Definition of timescales and related variables for passive and radioactive tracers in the World Ocean and its leaky funnel idealisation

Eric Deleersnijder, May 24th, 2013

Local and global variables for the World Ocean

Consider a radioactive tracer dissolved in the World Ocean waters. Its rate of decay is the constant parameter λ, implying that its mean and half lives are λ^{-1} and $(\log 2)\,\lambda^{-1} \approx 0.7\,\lambda^{-1}$, respectively. Obviously, a tracer for which $\lambda = 0$ is passive — or inert. Now assume that is is possible to evaluate the age τ of every particle of the tracer under study; the age is assumed to be positive definite, i.e. $0 \le \tau < \infty$. Further assume that the hydrodynamics and tracer concentration are at a steady state. Finally, the Boussinesq approximation is assumed to hold valid, so that the seawater density may be regarded as a constant, which is hereinafter denoted ρ.

At location $\mathbf{x}$, take an arbitrarily-small sample of seawater, whose volume is δV. In this seawater sample, the mass of the tracer particles whose age lies in the interval $[\tau, \tau + \delta\tau]$ tends to $\rho\, c_\lambda(\mathbf{x},\tau)\,\delta V\,\delta\tau$ as $\delta V \to 0$ and $\delta\tau \to 0$ (e.g. Delhez et al. 1999, Deleersnijder et al. 2001), where $c_\lambda(\mathbf{x},\tau)$ is termed the *tracer age distribution*. Then, the *concentration* of the tracer under study, which is defined as a mass fraction (i.e. a dimensionless variable), reads

$$(1) \qquad C_\lambda(\mathbf{x}) = \int_0^\infty c_\lambda(\mathbf{x},\tau)\, d\tau \ .$$

The tracer age distribution obeys the following partial differential equation

$$(2) \qquad \nabla \bullet (\mathbf{K} \bullet \nabla c_\lambda - \mathbf{u} c_\lambda) - \frac{\partial c_\lambda}{\partial \tau} - \lambda c_\lambda = 0 \ ,$$

where $\mathbf{u}$ and $\mathbf{K}$ denote the velocity, which is divergence-free, and the diffusivity tensor, which is symmetric and positive-definite, respectively. For a tracer to be used in ventilation studies, the age of every tracer particle must be prescribed to be zero at the ocean surface Γ^s. This leads to the following boundary condition:

$$(3) \qquad \left[c_\lambda(\mathbf{x},\tau)\right]_{\mathbf{x}\in\Gamma^s} = C_\lambda^s\, \delta(\tau - 0) \ ,$$

where δ denotes the Dirac function, while the constant C_λ^s denotes the value of the tracer concentration at the ocean surface. The time for a water particle to travel from the ocean surface to any point in the ocean interior is non zero. Therefore, in the ocean interior there is no water particle characterised by a zero age:

(4) $$\left[c_\lambda(\mathbf{x},\tau)\right]_{\tau=0} = 0 \ .$$

Since the diffusivity tensor is assumed to be positive definite, there is no region of the ocean that remains unventilated, no matter how slow the ventilation process may be. This is why, in the ocean interior, there is no tracer particle whose age is arbitrarily large:

(5) $$\left[c_\lambda(\mathbf{x},\tau)\right]_{\tau\to\infty} = 0 \ .$$

The ocean bottom and coastlines are impermeable. If this part of the domain boundary is denoted Γ^i, the associated boundary conditions are

(6) $$\left[\mathbf{u}\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0$$

and

(7) $$\left[(-\mathbf{K}\bullet\nabla c_\lambda + \mathbf{u}c_\lambda)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \ ,$$

where **n** is the outward unit normal vector to Γ^i.

Once the tracer age distribution is known from the solution of (2), the tracer concentration may be obtained from the integral (1). The tracer concentration may also be obtained by solving the partial diffential problem ensuing from the integration over the age of (2)-(7), i.e.

(8) $$\nabla\bullet(\mathbf{K}\bullet\nabla C_\lambda - \mathbf{u}C_\lambda) - \lambda C_\lambda = 0 \ ,$$

(9) $$\left[C_\lambda(\mathbf{x})\right]_{\mathbf{x}\in\Gamma^s} = C_\lambda^s \ ,$$

(10) $$\left[(-\mathbf{K}\bullet\nabla C_\lambda + \mathbf{u}C_\lambda)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \ .$$

The partial differential problem (8)-(10) is such that the concentration of passive tracer ($\lambda = 0$) is a constant,

(11) $$C_0(\mathbf{x}) = C_0^s \ ,$$

implying that, at any location, the tracer age distribution of such a constituent must satisfy the constraint

(12) $$C_0^s = \int_0^\infty c_0(\mathbf{x},\tau)\, d\tau \ .$$

The *mean tracer age*, i.e. the mean age of all the tracer particles present in the aforementioned seawater sample, is

$$(13) \qquad a_\lambda(\mathbf{x}) = \frac{\int_0^\infty \tau\, c_\lambda(\mathbf{x},\tau)\, d\tau}{\int_0^\infty c_\lambda(\mathbf{x},\tau)\, d\tau} = \frac{1}{C_\lambda(\mathbf{x})} \underbrace{\int_0^\infty \tau\, c_\lambda(\mathbf{x},\tau)\, d\tau}_{\alpha_\lambda(\mathbf{x})} = \frac{\alpha_\lambda(\mathbf{x})}{C_\lambda(\mathbf{x})} ,$$

where α_λ is termed *age concentration* in CART-related articles. The radio-age may also be estimated by means of the following formula:

$$(14) \qquad r_\lambda(\mathbf{x}) = \lambda^{-1} \log \frac{C_\lambda^0}{C_\lambda(\mathbf{x})} .$$

A series of global variables are worth estimating. Let Ω represent the domain of interest. Then, the mass of a tracer contained in the domain of interest is

$$(15) \qquad m_\lambda = \rho \int_\Omega C_\lambda(\mathbf{x})\, d\Omega .$$

Obviously, the mass of a passive tracer simplifies to

$$(16) \qquad m_0 = \rho \int_\Omega C_0(\mathbf{x})\, d\Omega = \rho C_0^s \underbrace{\int_\Omega d\Omega}_{=V} = \rho V C_0^s ,$$

where V is the volume of the domain.

The *global tracer age distribution*,

$$(17) \qquad \mu_\lambda(\tau) = \frac{\rho}{m_\lambda} \int_\Omega c_\lambda(\mathbf{x},\tau)\, d\Omega ,$$

is such that the fraction of the mass of the tracer particles whose age lies in the interval $[\tau, \tau + \delta\tau]$ tends to $\mu_\lambda(\tau)\delta\tau$ as $\delta\tau \to 0$. The global tracer age distribution satisfies the constraint

$$(18) \qquad \int_0^\infty \mu_\lambda(\tau)\, d\tau = \int_0^\infty \underbrace{\frac{\rho}{m_\lambda} \int_\Omega c_\lambda(\mathbf{x},\tau)\, d\Omega}_{=\mu_\lambda(\tau),\ \text{see (17)}}\, d\tau = \int_\Omega \frac{\rho}{m_\lambda} \underbrace{\int_0^\infty c_\lambda(\mathbf{x},\tau)\, d\tau}_{=C_\lambda(\mathbf{x}),\ \text{see (1)}}\, d\Omega = \frac{1}{m_\lambda} \underbrace{\rho \int_\Omega C_\lambda(\mathbf{x})\, d\Omega}_{m_\lambda,\ \text{see (15)}} = \frac{m_\lambda}{m_\lambda} = 1 .$$

The *global mean tracer age* is

(19)
$$\bar{a}_\lambda = \frac{\int_0^\infty \tau\, \mu_\lambda(\tau)\, d\tau}{\underbrace{\int_0^\infty \mu_\lambda(\tau)\, d\tau}_{=1,\ \text{see (18)}}} = \int_0^\infty \tau\, \mu_\lambda(\tau)\, d\tau = \int_0^\infty \tau \underbrace{\frac{\rho}{m_\lambda} \int_\Omega c_\lambda(\mathbf{x},\tau)\, d\Omega}_{=\mu_\lambda(\tau),\ \text{see (17)}} d\tau$$
$$= \frac{\rho}{m_\lambda} \int_\Omega \underbrace{\int_0^\infty \tau\, c_\lambda(\mathbf{x},\tau)\, d\tau}_{=\alpha_\lambda(\mathbf{x}),\ \text{see (13)}} d\Omega = \frac{\rho}{m_\lambda} \int_\Omega \alpha_\lambda(\mathbf{x})\, d\Omega = \frac{\int_\Omega \alpha_\lambda(\mathbf{x})\, d\Omega}{\int_\Omega C_\lambda(\mathbf{x})\, d\Omega}$$

Normalised variables

As will be seen, it is convenient to normalise the tracer age distribution by the prescribed surface concentration. Accordingly, the normalised tracer age distribution and tracer concentration are $\tilde{c}_\lambda = c_\lambda / C_\lambda^s$ and $\tilde{C}_\lambda = C_\lambda / C_\lambda^s$, respectively. The corresponding surface values are $\delta(\tau - 0)$ and 1. From here on, only normalised variables will be used. Therefore, for notational simplicity, the tildes will be dropped.

The tracer age distribution function obeys (Delhez et al. 2003, Waugh et al. 2003)

(20) $$c_\lambda(\mathbf{x},\tau) = e^{-\lambda\tau}\, c_0(\mathbf{x},\tau) \ .$$

Then, it is readily seen that the tracer distribution function and the tracer concentration decrease as the rate of decay increases. The mean tracer age is also a drecreasing function of the parameter λ. Demonstrating this property is, however, not straightforward (Delhez et al. 2003). The global tracer age distribution is

(21) $$\mu_\lambda(\tau) = \frac{m_0}{m_\lambda} e^{-\lambda\tau}\, \mu_0(\tau) \ .$$

The steady-state leaky funnel model

The leaky funnel introduced by Mouchet and Deleersnijder (2008) and further developed in Mouchet et al. (2012) is a semi infinite pipe whose section $S(x)$ decreases exponentially, i.e.

$$(22) \qquad S(x) = S_0 e^{-x/L}$$

where x is the distance to the entrance of the pipe ($0 \le x < \infty$), whereas the positive constants S_0 and L denote the section at the pipe entrance and the length scale characterising the exponential decrease of the pipe section, respectively. The water flows through the pipe at constant velocity U. The diffusion in the x-direction is parameterized by means of a Fourier-Fick expression involving the constant diffusivity K.

All variables are assumed to be constant over every section of the pipe and a steady state situation is assumed to prevail. In line with these hypotheses, the tracer age distribution in the leaky funnel, $c(x,\tau)$, obeys the partial differential equation

$$(23) \qquad \frac{\partial}{\partial x}\left(SK\frac{\partial c_\lambda}{\partial x} - USc_\lambda \right) + \frac{\partial (Sc_\lambda)}{\partial \tau} + \underbrace{\frac{\partial (SU)}{\partial x} c_\lambda}_{\text{leakage through porous walls}} + \underbrace{(-\lambda S) c_\lambda}_{\text{radioactive decay}} = 0 \ .$$

Substituting (22) into (23) yields

$$(24) \qquad K\frac{\partial^2 c_\lambda}{\partial x^2} - U'\frac{\partial c_\lambda}{\partial x} + \frac{\partial c_\lambda}{\partial \tau} - \lambda c_\lambda = 0 \ ,$$

where $U' = U + K / L$ may be regarded as a modified velocity. This equation is to be solved under the following boundary conditions:

$$(25) \qquad \left[c_\lambda(x,\tau) \right]_{x=0} = \delta(\tau - 0) \ ,$$

$$(26) \qquad \left[c_\lambda(x,\tau) \right]_{x \to \infty} = 0 \ ,$$

and

$$(27) \qquad \left[c_\lambda(x,\tau) \right]_{\tau=0} = 0 \ .$$

Upon using the solutions of Mouchet et al. (2012) for the water ($\lambda = 0$), and introducing the Peclet numbers, $Pe = UL/K$ and $Pe' = U'L/K$, and the modified Jenkins numbers, $Je = U^2/(K\lambda)$ $Je' = (U')^2/(K\lambda)$, the following expressions for a decaying tracer may be derived:

$$(28) \qquad c_\lambda(x,\tau) = e^{-\lambda\tau} \underbrace{\frac{x \exp\left[-\frac{(x - U'\tau)^2}{4K\tau} \right]}{\sqrt{4\pi K\tau^3}}}_{=c_0(x,\tau)} = \frac{x}{\sqrt{4\pi K \tau^3}} \exp\left[-\frac{x^2}{4K\tau} + \frac{U'x}{2K} - \frac{(U')^2}{4K}(1 + 4/Je')\tau \right] \quad \textbf{(tracer age distribution)}$$

$$(29)\qquad C_\lambda(x) = \int_0^\infty c_\lambda(x,\tau)\,d\tau = \exp\left[-\frac{U'x}{2K}\left(\sqrt{1+4/Je'}-1\right)\right] \quad \textbf{(tracer concentration)}$$

$$(30)\qquad \alpha_\lambda(x) = \int_0^\infty \tau\, c_\lambda(x,\tau)\,d\tau = \frac{x\,C_\lambda(x)}{U'\sqrt{1+4/Je'}} \quad \textbf{(tracer age concentration)}$$

$$(31)\qquad a_\lambda(x) = \frac{\alpha_\lambda(x)}{C_\lambda(x)} = \frac{1}{\sqrt{1+4/Je'}}\frac{x}{U'} = \frac{a_0(x)}{\sqrt{1+4/Je'}} \quad \textbf{(tracer age)}$$

$$(32)\qquad \frac{m_\lambda}{m_0} = \frac{\int_0^\infty e^{-x/L}\,C_\lambda(x)\,dx}{\int_0^\infty e^{-x/L}\,C_0(x)\,dx} = \frac{2}{(1-Pe)+(1+Pe)\sqrt{1+4/Je'}}$$

$$(33)\qquad \mu_\lambda(\tau) = \frac{m_0}{m_\lambda}\,e^{-\lambda\tau}\,\mu_0(\tau) = \frac{(1-Pe)+(1+Pe)\sqrt{1+4/Je'}}{2}\,e^{-\lambda\tau}\overbrace{\left[\sqrt{\frac{K}{\pi L^2\tau}}\exp\left(-\frac{U'^2\tau}{4K}\right)+\frac{1}{\theta}\left[1+\mathrm{erf}\left(\frac{1}{\theta}\sqrt{\frac{L^2\tau}{K}}\right)\right]\exp\left(-\frac{U\tau}{L}\right)\right]}^{=\mu_0(\tau)}$$

$$= \frac{(1-Pe)+(1+Pe)\sqrt{1+4/Je'}}{2}\left[\sqrt{\frac{K}{\pi L^2\tau}}\exp\left[-\frac{U'^2(1+4/Je')}{4K}\,\tau\right]+\frac{1}{\theta}\left[1+\mathrm{erf}\left(\frac{1}{\theta}\sqrt{\frac{L^2\tau}{K}}\right)\right]\exp\left[-\frac{U(1+Pe/Je)}{L}\,\tau\right]\right]$$

(global tracer age distribution)

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