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Intrinsic Sobolev maps between manifolds

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Introduction

Sobolev spaces between manifolds appear naturally in various contexts. They are natural tools to study variational problems for maps between manifolds arising in *nonlinear physical models* [11, 27, 65, 110] or in geometry while studying *geometrical objects* as harmonic maps [43, 83, 111], biharmonic maps [30, 56, 88, 109, 112] and polyharmonic maps [7, 50, 55, 78].

Given two Riemannian manifolds (M, g_M) and (N, g_N) , we could be interested for example in minimizing the *energy functional*

$$E(u) = \int_M |\nabla u|_{g_N}^2 d\mu_{g_M} \quad (1)$$

among all maps $u: M \rightarrow N$ in a suitable class which satisfy some fixed *boundary conditions*.

When M is an open set of $\mathbb{R}^m$ and $N = \mathbb{R}$, the functional could be defined on the space $C^1(\overline{M}, N)$ but this space lacks of a *weak compactness property* such that every bounded sequence has a convergent subsequence in some sense. As a consequence, the functional E is defined on a larger space which is the so-called *Sobolev space* $W^{1,2}(M)$ that involves a notion of weak derivative and which is a reflexive Banach space. With this idea in mind, in the first part of the present work, we suggest a robust intrinsic definition of Sobolev maps between manifolds in which the weak derivative plays a central role, and we endow with well-behaved intrinsic metrics the space of Sobolev maps.

Similar minimizing problems arise with higher-order derivatives, that is, for instance while minimizing the energy functional

$$E_k(u) = \int_M |\nabla^k u|_{g_N}^2 d\mu_{g_M}, \quad (2)$$

where $k \geq 2$ and $\nabla^k u$ denotes the k^{th} order covariant derivative of u with respect to both M and N , among all maps $u: M \rightarrow N$ in another suitable class, with some prescribed boundary conditions.

Once again, when M is an open set of $\mathbb{R}^m$ and $N = \mathbb{R}$, the appropriate space for the functional E_k is the k^{th} order Sobolev space $W^{k,2}(M)$. Higher-order Sobolev spaces between Euclidean spaces can be defined by

induction. In the second step, we therefore propose and study intrinsic higher-order derivatives and Sobolev spaces by iteration, in the same spirit of the first order.

1. Classical Sobolev spaces

When M is an open set of $\mathbb{R}^m$ and $N = \mathbb{R}$, classical Sobolev spaces appear in many variational problems [24, 44] and the notion of Sobolev maps between M and N is well-known.

Originally, if Ω is a smooth bounded open set of $\mathbb{R}^m$ and if $f \in C^\infty(\overline{\Omega}, \mathbb{R})$, Sobolev spaces appear to be a natural setting to minimize the energy functional $E_f: C_0^\infty(\overline{\Omega}, \mathbb{R}) \rightarrow \mathbb{R}$ defined for every $u \in C_0^\infty(\overline{\Omega}, \mathbb{R})$ by

$$E_f(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u. \quad (3)$$

P. G. L. Dirichlet and his contemporaries took for granted the existence of a minimizer of E_f until K. Weierstrass and his students pointed out that a rigorous proof was lacking [98, 115]. As a consequence, many mathematicians became interested in this issue and in some related questions [23, 25].

In particular, the idea due to D. Hilbert [69] to find a minimizer consists in considering a *minimizing sequence*, that is, a sequence $(u_n)_{n \in \mathbb{N}}$ in $C_0^\infty(\overline{\Omega}, \mathbb{R})$ such that the sequence $(E_f(u_n))_{n \in \mathbb{N}}$ converges to the infimum of E_f , which exists since the functional E_f is bounded from below. Since a minimizing sequence is a Cauchy sequence in the *Dirichlet norm* [81], that is,

$$\lim_{n, m \rightarrow \infty} \int_{\Omega} |\nabla u_n - \nabla u_m|^2 = 0,$$

if the minimizing sequence belongs to an appropriate complete space, it converges to a minimizer. In this context, the Sobolev space $W_0^{1,2}(\Omega)$ appears to be the natural setting where the energy is minimized.

More generally, the previous space belongs to the larger family of spaces which are nowadays called *Sobolev spaces* [81, 94, 108].

For every $p \in [1, +\infty)$, a map $u: M \rightarrow \mathbb{R}$ belongs to the *Sobolev space* $W^{1,p}(M)$ [24, §9.1; 44, §5.2.2; 120, definition 6.1.8] if u belongs to the Lebesgue space $L^p(M)$ and if there exists a measurable map $Du: M \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R})$ such that $|Du| \in L^p(M)$ and for every $\varphi \in C_c^\infty(M, \mathbb{R}^m)$,

$$\int_M u \operatorname{div} \varphi = - \int_M \langle Du, \varphi \rangle,$$

where for almost every $x \in M$, $\langle Du(x), \varphi(x) \rangle = Du(x)[\varphi(x)]$.

If u belongs to $W^{1,p}(M)$, the measurable map $Du: M \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R})$ is the *weak derivative* of u .

For every $k \in \mathbb{N}$ so that $k \geq 2$ and every $p \in [1, +\infty)$, the k^{th} order Sobolev space $W^{k,p}(M)$ is defined by iteration as the set of all maps $u \in W^{k-1,p}(M)$ such that $Du \in W^{k-1,p}(M, \mathbb{R}^m)$.

If the map $u: M \rightarrow \mathbb{R}$ belongs to $W^{k,p}(M)$, by induction, for every $j \in \{1, \dots, k\}$, there exists a measurable map $D^j u: M \rightarrow \mathcal{L}^j(\mathbb{R}^m, \mathbb{R})$ from M to the space of j -linear maps between $\mathbb{R}^m$ and $\mathbb{R}$ and it is the j^{th} order weak derivative of u .

A natural norm $\|\cdot\|_{k,p}$ on the Sobolev space $W^{k,p}(M)$ is defined for every $u \in W^{k,p}(M)$ by

$$\|u\|_{k,p} = \left(\int_M |u|^p + \sum_{j=1}^k \int_M |D^j u|^p \right)^{\frac{1}{p}},$$

and this space is complete with respect to this norm [24, proposition 9.1; 44, 5.2.3; 120, theorem 6.1.9].

Going back to our minimization problem, if the space $W_0^{1,2}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,2}(\Omega)$, since the functional $E_f: W_0^{1,2}(\Omega) \rightarrow \mathbb{R}$ is bounded from below, the infimum exists. Any minimizing sequence $(u_n)_{n \in \mathbb{N}}$ in $W_0^{1,2}(\Omega)$ is then a Cauchy sequence in the Dirichlet norm, and by the Poincaré inequality [24, corollary 9.19; 120, theorem 6.4.7], the sequence $(u_n)_{n \in \mathbb{N}}$ is also a Cauchy sequence in $L^2(\Omega)$. Hence, the sequence $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $W_0^{1,2}(\Omega)$ which is complete, and there thus exists $u \in W_0^{1,2}(\Omega)$ such that

$$E_f(u) = \inf\{E_f(v) : v \in W_0^{1,2}(\Omega)\}.$$

By regularity theory [44, §6.3.1 theorem 6], there exists a function $\tilde{u} \in C^\infty(\bar{\Omega}, \mathbb{R})$ such that $u = \tilde{u}$ almost everywhere in Ω and so the original minimization problem (3) is solved.

It is finally also worth noting that classical Sobolev spaces can be characterized in a different way. If M is an open bounded set of $\mathbb{R}^m$ and if we denote by $H^{k,p}(M)$ the completion of the set

$$\{u \in C^\infty(M, \mathbb{R}) : \|u\|_{k,p} < \infty\}$$

with respect to the norm $\|\cdot\|_{k,p}$, then $W^{k,p}(M) = H^{k,p}(M)$ [39, 85] (see also [44, §5.3.2]). In particular, the space $W^{k,p}(M)$ is the smallest space that contains $C^\infty(\bar{M}, \mathbb{R})$ and that is complete under the Sobolev norm $\|\cdot\|_{k,p}$.

2. Sobolev spaces between manifolds

2.1. State of the art

When (M, g_M) is a smooth Riemannian manifold and $N = \mathbb{R}$, the definition of classical Sobolev spaces can be easily extended to Sobolev maps from M to $\mathbb{R}$ [9, 67].

For every $p \in [1, +\infty)$, a map $u: M \rightarrow \mathbb{R}$ belongs to the *Sobolev space* $W^{1,p}(M)$ if u belongs to the Lebesgue space $L^p(M)$ and if there exists a measurable map $Du: M \rightarrow \mathcal{L}(TM, \mathbb{R})$ such that $|Du| \in L^p(M)$ and for every local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$ and every test function $\varphi \in C_c^\infty(\psi(V), \mathbb{R}^m)$,

$$\int_{\psi(V)} (u \circ \psi^{-1}) \operatorname{div} \varphi = - \int_{\psi(V)} \langle Du \circ D(\psi^{-1}), \varphi \rangle,$$

where for almost every $x \in V$,

$$\langle (Du \circ D(\psi^{-1}))(\psi(x)), \varphi(\psi(x)) \rangle = Du(x)[D(\psi^{-1})(\psi(x))[\varphi(\psi(x))]].$$

By induction, for every $k \in \mathbb{N}$ so that $k \geq 2$ and every $p \in [1, +\infty)$, a map $u: M \rightarrow \mathbb{R}$ belongs to the k^{th} order Sobolev space $W^{k,p}(M)$ if $u \in W^{k-1,p}(M)$ and $Du \in W^{k-1,p}(M)$.

However, if the target set is any Riemannian manifold, it is not possible to *use local charts directly* and there are several definitions of Sobolev maps into manifolds.

When (N, g_N) is a smooth Riemannian manifold, for every $k \in \mathbb{N}_*$ and every $p \in [1, +\infty)$, one way to define Sobolev maps between the manifolds M and N is to first embed the target manifold N in the Euclidean space $\mathbb{R}^\nu$ through an *isometric embedding* $\iota \in C^k(N, \mathbb{R}^\nu)$ and then to consider the set [7, 56, 88]

$$\begin{aligned} \dot{W}_\iota^{k,p}(M, N) = \{u: M \rightarrow N : \iota \circ u \in W_{\text{loc}}^{k,1}(M, \mathbb{R}^\nu) \\ \text{and } |D^k(\iota \circ u)| \in L^p(M)\}. \end{aligned} \quad (4)$$

Since every Riemannian manifold can be smoothly isometrically embedded in a Euclidean space [93, theorem 3], this definition (4) always makes sense. In contrast with $k = 1$ [34, proposition 2.7], in the higher-order case $k \geq 2$, this definition depends on the choice of the embedding ι and thus it *is not intrinsic* (see proposition 4.5.3 below). In addition, for any $k \geq 1$, the completeness of $\dot{W}_\iota^{k,p}(M, N)$ depends on the completeness of $\iota(N)$. When N is complete but not compact, the space $\dot{W}_\iota^{k,p}(M, N)$ might be not complete; however it is always possible to find an embedding ι such that $\dot{W}_\iota^{k,p}(M, N)$ is complete [90].

In the first-order case $k = 1$, another possibility to define Sobolev maps between the manifolds M and N is to use only the metric structure of N . There are various definitions of Sobolev spaces into metric spaces but they are all equivalent to each other [32, 73] and also equivalent to the definition by isometric embedding (4) [62, theorem 3.2; 63, theorem 2.17]. For example, Sobolev spaces into metric spaces can be defined for every $p \in [1, +\infty)$ as [3, 63, 99]

$$\begin{aligned} \dot{W}_m^{1,p}(M, N) = \{u: M \rightarrow N : \text{ there exists } w \in L^p(M) \text{ such that} \\ \forall f \in \text{Lip}(M, \mathbb{R}), f \circ u \in \dot{W}^{1,p}(M) \\ \text{and } |D(f \circ u)| \leq |f|_{\text{Lip}} w \text{ a.e in } M\}. \end{aligned} \quad (5)$$

The notions of Sobolev spaces into metric spaces are all *intrinsic* but do not provide any notion of weak derivative; they only provide a notion of Dirichlet integrand which might be different from the one given by embedding [32]. So these definitions do not seem suitable to a further definition of higher-order Sobolev spaces by iteration. Furthermore, in one of his article, D. Chiron suggests some distances on $\dot{W}_m^{1,p}(M, N)$, but the spaces are not complete for any of these distances [32, lemma 2].

2.2. Intrinsic definitions

In the present work, we propose intrinsic definitions of Sobolev maps between the manifolds M and N in the same spirit of classical Sobolev spaces. Furthermore, we endow these spaces with well-behaved metrics; since they do not form a vector space, we only equip these spaces with distances.

For first-order Sobolev maps between manifolds, we propose in chapter I an intrinsic definition in which weak differentiability plays a central role. For this purpose, we define *colocally weakly differentiable maps* as measurable maps $u: M \rightarrow N$ for which $f \circ u$ is weakly differentiable when $f \in C_c^1(N, \mathbb{R})$ (definition 1.2.1). The *colocal weak derivative* is defined as the unique measurable morphism of bundles Du such that the chain rule $D(f \circ u) = Df \circ Du$ holds (definition 1.2.2). In other words, if the map $u: M \rightarrow N$ is colocally weakly differentiable, then for every local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$, every test function $\varphi \in C_c^\infty(\psi(V), \mathbb{R}^m)$ and every $f \in C_c^1(N, \mathbb{R})$,

$$\int_{\psi(V)} (f \circ u \circ \psi^{-1}) \operatorname{div} \varphi = - \int_{\psi(V)} \langle Df \circ Du \circ D(\psi^{-1}), \varphi \rangle,$$

where for almost every $x \in V$,

$$(Df \circ Du \circ D(\psi^{-1}))(\psi(x)) = Df(u(x))[Du(x)[D(\psi^{-1})(\psi(x))]].$$

For every $p \in [1, +\infty)$, the *homogeneous Sobolev space* is then defined as the set (definition 1.3.1)

$$\begin{aligned} \dot{W}^{1,p}(M, N) = \{u: M \rightarrow N : u \text{ is colocally weakly differentiable} \\ \text{and } |Du|_{g_M^* \otimes g_N} \in L^p(M)\}, \end{aligned}$$

where the Euclidean norm $|\cdot|_{g_M^* \otimes g_N}$ is induced by the Riemannian metrics on M and N .

For the completion of smooth maps in Sobolev spaces between manifolds, it is essential to have a metric for which the space $\dot{W}^{1,p}(M, N)$ is complete. A natural candidate for an intrinsic distance is defined for every $u, v \in \dot{W}^{1,p}(M, N)$ by (section 1.5.3)

$$\delta_{1,p}(u, v) = \left(\int_M d^S(Du, Dv)^p \right)^{\frac{1}{p}},$$

where d^S is the geodesic distance on $T^*M \otimes TN$ associated to the *Sasaki metric* (section 1.5.1). Actually, the distance d^S belongs to the class of *concordant distances* (definition 2.2.1) which gives rise to a class of metrics on $\dot{W}^{1,p}(M, N)$ of the form $\delta_{1,p}$. These intrinsic distances characterize Sobolev maps (propositions 1.5.9 and 2.3.1) and also simultaneous convergence in Sobolev spaces (propositions 1.5.10 and 2.3.2). Moreover, Sobolev spaces are complete under all these distances (propositions 1.5.11 and 2.3.3).

For higher-order Sobolev spaces, the reader can refer to chapters IV and V. Although definitions, results and methods of chapter IV apply to arbitrary order Sobolev spaces, the details are in general made for the second-order case; and so chapter V is devoted to higher-order technicalities. In order to highlight the essential ideas and issues, the present introduction is focused on the second-order case.

For second-order Sobolev maps, we define by induction *twice colocally weakly differentiable maps* as colocally weakly differentiable maps $u: M \rightarrow N$ for which the colocal weak derivative $Du: M \rightarrow T^*M \otimes TN$ is also colocally weakly differentiable (definition 4.2.1). The colocal weak derivative of Du is a *double vector bundle morphism* almost everywhere in M (proposition 4.2.1), that is, for almost every $x \in M$, $T^2u(x): T_x^2M \rightarrow T_{u(x)}^2N$ is a bundle morphism over $Du(x)$ with respect to both natural vector bundle structures over TM and TN .

Whereas the second-order derivative T^2u is natural from the point of view of differential geometry, it is not a vector bundle morphism that covers u and this object does not seem to appear directly in geometrical analysis or in physical models; for example, while minimizing the

functional (2) with $k = 2$. In order to overcome this issue, we study the second-order derivative T^2u to understand how it is related to objects of affine and Riemannian geometry.

When M and N are Riemannian manifolds endowed with their Levi-Civita connections, the double colocal differentiability is equivalent to a *colocal weak covariant differentiability* (proposition 4.3.1). The notion of colocal weak covariant derivative consists of looking only at the vertical component associated to the connection in the chain rule (definition 4.3.1) and relying on the formula

$$(Df)(Du)[\text{Vert} \circ D_K^2 u] = D(f \circ Du) - (Df)(Du)[\text{Hor}_K \circ Du],$$

where Vert is the *vertical lifting* from $T^*M \otimes TN$ (see page 87) and Hor_K is the *horizontal lifting* associated to the connection K on $T^*M \otimes TN$, induced by the connections on TM and TN (section 4.3.1). In particular, if the map $u: M \rightarrow N$ is twice colocally weakly differentiable, its second-order weak covariant derivative $D_K^2 u$ and its second-order weak derivative T^2u are related by the formulas

$$T^2u = \text{Vert} \circ D_K^2 u + \text{Hor}_K \circ Du \quad \text{and} \quad D_K^2 u = K \circ T^2u.$$

As a consequence, for every $p \in [1, +\infty)$, we define the *intrinsic second-order Sobolev space* (definition 4.3.2)

$$\begin{aligned} \dot{W}^{2,p}(M, N) \\ = \{u: M \rightarrow N : u \text{ is twice colocally weakly differentiable} \\ \text{and } |D_K^2 u|_{g_M^* \otimes g_M^* \otimes g_N} \in L^p(M)\}, \end{aligned}$$

where the norm $|\cdot|_{g_M^* \otimes g_M^* \otimes g_N}$ is induced by the Riemannian metrics on M and N .

2.3. Comparison with other definitions

It is normal to compare the notions defined in the previous part with the other two definitions (4) and (5). For first-order Sobolev spaces, all definitions are equivalent. However, for higher-order case $k \geq 2$, the definition of *higher-order Sobolev spaces by embedding* (4) might be non equivalent to the intrinsic one (definitions 4.3.2 and 4.3.3).

In the first-order case $k = 1$, when N is isometrically embedded in $\mathbb{R}^\nu$, the definition by embedding is equivalent to the intrinsic one (proposition 1.3.6) — the definition (4) is thus *a posteriori* intrinsic — and the intrinsic definition is also equivalent to the one into metric spaces (proposition 1.3.1).

For spaces $\dot{W}^{1,p}(M, N)$, we define a notion of *strong convergence* in section 1.5.1. If N is isometrically embedded in $\mathbb{R}^\nu$, this notion of convergence coincides with the classical convergence defined by embedding in Euclidean spaces (proposition 1.5.3). Moreover, the notion of convergence in $\dot{W}_m^{1,p}(M, N)$ introduced by D. Chiron [32] is also equivalent to convergence in $\dot{W}^{1,p}(M, N)$ (proposition 1.5.5). So all the notions of convergence are equivalent but as mentioned before, the spaces $\dot{W}_\ell^{1,p}(M, N)$ are not always complete and the spaces $\dot{W}_m^{1,p}(M, N)$ are never complete (proposition 1.5.8), whereas the strong notion is induced by a distance $\delta_{1,p}$ that makes the spaces $\dot{W}^{1,p}(M, N)$ complete when N is complete (proposition 1.5.1).

The intrinsic second-order Sobolev spaces might be different from the one by embedding (proposition 4.5.4). The latter definition is not intrinsic and so we characterize it in terms of intrinsic spaces in various particular cases (propositions 4.5.1, 4.5.3 and 4.5.4): for every $p \in [1, +\infty)$,

- (i) if there exists $C > 0$ such that for every $v \in \dot{W}^{2,p}(M, \mathbb{R}) \cap L^\infty(M, \mathbb{R})$,

$$\begin{aligned} & \int_M |Dv|_{g_M^* \otimes g_1}^{2p} \\ & \leq C \left(\text{ess sup}\{|v(x) - v(y)| : x, y \in M\} \right)^p \int_M |D_K^2 v|_{g_M^* \otimes g_M^* \otimes g_1}^p \end{aligned}$$

and if N is compact, then for any isometric embedding $\iota \in C^2(N, \mathbb{R}^\nu)$, we have

$$\dot{W}_\ell^{2,p}(M, N) = \dot{W}^{1,2p}(M, N) \cap \dot{W}^{2,p}(M, N);$$

- (ii) for every isometric embedding $\iota \in C^2(N, \mathbb{R}^\nu)$,

$$\dot{W}_\ell^{2,p}(M, N) \subseteq \dot{W}^{2,p}(M, N);$$

- (iii) there exists an isometric embedding $\iota \in C^2(N, \mathbb{R}^\nu)$ such that

$$\dot{W}_\ell^{2,p}(M, N) \subseteq \dot{W}^{1,2p}(M, N) \cap \dot{W}^{2,p}(M, N).$$

In other words, if the manifold N is compact, we characterize higher-order Sobolev spaces by embedding. Without this assumption, assertion (i) is not valid. Indeed, classical Sobolev spaces between Euclidean spaces already do not satisfy this property. However, assertion (iii) implies that one inclusion always occurs for some isometric embedding and

so in the higher-order case $k \geq 2$, Sobolev spaces by isometric embedding depends on the choice of the embedding.

Furthermore, thanks to property (i), for any isometric embedding $\iota \in C^2(N, \mathbb{R}^\nu)$, if $1 \leq 2p < m$ and if the manifold N is compact, we prove that

$$\dot{W}_\iota^{2,p}(\mathbb{B}^m, N) \subsetneq \dot{W}^{2,p}(\mathbb{B}^m, N)$$

by exhibiting a map $u \in \dot{W}^{2,p}(\mathbb{B}^m, N)$ such that

$$\int_{\mathbb{B}^m} |D_K^2 u|_{g_m^* \otimes g_m^* \otimes g_N}^p < +\infty \quad \text{but} \quad \int_{\mathbb{B}^m} |Du|_{g_m^* \otimes g_N}^{2p} = +\infty,$$

that is, $u \notin \dot{W}_\iota^{2,p}(\mathbb{B}^m, N)$ (proposition 4.5.6). From this example, we can point out the failure of Gagliardo–Nirenberg interpolation inequalities [49, 95] for those spaces. More generally, if the target manifold N is not compact, this leads to an open question 4.5.1: does it exist a constant $C > 0$ such that for every $u \in \dot{W}^{2,p}(M, N) \cap L^\infty(M, N)$,

$$\begin{aligned} & \int_M |Du|_{g_M^* \otimes g_N}^{2p} \\ & \leq C \left(\text{ess sup} \{d_N(u(x), u(y)) : x, y \in M\} \right)^p \int_M |D_K^2 u|_{g_M^* \otimes g_M^* \otimes g_N}^p ? \end{aligned}$$

The answer should involve in particular the geometry of M and N .

When studying a variational problem for maps between manifolds, R. Moser mentions that if the functional E_2 (definition (2) with $k = 2$) is defined on $\dot{W}_\iota^{2,2}(M, N)$, where M is a smooth open bounded set of $\mathbb{R}^m$ and N is a smooth compact Riemannian manifold without boundary embedded in $\mathbb{R}^\nu$ with an isometric embedding $\iota \in C^2(N, \mathbb{R}^\nu)$, then the existence of minimizers, with some boundary conditions, is not obvious [88].

In order to obtain a more suitable space, for $p \in (1, 2)$, P. Hornung and R. Moser define a notion of weak covariant derivative [72, definition 2.5], which is essentially equivalent to our notion of second-order weak covariant derivative (proposition 4.3.2), and a notion of second-order Sobolev spaces [72, definition 3.1]. In this context, they prove the existence of a minimizer of the functional E_2 [72, theorem 3.3].

More generally, but without studying the theory of traces in Sobolev spaces between manifolds, it follows from propositions 4.3.3 and 4.3.4 that for every $p \in (1, +\infty)$ and every $v \in \dot{W}^{1,p}(M, N)$,

- (i) if (N, d_N) is complete,

- (ii) if $(u_\ell)_{\ell \in \mathbb{N}}$ is a sequence of twice colocally weakly differentiable maps from M to N such that

$$\limsup_{\ell \rightarrow \infty} \int_M \left(d_N(u_\ell, v)^p + |Du_\ell|_{g_M^* \otimes g_N}^p + |D_K^2 u_\ell|_{g_M^* \otimes g_M^* \otimes g_N}^p \right) < +\infty,$$

then there exists $u \in \dot{W}^{2,p}(M, N)$ such that up to a subsequence,

$$\int_M |D_K^2 u|_{g_M^* \otimes g_M^* \otimes g_N}^p \leq \liminf_{\ell \rightarrow \infty} \int_M |D_K^2 u_\ell|_{g_M^* \otimes g_M^* \otimes g_N}^p.$$

3. Chain rule for Sobolev maps

Composition is an essential tool in the theory of Sobolev maps. For instance, studying chain rule for intrinsic higher-order Sobolev maps between manifolds allows us to compare this notion with the one by embedding (4), which is defined by composition with a map $\iota \in C^k(N, \mathbb{R}^\nu)$. Furthermore, the chain rule is central in the definition of first-order weak differentiability. We could have defined naturally higher-order weak differentiability with some higher-order chain-rule condition; however such definition turns out to be stronger.

For first-order Sobolev maps, it is not surprising that the composition of a Lipschitz map from a manifold into another with a Sobolev map is also a Sobolev map (proposition 1.3.5).

In addition, for every $u \in W_{\text{loc}}^{1,1}(M, N)$ and every $f \in \text{Lip}(N, \mathbb{R}^k)$, the composition map $f \circ u$ satisfies some intrinsic chain-rule property. Indeed, for almost every $x \in M$, the map $f: N \rightarrow \mathbb{R}^k$ has a *tangential derivative* at $u(x)$ with respect to $Du(x)[T_x M] \subseteq T_{u(x)} N$ (proposition 3.5.2).

In fact, if M is an open set of $\mathbb{R}^m$ and $N = \mathbb{R}^n$, L. Ambrosio and G. Dal Maso prove that for almost every $x \in M$, the function f has a directional derivative at $u(x)$ with respect to the vector space $Du(x)[\mathbb{R}^m]$ [4, corollary 3.2]. But the notion of directional derivative does not make sense for maps on a differentiable manifold.

In order to obtain a similar property for Sobolev maps between manifolds, we define in chapter III a notion of tangential derivative (definition 3.2). This notion is invariant under diffeomorphisms (corollary 3.3.3) and thus it behaves well for maps on a differentiable manifold (definition 3.3.1). In the particular case of Lipschitz-continuous functions, the notions of directional and tangential derivatives are equivalent (proposition 3.2.2) and so we restate the Ambrosio–Dal Maso property for Sobolev maps between manifolds with the tangential derivative instead of the directional derivative.

For second-order Sobolev maps, given a twice colocally weakly differentiable map $u: M \rightarrow N$, we investigate whether for every $f \in C_c^2(N, \mathbb{R})$, the map $f \circ u$ belongs to $W_{\text{loc}}^{2,1}(M)$ and whether the chain rule $T^2(f \circ u) = T^2 f \circ T^2 u$ holds. The chain rule *does not hold* in general (example 4.4.1); however, it holds under an additional integrability condition on double vector bundle morphisms in terms of double norms (proposition 4.4.3).

For every twice colocally weakly differentiable map $u: M \rightarrow N$, the following statements are equivalent.

- (i) For every $f \in C_c^2(N, \mathbb{R})$, the map $f \circ u$ is twice weakly differentiable and almost everywhere in M

$$T^2(f \circ u) = T^2 f \circ T^2 u,$$

- (ii) for every compact sets $K \subseteq M$ and $L \subseteq N$,

$$\int_{u^{-1}(L) \cap K} \llbracket T^2 u \rrbracket < +\infty,$$

- (iii) for every $h \in C^1(T^*M \otimes TN, \mathbb{R})$ that is linear with respect to Du and has compact support with respect to u , the map $h \circ Du$ is weakly differentiable.

In assertion (ii), the condition involves a *double norm* $\llbracket \cdot \rrbracket$ of the double vector bundle morphism $T^2 u$ (definition 4.4.2). Such a double norm is a seminorm on fibers over both vector bundle structures of the double vector bundle.

In fact, the notion of double norm is defined on second-order tangent bundles in chapter IV. However, we only sketch the proofs related to this notion. In order to be more precise, we define in chapter VI a notion of *double seminorm* on *double vector spaces*, which are likely the local representation of a second-order tangent bundle at a point (section 6.2.1). The relevant properties of this seminorm are then useful to prove *rigorously* the propositions of section 4.4.2.

4. Problem of density of smooth maps

In contrast with classical Sobolev spaces [39, 85], the problem of density of smooth maps in Sobolev spaces between manifolds is more delicate [43, 104].

When M and N are two smooth compact Riemannian manifolds with N embedded in $\mathbb{R}^\nu$ through an isometric embedding $\iota \in C^k(N, \mathbb{R}^\nu)$, and $kp \geq \dim(M)$, R. Schoen and K. Uhlenbeck prove that every map of

Sobolev spaces $W^{k,p}_\ell(M, N)$ can be approximated by smooth maps [105]. But if $1 \leq kp < \dim(M)$, the density of smooth maps in $W^{k,p}_\ell(M, N)$ is equivalent to some homotopy invariant of the pair (M, N) (for the case $k = 1$, see [17; 19; 64, theorem 1.3] and for higher-order case, see [22, theorem 1, §10.1]).

In particular, the set of smooth maps $C^2(\mathbb{B}^m, N)$ is dense in the space $W^{2,p}_\ell(\mathbb{B}^m, N)$ if and only if the $[2p]^{\text{th}}$ homotopy group $\pi_{[2p]}(N)$ of N is trivial [22, theorem 1]. Since the intrinsic Sobolev space $\dot{W}^{2,p}(\mathbb{B}^m, N)$ is larger than $W^{2,p}_\ell(\mathbb{B}^m, N)$ (proposition 4.5.1), but has a weaker notion of convergence, the question of density is also relevant. Even if this question is still open, the necessary and sufficient condition that appears with the definition by embedding is also necessary for the intrinsic one (proposition 4.5.7).

The results of density of smooth maps in Sobolev spaces between manifolds may be useful for example in the study of *harmonic maps*, that is, smooth maps that are critical points of the energy functional E (definition (1)) [17, 18, 23, 51, 68].

One possibility to provide harmonic maps with prescribed boundary conditions is to use the classical variational method [43, §1.5]. If Ω is a smooth bounded open set of $\mathbb{R}^m$ and if $f \in C^\infty(\bar{\Omega}, \mathbb{R})$, as for the minimization problem (3), there exists $u \in W^{1,2}(\Omega)$ such that

$$E(u) = \inf \left\{ \int_{\Omega} |\nabla v|^2 : v \in W^{1,2}(\Omega) \text{ and } v = f \text{ on } \partial\Omega \right\}.$$

Moreover, the function u coincides with a harmonic function almost everywhere in Ω [44]. But for maps between manifolds, this method seems sometimes to be inappropriate to prove the existence of harmonic maps [18, 26].

For instance, in the theory of liquid crystals, the functional E appears as a simplified model with $M = \mathbb{B}^3$ the unit ball in $\mathbb{R}^3$ and $N = \mathbb{S}^2$ the 2-dimensional sphere. For every smooth map $f: \partial M = \mathbb{S}^2 \rightarrow \mathbb{S}^2$, if u is a minimizer of E in $W^{1,2}_\ell(M, N) \cap \{u = f \text{ on } \mathbb{S}^2\}$, then u is smooth except at a finite number of points in M [27, 65, 104, 105]. In fact, some topological condition on f forces the creation of singularities [23, 51], but singularities also allow to reduce the energy [66].

Indeed, in the mid 1980's, R. Hardt and F.-H. Lin established a *Lavrentiev phenomenon* [66]: there exists a map $f \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ with zero degree such that

$$\begin{aligned} \inf \{ E(u) : u \in W^{1,2}_\ell(\mathbb{B}^3, \mathbb{S}^2) \text{ and } u = f \text{ on } \mathbb{S}^2 \} \\ < \inf \{ E(u) : u \in C^\infty(\bar{\mathbb{B}}^3, \mathbb{S}^2) \text{ and } u = f \text{ on } \mathbb{S}^2 \}. \end{aligned}$$

Because of this *gap phenomenon*, and since smooth maps are dense in $W^{1,2}_\ell(\mathbb{B}^3, \mathbb{S}^2)$ for the weak topology [16, 17], F. Bethuel, H. Brezis and J.-M. Coron introduced then a notion of *relaxed energy* E_{rel} defined for every $u \in W^{1,2}_\ell(\mathbb{B}^3, \mathbb{S}^2)$ by [18]

$$E_{\text{rel}}(u) = \inf \left\{ \liminf_{\ell \rightarrow \infty} \int_{\mathbb{B}^3} |Du_\ell|_{g_3^* \otimes g_3}^2 : u_\ell \in C^\infty(\overline{\mathbb{B}^3}, \mathbb{S}^2) \right. \\ \left. \text{and } (u_\ell)_{\ell \in \mathbb{N}} \rightharpoonup u \text{ in } W^{1,2}(\mathbb{B}^3, \mathbb{R}^3) \right\}.$$

It is obvious that for every $u \in W^{1,2}_\ell(\mathbb{B}^3, \mathbb{S}^2)$,

$$E_{\text{rel}}(u) \geq E(u),$$

and the equality holds if and only if u can be approximated by smooth maps in $W^{1,2}_\ell(\mathbb{B}^3, \mathbb{S}^2)$. Since smooth maps are not dense in the space $W^{1,2}_\ell(\mathbb{B}^3, \mathbb{S}^2)$ [17, 19, 64], the equality does not hold in general. In particular, the absence of gap phenomenon is equivalent to density of the set $C^1(\overline{\mathbb{B}^3}, \mathbb{S}^2) \cap \{u = f \text{ on } \mathbb{S}^2\}$ in $W^{1,2}_\ell(\mathbb{B}^3, \mathbb{S}^2) \cap \{u = f \text{ on } \mathbb{S}^2\}$.

Finally, thanks to the relaxed energy [18], the authors prove that there exists $u \in W^{1,2}_\ell(\mathbb{B}^3, \mathbb{S}^2)$ such that

$$E_{\text{rel}}(u) = \inf \{ E_{\text{rel}}(v) : v \in W^{1,2}_\ell(\mathbb{B}^3, \mathbb{S}^2) \text{ and } v = f \text{ on } \mathbb{S}^2 \} \\ = \inf \{ E(w) : w \in C^\infty(\overline{\mathbb{B}^3}, \mathbb{S}^2) \text{ and } w = f \text{ on } \mathbb{S}^2 \};$$

but the problem of providing smooth harmonic maps is not solved since the minimizer need not be continuous. However, the map u satisfy the weak Euler–Lagrange equation associated to E [18].

Going back to the problem of density, for the first-order case $k = 1$, since the notion of intrinsic Sobolev maps is equivalent to the one by embedding, and since the notions of convergence are also equivalent, the density of smooth maps is equivalent to some topological invariant of the pair (M, N) [17, 19, 64].

For intrinsic second-order Sobolev spaces between manifolds, for the strong and weak-bounded approximation by smooth maps, we prove that the topological condition is still necessary (proposition 4.5.7), that is, for every $p \in [1, +\infty)$ such that $1 \leq 2p < m$,

- (i) if for every $u \in \dot{W}^{2,p}(\mathbb{B}^m, N)$, there exists a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ in $C^\infty(\overline{\mathbb{B}^m}, N)$ that is bounded in $\dot{W}^{2,p}(\mathbb{B}^m, N)$ and that converges almost everywhere to u , and if $2p \notin \mathbb{N}$, then $\pi_{[2p]}(N) \simeq \{0\}$;
- (ii) if for every $u \in \dot{W}^{2,p}(\mathbb{B}^m, N)$, there exists a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ in $C^\infty(\overline{\mathbb{B}^m}, N)$ that converges strongly to u in $\dot{W}^{2,p}(\mathbb{B}^m, N)$, and if $2p \in \mathbb{N}$, then $\pi_{2p}(N) \simeq \{0\}$.

In order to have some sufficient condition for the density problem, new intrinsic approximation methods seem to be needed; for instance, a notion of *intrinsic convolution* could be interesting [22]. This new perspective suggests further directions of study in the topic of intrinsic Sobolev spaces between manifolds.

Chapter I

Intrinsic colocal weak derivatives and Sobolev spaces between manifolds

1.1. Introduction

Sobolev spaces between manifolds are a natural tool to study variational problems for maps between manifolds, arising in geometry [43, 83, 111] or in nonlinear physical models [11, 27, 65]. A Sobolev space of maps between the manifolds M and N can be defined for every $p \in [1, \infty)$ by [11, 17, 27, 35, 43, 51–54, 62, 64, 65, 89]

$$\begin{aligned} \dot{W}^{1,p}(M, N) = \{u : M \rightarrow N : \iota \circ u \in W_{\text{loc}}^{1,1}(M, \mathbb{R}^\nu) \\ \text{and } |D(\iota \circ u)| \in L^p(M)\}. \end{aligned} \quad (1.1)$$

where $\iota : N \rightarrow \mathbb{R}^\nu$ is an isometric embedding of the target manifold N in the Euclidean space $\mathbb{R}^\nu$. This definition is always possible, since every Riemannian manifold is isometrically embedded in a Euclidean space [92, theorem 2; 93, theorem 3]. Since the embedding $\iota : N \rightarrow \mathbb{R}^\nu$ is not unique, this definition could in principle depend on the choice of the embedding ι .

This difficulty can be avoided by defining Sobolev spaces into N by using only the metric structure of N , either by composition with Lipschitz maps [3, 62, 99] or by oscillations on balls [73, 77]. These definitions are equivalent to each other [32, 73] and equivalent with the definition by isometric embedding (1.1) [62, theorem 3.2; 63, theorem 2.17]. They are all intrinsic but they do not have any notion of weak derivative; they only provide a notion of Dirichlet integrand $|Du|$ which might differ from the one given by isometric embedding and might depend on the integrability exponent p [32]. If the manifold N has a Riemannian structure, then an approximate derivative has been constructed a posteriori almost everywhere on M [48]; in contrast with the classical theory of Sobolev spaces between Euclidean spaces the derivative is a fine property of a

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function that plays no role in the definition of the Sobolev maps. Several distances have been proposed for spaces of Sobolev maps between metric spaces, but the spaces are not complete for any of these distances [32].

The goal of this work is to propose a robust intrinsic definition of Sobolev maps between manifolds in which the weak derivative plays a central role and to endow with well-behaved intrinsic metrics the space of Sobolev maps. We shall proceed in three steps: first we shall define a notion of differentiability and derivative, then we shall study the integrability of the derivative and finally we shall endow these spaces with convergence and metrics. Each of these steps will require additional structure on the manifolds: at the beginning we shall simply use the differentiable structure of the manifolds, then a Riemannian metric on the manifolds and finally a Riemannian metric on their tangent bundles. Defining the derivative before the space gives immediately the independence of the derivative from the Riemannian metric or the integrability exponent p . The primary role of the derivative in our approach will be quite handy to define complete intrinsic metrics.

In the first step we define *colocally weakly differentiable maps* as maps $u : M \rightarrow N$ for which $f \circ u$ is weakly differentiable when $f \in C_c^1(N, \mathbb{R})$ (definition 1.2.1). The *colocal weak derivative* is defined as the unique morphism of bundles Du such that the chain rule $D(f \circ u) = Df \circ Du$ holds (definition 1.2.2):

$$\begin{array}{ccc} TM & \xrightarrow{Du} & TN \\ & \searrow D(f \circ u) & \downarrow Df \\ & & \mathbb{R}. \end{array}$$

The colocal weak derivative has the usual nonlinear properties of a weak derivative; the definition extends previous definitions of the derivatives by truncation [5, 13]. The colocal weak derivative is an approximate derivative (proposition 1.2.9). This follows from the Euclidean counterpart. We recover thus without any Riemannian structure the derivative of Focardi and Spadaro [48].

In the second step, we define when M and N are Riemannian manifolds, for every $p \in [1, \infty)$, the *homogeneous Sobolev space* (definition 1.3.1)

$$\begin{aligned} \dot{W}^{1,p}(M, N) = \{u : M \rightarrow N : u \text{ is colocally weakly differentiable} \\ \text{and } |Du|_{g_M^* \otimes g_N} \in L^p(M)\}, \end{aligned}$$

where the Euclidean norm $|\cdot|_{g_M^* \otimes g_N}$ is induced by the Riemannian metrics on M and N . This definition is equivalent with (1.1) when N is

isometrically embedded in $\mathbb{R}^\nu$ (proposition 1.3.6) — (1.1) is thus a posteriori an intrinsic definition — and with the definition of Sobolev spaces into metric spaces (proposition 1.3.1). Given a colocally weakly differentiable map $u: M \rightarrow N$, we characterize the quantity $|Du|_{g_M^* \otimes g_N}$ as the smallest measurable function $w: M \rightarrow \mathbb{R}$ such that for every $f \in C_c^1(N, \mathbb{R}^{\min(\dim(M), \dim(N))})$,

$$|D(f \circ u)| \leq |f|_{\text{Lip}} w \quad \text{almost everywhere in } M. \quad (1.2)$$

This allows to define a robust Dirichlet integrand; previous definitions with scalar test function $f \in C_c^1(N, \mathbb{R})$ were quite unstable [3, 99]. Furthermore the inequality (1.2) might provide a robust definition of the Dirichlet integrand for Sobolev maps into metric spaces.

We study weakly convergent sequences in section 1.4. This part should be useful to obtain compactness and lower semi-continuity results in the calculus of variations for maps between manifolds.

We conclude our work by giving a natural notion of strong convergence and associated intrinsic distances $\delta_{1,p}$ and $\delta_{1,p}$. If the Riemannian manifold N is complete, Sobolev spaces are complete under such distances (propositions 1.5.1 – 1.5.11). This was not the case for distances proposed by Chiron [32] (proposition 1.5.8) and this opens the study of the completion of smooth maps in Sobolev spaces. Our notion of strong convergence is equivalent with existing ones for embedded manifolds and metric spaces (subsection 1.5.2).

1.2. Colocally weakly differentiable maps and colocal weak derivative

1.2.1. Weak differentiability on a differentiable manifold

We assume that M and N are differentiable manifolds of dimensions m and n which are Hausdorff and have a countable basis [42, §0.5; 70, §1.5].

We recall various definitions of local measure-theoretical notions on a manifold. A set $E \subset M$ is *negligible* if for every $x \in M$ there exists a local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$ – that is $\psi: V \subseteq M \rightarrow \psi(V) \subseteq \mathbb{R}^m$ is a diffeomorphism – such that $x \in V$ and the set $\psi(E \cap V) \subset \mathbb{R}^m$ is negligible. A map $u: M \rightarrow N$ is *measurable* if for every $x \in M$ there exists a local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and the map $u \circ \psi^{-1}$ is measurable [40, §3; 70, §3.1]. A function $u: M \rightarrow \mathbb{R}$ is *locally integrable* if for every $x \in M$ there exists a local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and $u \circ \psi^{-1}$ is integrable on $\psi(V)$ [71, §6.3]. Similarly,

a locally integrable map $u : M \rightarrow \mathbb{R}$ is *weakly differentiable* if for every $x \in M$ there exists a local chart $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and the map $u \circ \psi^{-1}$ is weakly differentiable. All these notions are independent on any particular metric or measure on the manifold M .

A Radon measure μ on M is *absolutely continuous* if for every $x \in M$ there exists a local chart $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and the image measure $\psi_*(\mu)$ defined by $\psi_*(\mu)(A) = \mu(\psi^{-1}(A))$ is absolutely continuous with respect to the Lebesgue measure. The measure μ is *positive* if for every $x \in M$ there exists a local chart $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and the Lebesgue measure on $\psi(V)$ is absolutely continuous with respect to $\psi_*(\mu)(A) = \mu(\psi^{-1}(A))$ [75, definition 5.1.1]. Every absolutely continuous measure on M is absolutely continuous with respect to every positive measure. A set $E \subset M$ is negligible if and only if $\mu(E) = 0$ for every positive absolutely continuous Radon measure μ . A measurable function $u : M \rightarrow \mathbb{R}$ is locally integrable if and only if there exists a positive absolutely continuous Radon measure μ such that $\int_M |u| d\mu < \infty$.

We first define the notion of colocally weakly differentiable map.

Definition 1.2.1. A map $u : M \rightarrow N$ is *colocally weakly differentiable* if u is measurable and for every $f \in C_c^1(N, \mathbb{R})$, $f \circ u$ is weakly differentiable.

When $N = \mathbb{R}$, definition 1.2.1 is related to the space $GBV(M, \mathbb{R})$ of functions of generalized bounded variation introduced by L. Ambrosio and E. De Giorgi [5; 6, definition 4.26] and to the space $\mathcal{T}_{\text{loc}}^{1,1}(M)$ of Sobolev functions by truncations from M to $\mathbb{R}$ [13]. Any function in $\mathcal{T}_{\text{loc}}^{1,1}(M)$ is colocally weakly differentiable. The converse is false as it can be observed by taking a function $u \in C^1(\mathbb{R} \setminus \{0\}, \mathbb{R})$ such that $\lim_{t \searrow 0} u(t) = -\infty$ and $\lim_{t \nearrow 0} u(t) = +\infty$.

When $N = \mathbb{R}^n$ and $n \geq 2$, colocally weakly differentiable functions are closely related to functions of generalized bounded variation [6, definition 4.26]. Indeed, if $f \in C^1(\mathbb{R}^n, \mathbb{R})$, then $\text{supp}(Df)$ is compact if and only if f is constant outside a compact set.

Finally, if $u : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is weakly differentiable, then u is colocally weakly differentiable. The converse is false: for example, the function $u : \mathbb{R}^m \rightarrow \mathbb{R}$ defined for every $x \in \mathbb{R}^m \setminus \{0\}$ by $u(x) = |x|^{-\alpha}$ is not weakly differentiable for any $\alpha > m - 1$, but is colocally weakly differentiable for every $\alpha \in \mathbb{R}$.

In order to define the colocal weak derivative, we denote by (TM, π_M, M) the *tangent bundle* over M , that is,

$$TM = \bigcup_{x \in M} \{x\} \times T_x M,$$

$\pi_M: TM \rightarrow M$ is the natural projection and for every $x \in M$, the fiber $\pi_M^{-1}(\{x\})$ is isomorphic to $\mathbb{R}^m$; a map $v: TM \rightarrow TN$ is a *bundle morphism* that covers $u: M \rightarrow N$ if

$$\begin{array}{ccc} TM & \xrightarrow{v} & TN \\ \pi_M \downarrow & & \downarrow \pi_N \\ M & \xrightarrow{u} & N \end{array}$$

commutes, that is, $\pi_N \circ v = u \circ \pi_M$, and for every $x \in M$, the map $v(x): \pi_M^{-1}(\{x\}) \rightarrow \pi_N^{-1}(\{u(x)\})$ is linear. The space of all bundle morphisms is denoted by $\text{Hom}(TM, TN)$. In particular, if $u: M \rightarrow N$ is a differentiable map, then $Du: TM \rightarrow TN$ is a bundle morphism that covers u . By a direct covering argument, if $u: M \rightarrow \mathbb{R}$ is weakly differentiable, then there exists a bundle morphism $Du: TM \rightarrow \mathbb{R}$ such that for every local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$,

$$D(u \circ \psi^{-1}) = Du \circ D\psi^{-1}$$

almost everywhere on $\psi(V)$. We have now all the ingredients to define the colocal weak derivative.

Definition 1.2.2. Let $u: M \rightarrow N$ be a colocally weakly differentiable map. A map $Du: TM \rightarrow TN$ is a *colocal weak derivative* of u if Du is a measurable bundle morphism that covers u and for every $f \in C_c^1(N, \mathbb{R})$,

$$D(f \circ u) = Df \circ Du$$

almost everywhere in M .

Consequently, if Du is a colocal weak derivative of u , for almost every $x \in M$, $Du(x) \in \mathcal{L}(T_x M, T_{u(x)} N)$ and for each $e \in T_x M$, $D(f \circ u)(x)[e] = Df(u(x))[Du(x)[e]]$.

We first observe that this notion extends the notion of classical differentiability:

Proposition 1.2.1 (Equivalence of classical and colocal weak derivatives). *Let $u \in C(M, N)$. Then u has a continuous colocal weak derivative if and only if $u \in C^1(M, N)$. Moreover, the colocal weak derivative and the classical derivative coincide almost everywhere.*

Proof. Since for every $f \in C_c^1(N, \mathbb{R})$, $f \circ u$ is weakly differentiable, we can apply the equivalence of classical and weak derivatives (Du Bois-Reymond lemma) [82, theorem 6.10; 120, theorem 6.1.4] and local charts on M to obtain that $f \circ u \in C^1(M, \mathbb{R})$. Since f is arbitrary, the map u is continuously differentiable. $\square$

Definition 1.2.3. A bundle morphism $v : TM \rightarrow TN$ that covers $u : M \rightarrow N$ is *bilocally integrable* on $A \subseteq M$ if for every $x \in A$ and every $y \in N$, there exist local charts $\psi : V \subseteq M \rightarrow \mathbb{R}^m$, $\varphi : U \subseteq N \rightarrow \mathbb{R}^n$ such that $x \in U$, $y \in V$ and if $L \subset V$ and $K \subset U$ are compact, then the function $D\varphi \circ v|_V \circ D(\psi^{-1})$ is integrable on $\psi(A \cap L \cap u^{-1}(K))$.

If μ is a positive absolutely continuous measure on M , the morphism v is bilocally integrable if and only if there exists a continuous norm $|\cdot|$ on $T^*M \otimes TN$ such that $\int_M |v| d\mu < \infty$.

The main result of the current section is that colocally weakly differentiable maps have a colocal weak derivative.

Proposition 1.2.2 (Existence and uniqueness of the colocal weak derivative). *If the map $u : M \rightarrow N$ is colocally weakly differentiable, then u has a unique colocal weak derivative $Du : TM \rightarrow TN$. Moreover, the bundle morphism Du is bilocally integrable and for every $f \in C^1(N, \mathbb{R})$ for which $f \circ u : M \rightarrow \mathbb{R}$ is weakly differentiable,*

$$D(f \circ u) = Df \circ Du$$

almost everywhere in M .

This result was already known when $N = \mathbb{R}$ for functions of generalized bounded variation [6, theorem 4.34] and for Sobolev functions by truncations [13, lemma 2.1]; as remarked there, the colocal weak derivative need not be locally integrable. The important geometric tool for proving proposition 1.2.2 is the existence of extended local charts. This construction is reminiscent of the patch mappings to the sphere $\varphi \in C^1(N, \mathbb{S}^n)$ [93, p. 60].

Lemma 1.2.3 (Extended local charts). *For every $y \in N$, there exist an open subset $U \subseteq N$ such that $y \in U$, and maps $\varphi \in C^1(N, \mathbb{R}^n)$ and $\varphi^* \in C^1(\mathbb{R}^n, N)$ such that*

- (i) *the set $\text{supp } \varphi$ is compact,*
- (ii) *the set $\overline{\{x \in \mathbb{R}^n : \varphi^*(x) \neq \varphi^*(\varphi(y))\}}$ is compact,*
- (iii) *the map $\varphi|_U$ is a diffeomorphism onto its image $\varphi(U)$,*
- (iv) *$\varphi^* \circ \varphi = \text{id}$ in U .*

Proof. By definition of differentiable manifold, there exists a local chart $\psi : V \subseteq N \rightarrow \psi(V) \subseteq \mathbb{R}^n$ around $y \in N$. Without loss of generality, we assume that $\psi(y) = 0$. Since the set $\psi(V) \subseteq \mathbb{R}^n$ is open, there exists $r > 0$ such that $B_{2r} \subseteq \psi(V)$. We choose a map $\theta \in C_c^1(\mathbb{R}^n, \mathbb{R})$ such that

$0 \leq \theta \leq 1$ on $\mathbb{R}^n$, $\theta = 1$ on B_r and $\text{supp}(\theta) \subset B_{2r}$. We take the set $U = (\psi|_V)^{-1}(B_r)$ and the maps $\varphi: N \rightarrow \mathbb{R}^n$ defined for every $z \in N$ by

$$\varphi(z) = \begin{cases} \theta(\psi(z))\psi(z) & \text{if } z \in (\psi|_V)^{-1}(B_{2r}), \\ 0 & \text{otherwise} \end{cases}$$

and $\varphi^*: \mathbb{R}^n \rightarrow N$ defined for every $x \in \mathbb{R}^n$ by $\varphi^*(x) = (\psi|_V)^{-1}(\theta(x)x)$. $\square$

We begin by proving a local counterpart of proposition 1.2.2.

Lemma 1.2.4. *If $u: M \rightarrow N$ is a colocally weakly differentiable map and $y \in N$, then there exist an open subset $U \subseteq N$ such that $y \in U$ and a unique measurable bundle morphism $D_U u: \pi_M^{-1}(u^{-1}(U)) \rightarrow TN$ such that for every $f \in C^1(N, \mathbb{R})$ for which $f \circ u: M \rightarrow \mathbb{R}$ is weakly differentiable,*

$$D(f \circ u) = Df \circ D_U u \quad \text{almost everywhere on } u^{-1}(U).$$

Moreover, $D_U u$ is bilocally integrable on $u^{-1}(U)$.

Proof. Let $U \subseteq N$, $\varphi \in C^1(N, \mathbb{R}^n)$ and $\varphi^* \in C^1(\mathbb{R}^n, N)$ be the extended local charts given by lemma 1.2.3. Since u is colocally weakly differentiable, the map $\varphi \circ u: M \rightarrow \mathbb{R}^n$ is weakly differentiable. Since for every $x \in u^{-1}(U)$, the linear map between tangent spaces $D\varphi(u(x)): T_{u(x)}N \rightarrow \mathbb{R}^n$ is invertible, the map $D_U u$ is uniquely defined for almost every $x \in u^{-1}(U)$ by

$$D_U u(x) = (D\varphi(u(x)))^{-1} \circ (D(\varphi \circ u)(x)).$$

If $f \in C^1(N, \mathbb{R})$ and $f \circ u: M \rightarrow \mathbb{R}$ is weakly differentiable, since $\varphi^*(\mathbb{R}^n)$ is compact and $f \circ \varphi^* \in C^1(\mathbb{R}^n, \mathbb{R})$, the chain rule for weakly differentiable functions implies (see for example [45, theorem 4.2.4 (ii); 120, theorem 6.1.13]) that $f \circ \varphi^* \circ \varphi \circ u$ is weakly differentiable and

$$D(f \circ \varphi^* \circ \varphi \circ u) = D(f \circ \varphi^*) \circ D(\varphi \circ u) = D(f \circ \varphi^*) \circ D\varphi \circ D_U u.$$

Since $f \circ u = f \circ \varphi^* \circ \varphi \circ u$ on $u^{-1}(U)$ and $D(f \circ \varphi^*) \circ D\varphi = Df$ on U , we have [45, theorem 4.2.4 (iv); 82, theorem 6.19; 120, corollary 6.1.14] almost everywhere on $u^{-1}(U)$

$$D(f \circ u) = D(f \circ \varphi^*) \circ D\varphi \circ D_U u = Df \circ D_U u. \quad \square$$

Proof of proposition 1.2.2. Let $(U_i)_{i \in I}$ be an open cover of N by sets given by lemma 1.2.4. Since the manifold N has a countable basis, we can assume that I is at most countable [91, theorem 3.30]. Let $D_{U_i} u$

and $D_{U_i \cap U_j} u$ be the derivatives given by lemma 1.2.4. By uniqueness, $D_{U_i \cap U_j} u = D_{U_i} u = D_{U_j} u$ almost everywhere on $u^{-1}(U_i \cap U_j)$. Since $\bigcup_{i \in I} u^{-1}(U_i) = M$ and I is countable, the bundle morphism $Du : TM \rightarrow TN$ can be defined uniquely almost everywhere by

$$Du = D_{U_i} u \text{ almost everywhere on } u^{-1}(U_i). \quad \square$$

1.2.2. Properties of the colocal weak derivative

The colocal weak derivative retains some properties of weak derivatives.

Proposition 1.2.5 (Chain rule). *Let $\tilde{N}$ be a differentiable manifold. If the maps $u : M \rightarrow N$ and $f \in C^1(N, \tilde{N})$ are such that u and $f \circ u$ are colocally weakly differentiable, then*

$$D(f \circ u) = Df \circ Du \text{ almost everywhere in } M.$$

Proof. For every $\varphi \in C_c^1(\tilde{N}, \mathbb{R})$, $\varphi \circ f \in C^1(N, \mathbb{R})$ and $\varphi \circ f$ is weakly differentiable. Therefore, by proposition 1.2.2 and the classical chain rule,

$$\begin{aligned} D\varphi \circ D(f \circ u) &= D(\varphi \circ f \circ u) = D((\varphi \circ f) \circ u) \\ &= D(\varphi \circ f) \circ Du = D\varphi \circ (Df \circ Du). \end{aligned} \quad \square$$

Proposition 1.2.6. *Let $\iota : N \rightarrow \tilde{N}$ be an embedding and let $u : M \rightarrow N$ be a map. If $\iota \circ u$ is colocally weakly differentiable, then u is colocally weakly differentiable and Du is the unique bundle morphism such that*

$$D(\iota \circ u) = D\iota \circ Du$$

almost everywhere in M .

If moreover $\iota(N)$ is closed, then the converse holds.

Proof. Since ι is an embedding, $\iota(N)$ has a tubular neighborhood in $\tilde{N}$: there exist a vector bundle (E, π_N, N) and an embedding $\tilde{\iota} : E \rightarrow \tilde{N}$ such that $\tilde{\iota}|_N = \iota$ and $\tilde{\iota}(E)$ is open in $\tilde{N}$ [70, theorem 4.5.2]. In particular, $\iota \circ \pi_N \circ \tilde{\iota}^{-1}$ is a retraction of the tubular neighborhood $\tilde{\iota}(E)$ on $\iota(N)$. Let $f \in C_c^1(N, \mathbb{R})$. We choose $\eta \in C_c^1(\tilde{N}, \mathbb{R})$ such that $\eta = 1$ on $\iota(\text{supp } f)$ and $\text{supp } \eta \subset \tilde{\iota}(E)$ and define $\tilde{f} = (f \circ \pi_N \circ \tilde{\iota}^{-1})\eta : \tilde{N} \rightarrow \mathbb{R}$. Since $\text{supp } \eta \subset \tilde{\iota}(E)$, the function $\tilde{f}$ is well-defined, $\tilde{f} \in C_c^1(\tilde{N}, \mathbb{R})$ and $\tilde{f} \circ \iota = f$ on N . In particular, $f \circ u = \tilde{f} \circ \iota \circ u$ is weakly differentiable.

Conversely, if $\iota(N)$ is closed, then for every $\varphi \in C_c^1(\tilde{N}, \mathbb{R})$, $\varphi \circ \iota \in C_c^1(N, \mathbb{R})$ and $\iota \circ u$ is thus colocally weakly differentiable. $\square$

By the Whitney embedding theorem [119], there always exists an embedding $\iota : N \rightarrow \mathbb{R}^\nu = \mathbb{R}^{2n+1}$ such that $\iota(N)$ is closed [1, theorem 2.6; 40, theorem 5]. Proposition 1.2.6 gives thus an equivalent definition of colocal weak differentiability; the drawback of this alternate approach to colocal weak differentiability is its dependence on the Whitney embedding theorem for differentiable manifolds.

Since differentiable manifolds do not have in general any algebraic structure and since the colocally weakly differentiable functions between Euclidean spaces do not form a vector space [6, 13], the colocal weak derivative does not have any algebraic properties of sum or product. There is however still a property of the Cartesian product of maps.

Proposition 1.2.7 (Product of manifolds). *Let N_1, N_2 be two differentiable manifolds. If $u_1 : M \rightarrow N_1$ and $u_2 : M \rightarrow N_2$ are colocally weakly differentiable, then the map $u = (u_1, u_2) : M \rightarrow N_1 \times N_2$ is colocally weakly differentiable and*

$$Du = (Du_1, Du_2) \quad \text{almost everywhere in } M.$$

The uniqueness property can be refined for maps that coincide on a set of positive measure:

Proposition 1.2.8 (Locality of derivatives). *If the maps $u, v : M \rightarrow N$ are colocally weakly differentiable, then $Du = Dv$ almost everywhere on the set $\{x \in M : u(x) = v(x)\}$.*

Proof. Let $A = \{x \in M : u(x) = v(x)\}$. Let $U \subseteq N$ and $\varphi \in C^1(N, \mathbb{R}^n)$ be the extended local chart given by lemma 1.2.3 and let $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ be a local chart. Since $\varphi \circ u \circ \psi^{-1}$ and $\varphi \circ v \circ \psi^{-1}$ are weakly differentiable and $\varphi \circ u \circ \psi^{-1} = \varphi \circ v \circ \psi^{-1}$ on $\psi(A \cap V)$, $D(\varphi \circ u \circ \psi^{-1}) = D(\varphi \circ v \circ \psi^{-1})$ almost everywhere on $\psi(A \cap V)$. By definition of the colocal weak derivative, $D\varphi \circ Du \circ D\psi^{-1} = D\varphi \circ Dv \circ D\psi^{-1}$ almost everywhere on $\psi(A \cap V)$. Since $D\varphi$ is invertible on $u^{-1}(U)$, $Du = Dv$ almost everywhere on $A \cap V \cap u^{-1}(U) \cap v^{-1}(U)$. Taking open countable covers $(U_i)_{i \in I}$ of N and $(V_j)_{j \in J}$ of M , we conclude that $Du = Dv$ almost everywhere on A . $\square$

1.2.3. Approximate differentiability

We study the relationship between the colocal weak derivative and the approximate derivative. For a map between differentiable manifolds, we generalize the classical definition of approximate derivative of maps between vector spaces [45, definition 6.1; 46, 3.1.2].

Definition 1.2.4. Let $u: M \rightarrow N$ and $x \in M$. The linear map $\xi: T_x M \rightarrow T_{u(x)} N$ is an *approximate derivative* of u at x if there exist local charts $\psi: V \rightarrow \psi(V) \subseteq \mathbb{R}^m$ with $x \in V$ and $V \subseteq M$ open, and $\varphi: U \rightarrow \varphi(U) \subseteq \mathbb{R}^n$ with $u(x) \in U$ and $U \subseteq N$ open such that for every $\varepsilon > 0$,

$$\begin{aligned} \lim_{\rho \rightarrow 0} \rho^{-m} \mathcal{L}^m(\psi(\{y \in V : u(y) \in U \\ |\varphi(u(y)) - \varphi(u(x)) - (D\varphi(u(x)) \circ \xi \circ (D\psi(x))^{-1})[\psi(y) - \psi(x)]| \\ > \varepsilon |\psi(y) - \psi(x)|\}) \cap B_\rho(\psi(x))) = 0. \end{aligned}$$

The approximate derivative is unique and it is sufficient to establish its existence for one pair of diffeomorphisms.

If M and N are endowed with Riemannian metrics g_M and g_N respectively, it is natural to take for φ and ψ the exponential coordinates and to use the Riemannian distances d_N and d_M and measure μ_M ; the approximate differentiability can be observed to be equivalent with requiring for every $\varepsilon > 0$,

$$\begin{aligned} \lim_{\rho \rightarrow 0} \rho^{-m} \mu_M(\{y \in B_\rho^M(x) : \\ d_N(u(x), \exp_{u(x)}(\xi(\exp_x^{-1}(y)))) > \varepsilon d_M(x, y)\}) = 0; \end{aligned}$$

we recover thus in this particular case the definition of Focardi and Spadaro for maps from the Euclidean space to a Riemannian manifold [48, definition 0.3].

Sobolev maps into Riemannian manifolds are known to have such an approximate derivative [48, corollary 1.3]. This property is also satisfied for Sobolev maps into other non-flat targets for which a notion of weak derivative is not yet available [37, lemma 1.4; 47, proposition 2.2].

Proposition 1.2.9 (Approximate differentiability of colocally weakly differentiable maps). *If $u: M \rightarrow N$ is colocally weakly differentiable, then for almost every $x \in M$, the colocal weak derivative $Du(x)$ is the approximate derivative of u at x .*

Proof. Let $\psi: V \subseteq M \rightarrow \mathbb{R}^m$ be a local chart around $x \in M$. Let $U \subseteq N$ and $\varphi \in C^1(N, \mathbb{R}^n)$ be the extended local chart given by lemma 1.2.3. Since u is colocally weakly differentiable, $\varphi \circ u \circ \psi^{-1}: \psi(V) \rightarrow \mathbb{R}^n$ is weakly differentiable and $D(\varphi \circ u \circ \psi^{-1}) = D\varphi \circ Du \circ D\psi^{-1}$ on $\psi(V)$. Since weakly differentiable maps between vector spaces are approximately differentiable [45, theorem 6.1.4], the map $\varphi \circ u \circ \psi^{-1}$ is approximately

differentiable almost everywhere on $\psi(V)$, that is,

$$\begin{aligned} \lim_{\rho \rightarrow 0} \rho^{-m} \mathcal{L}^m(\psi(\{y \in V : \\ |\varphi(u(y)) - \varphi(u(x)) - (D\varphi(u(x)) \circ Du(x) \circ (D\psi(x))^{-1})[\psi(y) - \psi(x)]| \\ > \varepsilon |\psi(y) - \psi(x)|\}) \cap B_\rho(\psi(x))) = 0. \end{aligned}$$

Next, we note that since u is measurable, almost every $x \in M$ is a Lebesgue point of u . Since the set U is open, for almost every $x \in u^{-1}(U) \cap V$,

$$\lim_{\rho \rightarrow 0} \rho^{-m} \mathcal{L}^m(\psi(\{y \in V : u(y) \notin U\}) \cap B_\rho(\psi(x))) = 0.$$

Therefore we have for almost every $x \in u^{-1}(U) \cap V$,

$$\begin{aligned} \lim_{\rho \rightarrow 0} \rho^{-m} \mathcal{L}^m(\psi(\{y \in V : u(y) \in U \\ |\varphi|_U(u(y)) - \varphi|_U(u(x)) - (D\varphi(u(x)) \circ Du(x) \circ (D\psi(x))^{-1})[\psi(y) - \psi(x)]| \\ > \varepsilon |\psi(y) - \psi(x)|\}) \cap B_\rho(\psi(x))) = 0, \end{aligned}$$

that is, $Du(x)$ is the approximate derivative of u for almost every $x \in u^{-1}(U) \cap V$. The conclusion follows by a countable covering argument. $\square$

1.3. Sobolev maps between Riemannian manifolds

Preliminaries. We assume now that (M, g_M) and (N, g_N) are Riemannian manifolds. In particular the metrics on vectors of TM and TN induce a metric $g_M^* \otimes g_N$ on $T^*M \otimes TN$. This metric can be computed for every $\xi \in T_x^*M \otimes T_yN = \mathcal{L}(T_xM, T_yN)$ by

$$(g_M^* \otimes g_N)(\xi, \xi) = \sum_{i=1}^m g_N(\xi(e_i), \xi(e_i)),$$

where $(e_i)_{1 \leq i \leq m}$ is an orthonormal basis in $\pi_M^{-1}(\{x\})$ with respect to the Riemannian metric g_M .

We are now in position to define the Sobolev spaces.

Definition 1.3.1. Let $p \in [1, \infty)$. A map $u: M \rightarrow N$ belongs to the Sobolev space $\dot{W}^{1,p}(M, N)$ if u is colocally weakly differentiable and $|Du|_{g_M^* \otimes g_N} \in L^p(M)$.

Characterization of the norm of the derivative. We characterize the quantity $|Du|_{g_M^* \otimes g_N}$ that appears in the definition of Sobolev spaces. We recall that the *operator norm* is defined for every $\xi \in T_x^*M \otimes T_yN = \mathcal{L}(T_xM, T_yN) = \text{Hom}(TM, TN)_{x,y}$ by

$$|\xi|_{\mathcal{L}} = \sup\{|\xi(e)|_{g_N} : e \in T_xM, |e|_{g_M} \leq 1\};$$

the *Lipschitz semi-norm* of $f: N \rightarrow \mathbb{R}^k$ is defined by

$$|f|_{\text{Lip}} = \sup \left\{ \frac{|f(y) - f(z)|}{d_N(y, z)} : y, z \in N \text{ and } y \neq z \right\},$$

where d_N is the distance on N induced by the Riemannian metric g_N . For every $k \geq 1$, we denote by g_k the standard Euclidean metric on $\mathbb{R}^k$.

Proposition 1.3.1. *Let $k \geq \min(m, n)$. Let $u: M \rightarrow N$, $w: M \rightarrow \mathbb{R}$ be measurable maps. The following statements are equivalent.*

- (i) *u is colocally weakly differentiable and $|Du|_{g_M^* \otimes g_N} \leq w$ almost everywhere in M ,*
- (ii) *for every $f \in C_c^1(N, \mathbb{R}^k)$, $f \circ u$ is weakly differentiable and almost everywhere in M*

$$|D(f \circ u)|_{g_M^* \otimes g_k} \leq |Df(u)|_{\mathcal{L}} w,$$

- (iii) *for every $f \in C_c^1(N, \mathbb{R}^k)$, $f \circ u$ is weakly differentiable and almost everywhere in M*

$$|D(f \circ u)|_{g_M^* \otimes g_k} \leq |f|_{\text{Lip}} w.$$

If moreover $w \in L_{\text{loc}}^1(M)$, then for every $f \in \text{Lip}(N, \mathbb{R}^k)$, $f \circ u$ is weakly differentiable and almost everywhere in M

$$|D(f \circ u)|_{g_M^* \otimes g_k} \leq |f|_{\text{Lip}} w.$$

Since the first assertion is independent of k , there is also equivalence between these statements for every $k \geq \min(m, n)$.

Proposition 1.3.1 implies in particular that if $p \in [1, \infty)$ and M is an open bounded subset of $\mathbb{R}^m$, definition 1.3.1 is equivalent with the notion of Sobolev spaces into metric spaces [99, theorem 5.1] and with the classical homogeneous Sobolev space when $N = \mathbb{R}^n$ (see also [13, lemma 2.1]).

In order to prove proposition 1.3.1, we shall use an approximation property of Lipschitz maps on manifolds.

Lemma 1.3.2 (Approximation of Lipschitz maps). *Let $f \in \text{Lip}(N, \mathbb{R}^k)$. There exists a sequence $(f_\ell)_{\ell \in \mathbb{N}}$ of maps in $C_c^1(N, \mathbb{R}^k)$ that converges uniformly over compact subsets of N and such that*

$$\limsup_{\ell \rightarrow \infty} |f_\ell|_{\text{Lip}} \leq |f|_{\text{Lip}}.$$

Sketch of the proof. Given $y \in N$ and $\theta \in C_c^1([0, \infty), \mathbb{R})$ such that $\theta = 1$ on $[0, 1]$, we define for every $\ell \in \mathbb{N}_*$,

$$\theta_\ell(z) = \theta\left(\sqrt[\ell^2]{d_N(y, z)}\right),$$

and observe that $\theta_\ell \in C_c^1(N, \mathbb{R})$,

$$|D\theta_\ell|_{\mathcal{L}} \leq \frac{\|\theta'\|_{L^\infty}}{\ell^2},$$

and $(\theta_\ell)_{\ell \in \mathbb{N}_*}$ converges to 1 uniformly over compact subsets of N . We also define $T_\ell : \mathbb{R}^k \rightarrow \mathbb{R}^k$ for $\ell \in \mathbb{N}_*$ and $x \in \mathbb{R}^k$ by

$$T_\ell(x) = \begin{cases} x & \text{if } |x| \leq \ell, \\ \ell x/|x| & \text{if } |x| > \ell. \end{cases}$$

For every $\ell \in \mathbb{N}_*$, T_ℓ is nonexpansive and bounded by ℓ and the sequence $(T_\ell)_{\ell \in \mathbb{N}_*}$ converges uniformly to the identity over compact subsets.

If we define $\tilde{f}_\ell = \theta_\ell \cdot (T_\ell \circ f)$, we observe for every $\ell \in \mathbb{N}_*$ that $\tilde{f}_\ell \in \text{Lip}(N, \mathbb{R}^k)$, that

$$|\tilde{f}_\ell|_{\text{Lip}} \leq |f|_{\text{Lip}} + \frac{\|\theta'\|_{L^\infty}}{\ell},$$

and that the support of $\tilde{f}_\ell$ is compact; the sequence $(\tilde{f}_\ell)_{\ell \in \mathbb{N}_*}$ converges uniformly over compact subsets. Hence the conclusion follows by approximating uniformly $\tilde{f}_\ell$ by differentiable functions with a control on the Lipschitz norm [8; 59, lemma 8]. $\square$

We shall also rely on a refined version of the extended local charts of lemma 1.2.3.

Lemma 1.3.3 (Almost isometric extended local charts). *For every $y \in N$ and every $\varepsilon > 0$, there exist an open subset $U \subseteq N$ such that $y \in U$, and maps $\varphi \in C^1(N, \mathbb{R}^n)$ and $\varphi^* \in C^1(\mathbb{R}^n, N)$ such that*

(i) *the set $\text{supp } \varphi$ is compact,*

(ii) *the set $\overline{\{x \in \mathbb{R}^n : \varphi^*(x) \neq \varphi^*(\varphi(y))\}}$ is compact,*

- (iii) the map $\varphi|_U$ is a diffeomorphism onto its image $\varphi(U)$,
- (iv) $\varphi^* \circ \varphi = \text{id}$ in U ,
- (v) for every $z \in N$,

$$|D\varphi(z)|_{\mathcal{L}} \leq 1 + \varepsilon \quad \text{and} \quad |D\varphi^*(\varphi(z))|_{\mathcal{L}} \leq 1 + \varepsilon.$$

The difference with lemma 1.2.3 lies in the control (v) on the operator norms of $D\varphi$ and $D\varphi^*$.

Proof of lemma 1.3.3. Since N is a differentiable manifold, there exists a local chart $\psi : V \subseteq N \rightarrow \mathbb{R}^n$ around $y \in N$. Up to an affine transformation on $\mathbb{R}^n$, we can assume that $\psi(y) = 0$ and $D\psi(y) \in \mathcal{L}(T_y N, \mathbb{R}^n)$ is an isometry.

By continuity of $D\psi$, there exists $\delta > 0$ such that if $z \in N$ and $d_N(y, z) \leq \delta$, then

$$|D\psi(z)|_{\mathcal{L}} \leq 1 + \varepsilon \quad \text{and} \quad |D(\psi|_V)^{-1}(\psi(z))|_{\mathcal{L}} \leq 1 + \varepsilon.$$

We choose $r > 0$ such that $B_{3r} \subseteq \psi(B_\delta(y))$. We take a map $\theta \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $\theta = \text{id}$ on B_r , $\text{supp}(\theta) \subset B_{3r}$ and $\sup_{x \in \mathbb{R}^n} |D\theta(x)|_{\mathcal{L}} \leq 1$. We finally define $\varphi = \theta \circ \psi$, $\varphi^* = (\psi|_V)^{-1} \circ \theta$ and $U = (\psi|_V)^{-1}(B_r)$ and we conclude as in the proof of lemma 1.2.3. $\square$

We shall also use the following lemma to compute Euclidean norms of maps.

Lemma 1.3.4 (Reduction of the Euclidean norm of operators). *Let $x \in M$ and let $\xi \in \mathcal{L}(T_x M, \mathbb{R}^n)$ be a linear map. If $k \geq \min(m, n)$, then*

$$|\xi|_{g_M^* \otimes g_n} = \sup\{|\rho \circ \xi|_{g_M^* \otimes g_k} : \rho \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k) \text{ and } |\rho|_{\mathcal{L}} \leq 1\}.$$

Proof. On the one hand, if $\rho \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k)$ and $|\rho|_{\mathcal{L}} \leq 1$, then

$$|\rho \circ \xi|_{g_M^* \otimes g_k} \leq |\rho|_{\mathcal{L}} |\xi|_{g_M^* \otimes g_n} \leq |\xi|_{g_M^* \otimes g_n}.$$

On the other hand, since $\dim(\text{Im}(\xi)) \leq \min(m, n) \leq k$, one can choose $\rho \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k)$ such that ρ is an isometry on $\text{Im}(\xi)$ and consequently $|\rho|_{\mathcal{L}} \leq 1$ and $|\rho \circ \xi|_{g_M^* \otimes g_k} = |\xi|_{g_M^* \otimes g_n}$. $\square$

Proof of proposition 1.3.1. Let us prove that (i) implies (ii). For every $f \in C_c^1(N, \mathbb{R}^k)$, since u is colocally weakly differentiable, $f \circ u$ is weakly differentiable. By proposition 1.2.2, $D(f \circ u) = Df \circ Du$ almost everywhere in M , and so

$$|D(f \circ u)|_{g_M^* \otimes g_k} \leq |Df(u)|_{\mathcal{L}} |Du|_{g_M^* \otimes g_N} \leq |Df(u)|_{\mathcal{L}} w.$$

Since for every $f \in C_c^1(N, \mathbb{R}^k)$ and for every $y \in N$, $|Df(y)|_{\mathcal{L}} \leq |f|_{\text{Lip}}$, the assertion (ii) implies directly (iii).

In order to prove that (iii) implies (i) we first note that the map u is colocally weakly differentiable, and, by proposition 1.2.2, has a unique colocal weak derivative $Du \in \text{Hom}(TM, TN)$. Secondly, let $U \subseteq N$, $\varphi \in C^1(N, \mathbb{R}^n)$ and $\varphi^* \in C^1(\mathbb{R}^n, N)$ be given by lemma 1.3.3 for $y \in N$ and $\varepsilon > 0$. Since $u = \varphi^* \circ \varphi \circ u$ on $u^{-1}(U)$, by proposition 1.2.8, $Du = D(\varphi^* \circ \varphi \circ u)$ almost everywhere on $u^{-1}(U)$. By lemma 1.3.3, almost everywhere on $u^{-1}(U)$

$$\begin{aligned} |Du|_{g_M^* \otimes g_N} &\leq |D\varphi^*(\varphi(u))|_{\mathcal{L}} |D(\varphi \circ u)|_{g_M^* \otimes g_n} \\ &\leq (1 + \varepsilon) |D(\varphi \circ u)|_{g_M^* \otimes g_n}. \end{aligned} \quad (1.3.3)$$

If $\rho \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k)$ and $|\rho|_{\mathcal{L}} \leq 1$, by assumption, $|D(\rho \circ \varphi \circ u)|_{g_M^* \otimes g_k} \leq \text{Lip}(\rho \circ \varphi) w$ almost everywhere in M . Since d_N is a geodesic distance, in view of lemma 1.3.3, $\text{Lip}(\varphi) = \sup_{z \in N} |D\varphi(z)|_{\mathcal{L}} \leq 1 + \varepsilon$. Hence, we have almost everywhere in M

$$|\rho \circ D(\varphi \circ u)|_{g_M^* \otimes g_k} = |D(\rho \circ \varphi \circ u)|_{g_M^* \otimes g_k} \leq (1 + \varepsilon) w.$$

Since the set of nonexpansive linear maps is separable, we deduce from lemma 1.3.4 that

$$|D(\varphi \circ u)|_{g_M^* \otimes g_n} \leq (1 + \varepsilon) w$$

almost everywhere in M . By inequality (1.3.3), we conclude that

$$|Du|_{g_M^* \otimes g_N} \leq (1 + \varepsilon)^2 w$$

almost everywhere on $u^{-1}(U)$. We conclude by countable covering of N .

We now prove the last assertion. Let $f \in \text{Lip}(N, \mathbb{R}^k)$. By the approximation lemma 1.3.2, there exists a sequence $(f_\ell)_{\ell \in \mathbb{N}}$ in $C_c^1(N, \mathbb{R}^k)$ such that $(f_\ell)_{\ell \in \mathbb{N}}$ converges uniformly to f over compact subsets and $\limsup_{\ell \rightarrow \infty} |f_\ell|_{\text{Lip}} \leq |f|_{\text{Lip}}$. Since the sequence $(D(f_\ell \circ u))_{\ell \in \mathbb{N}}$ is bounded and uniformly integrable and since the sequence $(f_\ell \circ u)_{\ell \in \mathbb{N}}$ converges almost everywhere to $f \circ u$, in view of the weak compactness criterion in $L^1(M)$ [20, corollary 4.7.19; 24, theorem 4.30], the sequence $(D(f_\ell \circ u))_{\ell \in \mathbb{N}}$ converges weakly to $D(f \circ u)$ in $L_{\text{loc}}^1(M)$ and $f \circ u \in W_{\text{loc}}^{1,1}(M, \mathbb{R}^k)$. Moreover, for every $v \in C_c^1(M, TM \otimes \mathbb{R}^k)$,

$$\begin{aligned} \left| \int_M \langle D(f \circ u), v \rangle \right| &= \lim_{\ell \rightarrow \infty} \left| \int_M \langle D(f_\ell \circ u), v \rangle \right| \\ &\leq \liminf_{\ell \rightarrow \infty} |f_\ell|_{\text{Lip}} \int_M |v|_{g_M^* \otimes g_k} w \leq |f|_{\text{Lip}} \int_M |v|_{g_M^* \otimes g_k} w, \end{aligned}$$

and therefore $|D(f \circ u)|_{g_M^* \otimes g_k} \leq |f|_{\text{Lip}} w$ almost everywhere in M . $\square$

Thanks to proposition 1.3.1, we can consider the composition of a Lipschitz map from a manifold into an other with a map of homogeneous Sobolev space.

Proposition 1.3.5 (Chain rule in Sobolev spaces). *Let $(\tilde{N}, g_{\tilde{N}})$ be a Riemannian manifold. Let $u \in \dot{W}^{1,p}(M, N)$ and let $f \in \text{Lip}(M, \tilde{N})$. Then $f \circ u \in \dot{W}^{1,p}(M, \tilde{N})$ and*

$$|D(f \circ u)|_{g_M^* \otimes g_{\tilde{N}}} \leq |f|_{\text{Lip}} |Du|_{g_M^* \otimes g_N} \text{ almost everywhere in } M.$$

This generalizes a well-known property ([24, proposition 9.5; 82, theorem 6.16; 120, proposition 6.1.13]); the corresponding chain rule is more delicate [4].

We also obtain a characterization of Sobolev spaces by embeddings.

Proposition 1.3.6. *Let $\iota : N \rightarrow \tilde{N}$ be an isometric embedding. For every $u : M \rightarrow N$, $u \in \dot{W}^{1,p}(M, N)$ if and only if $\iota \circ u \in \dot{W}^{1,p}(M, \tilde{N})$.*

In contrast with proposition 1.2.6, the equivalence does not require $\iota(N)$ to be closed.

Proof of proposition 1.3.6. If $u \in \dot{W}^{1,p}(M, N)$, then by the chain rule (proposition 1.3.5), $\iota \circ u \in \dot{W}^{1,p}(M, \tilde{N})$. Conversely, if $\iota \circ u \in \dot{W}^{1,p}(M, \tilde{N})$, then $\iota \circ u$ is colocally weakly differentiable. By the weakly differentiable embedding property (proposition 1.2.6), u is colocally weakly differentiable. By the chain rule (proposition 1.2.5), $D(\iota \circ u) = D\iota \circ Du$, and since the embedding ι is isometric, $|Du|_{g_M^* \otimes g_N} = |D(\iota \circ u)|_{g_M^* \otimes g_{\tilde{N}}}$. $\square$

In particular our intrinsic definition is equivalent with the definition given by any embedding of N in a Euclidean space; such an embedding is always possible by the Nash embedding theorem [92, 93].

1.4. Weak compactness and closure property

1.4.1. Weak compactness

A classical technique in the calculus of variations is to extract from a minimizing sequence a subsequence that converges almost everywhere.

To cover a rich spectrum of functional settings, we first give a compactness result independently of the Riemannian manifold or Sobolev space setting. Such a result should rely on a boundedness assumption on the derivatives and on the values of the mappings.

For the boundedness of the derivatives, we shall use the following assumption.

Definition 1.4.1. A sequence $(v_\ell)_{\ell \in \mathbb{N}}$ of bundle morphisms from TM to TN that covers a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ of maps from M to N is *bilocally L^1 -bounded* if for every $x \in M$ and every $y \in N$, there exist local charts $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ and $\varphi : U \subseteq N \rightarrow \mathbb{R}^n$ such that $x \in V$, $y \in U$ and

$$\sup_{\ell \in \mathbb{N}} \int_{\psi(V \cap u_\ell^{-1}(U))} |D\varphi \circ v_\ell \circ D(\psi^{-1})|_{g_m^* \otimes g_n} < \infty.$$

Equivalently, the sequence $(v_\ell)_{\ell \in \mathbb{N}}$ is bilocally L^1 -bounded if there exist a positive measure μ on M and a continuous norm $|\cdot|$ on the vector bundle $T^*M \otimes TN$ such that

$$\sup_{\ell \in \mathbb{N}} \int_M |v_\ell| d\mu < \infty.$$

The boundedness of the values is expressed in the next condition.

Definition 1.4.2. A sequence $(u_\ell)_{\ell \in \mathbb{N}}$ of maps from M to N is *locally compact in measure* if for every $x \in M$ there exists a local chart $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and for every $\varepsilon > 0$ there exists a compact set $K \subseteq N$ such that for every $\ell \in \mathbb{N}$,

$$\mathcal{L}^m(\psi(u_\ell^{-1}(N \setminus K))) \leq \varepsilon.$$

If N is compact, then this condition is trivially satisfied. In general, a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ of maps from M to N is locally compact in measure if and only if there exist a positive measure μ on N , $y \in N$ and a continuous distance d on N such that

$$\lim_{r \rightarrow \infty} \sup_{\ell \in \mathbb{N}} \mu(u_\ell^{-1}(N \setminus B_r^d(y))) = 0.$$

Proposition 1.4.1 (Compactness in measure). *Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of colocally weakly differentiable maps from M to N . If the sequence $(Du_\ell)_{\ell \in \mathbb{N}}$ is bilocally L^1 -bounded and the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ is locally compact in measure, then there exist a measurable map $u : M \rightarrow N$ and a subsequence $(u_{\ell_k})_{k \in \mathbb{N}}$ that converges to u almost everywhere in M .*

Proof. Let $((U_i, \varphi_i))_{i \in I}$ be a family of extended local charts satisfying the conclusion of lemma 1.2.3 such that $\bigcup_{i \in I} U_i = N$ and I is countable. Assume that $(\eta_i)_{i \in I}$ is a C^1 -partition of the unity subordinate to the covering $(U_i)_{i \in I}$. We observe that in view of lemma 1.2.3, the set $\bar{U}_i$ is compact and hence $\eta_i \in C_c^1(N, \mathbb{R})$.

By assumption and by definition of the colocally weakly differentiability, for every $i \in I$, the sequence $((\varphi_i \circ u_\ell, \eta_i \circ u_\ell))_{\ell \in \mathbb{N}}$ is bounded

in $W_{\text{loc}}^{1,1}(M, \mathbb{R}^{n+1})$. By the classical Rellich–Kondrashov compactness theorem [24, theorem 9.16; 82, theorem 8.9; 120, theorem 6.4.6] and a diagonal argument, there exist a subsequence $(u_{\ell_k})_{k \in \mathbb{N}}$ and a negligible set $E \subset M$ such that the sequence $((\varphi_i \circ u_{\ell_k}(x), \eta_i \circ u_{\ell_k}(x)))_{k \in \mathbb{N}}$ converges in $\mathbb{R}^{n+1}$ for every $x \in M \setminus E$ and every $i \in I$.

We define the set

$$F = \{x \in M \setminus E : \text{for every } i \in I, \lim_{k \rightarrow \infty} \eta_i(u_{\ell_k}(x)) = 0\}.$$

For every compact set $K \subseteq N$, we observe that

$$F \subseteq \bigcup_{j \in \mathbb{N}} \bigcap_{k=j}^{\infty} u_{\ell_k}^{-1}(N \setminus K).$$

Since the sequence $(u_{\ell})_{\ell \in \mathbb{N}}$ is compact in measure, for every $x \in M$, there is a local chart $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and for every $\varepsilon > 0$ there exists a compact set $K \subseteq N$ such that for every $k \in \mathbb{N}$,

$$\mathcal{L}^m(\psi(u_{\ell_k}^{-1}(N \setminus K))) \leq \varepsilon.$$

Therefore,

$$\mathcal{L}^m(\psi(F \cap V)) \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and M can be covered by countably many such charts, we conclude that the set F is negligible.

We conclude by showing that $(u_{\ell_k})_{k \in \mathbb{N}}$ converges everywhere in $M \setminus (E \cup F)$. For every $x \in M \setminus (E \cup F)$, by definition of the set F , there exists $i \in I$ such that $\lim_{k \rightarrow \infty} \eta_i(u_{\ell_k}(x)) > 0$. This implies that for $k \in \mathbb{N}$, large enough, $u_{\ell_k}(x) \in U_i$. Since φ_i is a diffeomorphism on U_i and since the sequence $(\varphi_i(u_{\ell_k}(x)))_{k \in \mathbb{N}}$ converges, we define $u(x) = \varphi_{i|U_i}^{-1}(\lim_{k \rightarrow \infty} \varphi_i(u_{\ell_k}(x)))$ and we conclude that $(u_{\ell_k}(x))_{k \in \mathbb{N}}$ converges to $u(x)$. $\square$

In particular, we have a Rellich–Kondrashov type compactness theorem.

Proposition 1.4.2 (Rellich–Kondrashov for Sobolev maps). *Let $(u_{\ell})_{\ell \in \mathbb{N}}$ be a sequence of colocally weakly differentiable maps from M to N , $v : M \rightarrow N$ be measurable. If (N, d) is complete, if there exist $p \in [1, \infty)$ such that*

$$\sup_{\ell \in \mathbb{N}} \int_M |Du_{\ell}|_{g_M^* \otimes g_N}^p < \infty$$

and $q \in [1, \infty)$ such that

$$\sup_{\ell \in \mathbb{N}} \int_M d(u_{\ell}, v)^q < \infty,$$

then there is a subsequence $(u_{\ell_k})_{k \in \mathbb{N}}$ that converges to $u : M \rightarrow N$ almost everywhere in M .

Proof. Since the metric space (N, d) is complete, for every $y \in N$ and $r \in \mathbb{R}$, the closed ball $\bar{B}_r^N(y)$ is compact. In particular the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ is locally compact in measure.

On the other hand, the sequence (Du_ℓ) is bilocally L^1 -bounded, and therefore by proposition 1.4.1, there exist a measurable map $u : M \rightarrow N$ and a subsequence $(u_{\ell_k})_{k \in \mathbb{N}}$ that converges to u almost everywhere in M . $\square$

1.4.2. Closure property

In order to study the lower semi-continuity properties of functionals, it is interesting to have some sufficient conditions for a limit of colocally weakly differentiable maps to be also colocally weakly differentiable. Such a condition will be useful in the study of stronger notion of convergence.

We shall consider sequences of maps converging in measure.

Definition 1.4.3. A sequence $(u_\ell)_{\ell \in \mathbb{N}}$ of maps from M to N converges locally in measure to a map $u : M \rightarrow N$ if for every $x \in M$ there exists a local chart $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and for every open set $U \subseteq N$,

$$\lim_{\ell \rightarrow \infty} \mathcal{L}^m(\psi((u^{-1}(U) \cap V) \setminus u_\ell^{-1}(U))) = 0.$$

If d is a continuous distance on N and if μ is an absolutely continuous positive finite measure on M , then the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges to u locally in measure if and only if for every $\varepsilon > 0$,

$$\lim_{\ell \rightarrow \infty} \mu(\{x \in M : d(u_\ell(x), u(x)) > \varepsilon\}) = 0.$$

This definition is consistent with the definition of convergence in measure of maps into a metric space [106, définition 5.6.17], which depends only on the topology of the space [106, théorème 5.6.21]. Since the topologies of M and of N have a countable basis, it suffices to check the condition for a countable set of charts $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ and a countable collection of open sets $U \subseteq N$. Hence, the convergence in measure for maps between manifolds induces a metrizable topology. Such a metric is given by

$$\delta(u, v) = \int_M \frac{d(u, v)}{1 + d(u, v)} d\mu. \quad (1.4.4)$$

Any Cauchy sequence with respect to δ has a subsequence which is Cauchy almost everywhere [20, exercise 4.7.60; 106, théorème 5.8.31].

Thus the space of measurable maps from M to N endowed with δ is complete if and only if the space (N, d) is complete.

The definition and the remarks apply directly to a sequence $(v_\ell)_{\ell \in \mathbb{N}}$ of bundle morphisms between TM and TN viewed as maps from M to $T^*M \otimes TN$.

A sufficient condition for colocally weakly differentiability is a uniform integrability assumption.

Definition 1.4.4. A sequence $(v_\ell)_{\ell \in \mathbb{N}}$ of bundle morphisms from TM to TN that covers a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ of maps from M to N is *bilocally uniformly integrable* if for every $x \in M$ and every $y \in N$, there exist local charts $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ and $\varphi : U \subseteq N \rightarrow \mathbb{R}^n$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $W \subseteq \psi(V)$, $\mathcal{L}^m(W) \leq \delta$ and $\ell \in \mathbb{N}$, then

$$\int_{W \cap \psi(u_\ell^{-1}(U))} |D\varphi \circ v_\ell \circ D(\psi^{-1})|_{g_m^* \otimes g_n} \leq \varepsilon.$$

Equivalently, the sequence $(v_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable if there exist a positive absolutely continuous measure μ on M and a continuous norm $|\cdot|$ on the vector bundle $T^*M \otimes TN$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $V \subseteq M$, $\mu(V) \leq \delta$ and $\ell \in \mathbb{N}$, then

$$\int_V |v_\ell| \leq \varepsilon.$$

Proposition 1.4.3 (Closure property). *Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of colocally weakly differentiable maps from M to N . If the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges to $u : M \rightarrow N$ locally in measure and if the sequence $(Du_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable, then the map u is colocally weakly differentiable, and for every map $f \in C_c^1(N, \mathbb{R})$, every local chart $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ and every test function $w \in C_c^1(\psi(V), \mathbb{R}^m)$,*

$$\lim_{\ell \rightarrow \infty} \int_{\psi(V)} \langle D(f \circ u_\ell \circ \psi^{-1}), w \rangle = \int_{\psi(V)} \langle D(f \circ u \circ \psi^{-1}), w \rangle.$$

If moreover the sequence $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to a bundle morphism $v : TM \rightarrow TN$ locally in measure, then $Du = v$.

In particular, under the additional condition of bilocally uniform integrability of the sequence of colocal weak derivatives, the map given by proposition 1.4.1 is colocally weakly differentiable.

Proof of proposition 1.4.3. Following classical argument [120, theorem 6.1.7], given a local chart $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ and $f \in C_c^1(N, \mathbb{R})$, we

define the linear functional $F_{f,\psi}$ on $C_c^1(\psi(V), \mathbb{R}^m)$ for every test function $w \in C_c^1(\psi(V), \mathbb{R}^m)$ by

$$\langle F_{f,\psi}, w \rangle = - \int_{\psi(V)} (f \circ u \circ \psi^{-1}) \operatorname{div} w.$$

Let $K \subset \psi(V)$ be compact. Since the sequence $(f \circ u_\ell \circ \psi^{-1})_{\ell \in \mathbb{N}}$ converges to $f \circ u \circ \psi^{-1}$ in $L^1(K)$, if $\operatorname{supp} w \subset K$,

$$\begin{aligned} |\langle F_{f,\psi}, w \rangle| &= \left| \int_K (f \circ u \circ \psi^{-1}) \operatorname{div} w \right| = \lim_{\ell \rightarrow \infty} \left| \int_K (f \circ u_\ell \circ \psi^{-1}) \operatorname{div} w \right| \\ &= \lim_{\ell \rightarrow \infty} \left| \int_K \langle D(f \circ u_\ell \circ \psi^{-1}), w \rangle \right| \\ &\leq \|w\|_{L^\infty} \liminf_{\ell \rightarrow \infty} \int_K |f|_{\operatorname{Lip}} |Du_\ell|. \end{aligned}$$

Therefore $F_{f,\psi}$ is represented by a vector-valued Radon measure $\mu_{f,\psi}$ on $\psi(V)$:

$$\langle F_{f,\psi}, w \rangle = \int_{\psi(V)} w \, d\mu_{f,\psi}.$$

We observe that for every open set $W \subseteq \psi(V)$,

$$|\mu_{f,\psi}|(W) \leq \liminf_{\ell \rightarrow \infty} \int_W |D(f \circ u_\ell \circ \psi^{-1})|_{g_m^* \otimes g_1}.$$

By the uniform integrability assumption, we conclude that the measure $|\mu_{f,\psi}|$ is absolutely continuous with respect to the Lebesgue measure on every compact set $K \subset \psi(V)$. The measure $\mu_{f,\psi}$ can thus be represented by a vector-field in $L_{\operatorname{loc}}^1(\psi(V))$. In particular the map $f \circ u \circ \psi^{-1}$ is weakly differentiable and

$$\lim_{\ell \rightarrow \infty} \int_{\psi(V)} \langle D(f \circ u_\ell \circ \psi^{-1}), w \rangle = \int_{\psi(V)} \langle D(f \circ u \circ \psi^{-1}), w \rangle.$$

Finally, if $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to a bundle morphism $v : TM \rightarrow TN$ locally in measure, then the sequence $(Df \circ Du_\ell \circ D\psi^{-1})_{\ell \in \mathbb{N}}$ converges to $Df \circ v \circ D\psi^{-1}$ in measure on $\psi(V)$. Since the sequence $(Df \circ Du_\ell \circ D\psi^{-1})_{\ell \in \mathbb{N}}$ is uniformly integrable on every compact set $K \subset \psi(V)$, we conclude that [20, theorem 4.5.4]

$$\lim_{\ell \rightarrow \infty} \int_{\psi(V)} \langle D(f \circ u_\ell \circ \psi^{-1}), w \rangle = \int_{\psi(V)} \langle Df \circ v \circ D\psi^{-1}, w \rangle.$$

Since $f \in C_c^1(N, \mathbb{R})$, the chart $\psi : V \rightarrow \mathbb{R}^m$ and $w \in C_c^1(\psi(V), \mathbb{R}^m)$ are arbitrary, $Du = v$. $\square$

In the particular case of bounded sequences in Sobolev spaces, we have the following closure property, which will play an important role in the completeness of Sobolev spaces.

Proposition 1.4.4 (Weak closure property for Sobolev maps). *Let $p \geq 1$ and let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of colocally weakly differentiable maps from M to N . Assume that the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges to $u : M \rightarrow N$ locally in measure, that*

$$\liminf_{\ell \rightarrow \infty} \int_M |Du_\ell|_{g_M^* \otimes g_N}^p < \infty,$$

and, if $p = 1$, that the sequence $(Du_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable. Then $u \in \dot{W}^{1,p}(M, N)$,

$$\int_M |Du|_{g_M^* \otimes g_N}^p \leq \liminf_{\ell \rightarrow \infty} \int_M |Du_\ell|_{g_M^* \otimes g_N}^p,$$

and for every $f \in C_c^1(N, \mathbb{R})$ and every section $w : M \rightarrow TM$, such that $|w|_{g_M} \in L^{p/(p-1)}(M)$,

$$\lim_{\ell \rightarrow \infty} \int_M \langle D(f \circ u_\ell), w \rangle = \int_M \langle D(f \circ u), w \rangle.$$

If moreover the sequence $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to a bundle morphism $v : TM \rightarrow TN$ locally in measure, then $Du = v$.

The Euclidean counterpart of this property is well-known [120, theorem 6.1.7]. The uniform integrability assumption is essential for $p = 1$: otherwise the closure property fails already for classical Sobolev maps between Euclidean spaces.

Proof of proposition 1.4.4. By the boundedness and bilocally uniform integrability assumptions, the sequence of bundle morphisms $(Du_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable. Proposition 1.4.3 applies and it remains to prove that $|Du|_{g_M^* \otimes g_N} \in L^p(M)$. By the boundedness and bilocally uniform integrability assumptions, up to a subsequence, the sequence of functions $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges weakly to some $w \in L^p(M)$. For every $f \in C_c^1(N, \mathbb{R}^k)$ with $k = \min(m, n)$ and $\ell \in \mathbb{N}$, we have

$$|D(f \circ u_\ell)|_{g_M^* \otimes g_k} \leq |f|_{\text{Lip}} |Du_\ell|_{g_M^* \otimes g_N}.$$

Hence, for every $v \in C_c^1(M, TM \otimes \mathbb{R}^k)$,

$$\begin{aligned} - \int_M \langle f \circ u_\ell, \text{div } v \rangle &= \int_M \langle D(f \circ u_\ell), v \rangle \leq \int_M |D(f \circ u_\ell)|_{g_M^* \otimes g_k} |v|_{g_M^* \otimes g_k} \\ &\leq |f|_{\text{Lip}} \int_M |Du_\ell|_{g_M^* \otimes g_N} |v|_{g_M^* \otimes g_k} \end{aligned}$$

and thus passing to the limit,

$$-\int_M \langle f \circ u, \operatorname{div} v \rangle \leq |f|_{\operatorname{Lip}} \int_M |v|_{g_M^* \otimes g_k} w.$$

Since the map u is colocally weakly differentiable, we deduce that

$$\int_M \langle D(f \circ u), v \rangle \leq |f|_{\operatorname{Lip}} \int_M |v|_{g_M^* \otimes g_k} w.$$

Since $v \in C_c^1(M, TM \otimes \mathbb{R}^k)$ is arbitrary we conclude that

$$|D(f \circ u)|_{g_M^* \otimes g_k} \leq |f|_{\operatorname{Lip}} w$$

almost everywhere in M . By the characterization of the norm of the derivative of proposition 1.3.1, $|Du|_{g_M^* \otimes g_N} \leq w$ almost everywhere in M and thus by lower semicontinuity of the norm under weak convergence [24, proposition 3.5; 82, theorem 2.11; 120, theorem 5.4.6]

$$\int_M |Du|_{g_M^* \otimes g_N}^p \leq \int_M w^p \leq \liminf_{\ell \rightarrow \infty} \int_M |Du_\ell|_{g_M^* \otimes g_N}^p. \quad \square$$

1.5. Strong convergence in Sobolev spaces

In order to characterize eventually $\dot{W}^{1,p}(M, N)$ as a completion of smooth maps in some cases, as it has been done for Sobolev spaces defined in embedded manifolds [17, 61, 64], it is essential to have a metric for which this space is complete.

1.5.1. Definition and properties

We first define a natural notion of convergence in homogeneous Sobolev spaces.

Definition 1.5.1. Let $p \in [1, \infty)$. The sequence $(u_\ell)_{\ell \in \mathbb{N}}$ in $\dot{W}^{1,p}(M, N)$ converges strongly to $u \in \dot{W}^{1,p}(M, N)$ in $\dot{W}^{1,p}(M, N)$ if $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du locally in measure and $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du|_{g_M^* \otimes g_N}$ in $L^p(M)$.

In order to define a distance that gives this convergence, we recall the definition of the Sasaki metric G^S on $T^*M \otimes TN$ when M and N are of class C^2 [102] (see also [42, chapter 3, exercise 2]): for every $\nu \in T(T^*M \otimes TN)$,

$$G^S(\nu) = (g_M \otimes g_N)(D\pi_{M \times N}(\nu)) + (g_M^* \otimes g_N)(K_{T^*M \otimes TN}(\nu)),$$

where $\pi_{M \times N}: T^*M \otimes TN \rightarrow M \times N$ is the canonical bundle morphism, and the connection $K_{T^*M \otimes TN}$ is defined for every $v \in T(T^*M)$ and $w \in T(TN)$ by $K_{T^*M \otimes TN}(v \otimes w) = K_{T^*M}(v) \otimes K_{TN}(w)$, where $K_{T^*M}: T(T^*M) \rightarrow T^*M$ and $K_{TN}: T(TN) \rightarrow TN$ are the respective Levi-Civita connection maps on T^*M and TN [116, definition 3.9]. The associated geodesic distance is denoted by d^S . Given two colocally weakly differentiable maps $u, v: M \rightarrow N$, we note that for almost every $x \in M$,

$$d(u, v)(x) = d^S((x, u(x)), (x, v(x))) \leq d^S(Du(x), Dv(x)), \quad (1.5.5)$$

with the usual identification of $M \times N$ with $M \times N \times \{0\} \subset T^*M \otimes TN$.

As a consequence, the convergence of definition 1.5.1 is induced by the distance

$$\dot{\delta}_{1,p}(u, v) = \delta(Du, Dv) + \| |Du|_{g_M^* \otimes g_N} - |Dv|_{g_M^* \otimes g_N} \|_{L^p(M)},$$

where δ is a distance of the form (1.4.4) taking the continuous distance d^S on $T^*M \otimes TN$. In fact, a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges strongly to u in $\dot{W}^{1,p}(M, N)$ if and only if $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du locally in measure and

$$\lim_{\ell \rightarrow \infty} \int_M |Du_\ell|_{g_M^* \otimes g_N}^p = \int_M |Du|_{g_M^* \otimes g_N}^p.$$

This follows from the Euclidean counterpart [20, proposition 4.7.30; 24, exercise 4.17.3; 120, theorem 4.2.6].

The Sobolev space with this distance is automatically complete.

Proposition 1.5.1. *If N is complete, then the space $(\dot{W}^{1,p}(M, N), \dot{\delta}_{1,p})$ is complete.*

Proof. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a Cauchy sequence for the distance $\dot{\delta}_{1,p}$. By the completeness of $L^p(M)$, there exists a map $w \in L^p(M)$ such that $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to w in $L^p(M)$. By the properties of the distance defined in (1.4.4), there is a subsequence $(Du_{\ell_k})_{k \in \mathbb{N}}$ which is Cauchy almost everywhere with respect to the distance d^S [20, exercise 4.7.60; 106, théorème 5.8.31] and $(|Du_{\ell_k}|_{g_M^* \otimes g_N})_{k \in \mathbb{N}}$ converges to w almost everywhere on M . By the nonexpansiveness property (1.5.5), $(u_{\ell_k})_{k \in \mathbb{N}}$ is a Cauchy sequence with respect to the distance d . Since N is complete, the sequence $(u_{\ell_k})_{k \in \mathbb{N}}$ converges almost everywhere to a measurable map $u: M \rightarrow N$. Since d^S is continuous, it is complete on every compact subset of $T^*M \otimes TN$ [91, theorem 45.1]. Therefore there exists a measurable bundle morphism $v: TM \rightarrow TN$ such that $(Du_{\ell_k})_{k \in \mathbb{N}}$ converges almost everywhere to v and $|v|_{g_M^* \otimes g_N} = w$. By the weak closure property (proposition 1.4.4), $v = Du$. Therefore, $(Du_{\ell_k})_{k \in \mathbb{N}}$ converges almost everywhere to Du and thus with respect to δ . Since the

sequence $(Du_\ell)_{\ell \in \mathbb{N}}$ is Cauchy with respect to the distance δ , the whole sequence $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du with respect to the distance δ . $\square$

This notion of convergence is strong enough to imply the continuity of the pointwise distance in Sobolev spaces.

Proposition 1.5.2. *Let $p \in [1, \infty)$. If the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ in $\dot{W}^{1,p}(M, N)$ converges strongly to $u \in \dot{W}^{1,p}(M, N)$ in $\dot{W}^{1,p}(M, N)$, then the sequence $(d(u_\ell, u))_{\ell \in \mathbb{N}}$ converges strongly to 0 in $\dot{W}^{1,p}(M)$.*

In practice, this proposition allows to deduce the continuity of non-linear Sobolev embedding from their classical linear counterpart.

Proof of proposition 1.5.2. We first observe that for the geodesic distance d , for every $\xi \in TN$,

$$\lim_{\substack{\zeta \rightarrow \xi \\ \pi_N(\zeta) \neq \pi_N(\xi)}} (Dd)(\xi, \zeta) = 0,$$

and that $Du_\ell = Du$ almost everywhere on the set where $u_\ell = u$ (proposition 1.2.8). Therefore, since $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du locally in measure, $(D(d(u_\ell, u)))_{\ell \in \mathbb{N}}$ converges to 0 in measure. Since for every $\ell \in \mathbb{N}$, almost everywhere in M

$$|D(d(u_\ell, u))| \leq |Du_\ell|_{g_M^* \otimes g_N} + |Du|_{g_M^* \otimes g_N},$$

and the sequence $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du|_{g_M^* \otimes g_N}$ in $L^p(M)$, by Lebesgue's dominated convergence theorem, $(D(d(u_\ell, u)))_{\ell \in \mathbb{N}}$ converges to 0 in $L^p(M)$. $\square$

1.5.2. Comparison with other notions of convergence

We first remark that the notion of convergence is stable under isometric embedding. Therefore the convergence of definition 1.5.1 coincides with the classical convergence defined by embedding into Euclidean spaces.

Proposition 1.5.3. *If $\iota : N \rightarrow \tilde{N}$ is an isometric embedding, then the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges strongly to $u \in \dot{W}^{1,p}(M, N)$ in $\dot{W}^{1,p}(M, N)$ if and only if the sequence $(\iota \circ u_\ell)_{\ell \in \mathbb{N}}$ converges strongly to $\iota \circ u \in \dot{W}^{1,p}(M, \tilde{N})$ in $\dot{W}^{1,p}(M, \tilde{N})$.*

Proof. Since ι is an embedding, $(D(\iota \circ u_\ell))_{\ell \in \mathbb{N}}$ converges to $D(\iota \circ u)$ in measure if and only if $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du locally in measure. As ι is isometric, for every $\ell \in \mathbb{N}$,

$$|D(\iota \circ u_\ell)|_{g_M^* \otimes g_{\tilde{N}}} - |D(\iota \circ u)|_{g_M^* \otimes g_{\tilde{N}}} = |Du_\ell|_{g_M^* \otimes g_N} - |Du|_{g_M^* \otimes g_N};$$

the conclusion follows from the definition of convergence. $\square$

The distance $\delta_{1,p}^{\iota}(u, v) = \delta_{1,p}(\iota \circ u, \iota \circ v)$ gives thus the same topology as $\delta_{1,p}$. However the completeness of $\dot{W}^{1,p}(M, N)$ will then depend on the completeness of $\iota(N)$; a necessary condition is that $\iota(N)$ should be closed. When N is complete but not compact, the original Nash embedding theorem will give $\iota(N)$ which is not closed [92, 93]; it is however always possible when N is complete to have a Nash embedding theorem with $\iota(N)$ closed [90].

We would also like to compare this notion with the metric of Chiron [32, §1.6]¹ for $u, v \in \dot{W}^{1,p}(M, N)$:

$$\delta_{1,p}^C(u, v) = \delta(u, v) + \left(\int_M |Du|_{g_M^* \otimes g_N} - |Dv|_{g_M^* \otimes g_N}|^p \right)^{\frac{1}{p}},$$

where δ is a distance of the form (1.4.4). In order to study the topological equivalence of these metrics, we give a criterion of convergence in measure of the derivative.

Proposition 1.5.4 (Criterion for convergence in measure). *Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of colocally weakly differentiable maps from M to N . If the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges locally in measure to a colocally weakly differentiable map $u: M \rightarrow N$ and if the sequence $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du|_{g_M^* \otimes g_N}$ in $L^1_{\text{loc}}(M)$, then the sequence $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du locally in measure.*

As an immediate consequence we have the topological equivalence between $\delta_{1,p}$ and $\delta_{1,p}^C$.

Proposition 1.5.5. *Let $p \in [1, \infty)$. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of maps in $\dot{W}^{1,p}(M, N)$ and $u \in \dot{W}^{1,p}(M, N)$. The sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges strongly to u in $\dot{W}^{1,p}(M, N)$ if and only if $\lim_{\ell \rightarrow \infty} \delta_{1,p}^C(u_\ell, u) = 0$.*

This proposition is due to Chiron when $N = \mathbb{R}$ and $p > 1$. The proof of proposition 1.5.4 relies on the Balder-Visintin criterion of strong convergence.

Proposition 1.5.6 (Balder-Visintin criterion of strong convergence [10, theorem 1; 113, corollary 2]). *Let $(f_\ell)_{\ell \in \mathbb{N}}$ be a sequence in $L^1(U, \mathbb{R}^k)$. If the sequence $(f_\ell)_{\ell \in \mathbb{N}}$ converges weakly to $f \in L^1(U, \mathbb{R}^k)$ and for almost every $x \in U$, the point $f(x)$ is an extreme point of $\bigcap_{j \in \mathbb{N}} \text{co}\{f_i(x) : i \geq j\}$, then the sequence $(f_\ell)_{\ell \in \mathbb{N}}$ converges to f in $L^1(U, \mathbb{R}^k)$.*

¹As the modulus of the derivative has several definitions, we have in fact a family of distances out of which we take the one that uses our notion of modulus of the derivative. Instead of introducing the notion of Lebesgue spaces into metric spaces, we consider convergence in measure for maps from M to N .

For a set $A \subseteq \mathbb{R}^k$, the set $\text{co } A$ denotes the *convex hull* of A , that is, the set of convex combinations of elements of A . A point c is an *extreme point* of a convex set C if $C \setminus \{c\}$ is convex.

Proof of proposition 1.5.4. Let $U \subseteq N$ and $\varphi \in C^1(N, \mathbb{R}^n)$ be the extended local chart given by lemma 1.2.3 and let $K \subseteq M$ be compact. In view of proposition 1.4.3, and the weak compactness criterion in $L^1_{\text{loc}}(M)$, the sequence $(D(\varphi \circ u_\ell))_{\ell \in \mathbb{N}}$ converges weakly to $D(\varphi \circ u)$ in $L^1(K)$.

By taking a subsequence, we can assume that the sequences $(u_\ell)_{\ell \in \mathbb{N}}$ and $(|Du_\ell|)_{\ell \in \mathbb{N}}$ converge almost everywhere in M . In order to apply the Balder–Visintin criterion of strong convergence, we note that for every $x \in M$,

$$D(\varphi \circ u_\ell)(x) \in D\varphi(u_\ell(x))(\bar{B}_{|Du_\ell(x)|_{g_M^* \otimes g_N}})$$

and so

$$\overline{\text{co}\{D(\varphi \circ u_k)(x) : k \geq \ell\}} \subseteq \overline{\text{co}\{D\varphi(u_k(x))(\bar{B}_{|Du_k(x)|_{g_M^* \otimes g_N}}) : k \geq \ell\}}.$$

Hence, since for almost every $x \in K$, $(D\varphi(u_\ell(x)))_{\ell \in \mathbb{N}}$ converges to $D\varphi(u(x))$ and $(|Du_\ell(x)|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du(x)|_{g_M^* \otimes g_N}$, we have

$$\bigcap_{\ell \in \mathbb{N}} \overline{\text{co}\{D(\varphi \circ u_k)(x) : k \geq \ell\}} \subseteq D\varphi(u(x))(\bar{B}_{|Du(x)|_{g_M^* \otimes g_N}}).$$

We finally observe that for every $x \in u^{-1}(U)$, $|Du(x)|_{g_M^* \otimes g_N}$ is an extremal point of $\bar{B}_{|Du(x)|_{g_M^* \otimes g_N}}$ and $D\varphi(u(x))$ is invertible, therefore $D(\varphi \circ u)(x)$ is an extremal point of $D\varphi(u(x))(\bar{B}_{|Du(x)|_{g_M^* \otimes g_N}})$ and hence of $\bigcap_{\ell \in \mathbb{N}} \overline{\text{co}\{D(\varphi \circ u_k)(x) : k \geq \ell\}}$. Hence by the Balder–Visintin criterion of strong convergence (proposition 1.5.6), the sequence $(D(\varphi \circ u_\ell))_{\ell \in \mathbb{N}}$ converges to $D(\varphi \circ u)$ in $L^1(K \cap u^{-1}(U))$ and thus in measure on $K \cap u^{-1}(U)$. By covering M and N by countably many such compact sets K and extended local charts (φ, U) , we obtain the conclusion. $\square$

The reader will observe that our argument relies on the structure of the norm that endows $T^*M \otimes TN$. More precisely our proof requires the norm on $T^*M \otimes TN$ to be uniformly convex; this is not the case when $\min(\dim(M), \dim(N)) \geq 2$ for the operator norm.

Proposition 1.5.7. *Let $p \in (1, \infty)$. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of maps in $\dot{W}^{1,p}(M, N)$ and $u \in \dot{W}^{1,p}(M, N)$. If $(u_\ell)_{\ell \in \mathbb{N}}$ converges to u locally in measure and*

$$\lim_{\ell \rightarrow \infty} \int_M |Du_\ell|_{g_M^* \otimes g_N}^p = \int_M |Du|_{g_M^* \otimes g_N}^p,$$

then $(|Du_\ell|_{g_M^ \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du|_{g_M^* \otimes g_N}$ in $L^p(M)$.*

In particular the two metrics introduced by Chiron have the same convergent sequences [32, lemma 2] for $p > 1$. By proposition 1.5.5, this notion of convergence is also equivalent with convergence in $\dot{W}^{1,p}(M, N)$. When $p = 1$, the equivalence does not hold already in the Euclidean case [32, lemma 2]. The proof relies on the proposition 1.3.1.

Proof of proposition 1.5.7. Since $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^p(M)$ and $p \in (1, \infty)$, by taking a subsequence, we can assume that the sequence $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges weakly to some w in $L^p(M)$. Since $(u_\ell)_{\ell \in \mathbb{N}}$ converges to u locally in measure, for every $f \in C_c^1(M, \mathbb{R}^k)$,

$$|D(f \circ u)|_{g_M^* \otimes g_k} \leq |f|_{\text{Lip}} w$$

almost everywhere in M . Hence, by proposition 1.3.1, $|Du|_{g_M^* \otimes g_N} \leq w$ almost everywhere in M . On the other hand, by lower semicontinuity of the norm under weak convergence [24, proposition 3.5; 82, theorem 2.11; 120, theorem 5.4.6] and by our assumption,

$$\int_M w^p \leq \liminf_{\ell \rightarrow \infty} \int_M |Du_\ell|_{g_M^* \otimes g_N}^p = \int_M |Du|_{g_M^* \otimes g_N}^p$$

and so $w = |Du|_{g_M^* \otimes g_N}$ almost everywhere in M . Thus the sequence $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges weakly to $|Du|_{g_M^* \otimes g_N}$ in $L^p(M)$. Since $p \in (1, \infty)$, we conclude that $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du|_{g_M^* \otimes g_N}$ in $L^p(M)$ [20, corollary 4.7.16; 24, exercise 4.19; 120, exercise 5.3]. $\square$

When $N = \mathbb{R}$, we already known that the Sobolev space is *not complete* under the Chiron distance [32, lemma 2]. We extend this result to Sobolev spaces between Riemannian manifolds.

Proposition 1.5.8. *If M and N are nonempty Riemannian manifolds and $p \in [1, \infty)$, then $(\dot{W}^{1,p}(M, N), \dot{\delta}_{1,p}^C)$ is not complete.*

Proof. We give the proof when $M = (0, 1)$. The reader will adapt the proof to the general case. Choose $\gamma \in C^1([0, L], N)$ such that for every $t \in [0, L]$, $|\gamma'(t)|_{g_N} = 1$ and define, following Chiron [32, lemma 2], for every $\ell \in \mathbb{N}$, the function $u_\ell : (0, 1) \rightarrow N$ for each $t \in (0, 1)$ by

$$u_\ell(t) = \gamma(\text{dist}(t, \mathbb{Z}/\ell)).$$

For every $\ell \in \mathbb{N}$, the function u_ℓ is Lipschitz and $|u'_\ell|_{g_N} = 1$ almost everywhere in $(0, 1)$. Moreover, the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges uniformly to the constant map $u = \gamma(0)$. Since

$$\lim_{\ell \rightarrow \infty} \int_0^1 |u'_\ell|_{g_N}^p = 1 \neq 0 = \int_0^1 |u'|^p,$$

the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ does not converge in $\dot{W}^{1,p}((0, 1), N)$. By proposition 1.5.5, the space $(\dot{W}^{1,p}(M, N), \dot{\delta}_{1,p}^C)$ is not complete. $\square$

1.5.3. Intrinsic distance

A natural candidate for an intrinsic distance is for every $u, v \in \dot{W}^{1,p}(M, N)$:

$$\delta_{1,p}(u, v) = \left(\int_M d^S(Du, Dv)^p \right)^{\frac{1}{p}} \in [0, \infty].$$

This distance can be infinite. This will happen for instance if M has infinite Riemannian volume and u, v are distinct constant maps.

We first note that this distance characterizes Sobolev maps.

Proposition 1.5.9. *If $v \in \dot{W}^{1,p}(M, N)$ then $u \in \dot{W}^{1,p}(M, N)$ and $d(u, v) \in L^p(M)$ if and only if the map u is colocally weakly differentiable and $d^S(Du, Dv) \in L^p(M)$.*

This distance characterizes also the simultaneous convergence in $\dot{W}^{1,p}(M, N)$. In the absence of a Poincaré inequality, the convergence associated to $\delta_{1,p}$ is stronger than the convergence of definition 1.5.1.

Proposition 1.5.10. *If $(u_\ell)_{\ell \in \mathbb{N}}$ is a sequence in $\dot{W}^{1,p}(M, N)$ and $u \in \dot{W}^{1,p}(M, N)$ then the sequence $(d^S(Du_\ell, Du))_{\ell \in \mathbb{N}}$ converges to 0 in $L^p(M)$ if and only if the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges strongly to u in $\dot{W}^{1,p}(M, N)$ and the sequence $(d(u_\ell, u))_{\ell \in \mathbb{N}}$ converges to 0 in $L^p(M)$.*

Proof. First assume that $(u_\ell)_{\ell \in \mathbb{N}}$ converges strongly to u in $\dot{W}^{1,p}(M, N)$ and $(d(u_\ell, u))_{\ell \in \mathbb{N}}$ converges to 0 in $L^p(M)$. By the definition of convergence in $\dot{W}^{1,p}(M, N)$ (definition 1.5.1), $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du|_{g_M^* \otimes g_N}$ in $L^p(M)$ and $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du locally in measure. The latter convergence implies that $(d^S(Du_\ell, Du))_{\ell \in \mathbb{N}}$ converges to 0 in measure. Since for every $\ell \in \mathbb{N}$,

$$d^S(Du_\ell, Du) \leq d(u_\ell, u) + |Du_\ell|_{g_M^* \otimes g_N} + |Du|_{g_M^* \otimes g_N}$$

almost everywhere in M , the conclusion follows from Lebesgue's dominated convergence theorem [20, theorem 2.8.5].

Conversely, if $(d^S(Du_\ell, Du))_{\ell \in \mathbb{N}}$ converges to 0 in $L^p(M)$, then the sequence $(d^S(Du_\ell, Du))_{\ell \in \mathbb{N}}$ converges to 0 in measure and thus $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du locally in measure. Moreover, $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du|_{g_M^* \otimes g_N}$ in measure. Since for every $\ell \in \mathbb{N}$,

$$|Du_\ell|_{g_M^* \otimes g_N} \leq d^S(Du_\ell, Du) + |Du|_{g_M^* \otimes g_N}$$

almost everywhere in M , by Lebesgue's dominated convergence theorem, the sequence $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du|_{g_M^* \otimes g_N}$ in $L^p(M)$. Finally, by the nonexpansiveness property (1.5.5), it is clear that the sequence $(d(u_\ell, u))_{\ell \in \mathbb{N}}$ converges to 0 in $L^p(M)$. $\square$

Finally, Sobolev spaces are complete under this metric.

Proposition 1.5.11. *If N is complete, then the space $(\dot{W}^{1,p}(M, N), \delta_{1,p})$ is complete.*

Proof. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a Cauchy sequence for the metric $\delta_{1,p}$. There exists a subsequence $(Du_{\ell_k})_{k \in \mathbb{N}}$ which is a Cauchy sequence for d^S almost everywhere in M . By the nonexpansiveness property (1.5.5), $(u_{\ell_k})_{k \in \mathbb{N}}$ is a Cauchy sequence for d almost everywhere in M . Since N is complete, the sequence $(u_{\ell_k})_{k \in \mathbb{N}}$ converges almost everywhere to a map $u : M \rightarrow N$.

Since for every $k \in \mathbb{N}$, $|Du_{\ell_k}|_{g_M^* \otimes g_N} \leq d^S(Du_{\ell_k}, Du_{\ell_0}) + |Du_{\ell_0}|_{g_M^* \otimes g_N}$ almost everywhere in M , we deduce that the sequence $(|Du_{\ell_k}|)_{k \in \mathbb{N}}$ is bounded almost everywhere in M and

$$\limsup_{k \rightarrow \infty} \int_M |Du_{\ell_k}|_{g_M^* \otimes g_N}^p < \infty.$$

Since the distance d^S is complete on any compact subset of $T^*M \otimes TN$, the sequence $(Du_{\ell_k})_{k \in \mathbb{N}}$ converges almost everywhere to a bundle morphism $v : TM \rightarrow TN$. By the closure property (proposition 1.4.4), we deduce that $u \in \dot{W}^{1,p}(M, N)$ and $Du = v$. By Fatou's lemma, for every $k \in \mathbb{N}$,

$$\delta_{1,p}(u_{\ell_k}, u) \leq \liminf_{j \rightarrow \infty} \delta_{1,p}(u_{\ell_k}, u_{\ell_j}),$$

and thus $\lim_{k \rightarrow \infty} \delta_{1,p}(u_{\ell_k}, u) = 0$. Since the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ is a Cauchy sequence for $\delta_{1,p}$, the conclusion follows. $\square$

Remark 1.5.1. In the above properties, the Sasaki metric can be generalized to *strongly concordant* (with the Euclidean structure $g_M^* \otimes g_N$) metrics, that is, any metric G on $T(T^*M \otimes TN)$ such that

(a) for every $\nu \in T(T^*M \otimes TN)$,

$$G(D\pi_{M \times N}(\nu)) \leq G(\nu);$$

(b) there exists $\kappa > 0$ such that for every $\nu \in T(T^*M \otimes TN)$,

$$(\kappa^{-1} D(g_M^* \otimes g_N)(\nu))^2 \leq 4(g_M^* \otimes g_N)(\pi_{T^*M \otimes TN}(\nu)) G(\nu),$$

where $\pi_{T^*M \otimes TN} : T(T^*M \otimes TN) \rightarrow T^*M \otimes TN$ is the canonical bundle morphism, and for every $e \in T^*M \otimes TN$,

$$G(\text{Vert}_e(e)) \leq \kappa^2 (g_M^* \otimes g_N)(e),$$

where Vert_e is the vertical lift defined for $\nu \in \pi_{M \times N}^{-1}(\{\pi_{M \times N}(e)\})$ by

$$\text{Vert}_e(\nu) = \left. \frac{d}{dt}(e + t\nu) \right|_{t=0} \in T_e(T^*M \otimes TN).$$

Such metrics are natural metrics on vector bundles [14, 60, 74]. An another example is the Cheeger-Gromoll metric [31].

Given $\iota: N \rightarrow \mathbb{R}^\nu$ an isometric embedding such that $\iota(N)$ is closed, we have two natural distances under the hand: the distance $\delta_{1,p}$ and the induced distance,

$$\delta_{1,p}^\iota(u, v) = \left(\int_M (|\iota \circ u - \iota \circ v|^2 + |D(\iota \circ u) - D(\iota \circ v)|^2)^{\frac{p}{2}} \right)^{\frac{1}{p}}.$$

(The distance on the right is in fact the Sasaki distance on $T^*M \otimes T\mathbb{R}^\nu$.) The next two propositions show that even if they induce the same topology and they are both complete, they are not uniformly equivalent in general.

Proposition 1.5.12. *If M is a nonempty Riemannian manifold, $p \in [1, \infty)$ and $n \geq 2$, then the identity map*

$$i: (\dot{W}^{1,p}(M, \mathbb{S}^n), \delta_{1,p}) \rightarrow (\dot{W}^{1,p}(M, \mathbb{S}^n), \delta_{1,p}^\iota)$$

is not uniformly continuous.

Proof. We begin by considering the case $M = \mathbb{R}$. Choose $y \in \mathbb{S}^n$ and $\rho: \mathbb{S}^n \rightarrow \mathbb{S}^n$ be a non-identical isometry such that $\rho(y) = y$ and $u \in C^1(\mathbb{R}, \mathbb{S}^n)$ such that for every $x \in \mathbb{R} \setminus [-1, 1]$, $\rho(u(x)) = y$ and $\rho(u(x)) \neq u(x)$ in $(-1, 1)$.

Let $u_\lambda: \mathbb{R} \rightarrow \mathbb{S}^n$ be defined for every $t \in \mathbb{R}$ by $u_\lambda(t) = u(t/\lambda)$ and let $v_\lambda = \rho \circ u_\lambda$. Since ρ is an isometry, $D\rho: T\mathbb{S}^n \rightarrow T\mathbb{S}^n$ is an isometry on tangent vectors. Since $n \geq 2$, for every $e \in T\mathbb{S}^n$ there exists a path γ such that $P^\gamma(e) = D\rho(e)$, where P^γ is the parallel transport along γ with respect to the connection $K_{T^*M \otimes T\mathbb{S}^n}$; the length of γ with respect to the Sasaki metric G^S can be bounded uniformly by 2π . Therefore, we have for every $e \in T\mathbb{S}^n$, $d^S(e, D\rho(e)) \leq 2\pi$ and we deduce that

$$\delta_{1,p}(u_\lambda, v_\lambda) = \left(\int_{\mathbb{R}} d^S(Du_\lambda, Dv_\lambda)^p \right)^{\frac{1}{p}} \leq 2\pi \lambda^{\frac{1}{p}}.$$

On the other hand,

$$\delta_{1,p}^\iota(u_\lambda, v_\lambda) = \left(\int_{\mathbb{R}} \lambda (|u - \rho \circ u|^2 + \lambda^{-2} |Du - D(\rho \circ u)|^2)^{\frac{p}{2}} \right)^{\frac{1}{p}},$$

Consequently, $\lim_{\lambda \rightarrow 0} \delta_{1,p}(u_\lambda, v_\lambda) = 0$ while $\liminf_{\lambda \rightarrow 0} \delta_{1,p}^\iota(u_\lambda, v_\lambda) = \infty$.

In the general case, let $a \in M$ and let $r \in (0, r_i(a))$ where $r_i(a)$ is the injectivity radius of the Riemannian manifold M at the point a . We

define then the maps $u_\lambda : M \rightarrow \mathbb{S}^n$ and $v_\lambda : M \rightarrow \mathbb{S}^n$ for $\lambda > 0$ and $x \in M$ by

$$u_\lambda(x) = u\left(\frac{d(x, a) - r}{\lambda}\right) \quad \text{and} \quad v_\lambda(x) = \rho \circ u\left(\frac{d(x, a) - r}{\lambda}\right);$$

the non uniform continuity follows as in the case $M = \mathbb{R}$ treated above. $\square$

Proposition 1.5.13. *If M is a nonempty Riemannian manifold and $p \in [1, \dim M)$, then the identity embedding map*

$$i: (\dot{W}^{1,p}(M, \mathbb{S}^1), \delta_{1,p}^\ell) \rightarrow (\dot{W}^{1,p}(M, \mathbb{S}^1), \delta_{1,p})$$

is not uniformly continuous.

Proof. We define $u_\lambda = (\cos(\varphi_\lambda), \sin(\varphi_\lambda))$ and $v_\lambda = (-\cos(\varphi_\lambda), \sin(\varphi_\lambda))$, where

$$\varphi_\lambda(x) = \begin{cases} \frac{\pi}{2} & \text{if } d(x, y) \geq 2\lambda, \\ \frac{d(x, y) - \lambda}{2\lambda} \pi & \text{if } \lambda \leq d(x, y) < 2\lambda, \\ \lambda \sin \frac{(d(x, y) - \lambda)\pi}{2\lambda^{1 + \frac{\dim M}{p}}} & \text{if } d(x, y) < \lambda, \end{cases}$$

and the point $y \in M$ is fixed. We observe that

$$\liminf_{\lambda \rightarrow 0} \delta_{1,p}(u_\lambda, v_\lambda) = 2 \liminf_{\lambda \rightarrow 0} \left(\int_M (\cos(\varphi_\lambda)^2 + |D\varphi_\lambda|^2)^{\frac{p}{2}} \right)^{\frac{1}{p}} > 0.$$

On the other hand, since $p < \dim M$,

$$\limsup_{\lambda \rightarrow 0} \delta_{1,p}^\ell(u_\lambda, v_\lambda) = 2 \limsup_{\lambda \rightarrow 0} \left(\int_M (\cos(\varphi_\lambda)^2 + \sin(\varphi_\lambda)^2 |D\varphi_\lambda|^2)^{\frac{p}{2}} \right)^{\frac{1}{p}} = 0. \quad \square$$

Chapter II

Concordant distances and metrics

2.1. Introduction

Given two Riemannian manifolds M and N of class C^2 , we define in the previous chapter a natural intrinsic distance on Sobolev spaces between the manifolds M and N (section 1.5.3). This intrinsic distance is defined from the geodesic distance induced by the Sasaki metric on $T^*M \otimes TN$ (section 1.5.1) which belongs to the class of *concordant distances*. This fact is mentioned in chapter I but we did not go further (remark 1.5.1). However, this notion allows us to define a class of intrinsic distances on Sobolev spaces that behave well.

As a consequence, in the present chapter, we first define and study *concordant distances* on any vector bundles (definition 2.2.1). We also give two examples of *strongly concordant metrics* (definition 2.2.2), namely the *Sasaki* and *Cheeger–Gromoll* metrics, that give rise to (strongly) concordant distances (definition 2.2.3).

In a second step, given a concordant distance d on the vector bundle $T^*M \otimes TN$, we define an intrinsic distance on Sobolev spaces $\dot{W}^{1,p}(M, N)$, that is, for every $u, v \in \dot{W}^{1,p}(M, N)$,

$$\delta_{1,p}(u, v) = \left(\int_M d(Du, Dv)^p \right)^{\frac{1}{p}}.$$

Concordant distances characterize Sobolev maps and also simultaneous convergence in Sobolev spaces (propositions 2.3.1 and 2.3.2). Furthermore, Sobolev spaces are complete under all these distances (proposition 2.3.3).

2.2. Definitions and examples

The notion of concordant distances is first defined for any vector bundles.

Definition 2.2.1. A continuous distance d_E on a normed topological vector bundle (E, π_M, M) is *concordant* with the norm $|\cdot|_E$ if

(a) (nonexpansiveness of the fiber projection) for every $e_1, e_2 \in E$,

$$d_E(\pi_M(e_1), \pi_M(e_2)) \leq d_E(e_1, e_2);$$

(b) (equivalence with the norm) there exists $\kappa > 0$ such that for every $e \in E$,

$$\kappa^{-1} |e|_E \leq d_E(e, \pi_M(e)) \leq \kappa |e|_E.$$

Here and in the sequel we identify M with the zero fiber subbundle $M \times \{0\} \subset E$.

Proposition 2.2.1 (Stability of subbundles). *Let (E, π_M, M) and $(\tilde{E}, \pi_{\tilde{M}}, \tilde{M})$ be vector bundles and let $\bar{\iota}: E \rightarrow \tilde{E}$ be an injective bundle morphism that covers an embedding $\iota: M \rightarrow \tilde{M}$. If $d_{\tilde{E}}$ is concordant with $|\cdot|_{\tilde{E}}$, then the distance d_E defined for $e_1, e_2 \in E$ by*

$$d_E(e_1, e_2) = d_{\tilde{E}}(\bar{\iota}(e_1), \bar{\iota}(e_2))$$

is concordant with $|\cdot|_E$ defined for $e \in E$ by $|e|_E = |\bar{\iota}(e)|_{\tilde{E}}$.

If E is a differentiable manifold, we study Riemannian metrics G_E on TE that give rise to a distance concordant with $|\cdot|_{g_E}$. We denote by G_E and g_E the quadratic forms associated to the corresponding metrics. We recall that for every $e \in E$, the *vertical lift* $\text{Vert}_e: E \rightarrow TE$ is defined for each $\nu \in \pi_M^{-1}(\{\pi_M(e)\})$ by

$$\text{Vert}_e(\nu) = \left. \frac{d}{dt}(e + t\nu) \right|_{t=0} \in T_e E.$$

Definition 2.2.2. A metric G_E is *strongly concordant* with the Euclidean structure g_E if

(a) for every $\nu \in TE$,

$$G_E(D\pi_M(\nu)) \leq G_E(\nu);$$

(b) there exists $\kappa > 0$ such that for every $\nu \in TE$,

$$(\kappa^{-1} Dg_E(\nu))^2 \leq 4g_E(\pi_E(\nu)) G_E(\nu),$$

and for every $e \in E$,

$$G_E(\text{Vert}_e(e)) \leq \kappa^2 g_E(e).$$

Strongly concordant metrics induce concordant distances. In fact, these distances have the stronger property of strong concordance.

Definition 2.2.3. A continuous distance d_E on a normed topological vector bundle (E, π_M, M) is *strongly concordant* with the norm $|\cdot|_E$ if

(a) (nonexpansiveness of the fiber projection) for every $e_1, e_2 \in E$,

$$d_M(\pi_M(e_1), \pi_M(e_2)) \leq d_E(e_1, e_2);$$

(b) (comparability with the norm) there exists $\kappa > 0$ such that for every $s, t \in \mathbb{R}$ and every $e \in E$,

$$d_E(te, se) \leq \kappa |te - se|_E,$$

and for every $e_1, e_2 \in E$,

$$||e_1|_E - |e_2|_E| \leq \kappa d_E(e_1, e_2).$$

Proposition 2.2.2. *The metric G_E is strongly concordant with g_E if and only if the geodesic distance d_E induced by the metric G_E is strongly concordant with $|\cdot|_{g_E}$.*

Proof. Let $e_1, e_2 \in E$ and let $\gamma \in C^1([0, 1], E)$ be a path such that $\gamma(0) = e_1$ and $\gamma(1) = e_2$. Since G_E is strongly concordant with g_E ,

$$\int_0^1 \sqrt{G_E(D\pi_M(\gamma'(\tau)))} d\tau \leq \int_0^1 \sqrt{G_E(\gamma'(\tau))} d\tau,$$

and the divergence property of the fibers follows.

Let $e \in E$ and let $\gamma \in C^1([0, 1], E)$ be defined for every $\tau \in [0, 1]$ by $\gamma(\tau) = (\tau t + (1 - \tau)s)e$. By definition of the vertical lift, for every $\tau \in [0, 1]$, we have $\gamma'(\tau) = \text{Vert}_{\gamma(\tau)}((t - s)e) = \tau \text{Vert}_{(t-s)e}((t - s)e)$. Consequently, for every $s, t \in \mathbb{R}$,

$$\begin{aligned} d_E(se, te) &\leq \int_0^1 \sqrt{G_E(\gamma'(\tau))} d\tau \\ &\leq \int_0^1 \sqrt{G_E(\text{Vert}_{(t-s)e}((t - s)e))} d\tau \leq \kappa \sqrt{g_E((t - s)e)}. \end{aligned}$$

Let $\gamma \in C^1([0, 1], E)$ be a path such that $\gamma(0) = e_1$ and $\gamma(1) = e_2$. By assumption,

$$\begin{aligned} \sqrt{g_E(e_1)} - \sqrt{g_E(e_2)} &= \int_0^1 \frac{d}{d\tau} \sqrt{g_E(\gamma(\tau))} d\tau = \int_0^1 \frac{Dg_E(\gamma'(\tau))}{2\sqrt{g_E(\gamma(\tau))}} d\tau \\ &\leq \kappa \int_0^1 \sqrt{G_E(\gamma'(\tau))} d\tau, \end{aligned}$$

so $||e_1|_{g_E} - |e_2|_{g_E}| \leq \kappa d_E(e_1, e_2)$. □

The Sasaki [102] and the Cheeger–Gromoll metrics [31], which are two classical constructions of natural metrics on bundles [60, 74], are strongly concordant.

In order to define these metrics, we endow M with a Riemannian metric g_M and we endow E with a metric connection $K_E: TE \rightarrow E$. We recall that K_E is a *connection* if K_E is a bundle morphism from (TE, π_E, E) to (E, π_M, M) that covers π_M and a bundle morphism from $(TE, D\pi_M, TM)$ to (E, π_M, M) that covers $\pi_M^{TM}: TM \rightarrow M$, and for every $e \in E$, $K_E \circ \text{Vert}_e = \text{id}$ on $\pi_M^{-1}(\{\pi_M(e)\})$ [116, definition 3.9]. The connection K_E is *metric* if for every $\nu \in TE$,

$$Dg_E(\nu) = 2g_E(K_E(\nu), \pi_E(\nu)).$$

The *Sasaki metric* G_E^S [102] (see also [42, chapter 3, exercise 2]) is defined for every $\nu \in TE$ by

$$G_E^S(\nu) = g_M(D\pi_M(\nu)) + g_E(K_E(\nu)).$$

Proposition 2.2.3. *The Sasaki metric G_E^S is strongly concordant with g_E .*

Proof. For every $\nu \in TE$, since $K_E(D\pi_M(\nu)) = 0$,

$$G_E^S(D\pi_M(\nu)) = g_M(D\pi_M(\nu)) \leq G_E^S(\nu).$$

Next, since the connection K_E is metric, for every $\nu \in TE$,

$$\begin{aligned} (Dg_E(\nu))^2 &= 4(g_E(K_E(\nu), \pi_E(\nu)))^2 \\ &\leq 4g_E(K_E(\nu))g_E(\pi_E(\nu)) \leq 4G_E^S(\nu)g_E(\pi_E(\nu)). \end{aligned}$$

Finally, for every $e \in E$, since $D\pi_M(\text{Vert}_e(e)) = 0$ and $K_E(\text{Vert}_e(e)) = e$, we conclude that

$$G_E^S(\text{Vert}_e(e)) = g_E(e). \quad \square$$

The *Cheeger–Gromoll metric* is defined for every $\nu \in TE$ by

$$G_E^{CG}(\nu) = g_M(D\pi_M(\nu)) + \frac{g_E(K_E(\nu)) + (g_E(K_E(\nu), \pi_E(\nu)))^2}{1 + g_E(\pi_E(\nu))}.$$

Proposition 2.2.4. *The Cheeger–Gromoll metric G_E^{CG} is strongly concordant with g_E .*

Proof. For every $\nu \in TE$, we have

$$G_E^{CG}(D\pi_M(\nu)) = g_M(D\pi_M(\nu)) \leq G_E^S(\nu),$$

and since the connection K_E is metric,

$$\begin{aligned}
 (Dg_E(\nu))^2 &= 4(g_E(K_E(\nu), \pi_E(\nu)))^2 \\
 &\leq 4\left(\frac{g_E(K_E(\nu))g_E(\pi_E(\nu))}{1 + g_E(\pi_E(\nu))} + \frac{(g_E(K_E(\nu), \pi_E(\nu)))^2 g_E(\pi_E(\nu))}{1 + g_E(\pi_E(\nu))}\right) \\
 &\leq 4G_E^{CG}(\nu)g_E(\pi_E(\nu)).
 \end{aligned}$$

Finally, for every $e \in E$, since $\pi_E(\text{Vert}_e(e)) = e$, $D\pi_M(\text{Vert}_e(e)) = 0$ and $K_E(\text{Vert}_e(e)) = e$, we obtain

$$G_E^{CG}(\text{Vert}_e(e)) = \frac{g_E(e) + g_E(e)^2}{1 + g_E(e)} = g_E(e). \quad \square$$

Finally, we compare the two previous strongly concordant metrics.

Lemma 2.2.5 (Comparison between the Cheeger–Gromoll and Sasaki metrics). *For every $\nu \in TE$,*

$$G_E^{CG}(\nu) \leq G_E^S(\nu).$$

Proof. By the Cauchy–Schwarz inequality, for every $\nu \in TE$,

$$(g_E(K_E(\nu), \pi_E(\nu)))^2 \leq g_E(K_E(\nu))g_E(\pi_E(\nu)). \quad \square$$

The Sasaki and Cheeger–Gromoll metrics are particular cases for $\lambda = 1$ and $\lambda = \frac{1}{2}$ of metrics defined for every $\nu \in TE$ by

$$G_E^\lambda(\nu) = g_M(D\pi_M(\nu)) + \frac{\lambda g_E(K_E(\nu)) + (1 - \lambda)(g_E(K_E(\nu), \pi_E(\nu)))^2}{\lambda + (1 - \lambda)g_E(\pi_E(\nu))}.$$

In the limit case $\lambda = 0$, the metric G_E^0 such that for every $\nu \in TE$,

$$G_E^0(\nu) = g_M(D\pi_M(\nu)) + \frac{(g_E(K_E(\nu), \pi_E(\nu)))^2}{g_E(\pi_E(\nu))},$$

is degenerate; the associated degenerate distance is defined for every $e_1, e_2 \in E$ by

$$d_E(e_1, e_2)^2 = d_M(\pi_M(e_1), \pi_M(e_2))^2 + ||e_1|_{g_E} - |e_2|_{g_E}|^2.$$

2.3. Intrinsic Sobolev distances

In our study of Sobolev spaces, we assume that (M, g_M) and (N, g_N) are Riemannian manifolds. In particular the metrics on vectors of TM and TN induce a metric $g_M^* \otimes g_N$ on $T^*M \otimes TN$. This metric can be computed for every $\xi \in T_x^*M \otimes T_yN = \mathcal{L}(T_xM, T_yN)$ by

$$(g_M^* \otimes g_N)(\xi, \xi) = \sum_{i=1}^m g_N(\xi(e_i), \xi(e_i)),$$

where $(e_i)_{1 \leq i \leq m}$ is an orthonormal basis in $\pi_M^{-1}(\{x\})$ with respect to the Riemannian metric g_M .

Given a concordant distance d , we define for every $p \in [1, +\infty)$, the distance $\delta_{1,p}$ for every $u, v \in \dot{W}^{1,p}(M, N)$ by

$$\delta_{1,p}(u, v) = \left(\int_M d(Du, Dv)^p \right)^{\frac{1}{p}} \in [0, +\infty].$$

This distance can be infinite. This will happen for instance if M has infinite Riemannian volume and u, v are distinct constant maps.

We first prove that concordant distances characterize Sobolev maps. Given two maps $u, v : M \rightarrow N$, we denote for every $x \in M$,

$$d(u, v)(x) = d((x, u(x)), (x, v(x))),$$

with the usual identification of $M \times N$ with $M \times N \times \{0\} \subset T^*M \otimes TN$.

Proposition 2.3.1. *Assume that d is concordant with $|\cdot|_{g_M^* \otimes g_N}$. If $v \in \dot{W}^{1,p}(M, N)$, then $u \in \dot{W}^{1,p}(M, N)$ and $d(u, v) \in L^p(M)$ if and only if the map u is colocally weakly differentiable and $d(Du, Dv) \in L^p(M)$.*

Proof. The first assumption follows from the fact that

$$\begin{aligned} d(Du, Dv) &\leq d(Du, u) + d(u, v) + d(v, Dv) \\ &\leq d(u, v) + \kappa(|Du|_{g_M^* \otimes g_N} + |Dv|_{g_M^* \otimes g_N}) \end{aligned}$$

almost everywhere in M . Conversely we observe that, by the nonexpansiveness property, $d(u, v) \leq d(Du, Dv)$ almost everywhere in M and that by the equivalence assumption,

$$\begin{aligned} |Du|_{g_M^* \otimes g_N} &\leq \kappa d(Du, u) \leq \kappa(d(Du, Dv) + d(Dv, v)) \\ &\leq \kappa d(Du, Dv) + \kappa^2 |Dv|_{g_M^* \otimes g_N} \end{aligned}$$

almost everywhere in M ; the converse statement follows. $\square$

Concordant distances characterize also the simultaneous convergence in Sobolev spaces.

Proposition 2.3.2. *Assume that d is concordant with $|\cdot|_{g_M^* \otimes g_N}$. If $(u_\ell)_{\ell \in \mathbb{N}}$ is a sequence in $\dot{W}^{1,p}(M, N)$ and $u \in \dot{W}^{1,p}(M, N)$, then the sequence $(d(Du_\ell, Du))_{\ell \in \mathbb{N}}$ converges to 0 in $L^p(M)$ if and only if the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges strongly to u in $\dot{W}^{1,p}(M, N)$ and the sequence $(d(u_\ell, u))_{\ell \in \mathbb{N}}$ converges to 0 in $L^p(M)$.*

Proof. First assume that $(u_\ell)_{\ell \in \mathbb{N}}$ converges strongly to u in $\dot{W}^{1,p}(M, N)$ and $(d(u_\ell, u))_{\ell \in \mathbb{N}}$ converges to 0 in $L^p(M)$. By the definition of convergence in $\dot{W}^{1,p}(M, N)$ (definition 1.5.1), $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du|_{g_M^* \otimes g_N}$ in $L^p(M)$ and $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du locally in measure. The latter convergence implies that $(d(Du_\ell, Du))_{\ell \in \mathbb{N}}$ converges to 0 in measure. Since for every $\ell \in \mathbb{N}$,

$$d(Du_\ell, Du) \leq d(u_\ell, u) + \kappa(|Du_\ell|_{g_M^* \otimes g_N} + |Du|_{g_M^* \otimes g_N})$$

almost everywhere in M , the conclusion follows from Lebesgue's dominated convergence theorem [20, theorem 2.8.5].

Conversely, if $(d(Du_\ell, Du))_{\ell \in \mathbb{N}}$ converges to 0 in $L^p(M)$, then the sequence $(d(Du_\ell, Du))_{\ell \in \mathbb{N}}$ converges to 0 in measure and thus $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du locally in measure. Moreover, $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du|_{g_M^* \otimes g_N}$ in measure. Since for every $\ell \in \mathbb{N}$,

$$|Du_\ell|_{g_M^* \otimes g_N} \leq \kappa d(Du_\ell, Du) + \kappa^2 |Du|_{g_M^* \otimes g_N}$$

almost everywhere in M , by Lebesgue's dominated convergence theorem, the sequence $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du|_{g_M^* \otimes g_N}$ in $L^p(M)$. Finally, since for every $\ell \in \mathbb{N}$, $d(u_\ell, u) \leq d(Du_\ell, Du)$, it is clear that the sequence $(d(u_\ell, u))_{\ell \in \mathbb{N}}$ converges to 0 in $L^p(M)$. $\square$

Sobolev spaces are complete under all these metrics.

Proposition 2.3.3. *If for almost every $x \in M$, the metric space $(\{x\} \times N, d)$ is complete and d is concordant with $|\cdot|_{g_M^* \otimes g_N}$, then the Sobolev space $(\dot{W}^{1,p}(M, N), \delta_{1,p})$ is complete.*

Proof. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a Cauchy sequence for the metric $\delta_{1,p}$. There exists a subsequence $(Du_{\ell_i})_{i \in \mathbb{N}}$ which is a Cauchy sequence for d almost everywhere in M . By the nonexpansiveness of the projection, $(u_{\ell_i})_{i \in \mathbb{N}}$ is a Cauchy sequence for d almost everywhere in M . Since for almost every $x \in M$, the metric space $(\{x\} \times N, d)$ is complete, the sequence $(u_{\ell_i})_{i \in \mathbb{N}}$ converges almost everywhere to a map $u : M \rightarrow N$.

Since the distance d is concordant, for every $i \in \mathbb{N}$,

$$|Du_{\ell_i}|_{g_M^* \otimes g_N} \leq \kappa d(Du_{\ell_i}, Du_{\ell_0}) + \kappa^2 |Du_{\ell_0}|_{g_M^* \otimes g_N}$$

almost everywhere in M and we deduce that $(|Du_{\ell_i}|_{g_M^* \otimes g_N})_{i \in \mathbb{N}}$ is bounded almost everywhere in M and

$$\limsup_{i \rightarrow \infty} \int_M |Du_{\ell_i}|_{g_M^* \otimes g_N}^p < +\infty.$$

Since the distance d is continuous, it is complete on any compact subset of $T^*M \otimes TN$ [91, theorem 45.1] and so the sequence $(Du_{\ell_i})_{i \in \mathbb{N}}$ converges almost everywhere to a bundle morphism $v : TM \rightarrow TN$. By the closure property (proposition 1.4.4), we deduce that $u \in \dot{W}^{1,p}(M, N)$ and $Du = v$. By Fatou's lemma, for every $i \in \mathbb{N}$,

$$\delta_{1,p}(u_{\ell_i}, u) \leq \liminf_{j \rightarrow \infty} \delta_{1,p}(u_{\ell_i}, u_{\ell_j}),$$

and thus $\lim_{i \rightarrow \infty} \delta_{1,p}(u_{\ell_i}, u) = 0$. Since the sequence $(u_{\ell})_{\ell \in \mathbb{N}}$ is a Cauchy sequence for $\delta_{1,p}$, the conclusion follows. $\square$

Finally, we assume that the Riemannian manifolds are of class C^2 and that we have metric connections K_{T^*M} and K_{TN} on T^*M and TN respectively. Such connections are given by the Levi-Civita connection. We can then define for every $v \in T(T^*M)$ and every $w \in T(TN)$,

$$K_{T^*M \otimes TN}(v \otimes w) = K_{T^*M}(v) \otimes K_{TN}(w).$$

Moreover, the connection $K_{T^*M \otimes TN}$ is metric with $g_M^* \otimes g_N$.

As a consequence, we have two examples of intrinsic distance on $\dot{W}^{1,p}(M, N)$ when $T^*M \otimes TN$ is equipped with a concordant distance: the Sasaki distance

$$\delta_{1,p}^S(u, v) = \left(\int_M (d^S(Du, Dv))^p \right)^{\frac{1}{p}},$$

where d^S is the Sasaki geodesic distance on $T^*M \otimes TN$, and the Cheeger–Gromoll distance

$$\delta_{1,p}^{CG}(u, v) = \left(\int_M (d^{CG}(Du, Dv))^p \right)^{\frac{1}{p}},$$

where d^{CG} is the Cheeger–Gromoll geodesic distance on $T^*M \otimes TN$. It follows from the previous results that the Sobolev space $\dot{W}^{1,p}(M, N)$ has the same topology for both distances and is complete. In view

of the comparison between the Cheeger–Gromoll and Sasaki metrics (lemma 2.2.5), we have

$$\delta_{1,p}^{CG}(u, v) \leq \delta_{1,p}^S(u, v),$$

that is, the identity is nonexpansive from the space $(\dot{W}^{1,p}(M, N), \delta_{1,p}^S)$ to $(\dot{W}^{1,p}(M, N), \delta_{1,p}^{CG})$. However, the distance $\delta_{1,p}^S$ is not uniformly equivalent with $\delta_{1,p}^{CG}$ in general.

Proposition 2.3.4. *If M is a nonempty Riemannian manifold, $p \in [1, +\infty)$ and $n \geq 2$, then the identity map*

$$i: (\dot{W}^{1,p}(M, \mathbb{R}^n), \delta_{1,p}^{CG}) \rightarrow (\dot{W}^{1,p}(M, \mathbb{R}^n), \delta_{1,p}^S)$$

is not uniformly continuous.

Proof. First we consider the case where $M = \mathbb{R}$. We choose a map $u \in C_c^1(\mathbb{R}, \mathbb{R}^n) \setminus \{0\}$ such that $\text{supp } u \subset (-1, 1)$, and we define for $\lambda > 0$ the maps $u_\lambda: \mathbb{R} \rightarrow \mathbb{R}^n$ and $v_\lambda: \mathbb{R} \rightarrow \mathbb{R}^n$ for each $t \in \mathbb{R}$ by $u_\lambda(t) = u(t/\lambda)$ and $v_\lambda(t) = -u_\lambda(t) = -u(t/\lambda)$. We have for every $t \in \mathbb{R}$, since $n \geq 2$ and $|u'_\lambda(t)| = |v'_\lambda(t)|$,

$$d^{CG}(u'_\lambda(t), v'_\lambda(t))^2 \leq 4|u_\lambda(t)|^2 + \frac{|u'_\lambda(t)|^2 \pi^2}{1 + |u'_\lambda(t)|^2} \leq 4|u_\lambda(t)|^2 + \pi^2.$$

Therefore, for every $\lambda > 0$,

$$\delta_{1,p}^{CG}(u_\lambda, v_\lambda) = \left(\int_{-\lambda}^{\lambda} d^{CG}(u'_\lambda, v'_\lambda)^p \right)^{\frac{1}{p}} \leq \sqrt[4]{4\|u\|_{L^\infty}^2 + \pi^2} (2\lambda)^{\frac{1}{p}}.$$

On the other hand, since $\mathbb{R}^n$ is flat, we have for each $\lambda > 0$,

$$\begin{aligned} \delta_{1,p}^S(u_\lambda, v_\lambda) &= \left(\int_{-\lambda}^{\lambda} d^S(u'_\lambda, v'_\lambda)^p \right)^{\frac{1}{p}} \\ &= \left(\int_{-\lambda}^{\lambda} (|u_\lambda - v_\lambda|^2 + |u'_\lambda - v'_\lambda|^2)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &= 2 \left(\int_{-1}^1 (\lambda^{\frac{2}{p}} |u|^2 + \lambda^{-2(1-\frac{1}{p})} |u'|^2)^{\frac{p}{2}} \right)^{\frac{1}{p}}. \end{aligned}$$

We have thus

$$\lim_{\lambda \rightarrow 0} \delta_{1,p}^{CG}(u_\lambda, v_\lambda) = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow 0} \delta_{1,p}^S(u_\lambda, v_\lambda) > 0,$$

which shows the non uniform continuity.

In the general case, we proceed as in the proof of proposition 1.5.12.

□

Chapter III

Geometric partial differentiability on manifolds: the tangential derivative and the chain rule

3.1. Introduction

Given a differentiable manifold M , we are interested in defining a geometric partial differentiation for functions from M to $\mathbb{R}$, that is, we would like to have a definition that implies classical directional differentiability when $M = \mathbb{R}^m$ and that *does not depend on local charts*. Equivalently we aim to define on $\mathbb{R}^m$ a notion of partial differentiability which is invariant under diffeomorphisms.

Let us first recall the classical definition of directional derivative which does not have this property.

Definition 3.1. Let $A \subseteq \mathbb{R}^m$ and let $V \subseteq \mathbb{R}^m$ be a linear subspace of $\mathbb{R}^m$. A linear map $L \in \mathcal{L}(V, \mathbb{R}^n)$ is a *directional derivative* of a function $f: A \rightarrow \mathbb{R}^n$ at $a \in A$ with respect to V if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in A \cap (V + a)$ and $\|x - a\| \leq \delta$, then

$$\|f(x) - f(a) - L[x - a]\| \leq \varepsilon \|x - a\|.$$

The definition of directional derivative can be rephrased in terms of limits as

$$\lim_{\substack{x \in V+a \\ x \rightarrow a}} \frac{\|f(x) - f(a) - L[x - a]\|}{\|x - a\|} = 0.$$

The class of differentiable maps $\psi: \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that the directional differentiability of each $f: A \rightarrow \mathbb{R}^n$ at $\psi(a)$ with respect to $D\psi(a)[V]$, where $D\psi(a)[V]$ is the image of V by the classical total derivative of ψ at a , implies the directional differentiability of $f \circ \psi$ at a with respect to V consists of the maps $\psi: \mathbb{R}^m \rightarrow \mathbb{R}^m$ for which there exists $\eta > 0$ such that $\psi((V + a) \cap B[a, \eta]) \subseteq \psi(a) + D\psi(a)[V]$.

The stronger notion that answers our question is provided by the tangential derivative that we define as follows.

Definition 3.2. Let $A \subseteq \mathbb{R}^m$ and let $V \subseteq \mathbb{R}^m$ be a linear subspace of $\mathbb{R}^m$. A linear map $L \in \mathcal{L}(V, \mathbb{R}^n)$ is a *tangential derivative* of a function $f: A \rightarrow \mathbb{R}^n$ at $a \in A$ with respect to V if for every $\varepsilon > 0$ there exist $\delta > 0$ and $\theta > 0$ such that if $x \in A$, $v \in V$, $\|x - a\| \leq \delta$ and $\|x - a - v\| \leq \theta\|x - a\|$, then

$$\|f(x) - f(a) - L[v]\| \leq \varepsilon\|x - a\|.$$

This new definition can be written in terms of limits as

$$\lim_{\substack{v \in V \\ x \rightarrow a \\ \frac{\|x - a - v\|}{\|x - a\|} \rightarrow 0}} \frac{\|f(x) - f(a) - L[v]\|}{\|x - a\|} = 0.$$

Geometrically, the tangential derivative considers the behavior of f in sharp cones around the plane V . In particular, if the function f is tangentially differentiable, then the function $f - f(a)$ grows linearly in a small sharp cone around the plane V :

$$\limsup_{\substack{x \rightarrow a \\ \frac{\text{dist}(x - a, V)}{\|x - a\|} \rightarrow 0}} \frac{\|f(x) - f(a)\|}{\|x - a\|} < \infty. \quad (3.1)$$

Condition (3.1) is reminiscent of, but weaker than, the notion of directionally Lipschitz functions [100, 101]. (If $V = \mathbb{R}^m$, this directional Lipschitz condition is equivalent to the classical local Lipschitz condition.)

Existing definitions either impose a uniform condition over all the directions (classical Fréchet–Stolz differentiability), or a condition in each direction individually (Hadamard differentiability [107, definition 2.2]). Here we impose a uniform condition in only certain directions. So here nothing is known in some directions. And we investigate the effect of those missing directions.

We show that this definition has the required invariance under diffeomorphisms (corollary 3.3.3) and allows thus to define a tangential derivative on a differentiable manifold (definition 3.3.1). Although directional differentiability does not imply tangential differentiability in general (example 3.2.1), it is the case in the class of Lipschitz continuous functions (proposition 3.2.2). This brings us to the original motivation of our work, which was to understand whether the directional derivative in the chain rule between Lipschitz continuous and weakly differentiable functions [4] was well defined when composing maps between manifolds (section 3.5).

We also show that the tangential derivative is the weakest notion of derivative which implies directional differentiability and which is invariant under diffeomorphisms. A serendipitous feature of this definition is that tangential differentiability with respect to V is equivalent to tangential differentiability with respect to $\langle v \rangle$ for every $v \in V$ and linearity of the directional derivative (proposition 3.4.1). This property is well-known to fail for directional derivatives.

Finally, the chain rule *does not hold in general* for the tangential derivatives (example 3.3.1). This is a striking consequence of the missing directions in definition 3.2. However, the chain rule holds under a necessary and sufficient condition which is not very stringent (proposition 3.3.1), and which covers for example the case where the function on the left is Lipschitz continuous or where the function on the right has an injective tangential derivative.

3.2. Examples and basic properties

In this section, we give various examples and some basic properties of tangentially differentiable functions.

We first consider the uniqueness of the tangential derivative at a with respect to V . For a set $A \subseteq \mathbb{R}^m$ and a linear subspace $V \subseteq \mathbb{R}^m$, a vector $v \in V$ is *tangent* to A at $a \in A$ if for every $\delta > 0$ and $\theta > 0$, there exists $x \in A$ such that $\|x - a\| \leq \delta$ and $\|x - a - v\| \leq \theta\|x - a\|$.

Proposition 3.2.1. *Let $A \subseteq \mathbb{R}^m$ and let $V \subseteq \mathbb{R}^m$ be a linear subspace of $\mathbb{R}^m$. If for every $v \in V$, v is tangent to A at a , then there exists at most one tangential derivative of $f: A \rightarrow \mathbb{R}^n$ at $a \in A$ with respect to V .*

In particular, if a is an interior point of the set A , or if a is interior to $A \cap (V + a)$ relative to $V + a$, then the tangential derivative is unique.

In the limiting cases the definition of tangential derivative covers known concepts. If $V = \{0\}$, one observes by taking $\theta = \frac{1}{2}$ that every function has a tangential derivative with respect to V and if $V = \mathbb{R}^m$, the notions of directional and tangential derivatives are both equivalent with the classical definition of total differentiability of Stolz and Fréchet. For any linear subspace $V \subseteq \mathbb{R}^m$, if a function f has a tangential derivative at a with respect to V such that for all $\varepsilon > 0$, we have $\theta \geq 1$, then, by taking $v = 0$, f is differentiable (with respect to $\mathbb{R}^m$) and $Df(a)$ is the 0 linear map.

In all the other cases, directional differentiability does not imply tangential differentiability.

Example 3.2.1. Given a linear subspace V such that $\{0\} \subsetneq V \subsetneq \mathbb{R}^m$, we take $K \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ such that $\ker(K) = V$ and $\alpha > 0$, and we define the function $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ for every $x \in \mathbb{R}^m$ by

$$f(x) = \begin{cases} \frac{K[x]}{\|x\|^\alpha} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Since $f = 0$ on V , the 0 linear map is a directional derivative at 0 with respect to V but since the boundedness condition (3.1) is not satisfied, the function f does not have a tangential derivative at 0 with respect to V .

It turns out however that for Lipschitz continuous functions, the notions of directional and tangential derivatives are equivalent. We recall that a function $f: A \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$ is *Lipschitz continuous* if there exists a constant $\kappa \geq 0$ such that for all $x, y \in A$, $\|f(x) - f(y)\| \leq \kappa\|x - y\|$. We also denote by $\langle v \rangle$ the *vector space spanned* by $v \in \mathbb{R}^m$, that is, $\langle v \rangle = \{rv: r \in \mathbb{R}\} \subseteq \mathbb{R}^m$.

Proposition 3.2.2. *Let $A \subseteq \mathbb{R}^m$, let $f: A \rightarrow \mathbb{R}^n$, let $V \subseteq \mathbb{R}^m$ be a linear subspace of $\mathbb{R}^m$ and let $L \in \mathcal{L}(V, \mathbb{R}^n)$. If the function f is Lipschitz continuous, then the following statements are equivalent:*

- (i) *the linear map L is a directional derivative of f at $a \in A$ with respect to V ,*
- (ii) *the linear map L is a tangential derivative of f at $a \in A$ with respect to V ,*
- (iii) *for every v in a dense subset of $V \cap \partial B[0, 1]$, the linear map $L|_{\langle v \rangle} \in \mathcal{L}(\langle v \rangle, \mathbb{R}^n)$ is a directional derivative of f at $a \in A$ with respect to $\langle v \rangle$.*

Proof. Let $\kappa \geq 0$ be given by the definition of Lipschitz continuity for the function f . First we prove that (i) implies (ii). For every $x \in A$ and every $v \in V$,

$$\begin{aligned} \|f(x) - f(a) - L[v]\| &\leq \|f(x) - f(v+a)\| + \|f(v+a) - f(a) - L[v]\| \\ &\leq \kappa\|x - a - v\| + \|f(v+a) - f(a) - L[v]\|. \end{aligned}$$

For each $\varepsilon > 0$, by definition of directional derivative, there exists $\delta > 0$ such that if $y \in A \cap (V + a)$ and $\|y - a\| \leq \delta$, then $\|f(y) - f(a) - L[y - a]\| \leq \frac{\varepsilon}{4}\|y - a\|$. Therefore, if $x \in A$, $v \in V$, $\|x - a\| \leq \frac{\delta}{2}$ and

$\|x - a - v\| \leq \min(1, \frac{\varepsilon}{2(1+\kappa)})\|x - a\|$, then $\|(v + a) - a\| \leq 2\|x - a\| \leq \delta$, and so $\|f(x) - f(a) - L[v]\| \leq \varepsilon\|x - a\|$.

Since it is obvious that (ii) implies (iii), it remains to prove that (iii) implies (i). For every $x \in A \cap (V + a)$ and every $v \in V$,

$$\begin{aligned} & \|f(x) - f(a) - L[x - a]\| \\ & \leq \|f(x) - f(\|x - a\|v + a) - L[x - a - \|x - a\|v]\| \\ & \quad + \|f(\|x - a\|v + a) - f(a) - L[\|x - a\|v]\| \\ & \leq (\kappa + \|L\|)\|x - a\|\left\|\frac{x - a}{\|x - a\|} - v\right\| \\ & \quad + \|f(\|x - a\|v + a) - f(a) - L[\|x - a\|v]\|. \end{aligned}$$

By assumption, for every v in a dense subset $S \subseteq V \cap \partial B[0, 1]$, there exists $\delta_v > 0$ such that if $y \in A \cap (\langle v \rangle + a)$ and $\|y - a\| \leq \delta$, then $\|f(y) - f(a) - L[y - a]\| \leq \frac{\varepsilon}{2}\|y - a\|$. Here and in the sequel $B[0, 1]$ denotes the closed ball of radius 1 centered at the point 0. Since $V \cap \partial B[0, 1]$ is compact and S is dense in $V \cap \partial B[0, 1]$, there exists $N \in \mathbb{N}_*$ and $v_1, \dots, v_N \in S$ such that

$$V \cap \partial B[0, 1] \subseteq \bigcup_{i=1}^N B[v_i, \frac{\varepsilon}{2(1+\kappa+\|L\|)}].$$

As a consequence, if $x \in A \cap (V + a)$ and $\|x - a\| \leq \min_{1 \leq i \leq N} \delta_{v_i}$, then

$$\|f(x) - f(a) - L[x - a]\| \leq \varepsilon\|x - a\|. \quad \square$$

The proof shows that the Lipschitz continuity assumption can be replaced by the weaker condition:

$$\limsup_{\substack{\frac{\text{dist}(x-a, V)}{\|x-a\|} \rightarrow 0 \\ \frac{\text{dist}(y-a, V)}{\|y-a\|} \rightarrow 0}} \frac{\|f(x) - f(y)\|}{\|x - y\|} < \infty.$$

3.3. The chain rule for tangential derivatives

We turn now on to the study of the chain rule for the tangential derivative. We first remark that the chain rule *does not* hold in general for the tangential derivative.

Example 3.3.1. Let $K \in \mathcal{L}(\mathbb{R}^m, \mathbb{R})$ be such that $\{0\} \subsetneq \ker(K) \subsetneq \mathbb{R}^m$ and let $\beta \geq 2$. We define the functions $f: \mathbb{R}^m \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ for

every $x \in \mathbb{R}^m$ and $t \in \mathbb{R}$ as

$$f(x) = \begin{cases} \frac{|K[x]|^\beta}{\|x\|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0; \end{cases}$$

$$g(t) = |t|^{\frac{1}{\beta}}.$$

Since the 0 linear map is the tangential derivative of f at 0 with respect to $\ker(K)$, g has a tangential derivative at $f(0) = 0$ with respect to $\{0\}$, but $g \circ f$ has the same form as the function in example 3.2.1 and so does not have a tangential derivative at 0 with respect to $\ker(K)$.

The chain rule holds however under an additional necessary and sufficient analytic assumption involving both functions in the composition.

Proposition 3.3.1. *Let $A \subseteq \mathbb{R}^m$ and let $V \subseteq \mathbb{R}^m$ be a linear subspace of $\mathbb{R}^m$. Let $L \in \mathcal{L}(V, \mathbb{R}^n)$ be a tangential derivative of $f: A \rightarrow \mathbb{R}^n$ at $a \in A$ with respect to V and let $K \in \mathcal{L}(L[V], \mathbb{R}^p)$ be a tangential derivative of $g: f(A) \rightarrow \mathbb{R}^p$ at $f(a)$ with respect to $L[V]$. The linear map $K \circ L \in \mathcal{L}(V, \mathbb{R}^p)$ is a tangential derivative of $g \circ f$ at a with respect to V if and only if*

$$\lim_{\substack{v \in V \\ x \rightarrow a \\ \frac{\|x-a-v\|}{\|x-a\|} \rightarrow 0 \\ \frac{\|f(x)-f(a)\|}{\|x-a\|} \rightarrow 0}} \frac{\|g(f(x)) - g(f(a))\|}{\|x-a\|} = 0. \quad (3.3.2)$$

Proof. We first assume that the function $g \circ f$ is tangentially differentiable. For every $x \in A$ and every $v \in V$,

$$\begin{aligned} & \|g(f(x)) - g(f(a))\| \\ & \leq \|(g \circ f)(x) - (g \circ f)(a) - (K \circ L)[v]\| \\ & \quad + \|K\| (\|f(x) - f(a) - L[v]\| + \|f(x) - f(a)\|). \end{aligned}$$

Fix $\varepsilon > 0$. By assumption, there exist $\delta, \theta > 0$ such that if $x \in A$, $v \in V$, $\|x - a\| \leq \delta$ and $\|x - a - v\| \leq \theta\|x - a\|$, then

$$\begin{aligned} & \|(g \circ f)(x) - (g \circ f)(a) - (K \circ L)[v]\| \leq \varepsilon\|x - a\|, \\ & \|f(x) - f(a) - L[v]\| \leq \varepsilon\|x - a\|. \end{aligned}$$

As a consequence, if $x \in A$, $v \in V$, $\|x - a\| \leq \delta$, $\|x - a - v\| \leq \theta\|x - a\|$ and $\|f(x) - f(a)\| \leq \varepsilon\|x - a\|$, then

$$\|g(f(x)) - g(f(a))\| \leq (1 + 2\|K\|) \varepsilon\|x - a\|,$$

that is, (3.3.2) is satisfied.

Conversely, we assume that (3.3.2) holds. For every $x \in A$ and every $v \in V$,

$$\begin{aligned} & \| (g \circ f)(x) - (g \circ f)(a) - (K \circ L)[v] \| \\ & \leq \| g(f(x)) - g(f(a)) \| \\ & \quad + \| K \| (\| f(x) - f(a) - L[v] \| + \| f(x) - f(a) \|). \end{aligned} \quad (3.3.3)$$

Fix $\varepsilon > 0$. By assumption, there exist $\delta_1, \theta_1 > 0$ and $\kappa > 0$ such that if $x \in A$, $v \in V$, $\|x - a\| \leq \delta_1$, $\|x - a - v\| \leq \theta_1 \|x - a\|$ and $\|f(x) - f(a)\| \leq \kappa \|x - a\|$, then

$$\|g(f(x)) - g(f(a))\| \leq \varepsilon \|x - a\| \text{ and } \|f(x) - f(a) - L[v]\| \leq \varepsilon \|x - a\|,$$

and thus by (3.3.3),

$$\| (g \circ f)(x) - (g \circ f)(a) - (K \circ L)[v] \| \leq (\varepsilon(1 + \|K\|) + \kappa) \|x - a\|.$$

Without loss of generality, we assume that $\kappa \leq \varepsilon$, and thus if $\|x - a\| \leq \delta_1$, $\|x - a - v\| \leq \theta_1 \|x - a\|$ and $\|f(x) - f(a)\| \leq \kappa \|x - a\|$, then

$$\| (g \circ f)(x) - (g \circ f)(a) - (K \circ L)[v] \| \leq (2 + \|K\|) \varepsilon \|x - a\|.$$

It remains to cover the case where $x \in A$ and $\|f(x) - f(a)\| > \kappa \|x - a\|$. First, for every $x \in A$ and every $v \in V$ such that $\|x - a\| \leq \delta_1$ and $\|x - a - v\| \leq \theta_1 \|x - a\|$, we have

$$\|f(x) - f(a)\| \leq (\varepsilon + \|L\|(1 + \theta_1)) \|x - a\|.$$

Let $\eta = \varepsilon + \|L\|(1 + \theta_1)$. Since g is tangentially differentiable, there exist $\tilde{\delta}, \tilde{\theta} > 0$ such that if $x \in A$, $v \in V$, $\|f(x) - f(a)\| \leq \tilde{\delta}$ and $\|f(x) - f(a) - L[v]\| \leq \tilde{\theta} \|f(x) - f(a)\|$, then

$$\|g(f(x)) - g(f(a)) - K[L[v]]\| \leq \frac{\varepsilon}{\eta} \|f(x) - f(a)\|.$$

Since L is a tangential derivative of f , there exist $\delta_2, \theta_2 > 0$ such that if $x \in A$, $v \in V$, $\|x - a\| \leq \delta_2$ and $\|x - a - v\| \leq \theta_2 \|x - a\|$, then

$$\|f(x) - f(a) - L[v]\| \leq \tilde{\theta} \kappa \|x - a\|.$$

Therefore, if $x \in A$, $v \in V$, $\|x - a\| \leq \min(\delta_1, \delta_2, \frac{\tilde{\delta}}{\eta})$ and $\|x - a - v\| \leq \min(\theta_1, \theta_2) \|x - a\|$, then $\|f(x) - f(a)\| \leq \tilde{\delta}$ and $\|f(x) - f(a) - L[v]\| \leq \tilde{\theta} \|f(x) - f(a)\|$, and so

$$\| (g \circ f)(x) - (g \circ f)(a) - (K \circ L)[v] \| \leq \frac{\varepsilon}{\eta} \|f(x) - f(a)\| \leq \varepsilon \|x - a\|,$$

that is, the function $g \circ f$ is tangentially differentiable at a with respect to V . $\square$

A first interesting consequence is a chain rule when the second function g in the composition is Lipschitz continuous.

Corollary 3.3.2. *Let $A \subseteq \mathbb{R}^m$ and let $V \subseteq \mathbb{R}^m$ be a linear subspace of $\mathbb{R}^m$. Let $L \in \mathcal{L}(V, \mathbb{R}^n)$ be a tangential derivative of $f: A \rightarrow \mathbb{R}^n$ at $a \in A$ with respect to V and let $K \in \mathcal{L}(L[V], \mathbb{R}^p)$ be a tangential derivative of $g: f(A) \rightarrow \mathbb{R}^p$ at $f(a)$ with respect to $L[V]$. If the function g is Lipschitz continuous, then $K \circ L \in \mathcal{L}(V, \mathbb{R}^p)$ is a tangential derivative of $g \circ f$ at a with respect to V .*

Proof. If $\kappa \geq 0$ is a Lipschitz constant for g , it is clear that condition (3.3.2) is satisfied since for every $x \in A \setminus \{a\}$,

$$\begin{aligned} 0 &\leq \frac{\|g(f(x)) - g(f(a))\|}{\|x - a\|} \\ &= \frac{\|g(f(x)) - g(f(a))\|}{\|f(x) - f(a)\|} \frac{\|f(x) - f(a)\|}{\|x - a\|} \leq \kappa \frac{\|f(x) - f(a)\|}{\|x - a\|}. \quad \square \end{aligned}$$

Another consequence which will be important in the sequel is the case where the tangential derivative of the first function f in the composition is injective.

Corollary 3.3.3. *Let $A \subseteq \mathbb{R}^m$ and let $V \subseteq \mathbb{R}^m$ be a linear subspace of $\mathbb{R}^m$. Let $L \in \mathcal{L}(V, \mathbb{R}^n)$ be a tangential derivative of $f: A \rightarrow \mathbb{R}^n$ at $a \in A$ with respect to V and let $K \in \mathcal{L}(L[V], \mathbb{R}^p)$ be a tangential derivative of $g: f(A) \rightarrow \mathbb{R}^p$ at $f(a)$ with respect to $L[V]$. If L is injective, then $K \circ L \in \mathcal{L}(V, \mathbb{R}^p)$ is a tangential derivative of $g \circ f$ at a with respect to V .*

Proof. Since the linear map L is injective on its domain V , there exists a constant $\eta > 0$ such that for every $v \in V$, $\|v\| \leq \eta \|L[v]\|$. For every $x \in A$ and every $v \in V$,

$$\begin{aligned} \|x - a\| &\leq \|x - a - v\| + \eta \|L[v]\| \\ &\leq \|x - a - v\| + \eta (\|f(x) - f(a)\| + \|f(x) - f(a) - L[v]\|). \end{aligned}$$

Since L is a tangential derivative of f , there exist $\delta > 0$ and $\theta > 0$ such that if $x \in A$, $v \in V$, $\|x - a\| \leq \delta$ and $\|x - a - v\| \leq \theta \|x - a\|$, then $\|f(x) - f(a) - L[v]\| \leq \frac{1}{4\eta} \|x - a\|$. Therefore, if $\|x - a\| \leq \delta$ and $\|x - a - v\| \leq \min(\theta, \frac{1}{4}) \|x - a\|$, then

$$\frac{3}{4} \|x - a\| \leq \eta \|f(x) - f(a)\| + \|x - a - v\|,$$

and so $\|x - a\| \leq 2\eta \|f(x) - f(a)\|$. Hence, the condition (3.3.2) is trivially satisfied. $\square$

The previous chain rule shows the invariance under diffeomorphisms. As a consequence, we extend the notion of tangential derivative to functions on differentiable manifolds. We recall that a *local chart* φ of a differentiable manifold M of dimension m is a map $\varphi: U \subseteq M \rightarrow \mathbb{R}^m$ such that U is an open subset of M and $\varphi: U \rightarrow \varphi(U)$ is a diffeomorphism. For every $p \in M$, we also denote by $T_p M$ the *tangent space at the point* p .

Definition 3.3.1. Let M be a differentiable manifold of dimension m . Let $p \in M$ and let $V \subseteq T_p M$ be a linear subspace of $T_p M$. A linear map $L \in \mathcal{L}(V, \mathbb{R})$ is a *tangential derivative* of a map $f: M \rightarrow \mathbb{R}$ at p with respect to V if for some local chart $\varphi: U \subseteq M \rightarrow \mathbb{R}^m$ such that $p \in U$, $L \circ (D\varphi(p))^{-1} \in \mathcal{L}(D\varphi(p)[V], \mathbb{R})$ is a tangential derivative of $f \circ \varphi^{-1}: \varphi(U) \rightarrow \mathbb{R}$ at $\varphi(p)$ with respect to $D\varphi(p)[V]$.

By corollary 3.3.3, the previous definition is independent of the local chart φ .

3.4. Characterizations of tangential differentiability

In this section we characterize tangential differentiability as the weakest notion which implies directional differentiability and which is invariant under diffeomorphisms. It is also the weakest notion for which there is a chain rule with respect to nondegenerate paths with a linear dependence on the direction. Finally, tangential differentiability with respect to a subspace is equivalent with tangential differentiability with respect to every direction.

Proposition 3.4.1. Let $a \in A \subseteq \mathbb{R}^m$, let $f: A \rightarrow \mathbb{R}^n$, let $V \subseteq \mathbb{R}^m$ be a linear subspace of $\mathbb{R}^m$ and $L \in \mathcal{L}(V, \mathbb{R}^n)$. The following statements are equivalent:

- (i) the linear map L is a tangential derivative of f at a with respect to V ,
- (ii) for each map $\psi: U \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that $a \in U \subseteq A$ is open and $\psi: U \rightarrow \psi(U)$ is a diffeomorphism, $L \circ (D\psi(a))^{-1} \in \mathcal{L}(D\psi(a)[V], \mathbb{R}^n)$ is a directional derivative of $f \circ \psi^{-1}$ at $\psi(a)$ with respect to $D\psi(a)[V]$,
- (iii) for each path $\gamma \in C^1((-1, 1), \mathbb{R}^m)$ such that $\gamma(0) = a$ and $\gamma'(0) \in V \setminus \{0\}$, the composite function $f \circ \gamma$ is differentiable at 0 and $(f \circ \gamma)'(0) = L[\gamma'(0)]$,

(iv) for each $v \in V \setminus \{0\}$,

$$\lim_{\substack{x \rightarrow a \\ \frac{x-a}{\|x-a\|} \rightarrow v}} \frac{f(x) - f(a) - \|x-a\|L[v]}{\|x-a\|} = 0.$$

If $V = \mathbb{R}^m$, the equivalence between statements (i), (iii) and (iv) are well-known in finite dimensional theory. Indeed, statement (iii) is equivalent to Hadamard differentiability ([29, teorema 1; 96; 107, definition 2.2]) and so it is equivalent — since we are working in finite dimension — to Fréchet–Stolz differentiability (i). Furthermore, statement (iv) is reminiscent to Bastiani–Michal differentiability [12, définition 3.2; 86; 97, definition 1.1] which is also equivalent to Hadamard differentiability. For any linear subspace $V \subseteq \mathbb{R}^m$, statement (iv) is also reminiscent to Schechter differentiability [103, §1].

In contrast with Lipschitz continuous functions (proposition 3.2.2 (i)), it is important in the previous statement (iv) to assume the property for *every* $v \in V \setminus \{0\}$ as it can be observed by taking $(v_n)_{n \in \mathbb{N}}$ a dense subset of $\partial B[0, 1] \subseteq \mathbb{R}^m$ and the function $f: \mathbb{R}^m \rightarrow \mathbb{R}$ defined for every $x \in \mathbb{R}^m$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \bigcup_{n \in \mathbb{N}} \{y \in \mathbb{R}^m : \text{dist}(y, \langle v_n \rangle) \leq \frac{1}{2^{n+2}} \|y\|\}, \\ 1 & \text{otherwise.} \end{cases}$$

Furthermore, property (iv) is well-known to fail for directional derivatives.

Given a differentiable manifold M of dimension m and a function $f: M \rightarrow \mathbb{R}$, the fact that (iv) implies (i) gives us a sufficient condition for f to be tangentially differentiable at $p \in M$ with respect to a linear subspace $V \subseteq T_p M$. Indeed, if there exists $L \in \mathcal{L}(V, \mathbb{R})$ which is a tangential derivative of $f: M \rightarrow \mathbb{R}$ at p with respect to $\langle v \rangle$ for every $v \in V$, that is, according to definition 3.3.1, for all local charts $\varphi: U \subseteq M \rightarrow \mathbb{R}^m$ such that $p \in U$ and for all $w \in D\varphi(p)[V]$,

$$\frac{d}{dt}(f \circ \varphi^{-1})(\varphi(p) + tw)|_{t=0} = (L \circ (D\varphi(p))^{-1})[w],$$

then f is tangentially differentiable at p with respect to V .

The proof of proposition 3.4.1 relies on the construction of an interpolation path between points of a sequence (for a similar construction, see [38, theorem 2.6]).

Lemma 3.4.2. *Let $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}^m$ be a sequence that converges to $a \in \mathbb{R}^m$. Let $(t_n)_{n \in \mathbb{N}} \subseteq (0, 1)$ be a decreasing sequence such that $\limsup_{n \rightarrow \infty} \frac{t_{n+1}}{t_n} < 1$. If $(\frac{x_n - a}{t_n})_{n \in \mathbb{N}}$ converges to $v \in \mathbb{R}^m$, then there exists a path $\gamma \in C^1([0, t_0], \mathbb{R}^m)$ such that $\gamma(0) = a$, $\gamma'(0) = v$ and for all $n \in \mathbb{N}$, $\gamma(t_n) = x_n$.*

Proof. We take p and q to be polynomials of degree 3 such that $p(0) = 0$, $p(1) = 1$ and $p'(0) = p'(1) = 0$, and $q(0) = q(1) = 0$ and $q'(0) = q'(1) = 1$, that is, for every $s \in \mathbb{R}$, $p(s) = 3s^2 - 2s^3$ and $q(s) = s - p(s)$. We define the path $\gamma: [0, t_0] \rightarrow \mathbb{R}^m$ for each $t \in [0, t_0]$ by

$$\gamma(t) = \begin{cases} x_{n+1} + (x_n - x_{n+1}) p\left(\frac{t-t_{n+1}}{t_n-t_{n+1}}\right) & \text{if } t_{n+1} < t \leq t_n, \\ + (t_n - t_{n+1}) q\left(\frac{t-t_{n+1}}{t_n-t_{n+1}}\right) v & \\ a & \text{if } t = 0. \end{cases}$$

It is clear that $\gamma \in C^1((0, t_0], \mathbb{R}^m)$. Since $0 \leq p \leq 1$ on $[0, 1]$ and $\sup_{s \in [0, 1]} |q(s)| \leq 2$, for every $t \in (t_{n+1}, t_n]$,

$$\begin{aligned} \|\gamma(t) - \gamma(0)\| &\leq \|x_{n+1} - a\| + \|x_n - x_{n+1}\| \left| p\left(\frac{t-t_{n+1}}{t_n-t_{n+1}}\right) \right| \\ &\quad + |t_n - t_{n+1}| \left| q\left(\frac{t-t_{n+1}}{t_n-t_{n+1}}\right) \right| \|v\| \\ &\leq \|x_{n+1} - a\| + \|x_n - x_{n+1}\| + 2|t_n - t_{n+1}| \|v\|. \end{aligned}$$

It follows that γ is continuous at 0. Next, since the sequence $(t_n)_{n \in \mathbb{N}}$ is decreasing and $\limsup_{n \rightarrow \infty} \frac{t_{n+1}}{t_n} < 1$, there exists $\alpha \in (0, 1)$ such that for every $n \in \mathbb{N}$, $0 < t_{n+1} \leq \alpha t_n$. For every $t \in (t_{n+1}, t_n]$, since $p' + q' = 1$ on $[0, 1]$,

$$\begin{aligned} \|\gamma'(t) - v\| &= \left| p'\left(\frac{t-t_{n+1}}{t_n-t_{n+1}}\right) \right| \left\| \frac{x_n - x_{n+1}}{t_n - t_{n+1}} - v \right\| \\ &\leq \left| p'\left(\frac{t-t_{n+1}}{t_n-t_{n+1}}\right) \right| \left\| \frac{x_{n+1} - a}{t_{n+1}} - v \right\| \\ &\quad + \frac{t_n}{t_n - t_{n+1}} \left\| \frac{x_n - a}{t_n} - \frac{x_{n+1} - a}{t_{n+1}} \right\| \\ &\leq \frac{3}{2} \left\| \frac{x_{n+1} - a}{t_{n+1}} - v \right\| + \frac{1}{1-\alpha} \left\| \frac{x_n - a}{t_n} - \frac{x_{n+1} - a}{t_{n+1}} \right\|. \end{aligned}$$

As a consequence, γ' has a limit at 0 and so γ is differentiable at 0. We have thus proved that $\gamma \in C^1([0, t_0], \mathbb{R}^m)$. $\square$

Proof of proposition 3.4.1. Assume that (i) holds. Since ψ is a local diffeomorphism, $D\psi^{-1}(\psi(a)) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ is a tangential derivative of ψ^{-1} at $\psi(a)$ with respect to $D\psi(a)[V]$ and is injective. By the chain rule (corollary 3.3.3), since L is a tangential derivative of f at $a = \psi^{-1}(\psi(a))$ with respect to $V = D\psi^{-1}(\psi(a))[D\psi(a)[V]]$, $L \circ D\psi^{-1}(\psi(a))$ is a tangential derivative of $f \circ \psi^{-1}$ at $\psi(a)$ with respect to $D\psi(a)[V]$, and so a directional derivative.

In order to prove that (ii) implies (iii), since $\gamma'(0) \in V \setminus \{0\}$, without loss of generality, if $\gamma = (\gamma_1, \dots, \gamma_m)$, we assume that there exists $\eta > 0$

such that $\gamma_1: (-\eta, \eta) \rightarrow \mathbb{R}$ is one-to-one. We consider the function $\tilde{\psi}: (-\eta, \eta) \times \mathbb{R}^{m-1} \rightarrow \mathbb{R}^m$ defined for all $(y_1, y'') \in (-\eta, \eta) \times \mathbb{R}^{m-1}$ as

$$\tilde{\psi}(y_1, y'') = \gamma(y_1) + (0, y'').$$

By construction, $\tilde{\psi}: (-\eta, \eta) \times \mathbb{R}^{m-1} \rightarrow \tilde{\psi}((-\eta, \eta) \times \mathbb{R}^{m-1})$ is a diffeomorphism. So we take the function $\psi = \tilde{\psi}^{-1}$. Since $f \circ \gamma = f \circ \psi^{-1}$ on $(-\eta, \eta) \times \{0\}$ and for every $t \in (-\eta, \eta)$, $D\psi^{-1}(\psi(a))[(t, 0)] = \gamma'(0)t \in V$, $f \circ \gamma$ is differentiable at 0 and $(f \circ \gamma)'(0) = L[\gamma'(0)]$.

We prove that (iii) implies (iv) by contradiction. Let $v \in V \setminus \{0\}$ be such that (iv) is not satisfied. Then there exist $\varepsilon > 0$ and a sequence $(x_n)_{n \in \mathbb{N}} \subseteq A$ that converges to a such that $\lim_{n \rightarrow \infty} \frac{x_n - a}{\|x_n - a\|} = v$ but for every $n \in \mathbb{N}$,

$$\|f(x_n) - f(a) - \|x_n - a\|L[v]\| > \varepsilon\|x_n - a\|.$$

Up to a subsequence, the sequence $(\|x_n - a\|)_{n \in \mathbb{N}}$ is decreasing and $\limsup_{n \rightarrow \infty} \frac{\|x_{n+1} - a\|}{\|x_n - a\|} < 1$. Let $\tilde{\gamma} \in C^1([0, t_0], \mathbb{R}^m)$ be given by lemma 4.2 with $(x_n)_{n \in \mathbb{N}}$ and $(t_n = \|x_n - a\|)_{n \in \mathbb{N}}$. Define $\gamma \in C^1((-1, t_0], \mathbb{R}^m)$ as $\gamma = \tilde{\gamma}$ on $[0, t_0]$ and $\gamma(t) = a + tv$ for all $t \in (-1, 0]$. As a consequence, $\gamma'(0) = v \in V$ and

$$\begin{aligned} \|(f \circ \gamma)(t_n) - (f \circ \gamma)(0) - t_n L[\gamma'(0)]\| &= \|f(x_n) - f(a) - \|x_n - a\|L[v]\| \\ &> \varepsilon\|x_n - a\| = \varepsilon|t_n|, \end{aligned}$$

and so $f \circ \gamma$ is not differentiable at 0.

Finally, we prove that (iv) implies (i). Fix $\varepsilon > 0$. By assumption, for every $v \in V \setminus \{0\}$, there exist $\delta_v, \theta_v > 0$ such that if $x \in A$, $\|x - a\| \leq \delta_v$ and $\|\frac{x-a}{\|x-a\|} - v\| \leq \theta_v$, then $\|f(x) - f(a) - L[\|x-a\|v]\| \leq \frac{\varepsilon}{2}\|x-a\|$. Then we define $\tilde{\theta}_v = \min(\frac{\varepsilon}{2(1+\|L\|)}, \frac{\theta_v}{2})$. Since the set $V \cap B[0, 2]$ is compact, there exists $N \in \mathbb{N}_*$ and $v_1, \dots, v_N \in V \setminus \{0\}$ such that

$$V \cap B[0, 2] \subseteq \bigcup_{i=1}^N B[v_i, \tilde{\theta}_{v_i}].$$

Finally, if $x \in A$, if $v \in V$, if $\|x - a\| \leq \min_{1 \leq i \leq N} \delta_{v_i}$ and if

$$\|x - a - v\| \leq \min(1, \min_{1 \leq i \leq N} \frac{\theta_{v_i}}{2})\|x - a\|,$$

then $\frac{\|v\|}{\|x-a\|} \leq 2$ and so there exists $v_i \in V \setminus \{0\}$ such that $\|\frac{v}{\|x-a\|} - v_i\| \leq \tilde{\theta}_{v_i}$ and

$$\|\frac{x-a}{\|x-a\|} - v_i\| \leq \|\frac{x-a}{\|x-a\|} - \frac{v}{\|x-a\|}\| + \|\frac{v}{\|x-a\|} - v_i\| \leq \theta_{v_i},$$

and thus

$$\begin{aligned}
& \|f(x) - f(a) - L[v]\| \\
& \leq \|f(x) - f(a) - \|x - a\|L[v_i]\| + \|L\| \|x - a\| \|v_i - \frac{v}{\|x - a\|}\| \\
& \leq \frac{\varepsilon}{2} \|x - a\| + \|L\| \|x - a\| \frac{\varepsilon}{2(1+\|L\|)} \leq \varepsilon \|x - a\|. \quad \square
\end{aligned}$$

3.5. Application to the chain rule

The notion of tangential derivative is useful to state and prove an intrinsic chain rule for derivatives of Sobolev maps between manifolds. The general chain rule of L. Ambrosio and G. Dal Maso applies as follows to weakly differentiable functions [4].

Proposition 3.5.1 ([4, corollary 3.2]). *Let Ω be an open subset of $\mathbb{R}^m$. If $u \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^n)$, and $f: \mathbb{R}^n \rightarrow \mathbb{R}^p$ is a Lipschitz continuous function, then $f \circ u \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^p)$ and for almost every $x \in \Omega$, the function f has a directional derivative at $u(x)$ with respect to $Du(x)[\mathbb{R}^m]$.*

Here, $W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^n)$ denotes the Sobolev space of all functions $u: \Omega \rightarrow \mathbb{R}^n$ such that the function u and its weak derivative $Du: \Omega \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ are locally integrable on Ω (see for example [24, §9.1; 120, definition 6.1.8]).

If we now consider the case where $u \in W_{\text{loc}}^{1,1}(\Omega, N)$ and $f: N \rightarrow \mathbb{R}^p$, with N a differentiable manifold, then the condition that f has a directional derivative does not have any geometric meaning. By proposition 3.2.2, since f is Lipschitz continuous, proposition 3.5.1 can be restated with tangential derivative instead of directional derivative in the conclusion. The proposition is then invariant under changes of local charts, and suggests the following generalization.

Proposition 3.5.2. *Let Ω be an open subset of $\mathbb{R}^m$ and let N be a Riemannian manifold of dimension n . If $u \in W_{\text{loc}}^{1,1}(\Omega, N)$, and $f: N \rightarrow \mathbb{R}^p$ is a Lipschitz continuous function, then $f \circ u \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^p)$ and for almost every $x \in \Omega$, the map f has a tangential derivative at $u(x)$ with respect to $Du(x)[\mathbb{R}^m] \subseteq T_{u(x)}N$.*

Proof. By the classical composition property in Sobolev spaces [34, proposition 2.5], $f \circ u \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^p)$.

For every $y \in N$, there exist an open subset $U \subseteq N$ such that $y \in U$, and some maps $\varphi \in C_c^1(N, \mathbb{R}^n)$ and $\varphi^* \in C^1(\mathbb{R}^n, N)$ such that (φ, φ^*) is an extended local chart: $\varphi|_U: U \subseteq N \rightarrow \mathbb{R}^n$ is a local chart and $\varphi^* \circ \varphi = \text{id}$ in U [34, lemma 1.3]. Furthermore, for almost every

$x \in \Omega$, $D(\varphi \circ u)(x) = D\varphi(u(x))[Du(x)]$ (existence and uniqueness of the colocal weak derivative [34, proposition 1.2]). If moreover $x \in u^{-1}(U)$, since $f \circ \varphi^*: \mathbb{R}^n \rightarrow \mathbb{R}^p$ is Lipschitz continuous and $\varphi \circ u \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^n)$, by proposition 3.5.1, the function $f \circ \varphi^*$ has a tangential derivative at $\varphi(u(x))$ with respect to $D(\varphi \circ u)(x)[\mathbb{R}^m] = D\varphi(u(x))[Du(x)[\mathbb{R}^m]]$. Hence, by definition 3.3.1, f has a tangential derivative at $u(x)$ with respect to $Du(x)[\mathbb{R}^m]$. $\square$

Chapter IV

Higher order weak differentiability and Sobolev spaces between manifolds

4.1. Introduction

Higher-order Sobolev maps between manifolds are measurable maps that are at least twice weakly differentiable whose derivatives satisfy some summability condition. As their first-order counterparts such maps are a natural framework to study *geometrical objects* as biharmonic [30, 56, 88, 109, 112] and polyharmonic [7, 50, 55, 78] maps and some *physical models* as the Schrödinger–Airy and third order Landau–Lifschitz equations [110].

For every $k \in \mathbb{N}_*$ and every $p \in [1, +\infty)$, Sobolev maps between the manifolds M and N can be defined by first embedding the target manifold N in a Euclidean space $\mathbb{R}^\nu$ for some $\nu \in \mathbb{N}$ through an *isometric embedding* $\iota \in C^k(N, \mathbb{R}^\nu)$ and then considering the set [7, 56, 88]

$$W_\iota^{k,p}(M, N) = \{u: M \rightarrow N : \iota \circ u \in W_{\text{loc}}^{k,1}(M, \mathbb{R}^\nu) \text{ and } |D^k(\iota \circ u)| \in L^p(M)\}. \quad (4.1)$$

Since every Riemannian manifold N can be smoothly isometrically embedded in a Euclidean space [93, theorem 3], this definition (4.1) is always possible. In the first-order case $k = 1$, this definition turns out to be independent of the embedding (see [34, proposition 2.7]). However in the higher-order case $k \geq 2$, this definition *is not intrinsic*: it depends on the choice of the embedding ι (see proposition 4.5.3 below). The goal of this work is to propose and study an intrinsic definition of higher-order Sobolev spaces.

In the first-order case $k = 1$, the definition by embedding (4.1) is also equivalent to the definition of Sobolev spaces into metric spaces [34, proposition 2.2; 62, theorem 3.2; 63, theorem 2.17]. However definitions of Sobolev spaces into metric spaces *do not provide any notion of weak derivative* and thus do not seem adapted to a further definition of higher-order Sobolev spaces by iteration.

For first-order Sobolev maps between manifolds, an *intrinsic definition* in which weak differentiability plays a central role has been proposed in our previous work [34]. We thus follow this strategy in our aim to define and study intrinsic higher-order derivatives and Sobolev spaces.

Our definitions, results and methods apply to arbitrary order weak differentiation and Sobolev spaces, but in order to highlight the essential ideas and issues the exposition in the present introduction is focused on the second-order case.

We begin by defining *twice colocally weakly differentiable maps* as colocally weakly differentiable maps for which the colocal weak derivative is also colocally weakly differentiable (definition 4.2.1). The notion of colocal weak differentiability was introduced in [34]: a measurable map $u: M \rightarrow N$ is *colocally weakly differentiable* if for every function $f \in C_c^1(N, \mathbb{R})$, the composite function $f \circ u: M \rightarrow \mathbb{R}$ is weakly differentiable [34, §1] and its *colocal weak derivative* $Tu: TM \rightarrow TN$ is a bundle morphism between the tangent spaces such that for every function $f \in C_c^1(N, \mathbb{R})$, the chain rule $T(f \circ u) = Tf \circ Tu$ holds. The colocal weak derivative of Tu is for almost every $x \in M$, a *double vector bundle morphism* $T^2u(x): T_x^2M \rightarrow T_{u(x)}^2N$ (proposition 4.2.1).

Although this second-order derivative T^2u is a natural object from the point of view of differential geometry, *it is not a vector bundle morphism* that covers u and it does not appear directly in affine or Riemannian geometry or in geometric analysis or in physical models. In order to remedy to this issue, we study how this object T^2u is related to objects of affine and Riemannian geometry.

When M and N are *affine manifolds*, that is differentiable manifolds whose tangent manifolds are endowed with a Koszul connection which induces notions of parallel displacement and covariant derivative, the double colocal derivative is described completely by the *colocal weak covariant derivative* $D_K(Tu)$ (proposition 4.3.1).

Proposition 4.1. *Let $K_{T^*M \otimes TN}$ be a connection on the vector bundle $T^*M \otimes TN$. Let $u: M \rightarrow N$ be a twice colocally weakly differentiable map. Then the map $Tu: M \rightarrow T^*M \otimes TN$ has a colocal weak covariant derivative $D_K(Tu): M \rightarrow T^*M \otimes T^*M \otimes TN$ if and only if the map $Tu: M \rightarrow T^*M \otimes TN$ has a colocal weak derivative $T^2u: TM \rightarrow T(T^*M \otimes TN)$. Furthermore, they are unique and almost everywhere in M*

$$T^2u = \text{Vert} \circ D_K(Tu) + \text{Hor}_K \circ Tu,$$

and

$$D_K(Tu) = K_{T^*M \otimes TN} \circ T^2u.$$

Roughly speaking, colocal weak covariant differentiability consists in looking only at the vertical component associated to the connection in the chain rule and relying on the formula

$$(Tf)(Tu)[\text{Vert} \circ D_K(Tu)] = T(f \circ Tu) - (Tf)(Tu)[\text{Hor}_K \circ Tu],$$

where Vert is the *vertical lifting* from $T^*M \otimes TN$ and Hor_K is the *horizontal lifting* associated to the connection $K_{T^*M \otimes TN}$ (see definition 4.3.1).

Proposition 4.1 states the equivalence between a purely differential geometric derivative and an affine geometric one; differentiability turns out to be independent of the choice of the connection.

When M and N are *Riemannian manifolds* endowed with their Levi-Civita connections, the colocal weak covariant derivative is a natural concept in geometry and physics and is essentially equivalent to the weak covariant derivative of P. Hornung and R. Moser [72, definition 2.5] (proposition 4.3.2).

Furthermore, this notion of covariant derivative brings us in position to define for every $p \in [1, +\infty)$, the *intrinsic second-order Sobolev space* (definition 4.3.2)

$$\begin{aligned} \dot{W}^{2,p}(M, N) \\ = \{u: M \rightarrow N : u \text{ is twice colocally weakly differentiable} \\ \text{and } |D_K(Tu)|_{g_M^* \otimes g_M^* \otimes g_N} \in L^p(M)\}, \end{aligned}$$

where the norm $|\cdot|_{g_M^* \otimes g_M^* \otimes g_N}$ is induced by the Riemannian metrics on M and N . Thanks to the iterative nature of our definition of the double colocal differentiability, we deduce immediately compactness (proposition 4.3.3) and closure (proposition 4.3.4) properties for second-order Sobolev maps.

The intrinsic second-order Sobolev spaces might be different from the one by embedding (see proposition 4.5.3 below). As a consequence, we compare the definition of *higher-order Sobolev spaces by embedding* (4.1) with the intrinsic one (definitions 4.3.2 and 4.3.3). The latter definition is not intrinsic and has characterizations in terms of intrinsic spaces in several particular cases (propositions 4.5.1, 4.5.3 and 4.5.4).

Proposition 4.2. *Let $p \in [1, +\infty)$.*

(i) *If there exists $C > 0$ such that for every $v \in \dot{W}^{2,p}(M, \mathbb{R}) \cap L^\infty(M, \mathbb{R})$,*

$$\begin{aligned} & \int_M |Tv|_{g_M^* \otimes g_1}^{2p} \\ & \leq C \left(\text{ess sup} \{ |v(x) - v(y)| : x, y \in M \} \right)^p \int_M |D_K(Tv)|_{g_M^* \otimes g_M^* \otimes g_1}^p \end{aligned}$$

and if N is compact, then for any isometric embedding $\iota \in C^2(N, \mathbb{R}^\nu)$, we have

$$\dot{W}_\iota^{2,p}(M, N) = \dot{W}^{1,2p}(M, N) \cap \dot{W}^{2,p}(M, N).$$

(ii) *For every isometric embedding $\iota \in C^2(N, \mathbb{R}^\nu)$,*

$$\dot{W}_\iota^{2,p}(M, N) \subseteq \dot{W}^{2,p}(M, N).$$

(iii) *There exists an isometric embedding $\iota \in C^2(N, \mathbb{R}^\nu)$ such that*

$$\dot{W}_\iota^{2,p}(M, N) \subseteq \dot{W}^{1,2p}(M, N) \cap \dot{W}^{2,p}(M, N).$$

As a consequence of the latter proposition, we prove that for every $r > 0$, if $1 \leq 2p < m$ and if the manifold N is compact,

$$\dot{W}_\iota^{2,p}(\mathbb{B}_r^m, N) \subsetneq \dot{W}^{2,p}(\mathbb{B}_r^m, N)$$

by exhibiting a map $u \in \dot{W}^{2,p}(\mathbb{B}_r^m, N)$ such that

$$\int_{\mathbb{B}_r^m} |D_K(Tu)|_{g_m^* \otimes g_m^* \otimes g_N}^p < +\infty \quad \text{but} \quad \int_{\mathbb{B}_r^m} |Tu|_{g_m^* \otimes g_N}^{2p} = +\infty,$$

that is, $u \notin \dot{W}_\iota^{2,p}(\mathbb{B}_r^m, N)$ (proposition 4.5.6). This example implies the failure of the Gagliardo–Nirenberg interpolation inequalities [49, 95] for those spaces. In the general case where the target manifold N is not compact, this leads to an open question 4.5.1: does it exist a constant $C > 0$ such that for every $u \in \dot{W}^{2,p}(M, N) \cap L^\infty(M, N)$,

$$\begin{aligned} & \int_M |Tu|_{g_M^* \otimes g_N}^{2p} \\ & \leq C \left(\text{ess sup} \{ d_N(u(x), u(y)) : x, y \in M \} \right)^p \int_M |D_K(Tu)|_{g_M^* \otimes g_M^* \otimes g_N}^p ? \end{aligned}$$

The answer should involve in particular the geometry of M and N .

We consider also the question of density of the set of smooth maps $C^k(M, N)$ in $\dot{W}^{k,p}(M, N)$ when M and N are compact manifolds. In the first-order case $k = 1$, the density of smooth maps is known to be equivalent to some homotopy invariant of the pair (M, N) [17, 64]. Although these results are proved for embedded manifolds, the Sobolev space and the convergence are intrinsic [34] and the density result is thus intrinsic. In the higher-order case on the ball, the set of smooth maps $C^2(\mathbb{B}^m, N)$ is dense in the space $\dot{W}_\iota^{2,p}(\mathbb{B}^m, N)$ if and only if $\pi_{[2p]}(N) \simeq \{0\}$ [22]. Since the set $\dot{W}^{2,p}(\mathbb{B}^m, N)$ is larger than $\dot{W}_\iota^{2,p}(\mathbb{B}^m, N)$ but is endowed with a weaker notion of convergence (proposition 4.5.4), it is not immediate that the topological condition is either necessary or sufficient for the density of smooth maps.

We prove that the topological condition is still a necessary condition for the strong and weak-bounded approximation by smooth maps (proposition 4.5.7).

Proposition 4.3. *Let N be a smooth connected compact Riemannian manifold and let $p \in [1, +\infty)$ so that $1 \leq 2p < m$.*

- (i) *If for every $u \in \dot{W}^{2,p}(\mathbb{B}^m, N)$, there exists a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ in $C^\infty(\mathbb{B}^m, N)$ that is bounded in $\dot{W}^{2,p}(\mathbb{B}^m, N)$ and that converges almost everywhere to u , and if $2p \notin \mathbb{N}$, then $\pi_{[2p]}(N) \simeq \{0\}$.*
- (ii) *If for every $u \in \dot{W}^{2,p}(\mathbb{B}^m, N)$, there exists a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ in $C^\infty(\mathbb{B}^m, N)$ that converges strongly to u in $\dot{W}^{2,p}(\mathbb{B}^m, N)$, and if $2p \in \mathbb{N}$, then $\pi_{2p}(N) \simeq \{0\}$.*

Giving some sufficient condition for the approximation problems seems to require new intrinsic approximation tools that would go beyond the scope of the present work.

Since Sobolev spaces by embedding (4.1) are defined by composition with a map $\iota \in C^2(N, \mathbb{R}^\nu)$ which is an immersion, we investigate for the twice colocal weak differentiability the chain rule in view of the comparison of the definition by embedding and the intrinsic one. Moreover, since the first-order colocal weak differentiability was defined by the chain rule, another natural candidate definition for the second-order differentiability could have been some second-order chain-rule condition. Such a definition turns out to be stronger and ill-behaved. Given a twice colocally weakly differentiable map $u: M \rightarrow N$, we ask whether for every $f \in C_c^2(N, \mathbb{R})$, the map $f \circ u$ is twice weakly differentiable and whether the chain rule $T^2(f \circ u) = T^2 f \circ T^2 u$ holds. The chain rule *does not hold in general* (example 4.4.1); however, it holds under an additional

necessary and sufficient integrability condition on colocal weak derivatives of bundle morphism (proposition 4.4.3).

Proposition 4.4. *Let $u: M \rightarrow N$ be a twice colocally weakly differentiable map. The following statements are equivalent.*

- (i) *For every $f \in C_c^2(N, \mathbb{R})$, the map $f \circ u$ is twice weakly differentiable and almost everywhere in M*

$$T^2(f \circ u) = T^2f \circ T^2u,$$

- (ii) *for every compact sets $K \subseteq M$ and $L \subseteq N$,*

$$\int_{u^{-1}(L) \cap K} \llbracket T^2u \rrbracket < +\infty,$$

- (iii) *for every $h \in C^1(T^*M \otimes TN, \mathbb{R})$ that is linear with respect to Tu and has compact support with respect to u , the map $h \circ Tu$ is weakly differentiable.*

The condition (ii) involves a *double norm* $\llbracket \cdot \rrbracket$ of the double vector bundle morphism T^2u (definition 4.4.2). Such a double norm is a seminorm on fibers over both vector bundle structures of the double vector bundle.

4.2. Colocal weak differentiability

4.2.1. Double colocal weak differentiability on differentiable manifolds

We assume that M and N are *differentiable manifolds* of class C^2 of respective dimensions m and n . As usual, the manifolds M and N satisfy the Hausdorff separation property and have a countable basis [42, §0.5; 70, §1.5]. We study the notion of second-order differentiability for measurable maps from M to N .

As a preliminary we recall various definitions of local measure-theoretical notions on a manifold. A set $E \subset M$ is *negligible* if for every $x \in M$ there exists a local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$ – that is $\psi: V \subseteq M \rightarrow \psi(V) \subseteq \mathbb{R}^m$ is a diffeomorphism – such that $x \in V$ and the set $\psi(E \cap V) \subseteq \mathbb{R}^m$ is negligible. A map $u: M \rightarrow N$ is *measurable* if for every point $x \in M$ there exists a local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and the map $u \circ \psi^{-1}$ is measurable [40, §3; 70, §3.1]. A function $u: M \rightarrow \mathbb{R}$ is *locally integrable* if for every $x \in M$ there exists a

local chart $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and $u \circ \psi^{-1}$ is integrable on $\psi(V)$ [71, §6.3]. Similarly, a locally integrable map $u : M \rightarrow \mathbb{R}$ is *weakly differentiable* if for every point $x \in M$ there exists a local chart $\psi : V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and the map $u \circ \psi^{-1}$ is weakly differentiable. All these notions are independent on any particular metric or measure on the manifold M and are invariant under diffeomorphisms of M .

Given a *colocally weakly differentiable map* $u : M \rightarrow N$ [34, definition 1.1], by existence and uniqueness of the colocal weak derivative [34, proposition 1.5], there exists a unique measurable bundle morphism $Tu : TM \rightarrow TN$ such that for every $f \in C_c^1(N, \mathbb{R})$, $T(f \circ u) = Tf \circ Tu$ almost everywhere in M , where (TM, π_M, M) is the *tangent bundle* over M , that is,

$$TM = \bigcup_{x \in M} \{x\} \times T_x M,$$

$\pi_M : TM \rightarrow M$ is the natural projection and for every point $x \in M$, the fiber $\pi_M^{-1}(\{x\})$ is isomorphic to the space $\mathbb{R}^m$, and where a map $v : TM \rightarrow TN$ is a *bundle morphism* that covers the map $u : M \rightarrow N$ if the diagram

$$\begin{array}{ccc} TM & \xrightarrow{v} & TN \\ \pi_M \downarrow & & \downarrow \pi_N \\ M & \xrightarrow{u} & N \end{array}$$

commutes, that is, we have $\pi_N \circ v = u \circ \pi_M$ on TM , and for every $x \in M$, the map $v(x) : \pi_M^{-1}(\{x\}) \simeq T_x M \rightarrow \pi_N^{-1}(\{u(x)\}) \simeq T_{u(x)} N$ is linear. As a consequence, the map Tu can be viewed as a measurable map from M to the bundle ${}^1 T^*M \otimes TN$, that is, for almost every $x \in M$, we have $Tu(x) \in T_x^*M \otimes T_{u(x)}N = \mathcal{L}(T_x M, T_{u(x)} N)$. Since both M and N are manifolds of class C^2 , the vector bundle $T^*M \otimes TN$ has a manifold structure of class C^1 and so we can define the notion of twice colocally weakly differentiable maps recursively.

Definition 4.2.1. A map $u : M \rightarrow N$ is *twice colocally weakly differentiable* whenever u is colocally weakly differentiable and its colocal weak derivative $Tu : M \rightarrow T^*M \otimes TN$ is itself colocally weakly differentiable.

If $u \in W_{\text{loc}}^{2,1}(\mathbb{R}^m, \mathbb{R}^n)$, that is, if the map $u : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is twice weakly differentiable, then u is twice colocally weakly differentiable. Indeed, the

¹The tensor product $(T^*M \otimes TN, \pi_{M \times N}, M \times N)$ of the bundles T^*M and TN is a vector bundle over $M \times N$ such that for every $(x, y) \in M \times N$, $\pi_{M \times N}^{-1}(\{(x, y)\}) = T_x^*M \otimes T_y N$.

weak derivative $Du: \mathbb{R}^m \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ is weakly differentiable and so for every $f \in C_c^1(\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n), \mathbb{R})$, the composite map $f \circ Du$ is weakly differentiable. But, as in the case of colocally weakly differentiable maps, *the converse is false*. For example, the function $u: \mathbb{R}^m \rightarrow \mathbb{R}$ defined for every $x \in \mathbb{R}^m \setminus \{0\}$ by $u(x) = |x|^{-\alpha}$ does not belong to $W_{\text{loc}}^{2,1}(\mathbb{R}^m)$ for any $\alpha > m - 2$, but is twice colocally weakly differentiable for every $\alpha \in \mathbb{R}$. The boundedness of u is not essential: indeed for every $\alpha < m - 1$, the function $(\cos \circ u, \sin \circ u): \mathbb{R}^m \rightarrow \mathbb{S}^1 \subseteq \mathbb{R}^2$ is twice colocally weakly differentiable but it does not belong to $W_{\text{loc}}^{2,1}(\mathbb{R}^m, \mathbb{R}^2)$ if $\alpha > \frac{m-2}{2}$.

The natural framework of the colocal weak derivative of a bundle morphism involves the *second-order tangent bundle* $T^2M := T(TM)$ [15, §7.1; 87, §8.13]; the natural projection $\pi_M^2: T^2M \rightarrow M$ is *not a vector bundle* but the tangent manifold T^2M has two natural vector bundle structures as a bundle over TM [15, §9.1; 41, §16.15.7; 84, example 1.3; 87, §8.12], that is,

- the canonical vector bundle structure of the tangent bundle $\pi_{TM}: T^2M \rightarrow TM$,
- the tangent vector bundle structure on the tangent fibration $T\pi_M: T(TM) \rightarrow TM$;

hence, T^2M is a *double vector bundle* [57, theorem 3.1; 84, definition 1.1], that is, a system of four vector bundle structures such that the following diagram commutes

$$\begin{array}{ccc} T^2M & \xrightarrow{\pi_{TM}} & TM \\ T\pi_M \downarrow & & \downarrow \pi_M \\ TM & \xrightarrow{\pi_M} & M \end{array}$$

and such that each of the four structure maps (namely, the bundle projection, the zero section, addition and scalar multiplication) of each vector bundle structure on T^2M is a morphism of vector bundles with respect to the other structure; the double vector bundle T^2M has one more natural structure, the *canonical flip* $\kappa_M: T^2M \rightarrow T^2M$ which is a linear isomorphism from the bundle $(T^2M, T\pi_M, TM)$ to the bundle (T^2M, π_{TM}, TM) [87, §8.13]; the space of all *double vector bundle morphisms* $\text{Mor}(T^2M, T^2N)$ is defined by requiring that for every $(x, y) \in M \times N$, the fiber $\text{Mor}(T^2M, T^2N)_{x,y}$ consists of maps $v: T_x^2M \rightarrow T_y^2N$ and $v_1, v_2 \in \mathcal{L}(T_xM, T_yN)$ such that $\pi_{TN} \circ v = v_1 \circ \pi_{TM}$ and $T\pi_N \circ v = v_2 \circ T\pi_M$; we note that $\text{Mor}(T^2M, T^2N)$ is a double vector bundle with the following commutative diagram

$$\begin{array}{ccc}
\text{Mor}(T^2M, T^2N) & \xrightarrow{\pi_{T^*M \otimes TN}^1} & T^*M \otimes TN \\
\pi_{T^*M \otimes TN}^2 \downarrow & & \downarrow \pi_{M \times N} \\
T^*M \otimes TN & \xrightarrow{\pi_{M \times N}} & M \times N
\end{array}$$

where for every $v \in \text{Mor}(T^2M, T^2N)$, $\pi_{T^*M \otimes TN}^1 \circ v = \pi_{TN} \circ v \circ 0_{T\pi_M}$ and $\pi_{T^*M \otimes TN}^2 \circ v = T\pi_N \circ v \circ 0_{\pi_{TM}}$, with the two zero sections $0_{\pi_{TM}}: TM \rightarrow T^2M$ and $0_{T\pi_M}: TM \rightarrow T^2M$. If $u \in C^2(M, N)$, then the second-order tangent map $T^2u: T^2M \rightarrow T^2N$ and the maps $Tu: TM \rightarrow TN$ and $u: M \rightarrow N$ define a *morphism of double vector bundles* [84, definition 1.2] such that the following diagrams commute.

$$\begin{array}{ccc}
\begin{array}{ccccc}
& & T^2M & \xrightarrow{T^2u} & T^2N \\
& \swarrow \pi_{TM} & \downarrow & \swarrow \pi_{TN} & \downarrow T\pi_N \\
TM & \xrightarrow{T\pi_M} & TN & & \\
\downarrow \pi_M & & \downarrow \pi_N & & \\
& TM & \xrightarrow{T\pi_M} & TN & \\
& \swarrow \pi_M & \downarrow \pi_N & \swarrow \pi_N & \\
M & \xrightarrow{u} & N & &
\end{array} &
\begin{array}{ccccc}
& & T^2M & \xrightarrow{T^2u} & T^2N \\
& \swarrow \kappa_M & \downarrow & \swarrow \kappa_N & \downarrow T\pi_N \\
T^2M & \xrightarrow{T^2u} & T^2N & & \\
\downarrow \pi_{TM} & & \downarrow \pi_{TN} & & \\
& TM & \xrightarrow{T\pi_M} & TN & \\
& \swarrow \pi_M & \downarrow \pi_N & \swarrow \pi_N & \\
TM & \xrightarrow{T\pi_M} & TN & &
\end{array}
\end{array}$$

Moreover, the map T^2u can be viewed as a section from M to $\text{Mor}(T^2M, u^*(T^2M))$, that is, for every $x \in M$, $T^2u(x) \in \text{Mor}(T^2M, T^2N)_{x, u(x)}$. However, twice colocally weakly differentiable maps are just measurable and so we say that a measurable map $v: M \rightarrow \text{Mor}(T^2M, T^2N)$ covers $u: M \rightarrow N$ whenever for almost every $x \in M$,

$$\pi_{M \times N} \circ \pi_{T^*M \otimes TN}^1 \circ v(x) = \pi_{M \times N} \circ \pi_{T^*M \otimes TN}^2 \circ v(x) = (x, u(x)).$$

If (E, π_M, M) is a vector bundle, the *vertical bundle* $VE \rightarrow E$ is the subbundle of $TE \rightarrow E$ defined by [116, definition 3.1]

$$VE = \{\nu \in TE: T\pi_M(\nu) = 0\} = \ker T\pi_M,$$

and for every $e \in E$, the *vertical lift* $\text{Vert}_e: E \rightarrow TE$ defined for $\nu \in \pi_M^{-1}(\{\pi_M(e)\})$ by

$$\text{Vert}_e(\nu) = \left. \frac{d}{dt}(e + t\nu) \right|_{t=0}$$

gives a natural isomorphism between $E_{\pi_M(e)}$ and $V_e E$. This construction is independent of any connection.

Proposition 4.2.1. *If the map $u: M \rightarrow N$ is twice colocally weakly differentiable, then there exists a unique measurable map $T^2u: M \rightarrow \text{Mor}(T^2M, T^2N)$ that covers the map u such that for every function $f \in C^1(T^*M \otimes TN, \mathbb{R})$ such that $f \circ Tu \in W_{\text{loc}}^{1,1}(M)$,*

$$T(f \circ Tu) = Tf \circ T^2u$$

almost everywhere in M . Moreover, $\kappa_N \circ T^2u = T^2u \circ \kappa_M$ and $T^2u \circ \text{Vert}_{TM} = \text{Vert}_{TN} \circ Tu$ almost everywhere in M .

In other words, the map T^2u is the colocal weak derivative of the bundle morphism Tu . Moreover, for almost every $x \in M$, the double vector bundle morphism $T^2u(x) \in \text{Mor}(T^2M, T^2N)_{x, u(x)}$ satisfies $\pi_{T^*M \otimes TN}^1 \circ T^2u(x) = \pi_{T^*M \otimes TN}^2 \circ T^2u(x) = Tu(x)$ and $T^2u(x) \circ \text{Vert}_{TM} = \text{Vert}_{TN} \circ Tu(x)$.

The first lemma recalls how these properties are obtained when the target manifold is a Euclidean space. We remind the reader that a locally integrable map $u: M \rightarrow \mathbb{R}$ is *twice weakly differentiable* if for every $x \in M$, there exists a local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and the map $u \circ \psi^{-1}$ belongs to $W_{\text{loc}}^{2,1}(\psi(V))$.

Lemma 4.2.2. *If the locally integrable map $v: M \rightarrow \mathbb{R}^n$ is twice weakly differentiable, then there exists a unique measurable map $T^2v: M \rightarrow \text{Mor}(T^2M, T^2\mathbb{R}^n)$ that covers the map v such that for every local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$,*

$$T^2(v \circ \psi^{-1}) = T^2v \circ T^2(\psi^{-1})$$

almost everywhere in $\psi(V)$. Moreover, $T\pi_{\mathbb{R}^n} \circ T^2v = Tv \circ T\pi_M$, $\kappa_{\mathbb{R}^n} \circ T^2v = T^2v \circ \kappa_M$ and $T^2v \circ \text{Vert}_{TM} = \text{Vert}_{T\mathbb{R}^n} \circ Tv$ almost everywhere in M .

Sketch of the proof. First, if M is an open set of $\mathbb{R}^m$, since the domain is flat, for almost every $x \in M$, $T^2v(x): T_x^2M \rightarrow T_{v(x)}^2\mathbb{R}^n$ is defined for every $(e, \mu, \nu) \in T_x^2M$ by

$$T^2v(x)[e, \mu, \nu] = (Dv(x)[e], Dv(x)[\mu], D^2v(x)[e, \mu] + Dv(x)[\nu]),$$

and the conclusion follows.

Next, for a general manifold M , for every local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$, the map $v \circ \psi^{-1}: \psi(V) \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$ is twice weakly differentiable. Since $\psi \in C^2(V, \mathbb{R}^m)$, we have $T^2\psi \in \text{Mor}(T^2V, \psi^*(T^2\mathbb{R}^m))$, $\kappa_{\mathbb{R}^m} \circ T^2\psi = T^2\psi \circ \kappa_M$ and $T^2\psi \circ \text{Vert}_{TM} = \text{Vert}_{T\mathbb{R}^m} \circ T\psi$ [87, §8.13]. As a consequence, since almost everywhere in V

$$T^2v = T^2(v \circ \psi^{-1}) \circ T^2\psi,$$

by the previous step, we can deduce that $T\pi_{\mathbb{R}^n} \circ T^2v = Tv \circ T\pi_M$, $\kappa_{\mathbb{R}^n} \circ T^2v = T^2v \circ \kappa_M$ and $T^2v \circ \text{Vert}_{TM} = \text{Vert}_{T\mathbb{R}^n} \circ Tv$ almost everywhere in V . Since $\pi_{TM} = T\pi_M \circ \kappa_M$ on T^2M and $\pi_{T\mathbb{R}^n} = T\pi_{\mathbb{R}^n} \circ \kappa_{\mathbb{R}^n}$ on $T^2\mathbb{R}^n$ [87, §8.13], then $\pi_{T\mathbb{R}^n} \circ T^2v = Tv \circ \pi_{TM}$ almost everywhere in V and so for almost every $x \in V$, $T^2v(x) \in \text{Mor}(T^2M, T^2N)_{x, v(x)}$. Finally, since the manifold M has a countable atlas of local charts, by a direct covering argument, the previous equalities are satisfied almost everywhere in M . $\square$

The following lemma 4.2.3 allows to embed a local chart into a compactly supported function; its proof and its statement is similar to the one for C^1 -manifolds [34, lemma 1.6].

Lemma 4.2.3 (Extended local charts). *For every $y \in N$, there exist an open subset $U \subseteq N$ such that $y \in U$, and maps $\varphi \in C^2(N, \mathbb{R}^n)$ and $\varphi^* \in C^2(\mathbb{R}^n, N)$ such that*

- (i) *the set $\text{supp}(\varphi)$ is compact,*
- (ii) *the set $\overline{\{x \in \mathbb{R}^n : \varphi^*(x) \neq y\}}$ is compact,*
- (iii) *the map $\varphi|_U$ is a diffeomorphism of class C^2 onto its image $\varphi(U)$,*
- (iv) *$\varphi^* \circ \varphi = \text{id}$ in U .*

Proof. By definition of manifold of class C^2 , there exists a local chart $\psi: V \subseteq N \rightarrow \psi(V) \subseteq \mathbb{R}^n$ around $y \in N$ such that $\psi: V \rightarrow \psi(V)$ is a diffeomorphism of class C^2 . Without loss of generality, we assume that $\psi(y) = 0$. Since the set $\psi(V) \subseteq \mathbb{R}^n$ is open, there exists $r > 0$ such that $B_{2r} \subseteq \psi(V)$. We choose a function $\theta \in C_c^2(\mathbb{R}^n, \mathbb{R})$ such that $0 \leq \theta \leq 1$ on $\mathbb{R}^n$, $\theta = 1$ on B_r and $\text{supp}(\theta) \subset B_{2r}$. We take the set $U = \psi^{-1}(B_r)$ and the maps $\varphi: N \rightarrow \mathbb{R}^n$ defined for every $z \in N$ by

$$\varphi(z) = \begin{cases} \theta(\psi(z)) \psi(z) & \text{if } z \in \psi^{-1}(B_{2r}), \\ 0 & \text{otherwise} \end{cases}$$

and $\varphi^*: \mathbb{R}^n \rightarrow N$ defined for every $x \in \mathbb{R}^n$ by $\varphi^*(x) = \psi^{-1}(\theta(x)x)$. $\square$

Proof of proposition 4.2.1. By the existence and uniqueness of the colocal weak derivative [34, proposition 1.5], there exists a unique measurable bundle morphism $T^2u: TM \rightarrow T(T^*M \otimes TN)$ such that for every $f \in C^1(T^*M \otimes TN, \mathbb{R})$ such that $f \circ Tu \in W_{\text{loc}}^{1,1}(M)$, we have $T(f \circ Tu) = Tf \circ T^2u$ almost everywhere in M . As for the colocal weak derivative, the map T^2u can be viewed as a measurable map from M to $T^*M \otimes T(T^*M \otimes TN)$, which is a subset of $\text{Mor}(T^2M, T^2N)$.

Let $U \subseteq N$, $\varphi \in C^2(N, \mathbb{R}^n)$ and $\varphi^* \in C^2(N, \mathbb{R}^n)$ be the extended local charts given by lemma 4.2.3. For every $y \in T^*M \otimes TU$, there exist two open sets $V, W \subseteq T^*M \otimes TU$ such that $y \in \bar{V} \subseteq W$ and then a map $\eta \in C_c^1(T^*M \otimes TN, \mathbb{R})$ such that $0 \leq \eta \leq 1$ on $T^*M \otimes TN$, $\eta = 1$ on V and $\text{supp}(\eta) \subset W$. As a consequence, the function h defined for every $z \in T^*M \otimes TN$ by

$$h(z) = \begin{cases} \eta(z)T\varphi(z) & \text{if } z \in W, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $C_c^1(T^*M \otimes TN, \mathbb{R}^n)$ and satisfies $T(h \circ Tu) = Th \circ T^2u$ almost everywhere in M .

Since $\eta = 1$ on V , we have in particular $T^2(\varphi \circ u) = T^2\varphi \circ T^2u$ almost everywhere in $u^{-1}(\pi_N(\pi_{M \times N}(V)))$. Since $T^2\varphi^* \circ T^2\varphi = \text{id}$ on $(\pi_N^2)^{-1}(U)$, by lemma 4.2.2 and since $\varphi^* \in C^2(\mathbb{R}^n, N)$ [87, §8.13], almost everywhere in $u^{-1}(\pi_N(\pi_{M \times N}(V)))$

$$\begin{aligned} \kappa_N \circ T^2u &= \kappa_N \circ T^2\varphi^* \circ T^2(\varphi \circ u) \\ &= T^2\varphi^* \circ \kappa_{\mathbb{R}^n} \circ T^2(\varphi \circ u) \\ &= T^2\varphi^* \circ T^2(\varphi \circ u) \circ \kappa_M = T^2u \circ \kappa_M. \end{aligned}$$

In particular, we have

$$T^2u = \kappa_N \circ T^2u \circ \kappa_M,$$

which gives a second bundle morphism structure to T^2u so that T^2u is a double vector bundle morphism in $u^{-1}(\pi_N(\pi_{M \times N}(V)))$.

The facts that $T^2u \circ \text{Vert}_{TM} = \text{Vert}_{TN} \circ Tu$ and $T\pi_N \circ T^2u = Tu \circ T\pi_M$ almost everywhere in $u^{-1}(\pi_N(\pi_{M \times N}(V)))$ can be proved along the same line. Since $\pi_{TM} = T\pi_M \circ \kappa_M$ on T^2M and $\pi_{TN} = T\pi_N \circ \kappa_N$ on T^2N [87, §8.13],

$$\begin{aligned} Tu \circ \pi_{TM} &= Tu \circ T\pi_M \circ \kappa_M = T\pi_N \circ T^2u \circ \kappa_M \\ &= T\pi_N \circ \kappa_N \circ T^2u = \pi_{TN} \circ T^2u \end{aligned}$$

almost everywhere in $u^{-1}(\pi_N(\pi_{M \times N}(V)))$. So for almost every $x \in u^{-1}(\pi_N(\pi_{M \times N}(V)))$,

$$\pi_{T^*M \otimes TN}^1 \circ T^2u(x) = \pi_{T^*M \otimes TN}^2 \circ T^2u(x) = Tu(x)$$

and $\pi_{M \times N} \circ Tu(x) = (x, u(x))$.

Finally, by a direct covering argument, since the manifolds M and N have a countable basis, the measurable map $T^2u: M \rightarrow \text{Mor}(T^2M, T^2N)$ satisfies the previous properties almost everywhere in M . $\square$

4.2.2. Sequences of twice colocally weakly differentiable maps

In order to study the lower semi-continuity of functionals, it is interesting to have some sufficient conditions in the calculus for a limit of twice colocally weakly differentiable maps to be also twice colocally weakly differentiable. In this part, we deduce from the closure property for first-order derivatives [34, proposition 3.7] a closure property for second-order derivatives.

We first recall a notion of convergence in measure [34, definition 3.5].

Definition 4.2.2. A sequence $(u_\ell)_{\ell \in \mathbb{N}}$ of maps from M to N *converges locally in measure* to a map $u: M \rightarrow N$ whenever for every $x \in M$ there exists a local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and for every open set $U \subseteq N$,

$$\lim_{\ell \rightarrow \infty} \mathcal{L}^m(\psi((u^{-1}(U) \cap V) \setminus u_\ell^{-1}(U))) = 0.$$

The definition applies directly to a sequence $(v_\ell)_{\ell \in \mathbb{N}}$ of bundle morphisms between TM and TN viewed as maps from M to $T^*M \otimes TN$.

We then generalize the notion of uniform integrability [34, definition 3.6] to any bundle morphisms.

Definition 4.2.3. Let (F, π_N, N) be a vector bundle. A sequence $(v_\ell)_{\ell \in \mathbb{N}}$ of bundle morphisms from TM to F that covers a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ of maps from M to N is *bilocally uniformly integrable* whenever for every $x \in M$ and every $y \in N$, there exist a local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$ and a bundle chart $\varphi: \pi_N^{-1}(U) \subseteq F \rightarrow \mathbb{R}^q$ ($\varphi: \pi_N^{-1}(U) \subseteq F \rightarrow \varphi(\pi_N^{-1}(U)) \subseteq \mathbb{R}^q$ is a diffeomorphism and $U \subseteq N$ is open) such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $W \subseteq \psi(V)$, $\mathcal{L}^m(W) \leq \delta$ and $\ell \in \mathbb{N}$,

$$\int_{W \cap \psi(u_\ell^{-1}(U))} |\varphi \circ v_\ell \circ T(\psi^{-1})| \leq \varepsilon.$$

This definition applies directly to a sequence $(v_\ell)_{\ell \in \mathbb{N}}$ from TM to TN or from TM to $T(T^*M \otimes TN)$.

Proposition 4.2.4. *Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of twice colocally weakly differentiable maps from M to N . If the sequence $(Tu_\ell)_{\ell \in \mathbb{N}}$ converges locally in measure to a measurable map $v: M \rightarrow T^*M \otimes TN$ that covers a map $u: M \rightarrow N$, if the sequences $(Tu_\ell)_{\ell \in \mathbb{N}}$ and $(T^2u_\ell)_{\ell \in \mathbb{N}}$ are bilocally uniformly integrable, then the map u is twice colocally weakly differentiable.*

This proposition is a direct consequence of the closure property for colocally weakly differentiable maps [34, proposition 3.7], applied to the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ and then to $(Tu_\ell)_{\ell \in \mathbb{N}}$.

4.2.3. Higher order colocal weak differentiability

In this part, we explain how previous notions and propositions extend to higher-order derivatives.

For $k \geq 3$, we assume that M and N are manifolds of class C^k . Since the set $T^*M \otimes TN$ is a manifold of class C^{k-1} , we also proceed by induction. A map $u: M \rightarrow N$ is 1 times colocally weakly differentiable whenever it is colocally weakly differentiable.

Definition 4.2.4. Let $k \in \mathbb{N}_*$. A map $u: M \rightarrow N$ is k times colocally weakly differentiable whenever u is $(k-1)$ times colocally weakly differentiable and $Tu: M \rightarrow T^*M \otimes TN$ is $(k-1)$ times colocally weakly differentiable.

As for the notion of twice weak differentiability, there is some natural framework to work with, that is, we denote by $T^k M$ the k^{th} order tangent bundle which is by induction $T^k M := T(T^{k-1} M)$ [15, §7.1]; this manifold has k natural vector bundle structures $\pi_{T^{k-1} M}^1, \dots, \pi_{T^{k-1} M}^k$ over $T^{k-1} M$ [15, §15.1] and it is a k^{tuple} vector bundle [57, definition 4.1 and theorem 5.1; 58, definition 3.1]; the canonical flips correspond to any permutation of the bundle structures and the space of k^{tuple} vector bundle morphisms $\text{Mor}(T^k M, T^k N)$ is defined by the condition that for every $(x, y) \in M \times N$, the fiber $\text{Mor}(T^k M, T^k N)_{x,y}$ consists of maps $v: T_x^k M \rightarrow T_y^k N$ and $v_1, \dots, v_k \in \text{Mor}(T^{k-1} M, T^{k-1} N)_{x,y}$ such that for every $j \in \{1, \dots, k\}$, $\pi_{T^{k-1} N}^j \circ v = v_j \circ \pi_{T^{k-1} M}^j$ [57, definition 4.1; 58, definition 3.1]; we note that $\text{Mor}(T^k M, T^k N)$ has also a k^{tuple} vector bundle structure.

If $u: M \rightarrow N$ is a k times colocally weakly differentiable map, then by existence and uniqueness of the colocal weak derivative [34, proposition 1.5], and by induction, there exist unique measurable maps $T^2 u, \dots, T^k u$ such that for every $j \in \{3, \dots, k\}$, $T^j u: TM \rightarrow T(\text{Mor}(T^{j-1} M, T^{j-1} N))$ is the colocal weak derivative of $T^{j-1} u$; this derivative can also be viewed as a map from M to $\text{Mor}(T^j M, T^j N)$ that covers u , that is, for almost every $x \in M$, $T^j u(x) \in \text{Mor}(T^j M, T^j N)_{x,u(x)}$.

We have a closure property for higher-order colocally weakly differentiable maps similar to proposition 4.2.4 for twice colocally weakly differentiable maps. Indeed, definition 4.2.2 applies directly to a sequence $(v_\ell)_{\ell \in \mathbb{N}}$ of maps from M to $\text{Mor}(T^k M, T^k N)$. Moreover, definition 4.2.3 applies also to a sequence $(v_\ell)_{\ell \in \mathbb{N}}$ from TM to $T(\text{Mor}(T^{k-1} M, T^{k-1} N))$ that covers a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ from M to $\text{Mor}(T^{k-1} M, T^{k-1} N)$.

Proposition 4.2.5. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of k times colocally weakly differentiable maps from M to N . If the sequence $(T^{k-1} u_\ell)_{\ell \in \mathbb{N}}$ converges

locally in measure to a measurable map $v: M \rightarrow \text{Mor}(T^{k-1}M, T^{k-1}N)$ that covers a map $u: M \rightarrow N$ and if for all $j \in \{1, \dots, k\}$, the sequence $(T^j u_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable, then the map u is k times colocally weakly differentiable.

Propositon 4.2.5 is a direct consequence of the closure property for colocally weakly differentiable maps [34, proposition 3.7], applied recursively to the sequences $(u_\ell)_{\ell \in \mathbb{N}}, \dots, (T^{k-1}u_\ell)_{\ell \in \mathbb{N}}$.

4.3. Colocal weak covariant derivatives and Sobolev spaces

In this section we study how differentiability can be characterized in covariant terms when the manifolds M and N have an affine structure.

4.3.1. Geometric preliminaries

We recall some concepts and tools of affine geometry of a vector bundle (E, π_M, M) . A map $K_E: TE \rightarrow E$ is a *connection* if K_E is a bundle morphism from (TE, π_E, E) to (E, π_M, M) that covers π_M and a bundle morphism from $(TE, T\pi_M, TM)$ to (E, π_M, M) that covers $\pi_M^{TM}: TM \rightarrow M$, and for every $e \in E$, $(K_E \circ \text{Vert})(e) = e$ [116, definition 3.9], where the map Vert is defined on page 87. If we endow E with a connection K_E , the *horizontal bundle* $H_K E \rightarrow E$ is the subbundle of $TE \rightarrow E$ defined by [60, definition 3.4]

$$H_K E = \{\nu \in TE: K_E(\nu) = 0\} = \ker K_E;$$

then the direct decomposition $TE = H_K E \oplus VE$ holds [60, proposition 3.5]. Equivalently, there exists a map called the *horizontal lift* $\text{Hor}_K: TM \rightarrow H_K E \subset TE$ such that for each $e \in TM$, $(T\pi_M \circ \text{Hor}_K)(e) = e$. Consequently, if $\text{id}_{TE}: TE \rightarrow TE$ is the identity map, then

$$\text{id}_{TE} = \text{Vert} \circ K_E + \text{Hor}_K \circ T\pi_M. \quad (4.3.2)$$

If (M, g_M) is a Riemannian manifold and if we endow E with a metric connection K_E , then the *Sasaki metric* G_E^S [102] (see also [42, chapter 3, exercise 2]) is defined for every $\nu \in TE$ by

$$G_E^S(\nu) = g_M(T\pi_M(\nu)) + g_E(K_E(\nu)).$$

If K_{TM} and K_{TN} are connections on TM and on TN respectively, a connection $K_{T^*M \otimes TN}: T(T^*M \otimes TN) \rightarrow T^*M \otimes TN$ can be defined for

every $v \in T(T^*M)$ and $w \in T(TN)$ by

$$K_{T^*M \otimes TN}(v \otimes w) = K_{T^*M}(v) \otimes K_{TN}(w),$$

where $K_{T^*M}: T(T^*M) \rightarrow T^*M$ is a connection on T^*M induced by K_{TM} . If $u: M \rightarrow N$ is a smooth map, the *second-covariant derivative* $D_K^2 u: M \rightarrow T^*M \otimes T^*M \otimes TN$ with respect to the connection $K_{T^*M \otimes TN}$ is defined by

$$D_K^2 u = K_{T^*M \otimes TN} \circ T^2 u.$$

More specifically, for every $x \in M$, the map $D_K^2 u(x): T_x M \times T_x M \rightarrow T_{u(x)} N$ is bilinear [116, equation (3.10)] and for every $e \in T_x M$, by equation (4.3.2), we have the decomposition

$$T^2 u(x)[e] = \text{Vert} \circ D_K^2 u(x)[e] + \text{Hor}_K \circ Tu(x)[e],$$

where $D_K^2 u(x)[e] = K_{T^*M \otimes TN}(T^2 u(x)[e]) \in \mathcal{L}(T_x M, T_{u(x)} N)$ and with the usual identification of $M \times TN$ with $M \times \{0\} \times TN \subseteq TM \times TN$.

4.3.2. Definition and properties of colocal weak covariant derivatives

In this part, we assume that M and N are affine manifolds.

We first define the notion of colocal weak covariant derivative.

Definition 4.3.1. Let $v: M \rightarrow T^*M \otimes TN$ be a colocally weakly differentiable map that covers a map $u: M \rightarrow N$. A map $D_K v: M \rightarrow T^*M \otimes T^*M \otimes TN$ is a *colocal weak covariant derivative* of v whenever $D_K v$ is measurable, for almost every $x \in M$, the map $D_K v(x): T_x M \times T_x M \rightarrow T_{u(x)} N$ is bilinear and for every $f \in C_c^1(T^*M \otimes TN, \mathbb{R})$,

$$(Tf)(v)[\text{Vert} \circ D_K v] = T(f \circ v) - (Tf)(v)[\text{Hor}_K \circ v]$$

almost everywhere in M .

Since the vertical lifting Vert is injective and since Tf can be taken to be injective at a given set of points, a map v has at most one colocal weak covariant derivative.

The previous definition 4.3.1 requires the colocal weak differentiability of the map v . The main result of the current section is that the colocal weak derivative and the colocal weak covariant derivative are equivalent objects.

Proposition 4.3.1. *Let $u: M \rightarrow N$ be a twice colocally weakly differentiable map. Then the map $Tu: M \rightarrow T^*M \otimes TN$ has a colocal*

weak covariant derivative $D_K(Tu): M \rightarrow T^*M \otimes T^*M \otimes TN$ if and only if the map $Tu: M \rightarrow T^*M \otimes TN$ has a colocal weak derivative $T^2u: TM \rightarrow T(T^*M \otimes TN)$. Furthermore, they are unique and almost everywhere in M

$$T^2u = \text{Vert} \circ D_K(Tu) + \text{Hor}_K \circ Tu,$$

and

$$D_K(Tu) = K_{T^*M \otimes TN} \circ T^2u.$$

Thus the colocal weak covariant derivatives for two connections K_1 and K_2 can be related to each others via the identity

$$\text{Vert} \circ D_{K_1}(Tu) + \text{Hor}_{K_1} \circ Tu = \text{Vert} \circ D_{K_2}(Tu) + \text{Hor}_{K_2} \circ Tu.$$

Proof of proposition 4.3.1. On the one hand, we assume that the colocal weak covariant derivative $D_K(Tu)$ exists. Then for every $f \in C_c^1(T^*M \otimes TN, \mathbb{R})$, almost everywhere in M

$$T(f \circ Tu) = (Tf)(Tu)[\text{Vert} \circ D_K(Tu) + \text{Hor}_K \circ Tu]$$

and we can take

$$T^2u = \text{Vert} \circ D_K(Tu) + \text{Hor}_K \circ Tu$$

for the second-order derivative of u .

Conversely, we assume that there is a colocal weak derivative $T^2u: TM \rightarrow T(T^*M \otimes TN)$ such that for every $f \in C_c^1(T^*M \otimes TN, \mathbb{R})$, $T(f \circ Tu) = Tf \circ T^2u$ almost everywhere in M . In view of the identity (4.3.2), almost everywhere in M

$$T(f \circ Tu) = (Tf)(Tu)[\text{Vert} \circ K_{T^*M \otimes TN} \circ T^2u + \text{Hor}_K \circ Tu]$$

and we can thus take $D_K(Tu) = K_{T^*M \otimes TN} \circ T^2u$. $\square$

We assume that (M, g_M) and (N, g_N) are Riemannian manifolds with the respective Levi-Civita connection maps on TM and on TN . Our concept of covariant derivative is, under some technical assumptions, equivalent to the notion of covariant derivative of P. Hornung and R. Moser [72, definition 2.5].

Indeed, the metrics on vectors of TM and TN induce a metric $g_M^* \otimes g_M^* \otimes g_N$ on $T^*M \otimes T^*M \otimes TN$. This metric can be computed for every bilinear map $\xi: T_x M \times T_x M \rightarrow T_y N$ by

$$(g_M^* \otimes g_M^* \otimes g_N)(\xi, \xi) = \sum_{1 \leq i, j \leq m} g_N(\xi(e_i, e_j), \xi(e_i, e_j)),$$

where $(e_i)_{1 \leq i \leq m}$ is any orthonormal basis in $\pi_M^{-1}(\{x\})$ with respect to the Riemannian metric g_M .

Proposition 4.3.2. *Let $u \in \dot{W}_{\text{loc}}^{1,1}(M, N)$ and let $f: T^*M \otimes TN \rightarrow T^*M \otimes TN$ be defined for every $\xi \in T^*M \otimes TN$ by*

$$f(\xi) = \frac{\xi}{\sqrt{1 + |\xi|_{g_M^* \otimes g_N}^2}}.$$

If $f \circ Tu$ is colocally weakly differentiable, then Tu is colocally weakly differentiable. Conversely if the map Tu is colocally weakly differentiable and if $|D_K(Tu)|_{g_M^ \otimes g_M^* \otimes g_N} \in L_{\text{loc}}^1(M)$, then the map $f \circ Tu$ is colocally weakly differentiable and almost everywhere in M*

$$D_K(f \circ Tu) = \left(\text{id} - \frac{Tu \otimes Tu}{\sqrt{1 + |Tu|_{g_M^* \otimes g_N}^2}} \right) \frac{1}{\sqrt{1 + |Tu|_{g_M^* \otimes g_N}^2}} D_K(Tu).$$

The advantage of this formulation is that $f \circ Tu$ is a bounded measurable bundle morphism, and thus if N is compact, its colocal weak differentiability is equivalent to its weak differentiability in local charts or in an isometric embedding. In Sobolev spaces, the additional integrability assumption for the converse implication is automatically satisfied.

Proof of proposition 4.3.2. Let us first assume that the composite map $f \circ Tu$ is colocally weakly differentiable. Since the map f is invertible on its image, for every $h \in C_c^1(T^*M \otimes TN, \mathbb{R})$,

$$h \circ Tu = (h \circ f^{-1}) \circ (f \circ Tu),$$

where the map $h \circ f^{-1}$ can be extended to a compactly supported map. By the definition of colocal weak differentiability of $f \circ Tu$, the right-hand side is weakly differentiable and thus the left-hand side is also weakly differentiable and the map Tu is colocally weakly differentiable by definition.

Conversely, almost everywhere in M

$$|T^2u|_{g_M^* \otimes G_{T^*M \otimes TN}^S}^2 = |Tu|_{g_M^* \otimes g_N}^2 + |D_K(Tu)|_{g_M^* \otimes g_M^* \otimes g_N}^2.$$

Hence, $Tu \in \dot{W}_{\text{loc}}^{1,1}(M, T^*M \otimes TN)$ and for every $h \in C_c^1(T^*M \otimes TN, \mathbb{R})$, since the map $h \circ f \in C^1(T^*M \otimes TN, \mathbb{R})$ is Lipschitz-continuous, by the chain rule in Sobolev spaces for maps between manifolds [34, proposition 2.6], the map $(h \circ f) \circ Tu$ is weakly differentiable, and so $f \circ Tu$ is colocally weakly differentiable. In particular we obtain that

$$T(f \circ Tu) = Tf \circ T^2u,$$

and the identity follows. Indeed, since the connection $K_{T^*M \otimes TN}$ is metric, by geometric properties of f and uniqueness of colocal weak covariant derivatives, we can deduce the desired formula. $\square$

4.3.3. Sequences of second order Sobolev maps

We now define intrinsic second-order Sobolev spaces.

Definition 4.3.2. Let $p \in [1, +\infty)$. A map $u: M \rightarrow N$ belongs to the *second-order Sobolev space* $\dot{W}^{2,p}(M, N)$ whenever the map u is twice colocally weakly differentiable and $|D_K(Tu)|_{g_M^* \otimes g_M^* \otimes g_N} \in L^p(M)$.

A classical technique in the calculus of variations is to extract from a minimizing sequence a subsequence that converges almost everywhere. For that, we have a Rellich–Kondrashov type compactness theorem as follows.

Proposition 4.3.3 (Rellich–Kondrashov for second-order Sobolev maps). *Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of twice colocally weakly differentiable maps from M to N and let $w \in \dot{W}^{1,p}(M, N)$. If (N, d) is complete, if there exists $p \in [1, +\infty)$ such that*

$$\sup_{\ell \in \mathbb{N}} \int_M d(u_\ell, w)^p + |Tu_\ell|_{g_M^* \otimes g_N}^p + |D_K(Tu_\ell)|_{g_M^* \otimes g_M^* \otimes g_N}^p < +\infty,$$

*then there is a subsequence $(Tu_{\ell_k})_{k \in \mathbb{N}}$ that converges to a measurable map $v: M \rightarrow T^*M \otimes TN$ almost everywhere in M .*

In particular, if the map v covers the map $u: M \rightarrow N$, the subsequence $(u_{\ell_k})_{k \in \mathbb{N}}$ converges to u almost everywhere in M .

For $p \in (1, 2)$, P. Hornung and R. Moser have a similar result [72, lemma 3.2] for their notion of second-order Sobolev spaces [72, definition 3.1].

Proof of proposition 4.3.3. Since (N, d) is complete, for every $x \in M$, the subset $(\pi_{M \times N})^{-1}(\{x\} \times N)$ of $T^*M \otimes TN$ equipped with the distance d^S , where d^S is induced by the Sasaki metric $G_{T^*M \otimes TN}^S$, is complete. Next, there exists $C > 0$ such that

$$\begin{aligned} \sup_{\ell \in \mathbb{N}} \int_M d^S(Tu, Tw)^p \\ \leq C \sup_{\ell \in \mathbb{N}} \int_M d(u_\ell, w)^p + |Tu_\ell|_{g_M^* \otimes g_N}^p + |Tw|_{g_M^* \otimes g_N}^p < +\infty. \end{aligned}$$

Since for every $\ell \in \mathbb{N}$,

$$|T^2u_\ell|_{g_M^* \otimes G_{T^*M \otimes TN}^S}^2 = |Tu_\ell|_{g_M^* \otimes g_N}^2 + |D_K(Tu_\ell)|_{g_M^* \otimes g_M^* \otimes g_N}^2$$

almost everywhere in M , we have

$$\sup_{\ell \in \mathbb{N}} \int_M |T^2u_\ell|^p < +\infty.$$

Hence, the sequence $(Tu_\ell)_{\ell \in \mathbb{N}}$ satisfies all the assumptions of the Rellich–Kondrashov compactness property for Sobolev maps [34, proposition 3.4] and so there exists a subsequence $(Tu_{\ell_k})_{k \in \mathbb{N}}$ that converges to a measurable map $v: M \rightarrow T^*M \otimes TN$ almost everywhere in M . $\square$

Before considering sequences of Sobolev maps, we rephrased the closure property (proposition 4.2.4) with the notion of colocal weak covariant derivative.

Proposition 4.3.4 (Closure property). *Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of twice colocally weakly differentiable maps from M to N . If the sequence $(Tu_\ell)_{\ell \in \mathbb{N}}$ converges locally in measure to a measurable map $v: M \rightarrow T^*M \otimes TN$ that covers a map $u: M \rightarrow N$, if the sequences $(Tu_\ell)_{\ell \in \mathbb{N}}$ and $(D_K(Tu_\ell))_{\ell \in \mathbb{N}}$ are bilocally uniformly integrable, then the map u is twice colocally weakly differentiable.*

It is a direct consequence of the following lemma 4.3.5 and proposition 4.2.4.

Lemma 4.3.5. *Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of twice colocally weakly differentiable maps from M to N . Then the sequence $(T^2u_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable if and only if the sequence $(D_K(Tu_\ell))_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable.*

Proof. For every $y \in T^*M \otimes TN$, there exist an open set $U \subseteq T^*M \otimes TN$ such that $y \in U$ and a map $\varphi \in C_c^1(T^*M \otimes TN, \mathbb{R}^q)$ such that $\varphi|_U: U \subseteq T^*M \otimes TN \rightarrow \mathbb{R}^q$ is a local chart [34, lemma 1.6] (see also lemma 4.2.3 above). Hence, for every local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$, every $\ell \in \mathbb{N}$ and almost everywhere on $\psi(V \cap (Tu_\ell)^{-1}(U))$, by the relation between T^2u_ℓ and $D_K(Tu_\ell)$ (proposition 4.3.1),

$$\begin{aligned} T\varphi \circ T^2u_\ell \circ T(\psi^{-1}) \\ = T\varphi \circ \text{Vert} \circ D_K(Tu_\ell) \circ T(\psi^{-1}) + T\varphi \circ \text{Hor}_K \circ Tu_\ell \circ T(\psi^{-1}). \end{aligned}$$

Since $\text{supp}(\varphi)$ is compact, the second term is uniformly bounded for $\ell \in \mathbb{N}$, and so the conclusion follows using this equality and one or another assumption. $\square$

Assuming that there exists a subsequence that converges almost everywhere, it is important to have some closure property in the particular case of bounded sequences in Sobolev spaces.

Proposition 4.3.6 (Weak closure property for second-order Sobolev spaces). *Let $p \in [1, +\infty)$. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of twice colocally*

weakly differentiable maps from M to N . Assume that the sequence $(Tu_\ell)_{\ell \in \mathbb{N}}$ converges locally in measure to a measurable map $v: M \rightarrow T^*M \otimes TN$ that covers a map $u: M \rightarrow N$, that

$$\limsup_{\ell \rightarrow \infty} \int_M \left(|Tu_\ell|_{g_M^* \otimes g_N}^p + |D_K(Tu_\ell)|_{g_M^* \otimes g_M^* \otimes g_N}^p \right) < +\infty,$$

and, if $p = 1$, that the sequences $(Tu_\ell)_{\ell \in \mathbb{N}}$ and $(D_K(Tu_\ell))_{\ell \in \mathbb{N}}$ are bilocally uniformly integrable. Then $u \in \dot{W}^{2,p}(M, N)$ and

$$\int_M |D_K(Tu)|_{g_M^* \otimes g_M^* \otimes g_N}^p \leq \liminf_{\ell \rightarrow \infty} \int_M |D_K(Tu_\ell)|_{g_M^* \otimes g_M^* \otimes g_N}^p.$$

For $p \in (1, 2)$, P. Hornung and R. Moser have a similar result [72, lemma 3.2] for their notion of second-order Sobolev spaces [72, definition 3.1].

We first prove the following lemma.

Lemma 4.3.7. *Let $p \in [1, +\infty)$. If $u \in \dot{W}^{1,p}(M, N) \cap \dot{W}^{2,p}(M, N)$, then the map $|Tu|_{g_M^* \otimes g_N}$ belongs to $W^{1,p}(M)$ and almost everywhere in M*

$$|T|Tu|_{g_M^* \otimes g_N}|_{g_M^* \otimes g_1} \leq |D_K(Tu)|_{g_M^* \otimes g_M^* \otimes g_N},$$

where g_1 is the Euclidean metric on $\mathbb{R}$.

Proof. Since $Tu \in \dot{W}^{1,p}(M, T^*M \otimes TN)$ (see proof of proposition 4.3.2 above) and since the map $|\cdot|_{g_M^* \otimes g_N}$ is Lipschitz-continuous with respect to the distance induced by the Sasaki metric $G_{T^*M \otimes TN}^S$, by the chain rule in Sobolev spaces between manifolds [34, proposition 2.6], $|Tu|_{g_M^* \otimes g_N} \in W^{1,p}(M)$. Since the connection $K_{T^*M \otimes TN}$ is metric, by the chain rule formula in Sobolev spaces [33, proposition 5.2] and by the relation between T^2u and $D_K(Tu)$ (proposition 4.3.1), we have the desired inequality. $\square$

Proof of proposition 4.3.6. By weak closure property for first-order Sobolev maps [34, proposition 3.8], $u \in \dot{W}^{1,p}(M, N)$ and $Tu = v$. Moreover, by closure property in terms of the colocal weak covariant derivative (proposition 4.3.4), the map u is twice colocally weakly differentiable. We now need to prove that $|D_K(Tu)|_{g_M^* \otimes g_M^* \otimes g_N} \in L^p(M)$.

By previous lemma 4.3.7, the sequence $(|Tu_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $W^{1,p}(M)$. By the classical Rellich–Kondrashov compactness theorem [9, theorem 2.34 (a); 24, theorem 9.16], and since $(|Tu_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Tu|_{g_M^* \otimes g_N}$ in measure, the sequence $(|Tu_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Tu|_{g_M^* \otimes g_N}$ in $L_{\text{loc}}^p(M)$. Up to a subsequence, the sequence $(|D_K(Tu_\ell)|_{g_M^* \otimes g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges weakly to some w in $L_{\text{loc}}^p(M)$. Since

the sequence $(Tu_\ell)_{\ell \in \mathbb{N}}$ converges to Tu locally in measure, for every $f \in C_c^1(T^*M \otimes TN, \mathbb{R}^q)$, with $q \geq \min(m, \dim(T^*M \otimes TN))$,

$$|T(f \circ Tu)|_{g_M^* \otimes g_q}^2 \leq |f|_{\text{Lip}}^2 (w^2 + |Tu|_{g_M^* \otimes g_N}^2),$$

where g_q is the Euclidean metric on $\mathbb{R}^q$, almost everywhere in M . By the characterization of the norm of the derivative [34, proposition 2.2],

$$|Tu|_{g_M^* \otimes g_N}^2 + |D_K(Tu)|_{g_M^* \otimes g_M^* \otimes g_N}^2 \leq w^2 + |Tu|_{g_M^* \otimes g_N}^2$$

almost everywhere in M and so $|D_K(Tu)|_{g_M^* \otimes g_M^* \otimes g_N} \leq w$. Consequently, by lower semi-continuity of the norm under weak convergence, for each compact set $Q \subseteq M$,

$$\int_Q |D_K(Tu)|_{g_M^* \otimes g_M^* \otimes g_N}^p \leq \int_Q w^p \leq \liminf_{\ell \rightarrow \infty} \int_Q |D_K(Tu_\ell)|_{g_M^* \otimes g_M^* \otimes g_N}^p.$$

Finally, since M has a countable basis, there exists a set $\{Q_i : i \in \mathbb{N}\}$ of compact sets such that for every $i \in \mathbb{N}$, $Q_i \subseteq Q_{i+1}$ and $M = \bigcup_{i \in \mathbb{N}} Q_i$ [80, proposition 4.76] and so by the monotone convergence theorem [20, theorem 2.8.2], we have the desired inequality. $\square$

4.3.4. Higher order weak covariant derivatives and Sobolev spaces

First, we assume that M and N are affine manifolds. In order to simplify the notations, we denote indifferently the connections K , the maps Vert and Hor_K related to the different bundles (section 4.3.1).

If $k = 2$ and u is twice colocally weakly differentiable, then $D_K^1 u = Tu$ and $D_K^2 u = D_K(Tu)$. As for the notion of twice colocally weakly differentiable maps, there is a relation between higher-order weak covariant derivative and higher-vector bundle morphism.

Proposition 4.3.8. *Let $u : M \rightarrow N$ be a k times colocally weakly differentiable map. Then the colocally weakly differentiable map $D_K^{k-1} u : M \rightarrow (\otimes^{k-1} T^*M) \otimes TN$ has a colocal weak covariant derivative $D_K^k u : M \rightarrow T^*M \otimes (\otimes^{k-1} T^*M) \otimes TN$ if and only if $D_K^{k-1} u : M \rightarrow (\otimes^{k-1} T^*M) \otimes TN$ has a colocal weak derivative $TD_K^{k-1} u : TM \rightarrow T((\otimes^{k-1} T^*M) \otimes TN)$, and almost everywhere in M*

$$TD_K^{k-1} u = \text{Vert} \circ D_K^k u + \text{Hor}_K \circ D_K^{k-1} u,$$

and

$$D_K^k u = K \circ TD_K^{k-1} u.$$

Moreover, for every $j \in \mathbb{N}_*$, the map $D_K^k u$ is j times colocally weakly differentiable if and only if the map $D_K^{k-1} u$ is $(j+1)$ times colocally weakly differentiable, and almost everywhere in M

$$T^{j+1} D_K^{k-1} u = (T^j \text{Vert}) \circ (T^j D_K^k u) + (T^j \text{Hor}_K) \circ (T^j D_K^{k-1} u), \quad (4.3.3)$$

and

$$T^j D_K^k u = (T^j K) \circ (T^{j+1} D_K^{k-1} u). \quad (4.3.4)$$

In a second step, we assume that (M, g_M) and (N, g_N) are Riemannian manifolds with Levi–Civita connection maps respectively. The metrics on vectors of TM and TN induce a metric $(\otimes^k g_M^*) \otimes g_N$ on $(\otimes^k T^* M) \otimes TN$. This metric can be computed for every k -linear map $\xi: \times^k T_x M \rightarrow T_y N$ by

$$((\otimes^k g_M^*) \otimes g_N)(\xi, \xi) = \sum_{1 \leq i_1, \dots, i_k \leq m} g_N(\xi(e_{i_1}, \dots, e_{i_k}), \xi(e_{i_1}, \dots, e_{i_k})),$$

where $(e_i)_{1 \leq i \leq m}$ is an orthonormal basis in $\pi_M^{-1}(\{x\})$ with respect to the Riemannian metric g_M .

We are now able to define higher-order Sobolev spaces.

Definition 4.3.3. Let $p \in [1, +\infty)$. A map $u: M \rightarrow N$ belongs to the k^{th} order Sobolev space $\dot{W}^{k,p}(M, N)$ whenever u is k times colocally weakly differentiable and $|D_K^k u|_{(\otimes^k g_M^*) \otimes g_N} \in L^p(M)$.

First, we have a Rellich–Kondrashov type compactness theorem.

Proposition 4.3.9. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of k times colocally weakly differentiable maps from M to N and let $w \in \bigcap_{j=1}^{k-1} \dot{W}^{j,p}(M, N)$. If (N, d) is complete, if there exists $p \in [1, +\infty)$ such that

$$\sup_{\ell \in \mathbb{N}} \int_M d(u_\ell, w)^p + \sum_{j=1}^k |D_K^j u_\ell|_{(\otimes^j g_M^*) \otimes g_N}^p < +\infty,$$

then there is a subsequence $(T^{k-1} u_{\ell_i})_{i \in \mathbb{N}}$ that converges to a measurable map $v: M \rightarrow \text{Mor}(T^{k-1} M, T^{k-1} N)$ almost everywhere in M .

By relying recursively on the formulas (4.3.3) and (4.3.4), the boundedness of $(D_K^k u_\ell)_{\ell \in \mathbb{N}}$ is related to the boundedness of the sequence $(T^k u_\ell)_{\ell \in \mathbb{N}}$ and thus, the sequence $(T^{k-1} u_\ell)_{\ell \in \mathbb{N}}$ satisfies all the assumptions of the Rellich–Kondrashov compactness property for Sobolev maps [34, proposition 3.4] and so the conclusion follows directly.

We also have a closure property in the particular case of bounded sequences in Sobolev spaces.

Proposition 4.3.10 (Weak closure property for higher-order Sobolev spaces). *Let $p \in [1, +\infty)$. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of k times colocally weakly differentiable maps from M to N . Assume that the sequence $(T^{k-1}u_\ell)_{\ell \in \mathbb{N}}$ converges locally in measure to a measurable map $v: M \rightarrow \text{Mor}(T^{k-1}M, T^{k-1}N)$ that covers a map $u: M \rightarrow N$, that*

$$\limsup_{\ell \rightarrow \infty} \int_M \sum_{j=1}^k |D_K^j u_\ell|_{(\otimes^j g_M^*) \otimes g_N}^p < +\infty,$$

and, if $p = 1$, that for every $j \in \{1, \dots, k\}$, the sequence $(D_K^j u_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable. Then $u \in \dot{W}^{k,p}(M, N)$ and

$$\int_M |D_K^k u|_{(\otimes^k g_M^*) \otimes g_N}^p \leq \liminf_{\ell \rightarrow \infty} \int_M |D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N}^p.$$

By relying recursively on the formulas (4.3.3) and (4.3.4) and applying the weak closure property for Sobolev maps [34, proposition 3.8] to the sequence $(T^{k-1}u_\ell)_{\ell \in \mathbb{N}}$, we have $u \in \dot{W}^{k-1,p}(M, N)$ and $T^{k-1}u = v$. For the last inequality, we can proceed as in the proof of proposition 4.3.6, and so by first proving a lemma similar to lemma 4.3.7.

Lemma 4.3.11. *Let $p \in [1, +\infty)$. If $u \in \dot{W}^{k-1,p}(M, N) \cap \dot{W}^{k,p}(M, N)$, then the map $|D_K^{k-1}u|_{(\otimes^{k-1}g_M^*) \otimes g_N}$ belongs to $W^{1,p}(M)$ and almost everywhere in M*

$$|T|D_K^{k-1}u|_{(\otimes^{k-1}g_M^*) \otimes g_N}|_{g_M^* \otimes g_1} \leq |D_K^k u|_{(\otimes^k g_M^*) \otimes g_N},$$

where g_1 is the Euclidean metric on $\mathbb{R}$.

4.4. Chain rule for higher order colocally weakly differentiable maps

Since the chain rule is central in the definition of first-order colocal weak differentiability and since composition is a crucial tool in the theory of Sobolev maps, in particular, in the definition by embedding (4.1), we investigate under which condition the chain rule holds for higher-order colocally weakly differentiable maps: whether for a k times colocally weakly differentiable map $u: M \rightarrow N$ and for a function $f \in C_c^k(N, \mathbb{R})$, the composite function $f \circ u$ is k times weakly differentiable and whether we have $T^k(f \circ u) = T^k f \circ T^k u$ almost everywhere on the manifold M .

4.4.1. Failure of the chain rule for twice colocally weakly differentiable maps

The starting point of the analysis is that the higher-order chain rule *does not hold* in general.

Example 4.4.1. Let $m \in \mathbb{N}$ with $m \geq 2$ and let $\alpha > 0$. We define the function $u: \mathbb{R}^m \setminus \{0\} \rightarrow \mathbb{R}^2$ for each $x \in \mathbb{R}^m \setminus \{0\}$ by

$$u(x) = (\cos(|x|^{-\alpha}), \sin(|x|^{-\alpha})).$$

For every $x \in \mathbb{R}^m \setminus \{0\}$, $|Du(x)|_{g_m^* \otimes g_2} = \alpha|x|^{-\alpha-1}$ and $|D^2u(x)|_{g_m^* \otimes g_m^* \otimes g_2} \geq \alpha|x|^{-2\alpha-2}$. If $\alpha - 1 < m < 2(\alpha + 1)$, then the map u is twice colocally weakly differentiable but if we take $f \in C_c^2(\mathbb{R}^2, \mathbb{R}^2)$ such that $f = \text{id}$ on B_2 , $f \circ u = u$ on $\mathbb{R}^m \setminus \{0\}$ and so $|D^2(f \circ u)|_{g_m^* \otimes g_m^* \otimes g_2} = |D^2u|_{g_m^* \otimes g_m^* \otimes g_2}$ does not belong to $L_{\text{loc}}^1(\mathbb{R}^m)$.

The crucial point in this example is that $|D^2u|$ is integrable on bounded sets on which the derivative $|Du|$ is bounded, but is not integrable on bounded sets on which the map u itself is bounded.

4.4.2. Double norms for double vector bundles

In order to characterize the maps for which a chain rule holds, we introduce a notion of norm for colocal weak derivatives of a bundle morphism. Since $\text{Mor}(T^2M, T^2N)$ does not carry a vector bundle structure over $M \times N$, we cannot define a notion of norm on this space with respect to each fiber over $M \times N$. However, we define a notion of *double norm* on this space which is compatible with the double vector bundle structure.

To fix the idea, since $\pi_M^2: T^2M \rightarrow M$ is not a vector bundle [15, §9.3; 41, §16.15.7] and T^2M is a double vector bundle, we first define this notion on T^2M .

Definition 4.4.1. A map $\llbracket \cdot \rrbracket: T^2M \rightarrow \mathbb{R}$ is a *double seminorm* on T^2M whenever

- (a) the map $x \in M \mapsto \llbracket \cdot \rrbracket_x$ is continuous, where for every $x \in M$, $\llbracket \cdot \rrbracket_x$ is the restriction of $\llbracket \cdot \rrbracket$ to the space T_x^2M ,
- (b) $\llbracket \cdot \rrbracket$ is a seminorm on fibers over both the bundles $(T^2M, T\pi_M, TM)$ and (T^2M, π_{TM}, TM) , that is, for every $e \in TM$, the restriction of $\llbracket \cdot \rrbracket$ to the fibers $(T\pi_M)^{-1}(\{e\})$ or $(\pi_{TM})^{-1}(\{e\})$ is a seminorm.

Consequently, if $\llbracket \cdot \rrbracket$ is a double seminorm on T^2M , for every $\nu, \mu \in T^2M$ such that $\pi_{TM}(\nu) = \pi_{TM}(\mu)$ and every $\lambda \in \mathbb{R}$,

$$\llbracket \lambda \cdot_{\pi_{TM}} \nu \rrbracket = |\lambda| \llbracket \nu \rrbracket,$$

where $\cdot_{\pi_{TM}}$ is the vector bundle multiplication of (T^2M, π_{TM}, TM) [84, §1], and

$$\llbracket \nu +_{\pi_{TM}} \mu \rrbracket \leq \llbracket \nu \rrbracket + \llbracket \mu \rrbracket,$$

where $+_{\pi_{TM}}$ is the vector bundle addition of (T^2M, π_{TM}, TM) [84, §1]; and it satisfies the same properties if we consider instead the vector bundle operations $\cdot_{T\pi_M}$ and $+_{T\pi_M}$ of $(T^2M, T\pi_M, TM)$.

Definition 4.4.2. A map $\llbracket \cdot \rrbracket: T^2M \rightarrow \mathbb{R}$ is a *double norm* on T^2M whenever

- (a) $\llbracket \cdot \rrbracket$ is a double seminorm on T^2M ,
- (b) $\llbracket \cdot \rrbracket$ is maximal: for every double seminorm $\llbracket \cdot \rrbracket'$ on T^2M , there exists a positive continuous function $\beta \in C(M, \mathbb{R})$ such that for every $\nu \in T^2M$,

$$\llbracket \nu \rrbracket' \leq \beta(\pi_M^2(\nu)) \llbracket \nu \rrbracket.$$

The maximality implies that all double norms are locally equivalent.

Proposition 4.4.1. *There exists a double norm on T^2M .*

Sketch of the proof. If $M = \mathbb{R}^m$, then $T^2M = \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ and a canonical double seminorm $\llbracket \cdot \rrbracket_*: T^2M \rightarrow \mathbb{R}$ can be defined for every $\nu = (x, e_1, e_2, e_{12}) \in T^2M$ by

$$\llbracket \nu \rrbracket_* = |e_1| |e_2| + |e_{12}|.$$

One concludes by noting that for any double seminorm $\llbracket \cdot \rrbracket$ on T^2M , there exists $C > 0$ such that for every $\nu \in T^2M$, $\llbracket \nu \rrbracket \leq C \llbracket \nu \rrbracket_*$.

For a general manifold M , since for every $x \in M$, there exists a local trivialization (V, ψ) such that $x \in V$, the set $V \subseteq M$ is open and

$$\psi: V \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow (\pi_M^2)^{-1}(V) \subseteq T^2M$$

is a diffeomorphism [41, §16.15.7], the argument above gives in each local trivialization a canonical double seminorm. These local double seminorms can then be patched together by a partition of unity. $\square$

Remark 4.4.1. The reader will observe that a double norm is a norm on fibers over non zero elements. This implies that if $\nu \in T^2M$ satisfies $\kappa_M(\nu) = \nu$ and if $\llbracket \nu \rrbracket = 0$, then $\nu = 0_{\pi_M^2(\nu)}$, where $0_{\pi_M^2(\nu)}$ is the double zero, that is, the vector such that $0_{\pi_{TM}}(\nu) = 0_{T\pi_M}(\nu)$ [84, definition 1.1]. We will not rely on these properties because they fail for k^{tuple} norms on k^{tuple} vector bundles, see section 4.4.5 below.

Since $\text{Mor}(T^2M, T^2N)$ has a double vector bundle structure, the definition and the existence of double norms can be obtained in a similar way on $\text{Mor}(T^2M, T^2N)$.

Motivated by the chain rule, we just remark how double norms behave under composition of double vector bundle morphisms.

Proposition 4.4.2. *Let K be a manifold of class C^2 . For every compact subsets $Q_M \subseteq M$, $Q_N \subseteq N$ and $Q_K \subseteq K$, there exists a constant $C > 0$ such that for every $v \in (\pi_{M \times N}^2)^{-1}(Q_M \times Q_N)$ and every $\xi \in (\pi_{N \times K}^2)^{-1}(Q_N \times Q_K)$ such that $\pi_N \circ \pi_{M \times N} \circ \pi_{T^*M \otimes TN}^1 \circ v = \pi_N \circ \pi_{N \times K} \circ \pi_{T^*N \otimes TK}^1 \circ \xi$, then*

$$\llbracket \xi \circ v \rrbracket \leq C \llbracket \xi \rrbracket \llbracket v \rrbracket.$$

Sketch of the proof. If $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$, a canonical double seminorm $\llbracket \cdot \rrbracket_*$ on $\text{Mor}(T^2M, T^2N)$ can be defined for every $f \in \text{Mor}(T^2M, T^2N)$ by

$$\llbracket f \rrbracket_* = |f_1| |f_2| + |f_{12}| + |\hat{f}_{12}|$$

since for every $\nu = (x, e_1, e_2, e_{12}) \in T^2M$, $f(\nu) = (y, f_1[e_1], f_2[e_2], f_{12}[e_{12}] + \hat{f}_{12}[e_1, e_2])$, where the maps $f_1, f_2, f_{12}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ are linear and $\hat{f}_{12}: \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is bilinear. Then if $K = \mathbb{R}^k$, for every $f \in \text{Mor}(T^2M, T^2N)_{x,y}$ and $h \in \text{Mor}(T^2N, T^2K)_{y,z}$, we have

$$\llbracket h \circ f \rrbracket_* \leq \llbracket h \rrbracket_* \llbracket f \rrbracket_*.$$

For any manifolds M, N and K , it suffices to prove the composition property in each local trivialization by using the one of the canonical double seminorm and then the maximality property of double norms. $\square$

4.4.3. Chain rule for second order colocally weakly differentiable maps

In order to have a chain rule for twice colocally weakly differentiable maps, we assume an additional integrability condition on colocal weak derivatives of bundle morphisms in terms of double norms. We also identify the class of maps to compose with the colocal weak derivative in order to have a chain rule.

For every map $h: T^*M \otimes TN \rightarrow \mathbb{R}^q$, we denote by $\text{supp}_N(h)$ the support of h with respect to N , that is,

$$\begin{aligned} \text{supp}_N(h) &= \overline{\{y \in N: \text{ there exists } x \in M \text{ such that } h|_{(\pi_{M \times N})^{-1}((x,y))} \neq 0\}} \\ &= \pi_N(\pi_{M \times N}(\text{supp}(h))). \end{aligned}$$

Proposition 4.4.3. *Let $u: M \rightarrow N$ be a colocally weakly differentiable map. The following statements are equivalent.*

- (i) *For every $f \in C_c^2(N, \mathbb{R})$, the map $f \circ u$ is twice weakly differentiable,*
- (ii) *for every bundle morphism $h \in C^1(T^*M \otimes TN, \mathbb{R})$ such that $\text{supp}_N(h)$ is compact, the map $h \circ Tu$ is weakly differentiable,*
- (iii) *the map u is twice colocally weakly differentiable and for all compact subsets $K \subseteq M$ and $L \subseteq N$ and every double norm $\llbracket \cdot \rrbracket$ on $\text{Mor}(T^2M, T^2N)$,*

$$\int_{u^{-1}(L) \cap K} \llbracket T^2u \rrbracket < +\infty,$$

We prove an intermediate result in order to prove this proposition.

Lemma 4.4.4. *If $v \in W_{\text{loc}}^{2,1}(M, \mathbb{R}^q) \cap L^\infty(M, \mathbb{R}^q)$, if the bundle morphism $w \in C^1(T^*M \otimes T\mathbb{R}^q, \mathbb{R})$ is such that $\text{supp}_{\mathbb{R}^q}(w)$ is compact, then $w \circ Tv$ is weakly differentiable.*

We recall that the *operator norm* is defined for every $\xi \in \mathcal{L}(T_xM, T_yN)$ by

$$|\xi|_{\mathcal{L}} = \sup\{|\xi(e)|_{g_N} : e \in T_xM, |e|_{g_M} \leq 1\},$$

where g_M and g_N are some Riemannian metrics on TM and TN respectively [42, §1.2 proposition 2.10].

Proof of lemma 4.4.4. Without loss of generality, we can assume that $q = 1$.

First, we assume that $M = \Omega$ is an open set of $\mathbb{R}^m$. By regularization theorem [20, theorem 4.2.4; 24, theorem 4.22; 120, theorem 4.3.9], there exists a sequence $(v_\ell)_{\ell \in \mathbb{N}}$ in $C_c^\infty(\Omega, \mathbb{R})$ that converges to v in $W_{\text{loc}}^{2,1}(\Omega)$. Moreover, for every compact set $K \subseteq \Omega$, there exists $C > 0$ such that for every $\ell \in \mathbb{N}$,

$$|T(w \circ Tv_\ell)|_{\mathcal{L}} \leq C(|Dv_\ell|_{g_m^* \otimes g_q}^2 + |D^2v_\ell|_{g_m^* \otimes g_m^* \otimes g_q} + |Dv_\ell|_{g_m^* \otimes g_q}),$$

where g_m and g_q are the Euclidean metrics on $\mathbb{R}^m$ and $\mathbb{R}^q$ respectively. By the classical Gagliardo–Nirenberg inequality [49, 95], we have $|Dv|_{g_m^* \otimes g_q}^2 \in L_{\text{loc}}^1(\Omega)$ and so the sequence $(|Dv_\ell|_{g_m^* \otimes g_q}^2)_{\ell \in \mathbb{N}}$ converges to $|Dv|_{g_m^* \otimes g_q}^2$ in $L_{\text{loc}}^1(\Omega)$. Hence, by Lebesgue's dominated convergence theorem, the sequence $(|T(w \circ Tv_\ell)|_{\mathcal{L}})_{\ell \in \mathbb{N}}$ converges in $L_{\text{loc}}^1(\Omega)$ and then by closing lemma [120, lemma 6.1.5], the map $w \circ Tv$ is weakly differentiable.

For a general manifold M , we apply the previous argument to every $v \circ \psi^{-1}: \psi(V) \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$ with any local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$.

By a direct covering argument, we have thus that $w \circ Tv$ is weakly differentiable. $\square$

Proof of proposition 4.4.3. We first prove that (i) implies (ii). Let $(U_i)_{i \in I}$ be an open cover of the target manifold N by sets given by lemma 4.2.3. Let $h \in C^1(T^*M \otimes TN, \mathbb{R})$ be a bundle morphism such that $\text{supp}_N(h)$ is compact. Then there exist $\ell \in \mathbb{N}_*$ such that $\text{supp}_N(h) \subseteq \bigcup_{i=1}^{\ell} U_i$ and a partition of unity $(\eta_i)_{1 \leq i \leq \ell}$ subordinate to the family $(U_i)_{1 \leq i \leq \ell}$ such that

$$h \circ Tu = \sum_{i=1}^{\ell} (\eta_i \circ u)(h \circ Tu) = \sum_{i=1}^{\ell} (\eta_i \circ u)((h \circ T\varphi_i^*) \circ T(\varphi_i \circ u))$$

almost everywhere in M . By lemma 4.4.4, every term in the right sum is weakly differentiable and so $h \circ Tu$ is weakly differentiable.

Next, we prove that (ii) implies (iii). First, let $(U_i)_{i \in I}$ be an open cover of N by sets given by lemma 4.2.3. Let $h \in C_c^1(T^*M \otimes TN, \mathbb{R})$. Since the set $\text{supp}(h)$ is compact, in particular, there exist $\ell \in \mathbb{N}_*$ and a finite partition of unity $(\eta_i)_{1 \leq i \leq \ell}$ subordinate to $(U_i)_{1 \leq i \leq \ell}$ such that almost everywhere in M

$$h \circ Tu = \sum_{i=1}^{\ell} (\eta_i \circ u)(h \circ Tu) = \sum_{i=1}^{\ell} (\eta_i \circ u)(h \circ T\varphi_i^* \circ (T\varphi_i \circ Tu)).$$

By assumption $T\varphi_i \circ Tu$ is weakly differentiable, and thus by the chain rule for weakly differentiable functions (see for example [45, theorem 4.2.4 (ii); 120, theorem 6.1.13]), each term in the right sum is weakly differentiable, and so $h \circ Tu$ is weakly differentiable. By proposition 4.2.1, there exists a measurable map $T^2u: M \rightarrow \text{Mor}(T^2M, T^2N)$ such that for every $h \in C^1(T^*M \otimes TN, \mathbb{R})$ such that $h \circ Tu \in W_{\text{loc}}^{1,1}(M)$, we have almost everywhere in M

$$T(h \circ Tu) = Th \circ T^2u.$$

We now prove that the integrability condition is satisfied. Let $(U_i)_{i \in I}$ be an open cover of N by sets given by lemma 4.2.3. Let $K \subseteq M$ and $L \subseteq N$ be two compact subsets. Since the set L is compact, there exists $\ell \in \mathbb{N}_*$ such that $L \subseteq \bigcup_{i=1}^{\ell} U_i$. By construction, for every $i \in \{1, \dots, \ell\}$, we have $T^2(\varphi_i \circ u) = T^2\varphi_i \circ T^2u$ almost everywhere in M and, by lemma 4.2.3, $T^2\varphi_i^* \circ T^2\varphi_i = \text{id}$ on $(\pi_N^2)^{-1}(U_i)$, and so $T^2u = T^2\varphi_i^* \circ T^2(\varphi_i \circ u)$ almost everywhere in $u^{-1}(U_i)$. Since for every $i \in \{1, \dots, \ell\}$ we have $\varphi_i \in C_c^2(N, \mathbb{R}^n)$, by proposition 4.4.2, there exist constants $C_1, \dots, C_{\ell} > 0$

such that almost everywhere in $u^{-1}(L) \cap K$

$$\begin{aligned} \llbracket T^2 u \rrbracket &\leq \sum_{i=1}^{\ell} \llbracket T^2 \varphi_i^* \circ T^2(\varphi_i \circ u) \rrbracket \leq \sum_{i=1}^{\ell} C_i \llbracket T^2 \varphi_i^* \rrbracket \llbracket T^2(\varphi_i \circ u) \rrbracket \\ &\leq (\max_{1 \leq i \leq \ell} C_i) (\max_{1 \leq i \leq \ell} \sup_{\mathbb{R}^n} \llbracket T^2 \varphi_i^* \rrbracket) \sum_{i=1}^{\ell} \llbracket T^2(\varphi_i \circ u) \rrbracket. \end{aligned}$$

If $(\eta_j)_{j \in J}$ is a partition of unity subordinate to an atlas $((\psi_j, V_j))_{j \in J}$ of local charts of M , by the maximality property of double norms, there exists a positive continuous function $\beta \in C(M \times \mathbb{R}^n, \mathbb{R})$ such that for every $i \in \{1, \dots, \ell\}$, almost everywhere in M

$$\begin{aligned} \llbracket T^2(\varphi_i \circ u) \rrbracket &\leq \beta(\cdot, \varphi_i \circ u) \sum_{j \in J} \eta_j (|D(\varphi_i \circ u \circ \psi_j^{-1}) \circ \psi_j|_{g_m^* \otimes g_n}^2 \\ &\quad + |D(\varphi_i \circ u \circ \psi_j^{-1}) \circ \psi_j|_{g_m^* \otimes g_n} + |D^2(\varphi_i \circ u \circ \psi_j^{-1}) \circ \psi_j|_{g_m^* \otimes g_m^* \otimes g_n}), \end{aligned}$$

where g_m and g_n are the Euclidean metrics on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Since for every $i \in \{1, \dots, \ell\}$, $\varphi_i \circ u \in W_{\text{loc}}^{2,1}(M, \mathbb{R}^n) \cap L^\infty(M, \mathbb{R}^n)$, by the Gagliardo–Nirenberg inequality [49, 95], $\llbracket T^2(\varphi_i \circ u) \rrbracket \in L^1(u^{-1}(L) \cap K)$ and so

$$\int_{u^{-1}(L) \cap K} \llbracket T^2 u \rrbracket < +\infty.$$

Finally, we prove that (iii) implies (i). We take a function $\theta \in C_c^1([0, +\infty), \mathbb{R})$ such that $0 \leq \theta \leq 1$ and $\theta = 1$ on $[0, 1]$. Then for every $\ell \in \mathbb{N}$, we define the map $\theta_\ell: T^*M \otimes TN \rightarrow \mathbb{R}$ for every $y \in T^*M \otimes TN$ by

$$\theta_\ell(y) = \theta\left(\frac{|y|_{g_{T^*M \otimes TN}}}{1 + \ell}\right),$$

where $|\cdot|_{g_{T^*M \otimes TN}}$ is a norm induced by a Riemannian metric on $T^*M \otimes TN$ [42, §1.2 proposition 2.10]. Let $f \in C_c^2(N, \mathbb{R})$ and let $K \subseteq M$ be a compact set. For every $\ell \in \mathbb{N}$, we define $\tilde{f}_\ell = \theta_\ell \cdot (Tf)$. Since the map $\tilde{f}_\ell: T^*M \otimes TN \rightarrow \mathbb{R}$ is Lipschitz-continuous and its support is compact, $\tilde{f}_\ell \circ Tu$ is weakly differentiable [34, proposition 2.2] and there exist constants $C_1, C_2, C_3 > 0$ such that for every $\ell \in \mathbb{N}$ and almost everywhere in $u^{-1}(\text{supp}(f)) \cap K$

$$\begin{aligned} |T(\tilde{f}_\ell \circ Tu)|_{\mathcal{L}} &\leq C_1 \left| \theta' \left(\frac{|Tu|_{g_{T^*M \otimes TN}}}{1 + \ell} \right) \right| \frac{1}{1 + \ell} \llbracket T^2 u \rrbracket |T(f \circ u)|_{\mathcal{L}} \\ &\quad + C_2 \theta \left(\frac{|Tu|_{g_{T^*M \otimes TN}}}{1 + \ell} \right) \llbracket T^2 f \rrbracket \llbracket T^2 u \rrbracket \\ &\leq C_3 \llbracket T^2 u \rrbracket. \end{aligned}$$

Since the sequence $(|T(\tilde{f}_\ell \circ Tu)|_{\mathcal{L}})_{\ell \in \mathbb{N}}$ is bounded and uniformly integrable and since $(\tilde{f}_\ell \circ Tu)_{\ell \in \mathbb{N}}$ converges almost everywhere to $Tf \circ Tu$ in $K \subseteq M$, in view of the weak compactness criterion in $L^1(K)$ [20, corollary 4.7.19; 24, theorem 4.30], the sequence $(T(\tilde{f}_\ell \circ Tu))_{\ell \in \mathbb{N}}$ converges weakly to $T^2(f \circ u)$ in $L^1(K)$ and $T(f \circ u)$ is weakly differentiable. Hence, the map $f \circ u$ is twice weakly differentiable. $\square$

4.4.4. Sequences of maps having the chain rule property

Since the property $f \circ u$ is twice weakly differentiable for every $f \in C_c^2(N, \mathbb{R})$ is stronger than double differentiability, one can wonder whether using this stronger property in the definition of Sobolev maps might not lead to better spaces.

Although there is a closure property for such maps in terms of double norms, the next example shows that there is no closure property in terms of covariant derivatives, which are the quantities that can be controlled and observed in Riemannian geometry.

Example 4.4.2. Let $m \in \mathbb{N}$ with $m \geq 3$ and let $\alpha > 0$. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence such that for every $\ell \in \mathbb{N}$, the map $u_\ell: \mathbb{R}^m \rightarrow \mathbb{S}^1 \subseteq \mathbb{R}^2$ is defined by $u_\ell = (\cos \circ v_\ell, \sin \circ v_\ell)$, where the function $v_\ell: \mathbb{R}^m \rightarrow \mathbb{R}$ is defined for every $x \in \mathbb{R}^m$ by

$$v_\ell(x) = \left(|x| + \frac{1}{1+\ell}\right)^{-\alpha}.$$

For every $\ell \in \mathbb{N}$, $|Du_\ell|_{g_m^* \otimes g_{\mathbb{S}^1}} = |Dv_\ell|_{g_m^* \otimes g_1}$ and $|D_K(Tu_\ell)|_{g_m^* \otimes g_m^* \otimes g_{\mathbb{S}^1}} = |D^2 v_\ell|_{g_m^* \otimes g_m^* \otimes g_1}$. The sequence $(v_\ell)_{\ell \in \mathbb{N}}$ converges locally in measure to the map $v: \mathbb{B}_r^m \rightarrow \mathbb{R}$ defined for every $x \in \mathbb{B}_r^m \setminus \{0\}$ by $v(x) = |x|^{-\alpha}$.

If $\alpha < m - 2$, then the sequences $(Du_\ell)_{\ell \in \mathbb{N}}$ and $(D_K(Tu_\ell))_{\ell \in \mathbb{N}}$ are bilocally uniformly integrable. Every term of the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ satisfies the second-order chain-rule property and the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges locally in measure to the map $u = (\cos \circ v, \sin \circ v): \mathbb{B}_r^m \rightarrow \mathbb{S}^1$. However, if $\alpha > \frac{m-2}{2}$, the map u does not satisfy the second-order chain rule. Indeed, if we take $f \in C^2(\mathbb{S}^1, \mathbb{R}^2)$ such that $f = \text{id}$ on $\mathbb{S}^1 \subseteq \mathbb{R}^2$, then for every $x \in \mathbb{R}^m \setminus \{0\}$, we have $|D^2(f \circ u)(x)|_{g_m^* \otimes g_m^* \otimes g_2} \geq \alpha|x|^{-2\alpha-2}$ and so $|D^2(f \circ u)|_{g_m^* \otimes g_m^* \otimes g_2}$ does not belong to $L_{\text{loc}}^1(\mathbb{B}_r^m)$.

In other words, the previous example exhibits a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ satisfying the same assumptions on the sequences $(Tu_\ell)_{\ell \in \mathbb{N}}$ and $(D_K(Tu_\ell))_{\ell \in \mathbb{N}}$ together with a second-order chain rule than those of proposition 4.3.4 but with a limit that does not have the second-order chain-rule property.

In fact, given a sequence of measurable maps $(u_\ell)_{\ell \in \mathbb{N}}$ such that for every $f \in C_c^2(N, \mathbb{R})$, $f \circ u_\ell$ is twice weakly differentiable, the sequence $(\|T^2 u_\ell\|)_{\ell \in \mathbb{N}}$ has to be uniformly integrable.

4.4.5. Chain rule for higher order colocally weakly differentiable maps

Given a k times colocally weakly differentiable map $u: M \rightarrow N$, we can also investigate whether for every $f \in C_c^k(N, \mathbb{R})$, $f \circ u$ is k times weakly differentiable.

In a first step, the notion of double norm can be generalized to a notion of k^{tuple} norm on k^{th} order tangent bundles. A map $[\![\cdot]\!]: T^k M \rightarrow \mathbb{R}$ is a k^{tuple} seminorm on $T^k M$ if

- (i) the map $x \in M \mapsto [\![\cdot]\!]_{|T_x^k M}$ is continuous,
- (ii) $[\![\cdot]\!]$ is a seminorm on fibers over each of the k vector bundle structure.

For example, if $M = \mathbb{R}^m$ and $I_k = 2^{\{1, \dots, k\}} \setminus \emptyset$, then the *canonical k^{tuple} seminorm* is defined for every $\nu = (x, (e_\lambda)_{\lambda \in I_k}) \in T^k M$ by

$$[\![\nu]\!]_* = \sum_{\ell=1}^k \sum_{\lambda \in \mathcal{P}_\ell(\{1, \dots, k\})} |e_{\lambda^1}| \cdots |e_{\lambda^\ell}|,$$

where $\mathcal{P}_\ell(\{1, \dots, k\}) = \{\lambda = \{\lambda^1, \dots, \lambda^\ell\} \in 2^{I_k} : \dot{\cup}_{i=1}^\ell \lambda^i = \{1, \dots, k\}, \emptyset \notin \lambda\}$ is the set of all partitions of $\{1, \dots, k\}$ of length ℓ . As for the canonical double seminorm, for every k^{tuple} seminorm $[\![\cdot]\!]$ on $T^k M$, there exists $C > 0$ such that for every $\nu \in T^k M$, $[\![\nu]\!] \leq C[\![\nu]\!]_*$. As a consequence, for any manifold M , there exists a k^{tuple} norm on $T^k M$ defined as follows.

Definition 4.4.3. A map $[\![\cdot]\!]: T^k M \rightarrow \mathbb{R}$ is a k^{tuple} norm on $T^k M$ whenever

- (i) $[\![\cdot]\!]$ is a seminorm on $T^k M$,
- (ii) $[\![\cdot]\!]$ is maximal: for every k^{tuple} seminorm $[\![\cdot]\!]'$ on $T^k M$, there exists a positive continuous function $\beta \in C(M, \mathbb{R})$ such that for every $\nu \in T^k M$,

$$[\![\nu]\!]' \leq \beta(\pi_M^k(\nu))[\![\nu]\!],$$

where $\pi_M^k: T^k M \rightarrow M$ is the canonical submersion.

Remark 4.4.2. Unlike the double norm, the k^{tuple} norm is not a norm on fibers over non zero elements. For example, if $M = \mathbb{R}^m$, for every $\nu = (x, e_1, e_2, e_3, e_{12}, e_{23}, e_{13}, e_{123}) \in T^3 M$,

$$[\![\nu]\!]_* = |e_1| |e_2| |e_3| + |e_1| |e_{23}| + |e_2| |e_{13}| + |e_3| |e_{12}| + |e_{123}|.$$

In particular, if $\nu = (x, 0, 0, 0, e, e, e, 0)$, then $\pi_{T^2 M}(\nu) = T\pi_{TM}(\nu) = T^2\pi_M(\nu)$ and $[\![\nu]\!]_* = 0$ but $\nu \neq 0_{\pi_M^3(\nu)}$ whenever $e \neq 0$.

Since $\text{Mor}(T^k M, T^k N)$ has a k^{tuple} vector bundle structure, we can also define a notion of k^{tuple} norm on this space. For example, if $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$, a *canonical k^{tuple} norm* $\llbracket \cdot \rrbracket_*$ on $\text{Mor}(T^k M, T^k N)$ can be defined for every $f \in \text{Mor}(T^k M, T^k N)$ by

$$\llbracket f \rrbracket_* = \sum_{\ell=1}^k \sum_{\Lambda \in \mathcal{P}_\ell(\{1, \dots, k\})} \left(\sum_{j=1}^{\ell} \sum_{\lambda \in \mathcal{P}_j(\Lambda)} |f_{\lambda^1}| \cdots |f_{\lambda^j}| \right)$$

since for every $\nu \in T^k M$,

$$f(\nu) = \left(y, \left(\sum_{i=1}^{|\Lambda|} \sum_{\lambda \in \mathcal{P}_i(\Lambda)} f_{\lambda}(e_{\lambda^1}, \dots, e_{\lambda^i}) \right)_{\Lambda \in I_k} \right),$$

where for every $\Lambda \in I_k$, for every $1 \leq i \leq |\Lambda|$ and every $\lambda \in \mathcal{P}_i(\Lambda)$, the map $f_{\lambda}: \times^i \mathbb{R}^m \rightarrow \mathbb{R}^n$ is i -linear.

In a second step, we state equivalent assertions to the chain rule for higher-order colocally weakly differentiable maps. We recall that a locally integrable map $u: M \rightarrow \mathbb{R}$ is *k times weakly differentiable* if for every $x \in M$, there exists a local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$ such that $x \in V$ and the map $u \circ \psi^{-1}$ belongs to $W_{\text{loc}}^{k,1}(\psi(V))$. For every map $h: \text{Mor}(T^k M, T^k N) \rightarrow \mathbb{R}^q$, we denote by $\text{supp}_N(h)$ the *support of h with respect to N* , that is, $\text{supp}_N(h) = \pi_N(\pi_{M \times N}^k(\text{supp}(h)))$, where $\pi_{M \times N}^k: \text{Mor}(T^k M, T^k N) \rightarrow M \times N$ is a natural submersion.

Proposition 4.4.5. *Let $j \in \{2, \dots, k\}$. Let $u: M \rightarrow N$ be a j times colocally weakly differentiable map. The following statements are equivalent.*

- (i) *For every $f \in C_c^k(N, \mathbb{R})$, $f \circ u$ is k times weakly differentiable,*
- (ii) *for every $f \in C_c^j(N, \mathbb{R})$, $f \circ u$ is j times weakly differentiable and for every $h \in \text{Mor}(\text{Mor}(T^j M, T^j N), \mathbb{R})$ of class C^{k-j} such that $\text{supp}_N(h)$ is compact, the map $h \circ T^j u$ is $(k - j)$ times weakly differentiable,*
- (iii) *the map u is k times colocally weakly differentiable and for all compact subsets $K \subseteq M$ and $L \subseteq N$, for all $\ell \in \{2, \dots, k\}$, for all ℓ^{tuple} norm $\llbracket \cdot \rrbracket$ on $\text{Mor}(T^\ell M, T^\ell N)$,*

$$\int_{u^{-1}(L) \cap K} \llbracket T^\ell u \rrbracket < +\infty.$$

Moreover, if for every $f \in C_c^k(N, \mathbb{R})$, $f \circ u$ is k times weakly differentiable, then for all compact subsets $K \subseteq M$ and $L \subseteq N$, for all j^{tuple} norm $\llbracket \cdot \rrbracket$ on $\text{Mor}(T^j M, T^j N)$,

$$\int_{u^{-1}(L) \cap K} \llbracket T^j u \rrbracket^{\frac{k}{j}} < +\infty.$$

To prove this proposition, we need similar intermediate results to those of proposition 4.4.3. In fact, we can state every intermediate results (lemmas 4.4.4 and 4.2.3) with all higher-order notions and then the proof of proposition 4.4.5 has the same structure.

If M is an open set of $\mathbb{R}^m$ and $N = \mathbb{R}$, it is well known that the left composition does not operate on higher-order Sobolev spaces [2; 21, théorème 3 (i); 36]. Thanks to proposition 4.4.5, we have some examples for which the chain rule does not hold by giving maps that belong to $W_{\text{loc}}^{k,1}(\mathbb{R}^m)$ but that do not satisfy the integrability condition in assertion (iii).

Example 4.4.3. Let $\beta > \alpha > 0$. The function $u: \mathbb{R}^m \rightarrow \mathbb{R}$ is defined for every $x \in \mathbb{R}^m$ by

$$u(x) = \begin{cases} \frac{1}{|x|^\alpha} \sin(|x|^\beta) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

If $0 < \alpha < m$ and $\frac{1}{3}(\alpha - m + 3) < \beta < \frac{1}{3}(3\alpha - m + 3)$, the map u belongs to $W_{\text{loc}}^{3,1}(\mathbb{R}^m)$ but there exist $\gamma > 0$ and $\delta > 0$ such that for almost every $x \in u^{-1}([-1, 1]) \cap \overline{\mathbb{B}}_\delta^m$,

$$|Du(x)|_{g_m^* \otimes g_1} \geq \gamma(\beta - \alpha)|x|^{\beta - \alpha - 1}.$$

Since there exists $C > 0$ such that $\llbracket T^3 u \rrbracket \geq C|Du|_{g_m^* \otimes g_1}^3$ and since the map $x \in \mathbb{R}^m \setminus \{0\} \mapsto |x|^{\beta - \alpha - 1}$ does not belong to $L^3(u^{-1}([-1, 1]) \cap \overline{\mathbb{B}}_\delta^m)$, the map u does not satisfy the integrability condition in assertion (iii).

4.5. Back to the definition by embedding

4.5.1. Comparison with the intrinsic definition

Given an isometric embedding of the target manifold, we compare the notion of higher-order Sobolev spaces by embedding (4.1) and the intrinsic one (definitions 4.3.2 and 4.3.3).

Let $k \geq 2$. We assume that (M, g_M) and (N, g_N) are Riemannian manifolds with the respective Levi-Civita connection maps. For every $q \geq 1$, we denote by g_q the Euclidean metric on $\mathbb{R}^q$.

If the target manifold is compact, we characterize higher-order Sobolev spaces by embedding (4.1). A measurable map $u: M \rightarrow N$ belongs to $L^\infty(M, N)$ if

$$\text{osc}(u) = \text{ess sup} \{d_N(u(x), u(y)) : x, y \in M\} < +\infty,$$

where d_N is the geodesic distance on N induced by the Riemannian metric g_N , and for every $p \in [1, +\infty)$, we say that M has the *Gagliardo–Nirenberg property* [49, 95] if there exists a constant $C > 0$ such that for every $v \in \dot{W}^{k,p}(M, \mathbb{R}) \cap L^\infty(M, \mathbb{R})$ and for every $j \in \{1, \dots, k-1\}$,

$$\int_M |D_K^j v|_{(\otimes^j g_M^*) \otimes g_1}^{\frac{kp}{j}} \leq C \text{osc}(v)^{\frac{k-j}{j}p} \int_M |D_K^k v|_{(\otimes^k g_M^*) \otimes g_1}^p. \quad (4.5.5)$$

Proposition 4.5.1. *Let $\iota \in C^k(N, \mathbb{R}^\nu)$ be an isometric embedding and let $p \in [1, +\infty)$. If M has the Gagliardo–Nirenberg property and if N is compact, then*

$$\dot{W}_\iota^{k,p}(M, N) = \bigcap_{j=1}^k \dot{W}^{j, \frac{kp}{j}}(M, N).$$

Without any assumptions on the manifolds, we begin by proving a lemma that concerns the notion of k times colocally weakly differentiable map.

Lemma 4.5.2. *Let $\iota \in C^k(N, \mathbb{R}^\nu)$ be an isometric embedding and let $u: M \rightarrow N$ be a measurable map. If the map $\iota \circ u$ is k times weakly differentiable, then the map u is k times colocally weakly differentiable.*

In contrast with proposition 4.4.3, the hypotheses involve only a single embedding of class C^k .

Proof of lemma 4.5.2. If the map $\iota \circ u$ is weakly differentiable, then u is colocally weakly differentiable and $T(\iota \circ u) = T\iota \circ Tu$ almost everywhere in M [34, proposition 1.9]. We then proceed by iteration. Let $j \in \{1, \dots, k-1\}$ and let $f \in C_c^1(\text{Mor}(T^j M, T^j N), \mathbb{R})$. Since ι is an embedding, $T^j \iota: T^j N \rightarrow T^j \mathbb{R}^\nu$ is also an embedding. Hence, $T^j \iota(T^j N)$ has a tubular neighborhood in $T^j \mathbb{R}^\nu$: there exists a vector bundle $(E, \pi_{T^j N}, T^j N)$ and an embedding $\tilde{\iota}: E \rightarrow T^j \mathbb{R}^\nu$ such that $\tilde{\iota}|_{T^j N} = T^j \iota$ and $\tilde{\iota}(E)$ is open in $T^j \mathbb{R}^\nu$ [70, theorem 4.5.2]. Thanks to the tubular neighborhood, there exists a map $\tilde{f} \in C_c^1(\text{Mor}(T^j M, T^j \mathbb{R}^\nu), \mathbb{R})$ such that $\tilde{f} \circ T^j \iota = f$ on $\text{Mor}(T^j M, T^j N)$ (see for example proof of proposition 1.9 in [34]). In particular, almost everywhere in M

$$f \circ T^j u = \tilde{f} \circ T^j \iota \circ T^j u = \tilde{f} \circ T^j(\iota \circ u).$$

Since $T^j(\iota \circ u)$ is weakly differentiable, the map $T^j(\iota \circ u)$ is colocally weakly differentiable and so $f \circ T^j u$ is weakly differentiable. Finally, since $T^j \iota \circ T^j u$ is weakly differentiable, $T^{j+1}(\iota \circ u) = T^{j+1} \iota \circ T^{j+1} u$ almost everywhere in M [34, proposition 1.5]. $\square$

Proof of proposition 4.5.1. On the one hand, let $u \in \dot{W}_\ell^{k,p}(M, N)$. By lemma 4.5.2, the map u is k times colocally weakly differentiable. For every $j \in \{2, \dots, k\}$, since $T^j(\iota \circ u) = T^j \iota \circ T^j u$ almost everywhere in M , there exists a constant $C_1 > 0$ such that

$$\begin{aligned} |D_K^j u|_{(\otimes^j g_M^*) \otimes g_N} &= |D\iota(u)[D_K^j u]|_{(\otimes^j g_M^*) \otimes g_N} \\ &\leq |D_K^j(\iota \circ u)|_{(\otimes^j g_M^*) \otimes g_N} \\ &\quad + C_1 \sum_{\ell=2}^j \sum_{i=1}^{j-\ell+1} |D_K^\ell \iota(u)|_{(\otimes^\ell g_N^*) \otimes g_N} |D_K^i u|_{(\otimes^i g_M^*) \otimes g_N}^{\frac{i}{j}} \end{aligned}$$

almost everywhere in M . Since $\iota \circ u \in \dot{W}^{k,p}(M, \mathbb{R}^\nu) \cap L^\infty(M, \mathbb{R}^\nu)$, by the Gagliardo–Nirenberg property (4.5.5), $|D_K^j(\iota \circ u)|_{(\otimes^j g_M^*) \otimes g_N} \in L^{\frac{kp}{j}}(M)$ for every $j \in \{1, \dots, k\}$. As a consequence, by iteration and the previous inequality, $u \in \bigcap_{j=1}^k \dot{W}^{j, \frac{kp}{j}}(M, N)$.

On the other hand, let $u \in \bigcap_{j=1}^k \dot{W}^{j, \frac{kp}{j}}(M, N)$. By definition 4.4.3 of k^{tuple} norms and by Young's inequality, for every $j \in \{2, \dots, k\}$ and every compact subset $K \subseteq M$, there exists a constant $C_2 > 0$ such that

$$\llbracket T^j u \rrbracket \leq C_2 \left(\sum_{\ell=1}^k |D_K^\ell u|_{(\otimes^\ell g_M^*) \otimes g_N}^{\frac{k}{\ell}} + 1 \right)$$

almost everywhere in K and so $\llbracket T^j u \rrbracket \in L_{\text{loc}}^1(M)$. Since the manifold N is compact by assumption, by the chain rule for higher-order colocally weakly differentiable maps (proposition 4.4.5), the map $\iota \circ u$ is k times weakly differentiable. Moreover, there exists a constant $C_3 > 0$ such that

$$|D_K^k(\iota \circ u)|_{(\otimes^k g_M^*) \otimes g_N} \leq C_3 \sum_{\ell=1}^k \sum_{i=1}^{k-\ell+1} |D_K^\ell \iota(u)|_{(\otimes^\ell g_N^*) \otimes g_N} |D_K^i u|_{(\otimes^i g_M^*) \otimes g_N}^{\frac{i}{k}}$$

almost everywhere in M and since N is compact, $\iota \circ u \in \dot{W}^{k,p}(M, \mathbb{R}^\nu)$. $\square$

Without the compactness and Gagliardo–Nirenberg assumption, proposition 4.5.1 is not true. Indeed, the embedded space in the intersection already fails for classical Sobolev maps between Euclidean spaces. However, there exists an embedding such one inclusion always occurs.

Proposition 4.5.3. *Let $p \in [1, +\infty)$. If either $k = 2$ or M has the Gagliardo–Nirenberg property, then there exists an isometric embedding $\iota \in C^k(N, \mathbb{R}^\nu)$ such that*

$$\dot{W}_\iota^{k,p}(M, N) \subseteq \bigcap_{j=1}^k \dot{W}^{j, \frac{kp}{j}}(M, N).$$

Proof. If M has the Gagliardo–Nirenberg property, and if $u \in \dot{W}_\iota^{k,p}(M, N)$, by lemma 4.5.2, the map u is k times colocally weakly differentiable. If $\iota \in C^k(N, \mathbb{R}^\nu)$ is an isometric embedding such that $\iota(N)$ is bounded, then $\iota \circ u \in \dot{W}^{k,p}(M, \mathbb{R}^\nu) \cap L^\infty(M, \mathbb{R}^\nu)$ and so $u \in \bigcap_{j=1}^k \dot{W}^{j, \frac{kp}{j}}(M, N)$ (as in the first part of the proof of proposition 4.5.1). For example, by the Nash embedding theorem [92, 93], such an embedding ι always exists.

If $k = 2$, let $\iota_1 \in C^2(N, \mathbb{R}^\nu)$ be an isometric embedding given by Nash embedding theorem [92, 93]. We define an isometric embedding $\iota_2: \mathbb{R}^\nu \rightarrow \mathbb{R}^{3\nu}$ for every $t \in \mathbb{R}^\nu$ by

$$\iota_2(t) = (\lambda t_1, \gamma \cos(\mu t_1), \gamma \sin(\mu t_1), \dots, \lambda t_\nu, \gamma \cos(\mu t_\nu), \gamma \sin(\mu t_\nu))$$

with $\lambda, \gamma, \mu \in \mathbb{R}$ such that $\lambda^2 + \gamma^2 \mu^2 = 1$. Then $\iota = \iota_2 \circ \iota_1: N \rightarrow \mathbb{R}^{3\nu}$ is an isometric embedding and the second fundamental form $A_\iota: TN \times TN \rightarrow \mathbb{R}^{3\nu}$ of ι [42, §6.2] satisfies for every $v_1, v_2 \in TN$,

$$\begin{aligned} |A_\iota[v_1, v_2]|^2 &= |A_{\iota_1}[v_1, v_2]|^2 + |A_{\iota_2}[D\iota_1(v_1), D\iota_1(v_2)]|^2 \\ &\geq |A_{\iota_2}[D\iota_1(v_1), D\iota_1(v_2)]|^2 \\ &= \gamma^2 \mu^4 \sum_{i=1}^\nu (\langle D\iota_1(v_1), e_i \rangle)^2 (\langle D\iota_1(v_2), e_i \rangle)^2, \end{aligned}$$

where $(e_i)_{1 \leq i \leq \nu}$ is the canonical basis of $\mathbb{R}^\nu$. So there exists $C_1 > 0$ such that for every $v \in TN$,

$$|v|_{g_N}^2 = |D\iota_1(v)|_{g_\nu}^2 \leq C_1 |A_\iota[v, v]|_{g_{3\nu}}.$$

Finally, for every $u \in \dot{W}_\iota^{2,p}(M, N)$, since ι is an isometric embedding, by orthogonal decomposition of $T\mathbb{R}^{3\nu}$ into TN and its orthogonal complement in $T\mathbb{R}^{3\nu}$ [42, §6.2], for almost every $x \in M$ and every $e_1, e_2 \in T_x M$,

$$\begin{aligned} |D_K^2(\iota \circ u)(x)[e_1, e_2]|_{g_{3\nu}}^2 \\ = |D_K^2 u(x)[e_1, e_2]|_{g_N}^2 + |A_\iota[Tu(x)[e_1], Tu(x)[e_2]]|_{g_{3\nu}}^2. \end{aligned}$$

Consequently, $u \in \dot{W}^{2,p}(M, N)$ and since for almost every $x \in M$,

$$\begin{aligned} |D_K^2(\iota \circ u)(x)|_{g_M^* \otimes g_M^* \otimes g_{3\nu}} &\geq \sum_{1 \leq i, j \leq m} |A_\iota[Tu(x)[e_i], Tu(x)[e_j]]|_{g_{3\nu}} \\ &\geq \sum_{1 \leq i \leq m} |A_\iota[Tu(x)[e_i], Tu(x)[e_i]]|_{g_{3\nu}} \\ &\geq \sum_{1 \leq i \leq m} \frac{1}{C_1} |Tu(x)[e_i]|_{g_N}^2 = \frac{1}{C_1} |Tu(x)|_{g_M^* \otimes g_N}^2, \end{aligned}$$

where $(e_i)_{1 \leq i \leq m}$ is an orthonormal basis in $\pi_M^{-1}(\{x\})$, $u \in \dot{W}^{1,2p}(M, N)$. $\square$

It turns out that the definition by embedding (4.1) and the intrinsic one may be different. In general, we do not even know if one or another inclusion does occur, except in the particular case of $k = 2$.

Proposition 4.5.4. *Let $\iota \in C^2(N, \mathbb{R}^\nu)$ be an isometric embedding and let $p \in [1, +\infty)$. Then*

$$\dot{W}_\iota^{2,p}(M, N) \subseteq \dot{W}^{2,p}(M, N).$$

Proof. For every $u \in \dot{W}_\iota^{2,p}(M, N)$, by lemma 4.5.2, u is twice colocally weakly differentiable. Since ι is an isometric embedding, by orthogonal decomposition of $T\mathbb{R}^\nu$ into TN and its orthogonal complement in $T\mathbb{R}^\nu$ [42, §6.2], $|D_K^2 u|_{g_M^* \otimes g_M^* \otimes g_N} \leq |D_K^2(\iota \circ u)|_{g_M^* \otimes g_M^* \otimes g_\nu}$ almost everywhere in M , and so $u \in \dot{W}^{2,p}(M, N)$. $\square$

4.5.2. Gagliardo–Nirenberg property

In view of proposition 4.5.1, if the target manifold is compact, we may ask if there exist some Gagliardo–Nirenberg inequalities [49, 95] for the spaces $\dot{W}^{k,p}(M, N)$ that can lead to

$$\dot{W}_\iota^{k,p}(M, N) = \dot{W}^{k,p}(M, N).$$

In general, even if the target manifold is compact, there are no such inequalities.

A first striking fact is that the second-order energy can vanish for a nontrivial map.

Proposition 4.5.5. *Let $p \in [1, +\infty)$. If N has at least one nontrivial closed geodesic, then there exists $u \in \dot{W}^{k,p}(\mathbb{S}^1, N) \cap L^\infty(\mathbb{S}^1, N)$ such that $|D_K^k u|_{(\otimes^k g_{\mathbb{S}^1}^*) \otimes g_N} = 0$ almost everywhere in M but $\int_{\mathbb{S}^1} |Tu|_{g_{\mathbb{S}^1}^* \otimes g_N}^{kp} > 0$.*

In particular, since a compact manifold has always at least one nontrivial closed geodesic [76], if N is compact, there are no Gagliardo–Nirenberg inequalities [49, 95] for the spaces $\dot{W}^{k,p}(\mathbb{S}^1, N)$.

Proof of proposition 4.5.5. Let $\ell \in \mathbb{N}_*$ and let $u_\ell: \mathbb{S}^1 \rightarrow \mathbb{S}^1 \subseteq \mathbb{C}$ be defined for every $\theta \in [0, 2\pi]$ by $u_\ell(e^{i\theta}) = e^{i\ell\theta}$. Let $\gamma: \mathbb{S}^1 \rightarrow N$ be a nontrivial closed geodesic. Since γ is a geodesic, we assume that for all $y \in \mathbb{S}^1$, $|T\gamma(y)|_{g_{\mathbb{S}^1}^* \otimes g_N} = 1$.

Then $u = \gamma \circ u_\ell$ is k times colocally weakly differentiable, it belongs to $L^\infty(\mathbb{S}^1, N)$ and $|D_K^k u|_{(\otimes^k g_{\mathbb{S}^1}^*) \otimes g_N} = 0$ almost everywhere in M but

$$\int_{\mathbb{S}^1} |Tu|_{g_{\mathbb{S}^1}^* \otimes g_N}^{kp} = \int_{\mathbb{S}^1} |Tu_\ell|_{g_{\mathbb{S}^1}^* \otimes g_{\mathbb{S}^1}}^{kp} = \ell^{kp} > 0. \quad \square$$

Proposition 4.5.6. *Let $p \in [1, +\infty)$ so that $1 \leq 2p < m$ and let $r > 0$. If N has at least one nontrivial bounded geodesic, then there exists $u \in \dot{W}^{2,p}(\mathbb{B}_r^m, N) \cap L^\infty(\mathbb{B}_r^m, N)$ such that $\int_{\mathbb{B}_r^m} |Tu|_{g_m^* \otimes g_N}^{2p} = +\infty$.*

If $\iota \in C^2(N, \mathbb{R}^\nu)$ is any isometric embedding given by proposition 4.5.3, it follows that

$$\dot{W}_\iota^{2,p}(\mathbb{B}_r^m, N) \subsetneq \dot{W}^{2,p}(\mathbb{B}_r^m, N),$$

and if N is compact, by propositions 4.5.1 and 4.5.4, the strict inclusion occurs for any embedding ι .

Proof of proposition 4.5.6. Let $\alpha > 0$. We define the function $v: \mathbb{B}_r^m \setminus \{0\} \rightarrow \mathbb{R}$ for every $x \in \mathbb{B}_r^m \setminus \{0\}$ by $v(x) = |x|^{-\alpha}$. Let $\gamma: \mathbb{R} \rightarrow N$ be a nontrivial bounded geodesic. Since γ is a geodesic, we assume that for all $y \in \mathbb{R}$, $|T\gamma(y)|_{g_1^* \otimes g_N} = 1$.

Then $u = \gamma \circ v$ is twice colocally weakly differentiable, $|Tu|_{g_m^* \otimes g_N} = |Tv|_{g_m^* \otimes g_1}$ and since γ is a geodesic, almost everywhere in $\mathbb{B}_r^m \setminus \{0\}$

$$|D_K^2 u|_{g_m^* \otimes g_m^* \otimes g_N} \leq |D^2 v|_{g_m^* \otimes g_m^* \otimes g_1}.$$

If $(2 + \alpha)p < m < 2p(\alpha + 1)$, then $u \in \dot{W}^{2,p}(\mathbb{B}_r^m, N) \setminus \dot{W}^{1,2p}(\mathbb{B}_r^m, N)$. $\square$

For other manifolds M and N and $p \in [1, +\infty)$, it leads to the following open question and the answer involves in particular the geometry of M and N .

Open question 4.5.1. If the manifolds M and N , $k \geq 2$ and $p \in [1, +\infty)$ do not satisfy the hypotheses of propositions 4.5.5 and 4.5.6, does it exist a constant $C > 0$ such that for every $u \in \dot{W}^{k,p}(M, N) \cap L^\infty(M, N)$, for every $j \in \{1, \dots, k-1\}$,

$$\int_M |D_K^j u|_{(\otimes^j g_M^*) \otimes g_N}^{\frac{kp}{j}} \leq C \operatorname{osc}(u)^{\frac{k-j}{j}p} \int_M |D_K^k u|_{(\otimes^k g_M^*) \otimes g_N}^p ?$$

4.5.3. Problem of density of smooth maps

In this part, we remark that intrinsic definitions 4.3.2 and 4.3.3 and proposition 4.5.1 give rise to new open questions whether the manifolds M and N are compact.

In view of proposition 4.5.1, given a map $u \in \dot{W}^{k,p}_\ell(M, N)$, we may ask whether there exists a sequence $(u_\ell)_{\ell \in \mathbb{N}} \subseteq \dot{W}^{k,p}_\ell(M, N)$ that *converges strongly* to u in $\dot{W}^{k,p}_\ell(M, N)$ [34, definition 4.1], that is, the sequence $(T^k u_\ell)_{\ell \in \mathbb{N}}$ converges to $T^k u$ locally in measure and the sequence $(|D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^k u|_{(\otimes^k g_M^*) \otimes g_N}$ in $L^p(M)$. In view of the strong density in Sobolev spaces between manifolds (see for example [17, 22, 64]), we may also ask whether there exists a map $u \in \dot{W}^{k,p}_\ell(M, N)$ that cannot be approximated by smooth maps in $\dot{W}^{k,p}_\ell(M, N)$ but that can be in $\dot{W}^{k,p}(M, N)$.

For instance, the hedgehog map $u_h: \mathbb{B}^m \rightarrow \mathbb{S}^{m-1}$ defined for every $x \in \mathbb{B}^m \setminus \{0\}$ by $u_h(x) = \frac{x}{|x|}$ belongs to $W^{k,p}_\ell(\mathbb{B}^m, \mathbb{S}^{m-1})$ for $kp < m$ but u_h cannot be strongly approximated by maps in $C^\infty(\overline{\mathbb{B}^m}, \mathbb{S}^{m-1})$ for $kp \geq m - 1$ since the identity map on $\mathbb{S}^{m-1}$ does not have a continuous extension to $\mathbb{B}^m$ with values into $\mathbb{S}^{m-1}$ [19, §II; 105, §4 example].

For the notion of intrinsic higher-order Sobolev maps, the space is larger than the one by embedding (proposition 4.5.1) but the notion of convergence is weaker. However, even if the question of strong density is still open, the necessary and sufficient condition that appears with definition by embedding [22, theorem 1] is necessary for the intrinsic one.

We prove the results for the case $k = 2$ but we note that those results extend to higher-order.

For $q \in [1, +\infty)$, we denote by $[q]$ the integer part of q and by $\pi_{[q]}(N)$ the $[q]^{\text{th}}$ homotopy group of N . For instance, if $\pi_{[q]}(N) \simeq \{0\}$, then every continuous map $f: \mathbb{S}^{[q]} \rightarrow N$ on the $[q]$ -dimensional sphere is homotopic to a constant map.

Proposition 4.5.7. *Let M and N be two smooth connected compact Riemannian manifolds and let $p \in [1, +\infty)$ so that $1 \leq 2p < m$.*

- (i) *If for every $u \in \dot{W}^{2,p}(M, N)$, there exists a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ in $C^\infty(M, N)$ such that the sequence $(Tu_\ell)_{\ell \in \mathbb{N}}$ converges to Tu locally in measure and such that the sequence $(|D_K^2 u_\ell|_{g_M^* \otimes g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^p(M)$, and if $2p \notin \mathbb{N}$, then $\pi_{[2p]}(N) \simeq \{0\}$.*
- (ii) *If for every $u \in \dot{W}^{2,p}(M, N)$, there exists a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ in $C^\infty(M, N)$ such that the sequence $(T^2 u_\ell)_{\ell \in \mathbb{N}}$ converges to $T^2 u$ locally in measure and such that the sequence $(|D_K^2 u_\ell|_{g_M^* \otimes g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$*

converges to $|D_K^2 u|_{g_M^* \otimes g_M^* \otimes g_N}$ in $L^p(M)$, and if $2p \in \mathbb{N}$, then $\pi_{2p}(N) \simeq \{0\}$.

As a consequence, if $m - 1 < 2p < m$, since $\pi_{m-1}(\mathbb{S}^{m-1}) \not\simeq \{0\}$, there is no weakly bounded sequence $(u_\ell)_{\ell \in \mathbb{N}}$ in $\dot{W}^{2,p}(\mathbb{B}^m, \mathbb{S}^{m-1})$ such that $(Tu_\ell)_{\ell \in \mathbb{N}}$ converges to Tu_h locally in measure and if $m - 1 \leq 2p < m$, the map u_h cannot be strongly approximated by smooth maps in $\dot{W}^{2,p}(\mathbb{B}^m, \mathbb{S}^{m-1})$.

Lemma 4.5.8. *Let N be a smooth connected compact Riemannian manifold and let $p \in [1, k]$. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence in $C^\infty(\mathbb{S}^k, N)$ and let $u \in \dot{W}^{2,p}(\mathbb{S}^k, N)$.*

- (i) *If $2p \geq k$, if the sequence $(Tu_\ell)_{\ell \in \mathbb{N}}$ converges to Tu locally in measure and if the sequence $(|D_K^2 u_\ell|_{g_{k+1}^* \otimes g_{k+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^p(\mathbb{S}^k)$, then the sequence $(|Tu_\ell|_{g_{k+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^{2p}(\mathbb{S}^k)$.*
- (ii) *If $2p \geq k$, if the sequence $(T^2 u_\ell)_{\ell \in \mathbb{N}}$ converges to $T^2 u$ locally in measure and if the sequence $(|D_K^2 u_\ell|_{g_{k+1}^* \otimes g_{k+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^2 u|_{g_{k+1}^* \otimes g_{k+1}^* \otimes g_N}$ in $L^p(\mathbb{S}^k)$, then the sequence $(|Tu_\ell|_{g_{k+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Tu|_{g_{k+1}^* \otimes g_N}$ in $L^{2p}(\mathbb{S}^k)$.*

Since Sobolev spaces between Riemannian manifolds do not form a vector space, it is not surprising that there is no some equivalence between both assertions in the previous lemma.

Lemma 4.5.9 (Poincaré inequalities). *Let M be a smooth connected compact Riemannian manifold and let μ be the measure associated to the Riemannian metric on M . Let $p \in [1, m)$ and let $q \in [1, \frac{mp}{m-p}]$. Let $\varepsilon > 0$. There exists $C > 0$ such that for every map $v \in W^{1,p}(M)$ and every measurable subset $A \subseteq M$ such that $\mu(A) > \varepsilon$,*

$$\|v\|_{L^q(M)} \leq C \left(\|\nabla v\|_{L^p(M)} + \frac{1}{\mu(A)} \int_A |v| \right).$$

Proof. By the classical Poincaré inequality on compact Riemannian manifolds [67, proposition 3.9], there exists a constant $C_1 > 0$ such that for every $v \in W^{1,p}(M)$,

$$\|v\|_{L^q(M)} \leq C_1 \|\nabla v\|_{L^p(M)} + \mu(M)^{\frac{1}{q}-1} \int_M |v|.$$

By Hölder's inequality applied to the second term, for every measurable subset $A \subseteq M$ such that $\mu(A) > \varepsilon$, we have

$$\|v\|_{L^q(M)} \leq C_1 \|\nabla v\|_{L^p(M)} + \frac{\mu(M \setminus A)^{1-\frac{1}{q}}}{\mu(M)^{1-\frac{1}{q}}} \|v\|_{L^q(M)} + \frac{\mu(M)^{\frac{1}{q}}}{\mu(A)} \int_A |v|,$$

and so

$$\begin{aligned}
& \|v\|_{L^q(M)} \\
& \leq \frac{\mu(M)}{\mu(M)^{1-\frac{1}{q}} - \mu(M \setminus A)^{1-\frac{1}{q}}} \left(C_1 \mu(M)^{-\frac{1}{q}} \|\nabla v\|_{L^q(M)} + \frac{1}{\mu(A)} \int_A |v| \right) \\
& \leq \frac{\mu(M)}{\mu(M)^{1-\frac{1}{q}} - (\mu(M) - \varepsilon)^{1-\frac{1}{q}}} \left(C_1 \mu(M)^{-\frac{1}{q}} \|\nabla v\|_{L^q(M)} + \frac{1}{\mu(A)} \int_A |v| \right).
\end{aligned}$$

□

Proof of lemma 4.5.8. Let μ be the measure associated to the Riemannian metric on the sphere $\mathbb{S}^k \subseteq \mathbb{R}^{k+1}$. Since the sequence $(Tu_\ell)_{\ell \in \mathbb{N}}$ converges to Tu locally in measure in assertions (i) and (ii), there exist $\varepsilon > 0$ and $\gamma > 0$ such that for every $\ell \in \mathbb{N}$,

$$\mu(\{x \in \mathbb{S}^k : |Tu_\ell(x)|_{g_{k+1}^* \otimes g_N} \leq \gamma\}) > \varepsilon.$$

For every $\ell \in \mathbb{N}$, by lemma 4.3.7, $|Tu_\ell|_{g_{k+1}^* \otimes g_N} \in W^{1,p}(\mathbb{S}^k)$ and almost everywhere in $\mathbb{S}^k$

$$|T|Tu_\ell|_{g_{k+1}^* \otimes g_N}|_{g_{k+1}^* \otimes g_1} \leq |D_K^2 u_\ell|_{g_{k+1}^* \otimes g_{k+1}^* \otimes g_N}.$$

So by the Poincaré inequality (lemma 4.5.9), there exists $C_1 > 0$ such that for every $\ell \in \mathbb{N}$, if $A_\ell = \{x \in \mathbb{S}^k : |Tu_\ell(x)|_{g_{k+1}^* \otimes g_N} \leq \gamma\}$,

$$\begin{aligned}
& \| |Tu_\ell|_{g_{k+1}^* \otimes g_N} \|_{L^{2p}(\mathbb{S}^k)} \\
& \leq C_1 \left(\| |T|Tu_\ell|_{g_{k+1}^* \otimes g_N}|_{g_{k+1}^* \otimes g_1} \|_{L^p(\mathbb{S}^k)} + \frac{1}{\mu(A_\ell)} \int_{A_\ell} |Tu_\ell|_{g_{k+1}^* \otimes g_N} \right) \\
& \leq C_1 \left(\| |D_K^2 u_\ell|_{g_{k+1}^* \otimes g_{k+1}^* \otimes g_N} \|_{L^p(\mathbb{S}^k)} + \gamma \right).
\end{aligned} \tag{4.5.6}$$

If assumptions of assertion (i) or (ii) are satisfied, by previous inequality (4.5.6), the sequence $(|Tu_\ell|_{g_{k+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^{2p}(\mathbb{S}^k)$.

We assume now that assumptions of assertion (ii) are satisfied. Since the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges to u locally in measure, by weak closure property in Sobolev spaces [34, proposition 3.8], $u \in \dot{W}^{1,2p}(\mathbb{S}^k, N)$. In particular, $|Tu|_{g_{k+1}^* \otimes g_N} \in L^{2p}(\mathbb{S}^k)$. Since $u \in \dot{W}^{1,p}(\mathbb{S}^k, N) \cap \dot{W}^{2,p}(\mathbb{S}^k, N)$, by lemma 4.3.7, $|Tu|_{g_{k+1}^* \otimes g_N} \in W^{1,p}(\mathbb{S}^k)$ and so by Sobolev inequalities on compact Riemannian manifolds [67, theorem 3.5], there exists $C_2 > 0$ such that for every $\ell \in \mathbb{N}$,

$$\begin{aligned}
& \| |Tu_\ell|_{g_{k+1}^* \otimes g_N} - |Tu|_{g_{k+1}^* \otimes g_N} \|_{L^{2p}(\mathbb{S}^k)} \\
& \leq C_2 \| |Tu_\ell|_{g_{k+1}^* \otimes g_N} - |Tu|_{g_{k+1}^* \otimes g_N} \|_{W^{1,p}(\mathbb{S}^k)}.
\end{aligned}$$

By inequality (4.5.6), the sequence $(|Tu_\ell|_{g_{k+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^{2p}(\mathbb{S}^k)$, and so by Hölder's inequality, bounded in $L^p(\mathbb{S}^k)$. Since the sequence $(|D_K^2 u_\ell|_{g_{k+1}^* \otimes g_{k+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^p(\mathbb{S}^k)$, by lemma 4.3.7, the sequence $(|Tu_\ell|_{g_{k+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $W^{1,p}(\mathbb{S}^k)$. By Rellich–Kondrashov embedding theorem for Sobolev maps on compact Riemannian manifolds [9, theorem 2.34 (a)], and since $(|Tu_\ell|_{g_{k+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Tu|_{g_{k+1}^* \otimes g_N}$ in measure, the sequence $(|Tu_\ell|_{g_{k+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to the map $|Tu|_{g_{k+1}^* \otimes g_N}$ in $L^p(\mathbb{S}^k)$. Since $(T^2 u_\ell)_{\ell \in \mathbb{N}}$ converges to $T^2 u$ locally in measure, the sequence $(T|Tu_\ell|_{g_{k+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $T|Tu|_{g_{k+1}^* \otimes g_N}$ locally in measure. Furthermore, for every $\ell \in \mathbb{N}$,

$$\begin{aligned} & |T(|Tu_\ell|_{g_{k+1}^* \otimes g_N} - |Tu|_{g_{k+1}^* \otimes g_N})|_{g_{k+1}^* \otimes g_1} \\ & \leq |D_K^2 u_\ell|_{g_{k+1}^* \otimes g_{k+1}^* \otimes g_N} + |D_K^2 u|_{g_{k+1}^* \otimes g_{k+1}^* \otimes g_N}. \end{aligned}$$

By Lebesgue's dominated convergence theorem [20, theorem 2.8.5], the sequence $(T(|Tu_\ell|_{g_{k+1}^* \otimes g_N} - |Tu|_{g_{k+1}^* \otimes g_N}))_{\ell \in \mathbb{N}}$ converges to 0 in $L^p(\mathbb{S}^k)$. Hence, the sequence $(|Tu_\ell|_{g_{k+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Tu|_{g_{k+1}^* \otimes g_N}$ in $W^{1,p}(\mathbb{S}^k)$, and so in $L^{2p}(\mathbb{S}^k)$ by the Sobolev inequality. $\square$

Proof of proposition 4.5.7. We first give the proof when $M = \mathbb{B}^{[2p]+1} \times L$. If $\pi_{[2p]}(N) \not\cong \{0\}$, then there exists a smooth map $f: \mathbb{S}^{[2p]} \rightarrow N$ which is not homotopic to a constant map in $C(\mathbb{S}^{[2p]}, N)$. We define the map $u: M \rightarrow N$ for every $x = (x', x'') \in \mathbb{B}^{[2p]+1} \times L$ by $u(x) = f(\frac{x'}{|x'|})$. We observe that there exists a constant $C > 0$ such that for every $x \in (\mathbb{B}^{[2p]+1} \setminus \{0\}) \times L$, $|D_K^2 u(x)|_{g_M^* \otimes g_M^* \otimes g_N} \leq \frac{C}{|x'|^2}$, and thus $u \in \dot{W}^{2,p}(M, N)$.

We first show that the assumptions of assertion (i) cannot be satisfied. Otherwise, since for every $\ell \in \mathbb{N}$,

$$\begin{aligned} \int_M |D_K^2 u_\ell|_{g_M^* \otimes g_M^* \otimes g_N}^p &= \int_L \int_{\mathbb{B}^{[2p]+1}} |D_K^2 u_\ell|_{g_M^* \otimes g_M^* \otimes g_N}^p \\ &= \int_L \left(\int_0^1 \left(\int_{\mathbb{S}_r^{[2p]}} |D_K^2 u_\ell|_{g_M^* \otimes g_M^* \otimes g_N}^p dr \right) \right), \end{aligned}$$

by Fatou's lemma, for almost every $x'' \in L$ and almost every $r \in (0, 1)$, up to a subsequence, the sequence $((|D_K^2 u_\ell|_{g_M^* \otimes g_M^* \otimes g_N})_{|\mathbb{S}_r^{[2p]} \times \{x''\}})_{\ell \in \mathbb{N}}$ is bounded in $L^p(\mathbb{S}_r^{[2p]})$. By lemma 4.5.8 (i), for almost every $x'' \in L$ and almost every $r \in (0, 1)$, the sequence $((|Tu_\ell|_{g_M^* \otimes g_N})_{|\mathbb{S}_r^{[2p]} \times \{x''\}})_{\ell \in \mathbb{N}}$ is bounded in $L^{2p}(\mathbb{S}_r^{[2p]})$. Hence, for almost every $x'' \in L$ and almost every $r \in (0, 1)$, the sequence $((u_\ell)_{|\mathbb{S}_r^{[2p]} \times \{x''\}})_{\ell \in \mathbb{N}}$ is bounded in $\dot{W}^{1,2p}(\mathbb{S}_r^{[2p]}, N)$

and since the homotopy classes are preserved by weakly bounded sequence in $\dot{W}^{1,2p}(\mathbb{S}_r^{[2p]}, N)$ [118, theorem 2.1], the map $u|_{\mathbb{S}_r^{[2p]} \times \{x''\}}$ is homotopic to a constant map, and so the map f also.

We now show that the assumptions of assertion (ii) cannot be satisfied. Otherwise, since for every $\ell \in \mathbb{N}$,

$$\begin{aligned} & \int_M ||D_K^2 u_\ell|_{g_M^* \otimes g_M^* \otimes g_N} - |D_K^2 u|_{g_M^* \otimes g_M^* \otimes g_N}|^p \\ &= \int_L \left(\int_0^1 \left(\int_{\mathbb{S}_r^{2p}} ||D_K^2 u_\ell|_{g_M^* \otimes g_M^* \otimes g_N} - |D_K^2 u|_{g_M^* \otimes g_M^* \otimes g_N}|^p \right) dr \right), \end{aligned}$$

up to a subsequence, for almost every $x'' \in L$ and almost every $r \in (0, 1)$, the sequence $((|D_K^2 u_\ell|_{g_M^* \otimes g_M^* \otimes g_N})|_{\mathbb{S}_r^{2p} \times \{x''\}})_{\ell \in \mathbb{N}}$ converges to the map $(|D_K^2 u|_{g_M^* \otimes g_M^* \otimes g_N})|_{\mathbb{S}_r^{2p} \times \{x''\}}$ in $L^p(\mathbb{S}_r^{2p})$. By lemma 4.5.8 (ii), for almost every $x'' \in L$ and almost every $r \in (0, 1)$, $((|Tu_\ell|_{g_M^* \otimes g_N})|_{\mathbb{S}_r^{2p} \times \{x''\}})_{\ell \in \mathbb{N}}$ converges to $(|Tu|_{g_M^* \otimes g_N})|_{\mathbb{S}_r^{2p} \times \{x''\}}$ in $L^{2p}(\mathbb{S}_r^{2p})$. Hence, for almost every $x'' \in L$ and almost every $r \in (0, 1)$, the sequence $((u_\ell)|_{\mathbb{S}_r^{2p} \times \{x''\}})_{\ell \in \mathbb{N}}$ converges to $u|_{\mathbb{S}_r^{2p} \times \{x''\}}$ in $\dot{W}^{1,2p}(\mathbb{S}_r^{2p}, N)$ [34, proposition 4.4] and since the homotopy classes are preserved by strong convergence in the space $\dot{W}^{1,2p}(\mathbb{S}_r^{2p}, N)$ [117], the map $u|_{\mathbb{S}_r^{2p} \times \{x''\}}$ is homotopic to a constant map, and so the map f also.

For a general manifold M , we start from a counterexample $u: \mathbb{B}^m \rightarrow N$ to a Sobolev map on the whole of M . Indeed, there is a map $\Phi: \mathbb{B}^m \rightarrow \mathbb{R}^m$ which maps diffeomorphically some subset $A \subset \mathbb{B}^m$ to $\mathbb{B}^{[2p]+1} \times \mathbb{S}^{m-[2p]-1}$ (A is a tubular neighborhood of an $(m - [2p] - 1)$ -dimensional sphere, with $\mathbb{S}^0 = \{-1, 1\}$), $\Phi(\mathbb{B}^m \setminus A) \subset (\mathbb{B}^{[2p]+1} \setminus \{0\}) \times \mathbb{R}^{m-[2p]-1}$ and Φ is constant near $\partial\mathbb{B}^m$. We then define the map $u: \mathbb{B}^m \rightarrow N$ for every $x \in \mathbb{B}^m$ by $u(x) = f(\frac{y'}{|y'|})$ if $\Phi(x) = (y', y'')$. The map u is constant near the boundary and can thus be transported and extended to any m -dimensional manifold, giving the required counterexample. $\square$

For the notion of intrinsic higher-order Sobolev spaces, if the target manifold is compact, the space is also larger than the one by embedding (proposition 4.5.1). In this case, the condition $\pi_{[kp]}(N) \simeq \{0\}$ which is necessary and sufficient in the definition by embedding [22, theorem 1] is also necessary.

Proposition 4.5.10. *Let M and N be two smooth connected compact Riemannian manifolds and let $p \in [1, +\infty)$ so that $1 \leq kp < m$.*

- (i) *If for every $u \in \dot{W}^{k,p}(M, N)$, there exists a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ in $C^\infty(M, N)$ such that the sequence $(T^{k-1}u_\ell)_{\ell \in \mathbb{N}}$ converges to $T^{k-1}u$*

locally in measure and such that the sequence $(|D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^p(M)$, and if $kp \notin \mathbb{N}$, then $\pi_{[kp]}(N) \simeq \{0\}$.

- (ii) If for every $u \in \dot{W}^{k,p}(M, N)$, there exists a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ in $C^\infty(M, N)$ such that the sequence $(T^k u_\ell)_{\ell \in \mathbb{N}}$ converges to $T^k u$ locally in measure and such that the sequence $(|D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^k u|_{(\otimes^k g_M^*) \otimes g_N}$ in $L^p(M)$, and if $kp \in \mathbb{N}$, then $\pi_{kp}(N) \simeq \{0\}$.

To prove this proposition, we can also state a similar lemma to lemma 4.5.8.

Lemma 4.5.11. *Let N be a smooth connected compact Riemannian manifold and let $p \in [1, +\infty)$ so that $(k-1)p < q$. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence in $C^\infty(\mathbb{S}^q, N)$ and let $u \in \dot{W}^{k,p}(\mathbb{S}^q, N)$.*

- (i) *If $kp \geq q$, if the sequence $(T^{k-1} u_\ell)_{\ell \in \mathbb{N}}$ converges to $T^{k-1} u$ locally in measure and if the sequence $(|D_K^k u_\ell|_{(\otimes^k g_{q+1}^*) \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^p(\mathbb{S}^q)$, then the sequence $(|Tu_\ell|_{g_{q+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^{kp}(\mathbb{S}^q)$.*
- (ii) *If $kp \geq q$, if the sequence $(T^k u_\ell)_{\ell \in \mathbb{N}}$ converges to $T^k u$ locally in measure and if the sequence $(|D_K^k u_\ell|_{(\otimes^k g_{q+1}^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^k u|_{(\otimes^k g_{q+1}^*) \otimes g_N}$ in $L^p(\mathbb{S}^q)$, then the sequence $(|Tu_\ell|_{g_{q+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Tu|_{g_{q+1}^* \otimes g_N}$ in $L^{kp}(\mathbb{S}^q)$.*

To prove such a lemma, we can use Poincaré inequalities recursively (lemma 4.5.9) and lemma 4.3.11. The structure of the proof of proposition 4.5.10 is then the same as the one of proposition 4.5.7.

Chapter V

Appendix of chapter IV

In the previous chapter, we stated definitions and results about higher-order weak differentiability and Sobolev spaces between manifolds. In general, the proofs concerned the second-order case although they applied to arbitrary order. In this chapter, we give the *missing details* of propositions reported in the previous chapter IV.

5.1. Higher order colocal weak differentiability

The k^{th} order tangent bundle $T^k M$ has k natural vector bundle structures $\pi_{T^{k-1}M}^1, \dots, \pi_{T^{k-1}M}^k$ over $T^{k-1}M$ (section 4.2.3). In fact, for every $i \in \{1, \dots, k-1\}$, there exists $\ell \in \{1, \dots, k-1\}$ such that $\pi_{T^{k-1}M}^i = T(\pi_{T^{k-2}M}^\ell): T(T^{k-1}M) \rightarrow T(T^{k-2}M)$ and so the corresponding *vertical bundle* $V_i(T^{k-1}M) \rightarrow T^{k-1}M$ is the subbundle of $T(T^{k-1}M) \rightarrow T^{k-1}M$ defined by

$$V_i(T^{k-1}M) = \{\nu \in T^k M : \pi_{T^{k-1}M}^i(\nu) = 0\}.$$

Furthermore, we denote by S_k the *symmetric group* which is the group of all bijections of $\{1, \dots, k\}$ and for every $\sigma \in S_k$, there exists a *canonical permutation* κ_M^σ which corresponds to some permutation of the bundle structures and which is a linear isomorphism from a vector bundle $(T^k M, \pi_{T^{k-1}M}^i, T^{k-1}M)$ to another bundle $(T^k M, \pi_{T^{k-1}M}^j, T^{k-1}M)$, with $i, j \in \{1, \dots, k\}$ [15, §7.2; 28, §2.5 (example 2.23)].

Proposition 5.1.1. *Let $k \in \mathbb{N}$ so that $k \geq 3$. If the map $u: M \rightarrow N$ is k times colocally weakly differentiable, then there exist unique measurable maps $T^2 u, \dots, T^k u$ such that for every $\ell \in \{3, \dots, k\}$, the map $T^\ell u: M \rightarrow \text{Mor}(T^\ell M, T^\ell N)$ covers the map u , for every $f \in C^1(\text{Mor}(T^{\ell-1}M, T^{\ell-1}N), \mathbb{R})$ such that $f \circ T^{\ell-1}u \in W_{\text{loc}}^{1,1}(M)$, almost everywhere in M*

$$T(f \circ T^{\ell-1}u) = T f \circ T^\ell u$$

and for every $\sigma \in S_\ell$, $\kappa_N^\sigma \circ T^\ell u = T^\ell u \circ \kappa_M^\sigma$ and for every $i \in \{1, \dots, \ell-1\}$, $T^\ell u \circ \text{Vert}_{T^{\ell-1}M}^i = \text{Vert}_{T^{\ell-1}N}^i \circ T^{\ell-1}u$ almost everywhere in M .

This proposition does not have its equivalent in the previous chapter; it was just a remark between two results (section 4.2.3).

Lemma 5.1.2. *Let $k \geq 3$. If the locally integrable map $v: M \rightarrow \mathbb{R}^n$ is k times weakly differentiable, then there exist unique measurable maps $T^2v, \dots, T^kv$ such that for every $\ell \in \{3, \dots, k\}$, the map $T^\ell v: M \rightarrow \text{Mor}(T^\ell M, T^\ell \mathbb{R}^n)$ covers the map v , for every local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$,*

$$T^\ell(v \circ \psi^{-1}) = T^\ell v \circ T^\ell(\psi^{-1})$$

almost everywhere in $\psi(V)$. Moreover, for every $\sigma \in S_\ell$,

$$\kappa_{\mathbb{R}^n}^\sigma \circ T^\ell v = T^\ell v \circ \kappa_M^\sigma,$$

there exists $i \in \{1, \dots, \ell\}$ such that $\pi_{T^{\ell-1}\mathbb{R}^n}^i \circ T^\ell v = T^{\ell-1}v \circ \pi_{T^{\ell-1}M}^i$, and for every $j \in \{1, \dots, \ell-1\}$, $T^\ell v \circ \text{Vert}_{T^{\ell-1}M}^j = \text{Vert}_{T^{\ell-1}\mathbb{R}^n}^j \circ T^{\ell-1}v$ almost everywhere in M .

Sketch of the proof. First, if M is an open set of $\mathbb{R}^m$, since the domain is flat, for every $\ell \in \{2, \dots, k\}$, for almost every $x \in M$, the map $T^\ell v(x): T_x^\ell M \rightarrow T_{v(x)}^\ell \mathbb{R}^n$ is defined for every $w = (e_\lambda)_{\lambda \in I_\ell}$ by [15, §15.3]

$$T^\ell v(x)[w] = \left(\sum_{i=1}^{|\Lambda|} \sum_{\lambda \in \mathcal{P}_i(\Lambda)} D^i v(x)[e_{\lambda^1}, \dots, e_{\lambda^i}] \right)_{\Lambda \in I_\ell},$$

and the conclusion follows.

For a general manifold M , for every local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$, the map $v \circ \psi^{-1}: \psi(V) \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$ is k times weakly differentiable. Since $\psi \in C^k(V, \mathbb{R}^m)$, for every $\ell \in \{3, \dots, k\}$, the map $T^\ell \psi$ is a section from M to $\text{Mor}(T^\ell V, \psi^*(T^\ell \mathbb{R}^m))$. Moreover, for every $\sigma \in S_\ell$, $\kappa_{\mathbb{R}^m}^\sigma \circ T^\ell \psi = T^\ell \psi \circ \kappa_M^\sigma$ and for every $j \in \{1, \dots, \ell-1\}$, $T^\ell \psi \circ \text{Vert}_{T^{\ell-1}M}^j = \text{Vert}_{T^{\ell-1}\mathbb{R}^m}^j \circ T^{\ell-1}\psi$. As a consequence, for every $\ell \in \{3, \dots, k\}$, since almost everywhere in V

$$T^\ell v = T^\ell(v \circ \psi^{-1}) \circ T^\ell \psi,$$

by the previous step, for every $\sigma \in S_\ell$, $\kappa_{\mathbb{R}^n}^\sigma \circ T^\ell v = T^\ell v \circ \kappa_M^\sigma$, there exists $i \in \{1, \dots, \ell\}$ such that $\pi_{T^{\ell-1}\mathbb{R}^n}^i \circ T^\ell v = T^{\ell-1}v \circ \pi_{T^{\ell-1}M}^i$, and for every $j \in \{1, \dots, \ell-1\}$, $T^\ell v \circ \text{Vert}_{T^{\ell-1}M}^j = \text{Vert}_{T^{\ell-1}\mathbb{R}^n}^j \circ T^{\ell-1}v$ almost everywhere in V . Since for every $j \in \{1, \dots, \ell\}$, there exists $\sigma \in S_\ell$ such that $\pi_{T^{\ell-1}M}^j = \pi_{T^{\ell-1}M}^i \circ \kappa_M^\sigma$ on $T^\ell M$ and $\pi_{T^{\ell-1}\mathbb{R}^n}^j = \pi_{T^{\ell-1}\mathbb{R}^n}^i \circ \kappa_{\mathbb{R}^n}^\sigma$ on $T^\ell \mathbb{R}^n$, then $\pi_{T^{\ell-1}\mathbb{R}^n}^j \circ T^\ell v = T^{\ell-1}v \circ \pi_{T^{\ell-1}M}^j$ almost everywhere in V , and so for almost every $x \in V$, $T^\ell v(x) \in \text{Mor}(T^\ell M, T^\ell \mathbb{R}^n)_{x, v(x)}$.

Finally, since the manifold M has a countable atlas of local charts, by a direct covering argument, the previous equalities are satisfied almost everywhere in M . $\square$

Lemma 5.1.3 (Extended local charts). *For every $y \in N$, there exist an open subset $U \subseteq N$ such that $y \in U$, and maps $\varphi \in C^k(N, \mathbb{R}^n)$ and $\varphi^* \in C^k(\mathbb{R}^n, N)$ such that*

- (i) *the set $\text{supp}(\varphi)$ is compact,*
- (ii) *the set $\overline{\{x \in \mathbb{R}^n : \varphi^*(x) \neq y\}}$ is compact,*
- (iii) *the map $\varphi|_U$ is a diffeomorphism of class C^k onto its image $\varphi(U)$,*
- (iv) *$\varphi^* \circ \varphi = \text{id}$ in U .*

The proof is similar to the case of C^1 -manifolds [34, lemma 1.6] but working with a manifold of class C^k (see also proposition 4.2.3).

Proof of proposition 5.1.1. By proposition 4.2.1, there exists a measurable map $T^2u: M \rightarrow \text{Mor}(T^2M, T^2N)$. We then proceed recursively. Let $T^{\ell-1}u: M \rightarrow \text{Mor}(T^{\ell-1}M, T^{\ell-1}N)$ be a measurable map for some $\ell \in \{3, \dots, k\}$. Since u is k times colocally weakly differentiable, by the existence and uniqueness of the colocal weak derivative [34, proposition 1.5], there exists a unique measurable bundle morphism $T^\ell u: TM \rightarrow T(\text{Mor}(T^{\ell-1}M, T^{\ell-1}N))$ such that for $f \in C^1(\text{Mor}(T^{\ell-1}M, T^{\ell-1}N), \mathbb{R})$ such that $f \circ T^{\ell-1}u \in W_{\text{loc}}^{1,1}(M)$, we have $T(f \circ T^{\ell-1}u) = Tf \circ T^\ell u$ almost everywhere in M . Furthermore, the map $T^\ell u$ can be viewed as a measurable map from M to $T^*M \otimes T(\text{Mor}(T^{\ell-1}M, T^{\ell-1}N))$, which is a subset of $\text{Mor}(T^\ell M, T^\ell N)$.

Let $U \subseteq N$, $\varphi \in C^k(N, \mathbb{R}^n)$ and $\varphi^* \in C^k(N, \mathbb{R}^n)$ be the extended local charts given by lemma 5.1.3. For every $y \in \text{Mor}(T^{\ell-1}M, T^{\ell-1}U)$, there exist two open sets $V, W \subseteq \text{Mor}(T^{\ell-1}M, T^{\ell-1}U)$ such that $y \in \bar{V} \subseteq W$ and then a map $\eta \in C_c^1(\text{Mor}(T^{\ell-1}M, T^{\ell-1}N), \mathbb{R})$ such that $0 \leq \eta \leq 1$ on $\text{Mor}(T^{\ell-1}M, T^{\ell-1}N)$, $\eta = 1$ on V and $\text{supp}(\eta) \subset W$. As a consequence, the function h defined for every $z \in \text{Mor}(T^{\ell-1}M, T^{\ell-1}N)$ by

$$h(z) = \begin{cases} \eta(z) T^{\ell-1}\varphi(z) & \text{if } z \in W, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $C_c^1(\text{Mor}(T^{\ell-1}M, T^{\ell-1}N), \mathbb{R}^n)$ and satisfies $T(h \circ T^{\ell-1}u) = Th \circ T^\ell u$ almost everywhere in M .

Since $\eta = 1$ on V , we have in particular $T^\ell(\varphi \circ u) = T^\ell\varphi \circ T^\ell u$ almost everywhere in $u^{-1}(\pi_N(\pi_{M \times N}^{\ell-1}(V)))$. Since $T^\ell\varphi^* \circ T^\ell\varphi = \text{id}$ on $(\pi_N^\ell)^{-1}(U)$,

by lemma 5.1.2 and since $\varphi^* \in C^k(\mathbb{R}^n, N)$, for every $\sigma \in S_\ell$, almost everywhere in $u^{-1}(\pi_N(\pi_{M \times N}^{\ell-1}(V)))$

$$\begin{aligned}\kappa_N^\sigma \circ T^\ell u &= \kappa_N^\sigma \circ T^\ell \varphi^* \circ T^\ell(\varphi \circ u) \\ &= T^\ell \varphi^* \circ \kappa_{\mathbb{R}^n}^\sigma \circ T^\ell(\varphi \circ u) \\ &= T^\ell \varphi^* \circ T^\ell(\varphi \circ u) \circ \kappa_M^\sigma = T^\ell u \circ \kappa_M^\sigma,\end{aligned}$$

which gives a ℓ^{th} bundle morphism structure to $T^\ell u$ so that $T^\ell u$ is a ℓ^{tuple} vector bundle morphism in $u^{-1}(\pi_N(\pi_{M \times N}^{\ell-1}(V)))$.

The facts that for every $j \in \{1, \dots, \ell-1\}$,

$$T^\ell u \circ \text{Vert}_{T^{\ell-1}M}^j = \text{Vert}_{T^{\ell-1}N}^j \circ T^{\ell-1}u$$

and there exists $i \in \{1, \dots, \ell\}$ such that $\pi_{T^{\ell-1}N}^i \circ T^\ell u = T^{\ell-1}u \circ \pi_{T^{\ell-1}M}^i$ almost everywhere in $u^{-1}(\pi_N(\pi_{M \times N}^{\ell-1}(V)))$ can be proved along the same line. Since for every $j \in \{1, \dots, \ell\}$, there exists $\sigma \in S_\ell$ such that $\pi_{T^{\ell-1}M}^j = \pi_{T^{\ell-1}M}^i \circ \kappa_M^\sigma$ on $T^\ell M$ and $\pi_{T^{\ell-1}N}^j = \pi_{T^{\ell-1}N}^i \circ \kappa_N^\sigma$ on $T^\ell N$,

$$\begin{aligned}T^{\ell-1}u \circ \pi_{T^{\ell-1}M}^j &= T^{\ell-1}u \circ \pi_{T^{\ell-1}M}^i \circ \kappa_M^\sigma = \pi_{T^{\ell-1}N}^i \circ T^\ell u \circ \kappa_M^\sigma \\ &= \pi_{T^{\ell-1}N}^i \circ \kappa_N^\sigma \circ T^\ell u = \pi_{T^{\ell-1}N}^j \circ T^\ell u\end{aligned}$$

almost everywhere in $u^{-1}(\pi_N(\pi_{M \times N}^{\ell-1}(V)))$. So for every $j \in \{1, \dots, \ell\}$, for almost every $x \in u^{-1}(\pi_N(\pi_{M \times N}^{\ell-1}(V)))$,

$$\pi_{\text{Mor}(T^{\ell-1}M, T^{\ell-1}N)}^j \circ T^\ell u(x) = T^{\ell-1}u(x)$$

and $\pi_{M \times N}^{\ell-1} \circ T^{\ell-1}u(x) = (x, u(x))$.

Finally, by a direct covering argument, since the manifolds M and N have a countable basis, the measurable map $T^\ell u: M \rightarrow \text{Mor}(T^\ell M, T^\ell N)$ satisfies the previous properties almost everywhere in M . $\square$

5.2. Higher order weak covariant derivatives

To be short, we denote by $(\otimes^k T^*M) \otimes TN$ the tensor product $\underbrace{T^*M \otimes \dots \otimes T^*M}_k \otimes TN$, and for every $x \in M$, we denote by $\times^k T_x M$ the product $\underbrace{T_x M \times \dots \times T_x M}_k$.

If the map $u: M \rightarrow N$ is smooth, then its k^{th} order covariant derivative $D_K^k u: M \rightarrow (\otimes^k T^*M) \otimes TN$ is defined recursively by [43, §2; 79, §12.8]

$$D_K^k u = K \circ T(D_K^{k-1} u).$$

More specifically, for every $x \in M$, the map $D_K^k u(x): \times^k T_x M \rightarrow T_{u(x)} N$ is k -linear and for every $e \in T_x M$, from the identity (4.3.2), we also have the decomposition

$$T(D_K^{k-1} u)(x)[e] = \text{Vert} \circ D_K^k u(x)[e] + \text{Hor}_K \circ D_K^{k-1} u(x)[e],$$

where $D_K^k u(x)[e]: \times^{k-1} T_x M \rightarrow T_{u(x)} N$ is $(k-1)$ -linear and the maps Vert and Hor_K are related to the vector bundle $((\otimes^{k-1} T^* M) \otimes TN, \pi, (\times^{k-1} M) \otimes N)$.

Proposition 4.3.8. *Let $u: M \rightarrow N$ be a k times colocally weakly differentiable map. Then the colocally weakly differentiable map $D_K^{k-1} u: M \rightarrow (\otimes^{k-1} T^* M) \otimes TN$ has a colocal weak covariant derivative $D_K^k u: M \rightarrow T^* M \otimes (\otimes^{k-1} T^* M) \otimes TN$ if and only if $D_K^{k-1} u: M \rightarrow (\otimes^{k-1} T^* M) \otimes TN$ has a colocal weak derivative $TD_K^{k-1} u: TM \rightarrow T((\otimes^{k-1} T^* M) \otimes TN)$, and almost everywhere in M*

$$TD_K^{k-1} u = \text{Vert} \circ D_K^k u + \text{Hor}_K \circ D_K^{k-1} u,$$

and

$$D_K^k u = K \circ TD_K^{k-1} u.$$

Moreover, for every $j \in \mathbb{N}_*$, the map $D_K^k u$ is j times colocally weakly differentiable if and only if the map $D_K^{k-1} u$ is $(j+1)$ times colocally weakly differentiable, and almost everywhere in M

$$T^{j+1} D_K^{k-1} u = (T^j \text{Vert}) \circ (T^j D_K^k u) + (T^j \text{Hor}_K) \circ (T^j D_K^{k-1} u), \quad (5.2.1)$$

and

$$T^j D_K^k u = (T^j K) \circ (T^{j+1} D_K^{k-1} u). \quad (5.2.2)$$

Proof. First, we prove recursively that the map $D_K^{k-1} u$ is colocally weakly differentiable. By definition 4.2.4 of the k times colocally weakly differentiability, the map $Tu = D_K^1 u$ is $(k-1)$ times colocally weakly differentiable. By proposition 4.3.1, the map $D_K^2 u$ exists and $D_K^2 u = K \circ T^2 u$ almost everywhere in M . Since $T^2 u$ is $(k-2)$ times colocally weakly differentiable, and since for every $j \in \{1, \dots, k-2\}$, $T^{j-1} K$ is a connection and a bundle morphism [15, §17.2], by induction, the map $D_K^2 u$ is $(k-2)$ times colocally weakly differentiable. Since for every $\ell \in \{2, \dots, k-1\}$, the map $D_K^\ell u$ is defined recursively by

$$D_K^\ell u = K \circ TD_K^{\ell-1} u$$

almost everywhere in M , as for the second-order weak covariant derivative, by induction, the map $D_K^\ell u$ is $(k-\ell)$ times colocally weakly differentiable. In particular, the map $D_K^{k-1} u$ is colocally weakly differentiable.

On the one hand, we assume that the colocal weak covariant derivative $D_K^k u: M \rightarrow T^*M \otimes (\otimes^{k-1} T^*M) \otimes TN$ exists. Then for every $f \in C_c^1((\otimes^{k-1} T^*M) \otimes TN, \mathbb{R})$,

$$T(f \circ D_K^{k-1} u) = (Tf)(D_K^{k-1} u)[\text{Vert} \circ D_K^k u + \text{Hor}_K \circ D_K^{k-1} u]$$

almost everywhere in M and we can take

$$TD_K^{k-1} u = \text{Vert} \circ D_K^k u + \text{Hor}_K \circ D_K^{k-1} u$$

for the colocal weak derivative of $D_K^{k-1} u$.

Conversely, let $TD_K^{k-1} u: TM \rightarrow T((\otimes^{k-1} T^*M) \otimes TN)$ be the colocal weak derivative such that for every $f \in C_c^1((\otimes^{k-1} T^*M) \otimes TN, \mathbb{R})$,

$$T(f \circ D_K^{k-1} u) = Tf \circ TD_K^{k-1} u$$

almost everywhere in M . In view of the identity (4.3.2), almost everywhere in M

$$T(f \circ D_K^{k-1} u) = (Tf)(D_K^{k-1} u)[\text{Vert} \circ K \circ TD_K^{k-1} u + \text{Hor}_K \circ D_K^{k-1} u]$$

and since Vert is an embedding, we can take $D_K^k u = K \circ TD_K^{k-1} u$.

Finally, on the one hand, we assume that $D_K^k u$ is j times colocally weakly differentiable. By the first steps of the proof, the map $D_K^{k-1} u$ is colocally weakly differentiable and almost everywhere in M

$$TD_K^{k-1} u = \text{Vert} \circ D_K^k u + \text{Hor}_K \circ D_K^{k-1} u,$$

and so the vector bundle $(E, \pi, (\times^{k-1} M) \times N)$, with $E = (\otimes^{k-1} T^*M) \otimes TN$, is endowed with a connection K . Since the maps Vert and Hor_K are vector bundle isomorphisms [41, §16.12.1; 60, definition 3.7; 79, definition 6.25] (see also section 4.3.1), they are proper and so the maps $\text{Vert} \circ D_K^k u: M \rightarrow VE$ and $\text{Hor}_K \circ D_K^{k-1} u: M \rightarrow H_K E$ are colocally weakly differentiable. By the product of manifolds property for colocally weakly differentiable maps (proposition 1.2.7), the map

$$(\text{Vert} \circ D_K^k u, \text{Hor}_K \circ D_K^{k-1} u): M \rightarrow VE \times H_K E$$

is colocally weakly differentiable. Since there exist vector bundle isomorphisms $\varphi_1: VE \oplus H_K E \rightarrow TE$ [60, proposition 3.5; 116, definition 3.4] and $\varphi_2: VE \times H_K E \rightarrow VE \oplus H_K E$ [41, §16.16.1; 79, example 6.35] such that almost everywhere in M

$$TD_K^{k-1} u = \varphi_1 \circ \varphi_2 \circ (\text{Vert} \circ D_K^k u, \text{Hor}_K \circ D_K^{k-1} u),$$

the map $TD_K^{k-1}u$ is colocally weakly differentiable, and so $D_K^{k-1}u$ is twice colocally weakly differentiable. Furthermore, by the existence and uniqueness of the colocal weak derivative [34, proposition 1.5], almost everywhere in M

$$T^2 D_K^{k-1}u = T\varphi_1 \circ T\varphi_2 \circ ((T \text{Vert}) \circ (TD_K^k u), (T \text{Hor}_K) \circ (TD_K^{k-1}u)),$$

that is, $T^2 D_K^{k-1}u = (T \text{Vert}) \circ (TD_K^k u) + (T \text{Hor}_K) \circ (TD_K^{k-1}u)$ almost everywhere in M . If $j \geq 2$, using similar arguments as before, since for every $i \in \{1, \dots, j-1\}$, $T^i \text{Vert}$, $T^i \text{Hor}_K$, $T^i \varphi_1$ and $T^i \varphi_2$ are proper maps, we can prove by induction that the map $D_K^{k-1}u$ is $(j+1)$ times colocally weakly differentiable and that we also have the desired formula.

On the other hand, we have $D_K^k u = K \circ TD_K^{k-1}u$ and the conclusion also follows if $D_K^{k-1}u$ is $(j+1)$ times colocally weakly differentiable, using the same arguments as in the first part of the proof. $\square$

5.3. Sequences of higher order Sobolev maps

Proposition 4.3.9. *Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of k times colocally weakly differentiable maps from M to N and let $w \in \cap_{j=1}^{k-1} \dot{W}^{j,p}(M, N)$. If (N, d) is complete, if there exists $p \in [1, +\infty)$ such that*

$$\sup_{\ell \in \mathbb{N}} \int_M d(u_\ell, w)^p + \sum_{j=1}^k |D_K^j u_\ell|_{(\otimes^j g_M^*) \otimes g_N}^p < +\infty,$$

then there is a subsequence $(T^{k-1}u_{\ell_i})_{i \in \mathbb{N}}$ that converges to a measurable map $v: M \rightarrow \text{Mor}(T^{k-1}M, T^{k-1}N)$ almost everywhere in M .

Proof. Since the space (N, d) is complete, for every $x \in M$, the subset $(\pi_{M \times N}^{k-1})^{-1}(\{x\} \times N)$ of $\text{Mor}(T^{k-1}M, T^{k-1}N)$ equipped with the distance d^S , where d^S is induced by the Sasaki metric $G_{\text{Mor}(T^{k-1}M, T^{k-1}N)}^S$, is complete. Since d^S is strongly concordant (proposition 2.2.3), and by relying recursively on the formulas (5.2.1) and (5.2.2), there exists a constants $C > 0$ such that for every $\ell \in \mathbb{N}$, almost everywhere in M

$$\begin{aligned} & d^S(T^{k-1}u_\ell, T^{k-1}w) \\ & \leq C \left(d(u_\ell, w) + \sum_{j=1}^{k-1} (|D_K^j u_\ell|_{(\otimes^j g_M^*) \otimes g_N} + |D_K^j w|_{(\otimes^j g_M^*) \otimes g_N}) \right). \end{aligned}$$

and there exists $C_1, \dots, C_k > 0$ such that for every $\ell \in \mathbb{N}$, almost everywhere in M

$$|T^k u_\ell|_{g_M^* \otimes G_{\text{Mor}(T^{k-1}M, T^{k-1}N)}^S}^2 = \sum_{j=1}^k C_j |D_K^j u_\ell|_{(\otimes^j g_M^*) \otimes g_N}^2.$$

As a consequence, all the assumptions of the Rellich–Kondrashov for Sobolev maps property [34, proposition 3.4] are satisfied by the sequence $(T^{k-1}u_\ell)_{\ell \in \mathbb{N}}$ and so the conclusion follows. $\square$

Proposition 4.3.10. *Let $p \in [1, +\infty)$. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of k times colocally weakly differentiable maps from M to N . Assume that the sequence $(T^{k-1}u_\ell)_{\ell \in \mathbb{N}}$ converges locally in measure to a measurable map $v: M \rightarrow \text{Mor}(T^{k-1}M, T^{k-1}N)$ that covers a map $u: M \rightarrow N$, that*

$$\limsup_{\ell \rightarrow \infty} \int_M \sum_{j=1}^k |D_K^j u_\ell|_{(\otimes^j g_M^*) \otimes g_N}^p < +\infty,$$

and, if $p = 1$, that for every $j \in \{1, \dots, k\}$, the sequence $(D_K^j u_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable. Then $u \in \dot{W}^{k,p}(M, N)$ and

$$\int_M |D_K^k u|_{(\otimes^k g_M^*) \otimes g_N}^p \leq \liminf_{\ell \rightarrow \infty} \int_M |D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N}^p.$$

We first rephrased the closure property 4.2.5 with the notion of higher-order weak covariant derivatives.

Lemma 5.3.4 (Closure property). *Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of k times colocally weakly differentiable maps from M to N . If the sequence $(T^{k-1}u_\ell)_{\ell \in \mathbb{N}}$ converges locally in measure to a measurable map $v: M \rightarrow \text{Mor}(T^{k-1}M, T^{k-1}N)$ that covers a map $u: M \rightarrow N$, if for every $j \in \{1, \dots, k\}$, the sequence $(D_K^j u_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable, then the map u is k times colocally weakly differentiable and $T^{k-1}u = v$ almost everywhere in M .*

Proof. We need to prove that for every $j \in \{2, \dots, k\}$, the sequence $(T^j u_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable. By lemma 4.3.5, the sequence $(T^2 u_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable. Let $j \in \{3, \dots, k\}$. By the formula (5.2.1), for every $\ell \in \mathbb{N}$, almost everywhere in M

$$\begin{aligned} T^j u_\ell &= T^{j-1}(T u_\ell) = T^{j-1}(D_K^1 u_\ell) \\ &= (T^{j-2} \text{Vert}) \circ (T^{j-2} D_K^2 u_\ell) + (T^{j-2} \text{Hor}_K) \circ (T^{j-2} D_K^1 u_\ell), \end{aligned}$$

and for every $i \in \{3, \dots, j\}$,

$$\begin{aligned} T^{j-(i-1)} D_K^{i-1} u_\ell &= (T^{j-i} \text{Vert}) \circ (T^{j-i} D_K^i u_\ell) + (T^{j-i} \text{Hor}_K) \circ (T^{j-i} D_K^{i-1} u_\ell). \end{aligned}$$

If $i = j$, since Vert is a bundle morphism and an embedding, and since the sequence $(D_K^j u_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable, the sequence

$(TD_K^{j-1}u_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable (see proof of lemma 4.3.5). By iteration and the previous formulas, since each T^{j-i} Vert is a bundle morphism and an embedding, we obtain finally that the sequence $(T^{j-2}D_K^2u_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable and so $(T^j u_\ell)_{\ell \in \mathbb{N}}$ too (see proof of lemma 4.3.5).

As a consequence, by the closure property for colocally weakly differentiable maps [34, proposition 3.7], applied recursively to the sequences $(u_\ell)_{\ell \in \mathbb{N}}, \dots, (T^{k-1}u_\ell)_{\ell \in \mathbb{N}}$, the map u is k times colocally weakly differentiable and $T^{k-1}u = v$ almost everywhere in M . $\square$

As mentioned in chapter IV, we also prove the following lemma.

Lemma 4.3.11. *Let $p \in [1, +\infty)$. If $u \in \dot{W}^{k-1,p}(M, N) \cap \dot{W}^{k,p}(M, N)$, then the map $|D_K^{k-1}u|_{(\otimes^{k-1}g_M^*) \otimes g_N}$ belongs to $W^{1,p}(M)$ and almost everywhere in M*

$$|T|D_K^{k-1}u|_{(\otimes^{k-1}g_M^*) \otimes g_N}|_{g_M^* \otimes g_1} \leq |D_K^k u|_{(\otimes^k g_M^*) \otimes g_N},$$

where g_1 is the Euclidean metric on $\mathbb{R}$.

Proof. By proposition 4.3.8, the map $D_K^{k-1}u: M \rightarrow (\otimes^{k-1}T^*M) \otimes TN$ is colocally weakly differentiable. Furthermore, almost everywhere in M

$$|TD_K^{k-1}u|_{g_M^* \otimes G_{(\otimes^{k-1}T^*M) \otimes TN}^S}^2 = |D_K^k u|_{(\otimes^k g_M^*) \otimes g_N}^2 + |D_K^{k-1}u|_{(\otimes^{k-1}g_M^*) \otimes g_N}^2$$

and so $D_K^{k-1}u \in W^{1,p}(M, (\otimes^{k-1}T^*M) \otimes TN)$. Since $|\cdot|_{(\otimes^{k-1}g_M^*) \otimes g_N}$ is Lipschitz-continuous with respect to the distance induced by the Sasaki metric $G_{(\otimes^{k-1}T^*M) \otimes TN}^S$, by the chain rule in Sobolev spaces between manifolds [34, proposition 2.6], $|D_K^{k-1}u|_{(\otimes^{k-1}g_M^*) \otimes g_N} \in W^{1,p}(M)$.

Since the connection K is metric, by the chain rule formula in Sobolev spaces [33, proposition 5.2] and by the relation between $TD_K^{k-1}u$ and $D_K^k u$ (proposition 4.3.8), we have the desired inequality. $\square$

Proof of proposition 4.3.10. By closure property (lemma 5.3.4), the map u is k times colocally weakly differentiable and $T^{k-1}u = v$. We now need to prove that $|D_K^k u|_{(\otimes^k g_M^*) \otimes g_N} \in L^p(M)$.

By lemma 4.3.11, the sequence $(|D_K^{k-1}u_\ell|_{(\otimes^{k-1}g_M^*) \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $W^{1,p}(M)$. By the classical Rellich–Kondrashov compactness theorem [9, theorem 2.34 (a); 24, theorem 9.16], and since $(|D_K^{k-1}u_\ell|_{(\otimes^{k-1}g_M^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^{k-1}u|_{(\otimes^{k-1}g_M^*) \otimes g_N}$ in measure, $(|D_K^{k-1}u_\ell|_{(\otimes^{k-1}g_M^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^{k-1}u|_{(\otimes^{k-1}g_M^*) \otimes g_N}$ in $L_{\text{loc}}^p(M)$. Up to a subsequence, the sequence $(|D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges weakly to some w in $L_{\text{loc}}^p(M)$.

Since the sequence $(D_K^{k-1}u_\ell)_{\ell \in \mathbb{N}}$ converges to $D_K^{k-1}u$ locally in measure, for every $f \in C_c^1((\otimes^{k-1}T^*M) \otimes TN, \mathbb{R}^q)$, with $q \geq \min(m, \dim((\otimes^{k-1}T^*M) \otimes TN))$,

$$|T(f \circ D_K^{k-1}u)|_{g_M^* \otimes g_q}^2 \leq |f|_{\text{Lip}}^2 (|D_K^{k-1}u|_{(\otimes^{k-1}g_M^*) \otimes g_N}^2 + w^2),$$

where g_q is the Euclidean metric on $\mathbb{R}^q$, almost everywhere in M . By the characterization of the norm of the derivative [34, proposition 2.2], almost everywhere in M

$$\begin{aligned} |TD_K^{k-1}u|_{g_M^* \otimes G_{(\otimes^{k-1}T^*M) \otimes TN}^S}^2 &= |D_K^{k-1}u|_{(\otimes^{k-1}g_M^*) \otimes g_N}^2 + |D_K^k u|_{(\otimes^k g_M^*) \otimes g_N}^2 \\ &\leq |D_K^{k-1}u|_{(\otimes^{k-1}g_M^*) \otimes g_N}^2 + w^2, \end{aligned}$$

and so $|D_K^k u|_{(\otimes^k g_M^*) \otimes g_N} \leq w$. Consequently, by lower semi-continuity of the norm under weak convergence, for each compact set $Q \subseteq M$,

$$\int_Q |D_K^k u|_{(\otimes^k g_M^*) \otimes g_N}^p \leq \int_Q w^p \leq \liminf_{\ell \rightarrow \infty} \int_Q |D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N}^p.$$

Finally, since M has a countable basis, there exists a set $\{Q_i: i \in \mathbb{N}\}$ of compact sets such that for every $i \in \mathbb{N}$, $Q_i \subseteq Q_{i+1}$ and $M = \bigcup_{i \in \mathbb{N}} Q_i$ [80, proposition 4.76] and so by monotone convergence theorem [20, theorem 2.8.2], we have the desired inequality. $\square$

5.4. Chain rule for higher order colocally weakly differentiable maps

Proposition 4.4.5. *Let $j \in \{2, \dots, k\}$. Let $u: M \rightarrow N$ be a j times colocally weakly differentiable map. The following statements are equivalent.*

- (i) *For every $f \in C_c^k(N, \mathbb{R})$, $f \circ u$ is k times weakly differentiable,*
- (ii) *for every $f \in C_c^j(N, \mathbb{R})$, $f \circ u$ is j times weakly differentiable and for every $h \in \text{Mor}(\text{Mor}(T^j M, T^j N), \mathbb{R})$ of class C^{k-j} such that $\text{supp}_N(h)$ is compact, the map $h \circ T^j u$ is $(k-j)$ times weakly differentiable,*
- (iii) *the map u is k times colocally weakly differentiable and for all compact subsets $K \subseteq M$ and $L \subseteq N$, for all $\ell \in \{2, \dots, k\}$, for all ℓ^{tuple} norm $\llbracket \cdot \rrbracket$ on $\text{Mor}(T^\ell M, T^\ell N)$,*

$$\int_{u^{-1}(L) \cap K} \llbracket T^\ell u \rrbracket < +\infty.$$

Moreover, if for every $f \in C_c^k(N, \mathbb{R})$, the map $f \circ u$ is k times weakly differentiable, then for all compact subsets $K \subseteq M$ and $L \subseteq N$, for all j^{tuple} norm $\llbracket \cdot \rrbracket$ on $\text{Mor}(T^j M, T^j N)$,

$$\int_{u^{-1}(L) \cap K} \llbracket T^j u \rrbracket_j^{\frac{k}{j}} < +\infty.$$

We prove intermediate results in order to prove this proposition.

We first remark how multiple norms behave under composition of multiple vector bundle morphisms.

Lemma 5.4.5. *Let K be a manifold of class C^k . For all compact subsets $Q_M \subseteq M$, $Q_N \subseteq N$ and $Q_K \subseteq K$, there exists a constant $C > 0$ such that for every $v \in (\pi_{M \times N}^k)^{-1}(Q_M \times Q_N)$ and every $\xi \in (\pi_{N \times K}^k)^{-1}(Q_N \times Q_K)$ such that $\pi_N \circ \pi_{M \times N}^k \circ v = \pi_N \circ \pi_{N \times K}^k \circ \xi$,*

$$\llbracket \xi \circ v \rrbracket \leq C \llbracket \xi \rrbracket \llbracket v \rrbracket.$$

Since the previous lemma involves the notion of k^{tuple} norm, the reader can find the proof in the next chapter VI (proposition 6.3.7).

Lemma 5.4.6. *Let $j \in \{2, \dots, k\}$. If $v \in W_{\text{loc}}^{k,1}(M, \mathbb{R}^q) \cap L^\infty(M, \mathbb{R}^q)$, if the morphism $w \in \text{Mor}(\text{Mor}(T^j M, T^j \mathbb{R}^q), \mathbb{R})$ of class C^{k-j} is such that $\text{supp}_{\mathbb{R}^q}(w)$ is compact, then the map $w \circ T^j v$ is $(k - j)$ times weakly differentiable.*

Since $T^k M$ is a k^{tuple} vector bundle, a morphism $w \in \text{Mor}(T^k M, \mathbb{R})$ means that it is linear in fibers over each of the k vector bundle structure [15, §22.2]. More precisely, if (V, ψ) is a local trivialization around $x \in M$, then the map $w \circ \psi$ is given by

$$\begin{aligned} (y, (e_\lambda)_{\lambda \in I_k}) &\in V \times \prod_{\lambda \in I_k} \mathbb{R}^m \\ &\mapsto \sum_{\ell=1}^k \sum_{\lambda \in \mathcal{P}_\ell(\{1, \dots, k\})} f_\lambda(y) [e_{\lambda^1}, \dots, e_{\lambda^\ell}] \in \mathbb{R} \end{aligned} \quad (5.4.3)$$

where for every $y \in V$, for every $\lambda \in \mathcal{P}_\ell(\{1, \dots, k\})$, the map $f_\lambda(y)$ is ℓ -linear and the map $y \in V \mapsto f_\lambda(y)$ is continuous. Since $\text{Mor}(T^k M, T^k \mathbb{R}^q)$ has also a k^{tuple} vector bundle structure, this notion applies directly to $w \in \text{Mor}(\text{Mor}(T^k M, T^k \mathbb{R}^q), \mathbb{R})$.

Proof of lemma 5.4.6. Without loss of generality, we assume that $q = 1$.

First, we assume that $M = \Omega$ is an open set of $\mathbb{R}^m$. By regularization theorem [20, theorem 4.2.4; 24, theorem 4.22; 120, theorem 4.3.9], there

exists a sequence $(v_\ell)_{\ell \in \mathbb{N}} \subseteq C_c^\infty(\Omega, \mathbb{R})$ that converges to v in $W_{\text{loc}}^{k,1}(\Omega)$. Since w is a morphism from a j^{tuple} vector bundle to $\mathbb{R}$, in view of the local representation (5.4.3), and since $\text{supp}_{\mathbb{R}}(w)$ is compact, for every compact set $K \subseteq \Omega$, there exists a constant $C_1 > 0$ such that for every $\ell \in \mathbb{N}$, for every $i \in \{1, \dots, k-j\}$ and $x \in K$,

$$|T^i(w \circ T^j v_\ell)(x)|_{\mathcal{L}} \leq C_1 \|T^{i+j} v_\ell(x)\|_*.$$

In view of the canonical $(i+j)^{\text{tuple}}$ norm for $(i+j)^{\text{tuple}}$ vector bundle morphisms (section 4.4.5), and by Young's inequality, there exists a constant $C_2 > 0$ such that for every $i \in \{1, \dots, k-j\}$ and almost every $x \in \Omega$,

$$\|T^{i+j} v(x)\|_* \leq C_2 \left(\sum_{r=1}^k |D^r v(x)|_{(\otimes^r g_m^*) \otimes g_1}^{\frac{k}{r}} + 1 \right),$$

where g_m and g_1 are the Euclidean metrics on $\mathbb{R}^m$ and $\mathbb{R}$ respectively. By Gagliardo–Nirenberg inequalities [49, 95], $\|T^{i+j} v\|_* \in L_{\text{loc}}^1(\Omega)$ and so the sequence $(\|T^{i+j} v_\ell\|_*)_{\ell \in \mathbb{N}}$ converges to $\|T^{i+j} v\|_*$ in $L_{\text{loc}}^1(\Omega)$. By Lebesgue's dominated convergence theorem, the sequence $(|T^i(w \circ T^j v_\ell)|_{\mathcal{L}})_{\ell \in \mathbb{N}}$ converges in $L_{\text{loc}}^1(\Omega)$ and then by closing lemma [120, lemma 6.1.5], the map $w \circ T^j v$ is $(k-j)$ times weakly differentiable.

For any manifold M , we can apply the previous argument to every $v \circ \psi^{-1}: \psi(V) \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$ with any local chart $\psi: V \subseteq M \rightarrow \mathbb{R}^m$. By a direct covering argument, we have thus that $w \circ T^j v$ is $(k-j)$ times weakly differentiable. $\square$

Proof of proposition 4.4.5. We first prove that (i) implies (ii). Let $(U_i)_{i \in I}$ be an open cover of N by sets given by lemma 5.1.3. Let $h \in \text{Mor}(\text{Mor}(T^j M, T^j N), \mathbb{R})$ be a morphism of class C^{k-j} such that $\text{supp}_N(h)$ is compact. There exist $\gamma \in \mathbb{N}_*$ such that $\text{supp}_N(h) \subseteq \bigcup_{i=1}^\gamma U_i$ and a partition of unity $(\eta_i)_{1 \leq i \leq \gamma}$ subordinate to $(U_i)_{1 \leq i \leq \gamma}$ such that almost everywhere in M

$$h \circ T^j u = \sum_{i=1}^\gamma (\eta_i \circ u)(h \circ T^j u) = \sum_{i=1}^\gamma (\eta_i \circ u)((h \circ T^j \varphi_i^*) \circ T^j(\varphi_i \circ u)),$$

since for every $i \in \{1, \dots, \gamma\}$, the maps $T^{j-1}(\varphi_i \circ u)$ and $T^{j-1} \varphi_i \circ T^{j-1} u$ are weakly differentiable by induction, and by proposition 5.1.1, $T^j(\varphi_i \circ u) = T^j \varphi_i \circ T^j u$ almost everywhere in M . By lemma 5.4.6, every term in the right sum is $(k-j)$ times weakly differentiable and so $h \circ T^j u$ is $(k-j)$ times weakly differentiable.

We then prove that (ii) implies (iii). First, let $(U_i)_{i \in I}$ be an open cover of N by sets given by lemma 5.1.3. We prove recursively that $T^j u$

is $(k - j)$ times colocally weakly differentiable. Let $\ell \in \{j, \dots, k - 1\}$ be such that the colocal weak derivative $T^\ell u: M \rightarrow \text{Mor}(T^\ell M, T^\ell N)$ exists. Let $h \in C_c^1(\text{Mor}(T^\ell M, T^\ell N), \mathbb{R})$. Since the set $\text{supp}(h)$ is compact, in particular, there exist $\gamma \in \mathbb{N}_*$ and a finite partition of unity $(\eta_i)_{1 \leq i \leq \gamma}$ subordinate to $(U_i)_{1 \leq i \leq \gamma}$ such that almost everywhere in M

$$\begin{aligned} h \circ T^\ell u &= \sum_{i=1}^{\gamma} (\eta_i \circ u)(h \circ T^\ell u) \\ &= \sum_{i=1}^{\gamma} (\eta_i \circ u)(h \circ T^\ell \varphi_i^* \circ (T^\ell \varphi_i \circ T^\ell u)). \end{aligned}$$

By assumption, $T^\ell \varphi_i \circ T^\ell u$ is weakly differentiable, and thus by the chain rule for weakly differentiable functions (see for example [45, theorem 4.2.4 (ii); 120, theorem 6.1.13]), each term in the right sum is weakly differentiable, and so $h \circ T^\ell u$ is weakly differentiable. By proposition 5.1.1, there exists a measurable map $T^{\ell+1}u: M \rightarrow \text{Mor}(T^{\ell+1}M, T^{\ell+1}N)$ such that for every $h \in C^1(\text{Mor}(T^\ell M, T^\ell N), \mathbb{R})$ such that $h \circ T^\ell u \in W_{\text{loc}}^{1,1}(M)$, we have almost everywhere in M

$$T(h \circ T^\ell u) = Th \circ T^{\ell+1}u.$$

We now prove that the integrability condition is satisfied. Let $(U_i)_{i \in I}$ be an open cover of N by sets given by lemma 5.1.3. Let $K \subseteq M$ and $L \subseteq N$ be two compact subsets. Let $\ell \in \{2, \dots, k\}$. Since L is compact, there exists $\gamma \in \mathbb{N}_*$ such that $L \subseteq \bigcup_{i=1}^{\gamma} U_i$. By construction, for every $i \in \{1, \dots, \gamma\}$, $T^\ell(\varphi_i \circ u) = T^\ell \varphi_i \circ T^\ell u$ almost everywhere in M and, by lemma 5.1.3, $T^\ell \varphi_i^* \circ T^\ell \varphi_i = \text{id}$ on $(\pi_N^\ell)^{-1}(U_i)$, and so

$$T^\ell u = T^\ell \varphi_i^* \circ T^\ell(\varphi_i \circ u)$$

almost everywhere on $u^{-1}(U_i)$. Since every $\varphi_i \in C_c^k(N, \mathbb{R}^n)$, by proposition 5.4.5, there exist constants $C_1, \dots, C_\gamma$ such that almost everywhere in $u^{-1}(L) \cap K$

$$\begin{aligned} \|T^\ell u\| &\leq \sum_{i=1}^{\gamma} \|T^\ell \varphi_i^* \circ T^\ell(\varphi_i \circ u)\| \leq \sum_{i=1}^{\gamma} C_i \|T^\ell \varphi_i^*\| \|T^\ell(\varphi_i \circ u)\| \\ &\leq \left(\max_{1 \leq i \leq \gamma} C_i \right) \left(\max_{1 \leq i \leq \gamma} \sup_{\mathbb{R}^n} \|T^\ell \varphi_i^*\| \right) \sum_{i=1}^{\gamma} \|T^\ell(\varphi_i \circ u)\|. \end{aligned} \tag{5.4.4}$$

If $(\eta_\lambda)_{\lambda \in \Lambda}$ is a partition of unity subordinate to an atlas $((\psi_\lambda, V_\lambda))_{\lambda \in \Lambda}$ of local charts of M , by the maximality property of ℓ^{tuple} norms, and by Young's inequality, there exists a positive continuous function $\beta \in$

$C(M \times \mathbb{R}^n, \mathbb{R})$ such that for every $i \in \{1, \dots, \gamma\}$, almost everywhere in M

$$\begin{aligned} & \llbracket T^\ell(\varphi_i \circ u) \rrbracket \\ & \leq \beta(\cdot, \varphi_i \circ u) \sum_{\lambda \in \Lambda} \eta_\lambda \left(\sum_{r=1}^k |D^r(\varphi_i \circ u \circ \psi_\lambda^{-1}) \circ \psi_\lambda|_{(\otimes^r g_m^*) \otimes g_n}^{\frac{k}{r}} + 1 \right), \quad (5.4.5) \end{aligned}$$

where g_m and g_n are the Euclidean metrics on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Since for every $i \in \{1, \dots, \gamma\}$, $\varphi_i \circ u \in W_{\text{loc}}^{k,1}(M, \mathbb{R}^n) \cap L^\infty(M, \mathbb{R}^n)$, by Gagliardo–Nirenberg inequalities [49, 95], $\llbracket T^\ell(\varphi_i \circ u) \rrbracket \in L^1(u^{-1}(L) \cap K)$ and so

$$\int_{u^{-1}(L) \cap K} \llbracket T^\ell u \rrbracket < +\infty.$$

Finally, we prove that (iii) implies (i). We proceed by induction. Let $\ell \in \{2, \dots, k-1\}$. For every $i \in \mathbb{N}$, we define the map $\theta_i: \text{Mor}(T^\ell M, T^\ell N) \rightarrow \mathbb{R}$ for every $y \in \text{Mor}(T^\ell M, T^\ell N)$ by

$$\theta_i(y) = \theta\left(\frac{|y|}{1+i}\right),$$

where $\theta \in C_c^1([0, \infty), \mathbb{R})$ is such that $0 \leq \theta \leq 1$, $\theta = 1$ on $[0, 1]$ and $|\cdot|$ is a norm induced by a Riemannian metric on $\text{Mor}(T^\ell M, T^\ell N)$ [42, §1.2 proposition 2.10]. Let $f \in C_c^k(N, \mathbb{R})$ and let $K \subseteq M$ be a compact set. For every $i \in \mathbb{N}$, we define $\tilde{f}_i = \theta_i \cdot (T^\ell f)$. Since each map $\tilde{f}_i$ is Lipschitz-continuous with compact support, $\tilde{f}_i \circ T^\ell u$ is weakly differentiable [34, proposition 2.2] and there exist constants $C_1, C_2, C_3 > 0$ such that for every $i \in \mathbb{N}$ and almost everywhere on $u^{-1}(\text{supp}(f)) \cap K$

$$\begin{aligned} |T(\tilde{f}_i \circ T^\ell u)|_{\mathcal{L}} & \leq C_1 \left| \theta' \left(\frac{|T^\ell u|}{1+i} \right) \right| \frac{1}{1+i} \llbracket T^{\ell+1} u \rrbracket |T^\ell f \circ T^\ell u|_{\mathcal{L}} \\ & \quad + C_2 \theta \left(\frac{|T^\ell u|}{1+i} \right) \llbracket T^{\ell+1} f \rrbracket \llbracket T^{\ell+1} u \rrbracket \\ & \leq C_3 \llbracket T^{\ell+1} u \rrbracket. \end{aligned}$$

Since the sequence $(|T(\tilde{f}_i \circ T^\ell u)|_{\mathcal{L}})_{i \in \mathbb{N}}$ is bounded and uniformly integrable and since $(\tilde{f}_i \circ T^\ell u)_{i \in \mathbb{N}}$ converges almost everywhere to $T^\ell f \circ T^\ell u$ in $K \subseteq M$, in view of the weak compactness criterion in $L^1(K)$ [20, corollary 4.7.19; 24, theorem 4.30], the sequence $(T(\tilde{f}_i \circ T^\ell u))_{i \in \mathbb{N}}$ converges weakly to $T^{\ell+1}(f \circ u)$ in $L^1(K)$ and $T^\ell(f \circ u)$ is weakly differentiable. Hence, $f \circ u$ is k times weakly differentiable.

Let $(U_i)_{i \in I}$ be an open cover of N by sets given by lemma 5.1.3. Let $K \subseteq M$ and $L \subseteq N$ be two compact subsets. If for every $f \in C_c^k(N, \mathbb{R})$,

$f \circ u$ is k times weakly differentiable, then there exist $C_1 > 0$ and $\gamma \in \mathbb{N}_*$ such that almost everywhere in $u^{-1}(L) \cap K$ (see inequality (5.4.4) above)

$$\|T^j u\|^{\frac{k}{j}} \leq C_1 \sum_{i=1}^{\gamma} \|T^j(\varphi_i \circ u)\|^{\frac{k}{j}}.$$

With the same notations as for inequality (5.4.5), by the maximality property of j^{tuple} norms, and by Young's inequality, for every $i \in \{1, \dots, \gamma\}$, almost everywhere in M

$$\begin{aligned} & \|T^j(\varphi_i \circ u)\| \\ & \leq \beta(\cdot, \varphi_i \circ u) \sum_{\lambda \in \Lambda} \eta_{\lambda} \left(\sum_{r=1}^j |D^r(\varphi_i \circ u \circ \psi_{\lambda}^{-1}) \circ \psi_{\lambda}|_{(\otimes^r g_m^*) \otimes g_n}^{\frac{j}{r}} + 1 \right). \end{aligned}$$

Since for every $i \in \{1, \dots, \gamma\}$, $\varphi_i \circ u \in W_{\text{loc}}^{k,1}(M, \mathbb{R}^n) \cap L^{\infty}(M, \mathbb{R}^n)$, by Gagliardo–Nirenberg inequalities [49, 95], $\|T^j(\varphi_i \circ u)\|^{\frac{k}{j}} \in L^1(u^{-1}(L) \cap K)$ and so $\int_{u^{-1}(L) \cap K} \|T^j u\|^{\frac{k}{j}} < +\infty$. $\square$

5.5. Problem of density of smooth maps

Proposition 4.5.10. *Let M and N be two smooth connected compact Riemannian manifolds and let $p \in [1, +\infty)$ so that $1 \leq kp < m$.*

- (i) *If for every $u \in \dot{W}^{k,p}(M, N)$, there exists a sequence $(u_{\ell})_{\ell \in \mathbb{N}}$ in $C^{\infty}(M, N)$ such that the sequence $(T^{k-1}u_{\ell})_{\ell \in \mathbb{N}}$ converges to $T^{k-1}u$ locally in measure and such that the sequence $(|D_K^k u_{\ell}|_{(\otimes^k g_M^*) \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^p(M)$, and if $kp \notin \mathbb{N}$, then $\pi_{[kp]}(N) \simeq \{0\}$.*
- (ii) *If for every $u \in \dot{W}^{k,p}(M, N)$, there exists a sequence $(u_{\ell})_{\ell \in \mathbb{N}}$ in $C^{\infty}(M, N)$ such that the sequence $(T^k u_{\ell})_{\ell \in \mathbb{N}}$ converges to $T^k u$ locally in measure and such that the sequence $(|D_K^k u_{\ell}|_{(\otimes^k g_M^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^k u|_{(\otimes^k g_M^*) \otimes g_N}$ in $L^p(M)$, and if $kp \in \mathbb{N}$, then $\pi_{kp}(N) \simeq \{0\}$.*

Lemma 4.5.11. *Let N be a smooth connected compact Riemannian manifold and let $p \in [1, +\infty)$ so that $(k-1)p < q$. Let $(u_{\ell})_{\ell \in \mathbb{N}}$ be a sequence in $C^{\infty}(\mathbb{S}^q, N)$ and let $u \in \dot{W}^{k,p}(\mathbb{S}^q, N)$.*

- (i) *If $kp \geq q$, if the sequence $(T^{k-1}u_{\ell})_{\ell \in \mathbb{N}}$ converges to $T^{k-1}u$ locally in measure and if the sequence $(|D_K^k u_{\ell}|_{(\otimes^k g_{q+1}^*) \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^p(\mathbb{S}^q)$, then the sequence $(|Tu_{\ell}|_{g_{q+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^{kp}(\mathbb{S}^q)$.*

(ii) If $kp \geq q$, if the sequence $(T^k u_\ell)_{\ell \in \mathbb{N}}$ converges to $T^k u$ locally in measure and if the sequence $(|D_K^k u_\ell|_{(\otimes^k g_{q+1}^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^k u|_{(\otimes^k g_{q+1}^*) \otimes g_N}$ in $L^p(\mathbb{S}^q)$, then the sequence $(|Tu_\ell|_{g_{q+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Tu|_{g_{q+1}^* \otimes g_N}$ in $L^{kp}(\mathbb{S}^q)$.

Proof. Let μ be the measure associated to the Riemannian metric on $\mathbb{S}^q \subseteq \mathbb{R}^{q+1}$. Since the sequence $(T^{k-1} u_\ell)_{\ell \in \mathbb{N}}$ converges to $T^{k-1} u$ locally in measure in assertions (i) and (ii), there exist $\varepsilon > 0$ and $\gamma > 0$ such that for every $\ell \in \mathbb{N}$, for every $j \in \{1, \dots, k-1\}$,

$$\mu(\{x \in \mathbb{S}^q : |D_K^j u_\ell(x)|_{(\otimes^j g_{q+1}^*) \otimes g_N} \leq \gamma\}) > \varepsilon.$$

For every $\ell \in \mathbb{N}$ and $j \in \{1, \dots, k-1\}$, by lemma 4.3.11, the sequence $|D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N} \in W^{1, \frac{qp}{q-(k-j-1)p}}(\mathbb{S}^q)$ and almost everywhere in M

$$|T|D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N}|_{g_{q+1}^* \otimes g_1} \leq |D_K^{j+1} u_\ell|_{(\otimes^{j+1} g_{q+1}^*) \otimes g_N}.$$

So by Poincaré inequalities (lemma 4.5.9), there exists $C_1 > 0$ such that for every $\ell \in \mathbb{N}$ and every $j \in \{1, \dots, k-1\}$,

$$\begin{aligned} & \| |D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N} \|_{L^{\frac{qp}{q-(k-j)p}}(\mathbb{S}^q)} \\ & \leq C_1 \left(\| |D_K^{j+1} u_\ell|_{(\otimes^{j+1} g_{q+1}^*) \otimes g_N} \|_{L^{\frac{qp}{q-(k-j-1)p}}(\mathbb{S}^q)} + \gamma \right). \end{aligned}$$

By iteration, there exists $C_2 > 0$ such that for every $j \in \{1, \dots, k-1\}$ and every $\ell \in \mathbb{N}$,

$$\begin{aligned} & \| |D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N} \|_{L^{\frac{qp}{q-(k-j)p}}(\mathbb{S}^q)} \\ & \leq C_2 \left(\| |D_K^k u_\ell|_{(\otimes^k g_{q+1}^*) \otimes g_N} \|_{L^p(\mathbb{S}^q)} + 1 \right). \end{aligned} \quad (5.5.6)$$

In particular, since $kp \geq q$, $1 \leq kp \leq \frac{qp}{q-(k-1)p}$, and by Hölder's inequality, there exists $C_3 > 0$ such that for every $\ell \in \mathbb{N}$,

$$\begin{aligned} & \| |Tu_\ell|_{g_{q+1}^* \otimes g_N} \|_{L^{kp}(\mathbb{S}^q)} \\ & \leq C_3 \left(\| |D_K^k u_\ell|_{(\otimes^k g_{q+1}^*) \otimes g_N} \|_{L^p(\mathbb{S}^q)} + 1 \right). \end{aligned} \quad (5.5.7)$$

So if assumptions of assertion (i) or (ii) are satisfied, by previous inequality (5.5.7), the sequence $(|Tu_\ell|_{g_{q+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^{kp}(\mathbb{S}^q)$.

We assume now that assumptions of assertion (ii) are satisfied. First, for every $j \in \{1, \dots, k-1\}$, $u \in \dot{W}^{j, \frac{kp}{j+1}}(\mathbb{S}^q, N)$. Indeed, since the sequence

$(u_\ell)_{\ell \in \mathbb{N}}$ converges to u locally in measure, by weak closure property in Sobolev spaces [34, proposition 3.8], $u \in \dot{W}^{1, kp}(\mathbb{S}^q, N)$. For every $j \in \{2, \dots, k-1\}$, by inequality (5.5.6), and Hölder's inequality, for every $1 \leq i \leq j+1$, the sequence $(|D_K^i u_\ell|_{(\otimes^i g_{q+1}^*) \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $L^{\frac{kp}{j+1}}(\mathbb{S}^q)$. Since the sequence $(T^{j-1} u_\ell)_{\ell \in \mathbb{N}}$ converges to $T^{j-1} u$ locally in measure, by weak closure property in higher-order Sobolev spaces (proposition 4.3.10), $u \in \dot{W}^{j, \frac{kp}{j+1}}(\mathbb{S}^q, N)$.

As a consequence, for every $j \in \{1, \dots, k-1\}$, by Hölder's inequality and by lemma 4.3.11, $|D_K^j u|_{(\otimes^j g_{q+1}^*) \otimes g_N} \in W^{1, \frac{kp}{j+1}}(\mathbb{S}^q)$. By Sobolev inequalities on compact Riemannian manifolds [67, theorem 3.5], there exists $C_4 > 0$ such that for every $\ell \in \mathbb{N}$ and every $j \in \{1, \dots, k-1\}$,

$$\begin{aligned} & \| |D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N} - |D_K^j u|_{(\otimes^j g_{q+1}^*) \otimes g_N} \|_{L^{\frac{kp}{j}}(\mathbb{S}^q)} \\ & \leq C_4 \| |D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N} - |D_K^j u|_{(\otimes^j g_{q+1}^*) \otimes g_N} \|_{W^{1, \frac{kp}{j+1}}(\mathbb{S}^q)}. \end{aligned}$$

Thanks to the previous inequality, we now prove that for every $j \in \{1, \dots, k-1\}$, if the sequence $(|D_K^{j+1} u_\ell|_{(\otimes^{j+1} g_{q+1}^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^{j+1} u|_{(\otimes^{j+1} g_{q+1}^*) \otimes g_N}$ in $L^{\frac{kp}{j+1}}(\mathbb{S}^q)$, then $(|D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^j u|_{(\otimes^j g_{q+1}^*) \otimes g_N}$ in $L^{\frac{kp}{j}}(\mathbb{S}^q)$. Let $j \in \{1, \dots, k-1\}$. By previous steps, the sequence $(|D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N})_{\ell \in \mathbb{N}}$ is bounded in $W^{1, \frac{kp}{j+1}}(\mathbb{S}^q)$. By Rellich–Kondrashov embedding theorem for Sobolev maps on compact Riemannian manifolds [9, theorem 2.34 (a)], and since the sequence $(|D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^j u|_{(\otimes^j g_{q+1}^*) \otimes g_N}$ in measure, $(|D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^j u|_{(\otimes^j g_{q+1}^*) \otimes g_N}$ in $L^{\frac{kp}{j+1}}(\mathbb{S}^q)$. Since $(T^{j+1} u_\ell)_{\ell \in \mathbb{N}}$ converges to $T^{j+1} u$ locally in measure, the sequence $(T|D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $T|D_K^j u|_{(\otimes^j g_{q+1}^*) \otimes g_N}$ locally in measure. Furthermore, for every $\ell \in \mathbb{N}$, almost everywhere in M

$$\begin{aligned} & |T(|D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N} - |D_K^j u|_{(\otimes^j g_{q+1}^*) \otimes g_N})|_{g_{q+1}^* \otimes g_1} \\ & \leq |D_K^{j+1} u_\ell|_{(\otimes^{j+1} g_{q+1}^*) \otimes g_N} + |D_K^{j+1} u|_{(\otimes^{j+1} g_{q+1}^*) \otimes g_N}. \end{aligned}$$

By Lebesgue's dominated convergence theorem [20, theorem 2.8.5], the sequence $(T(|D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N} - |D_K^j u|_{(\otimes^j g_{q+1}^*) \otimes g_N}))_{\ell \in \mathbb{N}}$ converges to 0 in $L^{\frac{kp}{j+1}}(\mathbb{S}^q)$. Hence, the sequence $(|D_K^j u_\ell|_{(\otimes^j g_{q+1}^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^j u|_{(\otimes^j g_{q+1}^*) \otimes g_N}$ in $W^{1, \frac{kp}{j+1}}(\mathbb{S}^q)$, and so in $L^{\frac{kp}{j}}(\mathbb{S}^q)$ by the Sobolev inequality.

Finally, since $(|D_K^k u_\ell|_{(\otimes^k g_{q+1}^*) \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|D_K^k u|_{(\otimes^k g_{q+1}^*) \otimes g_N}$ in $L^p(\mathbb{S}^q)$, we deduce by iteration that the sequence $(|Tu_\ell|_{g_{q+1}^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Tu|_{g_{q+1}^* \otimes g_N}$ in $L^{kp}(\mathbb{S}^q)$. $\square$

Proof of proposition 4.5.10. We first give the proof when $M = \mathbb{B}^{[kp]+1} \times L$. If $\pi_{[kp]}(N) \not\cong \{0\}$, then there exists a smooth map $f: \mathbb{S}^{[kp]} \rightarrow N$ which is not homotopic to a constant map in $C(\mathbb{S}^{[kp]}, N)$. We define the map $u: M \rightarrow N$ for every $x = (x', x'') \in \mathbb{B}^{[kp]+1} \times L$ by $u(x) = f(\frac{x'}{|x'|})$. We observe that there exists a constant $C > 0$ such that for every $x \in (\mathbb{B}^{[kp]+1} \setminus \{0\}) \times L$, $|D_K^k u(x)| \leq \frac{C}{|x'|^k}$, and thus $u \in \dot{W}^{k,p}(M, N)$.

We first show that the assumptions of assertion (i) cannot be satisfied. Otherwise, since for every $\ell \in \mathbb{N}$,

$$\begin{aligned} \int_M |D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N}^p &= \int_L \int_{\mathbb{B}^{[kp]+1}} |D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N}^p \\ &= \int_L \left(\int_0^1 \left(\int_{\mathbb{S}_r^{[kp]}} |D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N}^p dr \right) \right), \end{aligned}$$

by Fatou's lemma, for almost every $x'' \in L$ and almost every $r \in (0, 1)$, up to a subsequence, $((|D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N})_{|\mathbb{S}_r^{[kp]} \times \{x''\}})_{\ell \in \mathbb{N}}$ is bounded in $L^p(\mathbb{S}_r^{[kp]})$. By lemma 4.5.11 (i), for almost every $x'' \in L$ and almost every $r \in (0, 1)$, the sequence $((u_\ell)_{|\mathbb{S}_r^{[kp]} \times \{x''\}})_{\ell \in \mathbb{N}}$ is bounded in $\dot{W}^{1,kp}(\mathbb{S}_r^{[kp]}, N)$ and since the homotopy classes are preserved by weakly bounded sequence in $\dot{W}^{1,kp}(\mathbb{S}_r^{[kp]}, N)$ [118, theorem 2.1], the map $u_{|\mathbb{S}_r^{[kp]} \times \{x''\}}$ is homotopic to a constant map, and so the map f too.

We now show that the assumptions of assertion (ii) cannot be satisfied. Otherwise, since for every $\ell \in \mathbb{N}$,

$$\begin{aligned} \int_M ||D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N} - |D_K^k u|_{(\otimes^k g_M^*) \otimes g_N}|^p \\ = \int_L \left(\int_0^1 \left(\int_{\mathbb{S}_r^{kp}} ||D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N} - |D_K^k u|_{(\otimes^k g_M^*) \otimes g_N}|^p dr \right) \right), \end{aligned}$$

up to a subsequence, for almost every $x'' \in L$ and almost every $r \in (0, 1)$, the sequence $((|D_K^k u_\ell|_{(\otimes^k g_M^*) \otimes g_N})_{|\mathbb{S}_r^{kp} \times \{x''\}})_{\ell \in \mathbb{N}}$ converges to the map $(|D_K^k u|_{(\otimes^k g_M^*) \otimes g_N})_{|\mathbb{S}_r^{kp} \times \{x''\}}$ in $L^p(\mathbb{S}_r^{kp})$. By lemma 4.5.11 (ii), for almost every $x'' \in L$ and almost every $r \in (0, 1)$, $((u_\ell)_{|\mathbb{S}_r^{kp} \times \{x''\}})_{\ell \in \mathbb{N}}$ converges to $u_{|\mathbb{S}_r^{kp} \times \{x''\}}$ in $\dot{W}^{1,kp}(\mathbb{S}_r^{kp}, N)$ [34, proposition 4.4] and since the homotopy classes are preserved by strong convergence in $\dot{W}^{1,kp}(\mathbb{S}_r^{kp}, N)$ [117], the map $u_{|\mathbb{S}_r^{kp} \times \{x''\}}$ is homotopic to a constant map, and so the map f too.

For a general manifold M , we start from a counterexample $u: \mathbb{B}^m \rightarrow N$ to a Sobolev map on the whole of M . Indeed, there is a map $\Phi: \mathbb{B}^m \rightarrow \mathbb{R}^m$ which maps diffeomorphically some subset $A \subset \mathbb{B}^m$ to $\mathbb{B}^{[kp]+1} \times \mathbb{S}^{m-[kp]-1}$ (A is a tubular neighborhood of an $(m - [kp] - 1)$ -dimensional sphere, with $\mathbb{S}^0 = \{-1, 1\}$), $\Phi(\mathbb{B}^m \setminus A) \subset (\mathbb{B}^{[kp]+1} \setminus \{0\}) \times \mathbb{R}^{m-[kp]-1}$ and Φ is constant near $\partial\mathbb{B}^m$. We then define the map $u: \mathbb{B}^m \rightarrow N$ for every $x \in \mathbb{B}^m$ by $u(x) = f(\frac{y'}{|y'|})$ if $\Phi(x) = (y', y'')$. The map u is constant near the boundary and can thus be transported and extended to any m -dimensional manifold, giving the required counterexample. $\square$

Chapter VI

Multiple norms

6.1. Introduction

Motivated by the chain rule for higher-order colocally weakly differentiable maps, we define a notion of *multiple norm* for maps between higher-order tangent bundles of a manifold. In fact, since every higher-order tangent bundle does not carry a vector bundle structure over its manifold [15, §9.3, §15.1; 41, §16.15.7], we cannot define a notion of norm on this space with respect to each fiber over the manifold. However, higher-order tangent bundles are *multiple vector bundles* [57, 84] and so in a first step we define a notion of multiple norm which is compatible with the multiple structure. Similarly, it follows then a notion of multiple norm for maps between higher-order tangent bundles.

Actually, the notions of *double norm* and k^{tuple} *norm* were already defined in chapter IV (see sections 4.4.2 and 4.4.5). Furthermore, we mention some properties of those norms. However, we did not go into details.

In order to be more precise, we first define in the present chapter the notion of *double vector space* (definition 6.2.1) as spaces which are likely the local representation of a second-order tangent bundle at a point, and a notion of *double seminorm* on these spaces (definition 6.2.2). We also point out some properties of maps between double vector spaces. It results then more details about section 4.4.2 from chapter IV.

In the second step, we generalize the previous notions to any higher order. For every $k \in \mathbb{N}_*$, we first define a notion of k^{tuple} *seminorm* on k^{tuple} *vector spaces* (definitions 6.3.1 and 6.3.2). In particular, this notion includes the one of double seminorm. The corresponding properties are then useful to be more specific on that of the notion of k^{tuple} *norm* on k^{th} order tangent bundles.

6.2. Double (semi)norms

6.2.1. Double seminorms on double vector spaces

First, a *double vector bundle* consists of four manifolds E, E_1, E_2, M and a system of four vector bundle structures

$$\begin{array}{ccc} E & \xrightarrow{\pi_1} & E_1 \\ \pi_2 \downarrow & & \downarrow \pi_M^1 \\ E_2 & \xrightarrow{\pi_M^2} & M \end{array}$$

in which E has two vector bundle structures, on bases E_1 and E_2 , which are themselves vector bundles on M and such that each of the four structure maps (namely the bundle projection, zero section, addition and scalar multiplication) of each vector bundle structure on E is a morphism of vector bundles with respect to the other structure [84, definition 1.1].

In particular, we define a notion of double vector space.

Definition 6.2.1. A set W is a *double vector space* (of dimension m) if there exist three subspaces V_1, V_2 and V_{12} of $\mathbb{R}^m$ such that $W = V_1 \times V_2 \times V_{12}$ and such that the diagram

$$\begin{array}{ccc} (e_1, e_2, e_{12}) \in W & \xrightarrow{\pi_1} & e_1 \in V_1 \\ \pi_2 \downarrow & & \downarrow \pi_0^1 \\ e_2 \in V_2 & \xrightarrow{\pi_0^2} & 0 \in \mathbb{R}^m \end{array}$$

defines a double vector bundle.

We then define a notion of double seminorm on double vector spaces.

Definition 6.2.2. Let W be a double vector space. A map $\llbracket \cdot \rrbracket : W \rightarrow \mathbb{R}$ is a *double seminorm* on W if for every $i \in \{1, 2\}$, it is a seminorm on fibers over the vector bundle (W, π_i, V_i) .

As a consequence, if a map $\llbracket \cdot \rrbracket$ is a double seminorm on the double vector space W , then for every $i \in \{1, 2\}$ and every $w, w' \in W$ such that $\pi_i(w) = \pi_i(w')$,

$$\llbracket w +_i w' \rrbracket \leq \llbracket w \rrbracket + \llbracket w' \rrbracket,$$

where $+_i$ is the vector bundle addition of (W, π_i, V_i) [84, §1], and for every $\lambda \in \mathbb{R}$,

$$\llbracket \lambda \cdot_i w \rrbracket = |\lambda| \llbracket w \rrbracket,$$

where $\cdot_i$ is the vector bundle multiplication of (W, π_i, V_i) [84, §1].

Given a double vector space, there exists a nontrivial double seminorm on this space.

Proposition 6.2.1 (Canonical double seminorm). *Let W be a double vector space. The map $\llbracket \cdot \rrbracket_*: W \rightarrow \mathbb{R}$ defined for every $w = (e_1, e_2, e_{12}) \in W$ by*

$$\llbracket w \rrbracket_* = |e_1| |e_2| + |e_{12}|$$

is a double seminorm on W .

Given two double vector spaces, we define the notion of double vector map between them.

Definition 6.2.3. Let W and W' be two double vector spaces. A map $f: W \rightarrow W'$ is a *double vector map* if there exist linear maps $f_1: V_1 \rightarrow V'_1$, $f_2: V_2 \rightarrow V'_2$ and $f_{12}: V_{12} \rightarrow V'_{12}$ and a bilinear map $\hat{f}_{12}: V_1 \times V_2 \rightarrow V'_{12}$ such that for every $w = (e_1, e_2, e_{12}) \in W$,

$$f(w) = (f_1(e_1), f_2(e_2), f_{12}(e_{12}) + \hat{f}_{12}(e_1, e_2)).$$

A double vector map is bounded with respect to the canonical double seminorm given by proposition 6.2.1.

Proposition 6.2.2. *Let $f: W \rightarrow W'$ be a double vector map and let $\llbracket \cdot \rrbracket'$ be a double seminorm on W' . Then there exists a constant $C > 0$ such that for every $w \in W$,*

$$\llbracket f(w) \rrbracket' \leq C \llbracket w \rrbracket_*.$$

Proof. Let $w \in W$. If $(b_1^\ell)_{1 \leq \ell \leq \kappa}$ is a basis in V'_1 , then there exist $\lambda_1, \dots, \lambda_\kappa \in \mathbb{R}$ such that $f_1(e_1) = \sum_{\ell=1}^\kappa \lambda_\ell b_1^\ell$. Since the double seminorm is a seminorm on each fiber, there exist constants $C_1, \dots, C_\kappa > 0$ and then a constant $C > 0$ such that

$$\begin{aligned} \llbracket f(w) \rrbracket' &\leq \sum_{\ell=1}^\kappa |\lambda_\ell| \llbracket \left(b_1^\ell, f_2(e_2), \frac{f_{12}(e_{12}) + \hat{f}_{12}(e_1, e_2)}{\sum_{\ell=1}^\kappa |\lambda_\ell|} \right) \rrbracket' \\ &\leq \sum_{\ell=1}^\kappa |\lambda_\ell| C_\ell \left(|f_2(e_2)| + \frac{1}{\sum_{\ell=1}^\kappa |\lambda_\ell|} |f_{12}(e_{12}) + \hat{f}_{12}(e_1, e_2)| \right) \\ &\leq \left(\max_{1 \leq \ell \leq \kappa} C_\ell \right) \left(\sum_{\ell=1}^\kappa |\lambda_\ell| |f_2(e_2)| + |f_{12}(e_{12}) + \hat{f}_{12}(e_1, e_2)| \right) \\ &\leq C \left(|f_1(e_1)| |f_2(e_2)| + |f_{12}(e_{12})| + |\hat{f}_{12}(e_1, e_2)| \right). \end{aligned}$$

Since f_1, f_2, f_{12} are linear and $\hat{f}_{12}$ is bilinear, there exist constants $\tilde{C}_1, \tilde{C}_2 > 0$ such that

$$\llbracket f(w) \rrbracket' \leq \tilde{C}_1 |e_1| |e_2| + \tilde{C}_2 |e_{12}| \leq \max(\tilde{C}_1, \tilde{C}_2) \llbracket w \rrbracket_*. \quad \square$$

As a consequence, the canonical double seminorm given by proposition 6.2.1 is an upper bound for any double seminorms.

Corollary 6.2.3. *Let W be a double vector space. Let $\llbracket \cdot \rrbracket$ be a double seminorm on W . Then there exists a constant $C > 0$ such that for every $w \in W$, $\llbracket w \rrbracket \leq C \llbracket w \rrbracket_*$.*

The opposite inequality is not always true; for instance, if we take the double seminorm $\llbracket \cdot \rrbracket$ defined for every $w = (e_1, e_2, e_{12}) \in W$ by $\llbracket w \rrbracket = |e_1| |e_2|$.

Finally, if W and W' are both double vector spaces, we denote by $\text{Mor}(W, W')$ the space of all double vector maps between them.

It is also possible to define a notion of double seminorm on those spaces. However, motivated by the behavior of double norms under composition of double vector bundle morphisms, we just consider the following *canonical double seminorm*.

Definition 6.2.4. Let W and W' be double vector spaces. We define the map $\llbracket \cdot \rrbracket_* : \text{Mor}(W, W') \rightarrow \mathbb{R}$ for every $f \in \text{Mor}(W, W')$ by

$$\llbracket f \rrbracket_* = |f_1| |f_2| + |f_{12}| + |\hat{f}_{12}|,$$

with the corresponding norms on the space of linear maps and on the one of bilinear maps.

The previous double seminorm is compatible with the composition.

Proposition 6.2.4. *Let W, W' and W'' be double vector spaces. Let $f \in \text{Mor}(W, W')$ and $g \in \text{Mor}(W', W'')$. Then $g \circ f \in \text{Mor}(W, W'')$ and*

$$\llbracket g \circ f \rrbracket_* \leq \llbracket g \rrbracket_* \llbracket f \rrbracket_*.$$

Proof. For every $(e_1, e_2, e_{12}) \in W$,

$$\begin{aligned} (g \circ f)(e_1, e_2, e_{12}) &= ((g_1 \circ f_1)(e_1), (g_2 \circ f_2)(e_2), \\ &\quad (g_{12} \circ f_{12})(e_{12}) + ((g_{12} \circ \hat{f}_{12}) + (\hat{g}_{12} \circ (f_1, f_2)))(e_1, e_2)), \end{aligned}$$

and so

$$\begin{aligned} \llbracket g \circ f \rrbracket_* &\leq (|g_1| |g_2| + |\hat{g}_{12}|) |f_1| |f_2| + |g_{12}| (|f_{12}| + |\hat{f}_{12}|) \\ &\leq \llbracket g \rrbracket_* |f_1| |f_2| + \llbracket g \rrbracket_* (|f_{12}| + |\hat{f}_{12}|) = \llbracket g \rrbracket_* \llbracket f \rrbracket_*. \quad \square \end{aligned}$$

6.2.2. Double norms for double vector bundles

In this part, we follow the same structure as the one of section 4.4.2 but we give more details. So we assume that M and N are differentiable manifolds of class C^2 of respective dimensions m and n . As usually, they satisfy the Hausdorff separation property and have a countable basis [42, §0.5; 70, §1.5]

Some properties of the second-order tangent bundle T^2M are already mentioned in section 4.2. We also note that for every $x \in M$, there exists a *local trivialization* (V, ψ) such that $x \in V$, the set $V \subseteq M$ is open and

$$\psi: V \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow (\pi_M^2)^{-1}(V) \subseteq T^2M$$

is a diffeomorphism [41, §16.15.7]; hence, for local trivializations (V_i, ψ_i) and (V_j, ψ_j) , the transition diffeomorphism is given by [15, §9.2 – 9.3; 57, equation (20); 87, §8.12]

$$\begin{aligned} (x, e_1, e_2, e_{12}) &\in (V_i \cap V_j) \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \mapsto \\ &(x, \psi_{ij}(x)[e_1, e_2, e_{12}]) \in (V_i \cap V_j) \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m, \end{aligned}$$

where for every $x \in V_i \cap V_j$, $\psi_{ij}(x)$ is a double vector map and the map $x \in V_i \cap V_j \mapsto \psi_{ij}(x)$ is continuous; in other words, the transition maps are *non-linear* in (e_1, e_2, e_{12}) : it is linear in (e_1, e_{12}) or in (e_2, e_{12}) [15, §9.3].

The reader can then refer to the definition 4.4.2 for the notion of *double norm* and this notion relies on the one of *double seminorm* on second-order tangent bundles (definition 4.4.1).

Proposition 6.2.5. *There exists a double norm on T^2M .*

The previous proposition is already mentioned in chapter IV (proposition 4.4.1). However, we only sketched the proof.

Proof of proposition 6.2.5. For every local trivialization (V, ψ) , we define $\llbracket \cdot \rrbracket_V$ for every $\nu \in (\pi_M^2)^{-1}(V)$ such that $\psi^{-1}(\nu) = (x, e_1, e_2, e_{12})$ by

$$\llbracket \nu \rrbracket_V = |e_1| |e_2| + |e_{12}|.$$

Let $(V_i)_{i \in I}$ be an open cover of M and let $(\eta_i)_{i \in I}$ be a partition of unity subordinate to $(V_i)_{i \in I}$ [42, §0.5]. We define a double seminorm $\llbracket \cdot \rrbracket$ on T^2M for every $\nu \in T^2M$ by

$$\llbracket \nu \rrbracket = \sum_{i \in I} \eta_i(\pi_M^2(\nu)) \llbracket \nu \rrbracket_{V_i}.$$

To prove that $\llbracket \cdot \rrbracket$ is a double norm on T^2M , it suffices to prove that in each local trivialization [42, §0.5 remark 5.7]. Let $\llbracket \cdot \rrbracket'$ be a double

seminorm on T^2M . Let (V, ψ) be a local trivialization around $x \in M$. By definition 4.4.1 of double seminorm, the double seminorm $\llbracket \cdot \rrbracket'$ is represented for every $\nu \in (\pi_M^2)^{-1}(V)$ such that $\psi^{-1}(\nu) = (\pi_M^2(\nu), e_1, e_2, e_{12})$ by

$$\llbracket \nu \rrbracket'_V = \beta_V(\pi_M^2(\nu)) \llbracket (e_1, e_2, e_{12}) \rrbracket'_W,$$

where $\beta_V \in C(V, \mathbb{R})$ is a positive continuous function and $\llbracket \cdot \rrbracket'_W$ is a double seminorm on $W = \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$. There exists an open set $U \subseteq M$ such that $\overline{U}$ is compact and $x \in \overline{U} \subseteq V$. Since β_V is continuous, there exists a constant $C_1 > 0$ such that for every $\nu \in (\pi_M^2)^{-1}(\overline{U})$,

$$\llbracket \nu \rrbracket'_V \leq C_1 \llbracket (e_1, e_2, e_{12}) \rrbracket'_W,$$

and since the canonical double seminorm is an upper bound for any double seminorms (corollary 6.2.3), there exists a constant $C_2 > 0$ such that

$$\llbracket \nu \rrbracket'_V \leq C_1 C_2 \llbracket (e_1, e_2, e_{12}) \rrbracket_* = C_1 C_2 \llbracket \nu \rrbracket_V. \quad \square$$

Finally, we remark how double norms behave under composition of double vector bundle morphisms (proposition 4.4.2). For this purpose, we note that for every double vector bundle morphism $v \in \text{Mor}(T^2M, T^2N)$ such that $v \in \text{Mor}(T^2M, T^2N)_{x,y}$, if (V, ψ) and (U, φ) are local trivializations of T^2M and T^2N around $x \in M$ and $y \in N$ respectively, then $\varphi^{-1} \circ v \circ \psi: V \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow U \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ is such that for every $(e_1, e_2, e_{12}) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$,

$$(\varphi^{-1} \circ v \circ \psi)(x, e_1, e_2, e_{12}) = (y, v_1[e_1], v_2[e_2], v_{12}[e_{12}] + \hat{v}_{12}[e_1, e_2]),$$

where the maps $v_1, v_2, v_{12}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ are linear and the map $\hat{v}_{12}: \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is bilinear.

Proposition 6.2.6. *Let K be a manifold of class C^2 . For every compact subsets $Q_M \subseteq M$, $Q_N \subseteq N$ and $Q_K \subseteq K$, there exists a constant $C > 0$ such that for every $v \in (\pi_{M \times N}^2)^{-1}(Q_M \times Q_N)$ and every $\xi \in (\pi_{N \times K}^2)^{-1}(Q_N \times Q_K)$ such that $\pi_N \circ \pi_{M \times N}^2 \circ v = \pi_N \circ \pi_{N \times K}^2 \circ \xi$,*

$$\llbracket \xi \circ v \rrbracket \leq C \llbracket \xi \rrbracket \llbracket v \rrbracket.$$

In fact, the composition property 6.2.4 of the canonical double seminorm for double vector maps (definition 6.2.4) can be generalized to any double norms.

Proof of proposition 6.2.6. Let $Q_M \subseteq M$, $Q_N \subseteq N$ and $Q_K \subseteq K$ be compact sets. Let $v \in \text{Mor}(T^2M, T^2N)_{x,y}$ and $\xi \in \text{Mor}(T^2N, T^2K)_{y,z}$ such that $x \in Q_M$, $y \in Q_N$ and $z \in Q_K$.

We first consider double norms $\llbracket \cdot \rrbracket'$ on the space of double vector bundle morphisms similar to the one of proposition 4.4.1, but with canonical double seminorms for double vector maps (definition 6.2.4). If $(V_i \times U_i \times W_i)_{i \in I}$ is an open cover of $M \times N \times K$ and if $(\eta_i)_{i \in I}$, $(\varphi_i)_{i \in I}$ and $(\gamma_i)_{i \in I}$ are partitions of unity subordinate to $(V_i \times U_i)_{i \in I}$, $(U_i \times W_i)_{i \in I}$ and $(V_i \times W_i)_{i \in I}$ respectively [42, §0.5], then

$$\begin{aligned} \llbracket \xi \circ v \rrbracket' &= \sum_{i \in I} \gamma_i(x, z) \llbracket \xi \circ v \rrbracket'_{V_i \times W_i} \\ &= \sum_{i \in I} \gamma_i(x, z) (|\xi_1^i \circ v_1^i| |\xi_2^i \circ v_2^i| + |\xi_{12}^i \circ v_{12}^i| \\ &\quad + |\hat{\xi}_{12}^i \circ (v_1^i, v_2^i) + \xi_{12} \circ \hat{v}_{12}^i|), \\ \llbracket v \rrbracket' &= \sum_{i \in I} \eta_i(x, y) \llbracket v \rrbracket'_{V_i \times U_i} \\ &= \sum_{i \in I} \eta_i(x, y) (|v_1^i| |v_2^i| + |v_{12}^i| + |\hat{v}_{12}^i|), \end{aligned}$$

and similarly $\llbracket \xi \rrbracket' = \sum_{i \in I} \varphi_i(y, z) \llbracket \xi \rrbracket'_{U_i \times W_i}$. By the composition property of canonical double seminorms (proposition 6.2.4), and since Q_M , Q_N and Q_K are compact, there exists $\ell_1, \ell_2, \ell_3 > 1$ and constants $C_1, C_2 > 0$ such that

$$\begin{aligned} \llbracket \xi \circ v \rrbracket' &\leq \sum_{i=1}^{\ell_1} \gamma_i(x, z) \llbracket \xi \rrbracket'_{U_{\kappa_i} \times W_i} \llbracket v \rrbracket'_{V_i \times U_{\kappa_i}} \\ &= \sum_{i=1}^{\ell_1} \gamma_i(x, z) \left(\sum_{j=1}^{\ell_2} \varphi_j(y, z) \right) \llbracket \xi \rrbracket'_{U_{\kappa_i} \times W_i} \left(\sum_{k=1}^{\ell_3} \eta_k(x, y) \right) \llbracket v \rrbracket'_{V_i \times U_{\kappa_i}} \\ &\leq C_1 \sum_{i=1}^{\ell_1} \left(\sum_{j=1}^{\ell_2} \varphi_j(y, z) \llbracket \xi \rrbracket'_{U_j \times W_j} \right) \left(\sum_{k=1}^{\ell_3} \eta_k(x, y) \llbracket v \rrbracket'_{V_k \times U_k} \right) \\ &\leq C_2 \llbracket \xi \rrbracket' \llbracket v \rrbracket'. \end{aligned}$$

By the maximality property of double norms, there exist positive continuous functions $\beta_1 \in C(M \times K, \mathbb{R})$, $\beta_2 \in C(N \times K, \mathbb{R})$ and $\beta_3 \in C(M \times N, \mathbb{R})$ such that

$$\begin{aligned} \llbracket \xi \circ v \rrbracket &\leq \beta_1(x, z) \llbracket \xi \circ v \rrbracket' \leq \beta_1(x, z) C_2 \llbracket \xi \rrbracket' \llbracket v \rrbracket' \\ &\leq \beta_1(x, z) C_2 \beta_2(y, z) \llbracket \xi \rrbracket \beta_3(x, y) \llbracket v \rrbracket, \end{aligned}$$

and since Q_M , Q_N and Q_K are compact, there exists a constant $C > 0$ such that $\llbracket \xi \circ v \rrbracket \leq C \llbracket \xi \rrbracket \llbracket v \rrbracket$. $\square$

6.3. Multiple (semi)norms

6.3.1. Multiple seminorms on multiple vector spaces

For every $k \in \mathbb{N}_*$, we define the notion of k^{tuple} vector space. We denote by I_k the *index set* $2^{\{1, \dots, k\}} \setminus \emptyset$ [15, §MA.1].

Definition 6.3.1. Let $k \in \mathbb{N}_*$.

A set W is a 1^{tuple} *vector space* (of dimension m) if there exists a subspace V of $\mathbb{R}^m$ such that $W = V$ and for $\{0\} \in \mathbb{R}^m$, $(V, \pi_0, \{0\})$ is a vector bundle structure.

A set W is a 2^{tuple} *vector space* (of dimension m) if it is a double vector space of dimension m .

If $k \geq 3$, a set W is a k^{tuple} *vector space* (of dimension m) if there exists a set $(V_\lambda)_{\lambda \in I_k}$ of $2^k - 1$ subspaces of $\mathbb{R}^m$ such that

$$W = \prod_{\lambda \in I_k} V_\lambda$$

has k vector bundle structures, on bases $W_1, \dots, W_k$, such that for every $j \in \{1, \dots, k\}$, W_j is a $(k-1)^{\text{tuple}}$ vector space, and for all $i \neq j \in \{1, \dots, k\}$, if W_{ij} is a $(k-2)^{\text{tuple}}$ vector space such that there exist vector bundle structures $\pi_{ij}^i: W_i \rightarrow W_{ij}$ and $\pi_{ij}^j: W_j \rightarrow W_{ij}$, then the following diagram

$$\begin{array}{ccc} W & \xrightarrow{\pi_i} & W_i \\ \pi_j \downarrow & & \downarrow \pi_{ij}^i \\ W_j & \xrightarrow{\pi_{ij}^j} & W_{ij} \end{array}$$

is a double vector bundle.

This kind of spaces appears as to be a particular case of k^{tuple} vector bundles [57, definition 4.1; 84, definition 5.1; 114, §6.2 definition 3] and they are also studied in the context of k^{th} order tangent bundles of manifolds [15, §MA.2].

For instance, a triple vector space (or a 3^{tuple} vector space) is such that

$$W = V_1 \times V_2 \times V_3 \times V_{12} \times V_{23} \times V_{13} \times V_{123}$$

and each side of the following cube

$$\begin{array}{ccc}
(e_1, e_2, e_3, e_{12}, e_{23}, e_{13}, e_{123}) \in W & \xrightarrow{\pi_{23}} & (e_2, e_3, e_{23}) \in V_2 \times V_3 \times V_{23} \\
\swarrow \pi_{12} \quad \downarrow & & \swarrow \quad \downarrow \\
(e_1, e_2, e_{12}) \in V_1 \times V_2 \times V_{12} & \longrightarrow & e_2 \in V_2 \\
\downarrow \quad \downarrow \pi_{13} & & \downarrow \quad \downarrow \\
(e_1, e_3, e_{13}) \in V_1 \times V_3 \times V_{13} & \dashrightarrow & e_3 \in V_3 \\
\downarrow \quad \swarrow & & \downarrow \quad \swarrow \\
e_1 \in V_1 & \longrightarrow & 0 \in \mathbb{R}^m
\end{array}$$

is a double vector bundle.

We then define a notion of k^{tuple} seminorm on k^{tuple} vector spaces.

Definition 6.3.2. Let W be a k^{tuple} vector space. A map $\llbracket \cdot \rrbracket: W \rightarrow \mathbb{R}$ is a k^{tuple} seminorm on W if it is a seminorm on fibers over each of the k vector bundle structure of W .

A 1^{tuple} seminorm is just a seminorm on a subspace of $\mathbb{R}^m$ and a 2^{tuple} seminorm is a double seminorm on a double vector space.

Given a k^{tuple} vector space, there exists a nontrivial k^{tuple} seminorm on this space.

Proposition 6.3.1 (Canonical k^{tuple} seminorm). *Let W be a k^{tuple} vector space. The map $\llbracket \cdot \rrbracket_*: W \rightarrow \mathbb{R}$ defined for every $w = (e_\lambda)_{\lambda \in I_k} \in W$ by*

$$\llbracket w \rrbracket_* = \sum_{\ell=1}^k \sum_{\lambda \in \mathcal{P}_\ell(\{1, \dots, k\})} |e_{\lambda^1}| \cdots |e_{\lambda^\ell}|,$$

where $\mathcal{P}_\ell(\{1, \dots, k\}) = \{\lambda = \{\lambda^1, \dots, \lambda^\ell\} \in 2^{I_k} : \dot{\cup}_{i=1}^\ell \lambda^i = \{1, \dots, k\}, \emptyset \notin \lambda\}$ is the set of all partitions of $\{1, \dots, k\}$ of length ℓ , is a k^{tuple} seminorm on W .

For example, the canonical triple seminorm $\llbracket \cdot \rrbracket_*$ is defined for every $w = (e_1, e_2, e_3, e_{12}, e_{23}, e_{13}, e_{123}) \in W$ by

$$\llbracket w \rrbracket_* = |e_1| |e_2| |e_3| + |e_1| |e_{23}| + |e_2| |e_{13}| + |e_3| |e_{12}| + |e_{123}|.$$

Given two k^{tuple} vector spaces, we define k^{tuple} vector maps as follows.

Definition 6.3.3. Let W and W' be k^{tuple} vector spaces. A map $f: W \rightarrow W'$ is a k^{tuple} vector map if for every $w = (e_\lambda)_{\lambda \in I_k} \in W$,

$$f(w) = \left(\sum_{\ell=1}^{|\Lambda|} \sum_{\lambda \in \mathcal{P}_\ell(\Lambda)} f_\lambda(e_{\lambda^1}, \dots, e_{\lambda^\ell}) \right)_{\Lambda \in I_k},$$

where for every $\Lambda \in I_k$, for every $1 \leq \ell \leq |\Lambda|$ and every $\lambda \in \mathcal{P}_\ell(\Lambda)$, the map $f_\lambda: V_{\lambda^1} \times \cdots \times V_{\lambda^\ell} \rightarrow V'_\Lambda$ is ℓ -linear.

This kind of maps appears while studying maps between k^{th} order tangent bundles [15, §MA.5], and when considering change of coordinates for k^{tuple} vector bundles [57, equation (22); 114, §6.2 definition 3].

A k^{tuple} vector map is bounded with respect to the canonical k^{tuple} seminorm given by proposition 6.3.1.

Proposition 6.3.2. *Let $k \in \mathbb{N}$ so that $k \geq 2$. Let $f: W \rightarrow W'$ be a k^{tuple} vector map and let $\llbracket \cdot \rrbracket'$ be a k^{tuple} seminorm on W' . Then there exists a constant $C > 0$ such that for every $w \in W$,*

$$\llbracket f(w) \rrbracket' \leq C \llbracket w \rrbracket_*.$$

In order to prove the previous proposition, we introduce some notations. For every $j \in \{1, \dots, k\}$, the index set $J_j = 2^{\{1, \dots, k\} \setminus \{j\}} \setminus \emptyset$ has $2^{k-1} - 1$ elements. If W is a k^{tuple} vector space, then for every $\tilde{\lambda} \in I_k$, the map $\pi_{J_j, \tilde{\lambda}}: W \rightarrow W$ is defined for every $w = (e_\lambda)_{\lambda \in I_k} \in W$ by $\pi_{J_j, \tilde{\lambda}}(w) = ((\pi_{J_j, \tilde{\lambda}}(w))_\lambda)_{\lambda \in I_k}$ where for every $\lambda \in I_k$,

$$(\pi_{J_j, \tilde{\lambda}}(w))_\lambda = \begin{cases} e_\lambda & \text{if } \lambda \in J_j \cup \{\tilde{\lambda}\}, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, for every $j \in \{1, \dots, k\}$, if $\sum$ means the vector bundle addition of $(W, \pi_{J_j}, \prod_{\lambda \in J_j} V_\lambda)$, then for every $w \in W$, $w = \sum_{\lambda \in I_k \setminus J_j} \pi_{J_j, \lambda}(w)$.

We also prove the following lemma which is a particular case of proposition 6.3.2.

Lemma 6.3.3. *Let $k \geq 2$. Let W be a k^{tuple} vector space and let $\llbracket \cdot \rrbracket$ be a k^{tuple} seminorm on W . Let $\Lambda \in \mathcal{P}_\ell(\{1, \dots, k\})$. If the map $\pi_\Lambda: W \rightarrow W$ is defined for every $w = (e_\lambda)_{\lambda \in I_k} \in W$ by $\pi_\Lambda(w) = ((\pi_\Lambda(w))_\lambda)_{\lambda \in I_k}$ where for every $\lambda \in I_k$,*

$$(\pi_\Lambda(w))_\lambda = \begin{cases} e_\lambda & \text{if } \lambda \in \Lambda, \\ 0 & \text{otherwise,} \end{cases}$$

then there exists $C > 0$ such that for every $w \in W$,

$$\llbracket \pi_\Lambda(w) \rrbracket \leq C(|e_{\Lambda^1}| \cdots |e_{\Lambda^\ell}|).$$

Proof. Let $w = (e_\lambda)_{\lambda \in I_k} \in W$. If $\ell = 1$, then for every $j \in \{1, \dots, k\}$, $\pi_\Lambda(w) \in (\pi_{J_j})^{-1}(0)$. Since it is a seminorm on every fibers, there exists $C > 0$ such that $\llbracket \pi_\Lambda(w) \rrbracket \leq C|e_\Lambda|$.

If $\ell \geq 2$, since Λ is a partition of $\{1, \dots, k\}$, there exists $j \in \{1, \dots, k\}$ such that $\Lambda^1 \notin J_j$ but $\Lambda^2, \dots, \Lambda^\ell \in J_j$. If $\kappa_1 = \dim(V_{\Lambda^1})$ and if $(b_1^i)_{1 \leq i \leq \kappa_1}$ is a basis of V_{Λ^1} , there exists $\eta_1, \dots, \eta_{\kappa_1}$ such that $e_{\Lambda^1} = \sum_{i=1}^{\kappa_1} \eta_i b_1^i$. Since the k^{tuple} seminorm is a seminorm on the fiber $(W, \pi_{J_j}, \prod_{\lambda \in J_j} V_\lambda)$, there exists $C_1 > 0$ such that

$$\llbracket \pi_\Lambda(w) \rrbracket \leq \sum_{i=1}^{\kappa_1} |\eta_i| \llbracket w^i \rrbracket \leq C_1 |e_{\Lambda^1}| \sum_{i=1}^{\kappa_1} \llbracket w^i \rrbracket,$$

where for every $i \in \{1, \dots, \kappa_1\}$, $w^i \in W$ is defined by $(w^i)_\lambda = e_\lambda$ if $\lambda \in \Lambda \setminus \{\Lambda^1\}$, $(w^i)_{\Lambda^1} = b_1^i$ and $(w^i)_\lambda = 0$ otherwise.

Proceeding step by step, there exists $C_2, \dots, C_\ell > 0$ such that

$$\begin{aligned} \llbracket \pi_\Lambda(w) \rrbracket &\leq C_1 |e_{\Lambda^1}| \cdots C_\ell |e_{\Lambda^\ell}| \sum_{i_1=1}^{\kappa_1} \cdots \sum_{i_\ell=1}^{\kappa_\ell} \llbracket w^{i_1, \dots, i_\ell} \rrbracket \\ &= \left(C_1 \cdots C_\ell \left(\sum_{i_1=1}^{\kappa_1} \cdots \sum_{i_\ell=1}^{\kappa_\ell} \llbracket w^{i_1, \dots, i_\ell} \rrbracket \right) \right) |e_{\Lambda^1}| \cdots |e_{\Lambda^\ell}|, \end{aligned}$$

where for every $j \in \{1, \dots, \ell\}$, $\kappa_j = \dim(V_{\Lambda^j})$ and $(b_j^{i_j})_{1 \leq i_j \leq \kappa_j}$ is a basis of V_{Λ^j} , and for every $1 \leq i_1 \leq \kappa_1, \dots, 1 \leq i_\ell \leq \kappa_\ell$, $w^{i_1, \dots, i_\ell} \in W$ is defined by $(w^{i_1, \dots, i_\ell})_\lambda = b_j^{i_j}$ if $\lambda = \Lambda^j$ for some $j \in \{1, \dots, \ell\}$ and $(w^{i_1, \dots, i_\ell})_\lambda = 0$ if $\lambda \in I_k \setminus \Lambda$. $\square$

Proof of proposition 6.3.2. Let $w = (e_\lambda)_{\lambda \in I_k} \in W$. Since for every $j \in \{1, \dots, k\}$ and every $\tilde{\lambda} \in J_j$, $\llbracket \pi_{J_j, \tilde{\lambda}}(f(w)) \rrbracket' = 0$, and since $\llbracket \cdot \rrbracket'$ is a seminorm on every fibers,

$$\llbracket f(w) \rrbracket' \leq \sum_{\tilde{\lambda}_1 \in I_k \setminus J_1} \cdots \sum_{\tilde{\lambda}_k \in I_k \setminus J_k} \llbracket \pi_{J_k, \tilde{\lambda}_k} \circ \cdots \circ \pi_{J_1, \tilde{\lambda}_1}(f(w)) \rrbracket'.$$

For every $\tilde{\lambda}_1 \in I_k \setminus J_1, \dots, \tilde{\lambda}_k \in I_k \setminus J_k$, the set $\bigcap_{j=1}^k (J_j \cup \{\tilde{\lambda}_j\})$ is not empty and for every $\Lambda \in I_k$,

$$\begin{aligned} (\pi_{J_k, \tilde{\lambda}_k} \circ \cdots \circ \pi_{J_1, \tilde{\lambda}_1}(f(w)))_\Lambda &= \\ \begin{cases} \sum_{\ell=1}^{|\Lambda|} \sum_{\lambda \in \mathcal{P}_\ell(\Lambda)} f_\lambda(e_{\lambda^1}, \dots, e_{\lambda^\ell}) & \text{if } \Lambda \in \bigcap_{j=1}^k (J_j \cup \{\tilde{\lambda}_j\}), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If $\bigcap_{j=1}^k (J_j \cup \{\tilde{\lambda}_j\}) \notin \mathcal{P}(\{1, \dots, k\})$, then $\llbracket \pi_{J_k, \tilde{\lambda}_k} \circ \cdots \circ \pi_{J_1, \tilde{\lambda}_1}(f(w)) \rrbracket' = 0$. Hence,

$$\begin{aligned} \sum_{\tilde{\lambda}_1 \in I_k \setminus J_1} \cdots \sum_{\tilde{\lambda}_k \in I_k \setminus J_k} \llbracket \pi_{J_k, \tilde{\lambda}_k} \circ \cdots \circ \pi_{J_1, \tilde{\lambda}_1}(f(w)) \rrbracket' \\ = \sum_{\ell=1}^k \sum_{\lambda \in \mathcal{P}_\ell(\{1, \dots, k\})} \llbracket \pi_\lambda(f(w)) \rrbracket', \end{aligned}$$

where $\pi_\lambda: W' \rightarrow W'$ is defined as in lemma 6.3.3.

By lemma 6.3.3, and by properties of multilinear maps, there exists $C_1, C_2 > 0$ such that

$$\begin{aligned} \|f(w)\|' &\leq C_1 \sum_{\ell=1}^k \sum_{\Lambda \in \mathcal{P}_\ell(\{1, \dots, k\})} \left(\prod_{i=1}^{\ell} \left(\sum_{j_i=1}^{|\Lambda^i|} \sum_{\lambda_i \in \mathcal{P}_{j_i}(\Lambda^i)} |f_{\lambda_i}| |e_{\lambda_i^1}| \cdots |e_{\lambda_i^{j_i}}| \right) \right) \\ &\leq C_1 \sum_{\ell=1}^k \sum_{\Lambda \in \mathcal{P}_\ell(\{1, \dots, k\})} \left(\prod_{i=1}^{\ell} \left(\sum_{j_i=1}^{|\Lambda^i|} \sum_{\lambda_i \in \mathcal{P}_{j_i}(\Lambda^i)} |f_{\lambda_i}| \right) \right) \|w\|_* \\ &\leq C_2 \|w\|_*. \end{aligned} \quad \square$$

As a consequence, the canonical k^{tuple} seminorm given by proposition 6.3.1 is an upper bound for any k^{tuple} seminorms.

Corollary 6.3.4. *Let $\|\cdot\|$ be a k^{tuple} seminorm on a k^{tuple} vector space W . Then there exists a constant $C > 0$ such that for every $w \in W$, $\|w\| \leq C \|w\|_*$.*

Finally, if W and W' are both k^{tuple} vector spaces, we denote by $\text{Mor}(W, W')$ the space of all k^{tuple} vector maps between them. As for the space of double vector maps, we also just consider the following k^{tuple} seminorm on $\text{Mor}(W, W')$ which is compatible with the composition.

Definition 6.3.4. Let W and W' be k^{tuple} vector spaces. We define the map $\|\cdot\|_*: \text{Mor}(W, W') \rightarrow \mathbb{R}$ for every $f \in \text{Mor}(W, W')$ by

$$\|f\|_* = \sum_{\ell=1}^k \sum_{\Lambda \in \mathcal{P}_\ell(\{1, \dots, k\})} \left(\sum_{j=1}^{\ell} \sum_{\lambda \in \mathcal{P}_j(\Lambda)} |f_{\lambda^1}| \cdots |f_{\lambda^j}| \right).$$

Proposition 6.3.5. *Let W , W' and W'' be k^{tuple} vector spaces. Let $f \in \text{Mor}(W, W')$ and $g \in \text{Mor}(W', W'')$. Then $g \circ f \in \text{Mor}(W, W'')$ and*

$$\|g \circ f\|_* \leq \|g\|_* \|f\|_*.$$

Proof. For every $w = (e_\lambda)_{\lambda \in I_k} \in W$ and every $\lambda \in I_k$ [15, theorem MA.6],

$$\begin{aligned} ((g \circ f)(w))_\lambda &= \sum_{j=1}^{|\lambda|} \sum_{\Lambda \in \mathcal{P}_j(\lambda)} \sum_{\substack{\nu_1 \in \mathcal{P}(\Lambda^1) \\ \dots \\ \nu_j \in \mathcal{P}(\Lambda^j)}} g_\Lambda(f_{\nu_1}(e_{\nu_1^1}, \dots, e_{\nu_1^{\ell_1}}), \dots, f_{\nu_j}(e_{\nu_j^1}, \dots, e_{\nu_j^{\ell_j}})). \end{aligned}$$

By definition 6.3.4 of k^{tuple} seminorms, and the previous equality,

$$\begin{aligned}
& \llbracket g \circ f \rrbracket_* \\
& \leq \sum_{\ell=1}^k \sum_{\Lambda \in \mathcal{P}_\ell(\{1, \dots, k\})} \left(\prod_{i=1}^{\ell} \left(\sum_{j=1}^{|\Lambda^i|} \sum_{\lambda_i \in \mathcal{P}_{j_i}(\Lambda^i)} |g_{\lambda_i}| \sum_{\substack{\nu_1 \in \mathcal{P}(\lambda_i^1) \\ \dots \\ \nu_{j_i} \in \mathcal{P}(\lambda_i^{j_i})}} |f_{\nu_1}| \cdots |f_{\nu_{j_i}}| \right) \right) \\
& \leq \sum_{j=1}^k \sum_{\Lambda \in \mathcal{P}_j(\{1, \dots, k\})} \left(\sum_{i=1}^j \sum_{\lambda \in \mathcal{P}_i(\Lambda)} |g_{\lambda^1}| \cdots |g_{\lambda^i}| \right) \left(\sum_{i=1}^j \sum_{\lambda \in \mathcal{P}_i(\Lambda)} |f_{\lambda^1}| \cdots |f_{\lambda^i}| \right) \\
& \leq \sum_{j=1}^k \sum_{\Lambda \in \mathcal{P}_j(\{1, \dots, k\})} \llbracket g \rrbracket_* \left(\sum_{i=1}^j \sum_{\lambda \in \mathcal{P}_i(\Lambda)} |f_{\lambda^1}| \cdots |f_{\lambda^i}| \right) = \llbracket g \rrbracket_* \llbracket f \rrbracket_*.
\end{aligned}$$

□

6.3.2. Multiple norms for multiple vector bundles

For every $k \geq 3$, we assume that M and N are differentiable manifolds of class C^k of respective dimensions m and n . As usually, the manifolds M and N satisfy the Hausdorff separation property and have a countable basis [42, §0.5; 70, §1.5].

The k^{th} order tangent bundle $T^k M$ has k natural vector bundle structures over $T^{k-1} M$ [15, §15.1] and so it is a k^{tuple} vector bundle [57, definition 4.1, theorem 5.1; 114, §6.2 definition 3] (see also section 4.4.5 above).

In particular, as for the second-order tangent bundle, for every $x \in M$, there exists a local trivialization (V, ψ) such that $x \in V$, the set $V \subseteq M$ is open and

$$\psi: V \times \prod_{\lambda \in I_k} \mathbb{R}^m \rightarrow (\pi_M^k)^{-1}(V) \subseteq T^k M,$$

where $\pi_M^k: T^k M \rightarrow M$ is the canonical submersion, is a diffeomorphism [15, §15.3; 57, theorems 4.1 and 5.1]; and for local trivializations (V_i, ψ_i) and (V_j, ψ_j) , the transition diffeomorphism is given by

$$\begin{aligned}
& (x, (e_\lambda)_{\lambda \in I_k}) \in (V_i \cap V_j) \times \prod_{\lambda \in I_k} \mathbb{R}^m \\
& \mapsto (x, \psi_{ij}(x)[(e_\lambda)_{\lambda \in I_k}]) \in (V_i \cap V_j) \times \prod_{\lambda \in I_k} \mathbb{R}^m,
\end{aligned}$$

where for every $x \in V_i \cap V_j$, $\psi_{ij}(x)$ is a k^{tuple} vector map and the map $x \in V_i \cap V_j \mapsto \psi_{ij}(x)$ is continuous.

The notion of k^{tuple} norm on k^{th} order tangent bundles is defined in section 4.4.5 (definition 4.4.3). As for the second-order tangent bundle, there exists a nontrivial k^{tuple} norm on $T^k M$.

Proposition 6.3.6. *There exists a k^{tuple} norm on $T^k M$.*

Sketch of the proof. For every local trivialization (V, ψ) , we define $\llbracket \cdot \rrbracket_V$ for every $\nu \in (\pi_M^k)^{-1}(V)$ such that $\psi^{-1}(\nu) = (x, (e_\lambda)_{\lambda \in I_k})$ by

$$\llbracket \nu \rrbracket_V = \sum_{\ell=1}^k \sum_{\lambda \in \mathcal{P}_\ell(\{1, \dots, k\})} |e_{\lambda 1}| \cdots |e_{\lambda \ell}|.$$

If $(V_i)_{i \in I}$ is an open cover of M and if $(\eta_i)_{i \in I}$ is a partition of unity subordinate to $(V_i)_{i \in I}$, then we define a k^{tuple} seminorm $\llbracket \cdot \rrbracket$ on $T^k M$ for every $\nu \in T^k M$ by

$$\llbracket \nu \rrbracket = \sum_{i \in I} \eta_i(\pi_M^k(\nu)) \llbracket \nu \rrbracket_{V_i}.$$

To prove that $\llbracket \cdot \rrbracket$ is a k^{tuple} norm on $T^k M$, since we have an upper bound property (corollary 6.3.4), we can proceed as in the proof of proposition 4.4.1. $\square$

Finally, as for the double norms, we remark how k^{tuple} norms behave under composition of k^{tuple} vector bundle morphisms.

Proposition 6.3.7. *Let K be a manifold of class C^k . For all compact subsets $Q_M \subseteq M$, $Q_N \subseteq N$ and $Q_K \subseteq K$, there exists a constant $C > 0$ such that for every $v \in (\pi_{M \times N}^k)^{-1}(Q_M \times Q_N)$ and every $\xi \in (\pi_{N \times K}^k)^{-1}(Q_N \times Q_K)$ such that $\pi_N \circ \pi_{M \times N}^k \circ v = \pi_N \circ \pi_{N \times K}^k \circ \xi$,*

$$\llbracket \xi \circ v \rrbracket \leq C \llbracket \xi \rrbracket \llbracket v \rrbracket.$$

The structure of the proof is similar to the one of proposition 6.2.6.

Sketch of the proof of proposition 6.3.7. Let $Q_M \subseteq M$, $Q_N \subseteq N$ and $Q_K \subseteq K$ be compact subsets. Let $v \in (\pi_{M \times N}^k)^{-1}(Q_M \times Q_N)$ and $\xi \in (\pi_{N \times K}^k)^{-1}(Q_N \times Q_K)$ such that $v \in \text{Mor}(T^k M, T^k N)_{x,y}$ and $\xi \in \text{Mor}(T^k N, T^k K)_{y,z}$.

We first consider k^{tuple} norms $\llbracket \cdot \rrbracket'$ on k^{tuple} vector bundle morphisms similar to the one of proposition 6.3.6, but with canonical k^{tuple} seminorms for k^{tuple} vector maps (definition 6.3.4). For instance, if $(V_i \times U_i \times W_i)_{i \in I}$ is an open cover of $M \times N \times K$ and if $(\gamma_i)_{i \in I}$ is a partition of unity subordinate to $(V_i \times W_i)_{i \in I}$, then

$$\llbracket \xi \circ v \rrbracket' = \sum_{i \in I} \gamma_i(x, z) \llbracket \xi \circ v \rrbracket'_{V_i \times W_i} = \sum_{i \in I} \gamma_i(x, z) \llbracket g_i \circ f_i \rrbracket'_*,$$

where for every $i \in I$, the maps g_i and f_i are k^{tuple} vector maps as in definition 6.3.3 and are such that $\varphi_i^{-1} \circ \xi \circ v \circ \psi_i: V_i \times \prod_{\lambda \in I_k} \mathbb{R}^m \rightarrow W_i \times \prod_{\lambda \in I_k} \mathbb{R}^n$ is defined for every $w = (e_\lambda)_{\lambda \in I_k}$ by

$$\begin{aligned} (\varphi_i^{-1} \circ \xi \circ v \circ \psi_i)(x, w) &= (\varphi_i^{-1} \circ \xi \circ \tilde{\psi}_{j_i}) \circ (\tilde{\psi}_{j_i}^{-1} \circ v \circ \psi_i)(x, w) \\ &= (\varphi_i^{-1} \circ \xi \circ \tilde{\psi}_{j_i})(y, f_i(w)) \\ &= (z, (g_i \circ f_i)(w)). \end{aligned}$$

By the composition property of canonical k^{tuple} seminorms (proposition 6.3.5), and since the subsets Q_M, Q_N and Q_K are compact, we can prove that there exists a constant $C > 0$ such that (see proof of proposition 6.2.6)

$$\llbracket \xi \circ v \rrbracket' \leq C \llbracket \xi \rrbracket' \llbracket v \rrbracket'.$$

By the maximality property of k^{tuple} norms, and since the subsets Q_M, Q_N and Q_K are compact, there exists constants $C_1, C_2, C_3 > 0$ such that

$$\llbracket \xi \circ v \rrbracket \leq C_1 \llbracket \xi \circ v \rrbracket' \leq C_2 \llbracket \xi \rrbracket' \llbracket v \rrbracket' \leq C_3 \llbracket \xi \rrbracket \llbracket v \rrbracket. \quad \square$$

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