

SPARC: A new semiparametric correlation model

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We propose a new semi-parametric model for correlation dynamics in asset returns that nests the popular constant conditional correlation model of Bollerslev (1990). It is commonly found that correlations in financial markets are not constant over time. Models for dynamic correlations have been proposed recently, e.g. by Engle (2002). In particular, there is a growing literature reporting asymmetries in correlations, where correlations increase in bear markets but remain stable in bull markets, see e.g., Ang and Chen (2002), Butler and Joaquin (2002), Campbell et al. (2002), Cappiello et al. (2003) and Longin and Solnik (2001). Such asymmetries would have major consequences for risk management, as benefits from diversification seem to melt down when they are mostly needed. Our model can detect asymmetries in correlations with respect to a common factor, for example the lagged market return or volatility. Unlike many competing multivariate GARCH models (see Bauwens et al., 2005 for a review), it does not suffer from the curse of dimensionality.

The model

Suppose we observe an N -dimensional time series y_t which could be for example the returns on various stocks. For financial returns, the conditional mean is often found to have much less structure than the conditional second moments, so that we set the conditional mean to zero for simplicity. The conditional variances h_{it} of the components of y_t are often successfully modelled by univariate GARCH type models. We follow this approach and collect the conditional standard deviations, parameterized by the vector θ , in the matrix $D_t(\theta) = \text{diag}(\sqrt{h_{1t}}, \dots, \sqrt{h_{Nt}})$.

We now propose the following semi-parametric model:

$$(1) \quad y_t = D_t(\theta)\varepsilon_t,$$

where ε_t is a vector error term with conditional mean zero, conditional variances equal to one, and with correlation matrix

$$(2) \quad \mathbb{E}[\varepsilon_t \varepsilon_t' \mid x_t = x] = R(x),$$

where x_t is an observed variable like for example the market return at time $t - 1$. Assuming that we have a $\sqrt{T}$ -consistent estimator of θ , which we denote by $\hat{\theta}$, the estimated residuals are then defined by $\hat{\varepsilon}_t = D_t(\hat{\theta})^{-1}y_t$. These residuals are used in the second stage to estimate $R(x)$ nonparametrically, as

$$(3) \quad \hat{R}(x) = \hat{Q}^*(x)^{-1} \hat{Q}(x) \hat{Q}^*(x)^{-1}$$

where

$$(4) \quad \hat{Q}(x) = \frac{\sum_{t=1}^T \hat{\varepsilon}_t \hat{\varepsilon}_t' K_h(x_t - x)}{\sum_{t=1}^T K_h(x_t - x)}$$

where $K_h(\cdot) = (1/h)K(\cdot/h)$, K is a kernel function and $h > 0$ a bandwidth. The matrix $\hat{Q}^*(x)$ is diagonal with the square roots of the diagonal elements of $\hat{Q}(x)$ on its diagonal. Hence the

(i, j) -th element of $\widehat{R}(x)$ in (3) can be written as

$$\widehat{R}_{ij}(x) = \frac{\widehat{Q}_{ij}(x)}{\sqrt{\widehat{Q}_{ii}(x)\widehat{Q}_{jj}(x)}}.$$

The estimator $\widehat{R}(x)$ in (3) is essentially a transformed Nadaraya-Watson estimator applied to the elements of the matrix $\widehat{\varepsilon}_t \widehat{\varepsilon}_t'$. Of course, other nonparametric estimators such as local polynomials could be used as well, see e.g. Härdle and Linton (1994) for a review. Note that (4) does not suffer from the curse of dimensionality as it basically involves a univariate nonparametric regression applied to each component of the correlation matrix. Hence, estimation time is usually very short even in high dimensions, compared with parametric models where ill-conditioned likelihood functions are maximized. Using the same bandwidth for all components ensures that the estimate is positive definite. Note also that generalizations of the simple SPARC model to allow for more than one factor are possible but not pursued here.

To alleviate the problem of data sparsity in the tails of the density of x_t we suggest to use local bandwidths. In particular, we allow h to depend on the design density $f(x)$ such that $h(x) = bf(x)^{-a}$, where b is a positive constant and $0 \leq a \leq 1$. This can be viewed as encompassing a number of nonparametric estimators such as splines ($a = 1/4$) and nearest neighbor estimators ($a = 1$), see Jennen-Steinmetz and Gasser (1988).

Using the notation $\eta_t = \text{vech}(\varepsilon_t \varepsilon_t')$ and $r(x) = \text{vech}(R(x))$, we have

$$(5) \quad r(x) = \mathbb{E}[\eta_t \mid x_t = x]$$

$$(6) \quad \mathbb{V}(\eta_t \mid x_t = x) = \mathbb{E}[(\eta_t - r(x))(\eta_t - r(x))' \mid x_t = x]$$

Also, denote by $\widehat{f}(x)$ an estimator of $f(x)$, the density of x_t . As shown by Hafner et al. (2005), under $b \rightarrow 0$ as T increases such that $Tb^5 \rightarrow 0$ and $Tb^3 \rightarrow \infty$ and a few regularity conditions, it holds that

1. The diagonal elements of the matrix $\widehat{Q}$ in (4) converge in probability to 1:

$$\widehat{Q}_{ii}(x) \xrightarrow{p} 1, \quad i = 1, \dots, N.$$

2. The estimator $\widehat{R}(x)$ in (3) is consistent and asymptotically normal:

$$\sqrt{Tb}(\widehat{r}(x) - r(x)) \xrightarrow{\mathcal{L}} N(0, \Sigma(x))$$

where

$$\Sigma(x) = \frac{\int K^2(u) du}{\widehat{f}(x)^{1-a}} \mathbb{V}(\eta_t \mid x_t = x)$$

3. The asymptotic covariance matrix $V(x)$ can be estimated consistently: $\widehat{\Sigma}(x) \xrightarrow{p} \Sigma(x)$, where

$$\widehat{\Sigma}(x) = \frac{\int K^2(u) du}{\widehat{f}(x)^{1-a}} \frac{\sum_{t=1}^T (\eta_t - \widehat{r}(x))(\eta_t - \widehat{r}(x))' K_h(x_t - x)}{\sum_{t=1}^T K_h(x_t - x)}$$

We call our model SPARC, a short for semiparametric correlations. In the following we present an application to real data.

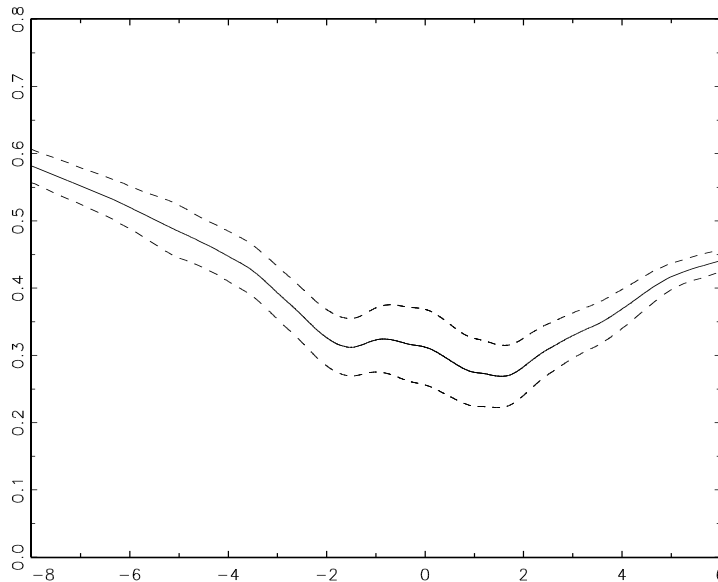


Figure 1: *Estimated correlation function for the pair (AT&T, Merck) using the SPARC model with local bandwidth ($a = 0.75$). The abscissa is the lagged weekly market return. The function represents the ‘typical’ shape. Pointwise 95% confidence bands are included.*

An empirical application

We apply the SPARC model to daily returns of the 30 stocks that constitute the Dow Jones Industrial Average (DJIA) between November 1999 and April 2004. The sample period covers the decade 1989-1998 (2528 observations). Hafner et al.(2005) provide a more exhaustive empirical analysis by considering an additional out-of-sample period for which alternative models are compared in terms of various criteria such as the accuracy to measure Value-at-Risk and the conditional variance of selected portfolios. For the common factor x_t we use here the one-day lagged weekly return on the DJIA. Hafner et al. (2005) also consider lagged market volatility as a factor and find increasing correlations in states of high market volatility.

Figure 1 shows an example of the altogether 435 correlation functions, the correlation between AT&T and Merck as a function of the factor. It represents the ‘typical’ shape. Pointwise 95% confidence bands are included. The estimation results suggest that correlations are not constant and, typically, are skewed so that indeed correlations increase in bear markets. However, this seems to hold for most correlations but not for all of them.

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RÉSUMÉ

Nous proposons un nouveau modèle semi-paramétrique de corrélations dynamiques qui emboîte le modèle populaire de Bollerslev, 1990, de corrélations conditionnelles constantes (CCC). L'évidence empirique montre que les corrélations sur les marchés financiers ne sont pas constantes. Des modèles de corrélations dynamiques ont été proposés récemment, p.e. Engle (2002). En particulier, une littérature croissante trouve des asymétries entre des corrélations et la direction du marché : des corrélations augmentent quand le marché est en baisse mais restent stables en hausse, voir Ang and Chen (2002), Butler and Joaquin (2002), Campbell et al. (2002), Cappiello et al. (2003) and Longin and Solnik (2001). De telles asymétries auraient d'importantes conséquences pour la gestion de risque, parce que les gains de diversification diminuent alors qu'il en faudrait plus. Notre modèle est capable de découvrir les asymétries de corrélations par rapport à un facteur commun, par exemple le rendement retardé du marché. Au contraire de la plupart des modèles GARCH multivariés (voir Bauwens et al, 2005, pour un survol), ce modèle reste facilement estimable quand la dimension du système est grande.