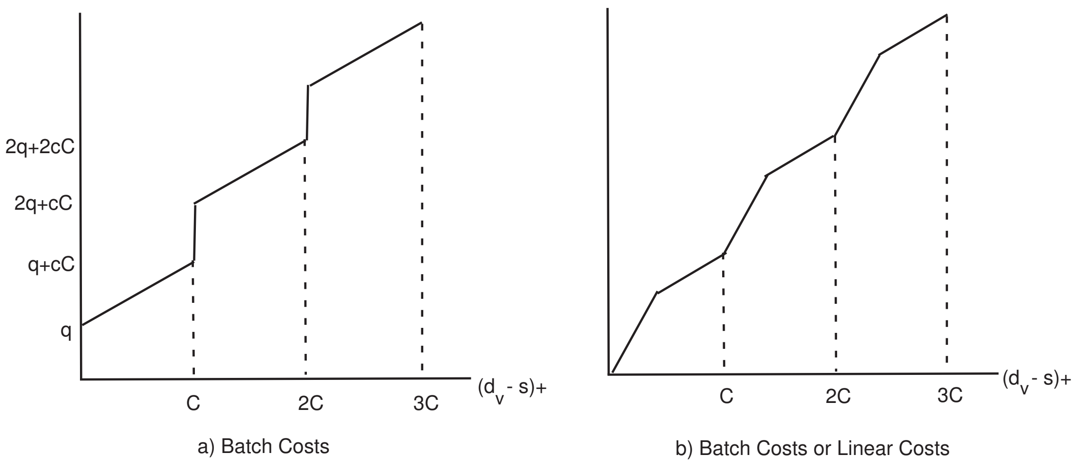

Figure 6: Production Recourse Functions

X^{N1}

$$\begin{aligned} s + x_v &\geq d_v \text{ for } v = 1, \ldots, n \\ x_v &\leq C y_v \text{ for } v = 1, \ldots, n \\ s \in \mathbb{R}^1_+, x &\in \mathbb{R}^n_+, y \in \mathbb{Z}^n_+. \end{aligned}$$

Proposition 9 *[4] In the uncapacitated case, when C is large and $y \in \{0,1\}^n$, $\text{conv}(X^{N1})$ is completely described by the initial constraints and (Q, S_Q) inequalities.*

For the case with constant C, the set X^{N1} has been studied recently by Conforti et al. under the name of mixing set with flows. They show that the initial constraints plus the convex hulls of the $n+1$ mixing set relaxations described in Proposition 1 give a tight extended formulation of the convex hull.

Proposition 10 *[1]*

$$\text{conv}(X^{N1}) = \text{proj}_{(s_0,x,y)} \cap_{v=0}^n \text{conv}(X_v^{MIX}) \cap \{(x,y) : 0 \leq x \leq Cy\}$$

where $\text{conv}(X_v^{MIX}) = \{(s_v, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^N_+ : s_v + C y_k \geq d_k - d_v$ for all k such that $d_k > d_v\}$ for $v = 0, \ldots, n$, with $s = s_0$, $d_0 = 0$ and $s + x_v = d_v + s_v$.

When y_v is integer corresponding to batches of size C, the recourse or amount that must be produced is $(d_v - s)^+$, and recourse or production cost function is of the form shown in Figure 6 a). One can also handle more general functions of the form shown in Figure 6 b) by allowing backlogging once the demands are known.

The formulation is now

$$\begin{aligned} \min hs + \textstyle\sum_{v=1}^n p_v[c_v x_v + q_v y_v + h_v s_v + b_v r_v] \\ s + x_v = d_v + s_v - r_v \text{ for } v = 1, \ldots, n \\ x_v \leq C y_v \text{ for } v = 1, \ldots, n \\ s \in \mathbb{R}^{n+1}_+, x, r \in \mathbb{R}^n_+, y \in \mathbb{Z}^n_+ \end{aligned}$$

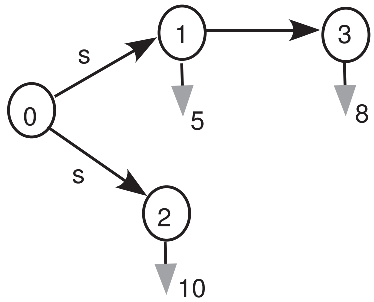

Figure 7: Two Period Instance

which, after elimination of the variables s_v for $v = 1, \dots, n$, gives the feasible region,

$$
\begin{aligned}
s + r_v + x_v \geq d_v &\text{ for } v = 1, \dots, n \\
x_v \leq C y_v &\text{ for } v = 1, \dots, n \\
s \in \mathbb{R}^1_+, r, x \in \mathbb{R}^n_+&, y \in \mathbb{Z}^n_+.
\end{aligned}
$$

denoted X^{N2}, known as a continuous mixing set with flows.

Proposition 11 *[2] In the constant capacity case, there is a polynomial size extended formulation for* $\mathrm{conv}(X^{N2})$.

To complete this section we present a small instance showing that mixing inequalities are insufficient to give a complete description of the convex hull of the solution set as soon as there is a second period with random outcomes. The instance is shown in Figure 7.

There are 15 nontrivial facets of which 10 are (Q, S_Q) inequalities, 2 are mixing inequalities but not (Q, S_Q) inequalities and three inequalities are not mixing inequalities.

The facets, other than the defining inequalities are:

s	$+\frac{3}{8}x_1$			$+\frac{25}{8}y_1$	$+5y_2$	$+3y_3$	$\geq$	13	
s	$+\frac{3}{8}x_1$			$+\frac{15}{8}y_1$	$+\frac{50}{8}y_2$	$+3y_3$	$\geq$	13	
s	$+\frac{3}{8}x_1$	$+\frac{1}{4}x_2$		$+\frac{15}{8}y_1$	$+\frac{30}{8}y_2$	$+3y_3$	$\geq$	13	
s				$+13y_1$		$+8y_3$	$\geq$	13	$Q = \{3\}$
s	$+x_1$					$+8y_3$	$\geq$	13	$Q = \{3\}$
s			$+x_3$	$+13y_1$			$\geq$	13	$Q = \{3\}$
s			$+x_3$	$+3y_1$	$+10y_2$		$\geq$	13	$Q = \{2, 3\}$
s	$+x_1$	$+x_2$				$+3y_3$	$\geq$	13	$Q = \{2, 3\}$
s		$+x_2$	$+x_3$	$3y_1$			$\geq$	13	$Q = \{2, 3\}$
s				$+3y_1$	$+10y_2$	$+3y_3$	$\geq$	13	$Q = \{2, 3\}$
s		$+x_2$		$+3y_1$		$+3y_3$	$\geq$	13	$Q = \{2, 3\}$
s					$+10y_2$		$\geq$	10	$Q = \{2\}$
s				$+5y_1$	$+5y_2$		$\geq$	10	$Q = \{1, 2\}$
s	$+x_1$				$+5y_2$	$+3y_3$	$\geq$	13	
s				$+8y_1$	$+5y_2$	$+3y_3$	$\geq$	13	

Observe that the last inequality is obtained by mixing the inequalities

$$s + 20y_1 \geq 5,\ s + 20y_2 \geq 10,\ s + 20(y_1 + y_3) \geq 13,$$

corresponding to the paths $P(0,1), P(0,2)$ and $P(0,3)$. So this inequality can be constructed using the technique we described in Section 3. Nevertheless, this inequality is not a (Q, S_Q) inequality. It is obtained by mixing inequalities of type (3) for the nodes in the set $\{1,2,3\}$, but there is no subset $V_Q \subseteq V$ such that the corresponding set W contains $\{1,2,3\}$.

Therefore the process we described in Section 3 gives all the (Q, S_Q) inequalities along with other inequalities which may be facet-defining.

The penultimate inequality is obtained by mixing the inequalities

$$s + x_1 - 5 + 20y_2 \geq 5,\ s + x_1 - 5 + 20y_3 \geq 8,$$

corresponding to the paths $P(0,2)$ and $P(0,3)$.

5 Computation

Here we report briefly on the effectiveness of the mixing set relaxations in solving the lot-sizing problem on a tree. For a problem with T periods, we take Δ outcomes in each period giving a Δ-ary tree with Δ^{T-1} scenarios (leaves) and a total of $N = \frac{\Delta^T - 1}{\Delta - 1}$ nodes. The data is randomly generated with d_t a random integer in [0,100], h_t a random integer in [1,11], p_t a random integer in [0,20], q_t 25 times a random integer in [0,80] and at each node the two random outcomes have equal probability $\frac{1}{\Delta}$ (in other words, the costs at distance k from the root are weighted by Δ^{-k}).

Below, for each value $\Delta \in \{2,3,4\}$, we report on four instances. For each instance we give the results for default Xpress-MP using the initial formulation in the first line and for the mixing set reformulation in the second line. Specifically we use an extended formulation for the mixing set described in Proposition 3 except that the set $W(v)$ is restricted to the nodes in the subtree rooted at v. For each Δ, two instances have the capacity fixed at $C = 100$, whereas for the other two $C = 500$ so the instances are essentially uncapacitated. For each Δ, we chose T so that the total number of nodes N (which is also the number of binary variables) was close to 1000. For the instances with $\Delta \in \{2,3\}$, a fixed run time was set at 120 seconds, while for the case of $\Delta = 4$ it was set at 300 seconds.

In each Table below, the first two columns indicate the capacity C and the total number of nodes (binary variables) N. The next two give the number r of rows and the number c of columns. The values LP, XLP, BIP and BLB indicate the linear programming value of the reformulation, the value XLP after the addition of the system cuts of Xpress-MP, the value BIP of the best integer solution found and BLB the value of the best lower bound on termination. *secs* gives the total run time, and *gap*, given by $\frac{BIP-BLB}{BIP} * 100$, is the percentage duality gap on termination.

For the 12 instances considered it appears that the mixing sets do indeed provide tight bounds. On the other hand the lower bounds obtained by the default optimizer are weak, even though the system cuts, which include path inequalities [9] that appear well-adapted to such problems, significantly strengthen the initial LP bounds.

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C	N	r	c	LP	XLP	BIP	BLB	secs	gap%
100	1023	2046	3070	9430	10526	11247	10617	120	5.6
100	1023	12287	2815	11120	11137	11162	11157	120	0.04
100	1023	2046	3070	9496	11060	11907	11114	120	6.7
100	1023	12287	2815	11716	11739	11757	11747	120	0.09
500	1023	2046	3070	5730	8825	10317	8983	120	12.9
500	1023	12287	2815	9897	9979	10045	9991	120	0.5
500	1023	2046	3070	5618	8402	9831	8529	120	13.2
500	1023	12287	2815	9472	9538	9594	9550	120	0.5

Table 1: An Instance of LS-TREE with $\Delta = 2$, $T = 10$

C	N	r	c	LP	XLP	BIP	BLB	secs	gap%
100	1093	2186	3280	6333	7348	8105	7416	120	8.50
100	1093	10017	11111	7925	7945	7973	7958	120	0.19
100	1093	2186	3280	7363	8775	9444	8804	120	6.78
100	1093	10017	11111	9305	9319	9327	9325	120	0.02
500	1093	2186	3280	3196	4806	6061	4910	120	18.99
500	1093	10020	11114	5808	5835	5852	5843	120	0.15
500	1093	2186	3280	3973	5978	7370	6133	120	16.78
500	1093	10022	11116	7181	7206	7237	7212	120	0.35

Table 2: An Instance of LS-TREE with $\Delta = 3$, $T = 7$

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C	N	r	c	LP	XLP	BIP	BLB	secs	gap%
100	1365	2730	4096	6141	7085	8121	7414	300	8.71
100	1365	11142	12508	8007	8017	8034	8031	300	0.04
100	1365	2730	4096	4013	4172	5356	4340	300	18.97
100	1365	11147	12513	5039	5054	5078	5071	300	0.14
500	1365	2730	4096	2975	4289	6153	4653	300	24.38
500	1365	11147	12513	5821	5849	5890	5870	300	0.34
500	1365	2730	4096	3251	4339	6620	4839	300	26.90
500	1365	11145	12511	6167	6189	6354	6204	300	2.36

Table 3: An Instance of LS-TREE with $\Delta = 4$, $T = 6$