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Innovative Applications of O.R.

A game-theoretic analysis of transmission-distribution system operator coordination[☆]

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ABSTRACT

In this paper, we formulate in a game-theoretic framework three coordination schemes for analyzing DSO-TSO interactions. This framework relies on a reformulation of the power flow equations by introducing linear mappings between the state and the decision variables. The first coordination scheme, used as a benchmark, is a co-optimization problem where an integrated market operator activates jointly resources connected at transmission and distribution levels. We formulate it as a standard constrained optimization problem. The second one, called shared balancing responsibility, assumes bounded rationality of TSO and DSOs which act simultaneously and is formulated as a non-cooperative game. The last one involves rational expectation from the DSOs which anticipate the clearing of the transmission market by the TSO, and is formulated as a Stackelberg game. For each coordination scheme, we determine conditions for existence and uniqueness of solutions. On a network instance from the NICTA NESTA test cases, we span the set of Generalized Nash Equilibria solutions of the decentralized coordination schemes. We determine that the decentralized coordination schemes are more profitable for the TSO and that rational expectations from the DSOs gives rise to a last-mover advantage for the TSO. Highest efficiency level is reached by the centralized co-optimization, followed very closely by the shared balancing responsibility. The mean social welfare is higher for the Stackelberg game than under shared balancing responsibility. Finally, under imperfect information, we check that the Price of Information, measured as the worst-case ratio of the optimal achievable social welfare with full information to the social welfare at an equilibrium with imperfect information, is a stepwise increasing function of the coefficient of variation of the TSO and reaches an upper bound.

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1. Introduction

Distributed energy resources are supply and demand-side resources that are connected to low-voltage electric power systems. On the supply side, these resources include distributed renewable resources such as solar photovoltaic panels deployed on rooftops, which are characterized by a significant amount of

uncertainty. On the demand side, demand response services that can be made available by commercial and residential consumers are characterized by a significant level of flexibility. The significant amount of Distributed Energy Resources (DERs) which have recently been integrated in power systems implies increasing degree of uncertainty but also increasing amount of flexibility in power system operations, thereby presenting both a challenge and an opportunity for power system operations and power market design. In particular, the active management of DERs raises a challenge about the extent of coordination between Transmission System Operators (TSOs), which are responsible for managing high-voltage transmission systems, and Distribution System Operators (DSOs), which are responsible for managing medium and low-voltage grids. Whereas DSO operations have traditionally been passive, there is a clear opportunity from proactively managing DERs in order to better deal with the unpredictable generation from Renewable Energy Sources (RES), whose penetration is constantly

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increasing (Hvelplund, 2006), and in order to offer a variety of other services to the high and low voltage grid.

At the European Union (EU) level, TSOs have been the only one procuring flexibility services connected to the distribution networks, while the role of DSOs is limited to validate that such flexibility can indeed be provided. The TSO dominant position (DSOs being only very weakly represented) has been the main trigger for the debates around TSO-DSO coordination within Europe (Hadush & Meeus, 2018). In a first attempt to provide solutions, DSOs have been proposing to improve TSO-DSO coordination by introducing concepts such as the 'traffic light' in Germany, that signals the distribution network state to the market (Hadush & Meeus, 2018). Many European projects have also proposed technical solutions to enhance TSO-DSO coordination (for example, *evolvdSO*¹ introduced an interval constrained and sequential power flow approach, *SmartNet*² has been studying potential TSO-DSO coordination schemes). Moreover, the Council of European Energy Regulators (CEER) has put forward principles that should set the trajectory of future TSO-DSO relationship and related regulated arrangements in the areas of governance, network planning and system operations. In a second attempt to provide solutions, DSOs have started to actively manage congestion in their networks. But, since the same flexibility resources could also be potentially used for congestion management and voltage control by the TSOs, conflicts might arise due to the misalignment of TSOs, DSOs, and market players' actions. Even though in many EU countries there are no rules in place that allow DSOs to activate flexibility services to redispatch the system at the distribution level, the Clean Energy Package³ presents clear provisions that will enable DSOs to procure flexibility services, and is expected to generate new schemes for TSO-DSO coordination.

This paper is specifically focused on the participation of DERs in Ancillary Services markets (Gerard, Rivero, & Six, 2017). This goal raises both computational as well as institutional challenges. From a computational perspective, the challenge is to optimize the real-time deployment of a huge number of resources connected to the distribution grid in order to provide balancing and congestion management services. From a market perspective, contracting DER-based ancillary services may complicate market clearing procedures and coordination due to the amount and complexity of the bids, as well as the associated coordination requirements and incentives of market participants (Bose, Cai, Low, & Wierman, 2014; Le Cadre, 2017; 2018; Xu, Cai, Bose, & Wierman, 2015). A simulator of TSO-DSO interactions over large-scale transmission and distribution networks, allowing nodes to provide complex bids for their flexibility activations, has been developed within *SmartNet*. The simulator outcome is not amenable to any analytical interpretation of the results; contrary to this paper, in which we provide analytical conditions for the existence and uniqueness of coordination schemes outcomes and quantify for each coordination scheme the profitability, efficiency and reserve activation levels, under full and imperfect information, on a simplified instance. To that extent, our work can be seen as complementary.

2. Literature review and contributions

In this section, we position our contribution with respect to the power system literature on decentralized power system oper-

ations and reserve provisioning, and to the equilibrium modeling literature that will provide us solution concepts that will guide our comparison of the efficiency of the decentralized coordination schemes with respect to the centralized scheme used as benchmark.

2.1. Hierarchical versus distributed power system operations

The need to integrate an increasing share of variable and unpredictable energy sources (such as wind and solar photovoltaic power) and the development of DERs is shifting the classical centralized market design to new market designs involving more decentralization and less communication between the agents (London, Chen, Vardi, & Wierman, 2017; Molzahn et al., 2017). In decentralized systems, operations/computations are performed in local markets and information based on local optimization problems output is shared only locally, generally in the form of messages exchanged between agents belonging to the same local market/energy community (Kraning, Chu, Lavaei, & Boyd, 2014; London et al., 2017; Molzahn et al., 2017). Two categories of decentralized designs emerge: hierarchical and distributed designs.

The first category, hierarchical design, involves agents in local markets which perform operations/computations independently and simultaneously and interact with other agents, known as centralized controllers, at a higher level in the hierarchical structure. Such a hierarchical interaction can be backwards in Stackelberg game settings (leader-follower type models) under the assumption that the leaders anticipate the rational reaction of the local market agents seen as followers (Dempe & Dutta, 2012; Dempe, Kalashnikov, Perez-Valdes, & Kalashnykova, 2015; Le Cadre, 2017; 2018; Xu et al., 2015; Yao, Adler, & Oren, 2008). In that case, the leaders incorporate explicitly in their optimization problems, the rational reaction functions of the followers. The closed form expression of these latter are obtained by solving first the followers' optimization problems at the lower level of the Stackelberg game, considering as fixed the decision variables of the leaders. The leaders, at the upper level, then incorporate the followers' rational reaction functions, expressed as functions of the leaders' decision variables only, directly in their optimization problems, therefore proceeding backwards. Alternatively, the hierarchical interaction can be forwards in case of decentralized control algorithms (Gregoratti & Matamoros, 2015; Matamoros, Gregoratti, & Dohler, 2012; Papavasiliou, 2017; Salehisadaghiani & Pavel, 2016), assuming that a centralized controller coordinates the outputs of the local optimization problems based on the locally reported information (Kraning et al., 2014; Sorin, Bobo, & Pinson, 2018).

The second category encompasses distributed designs where each agent communicates with its neighbors, but there is no centralized controller. This latter design is classically used to model peer-to-peer interactions in communication networks (Benjaafar, Kong, Li, & Courcoubetis, 2018; Le Cadre & Bedo, 2016; Sorin et al., 2018) or markets involving local energy communities (Gregoratti & Matamoros, 2015; Hvelplund, 2006; Le Cadre, Pagnoncelli, Homem-de Mello, & Beaudé, 2019). For this second category, community detection relying on structured data becomes an important issue.

Decentralized market designs may avoid costly communication between the agents. A drawback is that all the agents may not have access to the same information due to current privacy constraints which may limit data exchange. In the literature, a number of distributed approaches, characterized by the degree of access to information, have been developed. A distributed approach in which each local market solves a local optimization problem in an iterative fashion by exchanging some (limited) information with the others is proposed in Gregoratti and Matamoros (2015), Matamoros et al. (2012), and Sorin et al. (2018). From an infor-

¹ *evolvdSO*, FP7 funded project which aims to develop methodologies and tools for new DSO roles for efficient RES integration in distribution networks <https://www.edsoforsmartgrids.eu/projects/edso-projects/evolvdso/>

² *SmartNet*, H 2020 funded project, which aims to provide optimized instruments and modalities to improve the coordination between the grid operators at national and local level <http://smartnet-project.eu/>

³ Clean Energy Package, Eurelectric position <https://www.eurelectric.org/policy-areas/clean-energy-package/>

mation and communication technology (ICT) perspective, a fully decentralized market design provides a robust framework since if one node in a local market is attacked or in case of failures, the whole architecture should remain in place and information could find other paths to circulate from one point to another, avoiding malicious nodes/corrupted paths. From an algorithmic point of view, such a setting enables the implementation of algorithms that preserve privacy of the local market agents (requiring from them to not share more than their dual variables - e.g., local prices - updates). This also creates high computational challenges, especially if the number of local local markets/peers is high. Furthermore, such algorithms that derive from decomposition-coordination approaches (Gregoratti & Matamoros, 2015; Matamoros et al., 2012; Sorin et al., 2018) do not enable strategic behaviors of the local markets/peers. Depending on the market design, strategic behaviors of the local markets/peers can be quite complex; these latter can group together, self-organizing in complex coalitional structures, etc. The analysis of incentives for coalition formation and their relative stability is a current active research area in coalitional game theory (Le Cadre et al., 2019), which is out of the scope of this paper.

With the goal to typologize the optimization problems associated with new power system designs, we replace the contribution of this work with respect to Le Cadre (2017, 2018) and Le Cadre and Bedo (2016). In Le Cadre and Bedo (2016), we focus on the distribution level and consider a set of aggregators (service suppliers), which supply power to local energy communities. In the local energy community, the demand can be covered by the community own RES-based generation or by buying the missing quantity to an aggregator. The aggregator can buy energy from generators having a portfolio including a mix of RES-based and conventional generations, but faces double uncertainty coming from the uncertain demand level in the local energy community and the uncertain production levels from the generators. Local energy demand and RES-based generation learning strategies are implemented based on regret minimization. We prove in Le Cadre and Bedo (2016) that the aggregators have incentives to share their information and align on a single forecasting strategy instead of learning without communication with the other aggregators. In Le Cadre and Bedo (2016), we focus on establishing theoretical bounds on learning algorithms convergence, and distribution and transmission networks are not explicitly modeled. In Le Cadre (2017, 2018), we analytically compare centralized and decentralized market designs involving a national (global) and local market operators, strategic generators having market power and bidding sequentially in local markets, to determine which design is more efficient for the procurement of energy. In the centralized design, used as benchmark, the national market operator optimizes the exchanges between local markets and the generators' block bids. In the decentralized design, generators act as Stackelberg game leaders, anticipating the local market prices and the flows on the transmission lines. We determine that the decentralized design is as efficient as the centralized one with high share of RES-based generation and that information on local RES-based generation has a limited impact on the efficiency of the decentralized market design. In Le Cadre (2017, 2018), the transmission network is modeled through a simplified linear DC power flow model, which represents an approximation of Kirchhoff's laws. Distribution level constraints are ignored in the provision of DER-based generation. In this paper, similarly to Le Cadre and Bedo (2016) and Le Cadre (2017, 2018), a game-theoretic framework is introduced, and local energy communities are considered at the DSOs' levels. But, the focus is on the quantification of the inefficiency resulting from the decentralization of the TSO and DSO decisions. Furthermore, in contrast to our previous work, learning strategies of operational parameters and network topology that could be implemented by TSO and DSOs

are not considered, though they could constitute an interesting extension of this work. In this paper, distribution networks are modeled with more sophistication by approximating the distribution power flow equations using second order cone programming (SOCP) relaxation and explicitly incorporating operational network parameters in the constraints, which clearly complexify the TSO-DSO game resolution and is not amenable to analytical solution interpretation, by comparison with Le Cadre (2017, 2018).

2.2. Provision of reserves from DERs

Whereas the aforementioned literature focuses broadly on the dispatch of resources in power systems, there has been an increasing concern about the real-time balancing of the system through the activation of reserves (Caramanis, Ntakou, Hogan, Chakraborty, & Schoene, 2016; Gerard et al., 2017), which is the specific focus of this paper. Power system operations are characterized by a significant degree of uncertainty, which stems from forecast errors as well as component failures (Papavasiliou & Oren, 2013). The need of instantaneously balancing supply and demand in order to prevent system instabilities, compounded by the uncertainty involving real-time operations, implies that power systems need to carry a notable amount of spare capacity, referred to as reserve, which can be activated in short notice (a few seconds to minutes, depending on the specific definition of reserves in different markets) in order to ensure power balance. This capacity is reserved in sequential or multiproduct auctions in advance of real-time operations (Papavasiliou & Smeers, 2017). This process is referred to as reservation of reserve capacity. The capacity is then dispatched in real time in order to provide balancing services to the system. The latter process is referred to as reserve activation. Our interest in this paper is how this reserve can be offered by resources located at the distribution grid, given the inherent flexibility of these resources but also the significant degree of distributed renewable supply which suggests that local imbalances may better be dealt with by local flexible resources as opposed to centralized reserves made available by the TSO at the high-voltage grid.

From an optimization standpoint, the novel aspect of the provision of reserves from DERs is the fact that in so providing these reserve services, distributed resources need to respect power flow constraints for which a linear model is not adequate. Such constraints represent the fact that voltage limits on distribution nodes need to be respected, and current and complex power flow limits on lines also need to be within acceptable bounds.

An approach for overcoming this challenge was recently articulated by Caramanis et al. (2016). The major innovation of the authors is to introduce a reserve 'flow' variable which ensures the deliverability of reserves to the transmission system, while respecting the aforementioned voltage and flow constraints. The authors argue that a workable market for reserve provision from DERs requires the simultaneous clearing of real power, reactive power, and reserve capacity.

The representation of the aforementioned power flow constraints requires, in principle, a non-convex nonlinear model of power flow. Extensive research has recently been focused on developing and analyzing convex relaxations of the optimal power flow problem, with special focus on second order cone programming (SOCP) relaxations (Kocuk, Dey, & Sun, 2016) which provide an acceptable tradeoff between modeling accuracy and computational scalability. We apply specifically the branch flow SOCP relaxation (Farivar & Low, 2013), which was also employed in Caramanis et al. (2016). Our motivation for doing so is the fact that the relaxation is shown to be exact under fairly tenable conditions in radial networks. Given our focus on distribution systems, which are typically radial, we decide to follow (Caramanis et al., 2016) in employing the branch flow SOCP relaxation.

2.3. Equilibrium modeling of TSO-DSO coordination

Although the market design set forth by Caramanis et al. (2016) articulates a clear path towards the integration of DER in ancillary services markets, alternative hierarchical designs have been considered, with varying degrees of DSO involvement in market clearing, ranging from minimal to maximal DSO involvement in operations. These hierarchical designs are motivated by a variety of reasons, including communication bottlenecks, optimization bottlenecks, as well as institutional constraints (in particular, the fact that operators are not willing to share information and the trading of reactive power is not well understood among practitioners). Although computational and communication bottlenecks have largely been alleviated by recent work on distributed and peer to peer optimization of large-scale optimal power flow (Kraning et al., 2014; Peng & Low, 2014), institutional constraints prevail. Moreover, the introduction of binary activation variables in DER offers weakens the value of the aforementioned SOCP relaxations, and the resulting algorithms can then only be employed as heuristics. In either case, it then becomes relevant to investigate alternative models of TSO-DSO interactions that incorporate decentralized decision models.

Although it can be argued that a hierarchical organization of TSO-DSO coordination can closely or even perfectly replicate the outcome of full optimization if the market design is properly chosen, recent proposals of TSO-DSO coordination seem to permit the emergence of market incompleteness (Gerard et al., 2017). Such incompleteness results in operational inefficiencies, which may be inevitable in a realistic setting bound by institutional constraints. The relevant question, then, becomes which of these imperfect designs results in fewer efficiency losses. We focus on two specific schemes proposed in the literature (Gerard et al., 2017): shared balancing responsibility, and local (ancillary services) markets. We approach the first scheme as a simultaneous non-cooperative game, whereby we assume ‘bounded rationality’ on behalf of the TSO and DSO. Under ‘bounded rationality’, DSOs and TSO determine the reserves to activate on their networks simultaneously. We approach the second scheme as a sequential Stackelberg game, assuming DSOs with ‘rational expectations’ (Le Cadre, Papavasiliou, & Smeers, 2015; Yao et al., 2008) on the reaction of the TSO. In the first stage of the sequential Stackelberg game, the DSOs anticipate the future reaction of the follower (TSO) (which will play in the second stage) when determining the reserves to activate on their networks and send a signal based on this activation to the TSO, which reacts optimally in the second stage.

The solutions of both models of interaction are interpreted as Generalized Nash Equilibria. A Generalized Nash Equilibrium is the solution concept used to analyze non-cooperative games where the utility functions and the feasibility set of constraints of one agent depends on all the other agents’ actions (Facchini & Kanzow, 2007; Fudenberg & Tirole, 1991; Harker, 1981). In the context of our work, the utility functions of the DSOs and TSO are not coupled, in the sense that each utility function depends only on the agent’s own decision variables and random disturbances realizations, but the TSO and DSOs’ optimization problems are coupled through the shared physical constraints imposed by the interface nodes which belong to both transmission and distribution networks and the limits of the available resources. A similar approach has been employed by Oggioni, Smeers, Allevi, and Schaible (2012) in order to quantify the impact of the degree of coordination between two TSOs operating in interconnected areas in the case of congestion management. In our work, the focus is rather on balancing coordination in ancillary services markets. Also close to our work, the impact of different degrees of coordination both in time and in space (inter-regional) of day-ahead and balancing markets, operated by regional TSOs,

is studied in Delikaraoglou, Morales, and Pinson (2016). On the temporal dimension, cases of imperfect and full coordination for the market clearing are modeled as sequential and stochastic integrated optimization problems. Imperfect spatial coordination may arise in form of differentiated prices or quantities, in case TSOs can only activate resources that are physically located in their own networks. Market incompleteness, resulting from spatial quantity differentiation of even missing market for certain services, may constitute another source of imperfect coordination. In our work, we focus on TSO-DSO coordination, and introduce distribution network power flow and operational constraints.

2.4. Paper contributions

In this paper, we focus on three coordination schemes that present different ways of organizing the coordination between the TSO and DSOs in terms of activating reserves. We give an overview of these three coordination schemes:

- (i) The first scheme is a perfectly coordinated global market, where the TSO and DSOs jointly coordinate the activation of resources located in both the transmission as well as distribution grid, while taking into account both transmission and distribution grid constraints. The resulting co-optimization problem (Papavasiliou, 2017), formulated as a standard constrained optimization problem in Section 5.1, requires full coordination of the market parties and perfect information on the networks topology and operational parameters. It will be used as a benchmark to assess the performance of decentralized market designs.
- (ii) The second scheme is a decentralized market design with ‘bounded rational’ agents, in the sense of agents which do not anticipate the reactions of one another through an explicit reaction function. In this scheme, we assume that the DSO clears its local market by activating local reserves (solar PV power generations, demand response flexibilities) and assuming a desired injection by the TSO, taking into account local distribution grid constraints and offering a defined distribution grid capacity for the TSO needs. On its side, the TSO clears the global market by activating resources connected to the transmission grid and aggregated reserves activated by the DSOs, taking into account transmission grid constraints and distribution grid capacities allowed by the DSOs. We model this scheme as a (simultaneous) non-cooperative game in Section 5.2. The shared balancing responsibility game is analyzed under perfect and imperfect information on the operational parameters and network topology.
- (iii) The third scheme is a decentralized sequential market involving ‘rational expectation’ from the leaders. Under this design, the DSOs activate reserve strategically, with the aim of minimizing their activation costs, while forming rational expectations regarding the actions of the other DSOs and the TSO. Each DSO activates reserves taking into account local distribution grid constraints, and sends a signal based on their local activation to the TSO. The TSO then activates resources connected to the transmission grid and aggregated distribution system reserves, taking into account transmission grid constraints. In this coordination scheme, the DSOs act first, anticipating the behavior of the other DSOs and the TSO. This market design is formulated as a Stackelberg game involving DSOs (multi-leaders) and a TSO (follower) in Section 5.3.

The goal of this paper is to compare the efficiency of these three coordination schemes, relying on Generalized Nash Equilibrium as solution concept. There exists a wide literature in power

system economics that considers game theoretic approaches to analyze simple models of transmission markets (Borenstein, Bushnell, & Stoft, 2000; Boucher & Smeers, 2001; Oggioni et al., 2012; Yao et al., 2008). Following this trend, we propose in this paper to use game theoretical approaches to analyze transmission and distribution markets interactions, on simple models.

The implementation of these new TSO-DSO coordination schemes at the EU level, in a multi-area setting, will require to avoid the disclosure of sensitive intra-area data between the TSO and DSOs. Ideally, this coordination might be done by a centralized controller in case (i), or by the TSO and DSOs in cases (ii), (iii) assuming full information on operational parameters and network topologies, but the restriction imposed by new data privacy rules calls for methods with limited amount of information sharing (Halilbasic, Chatzivasileiadis, & Pinson, 2017). In a first approach, we will assume that the TSO and DSOs have full information on the operational parameters and network topology of all the TSO and DSOs, in cases (i), (ii), and (iii). Case (ii) will be refined by assuming that the agents have only imperfect information about the other agents' network characteristics, requiring to introduce forecasts of the state variables. The impact of incomplete information in the shared balancing responsibility game is formally quantified through the Price of Information (PoI), that we evaluate as a function of the coefficients of variation of the TSO and DSOs. We choose the PoI because it is an appropriate measure of the efficiency loss imposed by new data privacy rules, by comparison with the idealized paradigm (i) with full information and an integrated market operator coordinating the TSO and DSOs' decisions. We now highlight the practical and methodological contributions of our work.

From a practical point of view, our work provides game-theoretic approaches to model and interpret strategic interactions between TSO and DSO under full and imperfect information. Game-theoretic approaches have been applied in transmission markets (Borenstein et al., 2000; Boucher & Smeers, 2001; Oggioni et al., 2012) and our contribution is to extend what has been done by also considering distribution market. By using toy examples to illustrate our theoretical setting, we aim at furnishing quantitative insights on TSO-DSO coordination. Again, illustration through small test cases helped to understand crucial mechanisms when only transmission markets were examined (Borenstein et al., 2000; Boucher & Smeers, 2001; Oggioni et al., 2012). While centralized approaches where the TSO has control over all the network have to be dropped out (Kristov, De Martini, & Taft, 2016) and knowing the ongoing debate at EU level on the roles of TSOs and DSOs, the outcome of our game-theoretic models of TSO-DSO coordination schemes enters the scope of these issues of concern.

From a methodological point of view, we classify the power flow variables in state and decision (control) variables, the evolution of which determine the state outcome following the framework of (Başar & Olsder, 1999). To that purpose, we reformulate the power flow equations to identify linear mappings between the state and the decision (control) variables. We then prove formally that our power-flow equation reformulation is equivalent to the initial problem. The framework introduced by Başar and Olsder for dynamic games with information is not classically used in the bilevel optimization literature, the goal being here to consider the impact of information on the equilibrium output. This framework is mandatory for the introduction of more complex information structures for TSO-DSO interactions, as generically defined in Section 3.3, i)-ii). In the paper, we consider full and imperfect information. Finally, the framework is also mandatory for us to characterize analytical conditions for the existence and uniqueness of solutions for each coordination scheme, which constitutes another methodological contribution of our work.

2.5. Article organization

The article is organized as follows. In Section 3, we introduce the transmission and distribution network structures (3.1), the decision variables, states and utility functions of the agents (3.2), and the general setting of strategic form game with chance move (3.3). Power flow equations are reformulated in Section 4, highlighting relationships between state and decision variables. Coordination schemes are formulated as optimization problems in Section 5 and conditions for existence and uniqueness of solutions are detailed. In particular, the shared balancing responsibility game is analysed in a context of imperfect information on the operational parameters and network topology. The set of Generalized Nash Equilibria is spanned using random sampling and the impact of information is analyzed in Section 7. We conclude in Section 8.

Notation

Sets

$\mathbb{A}$	set of agents
a	generic agent
$\mathbb{A}_{-a}$	all the agents in $\mathbb{A}$ except a
$\mathbb{X}$	(full) state space
$\mathbb{U}_a$	control set of agent a
$\mathbb{I}_a$	information set of agent a
$\Gamma_a(\gamma_{-a})$	set of permissible strategies for agent a
Ω	action set of Nature
N	set of n local (distribution) markets
$\mathbb{DN}_k$	set of distribution nodes for local market k
$\mathbb{TN}$	set of transmission nodes
$\mathbb{L}$	set of transmission lines
$\mathbb{N}_\infty$	set of nodes at the interface
$\mathcal{C}_i$	set of children of node i
$\mathcal{F}S_a$	agent a constraint set
S_{GNE}	set of Generalized Nash Equilibria

Parameters and functions

$\pi_a(\cdot)$	agent a utility function
$p_n(\cdot)$	price in transmission node n
α_n, β_n	transmission node n price parameters
$p_i(\cdot)$	locational marginal price in distribution node i
α_i, β_i	locational marginal price $p_i(\cdot)$ parameters
$D_{n,i}$	day-ahead demand in node n, i
$C_n(\cdot)$	TSO activation cost in node n
$\bar{p}_n$	real power production cleared in day ahead in transmission node n
$\bar{p}_i^c$	real power consumption cleared in day ahead in distribution node i
$\bar{p}_i^g$	real power generation cleared in day ahead in distribution node i
$C_i^c(\cdot)$	demand-side activation cost in distribution node i
$C_i^g(\cdot)$	supply-side activation cost in distribution node i
$SW(\cdot)$	social welfare
κ	distribution tree depth
M_{adj}	adjacency matrix
M_{inc}	incidence matrix
G_i	shunt conductance of node i
B_i	shunt susceptance of node i
R_i	resistance of distribution line i
S_i	complex power flow limit over distribution line i
X_i	reactance of distribution line i
TC_l	flow thermal limit of transmission line l
R_i^g, R_i^c	reserve capacity for generator, consumer at distribution node i
$Q_i^{+/-}$	reactive power upper / lower capacity limit at distribution node i
R_n	reserve capacity at transmission node n
$g_a(\cdot)$	gradient of $\pi_a(\cdot)$
J_a	agent a Jacobian block matrix of $g_a(\cdot)$
μ^a, σ^a	mean, standard deviation of agent a forecast error on the set of agents in $\mathbb{A}_{-a}$ state variables

Variables

u_a	action of agent a
x	(full) state variable
y_a	observation of agent a
γ_a	agent a strategy

γ_{-a}	combination of strategies of all the agents in $\mathbb{A}$ except a
Δp_n	reserve activation in transmission node n
θ_n	bus angle of transmission node n
$\Delta p_i^g, \Delta p_i^c$	real power supply-side, demand-side reserve activation at distribution node i
q_i	net (unsigned) reactive power withdrawal at distribution node i
v_i	voltage magnitude squared at distribution or interface node i
f_l	real power flow over transmission line l
f_i^p	real power flow over distribution line i from i to its ancestor node
f_i^q	reactive power flow over distribution line i
l_i	current magnitude squared on distribution line i
$\Delta D_n(\omega)$	real power imbalance realization in node n
ϵ^a	agent a forecast error on the set of agents in $\mathbb{A}_{-a}$
	state variables

Conventions

M^T	transpose of matrix M
M^2	$M^T M$ for any matrix M
$\text{Ker}(M)$	matrix M Kernel
$\text{Im}(M)$	matrix M Image
$\text{card}(\mathbb{A})$	cardinality of set $\mathbb{A}$
x_n	n th element of vector x
$\text{diag}(x)$	square matrix having vector x on its diagonal and 0 for off-diagonal coefficients
I	Identity matrix

3. Market structure and agents

The focus of our paper is the activation of operating reserves. We therefore focus on the so-called balancing market, the role of which is to activate reserves. We will focus on upward reserve activation in our model, meaning that reserves are called to increase their real power production. Specifically, producers offer upward reserve by increasing their power production at a marginal cost of C_i^g , and consumers offer upward reserve by decreasing their power production at a marginal cost of C_i^c . The balancing market is preceded by a forward auctioning of energy and reserves⁴, which determines the set-point real power production/consumption of resources $\bar{p}$, as well as the amount of reserve capacity $R_i^{g/c}$ that each reserve resource can make available. In real time, random demand disturbances occur at the transmission network, $\Delta D_n(\omega)$, where n is a transmission node, as well as the distribution network, $\Delta D_i(\omega)$, where i is a distribution node.

We consider three categories of agents operating in the balancing market:

- DSOs which operate local distribution balancing markets.
- A TSO which operates a transmission balancing market.
- Fringe producers and consumers. These are represented on aggregate through marginal cost and marginal benefit functions respectively. For the sake of simplicity, we assume that only producers offer reserve at the transmission level, whereas both producers and consumers can offer reserve at the distribution level, thereby reflecting the fact that distribution systems may typically host flexible demand (e.g., electric vehicles).

In a centralized market, an integrated market operator contracts DERs directly from generators and consumers connected to the transmission and distribution grids, taking into account grid constraints. Such a centralized market can be formulated as a standard optimization problem under network constraints. This design will be used as a benchmark throughout the paper.

The motivation of a decentralized market structure, which we consider as an alternative to the above centralized design, is to minimize the amount of information that the TSO needs to account for when activating reserves. In a decentralized market

structure, each DSO seeks to activate local resources so as to resolve local grid issues at the lowest possible cost. In a sequential market design, each DSO activates its resources and communicates an aggregate signal to the TSO which indicates which of its local resources can also be activated by the TSO. The TSO, which is in charge of balancing market clearing at the transmission level and has no access to detailed distribution network information, determines which resources to activate in the transmission and distribution systems, taking the signal of the DSO as being fixed.

We propose to formalize such decentralized designs (either simultaneous or sequential) as a game in strategic form between the TSO and DSOs. Towards that end, we need to specify the network structure, the set of agents in the game, the set of options available to each agent, and the way that the payoffs of agents depend on the options that they choose.

3.1. Network structure

We consider a set $N := \{1, \dots, n\}$ of n local (distribution) markets. The set of distribution nodes in local market k is denoted as $\mathbb{DN}_k$, where $k \in N$.

The set of transmission nodes is denoted as $\mathbb{TN}$. The set of transmission lines is denoted as $\mathbb{L}$.

The set of nodes at the interface of the transmission and distribution grids is denoted as $\mathbb{N}_\infty$. Only transmission resources can bid in these nodes. We assume that there is no overlapping between the interface nodes of two DSOs, so that TSO can share resources with multiple DSOs, whereas two DSOs do not share any common resource. This network structure can be justified by assuming a 'local (distribution) market-to-grid' market structure, in which DSOs, by the intermediate of aggregators, provide services to a microgrid that is connected to a larger grid operated by the TSO (Parag & Sovacool, 2016). Other designs could be envisaged in an extension of our work, like 'peer-to-peer' models, in which DSOs interconnect directly with one another by the intermediate of aggregators, buying and selling energy services, e.g., sharing resources all together; 'islanded microgrids', in which local (distribution) markets behave as independent standalone microgrids, e.g., sharing resource neither with the TSO nor with the other DSOs, etc. (Parag & Sovacool, 2016).

The distribution networks follow a radial structure as pictured in Fig. 1. This means that each local network can be represented as a tree. Consequently, in each local market $k \in N$, we denote by i a line entering node $i \in \mathbb{DN}_k$.

3.2. Decision variables, states and utility functions

Our game-theoretic setting is inspired from the electrical engineering models (Farivar & Low, 2013; Papavasiliou, 2017), that we reformulate in the framework introduced by Başar and Olsder (1999). Adopting system theory terminology, we differentiate the variables into two categories: we call x the (full) state of the game, while $u_{\text{TSO}}, (u_{\text{DSO}, k})_{k \in N}$ are the TSO and DSOs' decision variables.

The TSO optimizes the reserve activation at each transmission node. Its decision variables are stored in a vector:

$$u_{\text{TSO}} = (\Delta p_n)_{n \in \mathbb{TN} \cup \mathbb{N}_\infty}.$$

The DSO optimizes the reserve activation, reactive power injection/consumption, and voltage at each distribution node. Similarly to the TSO, the decision variables of the DSO in the local market $k \in N$ are stored in a vector:

$$u_{\text{DSO}, k} = \begin{pmatrix} (\Delta p_i^g)_{i \in \mathbb{DN}_k} \\ (\Delta p_i^c)_{i \in \mathbb{DN}_k} \\ (q_i)_{i \in \mathbb{DN}_k} \\ (v_i)_{i \in \mathbb{DN}_k} \end{pmatrix}, \forall k \in N.$$

⁴ This auctioning may be performed simultaneously or sequentially, without any impact on our analysis.

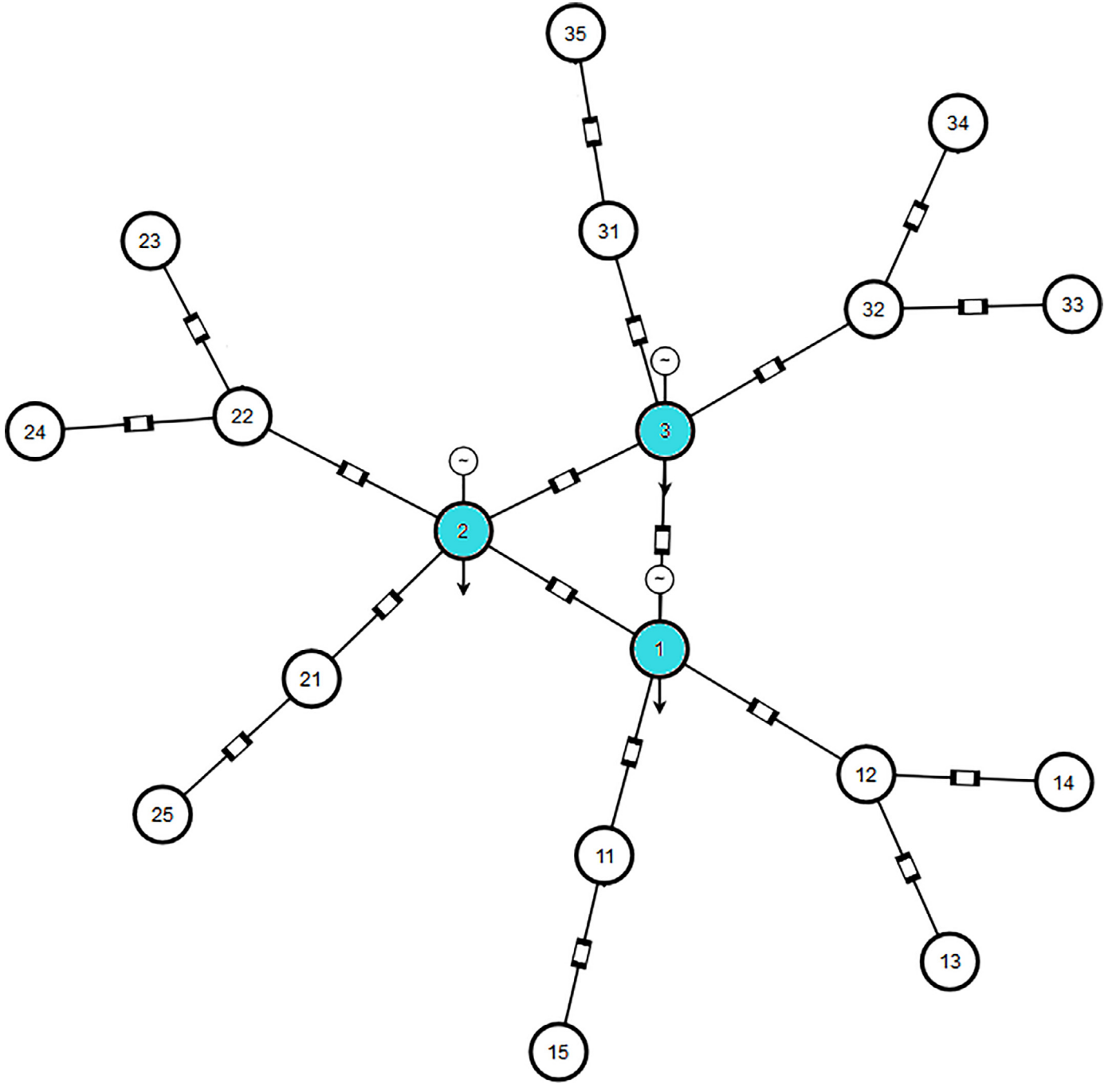


Fig. 1. Example of a meshed transmission network and distribution networks with radial structure. The transmission network consists of three nodes in blue (transmission network being restricted to its interface nodes only, $\mathbb{N}_\infty$), and each transmission node is the root of a distribution tree with 5 distribution nodes (source: NICTA NESTA test case Coffrin et al.). Note that there is no redundancy between the transmission and distribution node numbering. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The (full) state variable x contains state variable characterizing the TSO $(f_l)_{l \in \mathbb{L}}$, $(\theta_n)_{n \in \mathbb{T} \cup \mathbb{N}_\infty}$ and state variables characterizing the DSOs $(f_i^p)_{i \in \mathbb{DN}_k, k \in \mathbb{N}}$, $(f_i^q)_{i \in \mathbb{DN}_k, k \in \mathbb{N}}$, $(l_i)_{i \in \mathbb{DN}_k, k \in \mathbb{N}}$:

$$x = \begin{pmatrix} (f_l)_{l \in \mathbb{L}} \\ (f_i^p)_{i \in \mathbb{DN}_k, k \in \mathbb{N}} \\ (f_i^q)_{i \in \mathbb{DN}_k, k \in \mathbb{N}} \\ (l_i)_{i \in \mathbb{DN}_k, k \in \mathbb{N}} \\ (\theta_n)_{n \in \mathbb{T} \cup \mathbb{N}_\infty} \end{pmatrix}.$$

The analytical expressions of the state variables as functions of the TSO and DSOs' decision variables will be made explicit in Section 4. We can already say that the state variables contain:

- The power flows over transmission lines, $(f_l)_{l \in \mathbb{L}}$, which will be expressed through Eq. (25) as a linear matrix equation in the TSO and DSOs' decision variables.
- The real and reactive power flows over distribution lines, $(f_i^p)_{i \in \mathbb{DN}_k, k \in \mathbb{N}}$, and, $(f_i^q)_{i \in \mathbb{DN}_k, k \in \mathbb{N}}$, which will be obtained in Eqs. (18) and (19) as linear functions of the DSOs' decision variables.
- The current magnitude over distribution lines, $(l_i)_{i \in \mathbb{DN}_k, k \in \mathbb{N}}$, which will be expressed in Eq. (17) as a linear function of the DSOs' decision variables.
- The bus angles at transmission nodes, $(\theta_n)_{n \in \mathbb{T} \cup \mathbb{N}_\infty}$, which can be obtained as a linear function of the TSO and DSOs' decision variables in Eq. (31).

We assume that the energy unit price at each node in the transmission network $n \in \mathbb{TN} \cup \mathbb{N}_\infty$ is represented with $P_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that specifies how much the energy provider is willing to pay to supply one more power unit to the consumers. $D_n + \Delta D_n(\omega)$ is the aggregated demand in node n , D_n being the demand ordered in the day ahead market and $\Delta D_n(\omega)$ the imbalance in real time. We assume a linear relation between price in transmission and interface nodes and aggregated demand (Xu et al., 2015; Yao et al., 2008):

$$P_n(D_n + \Delta D_n(\omega)) = \alpha_n - \beta_n(D_n + \Delta D_n(\omega)),$$

with $\alpha_n > 0$ and $\beta_n \geq 0$.

Similarly to the transmission network, in the distribution nodes, we assume a linear relation between price at distribution nodes $P_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and the difference between the sum of real power consumption and imbalance, and real power reserve activation:

$$P_i(\bar{p}_i^c + \Delta D_i(\omega) - \Delta p_i^c) = \alpha_i - \beta_i(\bar{p}_i^c + \Delta D_i(\omega) - \Delta p_i^c),$$

with $\alpha_i > 0$ and $\beta_i \geq 0$.

Following Hobbs and Rijkers (2004) and Munoz, Wogrin, Oren, and Hobbs (2018), we model the TSO and DSOs as profit-maximizer agents which behave strategically by activating resources on their network, in real time (e.g., on the balancing market). The goal of the TSO is to maximize its profit defined as the difference between the revenue generated from the total demand at each of its nodes and the cost of reserve activation in the transmission network in real time. The objective function of the TSO takes the following form:

$$\begin{aligned} \pi_{TSO}(x, (u_a)_{a \in A}, \omega) \\ = \sum_{n \in \mathbb{TN} \cup \mathbb{N}_\infty} [P_n(D_n + \Delta D_n(\omega))(\bar{p}_n + \Delta p_n) - C_n(\Delta p_n)]. \end{aligned} \quad (1)$$

Similarly, the DSO operating in local balancing market $k \in N$ aims at maximizing its profit defined as the difference between the revenue paid by the energy provider to supply the total consumer demand at nodes in its distribution network and the cost resulting from the activation of generation reserves and demand response. Its objective function can be written as follows:

$$\begin{aligned} \pi_{DSO,k}(x, (u_a)_{a \in A}, \omega) \\ = \sum_{i \in \mathbb{DN}_k} [P_i(\bar{p}_i^c + \Delta D_i(\omega) - \Delta p_i^c)(\bar{p}_i^g + \Delta p_i^g) \\ - C_i^c(\Delta p_i^c) - C_i^g(\Delta p_i^g)]. \end{aligned} \quad (2)$$

Note that the agents' objective functions are independent of the state variables x , and that there is no coupling between their utility functions, in the sense that each utility function depends only on the agent's own decision variables and on the random disturbances realization ω . As such, in what follows, we will write $\pi_{TSO}(u_{TSO}, \omega)$ and $\pi_{DSO,k}(u_{DSO,k}, \omega)$, $\forall k \in N$. Also, note that the special case $\alpha_n = \beta_n = 0$, $\forall n \in \mathbb{TN} \cup \mathbb{N}_\infty$ and $\alpha_i = \beta_i$, $\forall i \in \mathbb{DN}_k$, $\forall k \in N$ coincides with a situation where TSO and DSOs minimize their activation costs (Mezghani, Papavasiliou, & Le Cadre, 2018).

We will assume that the cost functions are strictly convex and of the form: $C_n(\Delta p_n) = c_n \Delta p_n^2$, $c_n > 0$, $\forall n \in \mathbb{TN} \cup \mathbb{N}_\infty$, $C_i^c(\Delta p_i^c) = c_i^c (\Delta p_i^c)^2$, $c_i^c > 0$, $C_i^g(\Delta p_i^g) = c_i^g (\Delta p_i^g)^2$, $c_i^g > 0$, $\forall i \in \mathbb{DN}_k$, $\forall k \in N$.

3.3. Strategic form game with chance moves

The game incorporates a chance move, with possible alternatives for Nature being $\omega \in \Omega$. At the beginning of the game, Nature picks a state defined by a realization of the uncertain demand disturbances at transmission nodes and real time imbalances at distribution and interface nodes. Nature actions influence the evolution of the state of the game. In each state of Nature, the TSO and DSOs, whose information sets include the state of Nature,

compete through a discrete time game involving the N DSOs, each operating a local (distribution) market, and the TSO (Başar & Olsder, 1999), which involves:

- A set of agents $\mathbb{A} := \{DSO_k, k \in N\} \cup \{TSO\}$. A generic agent will be denoted as $a \in \mathbb{A}$.
- An infinite set $\mathbb{X}$, called the (full) state space of the game, to which the state of the game x belongs.
- An infinite set $\mathbb{U}_a$, defined for each agent $a \in \mathbb{A}$, which is called the action (control) set of agent $a \in \mathbb{A}$. Its elements are the permissible actions u_a of agent a .
- A set $\mathbb{Y}_a$, defined for each agent $a \in \mathbb{A}$, called the observation set of agent a , to which the observation y_a of agent a belongs.
- A function $h_a : \mathbb{X} \rightarrow \mathbb{Y}_a$, defined for each agent $a \in \mathbb{A}$, so that

$$y_a = h_a(x), a \in \mathbb{A}, \quad (3)$$

which is the state-measurement (observation) equation of agent a concerning the value of the (full) state x .

- A finite set $\mathbb{I}_a$ defined for each $a \in \mathbb{A}$ as a subset of $\{y_{a'}, a' \in \mathbb{A}\}$ which determines the information gained by agent $a \in \mathbb{A}$. Specification of $\mathbb{I}_a$ characterizes the information structure of agent a .
- A pre-specified class $\Gamma_a(\gamma_{-a})$ of mappings $\gamma_a : \mathbb{I}_a \rightarrow \mathbb{U}_a$ which are the permissible strategies of agent a . All the other agents have the possibility to influence agent a strategy through γ_{-a} . The class $\Gamma_a(\gamma_{-a})$ of all such mappings γ_a is the strategy set (space) of agent a .
- A finite or infinite set Ω , which denotes the action set of Nature. Any permissible action ω of Nature is an element of Ω .
- A utility function $\pi_a : \mathbb{X} \times \times_{a' \in \mathbb{A}} \mathbb{U}_{a'} \times \Omega \rightarrow \mathbb{R}$ defined for each agent $a \in \mathbb{A}$. Eqs. (1) and (2) give explicit expressions for TSO and DSOs' utility functions.

In the following, we will consider two information structures:

- Perfect information $\mathbb{I}_a = \{x\}$, $\forall a \in \mathbb{A}$,
- and, imperfect information $\mathbb{I}_a = \{y_a = h_a(x)\}$, $\forall a \in \mathbb{A}$.

For each fixed $\text{card}(\mathbb{A})$ -tuple of permissible strategies $\{\gamma_a \in \Gamma_a(\gamma_{-a}), a \in \mathbb{A}\}$, the strategic and extensive form game descriptions lead to a unique set of vectors $\{u_a := \gamma_a(I_a), I_a \in \mathbb{I}_a, a \in \mathbb{A}\}$ because of the causal nature of the information structure (Başar & Olsder, 1999). By abuse of notation, in the rest of the paper, we will refer to u_a and $\gamma_a(\cdot)$ without distinction.

4. Reformulation of power flow constraints

Later in the paper, we optimize the strategies of the agents so that the closed-loop system gives rise to an equilibrium. This requires first to introduce explicit relationships between state and decision variables (Başar & Olsder, 1999). To that purpose, we propose a reformulation of the power flow equations.

4.1. DSO distribution network power flows

For the sake of simplicity in what follows, we focus on a single distribution network $\mathbb{DN}$, but this can be easily extended to N independent local (distribution) markets coupled by the intermediate of the global grid, as assumed in Section 3.1.

In this section we will derive the DSO's active and reactive power flows and current magnitude squared as linear functions of the DSO's decision variables. We first state the SOCP relaxation of the power flow equations (Farivar & Low, 2013):

$$\begin{aligned} f_i^p - \sum_{j \in \mathcal{C}_i} f_j^p &= \bar{p}_i^g + \Delta p_i^g - \bar{p}_i^c + \Delta p_i^c - G_i v_i \\ &- \sum_{j \in \mathcal{C}_i} l_j R_j - \Delta D_i(\omega), \forall i \in \mathbb{DN}, \end{aligned} \quad (4)$$

$$f_i^q - \sum_{j \in \mathcal{C}_i} f_j^q = -q_i + B_i v_i - \sum_{j \in \mathcal{C}_i} l_j X_j, \forall i \in \mathbb{DN}, \quad (5)$$

$$v_i = v_{j|i \in \mathcal{C}_j} + 2(R_i f_i^p + X_i f_i^q) - l_i (R_i^2 + X_i^2), \forall i \in \mathbb{DN}, \quad (6)$$

$$(f_i^p)^2 + (f_i^q)^2 \leq S_i^2, \forall i \in \mathbb{DN}, \quad (\lambda_2)$$

$$(f_i^p)^2 + (f_i^q)^2 \leq v_i l_i, \forall i \in \mathbb{DN}, \quad (\lambda_3)$$

$$(f_i^p - l_i R_i)^2 + (f_i^q - l_i X_i)^2 \leq S_i^2, \forall i \in \mathbb{DN}. \quad (\lambda_4)$$

The variables of the so-called branch flow model presented above are real $(\Delta p_i^g, \Delta p_i^c)$ and reactive (q_i) power injections, real (f_i^p) and reactive (f_i^q) power flows, and the magnitude squared of voltage phasors (v_i) and angle phasors (l_i) . Constraints (4) and (5) express the balance of real and reactive power at each distribution node. Constraints (6) and (7), introduced by Farivar and Low (2013), correspond to the SOCP relaxation of the power flow constraints. Constraints (8) and (9) impose complex power flow limits along each distribution node.

Let $M_{adj} \in \text{Mat}(\mathbb{DN}, \mathbb{DN})$ denote the adjacency matrix of the oriented graph composed by the distribution network $\mathbb{DN}$, where $M_{adj}(i, j) = 1$ if there exists an oriented link connecting node $i \in \mathbb{DN}$ to node $j \in \mathbb{DN}$, and 0 otherwise. Then we can rewrite the system of Eqs. (4) into the matrix form:

$$(I - M_{adj})f^p = \Delta p^g + \Delta p^c - \text{diag}(G)v - M_{adj}\text{diag}(R)l + \bar{p}^g - \bar{p}^c - \Delta D(\omega), \quad (10)$$

where, by convention, $\text{diag}(G)$ is the diagonal matrix having on its diagonal elements $G_i, i \in \mathbb{DN}$, and $\text{diag}(R)$ is the diagonal matrix having on its diagonal elements $R_i, i \in \mathbb{DN}$.

Since the distribution network is an acyclic graph (more precisely a tree), there exists an integer $\kappa \in \mathbb{N}^*$ such that all paths in the distribution network have a length strictly lower than κ . Then we can prove easily that $M_{adj}^\kappa = 0$ (Hu & Shing, 2002). As a result, we have the following relation:

$$(I + \underbrace{\sum_{i=1}^{\kappa-1} M_{adj}^i}_M)(I - M_{adj}) = I, \quad (11)$$

and so, using Eq. (11) in Eq. (10) we can conclude that:

$$f^p = (I + M)(\Delta p^g + \Delta p^c - \text{diag}(G)v + \bar{p}^g - \bar{p}^c - \Delta D(\omega)) - M\text{diag}(R)l, \quad (12)$$

where $M := \sum_{i=1}^{\kappa-1} M_{adj}^i$ and using the relation $M = (I + M)M_{adj}$ inherited from Eq. (11).

Similarly, the reactive power flows can be derived from the system of Eqs. (5), that we recall below (Farivar & Low, 2013; Papavasiliou, 2017):

$$f_i^q - \sum_{j \in \mathcal{C}_i} f_j^q = -q_i + B_i v_i - \sum_{j \in \mathcal{C}_i} l_j X_j, \quad (13)$$

for each node i in the distribution network $\mathbb{DN}$. From Eqs. (5) and (11), we can then obtain the closed form of f^q :

$$f^q = (I + M)(-q + \text{diag}(B)v - M\text{diag}(X)l). \quad (14)$$

By replacing f_i^p and f_i^q by the expressions found in Eqs. (12) and (14), we can rewrite the system of Eqs. (6) as:

$$\begin{aligned} (I - M_{adj}^T + 2\Phi(R)\text{diag}(G) - 2\Phi(X)\text{diag}(B))v - Hv_\infty \\ = 2\Phi(R)(\Delta p^g + \Delta p^c + \bar{p}^g - \bar{p}^c - \Delta D(\omega)) - 2\Phi(X)q \\ - (\text{diag}(R)(2M + I)\text{diag}(R) + \text{diag}(X)(2M + I)\text{diag}(X))l, \end{aligned} \quad (15)$$

where $\Phi(\text{Mat}) := \text{diag}(\text{Mat})(I + M)$ for any matrix $\text{Mat} \in \text{Mat}(\mathbb{DN}, \mathbb{DN})$ and $H = ((\mathbf{1}_{i \in \mathcal{C}_j})_{i \in \mathbb{DN}, j \in \mathbb{N}_\infty})$ is the matrix that contains

as i, j -element $\mathbf{1}_{i \in \mathcal{C}_j} = 1$ if $i \in \mathcal{C}_j$ and 0 otherwise, for $i \in \mathbb{DN}$ and $j \in \mathbb{N}_\infty$. Note that the voltage is assumed to be the same at all the nodes in the interface, i.e., $v_i = v_\infty, \forall i \in \mathbb{DN}$.

Eq. (15) can be rewritten to give:

$$\begin{aligned} \Psi l = 2\Phi(R)(\Delta p^g + \Delta p^c + \bar{p}^g - \bar{p}^c - \Delta D(\omega)) \\ - 2\Phi(X)q - (I - M_{adj}^T + 2\Phi(R)\text{diag}(G) \\ - 2\Phi(X)\text{diag}(B))v + Hv_\infty. \end{aligned} \quad (16)$$

where we set $\Psi := (\text{diag}(R)(2M + I)\text{diag}(R) + \text{diag}(X)(2M + I)\text{diag}(X))$.

Proposition 1. Ψ is invertible.

Proof of Proposition 1. Using properties of the determinant operator, we have that: $\det(\Psi) \geq (\det(\text{diag}(R)^2) + \det(\text{diag}(X)^2))\det(2M + I)$. In addition, by def-

>0 by definition of R and X

inition of M , we have the relation: $\det(M) \geq \sum_{i=1}^{\kappa-1} \det(M_{adj})^i$. Since $\mathbb{DN}$ is an oriented tree, its adjacency matrix M_{adj} contains a line of zeros because there is no child connected to its leaves. This implies that $\det(M_{adj}) = 0$ and therefore that $\det(\Psi) \geq (\det(\text{diag}(R)^2) + \det(\text{diag}(X)^2)) > 0$. We conclude that Ψ is invertible. $\square$

As a corollary of Proposition 1, we can express l as a closed form in $\Delta p^g, \Delta p^c, q$, and v :

$$l = 2\Psi^{-1}\Phi(R)(\Delta p^g + \Delta p^c + \bar{p}^g - \bar{p}^c - \Delta D(\omega)) - 2\Psi^{-1}\Phi(X)q - \Upsilon v + \Psi^{-1}Hv_\infty, \quad (17)$$

where we set $\Upsilon := \Psi^{-1}(I - M_{adj}^T + 2\Phi(R)\text{diag}(G) - 2\Phi(X)\text{diag}(B))$.

By substitution of l obtained in Eq. (17) in f^p, f^q , we obtain expressions that depend only on $\Delta p^g, \Delta p^c, q$ and v :

$$\begin{aligned} f^p = [(I + M) - 2M\text{diag}(R)\Psi^{-1}\Phi(R)](\Delta p^g + \Delta p^c + \bar{p}^g - \bar{p}^c \\ - \Delta D(\omega)) + 2M\text{diag}(R)\Psi^{-1}\Phi(X)q \\ + [M\text{diag}(R)\Upsilon - (I + M)\text{diag}(G)]v \\ - M\text{diag}(R)\Psi^{-1}Hv_\infty, \end{aligned} \quad (18)$$

and

$$\begin{aligned} f^q = [-(I + M) + 2M\text{diag}(X)\Psi^{-1}\Phi(X)]q \\ + [(I + M)\text{diag}(B) + M\text{diag}(X)\Upsilon]v \\ - 2M\text{diag}(X)\Psi^{-1}\Phi(R)(\Delta p^g + \Delta p^c + \bar{p}^g - \bar{p}^c - \Delta D(\omega)) \\ - M\text{diag}(X)\Psi^{-1}Hv_\infty. \end{aligned} \quad (19)$$

By using the previous closed-form expressions of f^p, f^q and l , we can obtain easily an equivalent system of equations depending only on $\Delta p^g, \Delta p^c, q$ and v in matrix form. We can therefore conclude that by deciding on variables $\Delta p^g, \Delta p^c, q$ and v , the DSO is fixing all the parameters on its network. So we can see $\Delta p^g, \Delta p^c, q$ and v as decision variables, and the power flows and current magnitude as state variables. The results are summarized in the proposition below:

Proposition 2. Eqs. (4)–(6) enable to express DSO's active and reactive power flows f^p, f^q , and current magnitude squared l , as linear functions of the DSO's decision variables u_{DSO} defined in Section 3.2.

The physical intuition is that injections of real and reactive power into a circuit uniquely determine the full state of the circuit. But our results prove that we need to introduce the magnitude squared of the voltage phasors v_i in the action variables because the real and reactive power injections alone do not uniquely determine the state of the circuit. In particular, a unique choice of real and reactive power injections may imply multiple solutions of the constraints (4)–(9) (Papavasiliou, 2017). Instead, as we have

just demonstrated, additionally specifying v_i implies a unique value for the remaining state variables of the circuit.

Following Proposition 2, the DSO distribution network power flows can be reformulated as a simplified system of equations:

Proposition 3. The system of Eqs. (4)–(9) is equivalent to the system of Eqs. (7)–(9), where active and reactive power flows f^p , f^q , and current magnitude squared l , are replaced by their linear expressions as functions of the DSO's decision variables u_{DSO} . The closed-form expressions of f^p , f^q , l as functions of u_{DSO} are defined in Eqs. (17)–(19).

4.2. TSO transmission network power flows

Similar to the development of the previous section, in this section we derive the TSO state variables as functions of the TSO and DSOs' decision variables⁵. The linearized power flow approximation of the TSO transmission constraints can be expressed as follows:

$$(\bar{p}_n + \Delta p_n) + \left(\sum_{l|l=(m,n)} f_l - \sum_{l|l=(n,m)} f_l \right) = D_n + \Delta D_n^{TSO}(\omega), \forall n \in \mathbb{TN}, \quad (20)$$

$$(\bar{p}_n + \Delta p_n) + \sum_{j \in \mathcal{C}_n} (f_j^p - l_j R_j) + \left(\sum_{l|l=(m,n)} f_l - \sum_{l|l=(n,m)} f_l \right) = D_n + \Delta D_n^{TSO}(\omega) + G_n v_n, \forall n \in \mathbb{N}_\infty, \quad (21)$$

$$f_{n,m} = B_{n,m}(\theta_n - \theta_m), \forall l = (n, m) \in \mathbb{L}, \quad (22)$$

$$-TC_l \leq f_l \leq TC, \forall l \in \mathbb{L}. \quad (23)$$

In this linear approximation of power flow constraints, the variables are real power injections (Δp), bus voltage phase angles (θ) and real power flows along transmission lines (f). Reactive power flows are ignored in this approximation, following standard practice in the literature. Constraints (20), (21) express the TSO power balance constraints at each transmission node. Constraint (22) expresses the relationship between bus angles and power flows. Constraint (23) expresses the capacity constraints of transmission lines.

We define an interface connection matrix as:

$$M_{int} := ((\mathbf{1}_{j \in \mathcal{C}_i} \times \mathbf{1}_{i \in \mathbb{N}_\infty})_{i \in \mathbb{TN} \cup \mathbb{N}_\infty, j \in \mathbb{DN}}),$$

with its i, j element being equal to $\mathbf{1}_{j \in \mathcal{C}_i} \times \mathbf{1}_{i \in \mathbb{N}_\infty}$. It is equal to 1 if $j \in \mathcal{C}_i$ and $i \in \mathbb{N}_\infty$, i.e., if node i is an interface node and node j is one of its children, and 0 otherwise.

We also introduce the incidence matrix, M_{inc} , of size $\text{card}(\mathbb{TN} \cup \mathbb{N}_\infty) \times \text{card}(\mathbb{L})$, so that $M_{inc}(n, l) = -1$ if link l is leaving node n , $M_{inc}(n, l) = 1$ if link l is entering node n , and $M_{inc}(n, l) = 0$ otherwise.

Using these matrices, we can rewrite the TSO power balance constraints in matrix form as follows:

$$\bar{p} + \Delta p + M_{inc} f + M_{int} (f^p - \text{diag}(R) l) = D + \Delta D^{TSO}(\omega) + \text{diag}(G_\infty) v_\infty.$$

Using Eq. (12), $f^p - \text{diag}(R) l$ can be expressed in closed form as a function of the decision variables of the TSO and the DSOs:

$$f^p - \text{diag}(R) l = (I + \bar{M})[(I - 2\text{diag}(R)\Psi^{-1}\Phi(R))(\Delta p^g + \Delta p^c + \bar{p}^g - \bar{p}^c - \Delta D(\omega)) + 2\text{diag}(R)\Psi^{-1}\Phi(X)q + (\text{diag}(R)\Upsilon - \text{diag}(G))v - \text{diag}(R)\Psi^{-1}Hv_\infty]. \quad (24)$$

Then the TSO power balance constraints can be summarized through the single matrix equation:

$$\begin{aligned} & \bar{p} + \Delta p + M_{inc} f \\ & + M_{int} (I + \bar{M})[(I - 2\text{diag}(R)\Psi^{-1}\Phi(R))(\Delta p^g + \Delta p^c) \\ & + \text{diag}(R)\Psi^{-1}(2\Phi(X)q - Hv_\infty) + (\text{diag}(R)\Upsilon - \text{diag}(G))v] \\ & - \text{diag}(G_\infty)v_\infty \\ & = D + \Delta D^{TSO}(\omega) - M_{int} (I + \bar{M})[(I - 2\text{diag}(R)\Psi^{-1}\Phi(R))(\bar{p}^g \\ & - \bar{p}^c - \Delta D(\omega))]. \end{aligned}$$

By reshuffling the preceding equation, it is possible to express $M_{inc} f$ as a function of the decision variables of the TSO and DSOs:

$$\begin{aligned} M_{inc} f &= -\bar{p} - \Delta p \\ & - M_{int} (I + \bar{M})[(I - 2\text{diag}(R)\Psi^{-1}\Phi(R))(\Delta p^g + \Delta p^c) \\ & + \text{diag}(R)\Psi^{-1}(2\Phi(X)q - Hv_\infty) + (\text{diag}(R)\Upsilon - \text{diag}(G))v] \\ & + D + \Delta D^{TSO}(\omega) - M_{int} (I + \bar{M})[(I - 2\text{diag}(R)\Psi^{-1}\Phi(R))(\bar{p}^g \\ & - \bar{p}^c - \Delta D(\omega)) + \text{diag}(G_\infty)v_\infty. \end{aligned} \quad (25)$$

We have thus derived an explicit linear relation between the transmission flows, f , and the decision variables of the TSO and DSOs.

Since M_{inc} is in general not invertible, it is not possible to obtain a closed form expression of f as a function of TSO and DSO's decision variables in Eq. (25).

To express the TSO feasible set as a function of the TSO and DSO's decision variables, we introduce an additional reformulation. We define the function:

$$\begin{aligned} \rho(\Delta p, \Delta p^g, \Delta p^c, q, v) \\ := -\Delta p - M_{int} (I + \bar{M})[(I - 2\text{diag}(R)\Psi^{-1}\Phi(R))(\Delta p^g + \Delta p^c) \\ + 2\text{diag}(R)\Psi^{-1}\Phi(X)q + (\text{diag}(R)\Upsilon - \text{diag}(G))v], \end{aligned} \quad (26)$$

which is a linear combination of Δp , Δp^g , Δp^c , q and v . We further define the function:

$$\begin{aligned} \rho_C := \bar{p} - M_{int} (I + \bar{M})\text{diag}(R)\Psi^{-1}Hv_\infty - \text{diag}(G_\infty)v_\infty \\ - D - \Delta D^{TSO}(\omega + M_{int} (I + \bar{M})[(I - 2\text{diag}(R)\Psi^{-1}\Phi(R))(\bar{p}^g \\ - \bar{p}^c - \Delta D(\omega))]. \end{aligned} \quad (27)$$

such that $\rho(\cdot) - \rho_C = M_{inc} f$, following Eq. (25).

Define the matrix $B^\# \in \text{Mat}(\mathbb{L}, \mathbb{TN} \cup \mathbb{N}_\infty)$ such that for any $l = (n, m) \in \mathbb{L}$, $B^\#(l, n) = B_{n,m}$, $B^\#(l, m) = -B_{m,n}$ and $B^\#(l, n') = 0, \forall n' \in \mathbb{TN} \cup \mathbb{N}_\infty, n' \neq n, n' \neq m$.

We now arrive to the following result, where we reformulate the TSO feasible set as a function of the TSO and DSO's decision variables only:

Proposition 4. The TSO feasible set is equivalently described by the following set of inequalities:

$$0 \leq \Delta p \leq R_{\mathbb{TN} \cup \mathbb{N}_\infty}, \quad (\lambda_t^-, \lambda_t^+) \quad (28)$$

$$\rho(\Delta p, \Delta p^g, \Delta p^c, q, v) - \rho_C \in \text{Im}(\text{Minc} B^\#), \quad (\lambda_0) \quad (29)$$

$$\begin{aligned} -|M_{inc}|TC + \rho_C &\leq \rho(\Delta p, \Delta p^g, \Delta p^c, q, v) \\ &\leq |M_{inc}|TC + \rho_C. \quad (\lambda_1^-, \lambda_1^+) \end{aligned} \quad (30)$$

Proof of Proposition 4. The goal in this proof is to demonstrate that:

$$\begin{cases} f = B^\# \theta, \\ -TC \leq f \leq TC, \\ M_{inc} f = \rho - \rho_C \end{cases} \Leftrightarrow \begin{cases} (29), \\ (30). \end{cases}$$

The TSO power flow balance constraints have led us to the reformulation (25). With this reformulation, the feasible set of the TSO

⁵ Note that, because of the presence of interface nodes, both TSO and DSO decisions impact the TSO state variables. This couples the TSO and DSOs' optimization problems.

is delimited by the generator capacity constraints, the relationship between the bus angles and the power flows, and the transmission line capacity constraints. The generation capacity constraints correspond to Eq. (28), where R_n is the available reserve capacity of the generator located in transmission node n . Using the definition of B^π , the relationship between bus angles and power flows can be rewritten as $f = B^\pi \theta$. In addition, Eq. (25) implies a set of linear equations between decision variables and the vector of bus angles θ . More precisely:

$$M_{inc} B^\pi \theta = \rho(\Delta p, \Delta p^g, \Delta p^c, q, v) - \rho_C. \quad (31)$$

Therefore, there exists at least one feasible vector of bus angles solution of Eq. (31) for a given set of decision variables $\Delta p, \Delta p^g, \Delta p^c, q$ and v if, and only if, $\rho(\cdot) - \rho_C$ is in the image of $M_{inc} B^\pi$. Since the image of a matrix is orthogonal to the kernel of its transposed matrix and two orthogonal spaces have an intersection reduced to the vector 0, there exists one feasible vector of bus angles solution of Eq. (31) if, and only if, $\rho(\cdot) = \rho_C$ or $(\rho(\cdot) - \rho_C) \notin \text{Ker}(B^{\pi T} M_{inc}^T)$. The case $\rho(\cdot) = \rho_C$ would correspond to the very unlikely case where all transmission nodes do not need power flows from the transmission network to be balanced in real time, because from Eq. (31) we have that $\rho - \rho_C = M_{inc} f$, so $\rho - \rho_C = 0$ would imply $M_{inc} f = 0$. We will assume that this will not be the case. Then there exist bus angles solutions of Eq. (31) if $B^{\pi T} M_{inc}^T (\rho(\cdot) - \rho_C) \neq 0$, which is captured by condition (29). This latter condition can be rewritten under the form: $(B^{\pi T} M_{inc}^T (\rho(\Delta p, \Delta p^g, \Delta p^c, q, v) - \rho_C))^2 > 0$.

Now, we want to prove that Eq. (30) holds. From the capacity constraints (23) we have that:

$$-|M_{inc}|TC \leq M_{inc} f \leq |M_{inc}|TC, \quad (32)$$

where the absolute value of M_{inc} is taken element wise. By substitution of Eq. (25) in Eq. (32), we obtain lower and upper bounds on $\rho(\cdot)$ so that we have (30). This proves that the TSO feasible set is included in the set described by (28)–(30).

To prove that the reciprocal is true, we reason by contradiction. Suppose that we have found $\Delta p, \Delta p^g, \Delta p^c, q$ and v which verify (28)–(30). Eq. (29) means that there exists a vector of bus angles θ^* such that $M_{inc} B^\pi \theta^* = \rho(\cdot) - \rho_C$. By taking $f_0 = B^\pi \theta^*$ we obtain that all vectors f satisfying $M_{inc} f = \rho(\cdot) - \rho_C$ can be rewritten as $f = f_0 + k$ with $k \in \text{Ker}(M_{inc})$. Let assume that there is no vector $k \in \text{Ker}(M_{inc})$ such that $-TC \leq f_0 + k \leq TC$, i.e., for all $k \in \text{Ker}(M_{inc})$ either $-TC > f_0 + k$ or $f_0 + k > TC$. This is in contradiction with Eq. (30), because according to Weierstrass extreme value Theorem every continuous application on a closed interval is bounded and reaches its lower and upper bounds on this interval. As such $f_0 + k$ should belong to the interval $[-|M_{inc}|TC; |M_{inc}|TC]$. $\square$

As mentioned at the beginning of Sections 3.1 and 4.1, the local (distribution) markets are operated independently and are only interconnected through the global grid, sharing no common resources with one another. In the description of the optimization problems associated with each coordination scheme, we focus on a single local (distribution) market. Considering multiple DSOs operating local (distribution) markets requires to replicate N times the DSO optimization problems with parameters calibrated on each local (distribution) market. The TSO optimization problem remains the same except that it incorporates decision variables from all the DSOs and TSO in its coupling constraints. The resulting multi-leader Stackelberg game can be formulated as an equilibrium problem with equilibrium constraints (EPEC) (Le Cadre, 2018). KKT conditions would just incorporate new conditions corresponding to the new DSOs' decision variables and their own operational and power flow constraints, therefore increasing the computational complexity of the EPEC. Furthermore, difficulties might arise in case (a) the lower-level optimization problem admits multiple

solutions, requiring to consider optimistic or deterministic approaches (Dempe & Dutta, 2012; Dempe et al., 2015); in case (b), the leaders (DSOs) have different valuations of the dual variables associated with the follower (TSO) complementarity constraints (Leyffer & Munson, 2010). Regarding case (a), we will prove in Section 5.3, that the follower's problem admits a unique solution. Regarding case (b), various approaches exist like introducing normalized equilibrium as solution concept (Delikaraoglou et al., 2016), or duplicate the follower's complementary constraints in the KKT system of equations (Leyffer & Munson, 2010). Though interesting, both approaches generate difficulties of interpretation, and the second approach can be computationally quite expensive to deal with, especially if a large number of DSOs is involved. So, to keep the economic interpretations as simple as possible, we focus throughout this article on coordination schemes involving a single DSO. As explained, extensions to a multi-leader-common-follower game is straightforward, assuming that the follower's complementarity constraints are shared among the leaders.

5. TSO-DSO coordination schemes

In the formulation of the three coordination schemes that we will detail in Sections 5.1–5.3, we replace the state variables f^p, f^q, l by their linear mappings in the TSO and DSO's decision variables obtained in Eqs. (17)–(19), and recalled in Proposition 2. Mathematical description of TSO, DSO feasibility sets can be simplified relying on Propositions 3 and 4. We will come back to these simplifications later on in the text.

5.1. Centralized co-optimization problem

For coordination scheme (i), we start by discussing the motivation for its implementation in 5.1.1, then we detail its mathematical formulation in 5.1.2, before giving conditions for existence and uniqueness of a social welfare optimum in 5.1.3. KKT conditions that will be used to compute the social welfare optimum in the case study are explicitly given in 5.1.4.

5.1.1. Motivation

The centralized co-optimization problem aims at optimally coordinating the dispatch of all resources at both transmission and distribution levels. There is one common market operated jointly by the TSO and DSOs or by an integrated market operator acting as a coordinator, for both resources connected at transmission and distribution levels. This coordination scheme is modeled as a standard constrained optimization problem with perfect information on the state variables. We will use this scheme as a benchmark compared to our decentralized schemes. Indeed, due to privacy issues and practically large-scale size of the transmission and distribution networks, this scheme would be extremely difficult to implement in a real-life setting. The latter aspect requires the development of efficient decomposition algorithms capable to handle complex and large scale optimization problems, and to characterize the associated solution concepts as well as to deal with convergence issues. Complex bidding in the transmission and distribution nodes might also incorporate integer variables in the optimization problem, therefore resulting in challenging problems for the operations research community. Fig. 2 provides a graphical representation of TSO-DSO coordination scheme (i).

5.1.2. Formulation

We define the social welfare as the sum of the aggregated area under the nodal inverse demand functions $P_n(\cdot)$, $n \in \text{TN} \cup \text{N}_\infty$ and $P_i(\cdot)$, $i \in \text{DN}_k$, $k \in N$, which represents the total consumer willingness-to-pay, less the sum of all activation costs $C_n(\cdot)$ for the

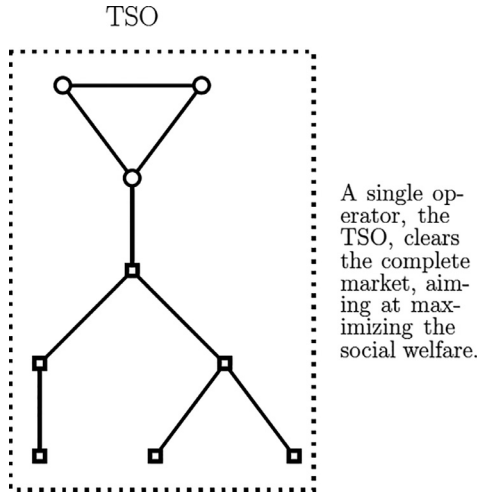


Fig. 2. Centralized co-optimization problem envisioned in TSO-DSO coordination scheme (i). The integrated market operator (that could be alternatively substituted by the TSO), which operates that scheme, has access to all the information on the DSOs' willingness-to-pay, activation cost functions, transmission and distribution network topologies and power flow equations. The optimal reserve activations at transmission and distribution nodes, reactive power injection/consumption, and voltage at each distribution nodes are determined by the integrated market operator.

TSO and $C_i^c(\Delta p_i^c) + C_i^g(\Delta p_i^g)$ for the DSOs:

$$\begin{aligned} & SW(x, (u_a)_{a \in A}, \omega) \\ &= \sum_{n \in \mathbb{T} \cup \mathbb{N}_{\infty}} \left[\int_0^{D_n + \Delta D_n(\omega)} P_n(\tau_n) d\tau_n - C_n(\Delta p_n) \right] \\ &+ \sum_{k \in \mathbb{N}} \sum_{i \in \mathbb{D} \cup \mathbb{N}_k} \left(\int_0^{p_i^c + \Delta D_i(\omega) - \Delta p_i^c} P_i(\tau_i) d\tau_i - C_i^c(\Delta p_i^c) - C_i^g(\Delta p_i^g) \right). \end{aligned} \quad (33)$$

The social welfare is independent of the state variables. As such, we will write $SW((u_a)_{a \in A}, \omega)$ in the rest of the paper.

We denote by $\mathcal{FS}_{TSO}$ and $\mathcal{FS}_{DSO}$ the sets defined by the upper and lower bounds for the decision variables of the TSO and the DSO respectively, and state variable l . More precisely, a vector $(\Delta p^q, \Delta p^c, q, v) \in \mathcal{FS}_{DSO}$ is such that:

$$\begin{aligned} 0 &\leq \Delta p_i^q \leq R_i^g, \forall i \in \mathbb{D}\mathbb{N}, & (\lambda_g^-, \lambda_g^+) \\ 0 &\leq \Delta p_i^c \leq R_i^c, \forall i \in \mathbb{D}\mathbb{N}, & (\lambda_c^-, \lambda_c^+) \\ Q_i^- &\leq q_i \leq Q_i^+, \forall i \in \mathbb{D}\mathbb{N}, & (\lambda_q^-, \lambda_q^+) \\ \nu_i^- &\leq \nu_i \leq \nu_i^+, & (\lambda_v^-, \lambda_v^+) \\ 0 &\leq l_i, \forall i \in \mathbb{D}\mathbb{N}, & (\lambda_l^-, \lambda_l^+) \end{aligned}$$

and a vector, $\Delta p \in \mathcal{FS}_{TSO}$, is such that constraint (28) is true. Note that this simplifying notation will be used in each coordination scheme. It is also important to note that $\mathcal{FS}_{TSO}$ and $\mathcal{FS}_{DSO}$ capture only the TSO and the DSO operational constraints. In other words, the power flow constraints are not included in $\mathcal{FS}_{TSO}$ and $\mathcal{FS}_{DSO}$ definition. So, this means that $\mathcal{FS}_{TSO}, \mathcal{FS}_{DSO}$ define only part of the TSO and DSO feasibility sets.

With centralized co-optimization as coordination scheme, the integrated market operator solves an integrated optimization problem activating reserves at transmission and distribution levels, determining reactive power injection/consumption, and voltage at each distribution node $u_{TSO} \in \mathcal{FS}_{TSO}, u_{DSO} \in \mathcal{FS}_{DSO}$ in state $x \in \mathbb{X}$:

$$\begin{aligned} & \max_{u_{TSO} \in \mathcal{FS}_{TSO}, u_{DSO} \in \mathcal{FS}_{DSO}, x \in \mathbb{X}} SW((u_a)_{a \in A}, \omega), \\ & \text{s.t. } f_l = B_l(\theta_n - \theta_m), \forall l = (n, m) \in \mathbb{L}, \end{aligned}$$

$$\begin{aligned} & (\bar{p}_n + \Delta p_n) + \left(\sum_{l \in \mathbb{L} | l=(m,n)} f_l - \sum_{l \in \mathbb{L} | l=(n,m)} f_l \right) \\ &= D_n + \Delta D_n(\omega), \forall n \in \mathbb{T}\mathbb{N}, \\ & (\bar{p}_n + \Delta p_n) + \sum_{j \in \mathbb{C}_n} (f_j^p - l_j R_j) + \left(\sum_{l \in \mathbb{L} | l=(m,n)} f_l - \sum_{l \in \mathbb{L} | l=(n,m)} f_l \right) = D_n \\ &+ \Delta D_n(\omega) + G_n v_n, \forall n \in \mathbb{N}_{\infty}, \\ & - \frac{TC_l}{B_l} \leq \theta_n - \theta_m \leq \frac{TC_l}{B_l}, \forall l = (n, m) \in \mathbb{L}, \\ & v_i = v_{A_i} + 2(R_i f_i^p + X_i f_i^q) - l_i(R_i^2 + X_i^2), \forall i \in \mathbb{D}\mathbb{N}, \\ & f_i^p - \sum_{j \in \mathbb{C}_i} (f_j^p - l_j R_j) - (\bar{p}_i^g + \Delta p_i^g) \\ &+ (\bar{p}_i^c + \Delta D_i(\omega) - \Delta p_i^c) + G_i v_i = 0, \\ & \forall i \in \mathbb{D}\mathbb{N}, \\ & f_i^q - \sum_{j \in \mathbb{C}_i} (f_j^q - l_j X_j) + q_i - B_i v_i = 0, \forall i \in \mathbb{D}\mathbb{N}, \\ & (f_i^p)^2 + (f_i^q)^2 \leq S_i^2, \forall i \in \mathbb{D}\mathbb{N}, \\ & (f_i^p)^2 + (f_i^q)^2 \leq v_i l_i, \forall i \in \mathbb{D}\mathbb{N}, \\ & (f_i^p - l_i R_i)^2 + (f_i^q - l_i X_i)^2 \leq S_i^2, \forall i \in \mathbb{D}\mathbb{N}. \end{aligned}$$

Propositions 2–4, can be used to reformulate the power flows in the market operator integrated optimization problem. With these reformulations, the integrated market operator solves an integrated optimization problem when activating reserves at transmission and distribution levels, and determining reactive power injection/consumption and voltage at each distribution node $u_{TSO} \in \mathcal{FS}_{TSO}, u_{DSO} \in \mathcal{FS}_{DSO}$:

$$\begin{aligned} & \max_{u_{TSO} \in \mathcal{FS}_{TSO}, u_{DSO} \in \mathcal{FS}_{DSO}} SW((u_a)_{a \in A}, \omega), \\ & \text{s.t. } (29), (30), \\ & (7), (8), (9). \end{aligned} \quad (34)$$

Note that in all the optimization problems associated with the coordination schemes in [Sections 5.1–5.3](#), the TSO and DSO decision variables are optimized over the sets $\mathcal{FS}_{TSO}$ and $\mathcal{FS}_{DSO}$, that appear as subscripts below the 'max' operator and capture the TSO and DSO operational constraints, defined at the beginning of [Section 5.1](#). Power flow equations are reported as explicit constraints in the optimization problems.

5.1.3. Existence and uniqueness of social welfare optimum

The reformulation of the power flow equations which is introduced in [Section 4](#) defines a new feasible set for the centralized co-optimization problem (34). This feasible set is defined by [Eqs. \(29\), \(30\), \(7\), \(8\), \(9\)](#). For the TSO co-optimization problem to be convex, we need to check whether the constraints of problem (34) remain convex.

Proposition 5. Using the reformulation of the power flow equations which is introduced in [Section 4](#), [Eqs. \(29\) and \(30\)](#) define a convex set for u_{TSO}, u_{DSO} .

Proof of Proposition 5. The proof can be found in [Appendix A.1](#). $\square$

Proposition 6. Using the reformulation of the power flow equations which are introduced in [Section 4](#), [Eqs. \(7\)–\(9\)](#) define a convex set for u_{TSO}, u_{DSO} .

Proof of Proposition 6. The proof can be found in [Appendix A.2](#). $\square$

We use the previous result in order to prove the uniqueness of the solution of the first coordination scheme.

Proposition 7. The centralized co-optimization problem in (34) admits a unique solution Δp^* , Δp^g , Δp^c .

Proof of Proposition 7. The proof can be found in Appendix A.3. $\square$

5.1.4. KKT Conditions

We introduce λ_0 as the Lagrange multiplier associated with Eq. (29), λ_1^-, λ_1^+ as the Lagrange multipliers associated respectively with the left inequality in Eq. (30), i.e., $-|M_{inc}|TC + \rho_C - \rho(\Delta p, \Delta p^g, \Delta p^c, q, v) \leq 0$ and with the right inequality in Eq. (30), i.e., $\rho(\Delta p, \Delta p^g, \Delta p^c, q, v) - |M_{inc}|TC - \rho_C \leq 0$, and we further introduce $\lambda_2, \lambda_3, \lambda_4$ as the Lagrange multipliers associated with Eqs. (7), (8), (9) respectively. Note that since Eq. (29) can be reformulated as $(B^{ST} M_{inc}^T (\rho(\Delta p, \Delta p^g, \Delta p^c, q, v) - \rho_C))^2 > 0$, it is always non-binding, therefore $\lambda_0 = 0$ (Dempe et al., 2015). Note that this does not necessarily implies that the TSO feasible set is open because it contains Eqs. (28), (30) which define closed sets. Furthermore, the objective functions that we optimize are (strictly) concave on their feasible sets, meaning that their optimum is reached inside the feasibility set and remains finite.

In the rest of this section, denote by ∇M the Jacobian matrix of M with respect to $\Delta p, \Delta p^g, \Delta p^c, q, v$, which means $\nabla M := (\nabla_{\Delta p} M \nabla_{\Delta p^g} M \nabla_{\Delta p^c} M \nabla_q M \nabla_v M)$.

The Jacobian matrix of $\rho(\cdot)$ with respect to $\Delta p, \Delta p^g, \Delta p^c, q, v$ is defined by:

$$\begin{aligned}\nabla_{\Delta p} \rho &= -I_{card(\mathbb{T}\mathbb{N}) + card(\mathbb{N}_\infty)}, \\ \nabla_{\Delta p^g} \rho &= -M_{int}(I + M)(I - 2diag(R)\Psi^{-1}\Phi(R)), \\ \nabla_{\Delta p^c} \rho &= -M_{int}(I + M)(I - 2diag(R)\Psi^{-1}\Phi(R)), \\ \nabla_q \rho &= -2M_{int}(I + M)diag(R)\Psi^{-1}\Phi(X), \\ \nabla_v \rho &= -M_{int}(I + M)(diag(R)\Upsilon - diag(G)).\end{aligned}$$

We also define the Jacobian matrices of f^p and f^q with respect to $\Delta p, \Delta p^g, \Delta p^c, q, v$:

$$\begin{aligned}\nabla_{\Delta p} f^p &= 0, \\ \nabla_{\Delta p^g} f^p &= (I + M) - 2Mdiag(R)\Psi^{-1}\Phi(R), \\ \nabla_{\Delta p^c} f^p &= (I + M) - 2Mdiag(R)\Psi^{-1}\Phi(R), \\ \nabla_q f^p &= 2Mdiag(R)\Psi^{-1}\Phi(X), \\ \nabla_v f^p &= Mdiag(R)\Upsilon - (I + M)diag(G),\end{aligned}$$

and

$$\begin{aligned}\nabla_{\Delta p} f^q &= 0, \\ \nabla_{\Delta p^g} f^q &= -2Mdiag(X)\Psi^{-1}\Phi(R), \\ \nabla_{\Delta p^c} f^q &= -2Mdiag(X)\Psi^{-1}\Phi(R), \\ \nabla_q f^q &= -(I + M) + 2Mdiag(X)\Psi^{-1}\Phi(X), \\ \nabla_v f^q &= (I + M)diag(B) + Mdiag(X)\Upsilon.\end{aligned}$$

We also need to introduce the KKT conditions associated with sets $\mathcal{F}_{SDSO}$, $\mathcal{F}_{STSO}$. For the $\mathcal{F}_{STSO}$, we have:

$$\begin{aligned}\Delta \mathcal{F}_{SDSO} &:= (\lambda_g^+ - \lambda_g^-)^T (I \ 0 \ 0 \ 0) + (\lambda_c^+ - \lambda_c^-)^T (0 \ I \ 0 \ 0) \\ &\quad + (\lambda_q^+ - \lambda_q^-)^T (0 \ 0 \ I \ 0) \\ &\quad + (\lambda_v^+ - \lambda_v^-)^T (0 \ 0 \ 0 \ I) - \lambda_l^T \nabla l,\end{aligned}$$

where $\lambda_i^-, \lambda_i^+, i \in \{g, c, q, v\}$, are the Lagrange multipliers associated with left and right parts respectively of the upper and lower bounds on the actions of the DSO and λ_l is the Lagrange multiplier associated with non-negativity constraint on state variable l . Similarly, for $\mathcal{F}_{SDSO}$, we obtain:

$$\Delta \mathcal{F}_{STSO} := (\lambda_t^+ - \lambda_t^-)^T,$$

where λ_t^-, λ_t^+ are the Lagrange multipliers associated with the left and right parts respectively of the upper and lower bounds on the decision variables of the TSO, in Eq. (28).

To simplify the notation, we define the set of complementarity constraints associated with each constraint in the set $\mathcal{F}_{SDSO}$ as follows:

$$\begin{aligned}0 &\leq \Delta p^g \perp \lambda_g^- \geq 0, \\ 0 &\leq R^g - \Delta p^g \perp \lambda_g^+ \geq 0, \\ 0 &\leq \Delta p^c \perp \lambda_c^- \geq 0, \\ 0 &\leq R^c - \Delta p^c \perp \lambda_c^+ \geq 0, \\ 0 &\leq q - Q^- \perp \lambda_q^- \geq 0, \\ 0 &\leq Q^+ - q \perp \lambda_q^+ \geq 0, \\ 0 &\leq v - v^- \perp \lambda_v^- \geq 0, \\ 0 &\leq v^+ - v \perp \lambda_v^+ \geq 0, \\ 0 &\leq l \perp \lambda_l \geq 0,\end{aligned}\tag{35}$$

Similarly, the set of complementarity constraints associated with each constraint in the set $\mathcal{F}_{STSO}$ writes down as follows:

$$\begin{aligned}0 &\leq \Delta p \perp \lambda_t^- \geq 0, \\ 0 &\leq R - \Delta p \perp \lambda_t^+ \geq 0.\end{aligned}\tag{36}$$

Recalling the analysis made in Appendix A.3, the differentiation of the social welfare function with respect to $\Delta p, \Delta p^g, \Delta p^c, q, v$ gives us:

$$\begin{aligned}\nabla_{\Delta p} SW^T &= -2(c_n \Delta p_n)_{n \in \mathbb{T}\mathbb{N} \cup \infty}, \\ \nabla_{\Delta p^g} SW^T &= -2(c_i^g \Delta p_i^g)_{i \in \mathbb{D}\mathbb{N}^+}, \\ \nabla_{\Delta p^c} SW^T &= -(P_i(\bar{p}_i^c + \Delta D_i(\omega) - \Delta p_i^c))_{i \in \mathbb{D}\mathbb{N}} - 2(c_i^c \Delta p_i^c)_{i \in \mathbb{D}\mathbb{N}^+}, \\ \nabla_q SW^T &= 0, \\ \nabla_v SW^T &= 0.\end{aligned}$$

To compute a social welfare optimum solution of problem (34), we derive first-order conditions (KKT conditions) for the centralized co-optimization problem:

$$\begin{aligned}-\nabla SW - (\lambda_1^- - \lambda_1^+)^T \nabla \rho + 2\lambda_2^T [diag(f^p) \nabla f^p + diag(f^q) \nabla f^q] \\ + \lambda_3^T [2diag(f^p) \nabla f^p + 2diag(f^q) \nabla f^q - diag(l) \nabla v - diag(v) \nabla l] \\ + \lambda_4^T [2diag(f^p - diag(R)l) (\nabla f^p - diag(R) \nabla l) \\ - 2diag(f^q - diag(X)l) (\nabla f^q - diag(X) \nabla l)] \\ + (\Delta \mathcal{F}_{STSO} \Delta \mathcal{F}_{SDSO}) = 0, \\ (B^{ST} M_{inc}^T (\rho - \rho_C))^2 > 0, \\ 0 &\leq \lambda_1^- \perp |M_{inc}|TC - \rho_C + \rho \geq 0, \\ 0 &\leq \lambda_1^+ \perp -\rho + |M_{inc}|TC + \rho_C \geq 0, \\ 0 &\leq \lambda_2 \perp -(f^p)^2 - (f^q)^2 + S^2 \geq 0, \\ 0 &\leq \lambda_3 \perp -(f^p)^2 - (f^q)^2 + diag(v)l \geq 0, \\ 0 &\leq \lambda_4 \perp -(f^p - diag(R)l)^2 - (f^q - diag(X)l)^2 + S^2 \geq 0,\end{aligned}\tag{35}, (36).$$

Note that the feasible sets being convex according to Propositions 5 and 6 and the social welfare being strictly concave in $\Delta p, \Delta p^g, \Delta p^c$, the KKT conditions are necessary and sufficient conditions for an optimum to exist. The TSO and DSO operational constraints are included in the optimization problem (34), by assuming that the TSO and DSO's decision variables are optimized over the sets $\mathcal{F}_{STSO}$ and $\mathcal{F}_{SDSO}$. Since $\mathcal{F}_{STSO}, \mathcal{F}_{SDSO}$ define part of the constraints of the optimization problem (34), the corresponding KKT conditions are considered in the set of KKT conditions reported just above. Complementarity constraints (35), (36) are not explicitly reported in the set of first-order conditions to keep the description at an acceptable level of details. In Section 7, the

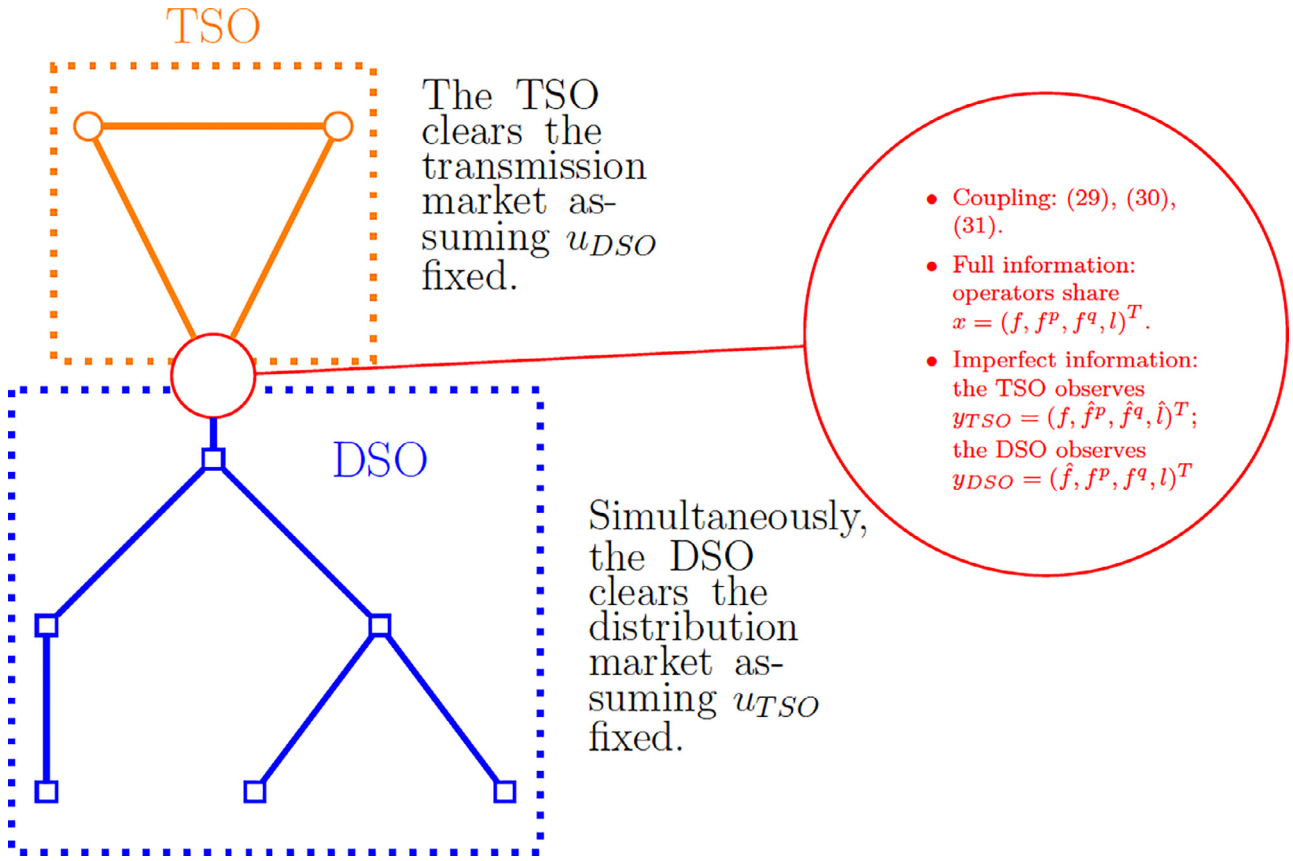


Fig. 3. Graphical representation of TSO-DSO coordination scheme (ii), called shared balancing responsibility (SBR). Each operator handles its own network. The only coupling comes from the interface with different settings depending on level of information. Decisions are taken simultaneously.

centralized co-optimization problem will be solved as a system of KKT conditions.

5.2. Shared balancing responsibility

For coordination scheme (ii), we start by discussing the motivation for its implementation in 5.2.1, then we detail its mathematical formulation in 5.2.2. Concept of Generalized Nash Equilibrium is introduced in 5.2.3. TSO and DSO KKT conditions that will be used to compute Generalized Nash Equilibria in the case study are explicitly given in 5.2.4 and 5.2.5. Conditions for existence and uniqueness of Generalized Nash Equilibrium are detailed in 5.2.6.

5.2.1. Motivation

Under shared balancing responsibility, there is a balancing market for resources connected at the transmission grid, managed by the TSO. There are separate local balancing markets for resources connected at the distribution grids, managed selfishly by each DSO. Resources from the distribution grids cannot be offered to the TSO. DSO grid constraints are integrated in the balancing market clearing process of the local market operated by each DSO. This is the simplest decentralized scheme where it is assumed that the operators have to take their decisions simultaneously knowing only border decisions (maybe even only partially) of the other operators. In practice, historical data might help operators to fix the flow of power at one interface. This scheme is explicitly mentioned as possible future DSO-TSO coordination scheme at the EU level, in the SmartNet project (Gerard et al., 2017).

5.2.2. Formulation

TSO and DSO problems remain coupled through the θ variables in the interface nodes, as expressed in Eq. (31). Fig. 3 provides

a graphical representation of the shared balancing responsibility scheme with one TSO and one DSO.

Formally, the TSO solves the following optimization problem in $u_{TSO} \in \mathcal{FS}_{TSO}$ and state $x \in \mathbb{X}$, assuming that the decision variables of the DSO, $u_{DSO} \in \mathcal{FS}_{DSO}$, are fixed:

$$\begin{aligned} & \max_{u_{TSO} \in \mathcal{FS}_{TSO}, x \in \mathbb{X}} \pi_{TSO}(u_{TSO}, \omega), \\ & \text{s.t. } f_l = B_l(\theta_n - \theta_m), \forall l = (n, m) \in \mathbb{L}, \\ & (\bar{p}_n + \Delta p_n) + \left(\sum_{l \in \mathbb{L} | l=(m,n)} f_l - \sum_{l \in \mathbb{L} | l=(n,m)} f_l \right) \\ & = D_n + \Delta D_n(\omega), \forall n \in \mathbb{TN}, \\ & (\bar{p}_n + \Delta p_n) + \sum_{j \in \mathbb{CN}} (f_j^p - l_j R_j) + \left(\sum_{l \in \mathbb{L} | l=(m,n)} f_l - \sum_{l \in \mathbb{L} | l=(n,m)} f_l \right) = D_n \\ & + \Delta D_n(\omega) + G_n v_n, \forall n \in \mathbb{N}_\infty, \\ & -\frac{TC_l}{B_l} \leq \theta_n - \theta_m \leq \frac{TC_l}{B_l}, \forall l = (n, m) \in \mathbb{L}. \end{aligned}$$

Simultaneously and independently, the DSO solves the following optimization problem in $u_{DSO} \in \mathcal{FS}_{DSO}$ and state $x \in \mathbb{X}$, assuming that the decision variables of the TSO, $u_{TSO} \in \mathcal{FS}_{TSO}$, are fixed:

$$\begin{aligned} & \max_{u_{DSO} \in \mathcal{FS}_{DSO}, x \in \mathbb{X}} \pi_{DSO}(u_{DSO}, \omega), \\ & \text{s.t. } v_i = v_{A_i} + 2(R_i f_i^p + X_i f_i^q) - l_i(R_i^2 + X_i^2), \forall i \in \mathbb{DN}, \\ & (\bar{p}_n + \Delta p_n) + \sum_{j \in \mathbb{CN}} (f_j^p - l_j R_j) + \left(\sum_{l \in \mathbb{L} | l=(m,n)} f_l - \sum_{l \in \mathbb{L} | l=(n,m)} f_l \right) = D_n \\ & + \Delta D_n(\omega) + G_n v_n, \forall n \in \mathbb{N}_\infty, \end{aligned}$$

$$\begin{aligned}
& f_i^p - \sum_{j \in \mathcal{C}_i} (f_j^p - l_j R_j) - (\bar{p}_i^g + \Delta p_i^g) + (\bar{p}_i^c + \Delta D_i(\omega) - \Delta p_i^c) \\
& + G_i v_i = 0, \forall i \in \mathbb{DN}, \\
& f_i^q - \sum_{j \in \mathcal{C}_i} (f_j^q - l_j X_j) + q_i - B_i v_i = 0, \forall i \in \mathbb{DN}, \\
& (f_i^p)^2 + (f_i^q)^2 \leq S_i^2, \forall i \in \mathbb{DN}, \\
& (f_i^p)^2 + (f_i^q)^2 \leq v_i l_i, \forall i \in \mathbb{DN}, \\
& (f_i^p - l_i R_i)^2 + (f_i^q - l_i X_i)^2 \leq S_i^2, \forall i \in \mathbb{DN}.
\end{aligned}$$

This second coordination scheme can be formulated as a simultaneous non-cooperative game involving TSO and DSOs, with perfect information on the (full) state variables, using Propositions 2–4.

Formally, the TSO solves the following optimization problem in $u_{TSO} \in \mathcal{FS}_{TSO}$, assuming that the decision variables of the DSO, $u_{DSO} \in \mathcal{FS}_{DSO}$, are fixed:

$$\begin{aligned}
& \max_{u_{TSO} \in \mathcal{FS}_{TSO}} \pi_{TSO}(u_{TSO}, \omega), \\
& \text{s.t. (29), (30)}.
\end{aligned} \quad (37)$$

Note that the dependence of the TSO constraints on DSO decisions is captured by the fact that the function $\rho(\cdot)$ appearing in constraints (29) and (30) is a function of both TSO as well as DSO decisions.

Simultaneously and independently, the DSO solves the following optimization problem in $u_{DSO} \in \mathcal{FS}_{DSO}$, assuming that the decision variables of the TSO, $u_{TSO} \in \mathcal{FS}_{TSO}$, are fixed:

$$\begin{aligned}
& \max_{u_{DSO} \in \mathcal{FS}_{DSO}} \pi_{DSO}(u_{DSO}, \omega), \\
& \text{s.t. (7), (8), (9)}, \\
& (31).
\end{aligned} \quad (38)$$

A classical formulation in non-cooperative game theory, is to assume that the players in competition operate simultaneously and independently (Fudenberg & Tirole, 1991; Osborne & Rubinstein, 1999). By ‘simultaneously’ is meant that the players operate with ‘bounded rationality’, e.g., without rational anticipation on the outcome of the adversaries decisions (by opposition to the Stackelberg game approach which will be detailed in Section 5.3). By ‘independently’ is meant that no pre-agreement which could have resulted in a coalition, is made between the players.

5.2.3. Solution computation: Introducing Generalized Nash Equilibrium (GNE)

The utility functions $\pi_a, a \in \mathbb{A}$, characterize the strategic form of the game, together with the strategy spaces $\Gamma_a := \{\gamma_a, a \in \mathbb{A}\}$.

Let $\Gamma_a(\gamma_{-a})$ be the output of a point to set map which represents the ability of agents in the set $\mathbb{A}_{-a}$ to influence the feasible strategy set of agent a .

We take a generic point of view to introduce the solution concept of Generalized Nash Equilibrium (GNE). In this setting, each agent $a \in \mathbb{A}$ faces the following optimization problem:

$$\begin{aligned}
& \max_{\gamma_a} \pi_a(\gamma_a, \gamma_{-a}), \\
& \text{s.t. } \gamma_a \in \Gamma_a(\gamma_{-a}).
\end{aligned}$$

We introduce the following definition which characterizes formally GNE:

Definition 1. Harker (1981) $(\gamma_a^{GNE})_{a \in \mathbb{A}}$ is a GNE if

$$\pi_a(\gamma_a^{GNE}, \gamma_{-a}^{GNE}) \geq \pi_a(\gamma_a, \gamma_{-a}^{GNE}), \forall \gamma_a \in \Gamma_a(\gamma_{-a}), a \in \mathbb{A}.$$

Under shared balancing responsibility, the non-cooperative game occurs simultaneously and the decisions of each agent determine the state of the game. This means that the information

space of each agent $a \in \mathbb{A}$ contains the state of the game, i.e., $\mathbb{I}_a = \{x\}, \forall a \in \mathbb{A}$.

To compute a GNE solution of the shared balancing responsibility game (37) and (38), we derive first-order necessary conditions (KKT conditions) for the TSO and DSO optimization problems.

5.2.4. TSO KKT conditions

Let $\lambda_0, \lambda_1^-, \lambda_1^+$ be the Lagrange multipliers associated with the TSO constraints in Eq. (29) and the left and right parts of Eq. (30) respectively. We first note that the TSO utility function $\pi_{TSO}(\cdot)$ is strictly concave in Δp , since its Hessian matrix is $\nabla_{\Delta p}^2 \pi_{TSO} = -2 \text{diag}(c_n)_{n \in \text{TN} \cup \mathbb{N}_\infty} < 0$ which is negative definite. Then the feasible sets being convex according to Propositions 5 and 6, the KKT conditions are necessary and sufficient conditions for an optimum to exist.

To determine solutions to the TSO optimization problem (37), we compute the stationarity conditions with respect to Δp assuming that the decision variables of the DSO, i.e., $\Delta p^g, \Delta p^c, q$ and v , are fixed. This gives us:

$$-\nabla_{\Delta p} \pi_{TSO} + (\lambda_1^- - \lambda_1^+)^T + \Delta \mathcal{FS}_{TSO} = 0, \quad (39)$$

$$0 \leq \lambda_1^- \perp |M_{inc}|TC - \rho_C + \rho(\Delta p, \Delta p^g, \Delta p^c, q, v) \geq 0, \quad (40)$$

$$0 \leq \lambda_1^+ \perp -\rho(\Delta p, \Delta p^g, \Delta p^c, q, v) + |M_{inc}|TC + \rho_C \geq 0, \quad (41)$$

where $\nabla_{\Delta p} \pi_{TSO} = (P_n(D_n + \Delta D_n(\omega))_{n \in \text{TN} \cup \mathbb{N}_\infty} - 2(c_n \Delta p_n)_{n \in \text{TN} \cup \mathbb{N}_\infty})$.

The combination of Eqs. (40) and (41) implies that for any $n \in \text{TN} \cup \mathbb{N}_\infty$ either $\lambda_1^-(n) = 0$, or $\lambda_1^+(n) = 0$, or $\lambda_1^-(n) = \lambda_1^+(n) = 0$. This leads us to distinguish between three cases:

- case (i) $\lambda_1^-(n) = 0$ and constraint (41) is binding, i.e., the line is congested and $\rho(\Delta p, \Delta p^g, \Delta p^c, q, v)(n) = (|M_{inc}|TC + \rho_C)(n)$. Then at the equilibrium $\Delta p_n^{GNE} = (\rho(0, \Delta p^g, \Delta p^c, q, v) - |M_{inc}|TC - \rho_C)_n \leq \frac{P_n(D_n + \Delta D_n(\omega))}{2c_n}$.
- case (ii) $\lambda_1^+(n) = 0$ and constraint (40) is binding, i.e., the line is congested and $\rho(\Delta p, \Delta p^g, \Delta p^c, q, v)(n) = (-|M_{inc}|TC + \rho_C)_n$. Then at the equilibrium $\Delta p_n^{GNE} = (\rho(0, \Delta p^g, \Delta p^c, q, v) + |M_{inc}|TC - \rho_C)_n \geq \frac{P_n(D_n + \Delta D_n(\omega))}{2c_n}$.
- case (iii) $\lambda_1^-(n) = \lambda_1^+(n) = 0$ (i.e., no constraint is binding, meaning that the line is not congested), then at the equilibrium $\Delta p_n^{GNE} = \frac{P_n(D_n + \Delta D_n(\omega))}{2c_n}$.

Therefore, at the equilibrium, depending on the position of $\frac{P_n(D_n + \Delta D_n(\omega))}{2c_n}$ with respect to $(\rho(0, \Delta p^g, \Delta p^c, q, v) - |M_{inc}|TC - \rho_C)_n$ and $(\rho(0, \Delta p^g, \Delta p^c, q, v) + |M_{inc}|TC - \rho_C)_n$, we can fall into one case or another, which also highlights the state of network regarding line congestion.

The first KKT condition in Eq. (39) also provides economic interpretation, as it implies that the demand in transmission node n is equal to the inverse demand of the marginal cost summed up with a price premium linked to the saturation of transmission lines:

$$D_n + \Delta D_n(\omega) = P_n^{-1} \left[\underbrace{2c_n \Delta p_n}_{\text{marginal cost}} + \underbrace{\lambda_1^+(n) - \lambda_1^-(n)}_{\text{price premium}} \right],$$

$$\forall n \in \text{TN} \cup \mathbb{N}_\infty.$$

5.2.5. DSO KKT Conditions

Let $\lambda_2, \lambda_3, \lambda_4, \lambda_5$ be the Lagrange multipliers associated with Eqs. (7), (8), (9), (31) respectively.

In the rest of this section, we denote by ∇X the Jacobian matrix of X with respect to $\Delta p^g, \Delta p^c, q, v$, which means $\nabla X = (\nabla_{\Delta p^g} X \ \nabla_{\Delta p^c} X \ \nabla_q X \ \nabla_v X)$.

To determine solutions of the DSO optimization problem (38), we compute the stationarity conditions with respect to Δp^g , Δp^c , q , v , assuming that Δp and θ are fixed:

$$\begin{aligned} & -\nabla \pi_{DSO} + 2\lambda_2^T [\text{diag}(f^p) \nabla f^p + \text{diag}(f^q) \nabla f^q] \\ & + \lambda_3^T [2\text{diag}(f^p) \nabla f^p + 2\text{diag}(f^q) \nabla f^q - \text{diag}(l) \nabla v - \text{diag}(v) \nabla l] \\ & + \lambda_4^T [2\text{diag}(f^p - \text{diag}(R)l) (\nabla f^p - \text{diag}(R) \nabla l) \\ & - 2\text{diag}(f^p - \text{diag}(X)l) (\nabla f^p - \text{diag}(X) \nabla l)] \\ & - \lambda_5^T \nabla \rho + \Delta \mathcal{F}_{DSO} = 0, \end{aligned} \quad (42)$$

$$0 \leq \lambda_2 \perp -(f^p)^2 - (f^q)^2 + S^2 \geq 0, \quad (43)$$

$$0 \leq \lambda_3 \perp -(f^p)^2 - (f^q)^2 + \text{diag}(v)l \geq 0, \quad (44)$$

$$0 \leq \lambda_4 \perp -(f^p - \text{diag}(R)l)^2 - (f^q - \text{diag}(X)l)^2 + S^2 \geq 0, \quad (45)$$

$$\lambda_5 \in \mathbb{R}^{\text{card}(\text{TNUN}_\infty)} \perp M_{\text{inc}} B^T \theta = \rho(\Delta p, \Delta p^g, \Delta p^c, q, v) - \rho_c. \quad (35), \quad (46)$$

where the gradient of the DSO utility function $\nabla \pi_{DSO}$ can be expressed as follows:

$$\begin{aligned} \nabla_{\Delta p^g} \pi_{DSO} &= (P_i(\bar{p}_i^c + \Delta D_i(\omega) - \Delta p_i^c))_{i \in \mathbb{DN}} - 2(c_i^g \Delta p_i^g)_{i \in \mathbb{DN}}, \\ \nabla_{\Delta p^c} \pi_{DSO} &= \beta_i(\bar{p}_i^g + \Delta p_i^g) - 2(c_i^c \Delta p_i^c)_{i \in \mathbb{DN}}, \\ \nabla_q \pi_{DSO} &= 0, \\ \nabla_v \pi_{DSO} &= 0. \end{aligned}$$

The stationarity conditions associated with the centralized co-optimization problem (detailed in Section 5.1.4) do not coincide with the concatenation of the stationarity conditions associated with the shared balancing responsibility game (that are detailed in Sections 5.2.4 and 5.2.5). To compute an equilibrium solution of the shared balancing responsibility game, we determine the best response function of each agent (TSO, DSO), assuming that the decision variable of the other's is fixed. This is equivalent to solving in the agents decision variables an optimization problem, parametrized in the decision variables of the other agent. As such, for each agent, the system of KKT conditions parametrized in the other agents decision variables can be used to determine the best response function of the agent. Then, the intersection of the best response functions of all the agents (TSO, DSO) give us the equilibrium solution of the shared balancing responsibility game.

5.2.6. Existence and Uniqueness of Generalized Nash Equilibrium

Proposition 8. Assuming that the TSO and DSO feasible sets are not empty, there exists a GNE solution of the shared balancing responsibility game (37), (38) if, and only if, $4c_i^g c_i^c \geq \beta_i$, $\forall i \in \mathbb{DN}$.

Proof of Proposition 8. Following Propositions 5 and 6, the feasible sets of the TSO and DSO are convex. Furthermore, these sets are compact. Indeed, constraint (31) implies that constraint (30) is satisfied. So we can consider the feasible set as the intersection of all constraints except (30). All these constraints are either closed balls or closed spaces. So their intersection is a compact set. The feasible sets of problems (37) and (38) are non-empty, convex, and compact. Furthermore, the TSO and DSO utility functions are continuous in u_{TSO} and u_{DSO} , and the TSO utility function is concave with respect to its own decision variable. To check that the DSO utility function is concave with respect to its own decision variables, we compute the Jacobian of the gradient of the DSO utility defined as $g_{DSO}(\cdot) := (\frac{\partial \pi_{DSO}(\cdot)}{\partial \Delta p^g}, \frac{\partial \pi_{DSO}(\cdot)}{\partial \Delta p^c})^T$, as follows:

$$J_{DSO} = \begin{pmatrix} -2\text{diag}(c_i^g)_{i \in \mathbb{DN}} & \text{diag}(\beta_i)_{i \in \mathbb{DN}} \\ \text{diag}(\beta_i)_{i \in \mathbb{DN}} & -2\text{diag}(c_i^c)_{i \in \mathbb{DN}} \end{pmatrix}.$$

Computing the determinant of the characteristic polynomial matrix associated with J_{DSO} in μ , we obtain $\prod_{i \in \mathbb{DN}} ((-2c_i^g - \mu) - \mu)(-2c_i^c - \mu) - \beta_i^2) = \prod_{i \in \mathbb{DN}} ((2c_i^g + \mu)(2c_i^c + \mu) - \beta_i^2) = \prod_{i \in \mathbb{DN}} (\mu^2 + 2(c_i^g + c_i^c)\mu + 4c_i^g c_i^c - \beta_i^2)$. The eigenvalues of J_{DSO} are the roots of the polynomial equation in μ . The minimum is reached in $\mu = -(c_i^g + c_i^c)$. We conclude that J_{DSO} is negative semi-definite if, and only if, $4c_i^g c_i^c \geq \beta_i$, $\forall i \in \mathbb{DN}$ ⁶.

Then, following (Fudenberg & Tirole, 1991), the strategic form game defined through problems (37) and (38) has a GNE. $\square$

Following (Rosen, 1965), we introduce the Jacobian block matrix J of the pseudo-gradient of the non-negative weighted sum of the TSO and DSO utility functions with weights equal to 1 defined as $g(\cdot) := (\frac{\partial \pi_{TSO}(\cdot)}{\partial \Delta p}, \frac{\partial \pi_{DSO}(\cdot)}{\partial \Delta p^g}, \frac{\partial \pi_{DSO}(\cdot)}{\partial \Delta p^c})^T$, as follows:

$$J = \begin{pmatrix} -2\text{diag}(c_n)_{n \in \text{TNUN}_\infty} & 0 & 0 \\ 0 & -2\text{diag}(c_i^g)_{i \in \mathbb{DN}} & \text{diag}(\beta_i)_{i \in \mathbb{DN}} \\ 0 & \text{diag}(\beta_i)_{i \in \mathbb{DN}} & -2\text{diag}(c_i^c)_{i \in \mathbb{DN}} \end{pmatrix}.$$

Proposition 9. If $4c_i^g c_i^c > \beta_i^2$, $\forall i \in \mathbb{DN}$, then the shared balancing responsibility game has a unique GNE.

Proof of Proposition 9. A sufficient condition guaranteeing that the positive weighted sum of the TSO and DSO utility functions is diagonally strictly concave is to check that the symmetric matrix $J + J^T$ is negative definite. Computing the determinant of the characteristic polynomial matrix associated with $J + J^T$ in μ , we obtain $(-1)^{\text{card}(\text{TNUN}_\infty)} \prod_{n=1}^{\text{card}(\text{TNUN}_\infty)} (4c_n + \mu) \prod_{i \in \mathbb{DN}} [(4c_i^g + \mu)(4c_i^c + \mu) - 4\beta_i^2]$. The eigenvalues of $J + J^T$ are the roots of the polynomial equation. We obtain two different types of values: $\mu_n = -4c_n$, $\forall n \in \text{TNUN}_\infty$, other μ are solutions of the polynomial equation $\mu^2 + 4(c_i^g + c_i^c)\mu + 16c_i^g c_i^c - 4\beta_i^2 = 0$. The minimum of this polynomial equation is reached in $\mu = -2(c_i^g + c_i^c)$. It admits two negative roots if, and only if, its value evaluated in zero is positive, i.e., $4c_i^g c_i^c > \beta_i^2$, $\forall i \in \mathbb{DN}$. Under this condition, the positive weighted sum of the TSO and DSO utility functions is diagonally strictly concave, which implies that there exists a unique GNE solution of the shared responsibility balancing game. $\square$

5.3. Local markets

For coordination scheme (iii), we start by discussing the motivation for its implementation in 5.3.1, then we detail its mathematical formulation in 5.3.2. The bilevel optimization problem is reformulated as a mathematical program with complementarity (equilibrium) constraints in 5.3.3. Relations with coordination scheme (ii) are highlighted in 5.3.4.

5.3.1. Motivation

In this coordination scheme, we assume that there are separate local markets, in each one of them operates a DSO. Resources from the DSO grids can only be offered to the TSO after the DSOs have selected resources needed to solve local imbalances within their periphery. The TSO is responsible for the operation of its own balancing market, where both resources from the transmission grid and resources from the distribution grids can participate. This is motivated by the fact that the RES-based DERs are not fully used for the moment. Giving the possibility to the TSO to activate directly DERs is then meaningful for completely using DERs, directly letting the TSO cover for the cost incurred by activating the resources it needs. In practice, it can be a way to avoid the waste of power into the distribution grid, as well as helping in congestion management, and can be coupled with flexibility mechanisms like demand response. This scheme makes

⁶ Note that we have the strict concavity, if $4c_i^g c_i^c > \beta_i^2$, $\forall i \in \mathbb{DN}$.

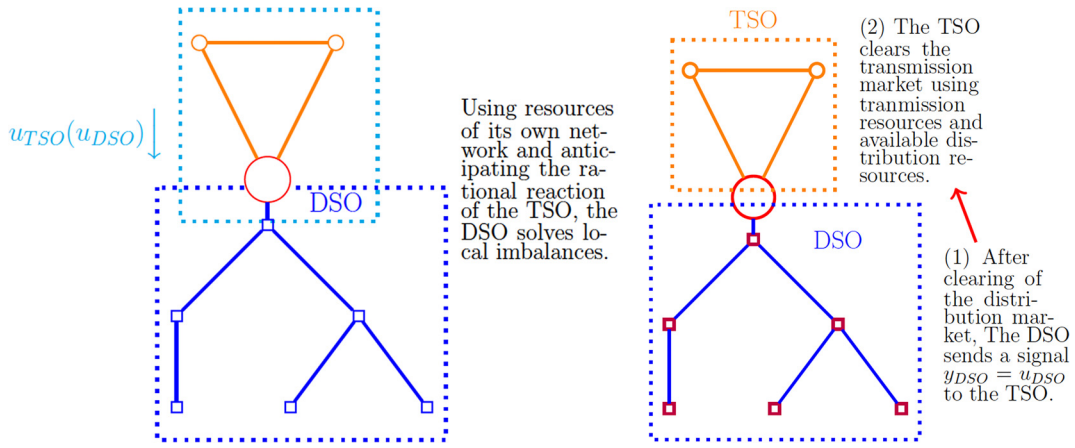


Fig. 4. Graphical representation of TSO-DSO coordination scheme (iii), called local markets. On the left part of the figure, the DSO anticipates the rational reaction function of the TSO by computing backwards $u_{TSO}(u_{DSO})$. The anticipation process takes place prior to the Stackelberg game. On the right part of the figure, the Stackelberg game involving the DSO and TSO takes place forward. First, the DSO sends a signal $y_{DSO} = u_{TSO}$ to the TSO which, in a second time, reacts rationally by activating transmission resources and possibly distribution resources if available. The DSO guarantees the feasibility of the dispatch on the distribution grid.

the mathematical link with the ‘local ancillary service market’ proposed in Gerard et al. (2017). A graphical representation of the scheme is available in Fig. 4. Contrary to the shared balancing responsibility game introduced in Section 5.2 that we interpret as a non-cooperative game, with bounded rationality of the TSO and DSO (implying that they both play at the same time), under the local markets coordination scheme that we describe in this section, the DSO is assumed to play first anticipating the rational reaction of the TSO, which reacts secondly to the signal sent by the DSO following its rational reaction function.

5.3.2. Formulation

This third coordination scheme can be interpreted as a sequential game involving the TSO and DSOs. We model the coordination scheme as a Stackelberg game with multiple leaders (DSOs) and one follower (TSO). The multi-leader Stackelberg game can be formulated as an EPEC (Le Cadre, 2018). As recalled just before the beginning of Section 5, since we assume that the local markets do not share any resource with each other and to keep the economic interpretations as simple as possible, we consider a single DSO. Stackelberg games are generally formulated as Bilevel mathematical Programming Problems (BLPPs) (Dempe & Dutta, 2012; Dempe et al., 2015). BLPPs are hierarchical optimization problems combining decisions of two decision makers, the so-called leader and the so-called follower. The leader acts first, and the follower reacts optimally on the action of the leader. The goal of the leader is to find such a selection which, together with the response of the follower, maximizes its utility function (Dempe et al., 2015).

In this class of problems, the set of decision variables is partitioned between $u_{DSO} \in \mathcal{FS}_{DSO}$ and $u_{TSO} \in \mathcal{FS}_{TSO}$. Given $u_{DSO} \in \mathcal{FS}_{DSO}$, the vector of TSO decisions u_{TSO} is to be chosen as an optimal solution $u_{TSO} = u_{TSO}(y_{DSO})$ of an optimization problem parametrized in $y_{DSO} \in \mathbb{Y}_{DSO}$, defined as the signal sent by the DSO to the TSO. This problem is the so-called lower-level problem of the TSO. The solution $u_{TSO}(y_{DSO})$ is called the rational reaction of the TSO on the signal of the DSO, $y_{DSO} \in \mathbb{Y}_{DSO}$ (Başar & Olsder, 1999; Dempe & Dutta, 2012; Dempe et al., 2015). Knowing this reaction, the bilevel problem reads as an optimization problem for the DSO in $u_{DSO} \in \mathcal{FS}_{DSO}$ only.

The general formulation of the BLPP problem in a local market coordination scheme, can be written down as follows:

$$\begin{aligned} & \max_{u_{DSO} \in \mathcal{FS}_{DSO}, u_{TSO}(y_{DSO}), x \in \mathbb{X}} \pi_{DSO}(u_{DSO}, \omega), \\ & \text{s.t. } v_i = v_{A_i} + 2(R_i f_i^p + X_i f_i^q) - l_i(R_i^2 + X_i^2), \forall i \in \mathbb{DN}, \end{aligned}$$

$$\begin{aligned} & (\bar{p}_n + \Delta p_n) + \sum_{j \in \mathcal{C}_n} (f_j^p - l_j R_j) + \left(\sum_{l \in \mathbb{L}, l=(m,n)} f_l - \sum_{l \in \mathbb{L}, l=(n,m)} f_l \right) = D_n \\ & + \Delta D_n(\omega) + G_n v_n, \forall n \in \mathbb{N}_\infty, \\ & f_i^p - \sum_{j \in \mathcal{C}_i} (f_j^p - l_j R_j) - (\bar{p}_i^g + \Delta p_i^g) + (\bar{p}_i^c + \Delta D_i(\omega) - \Delta p_i^c) \\ & + G_i v_i = 0, \\ & \forall i \in \mathbb{DN}, \\ & f_i^q - \sum_{j \in \mathcal{C}_i} (f_j^q - l_j X_j) + q_i - B_i v_i = 0, \forall i \in \mathbb{DN}, \\ & (f_i^p)^2 + (f_i^q)^2 \leq S_i^2, \forall i \in \mathbb{DN}, \\ & (f_i^p)^2 + (f_i^q)^2 \leq v_i l_i, \forall i \in \mathbb{DN}, \\ & (f_i^p - l_i R_i)^2 + (f_i^q - l_i X_i)^2 \leq S_i^2, \forall i \in \mathbb{DN}, \\ & u_{TSO}(y_{DSO}) = \arg \max_{u_{TSO} \in \mathcal{FS}_{TSO}, x \in \mathbb{X}} \pi_{TSO}(u_{TSO}, \omega), \\ & \text{s.t. } f_l = B_l(\theta_n - \theta_m), \forall l = (n, m) \in \mathbb{L}, \\ & (\bar{p}_n + \Delta p_n) + \left(\sum_{l \in \mathbb{L}, l=(m,n)} f_l - \sum_{l \in \mathbb{L}, l=(n,m)} f_l \right) = D_n + \Delta D_n(\omega), \\ & \forall n \in \mathbb{TN}, \\ & (\bar{p}_n + \Delta p_n) + \sum_{j \in \mathcal{C}_n} (f_j^p - l_j R_j) + \left(\sum_{l \in \mathbb{L}, l=(m,n)} f_l - \sum_{l \in \mathbb{L}, l=(n,m)} f_l \right) \\ & = D_n + \Delta D_n(\omega) + G_n v_n, \forall n \in \mathbb{N}_\infty. \end{aligned}$$

Note that in the BLPP formulation, we have kept $u_{TSO}(y_{DSO})$ in the decision variables at the upper-level. The reason for it is that, in all generalities, the lower-level problem may have multiple solutions. The leaders (DSOs) being not allowed to force the follower (TSO) to take the one or the others of its optimal solutions. Hence, the leaders cannot predict the true value of their utility functions until the follower has communicated its choice. To overcome this difficulty, two approaches have been suggested in Dempe and Dutta (2012) and Dempe et al. (2015): in an optimistic approach, the leader supposes that the follower is willing to support him, i.e., that the follower will select a solution $u_{TSO}(y_{DSO})$ which is the best from the point-of-view of the leader. On the contrary, in a pessimistic approach, the leader is to bound the damage resulting from an undesirable selection of the follower, i.e., it is assumed that the follower will select a solution $u_{TSO}(y_{DSO})$ which is the worst from the point-of-view of the leader.

In the following, we will assume that $y_{DSO} = u_{DSO}$, i.e., the TSO observes the actions chosen at the upper level by the DSO, but more complex formulations with feedback functions capturing partial observation of u_{DSO} might be considered.

Using Propositions 2–4, it is possible to reformulate the BLPP as the following bilevel optimization problem in $u_{DSO} \in \mathcal{FS}_{DSO}$, the TSO optimization problem being nested at the lower-level:

$$\begin{aligned} & \max_{u_{DSO} \in \mathcal{FS}_{DSO}, u_{TSO}(u_{DSO})} \pi_{DSO}(u_{DSO}, \omega), \\ & \text{s.t. (7), (8), (9),} \\ & (31), \\ & u_{TSO}(u_{DSO}) \in \arg \max_{u_{TSO} \in \mathcal{FS}_{TSO}} \pi_{TSO}(u_{TSO}, \omega), \\ & \text{s.t. (29), (30).} \end{aligned} \quad (47)$$

Bilevel optimization problems are non-concave programming problems with an implicitly determined feasible set. In our problem, assuming that the decision variables of the DSO are fixed, the feasibility set of the follower (TSO) optimization problem is convex according to Proposition 5. Furthermore, the TSO utility function is parametric in the DSO's decision variables Δp^g , Δp^c , q , v , and strictly concave in Δp . We can conclude that the lower-level problem admits a unique point-to-set solution $\Delta p^*(\Delta p^g, \Delta p^c, q, v)$ (also called rational reaction or reaction function) and the bilevel problem is well defined (Dempe & Dutta, 2012).

We replace the lower-level problem with its KKT conditions from Section 5.2.4. This results in a mathematical program with complementarity (equilibrium) constraints (MPCC) (Dempe & Dutta, 2012; Dempe et al., 2015; Yao et al., 2008) that we describe in the next section.

5.3.3. MPCC reformulation

The BLPP problem (47) is reformulated as an MPCC, replacing the TSO problem with its KKT conditions (39), (40), (41), (36), (29). The MPCC can be written down as:

$$\begin{aligned} & \max_{u_{DSO} \in \mathcal{FS}_{DSO}, u_{TSO}} \pi_{DSO}((u_a)_{a \in \mathbb{A}}, \omega), \\ & \text{s.t. (7), (8), (9),} \\ & (31) \\ & (39), (40), (41), (36), (29). \end{aligned} \quad (48)$$

Since the TSO and DSO utility functions are concave, the KKT conditions are necessary and sufficient to determine an optimum for the BLPP (47).

The constraints corresponding to the lower-level problem of BLPP provide closed form expressions for the TSO reaction function $\Delta p^*(.)$, which can be expressed as a parametric function of Δp^g , Δp^c , q , v . Following Section 5.2.4, we obtain Δp_n^* equals:

$$\begin{aligned} & \text{(i) } (\rho(0, \Delta p^g, \Delta p^c, q, v) - |M_{inc}|TC - \rho_C)_n \\ & \quad \text{if } \frac{P_n(D_n + \Delta D_n(\omega))}{2c_n} < (\rho(0, \Delta p^g, \Delta p^c, q, v) - |M_{inc}|TC - \rho_C)_n, \\ & \text{(ii) } (\rho(0, \Delta p^g, \Delta p^c, q, v) + |M_{inc}|TC - \rho_C)_n \\ & \quad \text{if } \frac{P_n(D_n + \Delta D_n(\omega))}{2c_n} > (\rho(0, \Delta p^g, \Delta p^c, q, v) + |M_{inc}|TC - \rho_C)_n, \\ & \text{(iii) } \frac{P_n(D_n + \Delta D_n(\omega))}{2c_n} \text{ otherwise.} \end{aligned}$$

Another difficulty is that bilevel programming BLPP is not a special case of MPCC in general (Dempe & Dutta, 2012). This is not obvious and concerns local solutions. For global and local optimal solutions of the MPCC to correspond to global and local optimal solutions of the BLPP (47), we need to check that the lower-level problem satisfies Slater's constraint qualification (Dempe & Dutta, 2012), i.e., that there exists a reaction function $\Delta p^*(.)$ such that all the nonlinear constraints for the nested optimization problem

are slack. Since the only nonlinear constraint is constraint (29), this is true in our case. For this case where Slater's constraint qualification is verified, global and local optimal solutions of MPCC (48) coincide with global and local optimal solutions of the BLPP (47). In other words, MPCC (48) is an equivalent reformulation of BLPP (47).

To solve the MPCC associated with the BLPP, we aggregate the KKT conditions associated with the DSO's and the TSO's optimization problems, which, in general, form a mixed non-linear complementarity problem. Similarly to Yao et al. (2008), we have assumed that the day-ahead demand and marginal activation cost functions are linear, so the problem becomes a linear complementarity problem, as soon as we remove constraint (29). The BLPP is then solved as a linear complementarity problem, assuming that constraint (29) is met. If, a posteriori, the equilibrium solution violates (29), the algorithm solving the linear complementarity problem is re-run with different initial conditions.

5.3.4. Relationship with the shared balancing responsibility game

To find the optimum for the DSO and TSO in this case, we have the same KKT conditions as in Section 5.2.5 except that $\nabla \rho_n$ is replaced by 0 when we are in case (i) and (ii) and that the constraints corresponding to the lower-level problem of BLPP are replaced by the closed form expression of $\Delta p^*(.)$.

Applying backwards induction to the local market game is the same as applying the concept of dominance, i.e., eliminating sequentially dominated strategies, taking into account sequential rationality. Refinements of GNE which incorporate sequential rationality are called Subgame Perfect GNE (Fudenberg & Tirole, 1991; Osborne & Rubinstein, 1999). So a Subgame Perfect GNE is a strategy profile that specifies a GNE in every subgame, i.e., part of the extensive form of the local market game that constitutes itself a well-defined extensive form game (Başar & Olsder, 1999).

We cannot obtain closed form expressions for the GNE solutions of the shared balancing responsibility and local market games because of the conic constraints introduced by the SOCP relaxations in the DSO optimization problems.

6. Imperfect information setting

All the coordination schemes introduced in Sections 5.1–5.3 are formulated in a perfect information setting, i.e., all the agents have access to the (full) state x . In many cases, however, not all information is common knowledge because some agents may choose to hide or only partially disclose their private information, to avoid the disclosure of sensitive intra-area data (Halilbasic et al., 2017). In these cases, information asymmetry might appear between the agents, which may impact their strategies. In this section, we consider a specific information structure. We assume that the state-measurement (observation) function defined in Section 3.3 takes the following form: $y_{TSO} = h_{TSO}(x) = x + \epsilon^{TSO}$ and $y_{DSO} = h_{DSO}(x) = x + \epsilon^{DSO}$, where ϵ^{TSO} , ϵ^{DSO} can be interpreted as noises; therefore falling in the imperfect information setting described in Section 3.3 ii). The fact that TSO and DSO get noisy observations of the state can come from errors in the sensor measurements on the transmission and distribution networks, or from strategic communication mechanisms through which the agents get incentives to bias their reported measures (Le Cadre & Bedo, 2016).

6.1. TSO forecast of the state variable

The TSO observes perfectly (i.e., without noise in the measurement) the power flows on its own transmission network, $(f_l)_{l \in L}$, but does not know a priori the operational parameters R , X , G , B and distribution network topology characteristics M_{adj} , κ of the DSO that determine the DSO state. This means that the TSO needs

to forecast the active and reactive power flows and current magnitude on the distribution network (DSO state), to solve the KKT conditions 5.2.4 that determine its best response to the DSO strategy.

From Eqs. (18) and (19), the DSO active and reactive power flows can be expressed as linear functions of the DSO decision variables. Since the TSO does not observe the DSO state, we introduce forecasts of the TSO related to the active and reactive power flows and current squared magnitude on the DSO distribution network. Let $\epsilon_i^{TSO} = (\epsilon_i^{TSO,p}, \epsilon_i^{TSO,q}, \epsilon_i^{TSO,l}) \sim \mathcal{N}(\mu_i^{TSO}; \sigma_i^{TSO})$ be the forecast error made by the TSO when estimating the DSO network state, and μ_i^{TSO} and σ_i^{TSO} are the mean and the standard deviation associated with random variable ϵ_i^{TSO} in distribution node i . We set:

$$\begin{aligned}\hat{f}^p &= f^p + \epsilon^{TSO,p} \\ &= \text{diag}(\nabla_{u_{DSO}} f^p) u_{DSO} \\ &\quad + [(I + M) - 2M \text{diag}(R) \Psi^{-1} \Phi(R)] (\bar{p}^g - \bar{p}^c - \Delta D(\omega)) \\ &\quad - M \text{diag}(R) \Psi^{-1} H v_\infty + \epsilon^{TSO,p},\end{aligned}\quad (49)$$

$$\begin{aligned}\hat{f}^q &= f^q + \epsilon^{TSO,q} \\ &= \text{diag}(\nabla_{u_{DSO}} f^q) u_{DSO} - (2M \text{diag}(X) \Psi^{-1} \Phi(R)) (\bar{p}^g \\ &\quad - \bar{p}^c - \Delta D(\omega)) + M \text{diag}(X) \Psi^{-1} H v_\infty + \epsilon^{TSO,q},\end{aligned}\quad (50)$$

$$\begin{aligned}\hat{l} &= l + \epsilon^{TSO,l} \\ &= \text{diag}(\nabla_{u_{DSO}} l) u_{DSO} \\ &\quad + [2\Psi^{-1} \Phi(R) (\bar{p}^g - \bar{p}^c - \Delta D(\omega)) + \Psi^{-1} H v_\infty] + \epsilon^{TSO,l},\end{aligned}\quad (51)$$

Note that $\nabla_{u_{DSO}} f^{p,q}$ is the gradient of $f^{p,q}$ restricted to the DSO decision variables, i.e., it does not contain $\nabla_{\Delta p} f^{p,q}$. The gradient of the DSO current squared magnitude $\nabla_{u_{DSO}} l$ is defined by:

$$\begin{aligned}\nabla_{\Delta p^g} l &= 2\Psi^{-1} \Phi(R), \\ \nabla_{\Delta p^c} l &= 2\Psi^{-1} \phi(R), \\ \nabla_q l &= -2\Psi^{-1} \Phi(X), \\ \nabla_v l &= -\Upsilon.\end{aligned}$$

This implies that the (full) state variables as observed by the TSO now take the form:

$$y_{TSO} = \begin{pmatrix} (f_l)_{l \in \mathbb{L}} \\ (\hat{f}_i^p)_{i \in \mathbb{DN}_k, k \in N} \\ (\hat{f}_i^q)_{i \in \mathbb{DN}_k, k \in N} \\ (\hat{l}_i)_{i \in \mathbb{DN}_k, k \in N} \end{pmatrix}.$$

6.2. DSO forecast of the state variable

Similarly to the TSO, the DSO may observe only partially the state variable. The DSO observes perfectly the active and reactive power and current magnitude on the distribution network (i.e., without noise measurement), f^p, f^q, l , but needs to forecast the flow on the transmission network (TSO state), to solve the KKT conditions 5.2.5 that determine its best response to the TSO strategy. To that purpose, we introduce the following form for the forecast of the DSO related to the power flow on the TSO transmission network:

$$\hat{f}_l = f_l + \epsilon_l^{DSO}, \forall l \in \mathbb{L}, \quad (52)$$

with $\epsilon_l^{DSO} \sim \mathcal{N}(\mu_l^{DSO}; \sigma_l^{DSO})$, μ_l^{DSO} and σ_l^{DSO} are the mean and the standard deviation associated with random variable ϵ_l^{DSO} in transmission line l .

This implies that the state variables as observed by the DSO now take the form

$$y_{DSO} = \begin{pmatrix} (\hat{f}_l)_{l \in \mathbb{L}} \\ (f_i^p)_{i \in \mathbb{DN}_k, k \in N} \\ (f_i^q)_{i \in \mathbb{DN}_k, k \in N} \\ (l_i)_{i \in \mathbb{DN}_k, k \in N} \end{pmatrix}.$$

6.3. Shared balancing responsibility with imperfect information

Under shared balancing responsibility, the TSO solves in Δp the system defined by the KKT conditions of Eqs. (39)–(41), (29), taking as parameters the decisions variables of the DSO $\Delta p^g, \Delta p^c, q, v$ obtained from the KKT conditions (42)–(46). Since the TSO has imperfect information about the (full) state variable, we need to consider these equations all together and to incorporate the forecasts of the TSO. To compute the TSO best response function, Eqs. (42)–(45) are updated by replacing f^p, f^q, l by $\hat{f}^p, \hat{f}^q, \hat{l}$ as defined in Eqs. (49)–(51). A graphical representation of the scheme is available in Fig. 3.

Simultaneously and independently, the DSO solves in $\Delta p^g, \Delta p^c, q, v$ the system defined by the KKT conditions Eqs. (42)–(46), taking as parameters the decisions variables of the TSO, Δp , obtained from the KKT conditions (39)–(41), (29). Since the DSO has imperfect information about the (full) state variable, we need to consider these equations all together and to incorporate the forecasts of the DSO. Eq. (25) is changed into:

$$M_{inc} f = \rho - \rho_C - M_{inc} \epsilon^{DSO},$$

which implies that Eq. (29) needs to be updated as follows:

$$(B^{ST} M_{inc}^T (\rho(\Delta p, \Delta p^g, \Delta p^c, q, v) - \rho_C - M_{inc} \epsilon^{DSO}))^2 > 0.$$

Eqs. (40) and (41) are updated as follows:

$$\begin{aligned}-|M_{inc}| TC + \rho_C + M_{inc} \epsilon^{DSO} &\leq \rho(\Delta p, \Delta p^g, \Delta p^c, q, v) \\ &\leq |M_{inc}| TC + \rho_C + M_{inc} \epsilon^{DSO}.\end{aligned}$$

The intersections of the best responses of the TSO, $u_{TSO}(u_{DSO}, \epsilon^{TSO})$, and DSO, $u_{DSO}(u_{TSO}, \epsilon^{DSO})$, furnish the GNE, which are parametrized in the errors of the TSO and DSO $\epsilon^{TSO}, \epsilon^{DSO}$. Let S_{GNE} be the set of GNE solutions of the shared balancing responsibility game with imperfect information. In Section 7 we will characterize the impact of TSO and DSO uncertainty on the efficiency of the shared balancing responsibility game.

7. Numerical illustrations

The three coordination schemes are tested on a meshed transmission network made of three interface nodes numbered from 1 to 3. Each one of these interface nodes is itself the root of a tree capturing a distribution network containing 5 nodes, as pictured in Fig. 1. Operational parameters are calibrated based on a NICTA NESTA test case (Coffrin, Gordon, & Scott, 2017). The data sets used in this section are available online⁷. We only run tests on this stylized example for different reasons: (a) equilibrium problems are computationally difficult to tackle, (b) we want to provide a preliminary efficiency analysis on each scheme before potentially selecting the ones that should be considered for large scale instances, (c) the assumptions made on the network (meshed transmission network with DC power flow and radial distribution network with SOCP relaxation) are common and largely used in the literature (Caramanis et al., 2016; Kocuk et al., 2016; Peng & Low, 2014). We aim at proving concepts and showing how the schemes work in practice in this work. Large scale instances are then not considered here.

7.1. GNE spanning using random sampling: Quantifying the TSO and DSOs' remunerations

In this section, we evaluate the remuneration of the TSO and DSOs. Since GNEs obtained as output of shared balancing responsibility and local market coordination schemes might not be unique, this requires to span the set of GNE solutions.

⁷ GitHub <https://github.com/helene83/CS-Games>

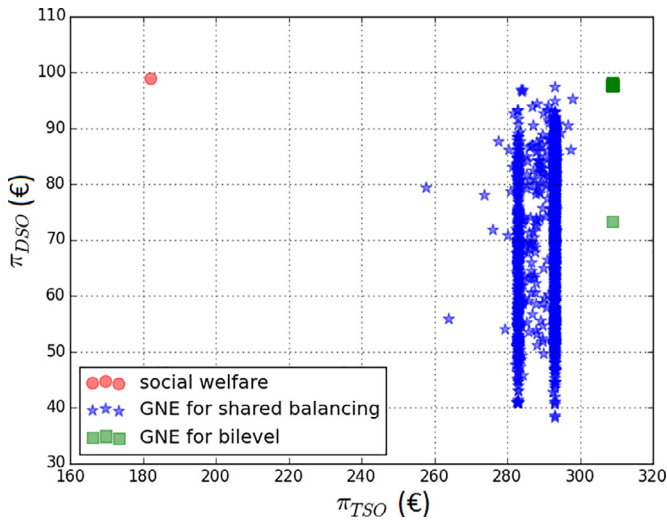


Fig. 5. GNE found by a parametrization approach, using random sampling. In red, we have represented π_{TSO} , π_{DSO} evaluated at the social optimum; in blue and green, π_{TSO} , π_{DSO} evaluated in the set of GNE solutions of the shared balancing responsibility and local market game respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Following the approach in [Nabetani, Tseng, and Fukushima \(2011\)](#), we span the set of GNE by randomly sampling the Lagrange multipliers $\lambda = (\lambda_0, \lambda_1^-, \lambda_1^+, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$.

We proceed as follows. We start with an initial guess $\lambda^0, u_{DSO}^0, u_{TSO}^0$. In practice, we set $\lambda^0 = 0$ and we define u_{DSO}^0 and u_{TSO}^0 as vectors containing the midpoints of the intervals defined by the feasibility sets $\mathcal{FS}_{DSO}$ and $\mathcal{FS}_{TSO}$. Then, by minimization of the gradient of the Lagrangian function under feasibility constraints in each coordination scheme, we update $\lambda^*, u_{DSO}^*, u_{TSO}^*$ and compute $\pi_{TSO}(u_{TSO}^*)$ and $\pi_{DSO}(u_{DSO}^*)$. The algorithm is repeated ([Facchinei & Kanzow, 2007](#)) for updated initial guesses $\lambda, u_{DSO}^*, u_{TSO}^*$, where λ is sampled randomly according to a uniform density function, until a stopping criterion is met.

In [Fig. 5](#), we have represented the TSO and DSO utility functions evaluated in each coordination scheme outcome. In red, the utility functions are evaluated in the social welfare optimum obtained as output of the centralized co-optimization problem (34). As proven in [Proposition 7](#), there exists a unique solution of the centralized co-optimization problem. In blue, the utility functions are evaluated in the set of GNE solutions of the shared balancing responsibility game (37) and (38)⁸. Finally, in green, the utility functions are evaluated in the set of GNE solutions of the local market game (47) formulated as a bilevel optimization problem. We observe that for the TSO it is more advantageous to behave as a follower when the DSO anticipates its follower resource activation strategy than to compete simultaneously with the TSO through a shared balancing responsibility game. This situation might be interpreted as a last-mover advantage for the TSO ([Mas-Colell, Whinston, & Green, 1995](#); [Osborne & Rubinstein, 1999](#)). Furthermore, the joint activation of reserves on transmission and distribution grids through co-optimization leads to a lower profit for the TSO than under the two other decentralized coordination schemes.

7.2. Social welfare and reserve activation levels

In this subsection, we compare the three coordination schemes based on other meaningful criteria to assess their relative efficiency, such as social welfare and reserve activation levels.

In [Fig. 6\(a\)](#), we have represented the TSO reserve activation level, quantified as the sum of the reserves activated on the TSO network and at the interface nodes $\sum_{n \in \text{TNUN}} \Delta p_n$, and the DSO reserve activation level, quantified as the sum of the reserves activated on the DSO network $\sum_{i \in \text{DN}} (\Delta p_i^g + \Delta p_i^c)$, evaluated at the social optimum in red, GNE solutions of the shared balancing responsibility game in blue, and local market in green. For all the coordination schemes, the level of activated reserves is higher on the DSO network than on the TSO network; this can be explained by the fact that in our NICTA NESTA test case, activation costs of DERs are assumed to very low (close to zero because coming from RES-based generators) whereas conventional generators' activation costs on the TSO network are quite high. Logically, under co-optimization, the integrated market operator activates a very low amount of reserves on the transmission network and a large quantity of reserves on the distribution network. Furthermore, more reserves are activated on the TSO network under local market coordination scheme than under shared balancing responsibility coordination scheme. This can be interpreted as a by-product of the last-mover advantage for the TSO, which gives rise to higher profitability for the TSO than the shared balancing responsibility game as highlighted in [Section 7.1](#). In [Fig. 6\(b\)](#), we have represented the social welfare as function of the total reserve activated by TSO and DSO evaluated in the social optimum in red, GNE solutions of the shared balancing responsibility game in blue and local market in green. We observe that the centralized co-optimization coordination scheme guarantees the highest level of efficiency in terms of resource allocation, giving rise to the highest social welfare with 200 €, followed very closely by the best equilibrium of the shared balancing responsibility game with 199 €. The local market coordination scheme gives rise to a lower social welfare than the centralized co-optimization coordination scheme with values between 147 € and 153 €. However, on average (e.g., with equiprobability of all the equilibria), the local market coordination scheme provides a higher social welfare with an average value of 150 €, than the shared balancing responsibility with 143 €.

Using our test case to assess the relative merits of our three coordination schemes, we summarize their comparison below:

- The local market coordination scheme gives rise to higher profitability for the TSO than the shared balancing responsibility coordination scheme.
- The joint activation of reserves on transmission and distribution networks through co-optimization leads to lower profit for the TSO than under the two other coordination schemes.
- More reserves are activated on the TSO network under local market coordination scheme than under the two other coordination schemes; the centralized co-optimization scheme giving rise to very low amount of reserves activated on the TSO network compared to the activation level on the DSO network.
- The social welfare is the highest when evaluated at the optimum of the centralized co-optimization problem, followed very closely by the highest value of the social welfare evaluated in the GNEs solutions of the shared balancing responsibility game. This means that the shared balancing responsibility game can reach an efficiency level very close to the centralized co-optimization problem, while enabling the introduction of strategic behaviors from the TSO and DSOs.
- The local market coordination scheme leads to a lower level of efficiency than the centralized co-optimization coordination scheme, which can be explained by the last-mover advantage of the TSO, which activates reserves on its network that are far more expensive than RES-based DERs in the distribution network. However, on average, the local market coordination scheme reaches a mean social welfare value higher than the shared balancing responsibility game.

⁸ Note that as the conditions of [Proposition 9](#) are not checked in our data set, there is no guarantee of uniqueness of GNE in this test case.

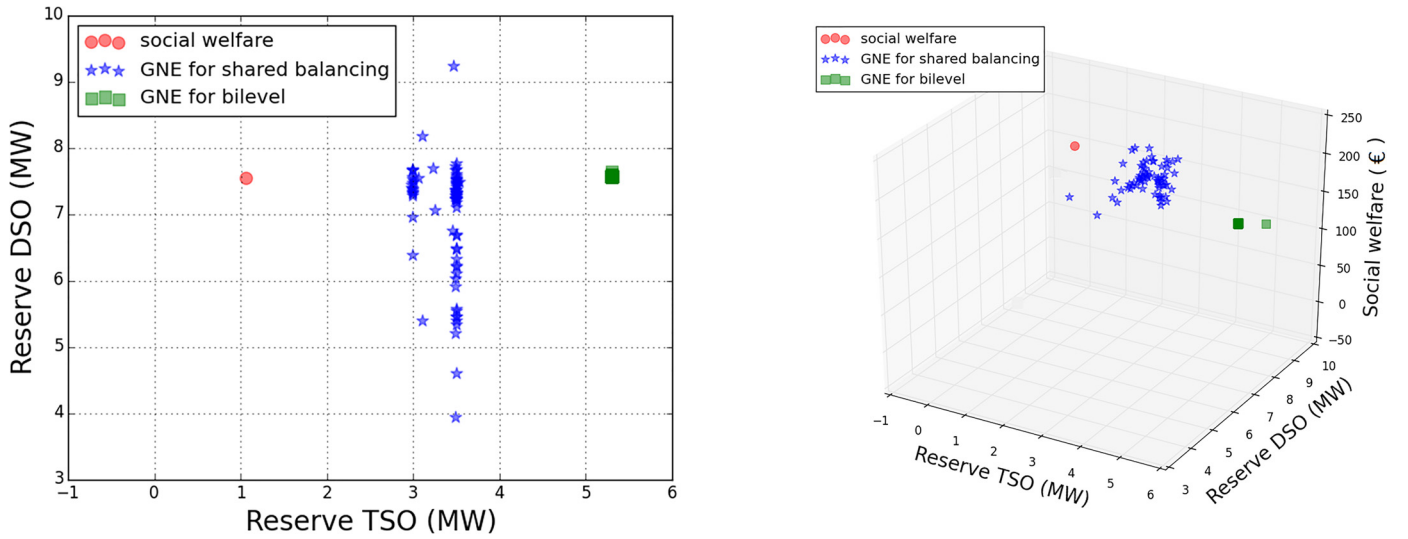


Fig. 6. Reserve activation levels measured as the sum of the reserves activated by the TSO and DSO on the transmission and distribution networks in Fig. 6(a) at the social optimum in red, GNE solutions of the shared balancing responsibility game in blue, and local market in green; social welfare as function of the total reserve activated by TSO and DSO evaluated in the social optimum in red, GNE solutions of the shared balancing responsibility game in blue and local market in green in Fig. 6(b). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

7.3. Impact of Information

Since the operational parameters and network topology are typically not shared (common knowledge) among TSO and DSOs, due to partial information disclosure or privacy constraints (Halilbasic et al., 2017), this leads the TSO and DSO to have their own forecasts of the state variable, i.e., y_{TSO} for the TSO and y_{DSO} for the DSO, which may not coincide between them and with the true (full) state value. In Eqs. (49)–(52), the uncertainty of the TSO and DSO on the state variable is captured through an error modeled as a random variable. Without loss of generality, we assume that the mean and standard deviation of the TSO error are the same for the active power, reactive power, and current magnitude forecasts, and that these parameters coincide at all nodes, i.e., $\mu_i^{TSO,p/q/l} = \mu^{TSO}, \forall i \in \mathbb{DN}$ and $\sigma_i^{TSO,p/q/l} = \sigma^{TSO}, \forall i \in \mathbb{DN}$. We also assume that the mean and standard deviation of the DSO forecast errors are the same at all lines, i.e., $\mu_l^{DSO} = \mu^{DSO}, \forall l \in \mathbb{L}$, $\sigma_l^{DSO} = \sigma^{DSO}, \forall l \in \mathbb{L}$. Following (Robu, Vinyals, Rogers, & Jennings, 2017), we characterize each agent by its coefficient of variation, which is the ratio of its mean over standard deviation. The coefficients of variation are thus $\frac{\sigma^{TSO}}{\mu^{TSO}}$ for the TSO, and $\frac{\sigma^{DSO}}{\mu^{DSO}}$ for the DSO. We consider $\mu^{TSO} > 0$ and $\mu^{DSO} > 0$ as being fixed in the numerical illustrations.

Given a fixed mean, a large standard deviation means that the distribution of the forecast errors in the state tends to be flat. It may decrease assuming that the other agents disclose some information on their own state.

We now want to quantify the impact of the coefficient of variation of each agent on the GNE set, S_{GNE} , in the context of imperfect information. To that end, we introduce the Price of Information (PoI) as a variant of the Price of Anarchy (Nisan, Roughgarden, Tardos, & Vazirani, 2007), as an efficiency measure. The PoI is the worst-case ratio of the optimal achievable social welfare with perfect information to the social welfare at an equilibrium with imperfect information:

$$PoI := \frac{SW(\gamma_{TSO}^*, \gamma_{DSO}^*)}{\min_{\gamma_{TSO}, \gamma_{DSO} \in S_{GNE}} SW(\gamma_{TSO}, \gamma_{DSO})}.$$

The PoI measures the worst-case loss arising from insufficient ability to control and coordinate the actions of selfish agents resulting from decentralization and lack of information disclosure

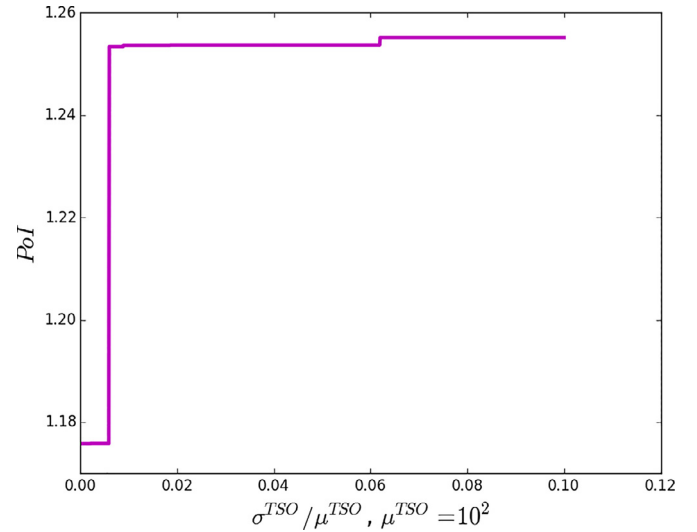


Fig. 7. Price of Information (PoI) as a function of the TSO coefficient of variation $\sigma^{TSO} / \mu^{TSO}, \mu^{TSO} = 10^2$.

caused by privacy constraints. The inefficiency loss is minimized when the PoI is the smallest and approaches 1.

Running simulations on the network considered in the previous section, we observe that the coefficient of variation of the DSO has a limited impact on the PoI. Intuition behind it is that in the DSO optimization problem, (7)–(9) contain only decision variables of the DSO while (31) is shared between TSO and DSO but the θ value does not affect directly the DSO utility function. So we focus on the coefficient of variation of the TSO. We notice in Fig. 7 that the PoI is a stepwise increasing function of the TSO coefficient of variation and that the PoI reaches an upper-bound value of 1.255. Furthermore, the loss of efficiency caused by decentralization is limited in full information ($PoI = 1.177$), i.e., for $\epsilon^{TSO} = \epsilon^{DSO} = 0$.

8. Conclusion

In this paper, we have formulated three coordination schemes involving DSOs and a TSO as mathematical programs. The first

coordination scheme is a centralized co-optimization problem where an integrated market operator activates jointly resources connected at the transmission and distribution levels. We formulate it as a standard constrained optimization problem. The second coordination scheme, called shared balancing responsibility, assumes bounded rationality of the DSOs and TSO, which activate simultaneously their resources taken as given the other agents' decision variables. It is formulated as a (simultaneous) non-cooperative game. In the last coordination scheme, we introduce some rational expectation from the DSOs which anticipate the TSO market clearing. We model this third local market scheme as a multi-leader Stackelberg game, that we formulate as a bilevel mathematical optimization problem. For each scheme, we determine conditions for existence and uniqueness of solutions. We also reformulate the multi-leader Stackelberg game as a mathematical program with complementarity constraints (MPCC), which does not coincide with the shared balancing responsibility game in general. We run a numerical illustration on a network calibrated on NICTA NESTA test cases and span the set of Generalized Nash equilibrium solutions of the decentralized coordination schemes. We observe that the decentralized coordination schemes are more advantageous in terms of profit maximization for the TSO than the centralized co-optimization, and that a Stackelberg game setting (with rational expectation from the DSOs) gives higher profits for the TSO than a non-cooperative game setting with bounded rational agents. Regarding the efficiency level in terms of resource allocation, the centralized co-optimization of transmission and distribution network resources reaches the highest value, followed very closely by the shared balancing responsibility game. The third coordination scheme gives lower values, which can be explained by the last-mover advantage of the TSO which activates conventional generation reserves on its network which are far more expensive than RES-based generations available on the distribution network. Finally, assuming that the agents have imperfect information on the (full) state variable, we check numerically that the Price of Information, measured as the worst-case ratio of the optimal achievable social welfare with perfect information to the social welfare at an equilibrium with imperfect information, is an increasing stepwise function of the TSO coefficient of variation and that it reaches an upper-bound threshold value.

Appendix A

A1. Proof of Proposition 5

We have proven that Eq. (29) is true if, and only if, $\rho(\cdot) - \rho_C$ is in the image of $M_{inc}^{B^2}$. By definition, the image of $M_{inc}^{B^2}$ is a convex space. This proves that Eq. (29) defines a convex set in u_{TSO}, u_{DSO} .

From Eq. (26), it is straightforward to check that $\rho(\cdot)$ is linear in $\Delta p, \Delta p^g, \Delta p^c, q, v$. Then, for any $\xi \in [0; 1]$, $\rho(\xi x + (1 - \xi)y) = \xi \rho(x) + (1 - \xi)\rho(y)$. So we can show easily that $\xi x + (1 - \xi)y$ satisfies Eq. (30). Therefore, Eq. (30) also defines a convex set in u_{TSO}, u_{DSO} .

Therefore, the Cartesian product of the convex sets defined by Eqs. (29) and (30), is itself a convex set.

A2. Proof of Proposition 6

Introduce the mapping $z \mapsto (f_i^p(z))^2 + (f_i^q(z))^2$. f_i^p and f_i^q are linear in each component of $z = (\Delta p^g, \Delta p^c, q, v)$. So the mapping is the sum of two convex functions, themselves compositions of convex functions and linear functions. Therefore, the mapping is convex in z . Consider z_1, z_2 which verify Eq. (7). We want to prove that for any $\xi \in [0; 1]$, $\xi z_1 + (1 - \xi)z_2$ also verifies Eq. (7). This is

straightforward, indeed:

$$\begin{aligned} & f_i^p(\xi z_1 + (1 - \xi)z_2)^2 + f_i^q(\xi z_1 + (1 - \xi)z_2)^2 \\ & \leq \xi f_i^p(z_1)^2 + (1 - \xi)f_i^p(z_2)^2 + \xi f_i^q(z_1)^2 + (1 - \xi)f_i^q(z_2)^2, \\ & \text{by convexity of the mapping } z \mapsto (f_i^p(z))^2 + (f_i^q(z))^2, \\ & \leq \xi S_i^2 + (1 - \xi)S_i^2 = S_i^2. \end{aligned}$$

This proves that Eq. (7) defines a convex set in z .

Similarly, consider the mapping $z \mapsto (f_i^p(z))^2 + (f_i^q(z))^2 - v_i l_i(z)$. The mapping $z \mapsto -v_i l_i(z)$ gives rise to a gradient function of the form $g_{vl}(z) := (-2\Psi^{-1}\Phi(R)v - 2\Psi^{-1}\Phi(R)v \ 2\Psi^{-1}\Phi(X)v \ 2\Upsilon v)^T$. For any $z_1, z_2 \in \mathcal{FSDSO}$, $g_{vl}(z_1)^T g_{vl}(z_2) = 8(\Psi^{-1}\Phi(R))^2 v_1 v_2 + 4(\Psi^{-1}\Phi(X))^2 v_1 v_2 + 4\Upsilon^2 v_1 v_2 \geq 0$ under the assumption that $v_i^- \geq 0, \forall i \in \mathbb{DN}$. This implies that the function $-g_{vl}$ is monotonic, i.e., that the mapping $z \mapsto -v_i l_i(z)$ is convex. Then the mapping $z \mapsto (f_i^p(z))^2 + (f_i^q(z))^2 - v_i l_i(z)$ is convex as the sum of two convex functions. It is straightforward to prove that Eq. (8) defines a convex set using the same reasoning as for Eq. (7).

To prove that Eq. (9) defines a convex set, introduce the mapping $z \mapsto (f_i^p(z) - R_i l_i(z))^2 + (f_i^q(z) - X_i l_i(z))^2$. $f_i^p(z) - R_i l_i(z)$ and $f_i^q(z) - X_i l_i(z)$ are linear in each component of $z = (\Delta p^g, \Delta p^c, q, v)$. We have recalled that the composition of a convex function with a linear function remains convex. Therefore, the mapping is convex in z . It is straightforward to prove that Eq. (8) defines a convex set using the same reasoning as for Eq. (7).

Therefore, the Cartesian product of the convex sets define by Eqs. (7)–(9) is a convex set.

A3. Proof of Proposition 7

Using the results of Propositions 5 and 6, the feasible set of the optimization problem (34) is convex. Furthermore, differentiating the social welfare function with respect to $\Delta p, \Delta p^g, \Delta p^c, q, v$ we obtain:

$$\begin{aligned} \nabla_{\Delta p} SW^T &= -2(c_n \Delta p_n)_{n \in \text{TNN} \cup \mathbb{N}_\infty}, \\ \nabla_{\Delta p^g} SW^T &= -2(c_i^g \Delta p_i^g)_{i \in \mathbb{DN}}, \\ \nabla_{\Delta p^c} SW^T &= -(P_i(\bar{p}_i^c + \Delta D_i(\omega) - \Delta p_i^c))_{i \in \mathbb{DN}} - 2(c_i^c \Delta p_i^c)_{i \in \mathbb{DN}}, \\ \nabla_q SW^T &= 0, \\ \nabla_v SW^T &= 0. \end{aligned}$$

Differentiating a second time the social welfare with respect to $\Delta p, \Delta p^g, \Delta p^c$, we obtain the Hessian matrix of the social welfare, which is null except on its diagonal which contains negative coefficients $-2(c_n)_{n \in \text{TNN} \cup \mathbb{N}_\infty}$, $-2(c_i^g)_{i \in \mathbb{DN}}$, $-(\beta_i + 2c_i^c)_{i \in \mathbb{DN}}$. We infer that SW is strictly concave with respect to these variables. Therefore, it admits a unique optimum $\Delta p^*(q, v)$, $\Delta p^{g*}(q, v)$, $\Delta p^{c*}(q, v)$. q and v are determined by the constraints of problem (34) but are not necessarily unique.

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